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MATE WITH THE TWO BISHOPS IN KRIEGSPIEL

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It is generally known that the kriegspiel endgame with a king and two bishops versus a king alone is a win for the player with the two bishops. This assumes, of course that the bishops are on opposite colored squares and that the king is initially guarding the two bishops, say king on d4, and bishops on d5 and e5 as in the diagram below. In the following, we take the player with the two bishops to be white and his opponent to be black.

1 White wins.

Interestingly, there does not exist a strategy for white that wins with probability one in this position when nothing is known as to the whereabouts of the black king. What we can say is that for every $\epsilon > 0$ there is a strategy for white that wins with probability at least $1 - \epsilon$ no matter where black starts and what strategy black uses. In the terminology of game theory, there exists an $\epsilon$-optimal strategy for white for every $\epsilon > 0$ and the value of the position is a win for white.

The reason there does not exist an optimal strategy for white is that white cannot reach a position in which he need guard the bishops only on one side. In particular, he cannot reach the side of the board without risking losing one of the bishops or allowing a possible stalemate. If he does reach the side of the board successfully, he has a strategy that mates surely in a finite number of moves without using randomization as will be seen later. He can safely move towards the edge, for example in the above position, by Bc4, Kd5, Bd4, Kc5, reaching the following critical position.

2 White wins.

To bring the king and bishops to the edge while still guarding them, white must play Bb5 which risks stalemate at a5. Therefore in an attempt to reach the edge, he may randomize
between Kd5 Kc5 and Kb4 Bc3, giving weight $1 - \epsilon$ to the former and weight $\epsilon$ to the latter. This is independently repeated indefinitely until at some future random time white plays the latter. White is successful if he receives a “no” from the referee at any time, or if when he finally plays Kb4 his bishop on d4 is not immediately captured by black. It is easy to see that white carries this out successfully with probability $1 - \epsilon$ no matter what strategy black uses.

Once white reaches the edge safely, he has a mate in a bounded number of moves that may be achieved by a nonrandomized strategy. The most efficient method of mating seems to be to set up a position with the bishops on b3 and c3 (or some rotated or mirror image of this). After playing Kb4 Bc3 in position 2, white merely plays Bb3 to set up the following position.

3 White wins.

Having set up a position in which the bishops cannot be attacked from the left, top or bottom, white sweeps the board in search of the black king making sure the bishops cannot be attacked from the right. This sweep begins

Kc4, Kd3, Kc3, Kf3, Kg2, Kf3, Kg4, Kf5, Kg6.

If any of these moves is a “no”, white traps the black king on the lower or right side of the board. Otherwise, black is known to be trapped in the upper left side of the board or on the bottom left. Once he has trapped the black king, white may proceed to mate by the methods detailed in the appendix. When the white king is at d3 in the above sweep, he may check immediately by the moves Kc2 and Kd3 whether the black king is at b1 or c1. If one is interested in minimizing the maximum number of moves this process takes, then it is more efficient to postpone such testing until the rest of the board has been swept clear. This reduces the maximum number of moves to mate by two. The strategy suggested in the appendix guarantees mate in at most 32 moves starting at position 3.

Section A of the appendix contains the elementary mates when the black king is already trapped in a corner. Section B describes the mates when black is trapped along the edge of the board. Section C and D contain the corresponding mates when black is trapped within two and three squares of the edge. Section E shows how to mate black when the sweep indicated above has been carried out, and Section F shows how to carry out the sweep.

For each diagram, the crucial variation, the one that takes the maximum number of moves, is starred. The crucial diagrams are also starred. These are the positions that occur in the most lengthy defense. The number of moves to mate for at least one of these positions will have to be improved if the number of moves to mate from position 3 is to be reduced below 32.

A general method that works on a quadrant. If the board is infinite in two directions, say north and east, there is a general method that white may use to mate provided the black king is known to be confined to a bounded region. This method proceeds as follows.

First, set up the bishops to bound the enemy king to a finite region in such a way that the bishops cannot be attacked. Suppose without loss of generality that the bishops are
stationed next to each other horizontally and that the black king is known to be in the region below.

Second, move the white king to the inside of this region, making sure the black king does not escape. This may require that the white king enter the region at the edge of the board.

Third, let the white king perform a random walk on the inside of this region, avoiding any squares controlled by the bishops and any squares that could lead to a possible stalemate. He will eventually, with probability one, encounter the black king by receiving a “no” from the referee.

Fourth, we say that white makes one line of progress if he can move the bishop pair one square to the left or one square diagonally to the right, still keeping the enemy king confined within the bounding region. Unless the “no” obtained in step 3 allows the black king to be on either of the squares two squares directly below the two bishops as in figure 5, or unless the information white has allows the black king can be at one of the two squares of figure 6, white can make at least one square of progress immediately. We treat these two cases separately.

The only difficult case is where the black king is on one of the two squares directly below the bishops, the white king is two squares directly below the black king, and white has the move. In this case black has the opposition and it is simplest for white to triangulate by moving one step back and one step forward diagonally to obtain the opposition. If black moves to prevent white from moving back, white makes immediate progress with right-bishop up and back to the left. Otherwise white has the opposition and can force the black king to retreat so that again white may make a least one square of progress.

The case of figure 6 may be treated by pushing up and to the right with the king. If this is impossible immediate progress may be made with the bishops. Otherwise, by further pushing up and to the left, white can make the black give way or position 5 can be reached.

Fifth, suppose the black king has been pushed down to the lower edge of the board with the bishops on the fifth rank, where the triangularization procedure described in the fourth
step does not work.

In this case, white may play his king one square to the right and one square up and then triangularize by moving the dark-squared bishop one square back to the right to obtain position D7. The method used from this position eventually forces the black king to be confined to two squares on the edge of the board as in position B2. From there, one repeatedly uses the procedure going from B2 to B1 to push the black king to the left along the edge to the corner, where he is mated as in position B1.

Sixth, assume that the white bishops have moved all the way to the left. White can make further progress by moving the white king to a position two or three squares below the bishops and then moving the bishops down to the right. If black tries to prevent by blocking the way, the black king must station himself two or three squares below the bishops or white can make immediate progress anyway.

From position 8, one can move as in position 6 and advance to position 9. There one may pin the king to the wall by moving the bishops to the third file and proceed as in position B2 to B1.

Mate on a finite board. White may mate with probability one on a larger rectangular board of any size, provided the king and two bishops are stationed together at an edge. One method to accomplish this is to move the pieces along one edge with king protecting both bishops at all times until one of the bishops gives check or the king receives a “no”. Then the bishops may be moved to bound the king in a known region and the method of the preceding section may be carried out.

More precisely, start the maneuver with bishops at b1 and b2, and the king at c1. Then in three moves, Bc2, Kd1, Be1, the formation has been moved over one file. Continue in this manner, Bd2, Ke1, Bd1, etc., until either (1) the bishop checks the black king on a move to the second rank, or (2) the bishop checks the king on a move to the first rank, or (3) White receives a “no” in reply to an attempted king move, or (4) the far corner is reached with (assuming z represents the last file) the moves Bx2, Ky2 (not Ky1, which may stalemate the black king at z3).

(1) If the move Bn2 checks the black king, then Bp4, Bp5 bounds the black king to the left side of the board.
(2) If the move Bn1 checks the black king, then Bℓ3, Bℓ4 bounds the black king to the right side of the board.

(3) If the white king receives a “no”, the black king is already confined to the right side of the board.

(4) If the far corner is reached, then the black king is known to be confined to the upper right half of the board.

The method of the previous section may now be used to mate with probability one.

Here is an unsolved problem. In how large a square board is it possible for the player with the two bishops to have a strategy that mates in a bounded number of moves, once the edge is reached? The method given in the appendix guarantees mate in 32 moves starting from position 3. On a larger board however, the method suggested in the previous section required that White perform a random walk until the black king was encountered and so no upper bound can be placed on the number of moves required.
§A The elementary mates.

A1* Mate in 4
Kb6 Ba6 Bb7 Be5 mate
Bd4 Bd5 mate

A2* Mate in 5
Be7 A1

A3* Mate in 7
Kc6 Kc7 A2

A4* Mate in 6
Kb6 mate
Bc4 Bd6 Kc7 Be7 Bc5 Bd5 mate
if+ Bd6 Kc7 Bc5 Bd5 mate

A5* Mate in 7
Kb6 A1
Bc8 Kc7 Bc5 Bb7 mate
if+ Kc7 Bb7 Bc5 mate

A6 Mate in 8

A7* Mate in 9

A8 Mate in 8

Bc7 Bd5 A4

Kc6 Bf6 A5*
Bc5 Ba6 A3*

Kc5 Kc6 A4
§B Black confined to the edge.

B1* Mate in 8

Bd5 Kc6 A4

B4* Mate in 16

Be6 Kd7 Bc4 A7
Kd6 Bd4 Bf6 B2*

B2* Mate in 12

Bd4 Kc5 Be5 Bc7 B1

B5 Mate in 14

Ke7 C6
Be6 Bf6 B2*

B3 Mate in 12

Kc4 Kc5 Be5 Bc7 B1
§C Black restricted to within two of the edge.

C1 Mate in 18

Kc3  C2
Kc2  Kc3  C2*

C2 Mate in 16

Kb3  C3*
Bc6  Kb2  Bc5  Bd5  Bd4 mate
if+  Bc5  Kc4  B2
Kc2  A1

C3 Mate in 15

Kc4  Bc5  Bc6  B2*
Bc3  Bc6  B3
Kb4  C6*

C4 Mate in 14

Kc4  C5
Kc3  Kc4  C5*

C5 Mate in 12

Kb4  C6*
Bc7  Kc3  Bc6  Bd6  Kc2  A1
if+  Bc6  Kc5  B1
if+  Kc3  Kc2  Bd6  Bd5  Be5  mate

C6 Mate in 11

Kc5  Bc6  Bc7  B1*
Be4  Bc7  A8

C7* Mate in 12

Kc5  C8
Kc4  Kc5  C8*

C8* Mate in 10

Kb5  A7*
Bc6  Bc7  B1*
§D Black restricted to be within three of the edge.

D1* Mate in 20
D3* Mate in 18
D4* Mate in 17

Ke7 D2
Ke8 Ke7 D2
Bc5 Bb4 D3*

Ke7 D2*
Bb5 Bc5 Ke6 C7

Ke7 D4*
Bb5 Bd7 Bd6 Ke6 Kd5 C7*

Ke7 D5
Kd8 Kd7 D5
Bb5 Bd7 Bd6 Ke6 Kd5 C7*

Ke7 D6
Bc4 Kd7 A1

Be3 Kd7 A7
Kd8 Bd4 Kd7 A7

Be3 Kd7 A7*
Bd5 Bb6 A8*

Be4 Bd4 B4

D6 Mate in 19
D7 Mate in 17
D8 Mate in 15

Kc2 Bc6 D7*

Kg4 C6
Kg3 Kg4 C6

Bc4 Bd4 B4
Be6 Be5 C4
Be4 Bf6 B3*
§E Black confined to the upper left of the board.

E1* Mate in 23

E2* Mate in 20

E3 Mate in 15

E4 Mate in 16
E41  Mate in 15
Kc6  Bc4  E43
if+  E45
Kc7  Kc6  Bc4  E43
Bc4  E42*

E42  Mate in 14
Kc6  E43
Bb4  D5*

E43  Mate in 9
Kb7  Kc6  Bc5  E44*
Kc7  A2

E44  Mate in 6
Bb3  Bb4  Kc7  Bc4  Bc5  Bd5  mate

E45  Mate in 8
Kb7  Kc6  E44*
Kc7  Be7  Bc5  Bd5  mate

E5  Mate in 13
Kc6  Kb6  Kc5  Bc4  Bb2  Bd3  Bc2  Bc3  Kc6  Kc7  Bd3  Bd4  Be4  mate
Kc7  Bc4  Bd4  Bd5  mate
Bc4  Kc7  Bb4  Bc5  Bd5  mate
Ba5  Kc6  A6
§F Sweeping out the right side of the board.

3 Mate in 32

\[
\begin{array}{c}
\text{Kc4 Kd3 Ke3 Kf3 Kg2 Kf3 Kg4 Kf5 Kg6 E1*} \\
\text{F9 F8 F7 F6 F5 F4 F3 C6 F1}
\end{array}
\]
F43 Mate in 16
Kg4 Bf7 Kf5 A8
Bd3 Bf6 Bf5 Kf4 B2*
if+ Kf2 A1

F44 Mate in 22
Kf3 Kg3 C3
Bc1 Be3 B4
Ke3 Kf3 Kg3 C3
Bc1 Be3 B4
Be5 Kf3 C2
Be4 Kf3 C2
if+ Kf2 C1*
Bg3 A8
Ba1 Be5 Be4 C1

F5 Mate in 13
Be6 A5
if+ Bf6 Bc3 A5
if+ Bf7 Kf4 Kf5 B1*

F6 Mate in 23
Bc2 Bd3 Be5 Kf3 Kg2 C1*
Kg3 A2
Bf6 Kf2 Bf5 Be5 Be4 mate
Bf4 Kf3 Be3 B4
F7 Mate in 25

Bd5 Ke3 Kf2 F71*

F72 Mate in 12

Kf3 Be6 A5
Bf3 Kg3 Bg4 Bh3 Bg2 Bd4 mate

F71 Mate in 22

Kf3 Bd4 D6
if+ F73
Ke3 Kf3 Bd4 D6*

F73 Mate in 9

Be6 Kf2 A2
Be3 Bb3 Be2 Bd2 Kg3 Bd3 Be3 Be4 mate
F8  Mate in 27

Bc2  Kd5  Ke6  Kf7  D1*

F9  Mate in 24

Bd5  Be5  Kc3  D6
   Kc4  Kc3  D6
       F91
   Kb3  Kc3  D6
       Kc4  Kc3  D6*
       Bc3  Kd3  Bb3  B1

F91  Mate in 16

Bf4  Kc3  Kd2  Bd6  A7
   C4*
   Be4  Be5  Bc3  Bc2  B1
   if+  Kc3  C4*
       Bf3  Kb3  A1
       Bd1  Kc3  A6