UNIVERSITY OF CALIFORNIA,
IRVINE

Motion of ice sheets at their margins: Modeling studies

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Earth System Science

by

Daniel Seneca Lindsey

Dissertation Committee:
Professor François Primeau, Chair
Professor Eric Rignot
Professor John Lowengrub

2015
DEDICATION

To my dear family Lynn, Mike, Krisean, and Calem. Your love, support, and endless encouragement have propelled me to such great heights. To my dear friend Yuting, the depth of your love, patience, and kindness is a beautiful wonder, thank you for everything.
TABLE OF CONTENTS

LIST OF FIGURES vi
LIST OF TABLES ix
ACKNOWLEDGMENTS x
CURRICULUM VITAE xi
ABSTRACT OF THE DISSERTATION xii

1 Introduction 1

1.1 Mechanisms resulting in dynamic motion of ice 3
1.2 Ross ice shelf 6
1.3 Prognostic modeling of ice shelves and their calving fronts 8
1.4 Potential for melange to affect calving front dynamics 10
1.5 Thesis contents 10

2 Methods for temporally evolving calving fronts 12

2.1 Introduction 12
2.2 Model equations 15
2.2.1 Calving-front treatment and initial ice-shelf geometry 15
2.2.2 Stress-equilibrium equations 17
2.2.3 Calving 20
2.2.4 Tracking the calving front: Level-set method 21
2.2.5 Mass continuity 22
2.3 Numerical treatment 23
2.3.1 Mesh 24
2.3.2 Initial geometry 24
2.3.3 Stress-equilibrium equations 26
2.3.4 Level-set method 29
2.3.5 Mass continuity 33
2.4 Numerical results 34
2.4.1 Front-motion accuracy - 1D experiment 35
2.4.2 Steady-state front 38
2.5 Discussion 42
2.6 Conclusions ................................................................. 44

3 Evolution of the Ross ice shelf calving front ........................................ 46
3.1 Introduction .................................................................. 46
3.2 Methods ..................................................................... 50
  3.2.1 Domain .................................................................. 51
  3.2.2 Mesh .................................................................... 51
  3.2.3 Initial condition, boundary conditions, ice hardness, and mass continuity 52
  3.2.4 Time step ............................................................... 55
3.3 Results ....................................................................... 55
  3.3.1 Diagnostic solution .................................................. 55
  3.3.2 Prognostic evolution of the Ross ice shelf ....................... 61
3.4 Discussion .................................................................... 62
3.5 Conclusions .................................................................. 65

4 Potential importance of melange for calving-front dynamics ................. 67
4.1 Introduction .................................................................. 67
4.2 Model description ................................................................ 70
4.3 Experiments and results .................................................. 79
4.4 Sensitivity analyses ........................................................ 83
  4.4.1 System sensitivity to material properties: $B_i$, $B_m$, $n$, and $B_e$ .... 84
  4.4.2 System sensitivity to embayment geometry properties: $L$ and $W$ .... 93
4.5 Discussion .................................................................... 99
4.6 Conclusions .................................................................. 101

5 Thesis conclusions .................................................................. 102
5.1 Principal conclusions ....................................................... 104
  5.1.1 Ice shelf calving front motion methods ................................ 104
  5.1.2 Simulation of the Ross ice shelf ...................................... 105
  5.1.3 Examination of potential importance of melange on calving front dynamics ................................................. 105
5.2 Future work .................................................................... 106
  5.2.1 Ice shelf calving front motion methods ............................ 106
  5.2.2 Simulation of the Ross ice shelf ...................................... 107
  5.2.3 Potential impact of melange on ice shelf dynamics .............. 108

Bibliography ........................................................................ 109

A Derivation of stress-equilibrium equations ............................................ 116
  A.1 Stress, strain, and stokes .................................................. 117
  A.2 Ice at the surface, ice at the base, and ice in-between ................... 117
    A.2.1 Surface boundary condition ........................................ 118
    A.2.2 Basal boundary condition ........................................... 119
    A.2.3 Depth integration ..................................................... 120
  A.3 Boundary conditions ..................................................... 125
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.3.1 Dirichlet condition</td>
<td>125</td>
</tr>
<tr>
<td>A.3.2 Neumann condition</td>
<td>125</td>
</tr>
<tr>
<td>A.4 Ice-ocean boundary</td>
<td>127</td>
</tr>
<tr>
<td>B Application of ocean pressure</td>
<td>129</td>
</tr>
<tr>
<td>B.1 Dynamic and glaciostatic components of ice pressure</td>
<td>130</td>
</tr>
<tr>
<td>B.1.1 Dynamic component of ice pressure</td>
<td>130</td>
</tr>
<tr>
<td>B.1.2 Glaciostatic component of ice pressure</td>
<td>131</td>
</tr>
<tr>
<td>B.2 Ocean pressure</td>
<td>131</td>
</tr>
<tr>
<td>C Kinematic boundary condition</td>
<td>133</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1.1</td>
<td>Location of the ANDRILL-1B sediment core near Ross island in the North West corner of the Ross ice shelf. Antarctic ice extent is presented at right ((Pollard and DeConto, 2009)). Image from ((McCay et al., 2012)).</td>
</tr>
<tr>
<td>2.1</td>
<td>Plan view and flow-line schematic diagrams for model domain. a) Plan-view diagram represents model domain with (\Gamma) the calving-front location separating ice shelf and ocean. b) Flow-line view of ice shelf and ocean.</td>
</tr>
<tr>
<td>2.2</td>
<td>Boundary conditions for depth-averaged stress-equilibrium equations. Boundaries ((5)) and ((6)) vary in time with boundary ((5)) covering the ocean/rock section and boundary ((6)) covering the ice/rock section.</td>
</tr>
<tr>
<td>2.3</td>
<td>Flow-line view schematic diagram: (a) Vertical-cliff ice-shelf. (b) Numerical treatment of calving front transition zone. As we decrease (2l), the length over which we transition from ice shelf to ocean, we approach a vertical face.</td>
</tr>
<tr>
<td>2.4</td>
<td>Flow-line schematic-diagram: Numerical treatment of calving-front transition-zone. As we decrease (2l), the length over which we transition from ice shelf to ocean, we approach a vertical face.</td>
</tr>
<tr>
<td>2.5</td>
<td>Convergence of numerical boundary condition to theoretical boundary condition with decreasing transition length (2l). A transition-zone length of 10 km has (\sim 3%) relative error.</td>
</tr>
<tr>
<td>2.6</td>
<td>A schematic mesh with (\Gamma) indicated. Open and closed black circles represent nodes within the NB, red circles represent contour points, ((x_c, y_c)), as computed in the algorithm. Closed circles represent nodes used to calculate contour points.</td>
</tr>
<tr>
<td>2.7</td>
<td>Ice shelf geometry and velocity field from initial starting geometry at (t = 0) years to (t = 300) years with front positions given for each 50 year increment. The green curve shows numerical results and the magenta curve shows the analytic values. The blue line is the location of the numerical calving-front-location, (\Gamma).</td>
</tr>
<tr>
<td>2.8</td>
<td>Convergence of calving to steady state from initial configuration over 1600 years for 1 km resolution ((2l = 10 \text{ km})). The calving-front location, (\Gamma), of the level-set function (\phi) is plotted. The dashed black line represents the embayment wall.</td>
</tr>
</tbody>
</table>
Schematic plan-view image showing how an ice parcel near the embayment wall shears after an increment of time. There is extension in one principal direction with $\dot{\epsilon}_+ > 0$. Since there is compression in the second principal direction then $\dot{\epsilon}_-$ is negative and calving shuts off for this particular calving treatment.

Domain image for the initial principal stress $\dot{\epsilon}_-$ that controls calving-front location.

Convergence of calving-front location to steady state from initial configuration over 1600 years for 0.5 km resolution ($2l = 5$ km). The dashed black line represents the embayment wall.

The Ross ice-shelf situated in Antarctica (Bentley, 1984).

Observed thickness field.

Initialized thickness field after running model for 200 years without calving. We’ve reset the calving front to an initial front position.

Relative error between initialized thickness and observed thickness.

Observed speed.

Initial numerical speed.

Relative error between numerical and observed speed.

Initial principal stress field $\dot{\epsilon}_+$. 

Initial principal stress field $\dot{\epsilon}_-$. 

Evolution of the calving front through time.

(a) Modeled plan-view ice shelf. Only half of ice shelf is modeled due to symmetry. (b) Side-view of model ice shelf. (c) Numerical treatment of calving front and melange showing smooth transition zone of tunable length $2l$.

(a) Plan-view schematic diagram with appropriate stress-equilibrium boundary-conditions for each boundary. (b) Boundary conditions contained within the depth-averaged stress-equilibrium equations where $T$ is the stress tensor, $n_b$ is the outward pointing normal at the base of the ice shelf, and hydrostatic equilibrium has been assumed.

Convergence of calving-front boundary-condition with transition width $2l$. Resolution is fixed for all transition widths at maximum triangular edge length 50 m (72,000 triangles with $\sim$62,000 nodes).
4.4 Thickness field, magnitude of flow, and x-directed longitudinal strain-rate. (a) From left to right: Uniform ice-shelf thickness with transition zone over 2 km from 15 km to 17 km along x-axis to uniform melange-thickness. In this case, the experiment corresponds to “no melange” with thickness of 1 cm. (b) From left to right: Inlet flow smoothly tapers in the y-direction from maximum at the centerline to minimum at embayment wall. In the x-direction, the flow accelerates to a maximum at centerline calving-front. The flow then slows as thickness is near zero and shear at embayment wall continues to slow ice. (c) From left to right: At the inlet, longitudinal strain-rate adjusts to kinematic boundary-condition. Away from the inlet, positive strain-rate increases up to a maximum at calving front. As thickness decreases toward minimal value, strain rate is negative due to shearing at wall. Strain rate is zero at far-right boundary as the stress-free condition dictates. 78

4.5 Strain rate sampled at \( x = 15 \) km and \( y = 3 \) km for various melange thicknesses. 81

4.6 Velocity and calving rate for a range of melange thicknesses. 82

4.7 Melange-induced back-stress of centerline calving-front. 83

4.8 Strain rate plotted as a function of ice hardness \( B_i \) and melange thickness. 85

4.9 Velocity minus calving rate plotted as a function of ice hardness \( B_i \) and melange thickness. 86

4.10 Strain rate plotted as a function of melange hardness \( B_m \) and melange thickness. 87

4.11 Velocity minus calving rate plotted as a function of melange hardness \( B_m \) and melange thickness. 88

4.12 Strain rate plotted as a function of melange-region flow-exponent \( n \) and melange thickness. 89

4.13 Velocity minus calving rate plotted as a function of melange-region flow-exponent \( n \) and melange thickness. 90

4.14 Strain rate plotted as a function of shear-zone hardness \( B_e \) and melange thickness. 92

4.15 Velocity minus calving rate plotted as a function of shear-zone hardness \( B_e \) and melange thickness. 93

4.16 Strain rate plotted as a function of length over which melange experiences shear \( L \) and melange thickness. 94

4.17 Velocity minus calving rate plotted as a function of length over which melange experiences shear \( L \) and melange thickness. 95

4.18 Strain rate plotted as a function of ice-shelf half-width \( W \) and melange thickness. 96

4.19 Velocity minus calving rate plotted as a function of ice-shelf half-width \( W \) and melange thickness. 97

4.20 Relative error for a range of \( W \) values. Each line corresponds to a position along the calving front from embayment wall to centerline. 98
**LIST OF TABLES**

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Model parameters and descriptions</td>
<td>16</td>
</tr>
<tr>
<td>2.2</td>
<td>Variables and base parameter values</td>
<td>41</td>
</tr>
<tr>
<td>4.1</td>
<td>Base parameter values used for idealized JI.</td>
<td>79</td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENTS

I would like to thank my thesis committee: François Primeau, Eric Rignot, John Lowengrub, and prior member Todd K. Dupont for all their help in these past years. François Primeau’s intuitive questions always quickly get to the heart of the scientific problem and his writing feedback, particularly for the melange chapter, greatly enhanced my writing skills and the quality of this dissertation. I thank my previous Ph.D. advisor Todd K. Dupont for advising me from 2008-2013. His drive to understand and validate all aspects of a glacier model has enabled me to develop a deeper and more intuitive understanding of model mechanics. I will always be thankful for his encouragement to take a holistic approach to the Ph.D., making sure to balance research with self-care. I would also like to thank John Lowengrub for his support and direction in implementing the level-set method and diffuse-domain method. His feedback has been instrumental in helping me develop my ice shelf model with time evolving calving front. He has always made himself available to discuss my latest results, usually within a few days of my request to meet. It’s been a pleasure to work with John.

The Cryosphere group at UCI is an active and growing research team, it’s been exciting to see this group grow throughout these past few years into such a diverse and talented team. I thank Erkut Aykutlug for many insightful conversations about glaciers and for feedback on research. I thank Yun Xu for always being available to talk about glacier flow and model equations. I also thank Mathieu Morlighem and the ISSM team for providing velocity data, thickness data, and the initial mesh for the Ross ice shelf.

I cannot speak high enough praise for the department of Earth System Science. Cynthia Dennis is wonderful. She is always available to talk with graduate students and goes out of her way to follow up on issues brought to her attention. I thank Morgan Sibley for all her help, especially in the last year as I’ve sought to wrap up my degree. All of the ESS department staff have been very helpful. John V. Croul hall is a beautiful place to work, we are truly lucky to have such a building. I would also like to acknowledge the faculty for their engagement and interest in graduate student well being. Their active support of the graduate student retreat and their responsiveness to feedback on the first year curriculum has been wonderful. I would like to thank my cohort and all ESS graduate students for being a truly great group of people.

I would like to thank my friends for all their support during this long process. Alex has been a great friend and I’m thankful for our many enlightening conversations. I never would have even applied for this Ph.D. program without his encouragement. Cameron has been instrumental in my pursuit of being a present, wise, and more caring human being. More than that he has been an important factor in developing my critical thinking skills and, just as importantly, my dance moves.

This work was performed at the University of California Irvine with support from the National Science Foundation grant ANT-0809106 and National Aeronautics and Space Administration grants NNX08AQ82G and NNX09AW02G.
CURRICULUM VITAE

Daniel Seneca Lindsey

EDUCATION

Doctor of Philosophy in Earth System Science 2015
University of California, Irvine Irvine, CA

Master of Science in Earth System Science 2011
University of California, Irvine Irvine, CA

Master of Science in Mathematics 2006
University of California, Riverside Riverside, CA

Bachelor of Science in Mathematics 2004
University of California, Los Angeles Los Angeles, CA

Minor in Teaching English as a Second or Other Language 2004
University of California, Los Angeles Los Angeles, CA

RESEARCH EXPERIENCE

Graduate Research Assistant Winter 2012 – Winter 2015
University of California, Irvine Irvine, California

Graduate Research Assistant Summer 2011
University of California, Irvine Irvine, California

Graduate Research Assistant Winter 2011
University of California, Irvine Irvine, California

Graduate Research Assistant Summer 2008 – Summer 2010
University of California, Irvine Irvine, California

TEACHING EXPERIENCE

Teaching Assistant Fall 2012
University of California, Irvine Irvine, CA

Teaching Assistant Spring 2011
University of California, Irvine Irvine, CA

Teaching Assistant Fall 2010
University of California, Irvine Irvine, CA
Ice sheets interact dynamically with the ocean, losing mass through ocean-induced melt and the calving of icebergs. At ice sheet margins, ice shelves buttress flow of tributary glaciers making them a critical component of the overall ice sheet stress balance and, consequently, mass budget. With retreat of ice shelves seen in both Antarctica and Greenland, accurate modeling of ice shelf motion is important for forecasting future sea-level rise. Presently, most glacier models leave the calving front fixed in time with the implicit assumption that flow at the calving front is balanced by calving resulting in a stationary front. Numerically evolving a calving front is challenging in two ways, the first is due to the difficulty in applying the correct calving front boundary condition as the front shifts in time and the second lies in utilizing an appropriate numerical treatment for the calving process. Many factors influence calving including ocean-induced melt, surface melt-water hydrofracturing of crevasses, and wave energy triggering calving events. Of importance at some Greenland glaciers, melange, a mixture of icebergs and sea ice, can act to suppress calving rates by either preventing calved bergs from rotating away from the front or directly buttressing front strain rates.

In this dissertation we utilize two mathematical techniques, the level-set method and diffuse-domain method, to develop an ice shelf model that allows for the temporal evolution of a calving front. Our ice-shelf model allows for mesh independent enforcement of the appro-
ropriate boundary condition thus allowing for continuous motion of the front within mesh elements. We validate this model in both a 1D domain and 2D idealized domain. We then apply these front-motion methods to the evolution of the Ross ice shelf and examine characteristics of flow through time. Our model captures key features of ice-shelf flow and produces a smooth evolution of the front. Finally, we use our ice shelf model to examine the potential for melange to buttress ice shelf calving front strain rates. We find that melange, if rheologically strong enough, of 100 m thickness shearing over 1 km can initiate a calving front advance of 490 m/a.
Chapter 1

Introduction

The Earth’s ice sheets serve as massive reservoirs of fresh water that also contain features that inform us about past climate including the presence of trapped air bubbles, crystal structure that tells us about past accumulation/ablation rates, and oxygen isotopes. At present, the Antarctic ice sheet contains a volume equivalent of roughly 58 m of sea-level rise with surface area (14 million km$^2$) larger than that of the United States and Mexico combined (Fretwell et al., 2013) while the Greenland ice sheet contains roughly 6 m of sea-level rise equivalent with surface area (1.7 million km$^2$) just smaller than Mexico (Bamber and Layberry, 2001). The ice sheets exchange mass with the Earth system, affecting ocean currents globally by facilitating formation of Antarctic deep water (e.g., Goldner et al., 2014) and modulating the supply of Greenland melt water to the Atlantic Meridional Overturning Circulation (e.g., Weijer et al., 2012). Ocean currents in turn significantly affect the mass budget of the Antarctic ice sheet (e.g., Rignot et al., 2013) and Greenland ice sheet (e.g., Holland et al., 2008; Rignot et al., 2010) through oceanic-induced melt at ice shelves and tidewater glaciers.
Mechanisms for ice sheet volume loss can be placed into three primary categories, the first is the ablation of mass at the ice sheet surface, second is the direct loss of mass through ocean-induced melt, and third is the calving of bergs, from relatively small bergs at tidewater glaciers to large tabular bergs characteristic of ice shelf calving. In addition, dynamic processes often serve to accelerate glacier flow and enhance loss via one of these mechanisms. All three modes of loss are significant in Greenland with \( \sim 60\% \) of total loss due to ocean-induced melt and calving while \( \sim 40\% \) of loss is due to surface ablation (Rignot and Kanagaratnam, 2006). The Antarctic ice sheet, on the other hand, loses little mass to surface melt and is primarily vulnerable to dynamic-enhanced loss through ocean-induced melt and calving. In seeking accuracy in forecasting future sea-level rise, dynamic processes that accelerate loss of ice remain the principal uncertainty (Lemke et al., 2007). Dynamic processes are an active area of research that includes investigation of surface-melt-water enhanced flow in Greenland (Zwally et al., 2002), rapid break up of shelves in Antarctica associated with the presence of surface crevasses (Scambos et al., 2000) and basal crevasses (McGrath et al., 2012; Bassis and Ma, 2015), reduced buttressing due to reduction/loss of shelves in Greenland (Thomas, 2004) and Antarctica (e.g., Rignot et al., 2004; Scambos et al., 2004), direct thinning and acceleration of tidewater glaciers in Greenland due to ocean-induced melt (e.g., Holland et al., 2008; Rignot et al., 2010), grounding line retreat through reverse-bed-slope instability (e.g., Weertman, 1957), and the sensitivity of the calving front to proglacial melange, a mixture of ice bergs and sea ice, both in terms of the ability of melange to restrain calved bergs from rotating away from the front (Amundson et al., 2010) and direct buttressing of strain rates and flow (Walter et al., 2012).

In this chapter we describe some of the important dynamic processes that affect ice sheet volume loss. We then discuss, in preparation for outlining the contents of this dissertation, current methods utilized for evolving 2D plan-view calving front motion, followed by background of the Ross ice shelf, and lastly discuss melange and its potential impact on glacier dynamics. Finally, we outline the contents of the thesis and discuss some of the key results.
1.1 Mechanisms resulting in dynamic motion of ice

Surface melt-water in Greenland has been shown to drain through a kilometer of ice at Swiss camp, located within the equilibrium zone, resulting in a measurable increase in surface velocity (Zwally et al., 2002). These accelerations are seasonal in nature and are closely correlated to drainage of surface meltwater ponds through moulins implying a lubricating effect at the bed. Observations show that the pre-drainage velocity accelerates as surface water drains and lubricates the bed. Once the water at the bed flushes downstream a lack of lubrication results in a slowdown of ice below the pre-drainage velocity before returning to normal. Seasonal acceleration of flow was overall found to be no greater than 8-10% of annual flow in Greenland indicating an overall small effect (Rignot and Kanagaratnam, 2006). It’s an open question as to whether increased melt due to rising surface temperatures will result in increased lubrication and faster glacial-flow or whether increased amounts of water draining to the bed will form more efficient drainage channels limiting the effect of enhanced flow while still resulting in associated post-drainage slow-down.

Mechanisms for shelf retreat vary between Antarctica and Greenland. In Antarctica, studies of the rapid break up of Larsen B focus on the presence of surface melt water ponding and its role in hydrofracturing and formation of crevasses that weaken the shelf. Indeed ice shelf breakup events occur during years with a longer melt season when ponding is visible (Scambos et al., 2000). Basal crevasses may also be important in weakening ice shelves by deforming the surface through alteration of local shelf buoyancy and reducing the distance that surface crevasses need to propagate before connecting to water beneath the shelf (e.g., McGrath et al., 2012). Banwell et al. (2013) suggests that the drainage of one supraglacial lake at Larsen B may have resulted in flexure in the ice shelf, triggering drainage of nearby lakes, resulting in widespread close proximity crevasses that then allowed for the disintegration of the entire ice shelf. At other Antarctic ice shelves formation of lateral fractures initiate the calving of tabular bergs such as at the Ronne ice shelf (Hulbe et al., 2010) and Ross ice shelf.
(Joughin and MacAyeal, 2005). MacAyeal et al. (2006) demonstrated that the arrival of a transoceanic wave at the Ross ice shelf resulted in the calving of a tabular berg indicating the importance of wave energy as trigger for shelf calving events. Greenlandic ice shelves are different in that they tend to be smaller in scale, fewer in number, and recessed away from open ocean in relatively narrow fjords. Ocean-induced melt is thought to be one primary factor in ice shelf retreat such as at the Jakobshavn ice shelf where the intrusion of warm ocean water led to subsequent ocean-induced melt and ice shelf retreat (Holland et al., 2008; Motyka et al., 2011).

Loss of ice shelves directly accelerate flow of tributary glaciers via loss of buttressing due to shearing along embayment walls, islands, and over ice rises. Modeling efforts have demonstrated the importance of ice shelves in increasing not only flow near the grounding line but tributary flow as well (Dupont and Alley, 2005). An examination of flow at the Ross ice shelf demonstrated the ability of the shelf to buttress flow resulting in reduced creep thinning rates both within the shelf and for inflowing ice as well (Thomas and MacAyeal, 1982). Observations after the rapid breakup of Larsen B ice shelf showed a speedup by 8 times or more for tributary glaciers (Rignot et al., 2004; Scambos et al., 2004). In Greenland, the thinning and break up of part of the Jakobshavn ice shelf correlates with observed acceleration of glacier flow (Thomas, 2004).

Ocean-induced melt at tidewater glacier calving faces is difficult to constrain due to sparse observations and challenges getting close to calving fronts to make measurements. A precursor to measurements in Greenland, melt rates were estimated at LeConte Glacier in Alaska at 12 m d\(^{-1}\) by utilizing oceanic measurements in the proglacial fjord (Motyka et al., 2003). They found that water emerging at the base of the calving front, acts as a pump that advects warm ocean water to the calving face enhancing melt. A similar process has been proposed to operate in Greenland where melt of 0.7 to 3.9 m d\(^{-1}\) was observed at several glaciers along Western Greenland (Rignot et al., 2010). At Helheim Glacier in Greenland a similar method
was employed to suggest melt rates of 1.8 md$^{-1}$ at the calving face (Sutherland and Straneo, 2012). Modeling efforts that seek to estimate ocean-induced melt rates are challenging due to the fleetingly small time steps needed to capture small scale ocean flow, however, one pioneering modeling study found that the calving face experiences the highest amount of melt just above outflowing fresh water channels. It was also found that melt rates are sensitive to both the flux of melt water and the ocean-water temperature (Xu et al., 2013). Melt rates at Antarctic ice shelves can be calculated differently since surface melt rates are small and, by estimating loss through calving and gain by surface accumulation, the resulting changes in thickness allow for quantification of ocean-induced melt. Melt rates have been estimated for ice shelves around Antarctica with the three largest ice shelves, Ross, Filchner, and Ronne, covering two thirds of total ice shelf surface area only account for 15% of total melt. Melt rates at other ice shelves, some previously undocumented, indicate large rates of melt even though their total surface area is relatively small (Rignot et al., 2013).

The slope of the bed rock upon which the glacier sits has implications for the dynamic response of a grounding line in retreat with consequences for flow. A reverse bed slope, one in which the basal elevation decreases upstream of the grounding line, is characteristic of outlet glaciers and ice streams in the West Antarctic ice sheet (WAIS) and certain portions of the East Antarctic ice sheet where the bed generally sits below sea level (Fretwell et al., 2013). This contrasts with Greenland outlet glaciers that are generally thought to have basal profiles that elevate upstream of the calving front. Of note, a new Greenland bed elevation map shows newly discovered deep glacial valleys extending back from the calving front, indicating there may be some reverse bed slope regions (Morlighem et al., 2014). Glaciers with reverse bed slope are vulnerable to what’s known as the “marine ice-sheet instability.” The instability is based on the simple idea that, for ungrounded ice, an increase in thickness leads to an increase in flow. In this way if the grounding line retreats, ungrounding thicker ice, velocity will consequently increase resulting in dynamic thinning and further grounding line retreat. Unless the grounding line can stabilize on an upstream ridge the instability will
result in a run-a-way feedback. One theoretical study of grounding line motion indicated grounding lines cannot be stable on reverse bed slopes (Schoof, 2007) while another modeling study of Thwaites Glacier in WAIS indicated unstable retreat is already underway (Joughin et al., 2014). This result is corroborated by observations at not only Thwaites glacier but Pine Island, Smith, and Kohler glaciers in WAIS where unstable retreat is underway (Rignot et al., 2014).

1.2 Ross ice shelf

The Ross ice shelf is the largest ice shelf in Antarctica covering a surface area of roughly 520,000 km$^2$, about the size of California and Maine put together (Bentley, 1984). An important component of the WAIS mass budget, the shelf serves to restrain flow of six ice streams (of which one is presently stagnant) draining ice from WAIS into the Ross sea (Thomas and MacAyeal, 1982). Given the potential for marine instability for WAIS, the future behavior of this ice shelf is of critical importance to forecasting sea-level rise in the next 90+ years. In the face of uncertain future stability, past behavior is a potential indicator of how the shelf may respond to a warming climate.

Historic shelf behavior has been studied in multiple ways on several time scales. Ice cores taken from the Ross ice shelf during the Ross Ice Shelf Geophysical and Glaciological Survey (RIGGS) during 1973-1978 tell us about shelf behavior in the recent past. Comparing ice-core internal structure to a theoretical structure derived assuming the shelf has been in steady state, Thomas and MacAyeal (1982) showed the shelf has been stable within the last 1500-2500 years. A sediment core, known as ANDRILL-1B, taken just in front of the current Ross calving front, near Ross island, gives evidence for past Ross ice shelf stability during the last 3 million years (Fig. 1.1). This is possible because sediment properties depend on whether the shelf was grounded (thickened to the point it is no longer at flotation), at flotation, or
had retreated leaving open water at the front. The ANDRILL-1B core suggests the Ross ice shelf alternated between grounded and ungrounded at least 7 times in the past 0.078 million years in response to Milankovitch orbital forcing (McCay et al., 2012). During the Pliocene period ∼3 Ma when atmospheric CO₂ levels were ∼400 ppm and temperatures were on average ∼3°C warmer than present, the core record indicates that the Ross ice shelf had periodically collapsed (Naish et al., 2009). Furthermore, Antarctic scale modeling suggests that WAIS had collapsed along with the Ross ice shelf ∼3 Ma (Pollard and DeConto, 2009). Interestingly sea-levels during the last interglacial period ∼11.7 ka were thought to be ∼6.6 m higher than present (Kopp et al., 2009) suggesting WAIS must have at least partially collapsed, however, the sediment core during this period suggests the Ross ice shelf was at least to some extent still present (McCay et al., 2012). If collapse (or partial collapse) of WAIS necessarily results in the collapse of the Ross ice shelf then there is a discrepancy in the two records. With present day CO₂ levels at ∼400 ppm future stability of the Ross ice shelf and the mechanisms for instability need to be explored.
over the AND-1B site during glacial periods originates from East Antarctica, previous studies have shown that the distal source of this East Antarctic ice, its reconstructed geometry and its thickness, grounded 835 m below modern sea level at the AND-1B drill site in the Ross Embayment, requires a significant contribution of ice from West Antarctica relative to the present-day (Fig. 1; Naish et al., 2009; Pollard and DeConto, 2009).

Last Glacial Maximum retreat reconstructions (Conway et al., 1999; Domack et al., 1999; Ship et al., 1999; Licht et al., 2005; Mosola and Anderson, 2006; McKay et al., 2008), mass balance considerations (Pollard and DeConto, 2009), and the overdeepened/reverse slope nature of the Ross Sea continental shelf since the Late Neogene (Weertman, 1974; De Santis et al., 1995) indicate that the record of grounded ice sheet deposition at the AND-1B drill site is expected to be intimately related to the overall state of past marine-based ice sheets in the Ross Embayment. That is, an ice sheet that is fed by a contribution from both the East and West Antarctic Ice Sheets, but subject to the same marine influences on its mass balance as the wider West Antarctic Ice Sheet while the high-altitude regions of the East Antarctic Ice Sheet remain relatively stable. This is because air temperatures never become warm enough to cause significant surface melting on the East Antarctic Ice Sheet, whereas variations in ocean-induced melt and sea level affect the marine-based West Antarctic Ice Sheet much more than the East Antarctic Ice Sheet.

This implies that once retreat was initiated for past configurations of these marine-based ice sheets, it was likely to have occurred within the deep basins across the Ross Embayment in patterns similar to those hypothesized for the last glacial/interglacial cycle. (e.g. Conway et al., 1999; Denton and Hughes, 2000; Mosola and Anderson, 2006). The presence of open water deposits at the AND-1B drill site indicates the ice shelf calving-line was south of its present position, and therefore, the ice sheet was at least partially reduced in extent, but this is not evidence of complete collapse of the West Antarctic Ice Sheet. Nevertheless, well-dated periods of open water at the AND-1B drill site during the last 5 Ma (Naish et al., 2009) all correspond to times of partially to completely deglaciated West Antarctic Ice Sheet based on model simulations (Pollard and DeConto, 2009). Therefore, in the context of documenting past periods of partial or complete collapse of the West Antarctic Ice Sheet during the Pleistocene, open water conditions at the AND-1B drill site are required. In this paper, we identify such deposits in AND-1B and examine these in the context of the hypothesized collapses of the marine-based sectors of the West Antarctic Ice Sheet, as inferred by far-field sea level records.

![Figure 1.1: Location of the ANDRILL-1B sediment core near Ross island in the North West corner of the Ross ice shelf. Antarctic ice extent is presented at right (Pollard and DeConto, 2009). Image from (McCay et al., 2012).](image)

1.3 Prognostic modeling of ice shelves and their calving fronts

A wide array of models simulate the evolution of ice shelves, from ice sheet scale to process-level regional-scale models. In many of these models the calving front remains fixed in time, essentially assuming that the calving rate balances flow at the front resulting in a stationary front. Given the importance of evolving calving fronts to dynamics of both ice
shelf and tributary glacier flow, ideally these calving fronts would be allowed to evolve in time. There are two primary reasons many models do not include motion of the front. The first reason is that the physical processes that control calving are very difficult to distill into a parameterization for use in a model. Calving depends on a wide variety of physical properties including rates of ocean melt, vertical distribution of melt rates at the front, preexisting damage in the ice, and the availability of surface water to promote crevasse propagation (Benn et al., 2007a). Present dynamic calving parameterizations are generally first order approximations dependent on strain rates, the first order control on calving (Alley et al., 2008; Amundson and Truffer, 2010; Levermann et al., 2012; Benn et al., 2007b).

The second reason is that in order to simulate the motion of an ice shelf, the correct calving front boundary condition needs to be applied through time. The calving front experiences an outward pointing stress with both a glaciostatic pressure component and dynamic component while experiencing a hydrostatic pressure from the ocean. These two stresses are not equal and the result is an acceleration of ice to the calving front. The biggest numerical challenge when evolving the calving front lies in the application of the ice-ocean stress-imbalance (SI) at the calving front. The boundary condition is usually applied at a set of specified nodes, but for an evolving front, the location of the calving front will most likely fall between nodes, making application of the SI condition less than straightforward. One solution to this issue was found by using a sub-grid-scale parameterization to account for the SI condition when the front falls between nodes (Winkelmann et al., 2011) while also conserving mass (Albrecht et al., 2011) for a fixed Cartesian grid in the Antarctic ice-sheet model PISM-PIK. Their sub-grid-scale parameterization of front motion tracks the amount of mass lost or gained when the front falls between nodes, occasionally reintroducing mass when the front advances from one element to the next. Their SI condition accounts for the front falling between nodes but still applies this condition at nodes, resulting in a jump in dynamics as the front shifts in time.
1.4 Potential for melange to affect calving front dynamics

In Greenland, the presence of melange has been correlated to dynamic changes in glacier behavior at Helheim, Kangerdlugssuaq, Store, and Jakobshavn Isbræ. Observations at Jakobshavn Isbræ between 2004 and 2007 demonstrate the presence of a “rigid” melange in winter months correlated to near cessation of calving events, an advance of the calving front, and corresponding slow down of glacial ice (Joughin et al., 2008b). More recently, observations at Store glacier found the presence of melange directly buttressed glacial flow (Walter et al., 2012). One theoretical study indicates that melange may play a role in preventing calved bergs from rotating away from the front (Amundson et al., 2010). Ultimately the presence of melange may have multiple roles in affecting ice dynamics including dampening wave energy that may play a role in triggering calving events, preventing calved bergs from rotating away from the calving front, and the direct buttressing of flow.

1.5 Thesis contents

The remainder of the thesis is organized into three chapters followed by concluding remarks in chapter 5. Chapter 2 outlines numerical techniques used to evolve an ice shelf calving front. These techniques, known as the level-set method and diffuse-domain method, are novel in that they allow for node independent application of the SI condition while also accounting for mass with no special procedure needed to reintroduce mass to the system. Furthermore, these methods are applicable across different mesh constructions including Cartesian, triangular, and hexagonal constructions. These methods are validated by application in idealized 1D and 2D plan-view domains.
Chapter 3 applies these numerical methods to the evolution of the Ross ice shelf examining front motion through time. We explore the characteristics of front evolution as well as discuss the potentiality of achieving a steady state calving front similar to the present location. We also discuss the importance of capturing the correct dynamic conditions at the ice/rock interface. Our techniques allow for the smooth motion of the front through time demonstrating their potential use in the scientific community.

Chapter 4 explores the potentiality that melange may, if strong enough, suppress calving front strain rates and hence calving rates. This is the first time that melange has been modeled within the interior of the domain. Our modeling study focuses on the potential for melange to reduce calving-front strain-rates in the limiting case that melange acts as a continuous medium. This is a theoretical limit and it’s uncertain melange will ever reach this level of strength, nevertheless, this is a useful starting point. We find that a front initially advances at 490 m/a for melange of 100 m thickness shearing over a 1 km embayment indicating melange can suppress strain rates and also calving rates. Our results suggest further modeling studies of melange are warranted. Our results are consistent with the hypothesis that melange not only plays a role in preventing calved bergs from rotating away from the front but also suppresses calving rates, allowing the front to advance. A seasonal front advance, at flotation as observed at Jakobshavn Isbørn (Joughin et al., 2008b), will generate additional shear consequently buttressing flow upstream. This may provide a piece of the explanation for observed winter slowdown at Jakobshavn Isbørn. A consequence of this dynamic is that, in a warming climate, melange will be weaker, limiting its ability to suppress calving rates. With a corresponding reduction in calving-front advance, slowdown at glaciers such as Jakobshavn Isbørn may decrease resulting in increased loss of ice. Finally we conclude in chapter 5 with a summary of results, description of limitations of current work, and a discussion of future work.
Chapter 2

Methods for temporally evolving calving fronts

2.1 Introduction

Marine ice-sheets are grounded below sea level and drain into relatively fast-flowing regions of ice that often extend into the ocean to form ice shelves. In turn, ice shelves experience shear at embayment walls, islands, or grounding on bathymetric shoals, that results in reduction of velocity for tributary glaciers. The resulting buttressing effect can be large such that a loss of the ice shelf leads to rapid acceleration of flow upstream (Dupont and Alley, 2005), by up to 8 times or more as in the case of the break up of the Larsen B ice-shelf (Rignot et al., 2004; Scambos et al., 2004). Capturing the motion of an ice-shelf calving-front is an important component of forecasting future sea-level rise.

Ice shelves terminate at calving fronts where the internal stresses at the calving front exert an outward force consisting of a dynamic deviatoric-stress and glaciostatic pressure while experiencing a hydrostatic back-pressure. These are not in balance and the net effect is an
increase in stress at the calving front resulting in the promotion of fracturing and calving of bergs, an important control on front position. In order to numerically incorporate the motion of calving fronts we need to accurately capture the stress imbalance in models that incorporate ice shelves.

Current 2D plan-view models often enforce this stress imbalance (SI) at boundary nodes via use of a depth-independent Neumann condition. This works well enough for diagnostic applications, however, in prognostic models that allow for a moving calving front, applying the boundary condition becomes a challenge when the front falls between nodes. While it is possible to simply chop ice off and move the front to the nearest set of nodes or add additional nodes, the former results in the need to reintroduce the mass somehow and the latter results in formidable numerical challenges for any substantial model run. Furthermore, reassigning the nodes at which the SI condition is applied results in a jump in the stress state near the front due to changes in calving-front geometry and possible shift in embayment-induced shear. Reassigning which nodes the SI condition is being applied at results in mesh-dependent dynamics. While mesh refinement can alleviate numerical dependence, numerical results ultimately should be mesh independent. Accurate treatment of both SI and calving-front motion is a prerequisite to incorporating calving into models.

A novel solution was found for applying the SI condition (Winkelmann et al., 2011) while also conserving mass (Albrecht et al., 2011) for a fixed Cartesian grid when moving calving fronts in the Antarctic ice-sheet model PISM-PIK. Their sub-grid scale parameterization of front motion does not simply chop ice when the front falls between nodes, rather, it tracks the partial mass within a grid cell and any additional small amount of mass that is gained or lost while the front is in advance or retreat is reintroduced into the system. Their techniques are currently applied in both the development of calving treatment (Levermann et al., 2012) and finite-difference ice-sheet models based on Cartesian grids (Martin et al., 2011; Pollard and DeConto, 2012; Winkelmann et al., 2011).
Here we present a suite of techniques that, together, allow for the motion of an ice-shelf calving-front on an unstructured triangular-mesh. Our treatment of the calving front is innovative in that the SI and dynamic calving-rate are not assigned at any specific nodes, rather, they are applied implicitly at the calving front which is free to evolve between nodes. To our knowledge this is the first time node independent motion of calving fronts with applied SI has been achieved. In this way there is no mesh dependence for numerical results for either the SI condition or the dynamic calving-rate. Furthermore, we utilize the traditional mass-continuity equation allowing for a continuous loss of mass due to calving with no further accounting or redistribution of mass needed. These techniques can be broadly categorized into two primary groups: the diffuse-domain methods and the level-set methods. The former allows us to implement the appropriate thickness field, SI condition, and calving rate. The latter allows us to track the evolution of the front through time. We present these techniques in a 1D domain and idealized 2D plan-view domain. Use of two spatial-dimensions allows us to capture lateral drag at the embayment wall as well as produce interesting ice-front geometries. The lack of a vertical dimension reflects negligible vertical shear in ungrounded ice-shelves. Our model is validated in three ways: By comparison to a 1D analytic solution, comparison to 1D results in Albrecht et al. (2011), and comparison to 2D results in Levermann et al. (2012). Our work suggests the calving-front motion techniques presented here are suitable for implementation in regional-scale models developed for any fixed mesh, be it: unstructured, Cartesian, or other formulation. Direct implementation in ice-sheet models is possible, but based on our implementation, computational cost of solving the model could increase by as much as 47% per time step, making these techniques more directly suitable for smaller scale models.

The outline of the remaining paper is given as follows. In section 2.2 we present the model equations and boundary conditions and in section 2.3 present the numerical implementation. Section 2.4 is devoted to model validation. Section 2.5 follows with discussion and we finish with concluding remarks in section 2.6.
2.2 Model equations

Our goal is to discuss the theoretical underpinnings of our model and layout relevant model equations, first those necessary for the diagnostic solution of the stress-equilibrium equations followed by those necessary for temporal evolution of the ice shelf and calving front. We start by discussing two mathematical tools needed to establish the initial thickness-field for the ice-shelf/ocean system. This is followed by a discussion of the relevant stress-equilibrium equations and associated boundary conditions. After, the selection of a dynamic calving-rate is discussed. Next we discuss the level-set method responsible for motion of the calving front and, finally, the mass-continuity equation responsible for updating the thickness field, again, outlining appropriate boundary conditions. In outlining domain geometry and boundary conditions we focus on the 2D geometry since the 1D geometry is relatively straightforward. Discussion of numerical implementation follows in section 2.3.

2.2.1 Calving-front treatment and initial ice-shelf geometry

Our model includes both an ice-shelf region and ocean region within the domain (Fig. 2.1a,b). The calving-front location, \( \Gamma \), demarcates the ice-shelf region and ocean region (Fig. 2.1a). Before we discuss the initial thickness-field we need to define two tools commonly utilized in the mathematics literature: The level-set function and a variant of the heaviside-step function (Osher and Fedkiw, 2003). To initialize the ice-shelf geometry it is useful for us to define the location of the calving front as the zero contour of a function \( \phi(x, y) \), often called the level-set function. That is to say we define the calving-front location as the set, \( \Gamma \), of points \( \vec{x}_c = (x_c, y_c) \) where \( \phi(\vec{x}_c) = 0 \). Values for \( \phi \) are negative in the ice-shelf region and positive in the ocean region. For now we leave the description of \( \phi \) without further specification, however, we will define this function more completely in section 2.2.4.
when we outline methods for front motion. Needing one more tool, we define a function, \( F(\phi) = 1 - H(\phi) \), where \( H \) is the usual Heaviside-step function such that:

\[
F(\phi) = \begin{cases} 
1 & \text{for } \phi \leq 0 \text{ (ice-shelf region)}, \\
0 & \text{for } \phi > 0 \text{ (ocean region)}. 
\end{cases}
\] (2.1)

The initial thickness-field over both ice shelf and ocean, \( h \), can now be defined by simply taking the product:

\[
h = h_o F, \tag{2.2}
\]
where \( h_o \) prescribes the desired initial thickness-field for the ice shelf and the product asserts zero thickness over the ocean.

Figure 2.1: Plan view and flow-line schematic diagrams for model domain. a) Plan-view diagram represents model domain with \( \Gamma \) the calving-front location separating ice shelf and ocean. b) Flow-line view of ice shelf and ocean.

### 2.2.2 Stress-equilibrium equations

To capture ice-shelf motion we utilize the Stokes equations that govern the motion of creep flow. Ungrounded ice-shelves experience lateral and longitudinal stresses due to the ice/ocean
stressed-imbalance at the calving front and shear generated at the embayment walls but experience little vertical stress. For these reasons we follow MacAyeal (1989) in using depth-integrated equations appropriate for ice-shelf flow. For clarity and relevance to discussion of boundary conditions, we derive these equations in appendix A. Since the scope of this present work involves the motion of glacial ice, we do not attempt to capture the fluid motion of ocean water. Capturing correct ocean dynamics would require prohibitively small time steps and an expanded set of governing equations to account for effects such as turbulence and the Coriolis effect. For these reasons our strategy is not to simulate ocean currents, rather, ensure motion in this region has minimal impact on glacier dynamics. With this in mind we utilize the momentum-balance equations appropriate for ungrounded ice-shelf flow given as:

\[ \frac{\partial}{\partial x} \left( 2\bar{\nu} h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left( 2\bar{\nu} h \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) - \rho gh \frac{\partial z_s}{\partial x} = 0, \quad (2.3) \]

\[ \frac{\partial}{\partial y} \left( 2\bar{\nu} h \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial x} \left( 2\bar{\nu} h \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) \right) - \rho gh \frac{\partial z_s}{\partial y} = 0, \quad (2.4) \]

where \( u \) is the \( x \)-directed velocity, \( v \) is the \( y \)-directed velocity, \( z_s \) is the freeboard, \( \rho \) is the density of ice, and \( \bar{\nu} \) is the depth-averaged effective viscosity. We use the Glen treatment for viscosity that is dependent on the second invariant of the stress tensor (Glen, 1955; Nye, 1957).

\[ \bar{\nu} = \frac{B}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \frac{1}{4} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right]^{\frac{\nu}{\bar{\nu}}}, \quad (2.5) \]
where \( n \) is the viscosity exponent taken as the usual value of 3 and \( B \) is the depth-integrated temperature-dependent hardness parameter.

For our 2D experiments, two boundary conditions along six boundaries are necessary to solve the stress-equilibrium equations. These are illustrated and numbered in Figure 2.2. At the inlet (1), we utilize a sine function to smoothly increase velocity from embayment wall, \( u_e \), to inlet centerline-flow, \( u_{ic} \):

\[
u(y) = u_e \sin \left( \frac{\pi y}{2W} \right) + u_{ic}, \tag{2.6}\]

where \( W \) is the glacier half-width. At the centerline (2), we do not allow cross flow but otherwise enforce a stress-free condition. At the downstream ocean-boundary (3) and bottom ocean-boundary (4) we allow for free flow, that is to say, stress-free conditions are applied. Boundaries (5) and (6) require special discussion. (5) refers to the ocean/rock boundary and (6) refers to ice-shelf/rock boundary. In our model we treat these as time varying boundaries, that is to say, as the front advances a boundary node that was ocean/rock will switch to become an ice-shelf/rock node. Our decision on what boundary condition to use for (5) derives from our desire to not allow stress conditions in the ocean region to affect ice-shelf flow. If we assign a fixed Dirichlet condition for the velocity in these regions, the flow field in the ice shelf would be numerically affected as the ice-shelf flow field shifts to meet these boundary conditions. For this reason, we use a stress-free condition in this region so that ice-shelf flow will be entirely controlled by ice dynamics. Along the embayment wall (6) we assign a fixed velocity \( u_o \) with no crossflow. The ice-shelf/ocean SI condition and associated conditions are not prescribed in the usual way and are discussed in section 2.3.3.
Figure 2.2: Boundary conditions for depth-averaged stress-equilibrium equations. Boundaries (5) and (6) vary in time with boundary (5) covering the ocean/rock section and boundary (6) covering the ice/rock section.

### 2.2.3 Calving

Calving can be applied as either a sequence of discrete calving-events in which each sudden loss of ice results in an abrupt shift in the calving-front location or as the time-averaged rate of these events for which the front motion responds to calving in a continuous way. Here we choose the latter and apply calving as a time-averaged rate of change, thus, calving-front motion is simply controlled by a competition between velocity and calving rate.

Calving is handled by application of the calving rate developed in Levermann et al. (2012). The calving rate is given as:
\[ \dot{c} = K \dot{\epsilon}_- \dot{\epsilon}_+, \]  

(2.7)

where, \(\dot{\epsilon}_\pm\) are the principal strain-rates and \(K\) is a coefficient that, to first order, incorporates the aggregate processes that influence calving such as tidal motion, wave energy, preexisting damage, etc. The calving rate is positive where both \(\dot{\epsilon}_\pm > 0\) and zero elsewhere. The \(x\)- and \(y\)- components of the calving-rate vector, \(\vec{c} = [c_x, c_y]\), are given by \(c_x = \dot{\epsilon}_- \frac{\mathbf{u}}{|\mathbf{u}|}\) and \(c_y = \dot{\epsilon}_+ \frac{\mathbf{v}}{|\mathbf{u}|}\) utilizing components of the unit vector \(\hat{\mathbf{u}} = \frac{[u, v]}{|\mathbf{u}|}\). In this way the calving vector has magnitude \(\dot{c}\) and the vector \(-\vec{c}\) applies the calving rate in the direction opposite to flow.

### 2.2.4 Tracking the calving front: Level-set method

In section 2.2.1 we loosely described the level-set function, \(\phi\), and in order to discuss motion of the calving front we must define \(\phi\) more precisely. There are many possible choices for \(\phi\) (any function with the desired zero-contour could be a candidate) and here we use a signed-distance function commonly used in the mathematical community for its numerical stability (Mulder et al., 1992) owing to its uniform gradient. This function defines the minimal distance from each point in the domain to the zero contour, \(\Gamma\), such that:

\[ \phi(\vec{x}) = \pm \min(|\vec{x} - \vec{x}_c| \text{ for all } x_c \text{ on } \Gamma), \]  

(2.8)

where \(|\cdot|\) denotes the Euclidean metric and, as described before, negative values designate the ice shelf and positive values, the ocean.
With velocity and calving rate in hand we can now evolve the calving front by solving the pure-advection equation,

\[ \frac{\partial \phi}{\partial t} + (\vec{u} - \vec{c}) \cdot \nabla \phi = 0. \]  

(2.9)

Here \( \phi \), and thus the front, evolve according to the rate \( \vec{u} - \vec{c} \). It’s quickly apparent that it doesn’t make sense to define \( \vec{u} - \vec{c} \) away from the calving front since \( \vec{c} \) is only defined at the calving front, yet we need this quantity defined over the entire domain to solve equation 2.9. This apparent contradiction is resolved by noting that we only really care about motion of the calving front, \( \Gamma \), and not motion of \( \phi \) away from the front. We’ll see in section 2.3.4 that we can construct a vector \( \vec{u} - \vec{c} \) that fulfills this purpose adequately.

The level-set equation needs one initial condition, \( \phi_{\text{init}} \), as well as one boundary condition for each boundary segment. The initial condition used here is a prescribed signed-distance function with a linear calving-front location, \( \Gamma \), at \( x = 39 \) km. At each boundary segment we prescribe a boundary condition such that the front can freely evolve out of the domain along segments (1), (3), and (4) in Figure 2.2. At all other boundaries we restrict the front from moving across the centerline or across the embayment wall.

### 2.2.5 Mass continuity

With the calving-front position, ice-shelf thickness, and velocity field in hand we can calculate the change in thickness governed by the mass-continuity equation. Equation 2.10 simply states that the change in thickness with respect to time is governed by the divergence of the ice flux plus accumulation rate, \( \dot{a} \), minus ablation rate, \( \dot{m} \),

\[ \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) + \dot{a} - \dot{m}, \]  

(2.10)
\[
\frac{\partial h}{\partial t} = -\nabla \cdot (\vec{u} h) + \dot{a} - \dot{m}.
\] (2.10)

Similar to the level-set equation, the mass-continuity equation needs an initial condition as well as a boundary condition for each boundary segment. For lack of a better starting geometry, we use a uniform initial-thickness for the ice shelf. At the upstream boundary (1) we utilize a fixed-flux condition (both \(\vec{u}\) and \(h\) are held constant in time). At the downstream boundaries (3) and (4) we assign a free-flux condition for which mass is allowed to freely leave the domain. At all other boundaries we set the flux to be zero (no mass flux in or out of embayment or across centerline). Note that evolution of the ice-shelf geometry must respond to calving, yet we do not add a calving term to equation 2.10. In section 2.3.5 we outline a numerical treatment for updating the thickness field at each time step that allows for dependence on the level-set function \(\phi\) and, in this way, response to calving rate.

### 2.3 Numerical treatment

Numerical treatment of the model equations require careful discussion, particularly with regard to implementation of the level-set method. Here we describe, step by step, the numerical implementation of each component of the model. We start with a brief discussion of the mesh with relevant considerations needed for mesh generation. This is followed by a discussion of numerical implementation of initial domain-geometry, stress-equilibrium equations, level-set method, and mass-continuity equation.
2.3.1 Mesh

We must make several considerations when generating the mesh. First, the motion of the ice-shelf must satisfy the Courant-Friedrichs-Lewy condition that assures ice does not travel more than one element per time step. Second, high strain-rates generated at the calving front, as well as our numerical treatment of the front (see section 2.3.2), require that we utilize a fine mesh near the front (relative to interior ice away from the front) to ensure both accuracy and numerical stability. Furthermore, we account for motion of the calving front by refining the mesh in regions where the front will evolve during the course of a model run. For the ease with which complex geometries are captured by unstructured meshes, we discretize the domain with a fixed unstructured triangular-mesh. The mesh is generated via the MATLAB© initmesh function which utilizes a Delaunay-triangulation algorithm.

2.3.2 Initial geometry

The initial modeled thickness described in equation 2.2 is not yet suitable for numerical implementation. Equation 2.1 needs modification because the calving front, Γ, will generally not fall on any mesh nodes except if prescribed for the initial configuration. This is the difficulty described in the introduction of enforcing the SI in 2D models. Whereas Albrecht et al. (2011) and Winkelmann et al. (2011) solved this sampling issue with the sub-grid scale parameterization of the SI, we utilize a technique developed for use with the level-set method called the diffuse-domain method (Li et al., 2009). The purpose of this method is to define a transition zone (TZ) between the ice shelf and ocean that physically represents a sloping cliff approximation of the calving face. In this way we turn the usual vertical cliff, modeled in 2D plan-view models (Fig. 2.3a), into a sloped sine-curve of length 2l (Fig. 2.3b). The sloped face allows us to approximate the correct SI condition without any further numerical treatment (see section 2.3.3). We will show that we do a better job approximating the
theoretical SI the smaller we make the TZ length $2l$ indicating that we are converging to the correct theoretical condition. In order to approximate the correct SI condition we solve the stress-equilibrium equations over the entire domain. To do this we assign a small thickness over the ocean that is thin enough not to affect ice-shelf dynamics.

![Flow-line view schematic diagram](image)

**Figure 2.3:** Flow-line view schematic diagram: (a) Vertical-cliff ice-shelf. (b) Numerical treatment of calving front transition zone. As we decrease $2l$, the length over which we transition from ice shelf to ocean, we approach a vertical face.

In order to construct the initial thickness-field we first define a smeared version of $F$ (Osher and Fedkiw, 2003) defined in Equation 2.1 for numerical implementation, $F^{sm}$:
\[ F^{sm}(\phi_{init}) = \begin{cases} 
1 & \text{for } \phi_{init} < 0, \\
1 - (1 - \alpha) \left( \frac{\phi_{init}}{2l} + \frac{1}{2\pi} \sin \left( \frac{\pi \phi_{init}}{l} \right) \right) & \text{for } 0 \leq \phi_{init} \leq 2l, \\
\alpha & \text{for } \phi_{init} > 2l, 
\end{cases} \]  
(2.11)

where \( \phi_{init} \) is the initial \( \phi \) described in section 2.2.4 and \( \alpha \) is a small quantity \( \sim 1\% - 10\% \). As an example, assigning an initial uniform-thickness over the domain, \( h_o \), of 600 m, and assigning to \( \alpha \) a value of 6\%, and taking the product:

\[ h = h_i F^{sm} \]  
(2.12)

we get a uniform thickness over the ice shelf and a smooth transition in the TZ from ice-shelf thickness to a uniform 10 m of thickness over the ocean. Having a smooth transition zone requires enough resolution to resolve the sine function in equation 2.11. Through trial and error we’ve determined we need roughly 10 nodes across the length of the TZ for adequate resolution. For example, we find that for a TZ length, \( 2l \), of 2000 m we require a mesh of triangular-edge length of no more than 200 m. Fewer nodes in the TZ leads to numerical instabilities rendering numerical results untenable.

### 2.3.3 Stress-equilibrium equations

The stress-equilibrium equations are implemented via the finite-element method with linear basis-functions. The nonlinear viscosity appropriate for ice rheology is accounted for via implementation of a Picard-iteration scheme. Convergence of viscosity is used as the criterion to stop iteration.
We have described the boundary conditions at all domain boundaries in section 2.2.2 and summarized in Figure 2.2. Here we discuss how our model captures the appropriate calving-front boundary-condition at the calving front and provide evidence for numerical convergence to the theoretical condition with decreasing $2l$ values.

Application of the correct SI condition begins with a discussion of the surface and basal boundary conditions that are integrated into Equations 2.3-2.4 (Fig. 2.4). Normally the theoretical boundary-condition prescribed at a vertical calving-front enforces the correct SI. For the outward pointing normal $\hat{n} = [n_x, n_y]$, in the case of an arbitrarily shaped calving front, the depth-averaged theoretical boundary-condition can be written as (see Appendix A for derivation):

\[
(2h\nu (2\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy})) n_x + 2h\nu \dot{\epsilon}_{xy} n_y = \frac{\rho gh^2}{2} \left( 1 - \frac{\rho}{\rho_w} \right) n_x, \tag{2.13}
\]

\[
(2h\nu (2\dot{\epsilon}_{yy} + \dot{\epsilon}_{xx})) n_y + 2h\nu \dot{\epsilon}_{xy} n_x = \frac{\rho gh^2}{2} \left( 1 - \frac{\rho}{\rho_w} \right) n_y, \tag{2.14}
\]
where the left hand side (LHS) denotes the dynamic deviatoric-stress and the right hand side (RHS) terms consist of both the glaciostatic ice-pressure and the hydrostatic pressure. Our approach approximates this boundary condition implicitly when we solve the stress-equilibrium equations. For a more detailed examination of how we capture the correct ocean pressure see appendix B.

To validate that the model is correctly approximating the theoretical boundary condition, we calculate the relative error between the theoretical centerline SI and the SI approximated in the model. We then reduce the TZ length $2l$ to ensure that we are converging to the theoretical value. The LHS can be calculated at element centroids using modeled velocities and strain rates and can then be compared to the RHS calculated using model thickness. Since there is no buttressing along the calving front except possibly right at the intersection with the embayment wall we expect the relative error between the LHS and RHS to be near
zero. Calculating the relative error at the centerline calving-front, we show in Figure 2.5 that this is not the case, however, we do see the relative error drop to zero with decreasing values of $2l$. Since we need enough resolution to capture the TZ, a shorter choice of $2l$ results in increased number of nodes and increased computational cost. Given these considerations we choose a value of 10 km for $2l$ thus placing the centerline relative-error near 3% (Fig. 2.5).

![Figure 2.5: Convergence of numerical boundary condition to theoretical boundary condition with decreasing transition length $2l$. A transition-zone length of 10 km has $\sim 3\%$ relative error.](image)

### 2.3.4 Level-set method

The key point to keep in mind when we talk about advection of the level-set function, $\phi$, by the level-set equation 2.9, is that we are only interested in accurate motion of the calving-
front location, $\Gamma$, and that we aren’t concerned at all about accurate motion of $\phi$ away from the front. While this strategy will most definitely result in numerical breakdown over time, we utilize a reinitialization scheme to reform $\phi$ periodically that prevents these numerical issues from occurring. While we could simply take the flow field, calculate the calving rate over the whole domain, and use these quantities in equation 2.9, this will not give optimal numerical results. A better approach was found in the construction of a flow field, $(\vec{u} - \vec{c})_{ext}$, that minimizes variation in velocity normal to the calving front (Malladi et al., 1993). This extended field comes out of the idea that the motion of the calving front depends solely on lateral variation in $\vec{u} - \vec{c}$ and is not at all controlled by variation normal to the calving front. For a nice discussion of this, see pages 25-28 in Osher and Fedkiw (2003). Since we only care about motion of the calving front we construct a field to advect the front within a narrow band (NB) of $\Gamma$ ensuring correct motion of the front and simply set the velocity field away from the front to be a small speed $\sim 1 \text{ m/yr}$. This technique is otherwise known as the narrow-band level-set method (Adalsteinsson and Sethian, 1995).

The algorithm we use to compute $(\vec{u} - \vec{c})_{ext}$ follows Malladi et al. (1993) in that we interpolate nodal calving values onto $\Gamma$ and then assign each node in the NB the closest value on the zero contour (Fig. 2.6). We lay out the algorithm in two stages: First we need to find coordinates for points on the contour $\Gamma$, then we interpolate nodal calving values onto these points and extend these values in directions normal to $\Gamma$.

1. First: We find coordinates of points along $\Gamma$.

   (a) We begin by finding the elements through which $\Gamma$ passes.

   (b) One element made up of three nodes consists of three edge segments. For each edge segment, we check if the zero contour crosses. If so, we compute the intercept for $\phi$ across the edge segment. The $x$- and $y$- coordinates of the contour point, $(x_c, y_c)$, are found by computing the $\phi$-intercept.
(c) Now it’s simply a matter of repeating this process for each edge segment through which \( \Gamma \) passes.

2. Second: We interpolate calving values onto contour points and extend them in the normal directions.

(a) We have coordinates for each contour point and the element(s) within which it lies. From this information, and use of the finite element linear basis-elements, we can compute the linearly-interpolated calving-value for each contour point.

(b) Repeating this process for each contour value, we now have a set of calving values along \( \Gamma \) and their respective coordinates (red circles in Fig. 2.6).

(c) In order to extend these values normal to the contour we pick a node in the NB and calculate the distance from this node to every point along the contour.

(d) With distances in hand, we choose the minimum distance as the closest point on the curve. Then it’s simply a matter of assigning the calving value of the closest point on the contour to the node in the NB.

(e) Repeating this process for each point in the NB gives us an approximation of the calving values extended normal to \( \Gamma \).
We utilize the finite-element method with linear-basis elements along with the streamline-upwind/Petrov-Galerkin (SUPG) scheme (Brooks and Hughes, 1982) to solve the level-set equation 2.9. The SUPG technique allows us to bias the weight functions in the direction of flow to improve the quality of numerical results without the traditional drawbacks of artificial-diffusion methods that can reduce numerical accuracy. Even with utilization of the SUPG method and the construction of $(\vec{u} - \vec{c})_{ext}$ which serves to help preserve $\phi$ as a signed-distance function (Zhao et al., 1996), $\phi$ will drift away from being a signed-distance function over time eventually rendering numerical results unusable.
Our solution to this issue follows Chopp (1993) in that we periodically reinitialize $\phi$ back to a signed-distance function. In order to preserve mass we reinitialize $\phi$ while preserving the calving-front location, $\Gamma$. We utilize the simplest approach to reinitializing $\phi$. First we locate contour points along $\Gamma$ in the same way as part 1 of the algorithm described above. Second, we repeat step 2c of the algorithm finding the distance from a node in the NB to the closest point on $\Gamma$. We simply assign this distance to be the node value of $\phi$. Since we are preserving the location of $\Gamma$, we know the correct sign of $\phi$ before reinitialization and it’s a simple matter to assign the correct sign to the newly reinitialized $\phi$. One assumption we make in this reinitialization scheme is that the location of the zero contour has been accurately advected for each time step, that is to say, the nodes that make up the elements through which $\Gamma$ passes (the closed circles in Fig. 2.6) are assumed to contain correct $\phi$ values and we do not alter these values during the reinitialization process. To save computational cost we need only conduct this reinitialization scheme within a NB region.

**2.3.5 Mass continuity**

When we update the thickness field at each time step we need our thickness field to respond to calving, and thus, be dependent on $\phi$. In the absence of calving, both $\phi$ and the thickness field evolve according to the velocity. We now need to use the fact that $\Gamma$ responds to the calving rate to update the thickness field. We begin by calculating the updated thickness field using equation 2.10. Since glacial ice away from the calving front doesn’t care about calving this will give us the updated thickness for the interior of the ice shelf. We compute the rest of the updated thickness field in the following way. The updated $\phi$ locates the new calving front. For the new transition zone we utilize the extension algorithm described above to extend the thickness field in directions normal to $\phi = 0$ across the TZ, $0 \leq \phi \leq 2l$, then multiply this extended thickness field by $F^{sm}$, equation 2.11, where $\phi_{init}$ is now the updated...
\( \phi \). This ensures a smooth transition from ice shelf to ocean. Nodes in the ocean, \( \phi > 2l \), are simply assigned to be a small value.

Similar to the level-set equation, we utilize the finite-element method with linear-basis elements along with the SUPG scheme to solve the mass-continuity equation. In addition, here we utilize a discontinuity-capturing scheme to limit numerical over- and under-shoots in the thickness field associated with the steep gradients in thickness within the TZ (Hughes and Mallet, 1986; Johnson, 2009). The discontinuity-capturing scheme solves the mass-continuity equation once, then minimizes the over- and under- shoots near the TZ along gradients of \( h \). This discontinuity-capturing scheme can be iterated until the change in thickness field is less than some tolerance, however, we have found most of the beneficial reduction in over- and under-shoots is achieved in the first time step and, to save computational cost, we do not iterate more than once. This step is useful in reducing noise in strain rates at the front.

### 2.4 Numerical results

Here we seek to validate our model and demonstrate its capacity to accurately capture the evolution of a calving front. We go about validation and demonstration of model capabilities in the following way:

- **Front-motion accuracy:** Here we validate our numerical front-motion by comparing to an analytic solution that’s based on Van der Veen (2013). We show how results converge under resolution and demonstrate the numerical benefits of the discontinuity-capturing scheme.

- **Steady state:** We seek to further validate the model for a 2D idealized glacier as outlined in the model description. Here, we utilize a dynamic-calving rate and find a steady-state calving-front configuration. We show the temporal evolution of the glacier to
demonstrate the transition from initial geometry to steady state. We also show the steady state converges under mesh refinement.

2.4.1 Front-motion accuracy - 1D experiment

Numerical dispersion of the thickness field at the calving front may result in an inaccurate enforcement of the SI condition (Albrecht et al., 2011). Here we show there is no dispersion of mass at the front and that the SI condition is applied correctly through time. Parameter values are given in Table 2.2. In order to compare our results to an analytic solution, we utilize a plan-view domain with parallel-sided embayment-walls, with a stress-free condition along the embayment wall, thus reducing the problem to an $x$-directed 1D problem. The analytic calving-front location based on work by Weertman (1957) and Van der Veen (2013) was found by Albrecht et al. (2011) to be:

$$x_c(t) = \frac{u_o h_o}{4C} \left[ \left( 3Ct + \frac{1}{h_o^3} \right)^{4/3} - \frac{1}{h_o^4} \right], \quad (2.15)$$

where $x_c$ is the location of the calving front assuming the initial front-position starts at the inlet $x = 0$, $h_o$ is the thickness at the inlet, $u_o$ is the velocity at the inlet, $C = \left( \frac{\rho g}{4B_o} \left( 1 - \frac{\rho}{\rho_w} \right) \right)^3$ with $B_o$ a uniform constant hardness, $\rho$ is the density of ice, and $\rho_w$ is the density of ocean water. We modify this equation slightly to account for the fact that we cannot start our front evolution from the inlet since we utilize a narrow band to evolve the location of our calving-front location $\Gamma$. Since the narrow band encompasses $\Gamma$ on each side over a length $l$ (for a total length of $2l$) we start with $\Gamma$ located at $x = l$. In this way we utilize the analytic front-location:
\[ x_{2}(t) = \frac{u_{l}h_{l}}{4C} \left[ \left( 3Ct + \frac{1}{h_{l}^{2}} \right)^{4/3} - \frac{1}{h_{l}^{4}} \right] + l, \]  

(2.16)

where \( t \) represents time, \( u_{l} \) is the analytic velocity at \( x = l \), and \( h_{l} \) is the analytic thickness at \( x = l \).

Given that we know the analytic thickness-profile, velocity profile, and theoretical calving-front location, it’s now a simple matter of comparing our numerical results to the analytic solution. We follow Albrecht et al. (2011) by running our model for 300 years. We show the centerline along-flow thickness profile, calving-front location, and velocity profile, at 50 year increments, in Figure 2.7. We see that the thickness field and calving front in 2.7a closely match those of the analytic solution with only a 0.1% relative error in front location between analytic (dashed magenta) and numerical (blue). Similarly the velocity profile for the ice shelf shown in Figure 2.7b is a near exact match indicating we are capturing the appropriate ice dynamics in this simplified scenario.
Next we examine how the system responds if we do not utilize the discontinuity-capturing scheme. Recall, the purpose of this scheme is to suppress overshooting in thickness at the calving front which tends to add spurious noise to the calving-front strain-rates which then feed back into our strain-rate-dependent calving-rate. To get a sense of how well this scheme is performing, we ran the model for 300 years, once with the discontinuity-capturing scheme and once without. Since we know the analytic thickness-profile, we can calculate the relative error between numerical thickness and analytic thickness. We find that the overshoot in thickness is reduced from 1.4% to 0.7% with the scheme. Given that strain rates are non-linearly dependent on thickness at the front, (Weertman, 1957) small deviations in thickness
are amplified in strain rates. Calculating the relative error between analytic and numerical strain-rates we find that the errors attributed to overshoots are reduced from 3.0% to 1.4%.

2.4.2 Steady-state front

Having verified that our methods work well in the absence of embayment-induced shearing, we seek to further verify that our method is producing correct front motion in a 2D setting. We make use of the fact that Levermann et al. (2012) produced a steady-state calving-front via implementation of their calving parameterization (discussed in section 2.3) and moving-front techniques from Albrecht et al. (2011) for an idealized rectangular-domain.

Here we present numerical results for this scenario with parameter values given in Table 2.2. In addition to reproducing a qualitatively similar steady-state result to that of Levermann et al. (2012), we show that our result converges under resolution and also show the spatiotemporal evolution of the front from an initial configuration. Figure 2.8 shows the temporal evolution of the calving-front location, \( \Gamma \), from initial-front position to final front-position. We show the front location for various times between \( t = 0 \) years and \( t = 1600 \) years. The final time corresponds to the glacier having reached steady state. The outgrowth near the embayment wall is a result of the fact that the calving rate is zero near the embayment wall since embayment-induced shear will result in compression of ice and hence one of the principal stresses being negative, thereby shutting off calving (Fig. 2.9). Once the front grows beyond the confines of the wall, spreading of ice initiates calving and a steady state is reached. Away from the wall, the steady state generally falls along the contour where \( \dot{\epsilon}_- \) is zero, this being the controlling principal stress since \( \epsilon_+ \) is positive throughout the domain (Fig. 2.10). Next we halve resolution and show a qualitatively similar result though the outgrowth is smaller indicating our results converge under refinement (Fig. 2.11).
Figure 2.8: Convergence of calving to steady state from initial configuration over 1600 years for 1 km resolution ($2l = 10$ km). The calving-front location, $\Gamma$, of the level-set function $\phi$ is plotted. The dashed black line represents the embayment wall.

Figure 2.9: Schematic plan-view image showing how an ice parcel near the embayment wall shears after an increment of time. There is extension in one principal direction with $\dot{\epsilon}_+ > 0$. Since there is compression in the second principal direction then $\dot{\epsilon}_-$ is negative and calving shuts off for this particular calving treatment.
Figure 2.10: Domain image for the initial principal stress $\dot{\epsilon}_-$ that controls calving-front location.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>1D Experiment</th>
<th>2D Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>50 km</td>
<td>25/50 km</td>
</tr>
<tr>
<td>$L$</td>
<td>400 km</td>
<td>80 km</td>
</tr>
<tr>
<td>$h_o$</td>
<td>600 m</td>
<td>600 m</td>
</tr>
<tr>
<td>$\rho_t$</td>
<td>917 kg m$^{-3}$</td>
<td>917 kg m$^{-3}$</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>1028 kg m$^{-3}$</td>
<td>1028 kg m$^{-3}$</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81 m s$^{-2}$</td>
<td>9.81 m s$^{-2}$</td>
</tr>
<tr>
<td>$B$</td>
<td>$1.9 \times 10^8$ Pa s$^{1/3}$</td>
<td>$1.9 \times 10^8$ Pa s$^{1/3}$</td>
</tr>
<tr>
<td>$n$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>$l$</td>
<td>10 km</td>
<td>10 km</td>
</tr>
<tr>
<td>$K$</td>
<td>N/A</td>
<td>$5 \times 10^9$ m yr</td>
</tr>
</tbody>
</table>

Table 2.2: Variables and base parameter values

![Figure 2.11](image.png)

Figure 2.11: Convergence of calving-front location to steady state from initial configuration over 1600 years for 0.5 km resolution ($2l = 5$ km). The dashed black line represents the embayment wall.
2.5 Discussion

Having compared our results to a 1D analytic solution and found a 2D steady state solution in an idealized rectangular geometry, we can now qualitatively compare our results to those for Albrecht et al. (2011) and Levermann et al. (2012). In the 1D case, comparing to Albrecht et al. (2011), we show our numerical front-location closely matches the analytic solution. Since we reinitialize the level-set function periodically, we do not have any dispersion of the calving-front thickness.

A direct comparison of 2D results to Levermann et al. (2012) is not possible since they do not give their exact boundary conditions or initial conditions, however, we can conduct a qualitative comparison for validation. We are in a position to give the transient response of the calving-front location $\Gamma$. Our 2D steady state result qualitatively matches that of Levermann et al. (2012), although, where our front advances beyond the mouth of the embayment, the shape of our solution differs from theirs. This difference may simply be a result of differing boundary conditions in this region or due to interactions between the calving front TZ and the embayment wall as discussed next. Both solutions are similar away from the embayment wall following the zero contour of $\dot{\epsilon}_\perp$.

Our method succeeds in naturally moving the calving front by removing node dependent dynamics. This allows for stresses to smoothly respond to front evolution. However, our modeled calving-front as a sloping cliff of length $2l$ has repercussions for the application of the calving-front boundary-condition near the embayment wall. For a calving front that intersects the embayment wall at a right angle or is sloping back and away from the front (such as a parabolic front opening towards the ocean in a parallel-sided embayment) the boundary condition is applied correctly since the sloped calving front is adequately represented. Only when the front is forward sloping near the embayment wall (such as an arcuate-shaped front bowing outward) does the calving front TZ come into contact with the embayment.
wall resulting in a reduction in the application of ocean pressure manifesting as a mismatch between theoretical spreading-rate and numerical spreading-rate. Insofar as our approximation of the calving front as a sloped face converges to a vertical front with decreasing $2l$, the reduced ocean pressure near the embayment wall is minimized as the TZ length decreases to zero. This numerical effect principally occurs when the front is in advance near the embayment wall akin to the physical phenomenon of a calving front coming into contact with the embayment wall.

The numerical accuracy of this suite of methods increases with mesh refinement. Our scheme to construct $(\vec{u} - \vec{c})_{\text{ext}}$, construct the $\phi$ dependent thickness field $h$, and also reinitialize the level-set function after advection all increase in accuracy as more elements are added resulting in more contour points $(x_c, y_c)$. Furthermore, increased resolution allows us to decrease the TZ length, $2l$, resulting in a better approximation of the SI.

The level-set methods and diffuse-domain methods add computational cost to our model runs above and beyond simply solving the stress-equilibrium equations and mass-continuity equation at each time step. We can break the overall computational cost into four components: Solving the level-set methods, utilizing the discontinuity-capturing scheme, solving the stress-equilibrium equations, and solving the mass-continuity equation. While many factors contribute to the overall computational cost of solving a multi-component model (such as code efficiency or utilizing relatively fast/slow numerical techniques), we can get a sense of the computational cost of each component by computing the relative run time of each to the overall run time for a single time step. The most computationally expensive component of the model is solving the stress-equilibrium equations with 33% of the run time. This is to be expected given the non-linearity of the viscosity requiring iteration. Solving the mass-continuity equation consumes 19% of total run time making it the least expensive component. Solving the level-set methods and applying the discontinuity-capturing scheme each account for 23% and 24% respectively and 47% together. Solving the level-set methods
includes: Solving the level-set equation 2.10, reinitializing $\phi$, and constructing $(\vec{u} - \vec{c})_{front}$ used to advect $\Gamma$. Of these, setting up $(\vec{u} - \vec{c})_{front}$ is the most expensive since we compute distances from the contour to each node in the narrow band. Furthermore, costs associated with solving the level-set methods will vary in time, as the length of the front grows, so too will the number of nodes in the narrowband region, increasing the computational cost of reinitializing the front and computing $(\vec{u} - \vec{c})_{ext}$ since these quantities need to be computed within the narrow band. Whether one decides to utilize the discontinuity-capturing scheme depends on how sensitive the results are to overshoots in thickness. Clearly computational saving are substantial when excluding this component from computation. The code here has not been optimized for parallelization beyond built-in MATLAB® functions that take advantage of multiple cores and these costs will shift with implementation of innovative numerical techniques.

2.6 Conclusions

We have demonstrated that the level-set methods combined with diffuse-domain methods effectively track and evolve the calving front of an ice shelf. We have implemented these methods in a finite-element framework on an unstructured triangular-mesh and we note these methods are mesh independent and can easily be transferred to other mesh geometries (e.g., Cartesian, hexagonal). These methods succeed in removing mesh-dependent dynamics allowing for a truly continuous motion of the calving front. Validation has taken two forms: First by comparison to a 1D analytic solution and second, the production of a 2D steady state calving front comparable to that of Levermann et al. (2012) that converges under mesh refinement.

The methods presented here succeed in achieving front motion but also have their numerical limitations. Careful thought must be given to boundary conditions where the calving-front
location $\Gamma$ intersects the embayment wall, for a front in advance near the embayment wall, the TZ will intersect the embayment wall thereby affecting application of the calving-front boundary-condition. Our need to resolve the calving front TZ puts a constraint on the coarseness of the mesh. While refinement of the mesh around the calving front, with coarser regions away from the front, is one potential avenue for reducing computational cost, our steady state results presented here utilize 1 km edge length mesh resolution significantly finer than many ice-sheet-scale models currently in use. Computational cost of these methods is significant when utilizing the techniques we’ve outlined. Other techniques exist that can further reduce computational cost of these front moving techniques such as the fast marching method (Sethian and Vladimirsky, 2000; Elias et al., 2007) that may improve cost of the reinitialization scheme. These methods should be explored in the future to make these methods more accessible for ice-sheet-scale modelers.

In the next chapter we look at an application of these techniques for the Ross ice-shelf. Our decision to model the Ross ice-shelf lies in the fact that it is generally stable and that steady state results have been presented before giving us a chance to further examine the numerical properties of level-set techniques. Given the present stability of the Ross ice-shelf, we explore achieving a steady-state front-position and its relationship to the present front position.
Chapter 3

Evolution of the Ross ice shelf calving front

3.1 Introduction

The Ross ice-shelf covers a surface area of roughly 520,000 km$^2$, about the size of France, making it the most expansive ice shelf in Antarctica (Fig. 3.1) (Bentley, 1984). The southwestern margin abuts the transantarctic mountain range with several inflowing glaciers distributing ice from the East Antarctic Ice Sheet into the Ross ice-shelf. Along the eastern margin six ice streams (of which one is presently stagnant) feed into the ice shelf draining ice from the West Antarctic Ice Sheet (WAIS) making the Ross ice-shelf an important component of the overall WAIS mass budget. Along the northern margin is the calving front, the interface between ice shelf and ocean. Several notable features slow the flow of ice within the ice shelf including Roosevelt island near the northeast section of the calving front and the Crary ice-rise and Steershead ice-rise at the South end of the shelf, the latter two being a result of the ice shelf grounding on a bathymetric shoal.
Ice shelves serve to stabilize tributary flow by providing a backstress generated by shearing at embayment walls, islands, and ice rises. This buttressing effect can be large such that loss of the ice shelf leads to rapid acceleration and thinning of flow upstream (Dupont and Alley, 2005), by up to 8 times speed increase or more as in the case of the break up of the Larsen B ice-shelf (Rignot et al., 2004; Scambos et al., 2004). The ability of the Ross ice-shelf to buttress flow is significant and one of the early quantities derived from extensive surveys conducted during the Ross Ice Shelf Geophysical and Glaciological Survey (RIGGS) between 1973 and 1978. Strain rates were measured, converted into stresses, and were then compared to theoretical stresses calculated as if all islands and ice rises were removed and shearing along embayment walls was non-existent (Thomas and MacAyeal, 1982). The difference between the two stresses is the resistive force provided by the shelf. The resistance offered by the shelf reduces thinning rates and it was shown that the removal of this resistive force would result in a 2-10 fold increase in creep-flow thinning rates of the ice shelf.

WAIS is largely grounded below sea level making it vulnerable to the reverse-bed slope instability (Weertman, 1957). In a reverse-bed slope scenario, the grounding line, the region in which ice transitions from grounded to floating, starts to retreat. With retreat, since the bed slope is reversed, thicker ice becomes separated from the bed, reducing buttressing, and allowing for a greater flux of ice across the grounding line. This leads to thinning and further retreat of the grounding line in a potential runaway positive feedback. With six ice streams draining WAIS into the Ross ice-shelf, mapped extensively during RIGGS (Shabtaie and Bentley, 1987), the stability of the shelf is important to determine sea-level projections.

Given the challenges of examining future stability of the Ross ice shelf, it’s useful to look to the past to get a sense of potential behavior. Several lines of evidence exist to inform us about past behavior, each on a different time scale. Current ice-core records tell us about stability of the shelf within the past 1500-2500 years. Records of sea level during the last interglacial (Eemian) period, ~120 ka, suggest whether WAIS and subsequently the Ross ice-
shelf were stable or not. At the calving front, oceanic sediment-core layers indicate whether there has been open ocean, floating shelf, or grounded ice at the sediment-core site over the past 5 million years.

Four ice cores taken from the Ross ice shelf during RIGGS give an indication of stability over the residence time of ice within the shelf. One approach to examine past stability is to assume steady state for the ice shelf, calculate the theoretical structured internal layers of the shelf, and compare to cores taken from the shelf. Thomas and MacAyeal (1982) were able to compare the theoretical steady state shelf structure to the RIGGS ice-cores and, finding good agreement, suggested that the shelf has been in steady state for the last 1500-2500 years.

Looking beyond this relatively short geologic time-scale the ANDRILL-1B sediment core, located at the Ross calving-front near Ross Island, provides information about the Holocene 0-0.0117 Ma, Pleistocene 0.0117-3 Ma, and Pliocene 3-5 Ma periods. Layering in the core suggests the Ross ice-shelf has transitioned between grounded and floating at least 7 times within the past 0.078 Ma indicating the shelf has responded to orbital Milankovitch forcing (McCay et al., 2012). Sea levels during the last interglacial ~0.0117 Ma were ~6.6 m higher than present day suggest WAIS may have partially or completely collapsed (Kopp et al., 2009). The ANDRILL-1B sediment record suggests the Ross ice-shelf was still present to some extent during the last interglacial but does not exclude the possibility of partial collapse.

Going back to the Pliocene, when carbon dioxide levels were ~400 ppm and temperatures were ~3° C warmer than today, the Ross ice-shelf periodically collapsed (Naish et al., 2009), allowing periods of open water, a result supported by Antarctic-continent-scale modeling (Pollard and DeConto, 2009).

With present day CO₂ levels just about 400 ppm it’s necessary to examine the susceptibility of present-day Ross ice-shelf to instability and the physical mechanisms through which deviations from steady state may occur. Observations and speculation can give an intuitive
sense of how the system will respond to a changing climate, numerical modeling can allow for hypothesis testing of physical mechanisms as well as examination of future behavior of the Ross ice-shelf. Moreover, expanding the toolset of numerical methods will allow us to examine new physical processes and obtain ever more realistic forecast level modeling of future sea levels.

Here we present an application of the front-motion techniques utilized in chapter 2 to the Ross ice-shelf. We look at the potentiality of achieving a steady state calving front utilizing the Levermann calving rate. In principal, steady-state calving-fronts achieved utilizing the Levermann calving rate require three conditions to be met. The first is a constant supply of mass to the front to maintain thickness and sustain strain rates necessary to maintain calving rates. The second condition requires divergent flow along the entire calving front, thus enabling a calving rate to balance flow. The third requires a no flow boundary condition at ice/rock interface since calving rate is necessarily zero. Our simulations of the Ross illustrate front motion over 300 years. Our front motion satisfies the first and third conditions but does not satisfy the second. For this reason our calving front does not reach steady state but instead continues a slow advance near rock faces. Ultimately, a calving parameterization that captures laterally-propagating fracturing processes found in ice shelves such as the Filchner-Ronne (Hulbe et al., 2010) and the Ross ice shelf (Joughin and MacAyeal, 2005) may be necessary to produce realistic steady state front positions for the Ross ice shelf.

The remaining material is organized as follows. In section 3.2 we discuss the model domain and differences in numerical methods between the 2D experiments in chapter 2 and Ross ice-shelf. In section 3.3, we present validation of the diagnostic solution and examine steady state results. This is followed in section 3.4 by a discussion of results. In section 3.5 we give some concluding remarks and discuss future work.
Figure 3.1: The Ross ice-shelf situated in Antarctica (Bentley, 1984).

3.2 Methods

The numerical methods used to evolve the Ross ice shelf are the same as those used in the idealized 2D domain described in chapter 2. For each time step we:

1. Establish the initial level-set function, $\phi_{init}$, and thickness-field, $h_{init}$

2. Solve the momentum-balance equations

3. Compute strain rates and corresponding calving rate
4. Construct $(\vec{u} - \vec{c})_{ext}$ and solve the level-set equation

5. Update the thickness field by solving the mass-continuity equation and adjust the thickness field for the updated $\phi$.

The differences between methods used here and those used in Chapter 2 will be outlined in this section, the most notable being the observation-based initial thickness-field and boundary conditions.

### 3.2.1 Domain

We have chosen to simulate the primary branch of the Ross ice shelf between Roosevelt island and Ross island while cutting out the section between Marie Bird Land and Roosevelt island and the section between Ross island and the transantarctic mountains (Fig. 3.2). This allows us to focus our computational resources on the primary branch of the Ross ice shelf and explore steady state within this context. Since we are primarily interested in motion of the front, we have removed both the Crary ice rise and Steershead ice rise but have left observed velocities at these locations as boundary conditions.

### 3.2.2 Mesh

In these experiments we utilize a mesh that is refined in a region near the calving front and is relatively coarse in regions away from the calving front. We find that, in order to prevent numerical instabilities from developing, a uniform mesh density of 1 km triangular edge length is needed to resolve the calving front. We utilize a uniform mesh density away from the front of 4 km triangular edge length. Our mesh utilizes $\sim 247,000$ nodes with $\sim 492,000$ triangular elements.
3.2.3 Initial condition, boundary conditions, ice hardness, and mass continuity

In solving the momentum-balance equations, we need an initial thickness field and set of boundary conditions. However we cannot simply take a data set of observed thickness field as our initial condition because the data set contains small scale features, such as rifts, that are generated through physics related to damage mechanics not captured by our model. Moreover, the data thickness field is formed from various physical processes different than those captured in the model (e.g., basal melt, accumulation, spatial variations in rheologic properties). In order to generate a thickness field consistent with the internal physics and velocity boundary conditions we start with a thickness data set (Fig. 3.2) derived from surface elevations measured by ERS-1 radar altimeter and ICESat laser altimeter satellite data (Griggs and Bamber, 2011). The thickness data set has already undergone a thickness correction owing to the fact that the surface layer of the ice shelf, called firn, has not yet compressed to the density that makes up most of the ice shelf. Rather than account for a depth-dependent density, the firn correction makes a small reduction in overall thickness for a spatially uniform density. With this data set we run the model for 200 years without calving before resetting the calving front to our initial calving front location (Fig. 3.3). This allows us to flush out the small scale features mostly seen near the calving front while also allowing the thickness field to respond to temporally-fixed inflowing velocities. We call this new thickness field our initialized thickness field. Differences between initialized thickness and thickness data fields will be discussed in section 3.3.1 and Figure 3.4.
Figure 3.2: Observed thickness field.
Velocity data utilized to prescribe our boundary conditions comes from InSAR data (Rignot et al., 2011). In the same way as chapter 2, we utilize these velocities for our ice shelf and calving front transition zone (TZ) boundary conditions. In the ocean region we simply apply a stress free condition to prevent these boundary conditions from affecting shelf flow.

As in chapter 2 we use Glen’s flow treatment to govern the relationship between stress and strain rate as appropriate for the Ross ice shelf (Rommelaere and MacAyeal, 1997) with an appropriate ice hardness parameter (MacAyeal et al., 1996). We use this value uniformly but a spatially variable hardness parameter should be implemented in the future (MacAyeal and Thomas, 1986).

In updating the thickness field at each time step we must consider the melt rate, both atmosphere- and ocean-induced, and the accumulation rate. At this point, given uncertain-
ties in spatio-temporal melt rates and added complications in interpreting model results, we set melt rate to balance accumulation rate both spatially and temporally. This effectively means that both of these quantities are zero at all time steps.

3.2.4 Time step

We utilize a variable time step when updating the level set and mass-continuity equations. There are two considerations when choosing a time step. The first is that we do not want mass to move more than one element edge-length per time-step. The second is that we need to ensure that the calving front does not move more than one triangular edge length per time step. We need our time step to satisfy two CFL conditions. We calculate both CFL conditions using the smallest edge length in the domain and then choose the time step to be one half the minimum of these two CFL conditions. This results in a time step size that ranges from 0.01 years to 0.001 years.

3.3 Results

We first compare the initialized numerical thickness with the observed thickness. We next validate the diagnostic flow field by comparison with InSAR-derived data. Finally, we allow the Ross ice shelf to evolve in time, examining characteristics of front evolution.

3.3.1 Diagnostic solution

For comparison we have computed the relative error (Fig. 3.4) between observed thickness data (Fig. 3.2) and our initialized numerical thickness (Fig. 3.3). While there is general agreement between the two, there are some notable differences. The first is that the numerical
thickness field has been smoothed considerably with fractures near the front removed and our numerical thickness is generally 10-30% thinner near the calving front. We observe that numerical ice is thicker at some of the inlet glaciers, possibly due to differences in ice rheology. This difference could also be due to the temporal mismatch between thickness data and inlet velocity data. If our inlet velocities are, on average, higher than those that formed the current observed thickness field then we will see thicker ice near the inlets. Numerical ice is also significantly thicker near Ross island indicating our model is either better able to advect ice into this region or there is some physical mechanism for ice loss, such as ocean melt, that is not captured in our model.

![Figure 3.4: Relative error between initialized thickness and observed thickness.](image)

With thickness field in hand, we compute the initial numerical speed and compare to the observed speed. For our initial model setup, our calving front location does not match that
of the observed Ross ice shelf, however, we still believe it’s useful to compare velocity fields. Even though there are differences between our modeled Ross ice shelf including thickness field, uniform ice hardness parameter, and isotropy assumption for ice, we are able to produce a realistic velocity field that closely resembles the observed velocities at the Ross ice shelf with only a 2% relative error between maximum numerical and observed velocities. Figure 3.5 shows the numerical speed using the observed velocities as boundary conditions, Figure 3.6 shows the observed speed. We see that we are able to reproduce many of the flow features within the ice shelf including the regions of faster flowing ice. This close match suggests that the gradients of thickness are dominant controls of the flow field. The relative error between numerical and observed speed are computed for each node in the domain (Fig. 3.7). For the broad swath of of the domain we are within ~ 10% indicating reasonably good agreement in our results.

![Figure 3.5: Observed speed](image-url)
Figure 3.6: Initial numerical speed.
The initial principal strain rate fields are shown in Figures 3.8 and 3.9. One principal stress, \( \dot{\varepsilon}_+ \), is positive throughout most of the domain and it’s role in determining the calving rate is largely modulating the magnitude of calving. The principal stress, \( \dot{\varepsilon}_- \), is negative in most regions. Since calving is zero when either principal stress is negative, \( \dot{\varepsilon}_- \) is the controlling stress that determines the calving front location. Figure 3.9 indicates how the calving front will initially respond to calving with retreat of interior ice back to the isocontour where \( \dot{\varepsilon}_- = 0 \).
Figure 3.8: Initial principal stress field $\dot{\epsilon}_+$. 
3.3.2 Prognostic evolution of the Ross ice shelf

We now describe the evolution of the ice shelf in time and examine the transient response of the calving front. We begin with the initialized thickness field and plot the calving front location, $\Gamma$, at 50 year intervals in Figure 3.10.

At time zero we start with a linear front. With calving enabled we see a rapid retreat of the interior portion of the front away from the embayment boundary with corresponding slow advance at the embayment walls corresponding to the absence of calving. Retreat of interior ice continues before transitioning to a slow advance. Ice near the Ross and Roosevelt islands continues to advance due to shearing at the wall resulting in compression of ice in one of the principal directions, and thus zero calving (as discussed in chapter 2 section 2.4.2). The
front continues to advance with the interior calving front lagging behind the advance of ice at embayment walls. The interior ice is responding to the zero contour of the controlling principal stress, $\dot{\epsilon}_-$, that advances as the front advances near ice/rock boundaries. While calving near the ice/rock interface increases in regions where the embayment slopes away the resulting flow is not divergent and calving does not balance flow.

![Figure 3.10: Evolution of the calving front through time.](image)

### 3.4 Discussion

Our diagnostic results show generally good agreement between observed thickness and numerical thickness with some notable differences near embayment margins and regions of particularly fast flow. Our prognostic results show a near steady state for the majority of
the calving front but a slow advance near the rock faces. Due to the inability of our model to maintain divergent flow near the rock/ice boundaries, steady state is not achieved.

This result stands in contrast to the steady state result achieved by Levermann et al. (2012) for not only the Ross ice shelf, but other shelves as well, including Larsen B. At this point it’s unclear why they were able to achieve steady state and we are not. It may have to do with differing boundary conditions, we utilize a thickness field based on BEDMAP2 data (Griggs and Bamber, 2011) and velocity from Rignot et al. (2011) while Levermann et al. (2012) utilize an earlier thickness and velocity dataset collected during RIGGS (MacAyeal et al., 1996). It may also have to do with treatment of mass, in which Levermann et al. (2012) must reintroduce ice to the system as part of their scheme that artificially shifts mass around. Another notable distinction lies in the difference in resolution of the model runs with Levermann et al. (2012) utilizing a 6.82 km resolution while we use a much finer 1 km resolution near the calving front and 4 km away from the front. Since their boundary conditions were not provided, it’s unclear how to reproduce their result. Whatever the reason, they were able to produce divergent flow at embayment walls while we were not.

Calving events at Antarctic ice shelves tend to result in the release of large tabular bergs often due to lateral propagation of crevasses such as at the Ross ice shelf (Joughin and MacAyeal, 2005) the Ronne ice shelf (Hulbe et al., 2010). In this way damage of ice, often at rock faces, and its role in forming crevasses is essential to the calving of ice. What unifies the process that governs the calving of large tabular bergs and the Levermann calving rate is their first-order dependence on the tensile stress of ice (Benn et al., 2007a). While there is no physical reason to expect the calving treatment used here to reproduce tabular-berg style calving behavior given the lack of damage mechanics in the model, we do see a qualitatively similar behavior in the response of the calving front to calving. That is to say, after the initial retreat of the interior portion of the calving front back to the $\dot{\epsilon}_- = 0$ contour, the slow advance of this region lags behind the advance near embayment wall. In this way, the
character of response is controlled by the migration of the $\dot{\epsilon}_- = 0$ contour which is in turn controlled by motion of ice near rock walls.

We found that results are sensitive to mesh construction. Many software packages that generate unstructured meshes seek to optimize the quality of the triangular mesh by prioritizing the construction of equilateral triangles. This priority makes sense when numerical methods are insensitive to node density. Here we must ensure that our mesh correctly resolves the thickness gradient in the transition zone. We therefore must take care that the trade-off in optimal triangular quality does not result in significantly varying triangular edge length. We have found that we require 10 nodes within the span of the transition zone to resolve the calving front adequately.

While we have successfully evolved the calving front through time, a number of model issues remain to be explored. We found that the front remains numerically stable for a longer period of time the more we refine the mesh. More specifically, we found that a resolution of 1 km for TZ length $2l$ of 10 km gives results that are numerically stable for 300 years but a coarser mesh, larger $K$ value, or longer TZ length results in a breakdown of numerical results at an earlier time. The theoretical limit for this result should be explored, what resolution is necessary for accurate numerical results over time and why? We find numerical results can degrade over time if $K$ is large which results in an amplification of noise in the numerical strain rates. These numerical degradations result in an artificial retreat of the front in very narrow regions producing finger-like features in the front. When and why these features develop should be explored in future work.
3.5 Conclusions

We have demonstrated that the mesh-independent front motion methods outlined in chapter 2 allow us to evolve the Ross ice-shelf calving-front. We found that the resulting calving front never settles into steady state due to the presence of convergent flow at rock/ice boundaries. Instead the front near rock faces continues a slow advance followed by the slow advance of the interior portion of the calving front.

The diagnostic model captures major features of flow but can still be improved to produce more realistic flow. In a study of Ross ice shelf flow, Rommelaere and MacAyeal (1997) found that attaining a correct basal melt rate and correct temperature profiles for inflowing ice were two important uncertainties that should be constrained in future modeling efforts. Attaining more realistic flow should therefore incorporate ice hardness of inflowing ice and accurate basal melt rates. In addition, incorporation of a thermal model would allow the temperature, and therefore hardness, to evolve in time. In this study we utilize an initialized thickness field that has been formed after 200 years of model run time. The choice of 200 years allowed us to clear out small scale fractures near the front of the shelf but the length of time was chosen in part due to constraints on computational cost. When creating a future initialized thickness field, the model should be run for a minimum of 1000 years, roughly the time it takes for ice entering the ice shelf at an inlet to travel the length of the ice shelf.

The retreat of ice shelves has been linked to ocean-induced melt and the formation of basal crevasses (e.g., McGrath et al., 2012; Bassis and Ma, 2015), propagation of surface crevasses (e.g., Scambos et al., 2000), surface water ponding and drainage (e.g., Banwell et al., 2013), and potential calving event triggers such as activation energy provided by transoceanic waves (MacAyeal et al., 2006). These processes are not currently captured in this modeling effort and examining ice shelf collapse will require better quantification of melt rates, both surface and oceanic, inclusion of wave energy at the calving front, improved mechanics responsible
for crevasse formation and propagation as well as surface meltwater pooling, drainage, and associated flexure damage to the shelf.

Future work should look at how Ross flow responds to different calving treatments and how these affect front motion. Characteristics of shelf flow should be examined with special attention given to which treatment(s), if any, allow steady state. Level-set methods lend themselves to treating calving as a rate but there is the potentiality for treating calving as a discrete process in which the front jumps to a new position due to loss of a berg and these calving treatments should be explored as well. Additional work should focus on improving both model stability and accuracy while reducing computational cost to make methods more accessible to the glaciological community at large.
Chapter 4

Potential importance of melange for calving-front dynamics

4.1 Introduction

A combination of satellite data and in situ measurements in Greenland show a seasonal calving-front advance/retreat cycle at marine-terminating glaciers: Helheim, Store, Kangerdlugssuaq, and Jakobshavn Isbræ (Howat et al., 2010; Joughin et al., 2008b; Luckman et al., 2006; Walter et al., 2012). At Jakobshavn Isbræ and Store glacier, the seasonal cycle has been correlated to the presence of melange, a medium made up of icebergs, marine ice, and sea ice. Observations at Jakobshavn Isbræ (JI) in Southwest Greenland show a strong correlation between periods when proglacial melange acts in a “rigid” manner and the calving-front advances (Joughin et al., 2008b). More recent observations at Store glacier indicate that the presence of melange provides a direct back stress to the calving front that acts to buttress glacial flow (Walter et al., 2012). Modeling efforts have sought to examine the impact of melange on glacier flow by applying a back stress as a boundary condition with results
presented in an idealized 1D flow-line model (Nick et al., 2010), a 2D plan-view model of Store glacier (Howat et al., 2010), and a 2D vertical flow-line model of Helheim glacier (Cook et al., 2014). Results for the former two models found melange could play an important role in calving-front dynamics while the third found that melange did not play a significant role in controlling front behavior. Other work suggests melange may play a role in preventing calved bergs from rotating away from the calving front (Joughin et al., 2008b; Amundson et al., 2010).

Several theories seek to explain why melange may affect calving-front dynamics including: The presence of melange altering ocean currents that play a role in melt/ freeze processes at the calving front, the suppression of wave energy thought to potentially trigger calving events, and the mechanical back stress communicated to the calving front. Each hypothesis is challenging to explore given limited data sets for melange rheology, melange thickness, and embayment ocean-temperature and flow. One step forward in understanding the third hypothesis is to explicitly simulate the melange and examine under what conditions and to what magnitude this buttressing effect can be produced. The choice of treatment for melange rheology depends on material properties that are somewhere between a geologic granular-medium, where there is no cohesion between clasts and encasing material, and a continuous medium that makes up much of the ice shelf itself.

The work presented here seeks to examine the potential importance of the effect melange can have on a calving front in the limiting case that melange can be treated as a continuous medium. We note that this is a theoretical limit and whether or not melange will reach this level of strength is uncertain, nevertheless it provides a valuable starting point for assessing the potential importance of melange-induced back-stress to glacier dynamics. Future research should examine a more realistic granular treatment. The other limiting case, that in which there is no cohesion within the melange, has been studied in the laboratory by Kuo and Dennin (2013). In their work they were looking at the ability for circular plastic blocks...
floating in water to episodically “jam”, communicating shear stress from the container sides to an advancing surface, before breaking apart after releasing a build up of stress. Here we are looking at the case in which icebergs come into contact, in the absence of slippage along grain boundaries, such that primary deformation occurs within the bergs in a continuous way. A calving front advancing into an embayment, in the absence of calving, would tend to compress melange. We assume that interstitial sea-ice that serves to connect these bergs may initially be ∼ 1 m thick but would buckle under compressive stress as bergs on the order of ∼ 0.1 km to ∼ 1 km come into contact. In this way it seems clear that melange should be treated as a medium significantly thicker than ∼ 1 m.

While other models simply apply the melange-induced back-stress as a boundary condition, we seek to model the melange itself, this being the first time to our knowledge that melange has been simulated within the domain of a model. We carry out this study by utilizing a 2D, depth-averaged, finite-element model that allows us to capture the communication of stresses within the melange. While the model is idealized, we select glacier parameters to produce calving-front velocity and calving-front strain-rate similar to those observed at the calving front of JI in the 1990s (Joughin et al., 2004; Csatho et al., 2008). The stress-equilibrium equations used here do not account for basal-shear stress assuming, in our JI-like scenario, that JI is weakly coupled to its bed or at flotation. Given that we’re not temporally evolving the glacier and that calving-front dynamics are largely controlled by properties near the calving front (Hindmarsh, 2012) this idealized setting is suitable for our purposes. Results indicate that melange will be able to buttress strain-rates while leaving velocity largely unaffected. Utilizing a heuristic empirical calving parameterization derived from a wide variety of ice-shelf data (Alley et al., 2008) that varies calving rate nearly linearly with strain rate we see a significant reduction in calving rate, thus enabling a calving-front advance.
4.2 Model description

The domain is split into two regions (Fig. 4.1a) with ice-shelf region and ocean region. Numerically, we smoothly transition the ice thickness from shelf to melange (Fig. 4.1c). For numerical reasons, melange cannot have zero ice thickness, thus we use a near-zero value of 1 cm to represent a melange-free condition.

Figure 4.1: (a) Modeled plan-view ice shelf. Only half of ice shelf is modeled due to symmetry. (b) Side-view of model ice shelf. (c) Numerical treatment of calving front and melange showing smooth transition zone of tunable length $2l$. 

70
We utilize stress-equilibrium equations appropriate for ice-shelf/glacier flow (MacAyeal, 1989) with negligible vertical shear. These equations are applied continuously across both glacial ice and melange. The equations are solved using the finite-element method with linear basis-functions across triangular elements. The primary variable solved for is the depth-integrated velocity \( \vec{u} = [u, v] \) where \( u \) is the \( x \)-directed velocity and \( v \) is the \( y \)-directed velocity. Other variables, such as strain rates, are calculated from these quantities.

The stress-equilibrium equations are:

\[
\frac{\partial}{\partial x} \left( 2\hat{\nu}h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left( \hat{\nu}h \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) - \rho gh \frac{\partial z_s}{\partial x} = 0, \tag{4.1}
\]

\[
\frac{\partial}{\partial y} \left( 2\hat{\nu}h \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) \right) + \frac{\partial}{\partial x} \left( \hat{\nu}h \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) - \rho gh \frac{\partial z_s}{\partial y} = 0, \tag{4.2}
\]

where, \( h \) is the thickness field, \( \rho \) is the density of ice or melange, \( g \) is the gravitational constant, \( \hat{\nu} \) is the depth-averaged effective viscosity and, assuming hydrostatic equilibrium, the surface elevation is \( z_s = (1 - \rho/\rho_{sw})h \) where \( \rho_{sw} \) is the density of sea water. The thickness field \( h \) is smoothly transitioned between ice thickness, \( h_i \), and melange thickness, \( h_m \), via:

\[
h = \begin{cases} 
  h_i, & \text{ice region,} \\
  h_i - (h_i - h_m) \left( \frac{\phi}{2\pi} - \frac{1}{2\pi} \sin \left( \frac{\phi}{\pi} \right) \right), & \text{transition zone,} \\
  h_m, & \text{melange region,}
\end{cases} \tag{4.3}
\]

where \( l \) is the transition half-length and \( \phi \) is the signed-distance function that specifies the location of the calving front. We define \( \phi(\vec{x}) \) as the signed distance from a point in the domain \( \vec{x} = [x, y] \) to the set of points \( \vec{x}_c = [x_c, y_c] \) that make up the curve \( \phi(\vec{x}_c) = 0 \).
\[ \phi(\vec{x}) = \pm \min(|\vec{x} - \vec{x}_c|) \]  

(4.4)

where the sign is prescribed as negative in the ice-shelf region and positive in the melange region. The curve made up of points, \( \vec{x}_c \), is the location of the top of the calving front. The sine function in equation 4.3 steepens to a cliff-like condition for smaller \( l \) values.

We use the Glen treatment for ice rheology with flow exponent, \( n = 3 \), based on laboratory compression of ice (Glen, 1955) generalized to glacier flow (Nye, 1957). The depth-averaged effective viscosity is defined as:

\[
\bar{\nu} = \frac{B^2}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \frac{1}{4} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right] \frac{1-n}{2n},
\]  

(4.5)

where, for lack of a better estimate, we make an isothermal assumption for the ice and melange regions with a uniform depth averaged ice-hardness parameter \( B = B_i \) in the ice region, \( B = B_m \) in the melange region, and a sine function transition between the two in the transition zone. \( B \) is given similarly to equation 4.3 as:

\[
B = \begin{cases} 
B_i & \text{ice region,} \\
B_i - (B_i - B_m) \left( \frac{\phi}{2\pi} - \frac{1}{2\pi} \sin(\frac{\pi \phi}{T}) \right) & \text{transition zone,} \\
B_m & \text{melange region.}
\end{cases}
\]  

(4.6)
Two boundary conditions along four boundaries are necessary to solve the stress-equilibrium equations. These are illustrated in Figure 4.2a. At the inlet, we utilize a sine function to smoothly increase velocity from embayment wall, $u_e$, to centerline flow, $u_{ic}$:

$$u(y) = u_e \sin \left( \frac{\pi y}{2W} \right) + u_{ic}, \quad (4.7)$$

where $W$ is the glacier half-width, $u_{ic}$ is the inlet centerline velocity, and $u_e$ is the inlet embayment velocity. At the embayment wall we assume shear dominates the stress within a narrow band and apply a non-linear boundary condition:

$$\sigma_{xy} = 4hB_e \left( \frac{u_{y_e}}{4y_e} \right)^{1/n}, \quad (4.8)$$

where $B_e$ is the hardness parameter within the shear zone, $y_e$ is the width of this boundary layer, and $u_{y_e}$ is the velocity along the interior edge of the boundary layer (for derivation see appendix B). This boundary condition allows the resistance to flow experienced at the embayment wall to depend on glacier velocity. A higher $B_e$ value provides more resistance and a lower $B_e$ value provides less resistance. At the downstream boundary, away from the ice shelf, we apply $x$-directed and $y$-directed stress-free conditions. At the centerline, we do not allow cross flow but otherwise enforce a stress-free condition.

Two additional boundary conditions, the surface and basal stresses, are incorporated into the flow equations when we vertically integrate them (Fig. 4.2b). For more see appendix A. It’s important to note that the stress condition at the base of the ice shelf allows us to apply the correct ocean pressure at the calving front without need for an additional
boundary-condition. Normally the theoretical boundary-condition prescribed at a vertical calving-front enforces the correct ice pressure and ocean pressure. For the outward pointing normal \( \hat{n} = [n_x, n_y] \), in the case when \( n_x = 1 \) and \( n_y = 0 \), the depth-averaged theoretical boundary-condition is given as:

\[
(2h\bar{\nu}(2\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy})) = \frac{\rho gh^2}{2} \left( 1 - \frac{\rho}{\rho_{sw}} \right), \tag{4.9}
\]

where the left hand side (LHS) denotes the dynamic deviatoric-stress and the right hand side (RHS) terms consist of both the glaciostatic component of the ice pressure and the ocean pressure. Our approach approximates this boundary condition implicitly when we solve the stress-equilibrium equations. A detailed discussion of this boundary condition is left to appendix C, however, we can calculate the LHS and RHS of equation 4.9 using model output and compute the relative error between the two. Figure 4.3 shows the relative error at the centerline calving-front decrease from \( \sim 28\% \) for 10 km transition width, 2\( l \), to \( \sim 1\% \) for 1 km transition width.
Figure 4.2: (a) Plan-view schematic diagram with appropriate stress-equilibrium boundary-conditions for each boundary. (b) Boundary conditions contained within the depth-averaged stress-equilibrium equations where $T$ is the stress tensor, $n_b$ is the outward pointing normal at the base of the ice shelf, and hydrostatic equilibrium has been assumed.
We utilize a calving parameterization to estimate the impact melange can have on calving-front motion. This parameterization was derived by use of a best-fit curve through a wide array of data measurements from ice shelves in both Antarctica and Greenland (Alley et al., 2008). The calving rate is effectively linearly dependent on strain rate and is given by:

\[
\dot{c} = c_a \left( hW \frac{\partial u_c}{\partial x} \right)^{0.98},
\]

where \( u_c \) is the centerline calving-front velocity and \( c_a \) is the Alley calving-coefficient.
We seek an estimate for the potential importance of melange-induced buttressing on calving-front dynamics and insight into potential calving-front advance for glaciers like JI. To motivate these experiments we choose parameters that allow centerline calving-front velocity and centerline calving-front strain-rate to match observations at the JI calving-front (Table 4.1). The glacier geometry, velocity, and longitudinal strain-rate fields, in the case when melange is absent, are shown in Figure 4.4. The centerline calving-front velocity, centerline calving-front strain rate, and centerline calving-front calving rate are sampled at the start of the calving front descent, $x = 15$ km and $y = 3$ km. For simplicity, these three quantities will simply be referred to as velocity, strain rate, and calving rate hereafter. The thickness field shows the transition from full ice-shelf thickness down to a nominal value of 1 cm. The velocity peaks at the calving front before slowing down as thickness decreases and shear slows flow. Similarly longitudinal strain-rate, $\frac{\partial u}{\partial x}$, increases to the front before becoming negative in the ocean region.
Figure 4.4: Thickness field, magnitude of flow, and x-directed longitudinal strain-rate. (a) From left to right: Uniform ice-shelf thickness with transition zone over 2 km from 15 km to 17 km along x-axis to uniform melange-thickness. In this case, the experiment corresponds to “no melange” with thickness of 1 cm. (b) From left to right: Inlet flow smoothly tapers in the y-direction from maximum at the centerline to minimum at embayment wall. In the x-direction, the flow accelerates to a maximum at centerline calving-front. The flow then slows as thickness is near zero and shear at embayment wall continues to slow ice. (c) From left to right: At the inlet, longitudinal strain-rate adjusts to kinematic boundary-condition. Away from the inlet, positive strain-rate increases up to a maximum at calving front. As thickness decreases toward minimal value, strain rate is negative due to shearing at wall. Strain rate is zero at far-right boundary as the stress-free condition dictates.
Parameter | Base value
---|---
$W$ | 3 km
$L$ | 15 km
$h_i$ | 1100 m
$h_m$ | 0.01 m
$\rho_i$ | 917 kg/m$^3$
$\rho_m$ | 917 kg/m$^3$
$\rho_{sw}$ | 1028 kg/m$^3$
$B_i$ | $1.36 \times 10^6$ Pa s$^{1/3}$
$B_m$ | $1.0245 \times 10^6$ Pa s$^{1/3}$
$n$ | 3
$g$ | 9.81 m s$^{-2}$
$B_e$ | $2.1 \times 10^6$ Pa s$^{1/3}$
$y_e$ | 10 m
$c_a$ | $0.011 m^{-0.96} a^{0.02}$
$l$ | 1 km
$u_{ie}$ | 4525 m a$^{-1}$
$u_c$ | 5025 m a$^{-1}$

Table 4.1: Base parameter values used for idealized JI.

### 4.3 Experiments and results

We examine the initial response of the strain rate and calving-front motion to the presence of melange. To do this we adjust the Alley coefficient to allow the calving rate to match the velocity for the melange-free state. We do this because the assumption that there is no change in centerline front-position was a key assumption in the derivation of the Alley calving-parameterization and generally true over a summer at JI. While evolving the glacier in time (and attempting to account for variables such as ocean freeze-on, ocean melt, surface melt, and accumulation) is left to future work, we do examine the initial calving-front response. The relative independence of strain rate and velocity to other glaciological parameters outside a calving-front boundary-layer (Hindmarsh, 2012) indicate our idealized glacier geometry is suitable for this purpose.

Starting with a calving front in which velocity balances calving rate under melange-free conditions, we examine how the velocity, strain rate, and calving rate respond to buttressing
produced by a broad range of melange thicknesses. Figure 4.5 shows the reduction in strain rate for a range of prescribed melange thicknesses and Figure 4.6 shows the nonlinear character of the calving rate response to melange thickness and the much more subdued response in front velocity. In both cases there is a higher reduction with greater melange-thickness. Having started with a balance between velocity and calving rate for the melange-free state, the preferential reduction in calving rate over velocity upon the introduction of melange implies a front advance, more ice is arriving than calving would remove.

If we adopt a melange thickness of 100 m, velocity is reduced by 5% while longitudinal strain rate is lowered by 28% with a corresponding suppression in calving rate of 28%. The front has an initial advance of 1100 m/a (3.1 m/d). Due to the non-linear response to melange-induced buttressing, thicker melange would lead to significantly higher buttressing. In Figure 4.7 we measure the buttressing stress for various melange thicknesses up to 550 m. We calculate the buttressing effect by measuring the stress at the centerline calving front, $2\nu\dot{\epsilon}_{xx}$, when no melange is present and then subtract the stress measured when melange is present. The difference is the reduction in stress at the calving front due to the presence of melange. These buttressing stresses are in the ballpark of those observed at Store glacier of 30 to 60 kPa (Walter et al., 2012).
Figure 4.5: Strain rate sampled at $x = 15$ km and $y = 3$ km for various melange thicknesses.
Figure 4.6: Velocity and calving rate for a range of melange thicknesses.
4.4 Sensitivity analyses

Our model utilizes a number of parameters for the material properties of ice and melange: $B_i$, $B_m$, $B_e$, and $n$, and specifications for glacier geometry: $L$ and $W$. Not only are the material properties a simplification of reality, such as Glen-flow parameters ($B_i$ and $n$) depending on isotropy assumption of glacier ice, but there are also poorly constrained dependencies of these parameters, such as the spatial dependence of $B_i$ on temperature. In order to gauge how the model depends on these parameters we perturb these values one at a time and rerun the model to get a sense of system sensitivity. We focus on two metrics for determining the system response. First, since we are interested in the response of strain rate, $\frac{\partial u}{\partial x}$, as the...
fundamental control for calving we provide figures illustrating that response. Second, since
we are interested in the initial response of the front position, velocity minus calving rate, we
also display figures showing the sensitivity of front position. Due to the fact that calving rate
depends on $W$, $B_e$, and $B_i$ and the fact that we want velocity to balance calving rate when
no melange is present, we adjust the Alley coefficient, $c_a$ in equation 4.10 to rebalance these
two quantities before increasing melange thickness. All parameters not being perturbed are
those given in Table 4.1.

4.4.1 System sensitivity to material properties: $B_i$, $B_m$, $n$, and $B_e$

We examine the model response to a range of ice-hardness parameter values corresponding
to temperatures appropriate for JI (Cuffey and Paterson, 2010). The ice hardness param-
eter, $B_i$, is a poorly constrained quantity due to sparse temperature measurements at JI.
Temperature data is limited at the calving front, however, upstream borehole measurements
indicate the temperature ranges from $-22^\circ C$ near the surface to $-10^\circ C$ at depth approach-
ing $0^\circ C$ at the bed (Iken et al., 1993; Luthi et al., 2002). In Figure 4.8, we vary strain rate
with respect to melange thickness from 0.01 m to 550 m along the x-axis and $B_i$ along the
y-axis. Results indicate a substantial reduction in strain rate for higher $B_i$ values, but no
matter what value for $B_i$ we set, the response of the calving front to the presence of melange
is similar. For example, when $B_i$ corresponds to $-10^\circ C$ on the y-axis, the percent change
in strain rate between melange free and 550 m thickness is 94.9% while the percent change
is 94.8% when $B_i$ corresponds to $-21^\circ C$. Since strain rate (and thus calving rate) has a
similar sensitivity to melange thickness for any $B_i$ value, then front motion is insensitive to
$B_i$ with front advance 1100 m/a (3.1 m/d) (Fig. 4.9).
Figure 4.8: Strain rate plotted as a function of ice hardness $B_i$ and melange thickness.
In order to gauge how strain rate and calving-front motion respond to variations in the material properties of the melange we vary the melange hardness $B_m$ in Figures 4.10, 4.11 and the flow exponent $n$ over the melange region in Figures 4.12, 4.13. Melange temperature was chosen to correspond to weak melange, $B_m$ corresponding to $0^\circ$ C, and hard melange equal to the upper bound $B_i$ value corresponding to $-21^\circ$ C. We see that the system is somewhat insensitive to variation in melange hardness. For example, for a melange thickness of 100 m, the buttressing of strain rate is 7% greater for $B_m$ corresponding to $-21^\circ$ C than $B_m$ corresponding to $0^\circ$ C while the front advances at a rate of 1300 m/a ($\sim 3.6$ m/day) and 960 m/a ($\sim 2.6$ m/d) respectively.
Figure 4.10: Strain rate plotted as a function of melange hardness $B_m$ and melange thickness.
In the sensitivity experiments involving $n$ we consistently use the standard value of 3 for the ice-shelf region but vary $n$ between 3 and 4 for the melange region. Increasing $n$ for the melange region creates a rheology that behaves with greater plasticity. For a fixed melange thickness, strain rate is more sensitive to the presence of melange when $n = 4$ than when $n = 3$. For 100 m melange thickness, strain rate decreases 26% for $n = 3$ and 48% for $n = 4$ suggesting a higher $n$ corresponds to a greater level of buttressing. Figure 4.13 similarly suggests a greater rate of advance for $n = 4$ than for $n = 3$ with the calving front advancing at a rate of 1200 m/a ($\sim 3.3$ m/day) for $n = 3$ and 2200 m/a ($\sim 6.1$ m/d) for $n = 4$. Interestingly, there is a maximum rate of advance for $n > 3.1$ that peaks and then begins to reduce for melange thickness values greater than 450 m (upper-right region of Fig. 4.13).
Since strain rate is negative in this region (Fig. 4.12), and calving is zero, this decrease is due to the direct reduction in velocity. Also of note, we see that when no melange is present in Figure 4.12 the strain rate slightly decreases as $n$ increases. This may seem counterintuitive since melange has a near zero thickness and should not affect glacier strain-rates. This reduction in strain rate for increasing $n$ is a consequence of the transition zone since the transition zone contains intermediate $n$ values transitioned similarly to equation 4.3. This decrease in strain rate corresponds to a mixture of materials of intermediate rheologic properties. These modeling choices are consistent with the fact that the calving front at JI is often difficult to distinguish from melange in winter months (Amundson et al., 2010) indicating intermediate states of continuous and plastic deformation.

Figure 4.12: Strain rate plotted as a function of melange-region flow-exponent $n$ and melange thickness.
Figure 4.13: Velocity minus calving rate plotted as a function of melange-region flow-exponent \( n \) and melange thickness.

The choice of value for \( B_e \) within the shear zone depends on the hardness of ice within this region. In turn, the hardness depends on material properties of the ice including temperature and crystal fabric structure. Both are expected to enhance flow given that frictional heating warms the ice and anisotropic crystal fabric orients in response to shear subsequently enhancing shear. Furthermore, fracturing of ice is often seen in the shear zone so that the standard assumption that ice behaves as a strain rate weakening material is tenuous. The shear zone choice of \( B_e \) reflects these considerations by being significantly smaller than that of ice outside this region and we choose a base value of \( B_e = 2.1 \times 10^6 \text{ Pa s}^{1/3} \) for a \( y_e = 10 \text{ m} \) shear zone. We choose this value in large part because it allows us to produce the correct centerline calving-front strain-rate as observed at JI. Indeed, we were not able to produce the
appropriate JI-like strain rate with a prescribed uniform embayment wall velocity (Dirichlet condition). We conduct a sensitivity analysis around the base $B_e$ value for a range of both weaker $B_e$ values and stronger $B_e$ values (Fig. 4.14, 4.15). We see that both strain rate and front motion are more sensitive to the presence of melange for higher $B_e$ values indicating that melange is communicating a larger back stress to the front when melange is more strongly coupled to the embayment wall. For 100 m melange thickness, strain rate decreases by 17% for $B_e = 1.17 \times 10^6$ Pa $s^{1/3}$ and 27% for $B_e = 2.57 \times 10^6$ Pa $s^{1/3}$ while the front advances at a rate of 820 m/a ($\sim 2.2$ m/day) and 1200 m/a ($\sim 3.3$ m/d) respectively.
Figure 4.14: Strain rate plotted as a function of shear-zone hardness $B_e$ and melange thickness.
Figure 4.15: Velocity minus calving rate plotted as a function of shear-zone hardness $B_e$ and melange thickness.

### 4.4.2 System sensitivity to embayment geometry properties: $L$ and $W$

Here we examine the sensitivity of results to variations in $L$ where $L$ is the length over which melange experiences shear. Examining a range of $L$ values, from 1 km to 25 km around the base value of 15 km, tells us how the length of melange affects the buttressing of calving-front dynamics. As expected, we see a markedly smaller buttressing effect for smaller values of $L$ for both Figures 4.16 and 4.17. Interestingly however, we see that melange still has a substantial buttressing effect for a small value of $L = 1$ km. In this case, we see longitudinal
strain rate is buttressed by 10% while for $L = 25$ km strain rate is buttressed by 30%. For a fixed melange thickness, the sensitivity of melange-induced buttressing is smaller for larger $L$ values indicating $L$ values larger than 25 km will not substantially increase the buttressing effect. For 100 m melange thickness inducing shear over 1 km and 25 km, we see an initial calving-front advance of 490 m/a ($\sim 1.3$ m/day) and 1400 m/a ($\sim 3.8$ m/d) respectively.

Figure 4.16: Strain rate plotted as a function of length over which melange experiences shear $L$ and melange thickness.
Figure 4.17: Velocity minus calving rate plotted as a function of length over which melange experiences shear $L$ and melange thickness.

Next we look at the system sensitivity for domain half-width, $W$, in Figures 4.18 and 4.19. Here we use the base value 3 km and examine larger half-widths of up to 9 km. As expected the ability for melange to buttress strain rate and affect calving-front motion decrease with increasing width since the centerline is farther from where melange generates buttressing. The sensitivity to a melange thickness of 100 m is 24% for $W = 3$ km and 15% for $W = 9$ km while the front advances at a rate of 1100 m/a ($\sim 3.1$ m/day) and 720 m/a ($\sim 2.0$ m/d) respectively.
Figure 4.18: Strain rate plotted as a function of ice-shelf half-width $W$ and melange thickness.
At this point, it’s useful to note a common issue of not only this model, but glacier models in general, where two boundary-conditions intersect. In Fig. 4.18, note that when no melange is present, the strain rate increases as $W$ increases. In theory, the value should remain constant due to the fact that centerline calving-front strain-rate is independent of how far or close it is to the shear zone along the embayment. Numerically, we have a mismatch between the boundary condition at the embayment wall and the strain rate being generated by the calving front. Along the calving front the ocean pressure should be balanced by the internal stresses made up of the glaciostatic component and dynamic deviatoric-stress component of pressure in the ice shelf. The depth-averaged theoretical boundary-condition was given in equation 4.9 and we can analyze how well we are capturing this boundary condition within

Figure 4.19: Velocity minus calving rate plotted as a function of ice-shelf half-width $W$ and melange thickness.
our model. The LHS can be calculated at element centroids using modeled strain rates and can then be compared to the RHS calculated using model glacier thickness. As discussed, the relative error between the computed LHS and RHS should be zero along the calving front since no buttressing occurs. In actuality, use of the kinematic boundary-condition for the embayment wall results in a mismatch between the LHS and RHS with a maximum error at the intersection of calving front and wall. To see this more clearly we plot the relative error for points along the calving front (Fig. 4.20) for a variety of half widths ranging from 3 km to 18 km. We see that as we increase $W$ from $W = 3$ km (blue) to $W = 18$ km (cyan) the model does a better job of producing the theoretical boundary-condition, however, it still does a poor job near the embayment wall. Since we are focused on the centerline measurements in this work, we can safely ignore this error in enforcement of the boundary condition. As shown in Figure 4.20, the relative error is just a few percent at the centerline calving-front for $W = 3$ km, small enough for present purposes.

![Figure 4.20: Relative error for a range of $W$ values. Each line corresponds to a position along the calving front from embayment wall to centerline.](image)
4.5 Discussion

Our results illustrate the immediate response of an idealized ice shelf with similar front dynamics, thickness, and embayment width as JI to the presence of melange-induced buttressing. We show a high sensitivity of strain rates to the presence of melange and relative insensitivity of velocity. For any melange thickness the calving front inevitably advances into the embayment. Despite the obvious differences in complexity between this model geometry and rheology to that of JI, we would expect the calving front at JI and associated strain rates to be similarly sensitive.

The Alley calving-rate parameterization has been critiqued as describing the viscous flow processes at the calving front rather than the processes involved in calving (such as ocean temperature, ocean currents, wave energy, etc.) (Amundson and Truffer, 2010; Hindmarsh, 2012). What the Alley parameterization describes is a relationship to centerline strain rate, the principal variable that controls fracture formation and hence calving (Benn et al., 2007b). Other dynamic calving treatments generally rely on the physical properties of the glacier, such as thickness, and strain rates at the front (Amundson and Truffer, 2010; Levermann et al., 2012) and also fracture filled water depth (Benn et al., 2007a). These parameterizations may fit observations better for specific situations and may be more useful in cases where surface melt is important, but until calving parameterizations account for processes such as ocean-induced melt, strain history, rift-propagation mechanics as seen in larger ice shelves like the Ronne ice-shelf (Hulbe et al., 2010), they are all, essentially, variations on functions of the principal control of calving: strain rates. For these reasons, we utilize the heuristic Alley treatment due to its derivation from data measurements.

Commentary on the continued evolution of the JI-like glacier past its initial response, based on the present work, is speculative. One possible response is that, as the front advances in the presence of melange, increased shear stress at the embayment wall would add a negative
feedback to the calving-front velocity and reduce subsequent advance. An advancing ice shelf will also have a tendency to thin, reducing the magnitude of strain rate generated at the front and further reduce velocity at the calving front. However, a thinner shelf will have reduced strain rates resulting in a reduction in calving rate and potentially allowing for further advance. Ultimately, a time evolving model, with moving calving front, will be needed to explore the various possibilities.

Incorporating melange within the domain, and modeling the potential effect it can have on a calving front, is a first step in the direction of a more realistic examination of these effects. The limitations in the model stem from a lack of observational data for melange and glaciers including JI. While some observations of glaciers show an annual calving-front advance/retreat cycle, winter velocity slowdown, and calving rate reduction correlated with the presence/absence of melange (e.g. Joughin et al., 2008b), measurements of melange are generally taken in summer when weather and sunlight availability make measurements more feasible (e.g. Walter et al., 2012). Further observations of melange, especially in winter, including more thickness data, velocity data, and derived strain rates, will better illuminate the rheology and geometry of melange. More detailed observations will inform more accurate numerical treatments of melange and further quantify the mechanical impact melange has on a calving front. In the limiting case where icebergs act as individual blocks with no cohesion, Kuo and Dennin (2013) show the shape of the bergs play an important role in the jamming mechanism. Further observations would inform the size and shape of bergs and the importance of this mechanism.

The sensitivity analyses give us a sense of weaknesses in the model and which parameters need to be better constrained. Our results indicate a high sensitivity of strain rate to $B_i$. A colder ice shelf experiences a greater viscosity, resisting flow, and reducing strain rates. Narrowing down the uncertainty in $B_i$ at JI and accounting for spatial variability is particularly important for these modeling efforts. Interestingly, strain rate is relatively
insensitive to $B_m$ indicating the hardness of the melange is not the critical quantity to constrain for our results. The strain rate is far more sensitive to melange thickness and this is an important quantity to constrain with field measurements. Calving-front motion is highly sensitive to Glen’s flow-exponent indicating that the rheological treatment of melange is crucial and that a more plastic rheology provides a greater back stress allowing for advance.

### 4.6 Conclusions

We have modeled the melange in the limiting case where it can be treated as a continuous medium to get a sense of its potential importance in buttressing: Velocity, strain rates, and calving rate. We find that melange is capable of suppressing longitudinal strain-rate while leaving velocity largely unaffected. Furthermore, we find melange can initiate a calving-front advance of 490 m/a for just 100 m melange thickness that shears over a 1 km length of embayment and up to 1200 m/a for melange that shears over a 15 km embayment. Likely the melange is weaker at JI and the advance that could be attributed to strain-rate suppression is smaller. Ultimately, the observed cumulative winter-advance of $\sim 5$ km (Joughin et al., 2008a) may involve a combination of suppressed calving-rate and the presence of calved bergs unable to rotate away from the front (Amundson et al., 2010). Other effects such as altered ocean tides and cessation of surface meltwater induced hydrofracturing need to be explored. Melange may be an important mechanism in allowing glaciers to rapidly advance into embayments during periods of cooler climate when melange would be stronger. In a warming climate the loss or weakening of melange will reduce these contributions to seasonal calving-front advance at glaciers where melange-induced buttressing is important such as at Store glacier. Loss or weakening of melange may consequently result in increased volume loss for these outlets.
Chapter 5

Thesis conclusions

Here we briefly summarize the contents of each chapter. We then list key results and ways this thesis has advanced upon past work. Finally, we discuss shortcomings of this thesis and outline potential future directions for research.

Antarctic and Greenland ice sheet margins interact dynamically with the ocean losing mass through both ocean-induced melt and iceberg calving. While observations, both satellite and in situ, advance our understanding of the relationship between ice sheet margins and oceanic forcings, modeling efforts to simulate these processes face numerous challenges. One particular challenge lies in simulating the temporal evolution of an ice shelf calving front while maintaining the appropriate boundary condition at the ice-ocean interface. The calving front may initially be defined at a set of nodes, however, as the ice shelf evolves through time the front will fall between nodes making enforcement of the boundary condition numerically difficult. One solution to this problem was found for utilizing a sub-grid-scale parameterization of the calving front location (Winkelmann et al., 2011) while also conserving mass (Albrecht et al., 2011) for the Antarctic ice sheet model PISM-PIK. For a Cartesian grid they were able to track the amount of partial ice within an element, redistributing small quantities
of ice as needed when the front passes from one element to another. While successful at conserving mass and limiting diffusion of the calving front they still apply the calving front boundary condition at nodes, jumping this condition as the front advances or retreats. Their method is also limited to Cartesian grids making application in triangular meshes less than straight-forward.

In chapter 2 we developed an ice shelf model utilizing techniques, the level set method and diffuse domain method, that allow us to evolve the calving front. Our methods allow for the continuous motion of the calving front without need for application of boundary condition at nodes. Additionally, we have chosen techniques that conserve mass and do not require redistribution of mass. Our techniques also maintain the shape of the calving front, without any front diffusion, thereby allowing for correct application of the calving front boundary condition through time. While our ice shelf model was developed for unstructured triangular meshes, the front-motion techniques are also applicable for other meshes such as Cartesian grids or hexagonal meshes. We validated the methods in both a 1D domain and idealized 2D model domain.

In chapter 3 we applied the front motion methods to the Ross ice shelf. Given the past collapse of the Ross ice shelf ∼3 Ma when CO₂ levels were ∼400 ppm, similar to present days levels, it is important to understand controls on front motion. We have compared our diagnostic speeds to observed speeds for the shelf and evolved the front through time to examine characteristics of front motion.

Seasonal calving front motion has been observed at a number of Greenlandic outlet glaciers with motion correlated to the presence of melange. While modeling efforts have sought to examine the response of the glacier to melange, the backstress of the melange has been applied as a boundary condition (Howat et al., 2010; Nick et al., 2010; Cook et al., 2014). One modeling study examined how melange may affect bergs that have already calved, suggesting
that melange may play a role in preventing these bergs from rotating away from the front (Amundson et al., 2010).

In chapter 4 we incorporated melange within the interior of the domain and examined whether and by how much melange could impact dynamics of a calving front. We set model parameters to reproduce calving front dynamics observed at Jakobshavn Isbræ. Given the fact that calving front dynamics are largely controlled by processes at the front (Hindmarsh, 2012) we use an idealized glacier to get a sense of how a Jakobshavn-Isbræ-like calving front might respond to the presence of melange. We seek to examine the potential importance of the effect melange can have on calving front dynamics in the limiting case that melange can be treated as a continuous medium. We found that melange could suppress strain rates while leaving velocities relatively unaffected. Even fairly thin 100 m thick melange shearing over a 1 km embayment wall initiated a calving front advance of 490 m/a.

5.1 Principal conclusions

5.1.1 Ice shelf calving front motion methods

1) Use of the diffuse domain method allows us to apply the appropriate calving front boundary condition independent of nodes, the first time to our knowledge this has been achieved. This allows the model to smoothly evolve the dynamics of the ice shelf, with out jumps in the stress state as application of the calving front boundary condition jumps from node to node.

2) Use of the level-set method allows us to continuously track the front between nodes. The front motion methods are independent of mesh structure and are readily applicable to Cartesian meshes, hexagonal meshes, or others.
3) Our front-motion methods prevent any dispersion of the calving front, preserving the geometry through time, allowing us to apply the calving front boundary condition correctly.

5.1.2 Simulation of the Ross ice shelf

4) Our diagnostic velocity captures the major flow features of the Ross ice shelf with only a 2% relative error between maximum numerical velocity and maximum observed velocity.

5) We evolve the Ross ice shelf calving front demonstrating our methods allow for smooth continuous motion of the front responding to ice velocity and calving rates.

6) Convergent flow at rock/ice interface shuts off calving, resulting in a slow advance along embayment walls. Without calving the front does not achieve steady state, however, ice in the interior slows with parts approaching a stationary position.

5.1.3 Examination of potential importance of melange on calving front dynamics

8) We find that melange, if strong enough, can have a significant impact on both calving front strain rates and calving rate. For 100 m of melange thickness shearing over 1 km embayment length the calving front will initially advance at a rate of 490 m/a. For 100 m melange thickness over a 15 km embayment the calving front starts advancing at 1200 m/a.

9) We find centerline calving front strain rate is significantly more sensitive to the presence of melange than velocity which is largely unaffected by the presence of melange. Since the calving rate depends on the strain rate, the presence of melange always results in front advance.
10) We find that melange-induced buttressing in embayments longer than 25 km does not significantly buttress the calving front more than if the embayment had a length of 25 km.

11) Our sensitivity studies demonstrate that the ice hardness, $B_i$, is one key parameter to constrain to ascertain the dynamic response of the calving front to melange. The results are also very sensitive to the rheology of melange, as studied in this thesis by altering the exponent of Glen’s flow treatment of ice, $n$.

## 5.2 Future work

### 5.2.1 Ice shelf calving front motion methods

1) Development of our front motion techniques utilized numerical algorithms that were straightforward and easy to understand as a starting point for initial development. For this reason we did not necessarily choose the most computational efficient algorithms. Future work should explore reducing numerical cost to make the methods more available to computationally expensive applications such as ice-sheet scale models.

2) Our choice for implementation of front-motion techniques did not seek out the most accurate numerical algorithms. In the case of the 1D and 2D validation experiments the present techniques performed adequately and did not numerically deteriorate over time. When evolving the Ross ice shelf calving front we found that numerical error accumulated over time resulted in the numerical breakdown of our calving front by producing narrow retreat at the calving front resulting in fingerlike features. While we found that increased mesh resolution alleviates these numerical issues, more accurate numerical methods may also prevent these features from developing.
3) The intersection of the calving front, which accelerates flow, and the embayment wall with fixed velocity creates a mismatch between the two boundary conditions at the calving front, this relationship and overall impact on front motion should be explored further.

5.2.2 Simulation of the Ross ice shelf

4) While we achieve smooth, continuous motion of the front, we see an unceasing slow advance of the front along the rock-ice interface. While the slow advance is due to convergent flow and a cessation of calving in this region, it’s not clear why we see convergent flow even when the front approaches regions where the rock face slopes away and one might expect divergent flow. This result may be a numerical artifact due to interaction between our transition zone, the region where ice thickness decreases from shelf to ocean, and the embayment wall. Further work should sort out whether it’s a correct feature of flow or a numerical artifact. Increasing resolution along with a decrease in transition zone length, $2l$, will increase method accuracy and potentially illuminate whether this is a correct feature of flow or not.

5) Future work should strive for more realistic flow of the ice shelf, particularly focusing on a more realistic spatially variable ice hardness as well as boundary conditions that account for hardness of ice flowing into the domain. Spatially variable ocean-induced melt rates should be estimated and implemented in model simulations.

6) Given the past collapse of the Ross ice shelf when atmospheric CO$_2$ was $\sim$400 ppm and temperatures were $3^\circ$C degrees warmer than today, accurate simulations of the ice shelf should be pursued and exploration of calving rates that better account for fracture formation and propagation should be developed.
5.2.3 Potential impact of melange on ice shelf dynamics

7) Measurements of melange are sparse, particularly in winter, and constraining physical properties of melange in our model requires spatially high resolution thickness measurements, measurements of velocity, and calculated strain rates to indicate how melange mechanically behaves during one of the seasonal glacier calving front advances such as at Jakobshavn Isbræ.

8) Our melange studies focus on exploring the potential for melange to impact ice shelf dynamics in the limiting case that melange can be treated as a continuous medium. Our results warrant further studies utilizing more realistic rheologic treatments of melange. This is particularly true given the sensitivity of our results to increase in the Glen flow exponent, \( n \), over the melange, which simulated a more plastic behavior.

9) More realistic flow and realistic embayment geometry will increase the accuracy of our results. More temperature profiles at Jakobshavn Isbræ will allow us to further constrain the ice hardness and attain more realistic results.

10) With the formation of melange, Jakobshavn Isbræ is observed to advance in winter along with a corresponding reduction in glacier velocity. Our work here simply explores the initial response of the front to the presence of melange. We do not temporally evolve the front. To further explore the seasonal advance with associated dynamic response, we suggest a synthesis of the thesis could be used to explore glacier behavior, including evolution of the calving front, over the course of a winter, from the time melange forms in winter to breakup in spring.
Bibliography


Brooks, A. N., and T. J. R. Hughes (1982), Streamline Upwind/Petrov-Galerkin formulations
for convection dominated flows with particular emphasis on the incompressible Navier-


Cook, S., I. Rutt, T. Murray, A. Luckman, T. Zwinger, N. Selmes, A. Goldsack, and T. James
(2014), Modelling environmental influences on calving at Helheim Glacier in eastern Green-

Csatho, B., T. Schenk, C. V. D. Veen, and W. Krabill (2008), Intermittent thinning of Jakob-
shavn Isbër, West Greenland, since the Little Ice Age, *Journal of Glaciology*, 54(184),
131–144.


Dupont, T. K., and R. B. Alley (2005), Assessment of the importance of ice-shelf buttressing

Elias, R. N. E., M. A. D. Martins, and A. L. G. A. Coutinho (2007), Simple finite element-
based computation of distance functions in unstructured grids, *International Journal for

Fretwell, P., H. D. Pritchard, D. G. Vaughan, J. L. Bamber, N. E. Barrand, R. Bell,
C. Bianchi, R. G. Bingham, D. D. Blankenship, G. Casassa, G. Catania, D. Callens,
H. Conway, A. J. Cook, H. F. J. Corr, D. Damaske, V. Damm, F. Ferraccioli, R. Fors-
berg, S. Fujita, Y. Gim, P. Gogineni, J. A. Griggs, R. C. A. Hindmarsh, P. Holmlund,
W. Krabill, M. Riger-Kusk, K. A. Langley, G. Leitchenkov, C. Leuschen, B. P. Luyendyk,
K. Matsuoka, J. Mouginot, F. O. Nitsche, Y. Nogi, O. A. Nost, S. V. Popov, E. Rignot,
D. M. Rippin, A. Rivera, J. Roberts, N. Ross, M. J. Siegert, A. M. Smith, D. Steinhage,
M. Studinger, B. Sun, B. K. Tinto, B. C. Welch, D. Wilson, D. A. Young, C. Xiangbin,
and A. Zirizzotti (2013), Bedmap2: improved ice bed, surface and thickness datasets for

228(1175), 519–538.

Goldner, A., N. Herold, and M. Huber (2014), Antarctic glaciation caused ocean circulation


Hindmarsh, R. (2012), An observationally validated theory of viscous flow dynamics at the
ice-shelf calving front, *Journal of Glaciology*, 58(208), 375–387.
Holland, D. M., R. H. Thomas, B. Young, M. Ribergaard, and B. Lyberth (2008), Accelera-

Howat, I., J. Box, Y. Ahn, A. Herrington, and E. Mcfadden (2010), Seasonal variability in

Hughes, T. J. R., and M. Mallet (1986), A new finite element formulation for computational
fluid dynamics: IV. a discontinuity-capturing operator for multidimensional adjective-

Hulbe, C. L., C. LeDoux, and K. Cruikshank (2010), Propagation of long fractures in the
Ronne Ice Shelf, Antarctica, investigated using a numerical model of fracture propagation, *Journal of Glaciology*, 56(197), 459–472.

Iken, A., K. Echelmeyer, W. Harrison, and M. Funk (1993), Mechanisms of fast flow in

Johnson, C. (2009), *Numerical Solutions of Partial Differential Equations by the Finite El-
ement Method*, Dover Publications.

Joughin, I., and D. R. MacAyeal (2005), Calving of large tabular icebergs from ice shelf rift

Joughin, I., W. Abdalati, and M. Fahnestock (2004), Large fluctuations in speed on Green-

Joughin, I., S. Das, M. King, B. E. Smith, I. Howat, and T. Moon (2008a), Seasonal Speedup

Joughin, I., I. Howat, M. Fahnestock, B. Smith, W. Krabill, R. Alley, H. Stern, and M. Truffer

Joughin, I., B. E. Smith, and B. Medley (2014), Marine Ice Sheet Collapse Potentially Under

Kopp, R. E., F. J. Simons, J. X. Mitrovica, A. C. Maloof, and M. Oppenheimer (2009),
Probabilistic assessment of sea level during the last interglacial stage, *Nature*, 462, 863–
867.

Kuo, C., and M. Dennin (2013), Buckling-induced jamming in channel flow of particle rafts,

Lemke, P., J. Ren, R. Alley, I. Allison, J. Carrasco, G. Flato, Y. Fujii, G. Kaser, P. Mote,
in *Climate Change 2007: The Physical Science Basis. Contribution of Working Group I to
the Fourth Assessment Report of the Intergovernmental Panel on Climate Change*, edited


Li, X., J. Lowengrub, A. Ratz, and A. Voigt (2009), Solving pdes in complex geometries: a diffuse domain approach, Communications in Mathematical Sciences, 7(1), 81–107.


Scambos, T., C. Hulbe, M. Fahnestock, and J. Bohlander (2004), Glacier acceleration and thinning after ice shelf collapse in the Larsen B embayment, Antarctica, *Geophysical Research Letters*, 31(18), L18,402.


Appendix A

Derivation of stress-equilibrium equations

Realistic motion of the calving front depends on capturing the appropriate stress imbalance at the calving front. Here I derive the depth-averaged stress-equilibrium equations appropriate for ice shelf flow (or flow over weak beds) (MacAyeal, 1989). The purpose of this appendix is to provide reference for discussion of our treatment of the appropriate boundary condition at the calving front. Additionally, one goal of this appendix is the derivation of the applied stress boundary condition at the theoretical vertical calving-front for any forcing function $f$. I first establish the Stokes equations for flow followed by a statement of the appropriate boundary conditions at the surface and base. Next, I depth integrate the equations utilizing surface and basal boundary conditions. This is followed by application of simplifications appropriate for ice-shelf flow and the derivation of the final stress-equilibrium equations. Lastly, I derive the appropriate theoretical boundary condition for a calving-front at flotation.
A.1  Stress, strain, and stokes

I begin with the viscous treatment for flow as described by Glen (1955):

\[ T' = 2\nu \dot{\varepsilon} \quad (A.1) \]

where \( T' \) is the deviatoric stress controlling the deformation of ice, \( T' = T + PI \) where \( T \) is the stress, \( P = -\frac{1}{3}T_{ii} \) is the pressure, and \( \dot{\varepsilon} \) is the \( 3 \times 3 \) strain rate tensor with entries

\[ \dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

The general equations for viscous flow are controlled by the Stokes equations in which the divergence of the stress tensor is balanced by the body forces:

\[ -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}(2\nu \dot{\varepsilon}_{xx}) + \frac{\partial}{\partial y}(2\nu \dot{\varepsilon}_{xy}) + \frac{\partial}{\partial z}(2\nu \dot{\varepsilon}_{xz}) = 0 \quad (A.2) \]
\[ -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}(2\nu \dot{\varepsilon}_{yx}) + \frac{\partial}{\partial y}(2\nu \dot{\varepsilon}_{yy}) + \frac{\partial}{\partial z}(2\nu \dot{\varepsilon}_{yz}) = 0 \quad (A.3) \]
\[ -\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}(2\nu \dot{\varepsilon}_{zx}) + \frac{\partial}{\partial y}(2\nu \dot{\varepsilon}_{zy}) + \frac{\partial}{\partial z}(2\nu \dot{\varepsilon}_{zz}) = \rho g \quad (A.4) \]

A.2  Ice at the surface, ice at the base, and ice in-between

In order to depth-integrate, I need to establish the appropriate boundary conditions at the surface and at the base of the ice shelf.
A.2.1 Surface boundary condition

At the surface elevation, \( z = z_s \), the stress vanishes:

\[
T \cdot \hat{n}_s = 0 \quad (A.5)
\]

The outward pointing normal for the surface \( z_s \) at point \((x_o, y_o)\) is given, in general by,

\[
\begin{bmatrix}
-\frac{\partial z_s}{\partial x} & -\frac{\partial z_s}{\partial y} & 1
\end{bmatrix}^T.
\]

The unit normal is then:

\[
\begin{bmatrix}
-\frac{\partial z_s}{\partial x} & -\frac{\partial z_s}{\partial y} & 1
\end{bmatrix}^T \frac{1}{\sqrt{\left(\frac{\partial z_s}{\partial x}\right)^2 + \left(\frac{\partial z_s}{\partial y}\right)^2 + 1}}
\]

where the sign dictates the direction of the normal. Expanding equation A.5 we get

\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_{yy} & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_{zz}
\end{bmatrix}
\begin{bmatrix}
-\frac{\partial z_s}{\partial x} \\
-\frac{\partial z_s}{\partial y} \\
1
\end{bmatrix}^T
= 0,
\]

(A.6)

and after matrix multiplication:

\[
\begin{align*}
\sigma_{xx} \frac{\partial z_s}{\partial x} + \tau_{xy} \frac{\partial z_s}{\partial y} + \tau_{xy} &= 0, \\
\tau_{xy} \frac{\partial z_s}{\partial x} + \sigma_{yy} \frac{\partial z_s}{\partial y} + \tau_{yz} &= 0, \\
\tau_{xz} \frac{\partial z_s}{\partial x} + \tau_{yz} \frac{\partial z_s}{\partial y} + \sigma_{zz} &= 0,
\end{align*}
\]

(A.7) (A.8) (A.9)
which, in terms of strain rate, becomes:

\[
(2\nu \dot{\epsilon}_{xx} - P) \frac{\partial z_s}{\partial x} + 2\nu \dot{\epsilon}_{xy} \frac{\partial z_s}{\partial y} + 2\nu \dot{\epsilon}_{xy} = 0, \tag{A.10}
\]

\[
2\nu \dot{\epsilon}_{xy} \frac{\partial z_s}{\partial x} + (2\nu \dot{\epsilon}_{yy} - P) \frac{\partial z_s}{\partial y} + 2\nu \dot{\epsilon}_{yz} = 0, \tag{A.11}
\]

\[
2\nu \dot{\epsilon}_{xz} \frac{\partial z_s}{\partial x} + 2\nu \dot{\epsilon}_{yz} \frac{\partial z_s}{\partial y} + (2\nu \dot{\epsilon}_{zz} - P) = 0. \tag{A.12}
\]

### A.2.2 Basal boundary condition

The basal boundary condition is not stress free. There is the pressure at depth \( \rho gh \) dotted onto the basal surface outward pointing unit normal \( \hat{n}_b \), written as

\[
T \cdot \hat{n}_b = -\rho gh \hat{n}_b. \tag{A.13}
\]

Here, assuming hydrostatic equilibrium, the ocean pressure is equal to ice pressure along \( z_b \). Similarly to the surface boundary equation we expand equation A.13 as

\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_{yy} & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_{zz}
\end{bmatrix}
\begin{bmatrix}
-\frac{\partial z_b}{\partial x} \\
\sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1 \\
\frac{1}{\sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1}
\end{bmatrix}
= \begin{bmatrix}
\frac{\rho gh}{\sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1} \\
\frac{\rho gh}{\sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1} \\
-\frac{\rho gh}{\sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1}
\end{bmatrix}, \tag{A.14}
\]

which, in terms of strain rate, results in:
\[ (2\nu \dot{\varepsilon}_{xx} - P) \frac{\partial z_b}{\partial x} + 2\nu \dot{\varepsilon}_{xy} \frac{\partial z_b}{\partial y} - 2\nu \dot{\varepsilon}_{xy} = -\rho gh \frac{\partial z_b}{\partial x}, \]  
(A.15)

\[ 2\nu \dot{\varepsilon}_{xy} \frac{\partial z_b}{\partial x} + (2\nu \dot{\varepsilon}_{yy} - P) \frac{\partial z_b}{\partial y} - 2\nu \dot{\varepsilon}_{yz} = -\rho gh \frac{\partial z_b}{\partial y}, \]  
(A.16)

\[ 2\nu \dot{\varepsilon}_{xz} \frac{\partial z_b}{\partial x} + 2\nu \dot{\varepsilon}_{yz} \frac{\partial z_b}{\partial y} - (2\nu \dot{\varepsilon}_{zz} - P) = \rho gh. \]  
(A.17)

### A.2.3 Depth integration

We assume the \(x\)- and \(y\)- directed strain rates are independent of the \(z\)- coordinate. This assumption allows us to simplify equations A.2-A.4. Vertically integrating, we get:

\[ -\int_{z_b}^{z_s} \frac{\partial P}{\partial x} \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial x} (2\nu \dot{\varepsilon}_{xx}) \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial y} (2\nu \dot{\varepsilon}_{xy}) \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{xz}) \, dz = 0, \]  
(A.18)

\[ -\int_{z_b}^{z_s} \frac{\partial P}{\partial y} \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial x} (2\nu \dot{\varepsilon}_{yx}) \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial y} (2\nu \dot{\varepsilon}_{yy}) \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{yz}) \, dz = 0, \]  
(A.19)

\[ -\int_{z_b}^{z_s} \frac{\partial P}{\partial z} \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial x} (2\nu \dot{\varepsilon}_{zx}) \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial y} (2\nu \dot{\varepsilon}_{zy}) \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{zz}) \, dz = \int_{z_b}^{z_s} \rho g \, dz. \]  
(A.20)

Utilizing Leibniz’ rule,
\[ \int_{z_b}^{z_s} \frac{\partial f(x, z, \ldots)}{\partial x} \, dz = \frac{\partial}{\partial x} \int_{z_b}^{z_s} f(x, z, \ldots) \, dz - f(x, z_s, \ldots) \frac{\partial z_s}{\partial x} + f(x, z_b, \ldots) \frac{\partial z_b}{\partial x}, \]

(A.21)

we get for the \( x \)-directed equation:

\[ -\frac{\partial}{\partial x} \int_{z_b}^{z_s} P \, dz + \frac{\partial}{\partial x} \int_{z_b}^{z_s} 2\nu \dot{\epsilon}_{xx} \, dz + \frac{\partial}{\partial y} \int_{z_b}^{z_s} 2\nu \dot{\epsilon}_{xy} \, dz + \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\epsilon}_{xz}) \, dz \\
- (2\nu \dot{\epsilon}_{xx} - P) \frac{\partial z_s}{\partial x} - 2\nu \dot{\epsilon}_{xy} \frac{\partial z_s}{\partial x} + 2\nu \dot{\epsilon}_{xz} \\
+ (2\nu \dot{\epsilon}_{xx} - P) \frac{\partial z_b}{\partial x} + 2\nu \dot{\epsilon}_{xy} \frac{\partial z_b}{\partial x} - 2\nu \dot{\epsilon}_{xz} = 0. \]  

(A.22)

Utilizing the surface and basal boundary conditions we get, along with the other two equations,
\[-\frac{\partial}{\partial x} \int_{z_b}^{z_s} P \, dz + \frac{\partial}{\partial x} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{xx} \, dz + \frac{\partial}{\partial y} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{xy} \, dz \]
\[
+ \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{xx}) \, dz - \rho gh \frac{\partial z_b}{\partial x} = 0, \tag{A.23}
\]

\[-\frac{\partial}{\partial y} \int_{z_b}^{z_s} P \, dz + \frac{\partial}{\partial y} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{yy} \, dz + \frac{\partial}{\partial y} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{xy} \, dz \]
\[
+ \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{yy}) \, dz - \rho gh \frac{\partial z_b}{\partial y} = 0, \tag{A.24}
\]

\[-\frac{\partial}{\partial z} \int_{z_b}^{z_s} P \, dz + \frac{\partial}{\partial x} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{xz} \, dz + \frac{\partial}{\partial y} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{yz} \, dz \]
\[
+ \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{zz}) \, dz + \rho g h = \int_{z_b}^{z_s} \rho g \, dz. \tag{A.25}
\]

Assuming there is no vertical shear then both \(\dot{\varepsilon}_{xz} = 0\) and \(\dot{\varepsilon}_{yz} = 0\). Vertically integrating equation A.4 from \(z\) to \(z_s\) we get:

\[-\int_z^{z_s} \frac{\partial P}{\partial z} \, dz + \int_z^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{zz}) \, dz = \int_z^{z_s} \rho g \, dz, \tag{A.26}\]

resulting, after integration, in

\[-P|_{z_s} + P + (2\nu \dot{\varepsilon}_{zz})|_{z_s} - 2\nu \dot{\varepsilon}_{zz} = \rho g (z - z_s). \tag{A.27}\]
Applying the simplified form of boundary condition A.12, \(-P|_{z_s} + (2\nu \dot{\epsilon}_{zz})|_{z_s} = 0\), at \(z_s\) we get:

\[
P = \rho g(z_s - z) + 2\nu \dot{\epsilon}_{zz}.
\]  

(A.28)

Equation A.28 will allow us to simplify the \(-\frac{\partial}{\partial x} \int_{z_b}^{z_s} P \, dz\) term in equation A.23 and eliminate the pressure term from equations A.23-A.25. We get:

\[
-\frac{\partial}{\partial x} \int_{z_b}^{z_s} P \, dz = -\frac{\partial}{\partial x} \int_{z_b}^{z_s} (\rho g(z_s - z) + 2\nu \dot{\epsilon}_{zz}) \, dz
\]

\[
= -\frac{\partial}{\partial x} \left( \rho g \left( z_s^2 - \frac{z_b^2}{2} - z_s z_b + \frac{z_b^2}{2} \right) - 2h\nu (\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) \right)
\]

\[
= -\frac{\partial}{\partial x} \left( \frac{\rho g h^2}{2} - 2h \tilde{\nu} (\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) \right),
\]  

(A.29)

where I have used the incompressibility of ice, \(\dot{\epsilon}_{zz} = -\dot{\epsilon}_{xx} - \dot{\epsilon}_{yy}\) and the assumption that \(\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}\) vary independently of \(z\). I’ve also replaced \(\nu\) by the depth-averaged quantity \(h \tilde{\nu} = \int_{z_b}^{z_s} \nu \, dz\).

With these assumptions, identities, and simplifications, the LHS of equation A.23 becomes:
\[ -\frac{\partial}{\partial x} \int_{z_b}^{z_s} P \, dz + \frac{\partial}{\partial x} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{xx} \, dz + \frac{\partial}{\partial y} \int_{z_b}^{z_s} 2\nu \dot{\varepsilon}_{xy} \, dz \\
+ \int_{z_b}^{z_s} \frac{\partial}{\partial z} (2\nu \dot{\varepsilon}_{xz}) \, dz - \rho gh \frac{\partial}{\partial x} \bar{z}_b \]

\[ = -\frac{\partial}{\partial x} \left( \rho g h^2 \frac{2}{2} - 2h \ddot{\nu}(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy}) \right) + \frac{\partial}{\partial x} (2h \ddot{v}\dot{\varepsilon}_{xx}) + \frac{\partial}{\partial y} (2h \ddot{v}\dot{\varepsilon}_{xy}) - \rho gh \frac{\partial}{\partial x} z_b \]

\[ = -\rho gh \frac{\partial}{\partial x} h + \frac{\partial}{\partial x} (2h \ddot{v}(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy})) + \frac{\partial}{\partial x} (2h \ddot{v}\dot{\varepsilon}_{xx}) + \frac{\partial}{\partial y} (2h \ddot{v}\dot{\varepsilon}_{xy}) - \rho gh \frac{\partial}{\partial x} z_b, \quad (A.30) \]

and finally we get the \( x \)-directed depth-averaged equation appropriate for ice shelf flow lacking vertical shear:

\[ \frac{\partial}{\partial x} (2h \ddot{v}(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy})) + \frac{\partial}{\partial y} 2h \ddot{v}\dot{\varepsilon}_{xy} - \rho gh \frac{\partial}{\partial x} z_s = 0, \quad (A.31) \]

where I’ve used the identity \( z_b + h = z_s \). The \( y \)-directed equation is derived similarly and gives:

\[ \frac{\partial}{\partial y} (2h \ddot{v}(\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{xx})) + \frac{\partial}{\partial x} 2h \ddot{v}\dot{\varepsilon}_{xy} - \rho gh \frac{\partial}{\partial y} z_s = 0. \quad (A.32) \]
A.3 Boundary conditions

The two common treatments at boundaries are a velocity or stress condition to describe motion of ice at it’s margin.

A.3.1 Dirichlet condition

Here we simply prescribe the ice velocity at the boundary.

A.3.2 Neumann condition

A Neumann condition assigns the appropriate stress condition at a boundary leaving the velocity itself to be determined by the solution of the stress-balance equations A.31 and A.32.

At any boundary, a stress condition may be applied. Let $T$ represent the 2D stress tensor defined as

$$
T = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} \\
\tau_{xy} & \sigma_{yy}
\end{bmatrix}
$$

where $\sigma_{xx}$ and $\sigma_{yy}$ are the $x$-directed and $y$-directed normal stresses respectively, $\tau_{xy}$ is the shear stress, and the identity $\tau_{yx} = \tau_{xy}$ holds. For the stress-equilibrium equations the deviatoric stress, that in which the pressure is removed, controls ice shelf flow, namely $T' = T + P \cdot I$. In this case, the deviatoric stress at the ice shelf margin, dotted onto the outward pointing unit normal $\hat{n} = [n_x \ n_y]^T$, is balanced by a presently unspecified forcing function,

$$
f = \begin{bmatrix}
f_{xx} & f_{xy} \\
f_{yx} & f_{yy}
\end{bmatrix},
$$

(A.33)
dotted onto the outward unit normal. The boundary condition can be written as

\[ T \cdot \hat{n} = f \cdot \hat{n}. \quad (A.34) \]

The LHS can be expanded to

\[ T' \cdot \hat{n} = (T + (P \cdot I)) \cdot \hat{n} = \begin{bmatrix} \sigma_{xx} - P & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - P \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}, \quad (A.35) \]

which, after substituting into (A.34) and depth integrating, gives us two equations utilizing
\[ \sigma_{ii} = 2\nu \dot{\epsilon}_{ii} \] and \[ \tau_{ij} = 2\nu \dot{\epsilon}_{ij} \]:

\[ \int_{z_b}^{z_s} (2\nu \dot{\epsilon}_{xx} - P) n_x + 2\nu \dot{\epsilon}_{xy} n_y \, dz \]
\[ = \int_{z_b}^{z_s} f_{xx} n_x + f_{xy} n_y \, dz, \quad (A.36) \]

\[ \int_{z_b}^{z_s} 2\nu \dot{\epsilon}_{xy} n_x + (2\nu \dot{\epsilon}_{yy} - P) n_y \, dz \]
\[ = \int_{z_b}^{z_s} f_{xy} n_x + f_{yy} n_y \, dz. \quad (A.37) \]

Depth integrating equation (A.28) we get

\[ \int_{z_b}^{z_s} P \, dz = \int_{z_b}^{z_s} \rho g (z_s - z) + 2\nu \dot{\epsilon}_{zz} \, dz, \quad (A.38) \]
and by incompressibility and replacing $\int_{z_b}^{z_s} \nu \, dz$ by depth averaged $h\bar{\nu}$ we get:

$$\int_{z_b}^{z_s} P \, dz = \frac{\rho gh^2}{2} - 2h\bar{\nu}(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}). \quad (A.39)$$

Thus A.36 and A.37 become after some rearrangement:

$$(2h\bar{\nu}(2\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy})) n_x + 2h\bar{\nu}\dot{\epsilon}_{xy} n_y - \frac{\rho gh^2}{2} n_x = \int_{z_b}^{z_s} f_{xx} n_x + f_{xy} n_y \, dz, \quad (A.40)$$

$$(2h\bar{\nu}(2\dot{\epsilon}_{yy} + \dot{\epsilon}_{xx})) n_y + 2h\bar{\nu}\dot{\epsilon}_{xy} n_x - \frac{\rho gh^2}{2} n_y = \int_{z_b}^{z_s} f_{xy} n_x + f_{yy} n_y \, dz. \quad (A.41)$$

### A.4 Ice-ocean boundary

In the case where the calving front meets the ocean, the appropriate $f$ is simply the sea water back pressure given by $f_{xx} = -\rho_{sw}gz$ and $f_{yy} = -\rho_{sw}gz$. Here, $f_{xy} = 0$ since we assume the ocean does not apply a shear stress. In vertically integrating the ocean back pressure I make the common assumption that the vertical structure of the back pressure has a negligible effect on ice shelf dynamics. Then
\[ \int_{z_b}^{z_s} f_{xx} n_x \, dz = -\rho_{sw} g \frac{z_s^2}{2} n_x + \rho_{sw} g \frac{z_b^2}{2} n_x \]
\[ = \rho_{sw} g \frac{z_b^2}{2} n_x \]
\[ = \rho_{sw} g \left( -\frac{\rho}{\rho_{sw}} h \right)^2 n_x, \quad (A.42) \]

where I’ve used the hydrostatic equilibrium assumption \( z_b = -\frac{\rho}{\rho_{sw}} h \). Then A.40 and A.41 become the final boundary condition after rearranging some terms:

\[ (2 h \bar{\nu} (2 \dot{\epsilon}_{xx} + \dot{\epsilon}_{yy})) n_x + 2 h \bar{\nu} \dot{\epsilon}_{xy} n_y = \frac{\rho g h^2}{2} \left( 1 - \frac{\rho}{\rho_{sw}} \right) n_x, \quad (A.43) \]
\[ (2 h \bar{\nu} (2 \dot{\epsilon}_{yy} + \dot{\epsilon}_{xx})) n_y + 2 h \bar{\nu} \dot{\epsilon}_{xy} n_x = \frac{\rho g h^2}{2} \left( 1 - \frac{\rho}{\rho_{sw}} \right) n_y, \quad (A.44) \]
Appendix B

Application of ocean pressure

In order to show that we capture the correct ice/ocean pressure boundary condition at the calving front, we show the necessary terms are present in equation 4.1 and, along with our numerical treatment of the calving front as a sloped face, that we are able to adequately approximate the theoretical condition. We see in the theoretical boundary condition 4.9 that the dynamic component of ice pressure, \(2h\ddot{v} (2\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy})\), is equal to the glaciostatic component of ice pressure, \(\rho gh\), minus the ocean pressure, \(\rho gh (\frac{\rho_{sw}}{\rho_{sw}})\), in the case where the x- and y-directed components of the normal vector are \(n_x = 1\) and \(n_y = 0\). We show these three terms perform correctly at the calving front in equation 4.1. The process is similar for equation 4.2.
B.1 Dynamic and glaciostatic components of ice pressure

We note that the boundary condition given in equation 4.9 is applied along a boundary contour, one spacial dimension lower than the governing equations. For this reason, the corresponding terms in equation 4.1 appear as derivatives $\frac{\partial}{\partial x}$ of the terms in equation 4.9.

As calculated in chapter 2, the depth-integrated ice pressure term relevant to Stokes flow is given by:

$$\frac{\partial}{\partial x} \int_{z_b}^{z_s} P \, dz = \frac{\partial}{\partial x} \left( \frac{\rho gh^2}{2} \right) + \frac{\partial}{\partial x} \left( 2h \nu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right).$$

(B.1)

B.1.1 Dynamic component of ice pressure

Starting with the second term on the RHS of equation C.1, we see it is accounted for in the flow equations since we can rewrite the first term in equation 4.1 as:

$$\frac{\partial}{\partial x} \left( 2\nu h \left( 2 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) = \frac{\partial}{\partial x} \left( 2\nu h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial x} \left( 2\nu h \left( \frac{\partial u}{\partial x} \right) \right),$$

(B.2)

where the first term on the RHS of equation C.2 is the dynamic component of ice pressure.
### B.1.2 Glaciostatic component of ice pressure

Next we show the glaciostatic component of ice pressure, \( \frac{\partial}{\partial x} \left( \frac{\rho gh^2}{2} \right) \), is contained within equation 4.1 by rewriting the third term of 1.1 using \( z_s = h + z_b \):

\[
-\rho gh \frac{\partial z_s}{\partial x} = -\rho gh \frac{\partial h}{\partial x} - \rho gh \frac{\partial z_b}{\partial x} = -\frac{\partial}{\partial x} \left( \frac{\rho gh^2}{2} \right) - \rho gh \frac{\partial z_b}{\partial x},
\]

and we see the first term in equation C.4 is the desired term.

### B.2 Ocean pressure

Up until now we haven’t made use of the fact that we approximate the calving front as a sloped face. That is to say, the ice pressure is applied in the standard way throughout the domain including the transition zone. However, the transition zone is essential in applying the correct ocean pressure. In applying the ocean pressure to the calving front we exploit the fact that the equation for flow, 4.1, already accounts for the ocean pressure along the base of the ice shelf. We simply use the fact that, as we increase the slope of the base, as is the case for our transition zone, the outward pointing normal approaches that of a vertical calving face. To see this in more detail we recall from chapter 2 that the outward pointing unit normal to the base of the ice shelf, \( \hat{n}_b \), is given by:

\[
\hat{n}_b = \left[ \frac{-\frac{\partial z_b}{\partial x}}{\sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1} \sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1} \sqrt{\left( \frac{\partial z_b}{\partial x} \right)^2 + \left( \frac{\partial z_b}{\partial y} \right)^2} + 1} \right]
\]

(B.5)
and that, assuming hydrostatic equilibrium, the pressure at the base of the ice shelf is given by $\rho gh$. During integration of the boundary condition into the flow equations the denominator of the normal vector in expression C.5 is cancelled and, for the $x$-directed equation, the ocean pressure is simply:

$$\rho gh \frac{\partial z_b}{\partial x},$$  \hspace{2cm} (B.6)

which we recognize in equation C.4. Thus, we see that $\frac{\partial z_b}{\partial x}$ approximates the $x$-directed outward pointing normal, $n_x$, of the calving face as the transition width, $2l$, approaches zero. We see this is the only term we approximate at the boundary condition and, as seen in figure 4.3, the error in the calving-front boundary-condition drops to near zero for sufficiently small transition widths.
Here we derive a treatment for the flow of ice near the embayment wall. The central assumption of this derivation is that the flow of ice near the embayment wall is dominated by shear, $\dot{\epsilon}_{xy}$, within a narrow band along the wall. Outside of this narrow band, ice flows according to internal deformation of ice as governed by the stress-equilibrium equations. Assuming $\dot{\epsilon}_{xy}$ dominates, all other strain rates are negligible, and we can write the depth-averaged lateral shear stress, $\sigma_{xy}$, as:

\[
\sigma_{xy} = 2h\bar{\nu}\dot{\epsilon}_{xy} \tag{C.1}
\]

\[
= 2hB_e \left( \frac{1}{2} \dot{\epsilon}_{xy} \right)^{1/n-1} \dot{\epsilon}_{xy} \tag{C.2}
\]

\[
= 4hB_e \left( \frac{1}{2} \epsilon_{xy} \right)^{1/n}, \tag{C.3}
\]

where we used Glen’s flow-treatment with $B_e$ as the ice hardness in the shear zone. By definition, $\dot{\epsilon}_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$, and assuming $v = 0$ near the embayment, we get $\sigma_{xy} =$
$4hB_e \left( \frac{1}{n} \frac{\partial u}{\partial y} \right)^{1/n}$. Defining the narrow-band length as $y_e$, we approximate $\frac{\partial u}{\partial y} \approx \frac{\delta u}{y_e}$. Here $\delta u = u_{y_e} - u_o$ the x-directed velocity at the embayment wall, $u_o$, and velocity where shear ceases to dominate, $u_{y_e}$. Assigning an x-directed no-flow condition at the rock/ice boundary, $u_o = 0$, we get equation 4.8:

$$\sigma_{xy} = 4hB_e \left( \frac{u_{y_e}}{4y_e} \right)^{1/n} \tag{C.4}$$