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Finite-action Configurations in Gauge Theory

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Abstract

Contrary to folklore, finite action does not require large-distance field strengths to vanish everywhere, nor must the potentials go everywhere to a pure gauge. Counterexamples are given.

I. Introduction

In the study of gauge fields on Euclidean spacetimes it is usually assumed that finite action configurations must satisfy strong boundary conditions:

1. The field strength $F_{\mu\nu} \to 0$ at large Euclidean radius (or time);

2. The potential $A_\mu \to -\frac{i}{g} S \partial_\mu S^\dagger$ (a pure gauge) at large Euclidean radius (or time);

3. In $A_0 = 0$ gauge (say) $A_1 \to -\frac{1}{g} S(x) \partial_1 S^\dagger(x)$ at large time, where $S(x)$ is a time independent gauge;

4. The winding number (in $1 + 3$ dimensions) is an integer.

It can be shown that all of these conditions follow from (1), refined to

$$F_{\mu\nu} \to 0 \text{ faster than } R^{-2} \text{ uniformly in all directions, for large Euclidean radius } R,$$

because any configuration satisfying (1) must be gauge equivalent to one brought to Euclidean spacetime via stereographic projection from a sphere.

The purpose of this note is to point out that the requirement of finite action is actually less stringent; in particular (1) can be relaxed along a set of directions of measure zero. To see this, write the action ($D$ is the number of spatial dimensions)

$$S = \int d^{D+1}x \ |F|^2(x) = \int_0^\infty dR \int_{D\text{-sphere}} R^D(d^D\tilde{S}) |F|^2(S,\tilde{S}) .$$

(1.1)
In order for the radial integral to be finite, the inner integral must go to zero faster than $R^{-1}$ (at least for almost every $R$). Elementary analysis then says that $F_{\mu \nu} (R, \hat{n}) \to 0$ as $R \to \infty$ for almost every $\hat{n}$. This leaves the possibility of almost arbitrary behavior at $\infty$, as long as such behavior lives on a set of directions of measure zero. Well-behaved sets of measure zero will be surfaces of dimension $D-1$ or less.

We present some simple examples of this phenomenon, including explicit counterexamples to (1), (2), (3) and of course (1'). The failure of (1') suggests the possibility of avoiding (4), but we have not been able to find a finite action configuration with nonintegral winding number.

II. EXAMPLES

(1) Abelian model with $F_{\mu \nu}$ strong near a point in space at all times

Consider the configurations

$$A_0 = g_1(t) \cdot g_2(t) r$$

and

$$A_1 = 0.$$  \hspace{1cm} (2.1)

The region of significant $F_{\mu \nu}$ should become small at large times so we want $g_2(t)$ to get large as $t \to \infty$. The finite action condition becomes, after a change of variables:

$$\int_{-\infty}^{\infty} dt \ g_1(t)^2 \ g_2(t)^2 = \int_{0}^{\infty} du \ u^{D-1} g'(u)^2 < \infty. \hspace{1cm} (2.2)$$

A simple choice which works for any $D > 1$ is

$$g(u) = u e^{-u}$$

and

$$g_2(t) = \frac{1}{g_1(t)} = 1 + t^2.$$  \hspace{1cm} (2.3)

With this choice $|F|^2(0, t)$ is nonzero and independent of $t$, so the field cannot be approaching a pure gauge near $r = 0$ as $t \to \infty$. The spatial region in which $F_{\mu \nu}$ is significantly different from zero shrinks as $t \to \infty$ with radius going as $t^{-2}$.

(2) Example (1) in $A_0 = 0$ gauge.

Up to a residual gauge transformation, the configuration of example (1) transformed to $A_0 = 0$ gauge is:

$$A_j(r, t) = -\frac{\partial}{\partial x^j} \left[ \frac{r^3}{2} e^{-r} - \frac{\rho^3}{2} e^{-r} \int_{r_1}^{\infty} \frac{1}{u^2} \ du e^{-u^2} \right]. \hspace{1cm} (2.4)$$

So outside any neighborhood of $r = 0$ the field tends, as $t \to \infty$, to a time-independent pure gauge:

$$A_j(r, t) \to -\frac{\partial}{\partial x^j} \left( \frac{r^3}{2} e^{-r} \right). \hspace{1cm} (2.5)$$

But, when the origin is included, the field, although always a spatial gauge (i.e., a gradient) no longer approaches a time-independent one.

(3) $F_{\mu \nu}$ strong on a surface of highest possible dimension as $t \to \infty$. 

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Consider, for \( r_0 > 0 \):

\[
A_0 = \mathcal{G}_1(t) \mathcal{G}(\mathcal{G}_2(t)(r-r_0))
\]

and

\[
A_1 = 0.
\]  

(2.6)

The finite action condition is

\[
\int_{-\infty}^{\infty} dt \, g(t)^2 \mathcal{G}_2(t)^2 \mathcal{G}_1(t) \int_{\mathcal{G}_2(t)r_0}^{\infty} dv (v + \mathcal{G}_2(t)r_0)^{D-1}(v'(v))^2 < \infty. \quad (2.7)
\]

Examining the large \( t \) behavior of the inner integral we see that the choices (2.5) work equally well here. Again, the field strength

\[
|F|^2(r = r_0, t)
\]

is nonzero and independent of \( t \). Clearly a similar construction is possible for any closed region of measure zero on the surface at \( \infty \).

III. CONCLUSIONS

These examples by no means exhaust the variety of possible asymptotic behaviors in the approach to the surface at \( \infty \). Also, our examples have all been abelian, for simplicity. It is easy to construct similar examples in the nonabelian case as well. In fact, such configurations exist in many theories (including free theories), not just in gauge theories. In general, then, their physical significance is not clear. However, if these configurations are significant in any theories, we expect the most likely candidates to be those showing infra-red instability.

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