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DOPPLER SHIFT IN FREQUENCY IN THE TRANSPORT
OF ELECTROMAGNETIC WAVES THROUGH AN UNDERDENSE PLASMA

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ABSTRACT

In an earlier publication the validity of the radiation transport theory was studied for the calculation of multiple scattering of electromagnetic waves by a turbulent plasma. In the present paper we extend the transport theory to include a description of the Doppler shift in frequency due to electron motion.
I. INTRODUCTION

In Part I of this series\textsuperscript{1} the classical radiation transport equation was derived from Maxwell's equations for the study of scattering of electromagnetic waves by a turbulent plasma. In Part II\textsuperscript{2} some techniques for using the transport equation were discussed. In both of these papers the Doppler shift in frequency due to the motion of the scattering electrons was neglected. In the present paper we extend the transport theory to include any frequency shift of the scattered waves.

An exhaustive analysis of the relation between a wave equation and the corresponding classical transport approximation has yet to be made. The first such analysis seems to have been given by Foldy\textsuperscript{3} who discussed the scattering of scalar waves by a set of uncorrelated point scatterers, obtaining a transport equation. The quantum theory of scattering by a "weakly bound medium" was related to a classical transport theory by Watson.\textsuperscript{4} It was an adaptation of the methods used in this work to Maxwell's equations which was given in I. A different approach was used by Barabanenkov and Finkel'berg,\textsuperscript{5} who derived a transport equation from the scalar wave equation using a "Bethe-Salpeter" type of equation.

In Section II we summarize the results derived in this paper. These lead to a radiation transport equation of conventional form,\textsuperscript{6} the scattering kernel being explicitly expressed in terms of plasma density fluctuations. The reader who is not interested in the details of the derivation will probably find the account in Section II adequate for using the transport equation.
II. SUMMARY OF RESULTS

The phenomena which we wish to describe are illustrated in Fig. 1. A plasma of finite extent is illuminated by an electromagnetic wave emitted by a distant source S and propagating in the direction $\hat{\kappa}$. The intensity of the waves scattered by the plasma is measured with a receiver R, also a great distance away. (The restriction to a distant source and receiver is of course not required for a derivation of the transport equation.)

Several assumptions concerning the plasma were introduced in I. We shall accept these here and, in addition, explicitly suppose the plasma electrons to have nonrelativistic energies. The assumed turbulence properties of the plasma will be reviewed later in this section. The non-relativistic assumption will be expressed by the inequality

$$(k R c)(v_e/c) \ll 1. \tag{NR}$$

Here $k/2\pi$ is the wave number of the radiation, $R_\text{c}$ a measure of the distance over which plasma motions are correlated, $v_e$ the mean speed of the plasma electrons, and $c$ the speed of light.

As in I, we suppose the plasma to be underdense and that

$$k R_S \gg 1, \tag{B4}$$

where $R_S$ is the "size" of the plasma. Assumption (B4) allows us to ignore diffraction scattering from the entire plasma (in all but a small cone with axis parallel to $\hat{\kappa}$).
In the classical theory of radiation transport the flow of radiant energy at a point \( \mathbf{x} \) per unit area, per unit time, and traveling in the direction \( \mathbf{\hat{p}} \) is

\[
I(\mathbf{x}, \mathbf{\hat{p}}, \omega) \, d\omega 
\]

(2.1)

The notation here implies that the radiation has an angular frequency \( \omega \), within the interval \( d\omega \), and is confined to propagation directions lying within the solid angle \( d\Omega \).

For waves which have some degree of polarization it is necessary to generalize (2.1). This was done by Chandrasekhar \(^6\) and, in a similar manner, in Ref. 1. To do this we shall follow the notation of I and introduce the two unit vectors \( \mathbf{\hat{e}}_{\mathbf{\hat{p}}}^{\wedge}(i), \, i = 1, 2 \), for a plane electromagnetic wave traveling in the direction \( \mathbf{\hat{p}} \). The electric field vector for such a wave is of the form

\[
E_{\mathbf{\hat{p}}} = \left[ E_{\mathbf{\hat{p}}}^{(1)} \mathbf{\hat{e}}_{\mathbf{\hat{p}}}^{\wedge}(1) + E_{\mathbf{\hat{p}}}^{(2)} \mathbf{\hat{e}}_{\mathbf{\hat{p}}}^{\wedge}(2) \right] e^{-i\omega t} .
\]

(2.2)

The unit vectors \( \mathbf{\hat{e}}_{\mathbf{\hat{p}}}^{\wedge}(i) \) are defined in terms of \( \mathbf{\hat{p}} \), the direction of propagation of the incident wave before entering the plasma. \(^7\) These are

\[
\mathbf{\hat{e}}_{\mathbf{\hat{p}}}^{\wedge}(2) = C(\mathbf{\hat{p}}) \mathbf{\hat{p}} \times \mathbf{\hat{p}},
\]

\[
\mathbf{\hat{e}}_{\mathbf{\hat{p}}}^{\wedge}(1) = \mathbf{\hat{e}}_{\mathbf{\hat{p}}}^{\wedge}(2) \times \mathbf{\hat{p}},
\]

(2.3a)

where

\[
C(\mathbf{\hat{p}}) = \left( |\mathbf{\hat{p}} \times \mathbf{\hat{p}}| \right)^{-1} .
\]

(2.3b)
To define the polarization vectors for the incident wave, we orient the 
z-axis of a rectangular coordinate system to be parallel to \( \hat{z} \) and 
choose \( \hat{e}_K(1) \) and \( \hat{e}_K(2) \) to be parallel to the \( x \)- and \( y \)-axes, 
respectively. For backscatter we define

\[
\hat{e}_\perp(1) = \hat{e}_K(1),
\]

\[
\hat{e}_\perp(2) = -\hat{e}_K(2). \tag{2.4}
\]

The electric field at any point can be represented as a sum of 
waves of the form (2.2). If we fix our attention on a single "bundle" 
of wavelets propagating within \( d\omega \) and \( d\omega \), we may define the intensity 
as

\[
I_{ij}(x, \hat{e}, \omega) = \text{constant} \times \left( \frac{E^*(i) E(j)}{E} \right), \tag{2.5}
\]

\( i, j = 1, 2 \). Here "\( \langle \cdots \rangle \)" represents an ensemble (or statistical) average 
over the plasma (and any source) fluctuations. The "constant" in Eq. (2.5) 
is defined by the following condition. We suppose that a filter at \( \hat{e} \) 
passes only the component of \( \hat{E}_\parallel \) parallel to some direction \( \hat{e} \). Then 
the power per unit area passed by the filter, corresponding to propagation 
within \( d\omega \) and frequency within \( d\omega \), is

\[
\hat{e} \cdot \left[ \sum_{i,j=1}^{2} \hat{e}_K(i) I_{ij}(x, \hat{e}, \omega) \hat{e}_K(j) \right] \cdot \hat{e} d\omega d\omega. \tag{2.6}
\]
We suppose the statistical properties of the plasma to be represented as a stationary random process. If the plasma contains $N$ free electrons with coordinates $z_1, z_2, \ldots, z_N$, we take the probability that electron 1 is at $z_1$ within $d^3z_1$ at time $t_1$, etc., to be

$$P_N(z_1, t_1; z_2, t_2; \ldots, z_N, t_N) d^3z_1 \cdots d^3z_N. \quad (2.7)$$

The statement that this is a stationary distribution function is equivalent to

$$P_N(z_1, t_1 + \tau; z_2, t_2 + \tau; \ldots, z_N, t_N + \tau) = P_N(z_1, t_1; z_2, t_2; \ldots, z_N, t_N). \quad (2.8)$$

We further suppose that from $P_N$ we can define a hierarchy of distribution functions as follows:

$$P_1(z_1) = \int P_N d^3z_2 \cdots d^3z_N. \quad (2.9a)$$

$$P_2(z_1, t_1; z_2, t_2) = \int P_N d^3z_3 \cdots d^3z_N, \quad (2.9b)$$

etc. Here

$$P_2(z_1, t_1; z_2, t_2) = P_2(z_1, 0; z_2, t_2 - t_1),$$

e tc.

Following the notation of I, we assume that $P_2, P_3, \ldots$ may be developed in terms of correlation functions. Thus, for example,

$$P_2(z_1, t_1; z_2, t_2) = P_1(z_1) P_1(z_2) \left[ 1 + g(z_1, t_1; z_2, t_2) \right]. \quad (2.10)$$
Here the "pair correlation function" $g$ is considered to vanish when

$$|z_1 - z_2| \gg R_c,$$

the "correlation range," or for $|t_1 - t_2| \gg t_c$, the
"correlation time." Again, we write $P_3$ in the form

$$P_3(z_1, t_1; z_2, t_2; z_3, t_3) = P_1(z_1) P_1(z_2) P_1(z_3)$$

$$\times \left[ 1 + g(z_1, t_1; z_2, t_2) + g(z_2, t_2; z_3, t_3) + g(z_3, t_3; z_1, t_1) + g_3(z_3, t_3; z_1, t_1; z_2, t_2; z_3, t_3) \right].$$

(2.11)

The "triplet correlation function" $g_3$ is assumed to vanish when any
pair of the three coordinates is separated by a distance large compared to
$R_c$ or any pair of times by an interval large compared to $t_c$.

Continuing as above, we can express the probability functions

$P_4, P_5, \ldots$ in terms of correlation functions.\(^{10}\) The $n$-particle correlation
function $g_n(z_1, t_1; \ldots; z_n, t_n)$ vanishes unless all $n$ coordinates lie
within a volume characterized by the linear dimension $R_c$ and all $n$
times within an interval of order $t_c$.

In the absence of significant effects from external magnetic
fields and/or Coriolis forces time reversal invariance implies several
symmetry relations for the $P$'s and $g$'s.\(^{11,12,13}\) For the pair

$g(z_1, t; z_2, 0) = g(z_1, -t; z_2, 0)$. \(2.12a\)

Because we have assumed a stationary random process, we may conclude that
\[ g(z_1, t; z_2; 0) = g(z_1, 0; z_2, -t) \]
\[ = g(z_1, 0; z_2, t), \quad (2.12b) \]

using (2.12a). On setting \( t_2 - t_1 = \tau \), we obtain

\[ g(z_1, t_1; z_2, t_2) = g(z_1, z_2; \tau) \]
\[ = g(z_1, z_2; -\tau). \quad (2.13) \]

We finally assume, following I, that

\[ g(z_1, z_2; \tau) = g(z_1; |z_1 - z_2|; \tau) \]
\[ = g(z_2; |z_1 - z_2|; \tau). \quad (2.14) \]

[The assumption \((2.14)\) is not required for the derivation of the transport equation. It does permit us to write the scattering kernel \((2.19)\) in "prettier" form, however.]

The mean plasma electron density at a point \( z \) is

\[ \rho(z) = N P_1(z). \quad (2.15) \]

The electron collision frequency at \( z \) will be written as \( \nu_c(z) \) and the plasma frequency as

\[ \omega_p(z) = \left[ 4\pi e^2 \rho(z)/m \right]^{1/2}. \quad (2.16) \]

The refractive index \( n(z) \) of the plasma was discussed in I.

The first approximation to this was written as \( n_1 \) and is given by the familiar expression
\[ n_1^2(z) = 1 - \omega_p^2 \left[ \frac{\omega}{\omega_p} + i\nu_c \right]^{-1}. \]  

(2.17a)

We shall, as in I, suppose the imaginary part of \( n(z) \) to be negligible for propagation over distances comparable to \( R_c \). This permits us to take

\[ n_1^2(z) \approx 1 - \omega_p^2 (\omega^2 + \nu_c^2)^{-1} \]  

(2.17b)

in Eqs. (2.20) and (2.22) below.

The absorption length \( t_c(z) \) due to electron collisions is expressed as

\[ \frac{1}{t_c(z)} \approx \frac{\omega_p^2}{(\omega^2 + \nu_c^2)} \left( \frac{\nu_c}{c} \right), \]  

(2.18)

where \( c \) is the speed of light.

We now define the scattering kernel \( M \) for scattering a wave from the direction \( \hat{\mathbf{p}}' \) to direction \( \hat{\mathbf{p}} \) as

\[ (i \ j | M(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \Omega) | s \ r) = \sigma_g(\hat{\mathbf{p}}', \hat{\mathbf{p}}; \Omega)(i \ j | m | s \ r) \]  

(2.19a)

where

\[ (i \ j | m | s \ r) = [\hat{\mathbf{p}}(i) \cdot \hat{\mathbf{p}}(s)] [\hat{\mathbf{p}}(j) \cdot \hat{\mathbf{p}}(r)] \]  

(2.19b)

and

\[
\sigma_g(\hat{\mathbf{p}}, \hat{\mathbf{p}}' ; \Omega) = \left[ \frac{r_0^2}{1 + \left( \frac{\nu_c}{\omega} \right)^2} \right] [\rho^2(g)] \frac{1}{2\pi} \int_{-\infty}^{\infty} \, d\tau \, e^{i\Omega\tau} \times \int d^3R \, g(z; R, \tau) \exp[i \, n_1(z) k(\hat{\mathbf{p}}' - \hat{\mathbf{p}}) \cdot R].
\]  

(2.20)
For later reference we observe that because of the time reversal invariance property (2.12a) \( \sigma_g \) is even in \( \Omega \).

The absorption length \( \ell_t(z) \) for scattering is defined by the equation

\[
\frac{1}{\ell_t(z)} = \frac{1}{2} \int_{\mathcal{P}} d\Omega_{\mathcal{P}} \sigma_g(\mathcal{P} \cdot \mathcal{P}') [1 + (\mathcal{P} \cdot \mathcal{P}')^2]
\]

(2.21)

where

\[
\sigma_g(\mathcal{P} \cdot \mathcal{P}') = \int_{-\infty}^{\infty} d\Omega \sigma_g(\mathcal{P}, \mathcal{P}'; \Omega)
\]

\[
= \left( \frac{\rho^2(z)}{\nu^2} \right) \rho^2(z) \int d^3R g(z; R)
\]

\[
\times \exp\left[ i \mathbf{k}_{\mathcal{P}} \cdot \mathbf{R} \right]
\]

(2.22)

and

\[
g(z; R) \equiv g(z; R, 0).
\]

(2.23)

An elementary calculation yields

\[
\sum_{s=1}^{2} \int_{-\infty}^{\infty} d\Omega \int d\Omega_{\mathcal{P}} \left( 1 \mid M(\mathcal{P}, \mathcal{P}' ; \Omega) \mid s s \right) = \frac{\delta_{ij}}{\ell_t}.
\]

(2.24)

The net absorption length \( \ell(z) \) is defined, finally, as

\[
\frac{1}{\ell} = \frac{1}{\ell_t} + \frac{1}{\ell_c}.
\]

(2.25a)
We note that this is equivalent to the equation

\[ \frac{1}{\lambda} = 2k \text{Im } n, \quad (2.25b) \]

where \( n \) is the refractive index given to the order of accuracy obtained in \( I \).

The transport equation for \( I_{ij} \) (to be derived in Section V) is

\[ \hat{p} \cdot \nabla \hat{I}(x, \hat{p}, \omega) + \frac{1}{\lambda(x)} \hat{I}(x, \hat{p}, \omega) = \int_0^\infty \int_0^\infty d\omega' \int d\omega'' \, M(\hat{p}, \hat{p}' ; \omega - \omega') \hat{I}(x, \hat{p}', \omega'). \quad (2.26) \]

Here we have written \( I_{ij} \) as a column matrix with four elements and \((i j | M | s r)\) as a \( 4 \times 4 \) square matrix [evaluated at the point \( x \), of course]. The product \( M I \) is then the column matrix with elements

\[ \sum_{s, r=1}^2 (i j | M | s r) I_{sr}, \quad i, j = 1, 2. \]

We see from Eq. (2.20) that \( M \) will vanish for \( |\omega - \omega'| \gg 1/t_c \).

If \( \hat{I} \) is nearly constant over a frequency range of order \( 1/t_c \) we can re-write Eq. (2.26) in the form

\[ \hat{p} \cdot \nabla \hat{I}(x, \hat{p}, \omega) + \frac{1}{\lambda(x)} \hat{I}(x, \hat{p}, \omega) = \int d\omega'' \, M(\hat{p}, \hat{p}' ; \omega - \omega') \hat{I}(x, \hat{p}', \omega). \quad (2.27) \]

Here

\[ M(\hat{p}, \hat{p}') = \int_{-\infty}^{\infty} d\omega M(\hat{p}, \hat{p}' ; \omega). \quad (2.28) \]
Alternatively, if the radiation is confined to a sufficiently narrow frequency interval $\Delta \omega$, we can integrate Eq. (2.26) over frequency to obtain Eq. (2.27) as satisfied by the integrated intensity

$$I(x, \hat{x}) = \int_0^\infty d\omega \, I(x, \hat{x}, \omega).$$  \hspace{1cm} (2.29)

It was this equation which was obtained in I.

The fundamental assumption required to derive the classical transport equation (2.26) is that

$$R_c << \ell_t, \hspace{1cm} (2.29a)$$

where (we recall) $R_c$ is the correlation length. When $R_c$ may be taken as $k^{-1}$, we may re-write (2.29a) as the condition that

$$\frac{\xi \omega}{\ell_t} \ll 1, \hspace{1cm} (2.29b)$$

where

$$\xi = 8 \rho^2 / \rho^2, \hspace{1cm} (2.30)$$

with $8 \rho^2$ the mean square electron density fluctuation.

In the derivation of Eq. (2.26) it was also assumed that the paths of geometrical optics for rays propagating with the refractive index $n(\hat{x})$ could be approximated by straight lines. More generally, Eq. (2.26) must be integrated along curved ray paths.
III. THE POWER SPECTRUM

We consider an electric field variable $E(t)$ defined over the "long" time interval $-(T/2) < t < (T/2)$ and vanishing outside this interval. In representing a scattered wave, $E$ will depend parameterically on the electron coordinates $z_1, \ldots, z_N$ and on any random variables characterizing the source. It will be convenient to use a complex representation for $E$, so the "power density" is

$$\mathcal{P}_0 = \frac{1}{2\pi} \langle E^*(t) E(t) \rangle,$$

(3.1)

in a suitable system of units. Here the average "\langle \cdots \rangle" represents an average over both plasma electron coordinates and over source fluctuations. That is,

$$\langle E^*(t) E(t) \rangle = \int P_N \, d^3z_1 \cdots d^3z_N \langle E^*(t) E(t) \rangle_S,$$

(3.2)

where "\langle \cdots \rangle_S" represents an average over source fluctuations only. We extend the assumption (2.8) that we are dealing with a stationary random process to include the source. Thus, for example,

$$\langle E^*(t) E(t) \rangle = \frac{1}{T} \int_{-(T/2)}^{T/2} dt \langle E^*(t) E(t) \rangle.$$

(3.3)

The field $E(t)$ is expressed in terms of its Fourier transform $\hat{E}(\omega)$ as

$$E(t) = \frac{1}{(2\pi)^{1/2}} \int_{-(T/2)}^{T/2} \hat{E}(\omega) e^{-i\omega t} \, d\omega.$$

(3.4)
The power spectrum of $E(t)$ is then

$$\mathcal{P}(\omega) = \frac{1}{T} \left\langle \frac{|\hat{E}(\omega)|^2}{\delta n} \right\rangle$$

$$= \frac{1}{\delta n} \int \left\langle E^*(t) E(t + \tau) \right\rangle \frac{e^{i\omega \tau}}{2\pi} d\tau, \quad (3.5)$$

normalized to

$$\int_{-\infty}^{\infty} \mathcal{P}(\omega) d\omega = \mathcal{P}_0. \quad (3.6)$$

It should be noted that we are here defining the power spectrum over the interval $-\infty < \omega < +\infty$. We shall see that our transport equation is even in $\omega$, so $I$ may be defined on the interval $0 < \omega < \infty$.

The incident plane wave emitted by the distant source (see Fig. 1) is assumed, for the present, to be plane polarized\(^\text{15}\) and of the form

$$E_I(x,t) = \hat{E}_e(1) E_I(x,t),$$

$$E_I(x,t) = \frac{1}{(2\pi)^{1/2}} \int \hat{E}_o(\omega) e^{i(k \cdot x - \omega t)} d\omega. \quad (3.7)$$

The power spectrum of the incident wave is

$$\mathcal{P}_I(\omega) = \frac{1}{T} \left\langle \frac{|\hat{E}_o(\omega)|^2}{\delta n} \right\rangle = \mathcal{P}_I(-\omega), \quad (3.8)$$

which must be even in $\omega$ if it is to correspond to a physical wave. The incident intensity, representing flow of power per unit area, is then [here $c$ is again the speed of light]
The total intensity is then

$$I^0(\omega) = 2\epsilon \Theta I(\omega), \quad 0 < \omega < \infty.$$  \hspace{1cm} (3.9)

The total intensity is then

$$I^0 = \int_0^\infty I^0(\omega) \, d\omega.$$  \hspace{1cm} (3.10)
IV. THE BORN APPROXIMATION

It is instructive to first calculate the scattered power in the Born approximation.\(^{16}\) The scattered waves at a point \(r\) far from the plasma can then be written in the form [see Eqs. (2.3)]

\[
E_{\text{sc}}(r,t) = \sum_{j=1}^{2} \hat{E}(j,t) E_{\text{sc}}(j,t),
\]

\[
E_{\text{sc}}(j,t) = \sum_{\alpha=1}^{N} G_{\alpha}^{0} f_{j1}(\hat{r},\hat{k}) E_{I}(\vec{z}_{\alpha},t).
\]  \hspace{1cm} (4.1)

Here \(E_{I}\) is the incident field (3.7) and \(\vec{z}_{\alpha} = \vec{z}_{\alpha}(t_{\alpha})\), \(\alpha = 1,2,\ldots,N\), is an electron coordinate evaluated at the retarded time

\[
t_{\alpha} = t - \frac{R_{\alpha}}{c},
\]  \hspace{1cm} (4.2)

where

\[
R_{\alpha} = r - \vec{z}_{\alpha}(t_{\alpha}),
\]  \hspace{1cm} (4.3)

and \(\hat{r} = \hat{\vec{z}}\).

For a plane wave having wave number \(k/2\pi\) and angular frequency \(\omega = kc\),

\[
G_{\alpha}^{0} = \frac{1}{R_{\alpha}},
\]  \hspace{1cm} (4.4)

and

\[
f_{j1}(\hat{r},\hat{k},\omega) = -r_{0}(1 + \frac{v}{c})^{-1} \hat{E}(j) \cdot \hat{E}_{k}(1)
\]  \hspace{1cm} (4.5)

is the Thomson amplitude (here \(r_{0}\) is the classical electron radius).
Since $E_i$ contains a spectrum of plane waves, we interpret (4.1) as follows:

$$G_{\alpha}^{0} f_{j1}(\hat{p},\hat{k}) E_{j}(z, t) = \int \frac{d\omega}{(2\pi)^{1/2}} \frac{e^{i\mathbf{R}}}{iR_\alpha} f_{j1}(\hat{p},\hat{k},\omega) E_0(\omega) e^{i(k'z - \omega t)}. \quad (4.7)$$

Here $k' = k(k')$ is the wave number after scattering for an incident wave number $k$. Now, $|k' - k| = \mathcal{O}(k v/c) \ll k$ by assumption NR made at the beginning of Section II. We shall interpret this to mean, for example, that

$$f_{ji}(\hat{p},\hat{k},\omega) \approx f_{ji}(\hat{p},\hat{k},\omega), \quad (4.7)$$

where $\omega = \omega'c$.

On setting $t' = t + r$, using Eqs. (4.1) and (4.6), and writing $R_{\alpha} \equiv r - \hat{p} \cdot \hat{z}_{\alpha}$, etc., we obtain

$$\langle E_{sc}^{*}(j,t) E_{sc}(t',t') \rangle = \frac{1}{T} \int_{-T/2}^{T/2} dt \langle E_{sc}^{*}(j,t) E_{sc}(t,t') \rangle$$

$$= \sum_{\alpha,\beta=1}^{N} \frac{1}{r^2} \int_{-\infty}^{\infty} d\omega \ f_{j1}(\hat{p},\hat{k},\omega) f_{j1}(\hat{p},\hat{k},\omega)$$

$$\times \langle [\frac{1}{T} |\mathbf{P}_0(\omega)|^2 \exp[-i\omega t] \times \exp\{i[(\mathbf{p}_\beta(t_\beta) - \mathbf{z}_\alpha(t'_\alpha)) \cdot (\mathbf{k} - \mathbf{k})] \} \rangle. \quad (4.8)$$

According to Eq. (3.2) the average "$\langle \cdots \rangle$" here implies the integration
From Eq. (2.10) we see that the term not involving the pair correlation function does not involve the times \( t'_{\alpha} \) and \( t_{\beta} \), which could be taken to be any two times in the interval \( -(T/2) < t < T/2 \). Also, for this term we have \( \bar{k} = k \), corresponding to coherent scattering.

For the other term, involving the pair correlation function, we have \( |z_{\alpha} - z_{\beta}| \approx O(R_c) \). Since the distribution function is stationary, we may set

\[
z_{\alpha}(t'_{\alpha}) - z_{\beta}(t_{\beta}) = z_{\alpha}(t'_{\alpha} + r/c) - z_{\beta}(t_{\beta} + r/c)
\]

\[
= z_{\alpha}(t') - z_{\beta}(t) + O\left(\frac{v_e}{c} R_c\right)
\]

within the average in Eq. (4.8). The term of \( O\left(\frac{v_e}{c} R_c k\right) \) may be dropped in the exponential, using assumption NR. Also, since \( \bar{k} - k \approx (k v_e)/c \), we may use assumption NR to set \( \bar{k} = k \) in (4.8).

Finally, then, we may write this equation in the form

\[
\langle E_{sc}^*(j, t) E_{sc}(k, t') \rangle = \sum_{\alpha, \beta = 1}^{N} \frac{1}{r^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega' f_{j1}^*(\hat{p}, \hat{k}, \omega')
\]

\[
\times f_{j1}(\hat{p}', \hat{k}', \omega') [8\pi P_{1}(\omega')]
\]

\[
\times e^{-i\omega' t} \int \int P_{1}(\hat{x}) P_{1}(\hat{x}') [1 + g(\hat{x}, \hat{x}'; \tau)]
\]

\[
\times \exp[i(\hat{x}' - \hat{p}')(\hat{x} - \hat{x}')] d^3x d^3x', \ (4.11)
\]

where \( \hat{p}' = k' \hat{x} \).
The power spectrum of the scattered waves is then

\[ \mathcal{P}_{j,t}(\omega) = \int (2\pi)^{-1} \langle E_{sc}^*(j,t) E_{sc}(t',t') \rangle e^{i\omega t} \frac{d\tau}{2\pi} \]

\[ = \frac{1}{r^2} \int f^*_{j,l}(\mathbf{\hat{r}}, \mathbf{\hat{k}}, \omega) f_{l,1}(\mathbf{\hat{r}}, \mathbf{\hat{k}}, \omega) \int d\omega' \mathcal{I}_{\mathbf{\hat{r}}}(\omega') \int \frac{d\tau}{2\pi} \]

\[ \times \int d^3x d^3x' \rho(\mathbf{x}) \rho(\mathbf{x}') [1 + g(\mathbf{x}, \mathbf{x}'; \tau)] \]

\[ \times e^{i(\omega - \omega') \tau} \exp[i(k - p) \cdot (x - x')] \]  \hspace{1cm} (4.12)

We have here used the relation (4.7) to remove the scattering amplitudes from the \( \omega' \)-integrand.

The coherent scattering in Eq. (4.12) is given by the term that does not involve \( g \). This is immediately seen to reduce to

\[ \mathcal{P}_{j,t}(\omega) \bigg|_{coh} = \frac{1}{r^2} \int f^*_{j,l}(\mathbf{\hat{r}}, \mathbf{\hat{k}}, \omega) f_{l,1}(\mathbf{\hat{r}}, \mathbf{\hat{k}}, \omega) \mathcal{I}_{\mathbf{\hat{r}}}(\omega) \]

\[ \times \left| \int d^3x \exp[i(k - p) \cdot x] \rho(x) \right|^2 , \hspace{1cm} (4.13) \]

where \( p = k \mathbf{\hat{r}} \).

The remaining part of (4.12) represents the incoherent scattered power. This may be written in the form

\[ \mathcal{P}_{j,t}(\omega) \bigg|_{inc} = \frac{1}{r^2} \int d\omega' \langle j_t | M(\mathbf{\hat{r}}, k; \omega - \omega') | 11 \rangle \mathcal{P}_{i,\omega'}(\omega'), \hspace{1cm} (4.14) \]

where \( M \) is defined by Eqs. (2.19) and (2.20) and the refractive index \( n_1 \) is replaced by unity.
To derive the transport equation we must consider a sequence of scatterings, just like the single one just described. In I it was shown that all coherent scatterings result in propagation with the refractive index $n$. The incoherent scatterings lead to the transport equation.

A sequence of coherent scatterings will not lead to a frequency shift. On the other hand, a long sequence of incoherent scatterings may lead to a large frequency shift in the wave. For each single scattering in such a sequence, we can continue to assume that the frequency shift $(\bar{\omega} - \omega) = \mathcal{O}(\omega v/c)$ is small, because of the assumption NR. In particular, we can continue to use the relation (4.7), where $\omega$ and $\bar{\omega}$ are the respective frequencies before and after a given single scattering.
V. DERIVATION OF THE TRANSPORT EQUATION

The derivation of the transport equation as given in I needs only minor modifications to take account of the frequency shift. In this section we shall therefore rely heavily on the development given in I.

Following the discussion given in I, we write a particular component of the scattered electric field vector in the form

\[ E(r) = \sum_{n=1}^{\infty} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n} Q_n(r; z_{\alpha_1} \cdots z_{\alpha_n}) \]  

(5.1)

Here \( Q_n \) represents the contribution from a wave multiply scattered by electrons at \( z_{\alpha_1} \cdots z_{\alpha_n} \) and the sum is over all electrons and numbers of scatterings.

To find the scattered power we must evaluate such quantities as

\[ P_{nm} = \langle Q_m^*(r; z_{\beta_1} \cdots z_{\beta_m}) Q_n(r; z_{\alpha_1} \cdots z_{\alpha_n}) \rangle \]

\[ = \langle \int P_{m+n} Q_m^* Q_n \ d^3z_{\beta_1} \cdots d^3z_{\alpha_n} \rangle \]  

(5.2)

using the notation of Eq. (3.2). We suppose the probability function \( P_{m+n} \) to be decomposed into a cluster expansion of correlated coordinates, as in Eqs. (2.10) and (2.11). For each term of this expansion each coordinate is a member of a correlated cluster of coordinates. First, a given coordinate \( z_{\alpha} \) may be uncorrelated with another coordinate. If not uncorrelated, \( z_{\alpha} \) is correlated with other members of the set \( z_{\beta_1} \cdots z_{\alpha_n} \) in (5.2).
Let us suppose that $z_\alpha$ belongs to the correlated cluster set $z_\alpha^c \cdots z_\alpha^d$, which consist of only $z_\alpha$. In this case the integral over $z_\alpha^c \cdots z_\alpha^d$ involves only $Q_n$. This was called a "coherent part" of the average in I. Such "coherent part" averages may clearly be performed on each factor of $E(x)$ before squaring. It was shown in I that the effect of the "coherent part" averages is to give the plasma a refractive index. This result may be taken unchanged for our present analysis.

To see this, we note that the introduction of the time-dependent correlation functions does not modify the expressions obtained in I for the refractive index. This is obvious [because of the stationary property (2.3)] for scatterings which are uncorrelated. Scatterings which are correlated are separated by distances of the order of $R_c$. During the time $R_c/c$ required for propagation across a correlated cluster a typical electron will have moved a distance $(R_c v_e)/c$. The resulting change of phase in the exponentials is therefore of order

$$ (k R_c) \frac{v_e}{c} \ll 1, \quad (5.3) $$

by assumption NR, and can be neglected.

The resulting equations for the multiply scattered waves are [see Eqs. (I 3.31), (I 3.32), (I 3.33)]

$$ E(z_\alpha, t) = E(z_\alpha^c, t) + \sum_{\beta(\neq \alpha) = 1}^{N} \sum_{j=1}^{2} \hat{E}_{\alpha\beta}(j) E_{\alpha\beta}(j, t), \quad (5.4) $$
\[
E_{\alpha\beta}(i,t) = G_{\alpha\beta} f_{11}(\alpha\beta, \beta\sigma) E_c(z_{\beta}) + \sum_{\sigma(\neq \beta) = 1}^{N} \sum_{j=1}^{2} G_{\alpha\beta} f_{ij}(\alpha\beta, \beta\sigma) E_{\beta\sigma}(j,t).
\]

(5.5)

Here

\[
\hat{E}_{\alpha\beta}(i) = \hat{E}_{\alpha}(i),
\]

(5.6)

where \(\hat{\mathbf{q}}\) is the unit vector parallel to \((z_\alpha - z_\beta)\). The quantity \(E_c(z_\alpha, t)\) represents the coherent wave [see Eqs. (3.7)]

\[
E_c(z_\alpha, t) = \hat{\mathbf{e}}(1) E_c(z_\alpha, t),
\]

\[
E_c(z_\alpha, t) = \frac{1}{(2\pi)^{1/2}} \int d\omega \hat{E}_0(\omega) e^{i(kS_\alpha - \omega t)}.
\]

(5.7)

where \(S_\alpha\) is the eikonal for the coherent wave [Eq. (3.34)]

\[
S_\alpha = S(z_\alpha) = \int_{-\infty}^{z_\alpha} [n(\chi) - 1]d\chi + \hat{\mathbf{e}} \cdot z_\alpha.
\]

(5.8)

Here \(n(\chi)\) is the refractive index and the constant of integration has been chosen to be consistent with Eq. (3.7). The Thomson amplitude (4.5) for scattering has been re-written in Eqs. (5.5) to indicate scattering from the direction of \((z_\beta - z_\sigma)\) to that of \((z_\alpha - z_\beta)\), etc. For a monochromatic wave \(G_{\alpha\beta}\) is the Green's function [see Eq. (I 3.29)]

\[
G_{\alpha\beta} = \frac{e^{ikS_{\alpha\beta}}}{R_{\alpha\beta}},
\]

(5.9)
\[ S_{\alpha\beta} = \int_{z_\beta}^{z_\alpha} n(\xi) \, ds , \quad (5.10) \]

the integral being taken along the straight line path from \( z_\beta \) to \( z_\alpha \) and

\[ R_{\alpha\beta} = z_\alpha - z_\beta \quad (5.11) \]

We have used the notation of Eq. (4.6) on the right-hand side of Eq. (5.5), writing

\[ G_{\alpha\beta} f_{ij}(\alpha\beta, \beta\sigma) E_{\beta\sigma}(j,t) = f_{ij}(\alpha\beta, \beta\sigma; \omega) E_{\beta\sigma}(j,\omega) e^{-i\omega t} \quad (5.12) \]

in terms of the Fourier transform \( \hat{E}_{\beta\sigma} \) of \( E_{\beta\sigma} \).

The argument given following Eq. (4.6) lets us set \( k = k \) in the exponential in (5.12).

The "coherent part" averages in the expressions (5.2) permitted us to derive the multiple scattering equations (5.4) and (5.5). On performing the remaining averages, after using Eqs. (5.4) and (5.5), we must omit "coherent part" averages. This means that every coordinate \( z_{\alpha i} \) must be now correlated with at least one of the \( z_{\beta j} \)'s in (5.2).

Continuing to follow I, we define [a generalization of Eq. (I 5.6)]

\[ U_{ij}(\alpha, \beta; \omega) = \frac{1}{2\pi} \int_0^{2\pi} dr e^{i\omega r} \int d^3z_\gamma d^3z_\sigma \delta \left[ \frac{1}{2} (z_\gamma + z_\sigma) - z_\beta \right] \chi \rho(z_\gamma) \rho(z_\sigma) g(z_\gamma z_\sigma; \tau) \frac{1}{\delta_{\omega \gamma}} \left\langle E_{\alpha\gamma}^*(1, t) E_{\alpha\sigma}(j, t + \tau) \right\rangle_{\alpha\gamma} \quad (5.13) \]
Here the notation \( \langle \cdots \rangle_{q r s} \) means an average over all coordinates except for \( z_q \), \( z_r \), and \( z_s \), which are held fixed.\(^{17}\)

We now follow the derivation of Eq. (5.11). Equations (5.5) are substituted into the right-hand side of Eq. (5.13). Equations (5.9) and (5.10) are used to write \( E_{\beta \sigma} \) in terms of \( E_{\beta \sigma} \), etc., for \( |z_\beta - z_\sigma| = O(R_c) \). There finally results

\[
u_{ij}(\alpha, \beta; \omega) = (2\pi)^{-1} \int d\tau e^{i\omega \tau} \int d^3 z_\gamma d^3 z_\delta, \delta \left( \frac{1}{2}(z_\gamma + z_\delta) - z_\beta \right) \]

\[
\times \rho(z_\gamma) \rho(z_\delta) g(z_\gamma, z_\delta; \tau) |G_{\alpha \beta}|^2
\]

\[
\times \left\{ \sum_{i, j} (\alpha, \beta; \sigma) f_{ij}^\dagger(\alpha, \beta; \sigma) (\beta \sigma)^{1/2} \left( E_{\alpha \beta}^*(z_\beta, t) E_{\beta \sigma}^*(z_\delta, t + \tau) \right) \right\}
\]

\[
\times \exp \left[ i n_1(z_\beta) (k_{\alpha \beta} - k) \cdot (R_{\gamma \beta} - R_{\gamma \beta}') - i n_1(z_\delta) (k_{\beta \delta} - k) \cdot (R_{\gamma \delta} - R_{\gamma \delta}') \right]
\]

\[
+ \sum_{s, s'=1}^2 \int d^3 z_\sigma d^3 z_\sigma', \rho(z_\sigma) \rho(z_\sigma') g(z_\sigma, z_\sigma'; \tau) \]

\[
\times \left\{ \sum_{i, s} (\alpha, \beta; \sigma) f_{is}^\dagger(\alpha, \beta; \sigma) \exp \left[ i n_1(z_\beta) (k_{\alpha \beta} - k_{\beta \sigma}) \cdot R_{\gamma \gamma'} \right] \right\}
\]

\[
\times (\delta \pi)^{-1} \left( E_{\beta \sigma}^*(s, t) E_{\beta \sigma}^*(s', t + \tau) \right)_{\beta \sigma}, + \text{cross terms}
\]

To simplify the first term above we write
\( \left( \frac{2\pi}{1} \right)^{-1} \int dt \, e^{i\omega t} \left( \delta\omega \right)^{-1} \langle E_c^* (z', t) E_c (z', t + \tau) \rangle_s \)
\[
= \left[ \frac{1}{2c}\, I_0\omega \right] \, e^{ik(S_x - S^*_x)}
\]
\[
= \left[ \frac{1}{2c}\, I_c (z', \omega) \right],
\]

where [see Eq. (3.9)],
\[
I_c (z', \omega) = I_0 (\omega) \exp \left[ - \int_{z'} \frac{ds}{V(x)} \right]
\]

and the integral is taken along the straight line parallel to \( \hat{k} \).

On making use of the assumption that \( R_c / \ell \ll 1 \), we may neglect the cross terms in Eq. (5.14) and express the second term in terms of \( U \). In so doing we make use of our conclusion of Section IV that the change in frequency on a single scattering may be neglected in the Thomson amplitude and in the exponentials.

To simplify the second term in Eq. (5.14), we write it in the form
\[
(2\pi)^{-1} \int \! \! \! \sum_{s,s'=1}^{2} \int \! \! \! d\omega' \int \frac{d^3 z_\gamma d^3 z_{\gamma'}}{2\pi} e^{-i\omega'\tau} \int \! \! \! d\tau' \! \! \! e^{i\omega'\tau'} \\
\times \rho(z_\gamma) \rho(z_{\gamma'}) \mathcal{g}(z_\gamma, z_{\gamma'}; \tau) |c_{\alpha \beta}|^2 \sum_{s,s'=1}^{2} \int \! \! \! \frac{d\omega'}{2\pi} e^{-i\omega'\tau} \int \! \! \! d\tau' \! \! \! e^{i\omega'\tau'} \\
\times \int \! \! \! \frac{d^3 z_\sigma d^3 z_{\sigma'}}{2\pi} \int \! \! \! \frac{d^3 z_{\lambda} d^3 z_{\lambda'}}{2\pi} \delta \left( \frac{1}{2}(z_\sigma + z_{\sigma'}) - z_{\lambda} \right) \rho(z_\sigma) \rho(z_{\sigma'}) \\
\times \mathcal{g}(z_\sigma, z_{\sigma'}; \tau') \left[ f^*_{is}(\alpha_\beta, \beta_\sigma) f^*_{js}(\alpha_\beta, \beta_\sigma) \right] \text{exp} \left[ i \eta(z_\rho) (k_{\alpha \beta} - k_{\beta \sigma}) \cdot \mathbf{R}_{\gamma'} \right] \\
\times \langle (8\pi)^{-1} E^*_{\beta_\sigma}(s,t) E_{\beta_\sigma'}(s', t + \tau') \rangle_{\beta_\sigma'} \]

\[= \sum_{s,s'=1}^{2} \int \! \! \! d^3 z_\lambda \int \! \! \! \frac{d\omega'}{2\pi} |c_{\alpha \beta}|^2 \left( 1j | M(\alpha_\beta, \beta_\lambda; \omega-\omega') | ss' \right) \\
\times U_{ss'}(\beta, \lambda; \omega'). \]

\[ (5.17) \]

In writing the exponential involving \( R_{\gamma'} \), here we have replaced \( k_{\beta_\sigma} \) in Eq. (5.14) by \( k_{\beta_\lambda} \). This is permissible since we assume \( \epsilon \gg \mathcal{R}_c \). The quantity \( M \) in Eq. (5.17) is defined by Eqs. (2.19) with the obvious notational change of indicating directions of propagation as "\( \beta_\lambda \)" and "\( \alpha_\beta \)."

The results (5.15) and (5.17) permit us to write (5.14) in the form
\[ U_{ij}(\alpha, \beta; \omega) = \int d\omega' \left| G_{\alpha \beta} \right|^2 \left\{ (ij \mid M(\alpha \beta, \beta 0; \omega - \omega') \mid 11) \frac{1}{2c} I_0(z_{\beta}', \omega') \right. \\
+ \sum_{s, s' = 1}^{2} \int d^3 z_{\sigma} \delta_{ij} M(\alpha \beta, \beta \sigma; \omega - \omega') M_{ss'}(\beta, \sigma; \omega') \left\} \right. \\
\] 
\[ (5.18) \]

Since \( M \) is even in \((\omega - \omega')\) [see remark following Eq. (2.20)] and \( I_c \) is even in \( \omega' \), it follows that \( U \) is even in \( \omega \). This lets us define the intensity \( I_{ij} \) for \( \omega > 0 \) with the equation

\[ I_{ij}(z_{\alpha} \hat{z}, \omega) = I_c(z_{\alpha}, \omega) \delta_{il} \delta_{jl} \delta_{\hat{k}, \hat{z}} + 2c \int_{\hat{z}} d\alpha_{ij} R_{\alpha \beta} \ U_{ij}(\alpha, \beta; \omega). \]
\[ (5.19) \]

The \( \delta \)-function here is defined by the condition that

\[ \int d\hat{z} \ f(\hat{p}) \delta_{\hat{z}, \hat{p}} = f(\hat{k}) \]

for a function \( f(\hat{p}) \) which is regular at \( \hat{z} = \hat{p} \). The integration in Eq. (5.19) is performed over \( z_{\beta} \) along the semi-infinite straight line beginning at \( z_{\alpha} \) and directed parallel to \( \hat{z} \).

Using Eq. (5.19) we can express (5.18) in terms of \( I_{ij} \). If we write

\[ |G_{\alpha \beta}|^2 = \frac{1}{R_{\alpha \beta}} \exp \left[ - \int_{\hat{z}} \frac{dz}{\xi(\hat{z})} \right] \]

we obtain [in the matrix notation of Eq. (2.26)]
\[ I(x, \hat{z}, \omega) = I_c(x, \omega) \exp \left( \frac{-i}{\hbar} \int_{-\hat{z}}^{z} ds(\chi') \right) \]

\[ \times \int_{0}^{\infty} d\omega' \int d\omega_{p}, \quad M(\hat{z}, \hat{z}'; \omega - \omega') I(z, \hat{z}', \omega'), \quad (5.20) \]

where now \( \omega > 0 \) and

\[ I_c(x, \omega) = I^0(\omega) \exp \left[ -\int_{-\hat{z}}^{z} \frac{ds}{\ell} \right]. \quad (5.21) \]

The path integral in Eq. (5.20) extends along the straight line from \( \hat{z} \) to \( \infty \) in the direction \( -\hat{z} \).

Differentiation of Eq. (5.20) along a ray path leads to Eq. (2.26).

Equation (5.20) is evidently valid for arbitrary incident polarization, to be specified by the choice of \( I^0(\omega) \).
VI. RADAR BACKSCATTER

It was pointed out in I that the transport equation is not valid for backscatter. The reason for this is illustrated in Fig. 2. To each ray path defined by a particular sequence of multiple scatterings there corresponds a path obtained by reversing all propagation vectors. These pairs of paths can interfere coherently, and this is not included in the transport equation. As was shown in I this effect can be accounted for, however, by choosing a certain linear combination of solutions of the transport equation. The specific expression for backscatter was given in Eqs. (I 7.7) and (I 7.10).

When there is a frequency shift, Eqs. (I 7.5a) and (I 7.5b) are modified. These now read, respectively,

\[
Q_n(i,s) = \sum_{j_1 \ldots j_{n-1}} \frac{e^{ik_1S_{ij_1j_2 \ldots j_{n-1}}} f_{ij_1j_2 \ldots j_{n-1}}}{r} \cdot \hat{\mathbf{r}}(\hat{x}_n \hat{x}_{n-1}) \cdots \hat{\mathbf{r}}(\hat{x}_1 \hat{x}) e^{ik_{n+1}Sr},
\]

(6.1a)

\[
\tilde{Q}_n(i,s) = \sum_{j_1 \ldots j_{n-1}} \frac{e^{ik_1S_{ij_1j_2 \ldots j_{n-1}}} f_{ij_1j_2 \ldots j_{n-1}}}{r} \cdot \hat{\mathbf{r}}(\hat{x}_n \hat{x}_{n-1}) \cdots \hat{\mathbf{r}}(\hat{x}_1 \hat{x}) e^{ik_{n+1}Sr},
\]

(6.1b)

Here \(k_1\) is the incident wave number, \(k_2\) that after the first scattering, \(\ldots\), and \(k_{n+1}\) that after the \(n\)th scattering.

For \(Q_n\) and \(\tilde{Q}_n\) to interfere coherently, the frequency (wave-number) spread must be small enough that the phase differences 

\[
[S_{\ell \ell-1}(k_\ell - k_{n-\ell+2})]
\]

are small compared to unity. The criterion for this is that
where $\Delta \omega$ is the total spread in frequency due to scattering. If condition (6.2) is satisfied the expressions (I 7.7) and (I 7.10) may be used.

On the other hand, when

$$\frac{(\Delta \omega \ell)}{c} \ll 1,$$

(6.2)

$Q_n$ and $\tilde{Q}_n$ will not interfere. Then the transport equation (2.26) [or (5.20)] does tend to be valid for backscatter, without the special correction of Eqs. (I 7.7) and (I 7.10).

In intermediate cases it is not anticipated that the transport equation will be applicable to the calculation of backscatter.
FOOTNOTES AND REFERENCES

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1. K. M. Watson, J. Math. Phys. 10, 688 (1969). This paper will hereafter be referred to as I. Equations in I will be referred to as "Eq. (I 3.2)," etc.

2. K. M. Watson, Electromagnetic Wave Scattering Within a Plasma in the Transport Approximation (to be published in Phys. of Fluids). We shall refer to this paper as II.


7. Any fixed unit vector $\hat{k}$ would do.

8. The representation of polarization is discussed in II in some detail.

9. As a practical condition, we require that (2.8) be valid for

$$\tau \leq \mathcal{O}(\Delta \omega^{-1}),$$

where $\Delta \omega$ is the smallest frequency interval of interest in the intensity spectrum.

10. This was described in more detail in I.

14. Since Eq. (2.26) is homogeneous in the intensity $\mathcal{J}$, the actual system of units used is not significant. We are here using unrationalized Gaussian units, as in I.
15. We shall later extend the result to include arbitrary polarization of the incident wave, as was done in I.
17. This notation was described in more detail in I.
Fig. 1. Illustration of scattering from a plasma.
Fig. 2. Illustration of backscattering.
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