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Publication Date
1983-04-01
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Ghassan George Batrouni
(Ph.D. Thesis)

April 1983

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Plaquette Formulation of Lattice Gauge Theories*

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Ph.D. Thesis

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* This research was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.
ABSTRACT

This thesis deals with the plaquette formulation of lattice gauge theories and illustrates some of the applications of this formulation. In chapter (I) we show how such a formulation is implemented for abelian and non-abelian theories. We show that when we change the variables of integration from links to plaquettes, in the generating functional, the Jacobian is just a delta function of the lattice Bianchi identity. The lattice Bianchi identity is the only source of correlations among the plaquettes: without it the theory is trivial to solve.

We then give the geometrical interpretation of the lattice Bianchi identity. From this, the plaquette formulation of lattice gauge theories yields new geometrical interpretation of old tools on the lattice: the strong coupling expansion, the duality transformation, and the Coulomb gas representation. The strong coupling expansion is found to be a systematic expansion towards restoring the lattice Bianchi identity. The dual potential is the Fourier conjugate to the lattice Bianchi identity and the Coulomb gas is given trivially by it. For non-abelian theories, we discuss gauge invariant plaquettes and their role in the lattice Bianchi identity.
In chapter (II) we use the formalism and interpretations developed in chapter (I) to discuss a new gauge-invariant mean-plaquette method for lattice gauge theories. Previous mean field methods face some problems when applied to lattice gauge theories. First, one has to work quite hard to show gauge invariance of the method. More seriously, this method always predicts a first order phase transition whether or not there is one. It is also quite hard to take into account the effect of fluctuations. These difficulties are easily overcome by our method. It is manifestly gauge-invariant, and it is straightforward to take fluctuations into account. Moreover, it seems to predict a phase transition only when there is one. The order of the transition is not always clear, but the most gratifying result is the absence of a phase transition for lattice QCD$_4$ where our method predicts a crossover, in agreement with Monte Carlo simulations.
ACKNOWLEDGMENTS

This work was done under the guidance of M. B. Halpern. I wish to thank him for all his help. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U. S. Department of Energy under Contract DE-AC03-76SF00098.
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I. INTRODUCTION

Some time ago, Halpern proposed two related new formulations of continuum gauge theories: A field strength formulation and a dual potential formulation. It is the purpose of this paper to extend these formulations onto the lattice and study the connections with other lattice formalisms.

Following Halpern's steps, we achieve then a plaquette variable formulation for all lattice gauge theories. For Abelian theories we also show how the dual potential and the Coulomb gas are related to the plaquette formulation.

As in the continuum, the central step in the formulation is the lattice Bianchi identity among the plaquettes. For Abelian lattice gauge theories, the Bianchi identity forms the pivot point from which springs easily the three major known tools for analyzing these theories (Coulomb gas representation, duality transformation, and strong coupling expansion). 1) The lattice Bianchi identity contains the monopole currents explicitly and independent of the action. 2) The dual links are conjugate to the Bianchi identity. 3) The strong coupling expansion is seen to be an expansion towards restoring the lattice Bianchi identity: In leading order of the strong coupling expansion the Bianchi identity is ignored. Higher order terms in the strong coupling expansion correspond to restoring the Bianchi identity in a systematic fashion. Point (3) above, is also shown for non-Abelian theories. For these theories, we also discuss "Abelianization" of the lattice Bianchi identity, i.e. putting it in the form of the product of the six gauge invariant plaquettes forming the surface of a cube.
The techniques and results presented in this paper will be useful for a gauge invariant mean-plaquette formulation of lattice gauge theories to be presented in another paper.

The paper is organized as follows, in Sections II and III we present respectively the plaquette formulation of Abelian and non-Abelian lattice gauge theories. In Section IV we discuss the Abelianization of the non-Abelian lattice Bianchi identity and its geometrical interpretation. Duality transformations and Coulomb gas representations for Abelian theories are presented in Section V. Sections VI and VII discuss the strong coupling expansion for Abelian and non-Abelian theories respectively. Conclusions and some comments are in Section VIII.
II. PLAQUETTE FORMULATION OF ABELIAN LATTICE GAUGE THEORIES

In this section we express lattice QED\textsubscript{4} in terms of plaquette variables. This same derivation holds for other Abelian theories with any action but for illustration we will use the Wilson action.

The partition function is

\[ Z = \int DU_\mu(r) e^{\frac{1}{4g^2} \sum [P_{\mu\nu}(r) + P_{\mu\nu}^+(r)]} \]  

(II.1)

where \( U_\mu(r) \) is the \( U(1) \) link variable and \( P_{\mu\nu}(r) \) is a plaquette

\[ P_{\mu\nu}(r) = U_\mu(r)U_\nu(r + \hat{\mu})U_\mu^+(r + \hat{\nu})U_\nu^+(r). \]  

(II.2)

In order to express the link variables, \( U_\mu(r) \), in terms of the plaquette variables, \( P_{\mu\nu}(r) \), we use the completely fixed "path gauge". A path gauge is defined as follows. Choose an origin \((x_0, y_0, z_0, t_0)\) and construct a path to an arbitrary point \((x, y, z, t)\). For example, we start along the \(t\)-axis to the point \((x_0, y_0, z_0, t)\), then parallel to the \(y\)-axis to \((x_0, y, z_0, t)\), then parallel to the \(x\)-axis to \((x, y, z_0, t)\) and finally to \((x, y, z, t)\). All links lying on this path are gauge fixed to unity. This is done with paths to all sites on the lattice. The dotted lines in Figure 1(a) show such paths (in 3 dimensions). The above path gauge is therefore

\[ U_3(txyz) = U_1(txyz_0) = U_2(tx_0yz_0) = U_0(tx_0y_0z_0) = 1 \]  

(II.3)

it is then easy to express links in terms of plaquettes
These equations have a very simple geometrical interpretation in the above gauge, each link is a closed Wilson loop, formed by the path from \((x_0'y_0'z_0't_0')\), along the gauge lines to and through the link and back to \((x_0'y_0'z_0't_0')\) along the gauge lines. This is shown for \(U_2(\text{xyz})\) (in 3 dimensions) in Figure 1(a). Dotted lines denote gauge lines. Each of these Wilson loops may be filled with plaquettes as is also shown in the figure. This is the geometrical content of (II. 4a,b,c).

To change from link to plaquette variables in (II. 1), we fix the gauge by inserting

\[
\delta(\text{CGF}) = \delta[U_3^\text{(xyz)} - 1] \delta[U_1^\text{(txyz)}] \delta[U_2^\text{(txyz)}] \delta[U_0^\text{(txyz)}]
\]

where \(\text{CGF}\) stands for complete gauge fixing. Then insert in (II.1)

\[
1 = \int \mathcal{D}_{\mu \nu}(r) \delta_{\mu \nu}(r) \delta[U_\mu^\text{(r)} U_\nu^+(r)] \delta[U_\mu^+(r) U_\nu(r)] \delta[U_\mu^+(r) U_\nu^+(r)]
\]

for every plaquette and do the \(U_\mu(r)\) integrals. The crucial result is
\[ \int D U_\mu (r) \delta (CGF) \prod_{\mu \nu} \delta [P_{\mu \nu} (r) (U_\mu (r) U_\nu (r+\mu) U_\mu^{+} (r+\nu) U_\nu^{+} (r))^{+} - 1] \]
\[ = \prod_{r, \nu} \delta [e^{\theta_{\mu \nu} (r)} - 1] \]

where
\[ P_{\mu \nu} (r) \equiv e^{\theta_{\mu \nu} (r)} \]  \hspace{1cm} (II.8a)
and
\[ \tilde{\theta}_{\mu \nu} (r) \equiv \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \theta_{\rho \sigma} (r). \]  \hspace{1cm} (II.8b)

The proof of (II.7) is left for an appendix. So, the lattice Bianchi identity is given by

\[ i \Delta \tilde{\theta}_{\mu \nu} (r) e^{\mu \nu} = 1 \]  \hspace{1cm} (II.9)

and is the product of the six plaquettes forming the six faces of a three dimensional cube.

Equation (II.1) thus becomes

\[ Z = \int_{-\pi}^{+\pi} D(\theta_{\mu \nu} (r)) \prod_{r, \nu} \delta \left[ e^{\mu \nu \tilde{\theta}_{\mu \nu} (r)} - 1 \right] \frac{1}{e^{2g^2}} \sum \cos \theta_{\mu \nu} (r) \]

We have, therefore, obtained a result similar to the continuum: The integral over plaquettes is constrained only by the lattice Bianchi identity.

The same derivation is applicable to other gauge theories like Z(N) in any number of dimensions. The number of Bianchi identities will depend on the dimensionality of the theory: it is the number of different types of three dimensional cubes one can construct in the space. For example, in two dimensional theories there are no
cubes (and therefore no Bianchi identity), in three dimensional (xyz) theories, there is only one kind of cube: xyz (and thus one Bianchi identity). In general, for an N dimensional theory there are \( N(N-1) (N-2)/6 \) different types of three dimensional cubes and thus the same number of Bianchi identities.
III. PLAQUETTE FORMULATION OF LATTICE QCD

For illustration, we will use QCD₃. QCD₄ has four types of cubes (and therefore Bianchi identities), each of which is handled the same way.

The partition function for lattice QCD₃ is given by

\[
Z = \int DU e^{\frac{\beta}{4} \sum_{\mu \nu} \text{Tr}[P_{\mu\nu} (r) + P^+_{\mu\nu} (r)]}
\]

(III.1) where \( P_{\mu\nu} (r) \) is given by (II.2).

We go through exactly the same steps as in Section II to change the variables of integration in \( Z \) from links to plaquettes. The result is

\[
Z = \int D_P \left[ \prod_\zeta \delta(P_c - 1) \right] e^{\frac{\beta}{4} \sum_{\mu \nu} \text{Tr}[P_{\mu\nu} (r) + P^+_{\mu\nu} (r)]}
\]

(III.2)

where

\[
\delta(P_c - 1) = \delta(P_{12}(xyz)P_{23}(xyz)U_2(xy+1z)U_1(xy+1z+1)
\]

\[
\times U_2^+(x+1yz+1)P^+_{23}(x+1yz)U_1^+(xy+1z)P^+_{13}(xyz)U_1(xy+1)
\]

\[
\times U_2(x+1yz+1)U_1^+(xy+1z+1)U_2^+(xyz+1)P^+_{12}(xyz+1)-1).\] (III.3)

The links \( U_1 \) and \( U_2 \) are given in terms of the plaquettes by (II.4a-b). \( \delta(P_c - 1) \) is a \( \delta \) function of the non-Abelian lattice Bianchi identity associated with a given cube \( C \). This is the lattice version of Halpern's result in the continuum.\(^1\) \( \prod \) denotes a product over all cubes of the lattice.
Notice that for every $U_1$ that appears in the non-Abelian lattice Bianchi identity (III.3), there also appears its conjugate. However, because the group is non-Abelian, $U_1$ and $U_1^+$ cannot be made to cancel. In the Abelian case, the $U_1$ terms cancel out and we are left with the product of the six plaquettes forming a cube.

We can also express the partition function in terms of unconstrained plaquette variable by solving the lattice Bianchi identities as Halpern did in the continuum.¹ Using equations (II.4a-b) we get

\[ P_{12}(xyz) = U_1(xyz)U_2(x+lyz)U_1^+(xy+lz)U_2^+(xyz) \]

\[
= \begin{bmatrix}
  z_0^{-1} \\
  \prod_{z' = z} P_{13}(xyz')
\end{bmatrix}
\begin{bmatrix}
  x_0^{-1} \\
  \prod_{z' = z} P_{23}(x+lyz') \prod_{x' = x} P_{12}(x'yz_0')^+
\end{bmatrix}
\begin{bmatrix}
  z_0^{-1} \\
  \prod_{z' = z} P_{13}(xy+lz')
\end{bmatrix}
\begin{bmatrix}
  x_0^{-1} \\
  \prod_{z' = z} P_{23}(xyz') \prod_{x' = x} P_{12}(x'yz_0')^+
\end{bmatrix}
\]  

(III.4)

It is trivial to check that this is the solution to the lattice Bianchi identity. Thus

\[ Z = \int_{D_{13}D_{23}D_{12}(xyz_0)} e^{\frac{\beta}{4} \sum i, j \text{Tr}[P_{13}(r) + P_{13}^+(r)] + \frac{\beta}{4} \text{Tr}[P_{12}(r) + P_{12}^+(r) ]} \]

(III.5)

where $P_{12}(r)$ is given by equation (III.4). This form of the partition function, and others like it (from different gauges), involve no plaquette constraints, but a complicated effective action. We mention that these remaining plaquettes can be thought of as gauge invariant plaquettes (in analogy with Mandelstam's gauge-invariant field strengths²), with their tails along the gauge lines. This is shown in Figure 1(b).

Finally we mention that if the point $Z_0$ is chosen at infinity. The unconstrained partition function (III.5) may be formally considered as over only $D_{13}D_{23}$. 

IV. ABELIANIZATION OF THE NON-ABELIAN LATTICE BIANCHI IDENTITY

We have noticed that under the following change of variables

\[ P'_{23}(x+lyz) = U_1(xyz+1)P_{23}(x+lyz)U_1^+(xyz+1) \]  \hspace{1cm} (IV.1a)

\[ P'_{13}(xy+lz) = U_2(xyz+1)P_{13}(xy+lz)U_2^+(xyz+1) \]  \hspace{1cm} (IV.1b)

the lattice Bianchi identity (III.3), for a given cube becomes

\[ \delta[P_{12}(xyz)P_{23}(xyz)P_{13}^+(xy+lz)P_{12}^+(xyz+1)P_{23}^+(x+lyz)P_{13}^+(xyz)-1] \]  \hspace{1cm} (IV.2)

which we call "Abelianization" because it has the form of the lattice Bianchi identity for an Abelian theory. This simplification apparently corresponds to Mandelstam's 2 "Abelianized" non-Abelian Bianchi identity

\[ \nabla \cdot \mathbf{B}(x,P) = 0 \]  \hspace{1cm} (IV.3)

satisfied by his path dependent magnetic fields.

The important question arises, whether this Abelianization can be globally implemented, and variables changed from \( P_{\mu\nu} \) to \( P'_{\mu\nu} \) (as in the above example) such that \( P_{\mu\nu} \) and \( P'_{\mu\nu} \) are in one to one correspondence. If possible, this would result in a "totally Abelianized" or local QCD, integrated over the new variables \( P'_{\mu\nu} \).

As it stands, we have failed to do this. We have however, found variable changes over large but not complete subspaces of the lattice, for example a two by two by infinite sublattice, where this can be accomplished.
After the completion of this algebraic approach, J. Kiskis informed us of the geometrical interpretation of identities such as (IV.2), and it is easiest to explain the more complicated variable changes that we found in that geometrical language.

The lattice Bianchi identity (IV.2) is equivalent to Figure 2 in terms of the indicated gauge invariant plaquettes. The change of variables (IV.1a-b) is precisely the change to these gauge invariant plaquettes in the U_3 = 1 gauge. In addition, Equations (III.3) and (IV.2) and Figure 2 are unitarily equivalent to Figure 3 and to many others one can draw. So, Abelianization is expressing the lattice Bianchi identity for a given cube as a product of the six gauge invariant plaquettes forming that cube. Furthermore, the question of the "totally Abelianized QCD" is the question whether each geometrical plaquette corresponds to just one gauge-invariant plaquette.

Now notice that if, in Figure 4, we use Figure 2 for cubes 1, 3, 5, and 7 and Figure 3 for cubes 2, 4, 6 and 8, all the gauge invariant plaquettes shared by these cubes are the same. This can be continued to infinity in the z direction. This is the geometrical interpretation of the algebraic Abelianization mentioned above for large but incomplete regions of the lattice.

It is also easy to see that this Abelianization works for the 2 x 2 x infinite lattice by reflecting the figures associated with cubes 1 to 7 in the yz plane at x(Fig. 4). To see what is meant by reflection, imagine glueing the cube in Figure 3 to the left xz face.
of the cube in Figure 2. The gauge invariant plaquettes of cubes 2 and 3 are reflections of each other (except for the sense of the arrows) in the common xz plaquette. The sense of the arrows can be reversed by taking the conjugate of the Bianchi identity.

We can also show that we can Abelianize the lattice Bianchi identity for cubes 9 and 10 (Fig. 4) and so on in the y direction (and for many other paths). However, in the presence of cubes 9 and 10 we have not managed to Abelianize the identity for cube 11. We found that it will involve plaquettes from nearby cubes, and thus not have an Abelianized (local) form.

The overall pattern seems to be the following. Space can be filled with cubic (i.e. Abelianized) lattice Bianchi identities sharing the same gauge invariant plaquettes, except that cavities will develop in which the identities are not cubic—as for cube 11 in Figure 4. This seems to correspond to the types of regions over which Mandelstam's path dependent phase can be defined by parallel transport in such a way as to satisfy (IV.3).

The fact that the lattice Bianchi identity can be Abelianized for some regions of the lattice, as described above, provides a simplification both in the strong coupling expansion described in Section VII, and in a new gauge-invariant mean-plaquette formulation of lattice gauge theories which we will describe elsewhere.
V. DUALITY TRANSFORMATION AND COULOMB GAS
REPRESENTATION FOR ABELIAN THEORIES

We will show in this section how the duality transformation and the Coulomb gas representation arise from two different ways of writing the $\delta$function of the lattice Bianchi identity. The form that gives the Coulomb gas is

\[
\Pi_\nu \delta[e^{-\mu \delta_{\mu \nu}} - 1] = \sum_{\{m_\nu(\rho) = -\infty\}} \prod_{\nu} 2\pi \delta[\Delta_{\mu \nu} (\tau) - 2\pi m_\nu (\rho)]
\]

\[
= \sum_{\{m_\nu(\rho) = -\infty\}} \left[ \prod \delta(\Delta_{\nu m_\nu (\rho)}) \right] \int_{-\infty}^{+\infty} D\chi_{\nu} e^{i \sum \chi_{\nu}(\rho) (\Delta_{\mu \nu} (\tau) + 2\pi m_\nu (\rho))}
\]

(V.1)

It is clear from (V.1) that the Coulomb gas arises from the lattice Bianchi identity independent of the action being used. It is easy to see that substituting (V.1) in the lattice QED$_4$ partition function with Villain action and doing the plaquette and $\chi$ integrals immediately yields the well known Coulomb gas representation

\[
Z = \prod_{\{m_\nu(\rho) = -\infty\}} \left[ \prod \delta(\Delta m_\nu (\rho)) \right] e^{\frac{-2\pi^2}{\lambda} \sum m_\nu (\rho) G(\rho - \rho')} \tag{V.2}
\]

where

\[
\Delta_{\mu \nu} G(\rho - \rho') = \delta_{\rho \rho'} \tag{V.3a}
\]

\[
\Delta_{-\mu} f(\tau) \equiv f(\tau - \hat{\mu}). \tag{V.3b}
\]

For the duality transformation, we use the Fourier series expansion of the periodic $\delta$ function

\[
\Pi_\nu \delta[e^{-\mu \delta_{\mu \nu}} - 1] = \sum_{\{n_\mu(\rho) = -\infty\}} e^{i \sum \delta_{\mu \nu}(\rho) \Delta_{\mu \nu} (\tau)} \tag{V.4}
\]
\( n_\nu(\rho) \) and \( \chi_\nu(\rho) \) are both lattice versions of Halpern's continuum dual potential \( \tilde{A}_\mu \), and both lie on the link dual to the cube formed by the six plaquettes in \( \exp i \Delta_{\mu \nu}(\tau) \). In general, in D dimensions the dual potential has as many components as there are Bianchi identities.

Substituting (V.4) in (II.10) and doing the plaquette integrals gives the dual to lattice QED\(_4\) with a Wilson action

\[
Z = \sum_{\{n_{\mu}(\rho) = -\infty\}} \left[ \frac{2}{\mu \rho} I_{n_{\mu \rho}}(\rho) (g^{-2}) \right] \tag{V.5}
\]

where \( I_n(\beta) \) is a modified Bessel function and

\[
\tilde{n}_{\mu \nu}(\rho) = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} [\Delta_{\alpha \beta}(\rho) - \Delta_{\beta \alpha}(\rho)]. \tag{V.6}
\]

This method of doing the duality transformation shows that the dual potential \( n_{\mu}(\rho) \) is the Fourier conjugate to the lattice Bianchi identity. This is the same as Halpern's result in the continuum.
VI. STRONG COUPLING EXPANSION AS A RESTORATION OF THE BIANCHI IDENTITY: ABELIAN CASE

From equation (V.5) it is clear that the strong coupling expansion is a perturbation expansion in small $n_{\mu\nu}$. This means that in such an expansion, we keep only some of the terms in the expansion of the lattice Bianchi identity (V.4).

This leads to the interpretation of the strong coupling expansion as an expansion towards restoring the lattice Bianchi identity: To leading order we ignore it completely. The higher the order of the terms we keep, the closer the partition function is to accepting contributions only from plaquette configurations that satisfy the lattice Bianchi identity.

It should be clear that the strong coupling expansion obtained in this formalism gives the same results as those obtained by, say, cluster expansions because both methods give the same dual.

To illustrate these points, we will calculate the first two terms in the strong coupling expansion of the string tension in lattice $\text{QED}_4$.

Consider a Wilson loop of minimal surface area $A$ and lying in the xy plane. The loop can be expressed in terms of the plaquettes forming the minimal surface giving for the expectation value

$$<W[C]> = z^{-1} \sum_{\Delta \Theta_\nu} \int_{D_\Omega} e^{\frac{i}{2g^2} \sum_{\mu\nu}(r) \Delta \Theta_\nu(r)}$$

$$\times e^{\frac{i}{2g^2} \sum_{\mu\nu}(r) + i \sum_{A,xy}(r)}$$

(VI.1)
here \( \exp i \sum_{A} \theta_{xy}(r) \) is the Wilson loop expressed in terms of its minimal plaquettes. Note that \( \exp \prod_{\nu} (\rho) \Delta \tilde{\theta}_{\mu \nu}(r) \) is the product of six plaquettes forming a cube raised to the power \( n_{\nu}(\rho) \). Thus it is clear that keeping such terms with nonzero \( n_{\nu} \) amounts to including contributions from nonminimal surfaces.

To obtain the first two terms in the strong coupling expansion, we ignore the Bianchi identity everywhere except at cubes in contact with the minimal surface of the loop. In other words, the expansion (V.4) of the \( \delta \) function of the Bianchi identity is replaced by 1 everywhere except at the cubes mentioned above. At these cubes, we ignore most of the Bianchi identity by replacing (V.4) by the \( n_{\nu} = 0, \pm 1 \) terms of the expansion. With this approximation to the Bianchi identity, Eq. (VI.1) becomes

\[
<\mathcal{W}[C]> = Z^{-1} \int_{-\pi}^{+\pi} d\theta_{\mu \nu} \prod_{\nu = t, z} \left[ 1 + e^{\pm i \Delta \tilde{\theta}_{\mu \nu}(r')} \right] \\
\times \frac{1}{e^{2g^2}} \sum_{x} \cos \theta_{\mu \nu}(r) + i \sum_{A} \theta_{xy}(r)
\]

(\text{VI.2})

\( \prod' \) is defined to mean that we take the product over all \( r' \) such that \( \exp \pm i \Delta \tilde{\theta}_{\mu \nu}(r') \) has a face in common with the minimal surface, and the sign of the exponent is chosen so that this common face cancels. \( \nu \) is restricted to be \( t \) or \( z \) because when \( \nu = t \) the cube formed by the exponential is an \( xyz \) cube and when \( \nu = z \) it is an \( xyt \) cube, and these are the only cubes that have a face in the \( xy \) plane.

The integrals are easy to do giving
\[ <W^{(C)}> = \left[ \frac{I_1(g^{-2})}{I_0(g^{-2})} \right] \frac{A}{a^2} + 4 \left[ \frac{I_1(g^{-2})}{I_0(g^{-2})} \right] \frac{A}{a^2} + 4 \left[ \frac{I_1(g^{-2})}{I_0(g^{-2})} \right] \frac{A}{a^2} + 6 \]

(VI.3)

where \( a \) is the lattice spacing.

The first term in (VI.3) results from the 1 (Bianchi identity gone) in the product in (VI.2), and by itself gives the familiar leading term in the strong coupling string tension

\[ \sigma = \frac{1}{2} \frac{a^2}{a^2} \ln \left( \frac{I_0(g^{-2})}{I_1(g^{-2})} \right) \]

(VI.4)

Therefore, we see that ignoring the Bianchi identity everywhere gives the correct leading term for the string tension. This of course amounts to ignoring all correlations among the plaquettes because, as is clear from (II.10), the lattice Bianchi identity is the only source of such correlations. Moreover, this contribution of the \( n = 0 \) term is the contribution of the minimal surface of the Wilson loop.

The second term in (VI.3) arises from the cross terms in the product in (VI.2) that contain only one exponential. This says that the first correction to the leading term in (VI.3) comes from putting all dual links \( (n^v) \) except one equal to zero. The nonzero link (whose value and sign were discussed above) is dual to a cube in contact with the minimal surface. The \( 4A/a^2 \) factor counts the number of allowed locations for such a cube. Clearly this is the contribution of the smallest nonminimal surface.

Thus, the first step towards restoring the Bianchi identity, Eq. VI.3, gives the string tension
\[
\sigma = \frac{1}{a^2} \left| \ln \left( \frac{I_0(g^{-2})}{I_1(g^{-2})} \right) - 4 \left( \frac{I_1(g^{-2})}{I_0(g^{-2})} \right) \right|
\]  
(VI.5)

which agrees with cluster expansion results.

The procedure for calculating higher order corrections (i.e. contributions from more non minimal surfaces) to (VI.3) is now clear: We keep more Fourier components of the \( \delta \) function of the lattice Bianchi identity. Clearly, the more Fourier components we keep, the closer we get to restoring the Bianchi identity.

We therefore reach quite a striking picture of strong coupling confinement. The strong coupling limit corresponds to totally ignoring the Bianchi identity, and is thus a state of maximal disorder among the plaquettes. The strong coupling expansion is a gradual restoration of the Bianchi identity, i.e. a gradual restoration of a certain degree of order among the plaquettes.
VII. STRONG COUPLING EXPANSION AS A RESTORATION OF THE BIANCHI IDENTITY:
NON-ABELIAN CASE

In this section we will show that the strong coupling expansion for QCD_3, as for Abelian theories, is an expansion towards restoring the non-Abelian lattice Bianchi identity. The same method applies to QCD_4 and other non-Abelian theories.

Consider a rectangular Wilson loop \( \frac{1}{2} \text{Tr} W \), where \( W \) is the ordered product of the links forming the boundary of the loop. We need to express \( \text{Tr} W \) in terms of plaquettes, and since it is gauge invariant, we are free to choose the gauge in such a way as to make this expression as simple as possible. Following Halpern, we choose the Wilson loop in the xy gauge plane at \( z_0 \). This makes both of its sides that are parallel to the x-axis equal to unity. Furthermore, we can choose one of the y sides along the gauge line \( x = x_0, z = z_0 \) (the y-axis in Fig. 1). This reduces the Wilson loop to

\[
\frac{1}{2} \text{Tr} W = \frac{1}{2} \text{Tr} \prod_{y' = y_1}^{y_2} U_2(x_1'y'_z0) \\
= \frac{1}{2} \text{Tr} \prod_{y' = y_1}^{y_2} \left[x_0^{-1} \prod_{x' = x_1}^{x_0} P^{+}_{12}(x'y'_z0) \right] \quad \text{(VII.1)}
\]

where \( (x_1'y_1z_0) \) and \( (x_1'y_2z_0) \) are, respectively, the beginning and end of the remaining side of the Wilson loop. This is Halpern's "Abelianized" Wilson loop (involving only the plaquettes of the minimal area).

The expectation value of this loop is given by
where \( \text{Tr} \) is given by (VII.1). The \( \delta \) function of the Bianchi identity has the form \( \delta(P_c - 1) \) where \( P_c \) is an SU(2) matrix. \( \delta(P_c - 1) \) is invariant under similarity transformations and may therefore be character expanded

\[
\delta(P_c - 1) = \sum_{J_c} (2J_c + 1) \chi_{J_c}(P_c)
\]

\( J_c = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots \)

The subscript \( c \) refers, as before, to the cube which is associated with the Bianchi identity we are considering. \( \chi_{J_c}(P_c) \) is the trace of \( P_c \) in the \((2J_c + 1)\) dimensional representation.

\( J_c \) is the dual potential for lattice QCD. This is the non-Abelian generalization of the Abelian case where the dual potential was shown to be the Fourier conjugate to the lattice Bianchi identity.

Substituting (VII.3) in (VII.2) gives

\[
< \frac{\text{Tr}W}{2} > = Z^{-1} \int \text{D} P \{ \prod_{c} \delta(P_c - 1) \} \text{Tr} \left[ \frac{\beta}{4} \sum_{\mu < \nu} \text{Tr}[P_{\mu \nu}(r) + P_{\mu \nu}^+(r)] \right] e^{\sum_{\mu < \nu} \text{Tr}[P_{\mu \nu}(r) + P_{\mu \nu}^+(r)]}
\]

(VII.4)

For \( \beta \ll 1 \), the dominant contribution to \( \frac{1}{2} < \text{Tr}W > \) comes from all \( J_c = 0 \), i.e. we ignore the Bianchi identity, which means we have plaquette disorder. This contribution is the familiar leading term

\[
< \frac{\text{Tr}W}{2} > = \left[ \frac{I_2(\beta)}{I_1(\beta)} \right] A/a^2
\]

(VII.5)
where $A$ is the minimal area of the loop and $a$ the lattice spacing.

The next contribution comes from putting $J_c = 0$ for all cubes except for one cube in contact with the minimal surface of the loop. In other words, the next contribution comes from ignoring the Bianchi identity at all cubes except one which is in contact with the minimal surface. At this cube we take $J_c = \frac{1}{2}$, i.e. at this cube we still ignore most of the lattice Bianchi identity by ignoring most of the characters in its expansion. The term that we keep from this expansion has the form

$$2\text{Tr}[P_{12}(xyz_0)P_{23}(xyz_0)U_{13}(xyz_0)U_{01}(xyz_0+1)P_{23}(x+lyz_0+1)P_{12}(xyz_0+1)P_{12}(xyz_0+1)].$$

(VII.6)

Applying the change of variable (IV.1a-b) Abelianizes it into the form

$$2\text{Tr}[P_{12}(xyz_0)P_{23}(xyz_0)P'_{13}(xyz_0)P_{12}'(xyz_0+1)P_{12}'(xyz_0+1)P_{12}'(xyz_0+1)].$$

(VII.7)

Now, by applying the Gross and Witten trick, we change the trace in (VII.7), and $\text{Tr}W$, into a product of the traces of single plaquettes. It is simplest to apply this trick to all plaquettes appearing in (VII.7) and $\text{Tr}W$ except $P_{12}(xyz_0)$ because it is the only one that appears in both. The plaquette integrals are then easy to do, and up to this order we get

$$<\frac{\text{Tr}W}{2}> = \left[\frac{I_2(\beta)}{I_1(\beta)}\right]^{A/a^2} \left[1 + \frac{2A}{a^2} \left(\frac{I_2(\beta)}{I_1(\beta)}\right)^4 + 0 \left(\frac{I_2(\beta)}{I_1(\beta)}\right)^6 \right]$$

(VII.8)

In fact it would have been just as simple, in this case, to use the Gross and Witten trick directly on the plaquettes $P_{23}(xyz_0)$, $P_{13}(x+lyz_0)$, $P_{23}(x+lyz_0)$, $P_{13}(xyz_0)$ and $P_{12}(xyz_0+1)$ in (VII.6)
which case, the rest of the $U_4$ variables cancel. However, we wanted to demonstrate the use of Abelianization because for higher order terms (such as two cubes sharing a plaquette and lying on the surface of the Wilson loop, to be discussed below), we found it much simpler to Abelianize before using the Gross and Witten trick.

The next contribution comes from $J_c = 0$ for all cubes except one cube not touching the minimal surface. For this cube $J_c = \frac{1}{2}$. The contribution after this comes from $J_c = 0$ for all cubes except two that lie on the minimal surface and at the same time share a face. For each of these two cubes $J_c = \frac{1}{2}$. The string tension up to this order is

$$\sigma = -\ln U - 2U^4 + 4U^6 - 6U^4V$$

and

$$U \equiv \frac{I_2(\delta)}{I_1(\delta)}, \quad V \equiv \frac{I_3(\delta)}{I_1(\delta)}$$

This result agrees with the result from cluster expansions. 5

We therefore see that, as in the Abelian case, the strong coupling expansion is an expansion towards restoring the lattice Bianchi identity.

Finally a comment about the use of Abelianization in this calculation. Since we carried the strong coupling expansion only to low order, we only needed to consider the contributions from a few Bianchi identities. This enabled us to Abelianize them as discussed in Section
IV and thus simplify the calculation. However, going to higher orders, one will encounter situations where not all Bianchi identities can be Abelianized. Moreover, the lower the temperature, the more of these non-Abelianized Bianchi identities will contribute. So, it seems that the high temperature region is adequately described by Abelianized Bianchi identities while in the low temperature region non-Abelianized identities will play an important role.
VIII CONCLUSIONS

We have accomplished in this paper a lattice analogue of Halpern's field strength and dual potentials for gauge theories. As in the continuum, the lattice Bianchi identity is the pivot point for the structure of these theories.

One of our most interesting results is the interpretation of the strong coupling expansion as a restoration of the Bianchi identity. In a gauge theory, the Bianchi identity is the only source of plaquette correlations, and it is ignored in the strong coupling limit. This corresponds to maximal plaquette disorder. As the strong coupling expansion proceeds, the Bianchi identity, and thus plaquette order, are gradually restored.

We can, therefore, say that the confining phase in lattice gauge theories (Abelian and non-Abelian) corresponds to disordered plaquettes. This is seen very clearly in two dimensional theories. In such theories, there are no Bianchi identities and thus no correlations among the plaquettes; the plaquettes are disordered at all couplings, and therefore these theories always confine. Another way of saying what we have seen here is then that the only kind of confinement we have so far seen in models is "plaquette disorder confinement"—the common ingredient in all gauge theories in any number of dimensions.

Another very interesting result is the "Abelianization" of the non-Abelian lattice Bianchi identity and the connection of this procedure to gauge invariant plaquettes. This is interpreted as the lattice version of Mandelstam's Abelian Bianchi identity satisfied by gauge invariant non-Abelian field-strengths.
Finally, we mention that everything discussed in this paper can be applied to spin systems. There one changes from site to link variables and the Bianchi identity is a plaquette.
Appendix: Proving Equation (II.7).

Written out in full, the left hand side of (II.7) is

\[
\int dU_1 dU_2 dU_0 \delta[U_1(txyz_0)-1] \delta[U_2(tx_0y_0z_0)-1] \delta[U_0(tx_0y_0z_0)-1] \\
\times \delta[P_{01}(txyz)U_1(txyz)U_0(tx+lyz)U_{1+}(t+1xyz)U_{0+}(txyz)-1] \\
\times \delta[P_{02}(txyz)U_2(txyz)U_0(txy+1lz)U_{2+}(t+1lyz)U_{0+}(txyz)-1] \\
\times \delta[P_{03}(txyz)U_0(txyz+DU_{0+}(txyz)-1] \\
\times \delta[P_{12}(txyz)U_2(txyz)U_1(txy+1lz)U_{2+}(t+1lyz)U_{1+}(txyz)-1] \\
\times \delta[P_{13}(txyz)U_1(txyz+1l)U_{1+}(txyz)-1] \delta[P_{23}(txyz)U_2(tx+yz+1) \\
\times U_{2+}(txyz)-1]
\]

(A.1)

where we have already performed the trivial integration over \( U_3(r) \).

The \( U_1(r) \) integral can be easily done using the first and eighth \( \delta \) functions. It gives Equation (II.4a) for \( U_1 \) in terms of \( P_{13} \).

To do the \( U_2 \) integral, I change variables from \( U_2 \) to \( V_{21}(txyz) \) according to

\[
U_2(txyz) = \prod_{z'=z}^{z_0-1} P_{23}(txyz') V_{21}(txyz)
\]

(A.2)

The second \( \delta \) function simply becomes \( \delta(V_{21}(tx_0yz_0)-1) \). Making this substitution (A.1) becomes
\[
\delta(A(txyz)-1) = \delta[A(txyz_0)-1] \delta[A(txyz+1)A^+(txyz)-1]
\]  
(A.4)

i.e. we split it into a \(\delta\) function of the initial condition at \(z = z_0\) and a relation between \(A\) and \(A^+\) at two neighboring \(z\). It is trivial to see that the R.H.S. of (A.4) is the same as the L.H.S. Working out \(A(txyz_0)\) and \(A(txyz+1) A^+(txyz)\) (A.4) becomes

\[
\delta(A(txyz)-1) = \delta[P_{12}(txyz_0)V_{21}(txyz_0)V^+_21(txyz_0+1z_0)-1]\delta[e^{i\Delta \delta_0(r)}] -1]
\]  
(A.5)
where I put $P_\mu^\nu(r) = e^{i\theta_{\mu\nu}(r)}$ and I used $V_{21}(txyz+1) = V_{21}(txyz)$ as demanded by one of the $\delta$ functions. So, replacing (A.5) in (A.3) and doing the $V_{21}$ integral gives

$$V_{21}(txyz_0) = \frac{x_0-1}{x'} P_{21}(tx'yz_0)$$

(A.6)

and (A.3) becomes

$$\int DU_0(txyz) \delta[U_0(txyz_0'z_0) - 1] \delta[P_{01}(txyz) \prod_{z'} P_{13}(txyz') U_0(tx+lyz)$$

$$\times \prod_{z''=z} P_{13}^+(t+1xyz'') U_0^+(txyz)-1]$$

$$\times \delta[P_{02}(txyz) \prod_{z'} P_{23}(txyz') \prod_{x'} P_{21}(tx'yz_0) U_0(tx+yz) \prod_{z''=z} P_{23}^+(t+1xyz'')]$$

$$\times \prod_{x''=x} P_{21}^+(t+1x''yz_0) U_0^+(txyz)-1]$$

$$\times \delta[P_{03}(txyz) U_0(txyz+1) U_0^+(txyz)-1] \times \delta[e^{i\Delta \theta_0}(r) - 1]$$

(A.7)

To do the $U_0$ integral we again change variables to $V_0(txyz)$

$$U_0(txyz) = \prod_{z'} P_{03}(txyz') \prod_{x'} P_{01}(tx'yz_0) \prod_{y'} P_{02}(tx'y'z_0) V_0(txyz)$$

(A.8)

(A.7) becomes
\[ \delta[e^{-i\Delta \mu_0(r)} - 1] \delta[V_0(t x_0 y_0 z_0)] \delta[P_{01}(txyz)U_1(txyz)U_0(tx+lyz) - 1] \]

\[ \times \delta[P_{02}(txyz)U_2(txyz)U_0(txyz+1z)U_2(t+1xyz)U_0^+(txyz) - 1] \]

\[ \times \delta[V_0^+(txyz+1)U_0^+(txyz) - 1] \]

(A.9)

where we wrote variables in terms of \( U_1, U_0 \) and \( U_2 \) because the expressions were becoming cumbersome. \( U_1, U_2 \) and \( U_0 \) are given by (II.4a), (II.4b) and (A.8) respectively. As we did before, if we call the product of variables in the second \( \delta \) function of (A.9) \( A(txyz) \), we can split this \( \delta \) as we did in (A.4). The result is

\[ \delta[P_{01}(txyz)U_1(txyz)U_0(tx+lyz)U_1^+(t+1xyz)U_0^+(txyz) - 1] \]

\[ = \delta[e^{-i\Delta \mu_2(r)} - 1] \delta[V_0(tx+lyz)U_0^+(txyz) - 1] \]

(A.10)

therefore (A.9) becomes

\[ \delta[e^{-i\Delta \mu_0(r)} - 1] \delta[e^{-i\Delta \mu_2(r)} - 1] \delta[V_0(t x_0 y_0 z_0) - 1] \delta[V_0(txyz+1)U_0^+(txyz) - 1] \]

\[ \times \delta[V_0^+(txyz)U_0(tx+lyz) - 1] \]

\[ \times \delta[P_{02}(txyz)U_2(txyz)U_0(txyz+1z)U_2^+(t+1xyz)U_0^+(txyz) - 1] \]

(A.11)

Again splitting the last \( \delta \) at \( Z_0 \) and \( X_0 \) (in the same manner as before) we finally end up with
\[
\delta[e_{\mu}] \mu_0(r) \delta[e_{\mu}] \mu_1(r) \delta[e_{\mu}] \mu_2(r) \delta[e_{\mu}] \mu_3(txyz_0)
\]
\[
\times \int \text{DV}_0 \delta[V_0(txyz_0)] \delta[V_0(txyz+1)] \delta[V_0(txyz_0)V_0(tx+1yz_0)]
\]
\[
\times \delta[V_0(txyz_0)V_0(txyz_0)-1]
\]

It is trivial to see that the solution to all the \( \delta \) functions in the integrand is \( V_0(txyz) = 1 \) because the second \( \delta \) function tells us that \( V_0(txyz) = V_0(txyz_0) \). The third \( \delta \) says \( V_0(txyz_0) = V_0(tx_0yz_0) \) and the last that: \( V_0(tx_0yz_0) = V_0(tx_0yz_0) \). But by the first \( \delta \) this is 1. Therefore \( V_0(txyz) = 1 \).

Thus, we have shown that (A.1) is equal to:

\[
\delta[e_{\mu}] \mu_0(r) \delta[e_{\mu}] \mu_1(r) \delta[e_{\mu}] \mu_2(r) \delta[e_{\mu}] \mu_3(txyz_0)
\]

These are the "3.1" Bianchi identities which can be easily shown to imply the full 4 Bianchi identities.

So, finally, the Jacobian of the variable change from links to plaquetts is

\[
\delta[e_{\mu}] \mu_0(r) \delta[e_{\mu}] \mu_0(r) \delta[e_{\mu}] \mu_0(r)
\]

(A.14)
REFERENCES

3. J. Kiskis, private communication.
FIGURE CAPTIONS

Figure 1-a. $U_2(xyz)$ as a Wilson loop filled with plaquettes. The dotted lines are gauge lines.

Figure 1-b. A gauge invariant plaquette. The tail runs along gauge (dotted) lines and is therefore invisible.

Figure 2. The gauge invariant plaquettes whose product, in the order abcdef, gives the lattice Bianchi identity for the cube.

Figure 3. The product of these gauge invariant plaquettes, in the order afedcb, gives a lattice Bianchi identity which is unitarily equivalent to that of Fig. 2.

Figure 4. The lattice Bianchi identity can be Abelianized for cubes 1 through 10 but not 11.
Figure 3
CHAPTER II
The purpose of this paper is to present a new gauge-invariant mean-plaquette calculation method for lattice gauge theories. The starting point of this method is our plaquette formulation of lattice gauge theories\(^1,2\), which is an application of Halpern's field strength and dual potential formulation in the continuum.\(^3\)

The method is quite straightforward for all lattice gauge theories, Abelian and non-Abelian. The strong coupling plaquette energies it gives are in excellent agreement with Monte Carlo data, for all theories we examined, up to the critical point if there is one. For lattice QCD\(_4\)(SU(2)), which is known to have a continuous crossover instead of a phase transition, our mean-plaquette results are in remarkable agreement with Monte Carlo data from \(\beta = 0\) to \(\beta = 5\). Our calculation tracks the crossover with high accuracy and does not predict a phase transition. Abelian theories that we examined have phase transitions and their critical couplings are predicted quite accurately by our method. However, we have not yet succeeded in solving our mean-plaquette equations in the weak coupling phases of the Abelian theories. These difficulties will be discussed in more detail.

Our most complete and impressive results are for lattice QCD\(_4\). We will, however, illustrate our method for lattice QED\(_4\) first because it is easiest to understand for Abelian theories, and because we want to motivate some short cuts we will take when doing the calculation for the non-Abelian theories.
It is worth noting, that in low order, the method yields equations that are simple enough to solve on a programmable calculator. All calculations in this paper were done with an HP29C calculator. We should mention, however, that numerical results, especially for the non-Abelian theories, can be easily improved by using a computer. We did not do that here because our purpose is to explain the method and show how it works. Our low order calculation is quite good enough for that.

The paper is organized as follows. In Section II we obtain the mean-plaquette equation for lattice QED\(_4\). In Section III we show how to find approximate solutions to this equation in the strong coupling phase. We also discuss our difficulties with the weak coupling phase. In Section IV we present our results for lattice QED\(_5\) and Z(2) gauge theory in 4 dimensions. We obtain the approximate mean plaquette equation for lattice QCD\(_4\) (SU(2)) in Section V and find approximate solutions in Section VI. In Section VII we present the results for SO(3) lattice gauge theory and compare it with QCD\(_4\). Conclusions and final remarks are in Section VIII.
In this section we will show, in detail, how to obtain the mean-plaquette equation for lattice QED\textsubscript{4}.

Consider the expectation value of a plaquette \( P_{\alpha\beta}(r_0) \)

\[
\langle P_{\alpha\beta}(r_0) \rangle = Z^{-1} \int DU_{\mu} P_{\alpha\beta}(r_0) e^{\frac{\beta}{4} \sum [P_{\mu\nu}(r) + P_{\mu\nu}^+(r)]} \tag{II.1}
\]

Where \( P_{\mu\nu}(r) \) is the product of the four link variables, \( U_{\mu} \), forming the plaquette. \( Z \) is the partition function. Using the notation and results of Reference 1 we change the variable of integration to plaquettes

\[
\langle P_{\alpha\beta}(r_0) \rangle = Z^{-1} \int DP_{\mu\nu} \left[ \prod_{r, \nu} \delta(e^{\theta_{\mu\nu}(r)} - 1) \right] P_{\alpha\beta}(r_0) e^{\frac{\beta}{4} \sum [P_{\mu\nu}(r) + P_{\mu\nu}^+(r)]} \tag{II.2}
\]

where \( P_{\mu\nu}(r) = \exp(\theta_{\mu\nu}(r)) \). The \( \delta \) function of the lattice Bianchi identity is periodic and can, therefore, be expanded in a Fourier series giving

\[
\langle P_{\alpha\beta}(r_0) \rangle = Z^{-1} \sum_{\{v(\rho)\}} \int DP_{\mu\nu} P_{\alpha\beta}(r_0) e^{\frac{\beta}{4} \sum [P_{\mu\nu}(r) + P_{\mu\nu}^+(r)] + \sum_{\mu, \nu} \sum_{v(\rho)} \bar{n}_{\mu\nu}(v) \theta_{\mu\nu}(r)} \tag{II.3}
\]

where

\[
\sum_{\{v(\rho)\}} \prod_{\rho, \nu} \sum_{v(\rho)} = \infty \tag{II.4a}
\]
\[ \tilde{n}_{\mu\nu}(\rho) = \frac{1}{2} \epsilon_{\mu\nu\rho \alpha} [\Delta_{\mu\rho} n_{\alpha}(\rho) - \Delta_{\nu\rho} n_{\alpha}(\rho)] \]  

(II.4b)

And as discussed in Reference 1, \( n_{\mu}(\rho) \) is the dual potential, it lies along a link in the dual lattice. \( \rho \) and \( r \) are, respectively, the coordinates on the dual and original lattices, and since they are in one to one correspondence, a product over \( r \) also means a product over \( \rho \).

Now we make the mean-plaquette approximation. Make the following substitution for all plaquettes except \( P_{\alpha\beta}(r_0) \)

\[ P_{\mu\nu}(r), P_{\mu\nu}^+(r) = P_{\mu\nu}^{-1}(r) \rightarrow <P> \]  

(II.5a)

which means

\[ [P_{\mu\nu}(r)]^n \rightarrow <P>n \]  

(II.5b)

We dropped the indices from \( <P> \) because all plaquettes have the same expectation value. Equation (II.3) becomes

\[ <P> = z^{-1} \sum_{\{n_{\nu}(\rho)\}} \langle P \rangle \sum_{\nu} |\tilde{n}_{\mu\nu}(\rho)| \int dP_{\alpha\beta}(r_0) [P_{\alpha\beta}(r_0)]^{1+\tilde{n}_{\alpha\beta}(r_0)} e^{\frac{1}{2}[P_{\alpha\beta}(r_0)+P_{\alpha\beta}^+(r_0)]} \]  

(II.6)

where \( \Sigma' \) means the summation does not include \( \tilde{n}_{\alpha\beta}(r_0) \). Doing the integral finally gives

\[ <P> = \sum_{\{n_{\nu}(\rho)\}} \langle P \rangle \sum_{\mu \nu} |\tilde{n}_{\mu\nu}(\rho)| \frac{I(\beta)}{(1+\tilde{n}_{\alpha\beta}(r_0))} \]  

(II.7)

where \( I_n(\beta) \) is a modified Bessel function, and \( \tilde{n}_{\alpha\beta}(r_0) \) is the plaquette dual to \( P_{\alpha\beta}(r_0) \).
Equation (II.7) is the mean plaquette self-consistency equation for the plaquette energy in lattice QED$_4$. It is obviously invariant under gauge transformations of the dual link variable $n_{\mu}(\rho)$. Moreover, the mean-plaquette substitution (II.5) is invariant under gauge transformations of the original variables. So, this mean-plaquette method preserves the gauge invariance of the theory and its dual.
III. APPROXIMATE SOLUTION OF THE MEAN-PLAQUETTE EQUATION FOR QED$_4$

We do not know how to solve Equation (II.7) because there is a very large number of dual links being summed over. However, we can make approximations that will simplify (II.7) in the strong coupling phase. We will make some comments concerning the weak coupling phase at the end of this section.

To motivate the strong coupling ansatz that we will use for the dual potential, $n_\mu (\rho)$, we first recall some of our results from Reference 1. In that paper, we showed that the strong coupling expansion is an expansion towards restoring the lattice Bianchi identity. In other words, in the strong coupling phase, only small fluctuations of the dual potential are important. This means that in this phase, the plaquettes are very weakly correlated, and that a given plaquette is more strongly correlated to plaquettes that share a cube with it than to others that do not.

For these reasons we use the following ansatz

$$n_\mu (\rho) = \Delta_\mu \Lambda (\rho)$$  \hspace{1cm} (III.1)

for all dual links except the four that make the dual plaquette $\tilde{n}_{\alpha\beta} (\rho_0)$. $\Lambda (\rho)$ is an integer scalar field. Equation (III.1) says that all, but four, of the dual links are pure gauge. Looking at Equation (II.3) we see that this ansatz has the effect of ignoring the lattice Bianchi identity everywhere except for the four cubes that share the plaquette $P_{\alpha\beta} (r_0)$. The ansatz (III.1) makes all $\tilde{n}_{\mu\nu} = 0$ except $\tilde{n}_{\alpha\beta} (\rho_0)$ and those dual plaquettes that share a link
with it. Equation (II.7) becomes

\[
\langle p \rangle = Z^{-1} \sum_{\{\Lambda(p)\}\{n_0(p)\}} \sum_{\mu,\nu} <p> \sum_{\mu,\nu} |n_{\mu}(p) - \Delta_{\mu} \Lambda(p)| ^{I^\prime}(\beta) (1 + \tilde{n}_{ab}(p_0))^{I^\prime}(\beta)
\]

where here \(\sum_{\{n_0(p)\}}\) is a summation over the values of the four nongauge dual links that form the plaquette \(\tilde{n}_{ab}(p_0)\). Also, \(\Sigma'\) includes only these four dual links. The 5 in the exponent of \(\langle p \rangle\) comes from the fact that each of the surviving "live" dual links that form \(\tilde{n}_{ab}(p_0)\) belongs to five other dual plaquettes. The summation over \(\Lambda(p)\) is just a sum over all gauge equivalent classes and will only result in a multiplicative infinity that cancels between numerator and denominator. So we only need to do the calculation for one gauge class and we do it for \(\Delta_{\mu} \Lambda(p) = 0\). This finally gives

\[
\langle p \rangle = \frac{\sum_{\{n_{1,2,3,4}\}} <p> \sum_{i=1}^{4} |n_i| ^{I^\prime}(\beta) (1+n)} {\sum_{\{n_{1,2,3,4}\}} <p> \sum_{i=1}^{4} |n_i| ^{I^\prime}(\beta)}
\]

(III.3)

where we called the four live dual links \(n_1, n_2, n_3, n_4\) and

\[
\tilde{n}_{ab}(p_0) = n = n_1 + n_2 - n_3 - n_4
\]

(III.4)

Equation (III.3) generates a polynomial in \(\langle p \rangle\) which can be solved numerically. This polynomial is of infinite order, but we can truncate it at a suitably high power because \(\langle p \rangle\) is always less than 1.
Notice that the leading term given by (III.3), all $n_i = 0$, is just the leading term in the strong coupling expansion

\[
\langle P \rangle = \frac{I_1(\beta)}{I_0(\beta)} \tag{III.5}
\]

This is plotted as curve 1 in Figure 1. This term is known to agree with Monte Carlo data at large coupling but starts to deviate appreciably at $\beta \approx 0.5$.

If we include the first nonleading term, one of the four $n_i = \pm 1$, we get the polynomial

\[
\langle P \rangle = \frac{I_1(\beta)}{I_0(\beta)} - 4 \langle P \rangle^5 \left( 1 + \frac{I_2(\beta)}{I_0(\beta)} \right) + \frac{8I_1(\beta)}{I_0(\beta)} \langle P \rangle^6 = 0 \tag{III.6}
\]

This can be easily solved numerically. The solution is plotted as curve 2 in Figure 1. It is very interesting to note that already, at this very low order, agreement between our calculation and Monte Carlo has improved dramatically. Moreover, equation (III.6) has no roots less than one for $\beta > 0.93^*$. We interpret this as an indication of a phase transition at $\beta_c = 0.93$. We will have more to say about this interpretation later.

Moreover, writing (III.6) in the form

\[
\langle P \rangle = \frac{I_1(\beta) + 4 \langle P \rangle^5 (I_0(\beta) + I_2(\beta))}{I_0(\beta) + 8I_1(\beta) \langle P \rangle^5}
\]

* Notice that $\langle P \rangle = 1$ is always a solution of (III.3) for any $\xi$. For this reason we do not attach any physical significance to it. The same is true for the mean-plaquette equations of all the theories we examined, Abelian and non-Abelian.
it is easy to see that solving it by iteration will immediately yield the first 4 terms in the strong coupling expansion plus a number of higher order terms. Although these higher order are present in the strong coupling expansion, they are not complete. To generate a more complete set of strong coupling expansion terms, one needs to keep higher powers of \( <P> \) in the numerator and denominator.

So, already at the level of the first nonleading term, our method yields remarkably accurate results for the strong coupling phase. To estimate the importance of higher order terms we do the following. We find the expression for the next higher order term, for example the next term that contributes in (III.6) is

\[
- \left\{ \frac{22}{I_0(\beta)} I_1(\beta) + 10 \frac{I_3(\beta)}{I_0(\beta)} \right\} \langle P \rangle^{10} + \left\{ 12 + 20 \frac{I_2(\beta)}{I_0(\beta)} \right\} \langle P \rangle^{11}
\]

(III.7)

Then we use the solution of (III.6) to evaluate (III.7) at the coupling of interest. For example, at \( \beta = 0.9 \), the value of (III.7) is 0.4% of the value of \( <P> \) as given by (III.6), and so we do not expect it to change the value of \( <P> \) by much. Indeed, if we include this term in (III.6) and solve for \( <P> \), we find its value to be shifted up by 1.2%.

Using this method, we find that the combined values of the terms of order 20 and 21 is 0.003% of \( <P> \) at \( \beta = 0.9 \). This is very small and the contributions of higher order terms are even smaller and can be ignored. When the solution of the polynomial of order 21 generated by (III.3) is plotted in Figure 1, it is indistinguishable from curve 2, but stops at \( \beta_c = 0.91 \).
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So the mean-plaquette curve in Figure 1 stops at $\beta_c = 0.91$ because the polynomial that we solve has no roots less than one for $\beta > 0.91$. This is interpreted as being due to the complete invalidity of the strong coupling ansatz (III.1) for $\beta > 0.91$. In other words, the strong coupling vacuum, as approximated by the ansatz (III.1), is no longer the vacuum of the theory for $\beta > 0.91$. The theory has now a different vacuum that needs to be approximated by a different ansatz. So, a phase transition occurs at $\beta_c = 0.91$.

Notice also, that in terms of monopoles, (III.1) says that the strong coupling vacuum is a condensate of these monopoles with small fluctuations in their density.

A suitable weak coupling ansatz for the dual potential, $n_\nu$, has to take into account the fact that for $\beta > \beta_c$, all fluctuations of this potential are important. This can be done by changing the summation over $n_\nu$ in (II.7) into an integration by using the Poisson sum formula

$$
\Sigma f(n) = \Sigma \int_{-\infty}^{+\infty} dx f(x)e^{2\pi imx}
$$

However, we could not do the integrals exactly. Work on an approximation scheme is underway.

Another question that needs to be settled is the order of the phase transition. At the moment it is not clear how to determine that. We feel however, that the weak coupling calculation might give an indication of how to answer this question.

The results obtained for the strong coupling phase are in excellent agreement with Monte Carlo. There are two ways to improve
this agreement near the critical point. The first is to keep higher order terms in the polynomial generated by (III.3). The second way, which would require the use of computers, is to keep more "live" dual links. In (III.1) we kept live only the four dual links that formed the plaquette dual to $P_{\alpha\beta}(r_0)$. We should be able to do better by keeping live, in addition to that dual plaquette, all other dual plaquettes that share a link with it. This will give 64 dual links to sum over. It has the effect of taking into account longer correlation lengths which become important near the critical coupling. Another way of saying this is that keeping more live links takes into account larger fluctuations in the density of the monopole condensate that forms the strong coupling vacuum. Such fluctuations are important near the critical coupling.
IV. OTHER ABELIAN LATTICE GAUGE THEORIES

The mean plaquette method can be applied, in exactly the same way as for lattice QED$_4$, to other Abelian lattice gauge theories. For example, the analog of (II.7) for lattice QED$_5$ is derived in exactly the same way as (II.7) (but here there are ten types of Bianchi identities) and has the same form. The difference is that the dual variable in QED$_4$ was a link $n_\nu$, and for 5 dimensions it is a dual plaquette $n_{\mu \nu}$. Also, $\tilde{n}_{\mu \nu}(\rho)$ in (II.7) is replaced by

$$\tilde{n}_{\mu \nu}(\rho) = \frac{1}{2} \varepsilon_{\alpha \beta \gamma} \Delta_{\alpha} n_{\beta \gamma}(\rho)$$

(IV.1)

where the indices take the values 1, 2, 3, 4, 5. So, $\tilde{n}_{\mu \nu}(\rho)$ is made from the six dual plaquettes that form a 3 dimensional cube.

The reasoning that led from (II.7) to (III.3) was as follows. The only dual variables (links in QED$_4$) that are kept live, are those that form the geometrical object (plaquette in QED$_4$) which is dual to $P_{\alpha \beta}(r_0)$. Applying the same reasoning to QED$_5$ we find that we keep live only the six dual plaquettes that form the cube dual to $P_{\alpha \beta}(r_0)$. All other dual plaquettes are put equal to pure gauge. And since each live plaquette is shared by 5 cubes, other than the cube appearing in the order of the Bessel function, the analog of Equation (III.3) for QED$_5$ is

$$\langle p \rangle = \frac{\sum_{-\infty}^{+\infty} 5_{i=1}^{6} |n_{i}|}{\sum_{-\infty}^{+\infty} 5_{i=1}^{6} |n_{i}|}$$

(IV.2)
where $n_i$ are the six live dual plaquette variables that form the cube $n$

$$n = n_1 + n_2 + n_3 - n_4 - n_5 - n_6$$

It is straightforward to convince one's self that in $d$ dimensions Equation (IV.2) still holds but where instead of 6 live dual variables, there are $2(d-2)$ of them. And

$$n = \sum_{i=1}^{d-2} n_i - \sum_{i=d-1}^{2(d-2)} n_i$$

In particular, for $d = 2$ there are no dual variables and Equation (IV.2) yields the exact solution as it should since there are no Bianchi identities.

For $Z(2)$ lattice gauge theory in four dimensions, the analog of (III.3) is

$$<p> = \frac{1}{2\pi} \sum_{n=0}^{4} \frac{1}{\sin n} \sum_{i=1}^{5} |n_i| e^{\beta \cos(n-1)}$$

$$<p> = \frac{1}{\sin n} \sum_{i=1}^{5} |n_i| e^{\beta \cos n}$$

where

$$n = n_1 + n_2 - n_3 - n_4$$

$$\beta = \frac{1}{2} \ln \left( \frac{1 + e^{-2\beta}}{1 - e^{-2\beta}} \right)$$

The solutions to (IV.2) and (IV.3) are plotted in Figures 2 and 3 respectively. We have in these theories the same problems in the weak coupling phase as we do in QED$_4$, and the same discussion presented in
the previous section holds here too. However, $Z(2)$ lattice gauge theory in four dimensions is self dual, and this enables us to calculate the weak coupling plaquette energy by using

$$< P > (\beta) = \cosh 2\tilde{\beta} - < P >(\tilde{\beta}) \sinh 2\tilde{\beta}$$  \hspace{1cm} (IV.5)

where $\tilde{\beta}$ is given by (IV.4).
In this section we will apply to lattice QCD\textsubscript{4} the same mean-plaquette method we applied to Abelian lattice gauge theories and derive the analog of Equation (III.3). Because the Bianchi identities for lattice QCD\textsubscript{4} are very complicated, we will avoid deriving the analog of Equation (II.7) by using the Abelian results to motivate some short cuts that we will take.

The plaquette energy for lattice QCD\textsubscript{4} is given by
\[
\frac{\langle \text{Tr} P_{\alpha\beta}(r_1) \rangle}{2} = z^{-1} \int_{\text{D}U_{\mu}} \frac{\text{Tr} P_{\alpha\beta}(r_1)}{2} e^{\frac{8}{4} \sum_{\mu<\nu} \text{Tr} \{P_{\mu\nu}(r)+P_{\mu\nu}^{+}(r)\}}
\]
\[
= \langle p \rangle
\]
(V.1)

where $P_{\mu\nu}(r)$ is the product of link variables around a plaquette. As we showed for QCD\textsubscript{3} in Reference 2, we can change the variable of integration in (V.1) from links to plaquettes.

\[
\langle p \rangle = z^{-1} \int_{\text{D}P_{\mu\nu}[\prod_{c=1}^{P_{c}} \delta(P_{c} - 1)]} \frac{\text{Tr} P_{\alpha\beta}(r_1)}{2} e^{\frac{8}{4} \sum_{\mu<\nu} \text{Tr} \{P_{\mu\nu}(r)+P_{\mu\nu}^{+}(r)\}}
\]
(V.2)

Where, following the notation of Reference 2, $\delta(P_{c} - 1)$ is the lattice Bianchi identity associated with a given cube, and $\prod_{c}$ is a product over all cubes in the lattice. Here, there are four types of lattice Bianchi identities because there are four types of three dimensional cubes. These Bianchi identities are derived in exactly the same way as in the appendix of Reference 1.

As we saw for QCD\textsubscript{3},\textsuperscript{2} the non-Abelian lattice Bianchi identities are nonlocal and somewhat messy to write down and work with. However,
here we do not need to write them out in detail. From the results of the mean-plaquette calculation for Abelian theories, we expect only the four Bianchi identities associated with the 4 cubes that share $P_{\alpha \beta} (r_1)$, to be important in a mean-plaquette calculation for QCD$_4$. This should be true at least in the strong coupling region.

Dropping all the Bianchi identities, except the four mentioned above, Equation (V.2) becomes

$$<P> = z^{-1} \int_{\mathcal{D}P_{\mu \nu}} \left[ \prod_{c=1}^{4} \frac{\text{Tr} P_{\alpha \beta} (r_1)}{\delta (P - 1)} \right] \frac{1}{4} \sum_{\mu < \nu} \text{Tr} (P_{\mu \nu} (r) + P_{\mu \nu}^+ (r))$$

$$= \frac{1}{2}$$

(V.3)

where $c = 1$ to 4 labels the four cubes that share $P_{\alpha \beta} (r_1)$. We emphasize, that Equation (V.3) is expected to give accurate results for the whole strong coupling region after we make the mean-plaquette approximation not before. $\delta (P - 1)$ can be character expanded

$$\delta (P - 1) = \sum_{J_c} (2J_c + 1) \chi_{J_c} (P_c)$$

$$J_c = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

(V.4)

where $\chi_{J_c} (P)$ is the trace of $P$ in the $(2J_c + 1)$ dimensional representation. Moreover, because we have only 4 non-Abelian lattice Bianchi identities that share one plaquette, we can easily "Abelianize" them. Then, each $P_c$ is just the product of the six gauge invariant plaquettes that form the cube. To fix ideas, let $P_{\alpha \beta} (r_1)$ be $P_{12} (r_1)$, then the Abelianized Bianchi identity for the (xyz) cube sitting on top of $P_{12} (r_1)$ is given by

$$\chi_{J_c} (P_{12} (r_1))$$
\[ \delta[P_{12}(r_1)P_{23}(r_1)P_{13}(r_1 + \hat{2})P_{12}^+(r_1 + \hat{3})P_{23}(r_1 + \hat{i})P_{13}^+(r_1) - 1] \]

\[ = \sum_{J_c = 0, \frac{1}{2}}^{\infty} (2J_c + 1) xJ_c xJ_c [P_{12}(r_1)P_{23}(r_1)P_{13}(r_1 + \hat{2})P_{12}^+(r_1 + \hat{3})P_{23}(r_1 + \hat{i})P_{13}^+(r_1)] \]

(V.5)

where \( \hat{1}, \hat{2}, \) and \( \hat{3} \) are unit vectors in the 1, 2 and 3 directions. Using the Gross and Witten trick, the right hand side of Equation (V.5) becomes

\[ \sum_{J_c = 0, \frac{1}{2}}^{\infty} (2J_c + 1)^{-4} xJ_c xJ_c (P_{12}(r_1))^J_c (P_{23}(r_1)^J_c xJ_c (P_{13}(r_1 + \hat{2})) xJ_c (P_{12}(r_1 + \hat{3})) \]

\[ \times xJ_c (P_{23}(r_1 + \hat{i})) xJ_c (P_{13}(r_1)) \]

\[ = \sum_{J_c = 0, \frac{1}{2}}^{\infty} (2J_c + 1)^{-4} U_{2J_c}^2 \frac{\text{Tr}P_{12}(r_1)}{2} U_{2J_c}^2 \frac{\text{Tr}P_{23}(r_1)}{2} U_{2J_c}^2 \frac{\text{Tr}P_{13}(r_1 + \hat{2})}{2} \]

\[ \times U_{2J_c}^2 \frac{\text{Tr}P_{12}(r_1 + \hat{3})}{2} U_{2J_c}^2 \frac{\text{Tr}P_{23}(r_1 + \hat{i})}{2} U_{2J_c}^2 \frac{\text{Tr}P_{13}(r_1)}{2} \]

(V.6)

where we used

\[ xJ_c (P) = U_{2J_c} \left( \frac{\text{Tr}P}{2} \right) \]

(V.7)

and \( U_n(x) \) is the Chebyshev polynomial of the second kind and order \( n \).

The other three Bianchi identities are treated similarly. Substituting (V. 4, 5, 6) in (V.3), and making the mean-plaquette approximation

\[ \frac{1}{2} \text{Tr}P_{\mu\nu}(r) \rightarrow \langle P \rangle \]

(V.8)

for all plaquettes except \( P_{12}(r_1) \) gives
\[<P> = z^{-1} \sum_{J_1, J_2, J_3, J_4} \left[ \frac{1}{4} (2J_1 + 1)^4 U_{2J_1} (\langle P \rangle) \right] \int dP_{12} (r_1) [U_{2J_1} \left( \frac{\text{Tr} P_{12} (r_1)}{2} \right) ] \]

\[x \frac{\text{Tr} P_{12} (r_1)}{2} \frac{6}{4} \text{Tr} [P_{12} (r_1) + P_{12}^+ (r_1)] \]  \quad (V.9)

For each configuration of the \( J_1 \)'s, the integral over \( P_{12} (r_1) \) is easy to do. Equation (V.9) is the approximate mean-plaquette equation for lattice QCD. It is the analog of Equation (III.3) for lattice QED. It can be solved numerically by truncating the summation at high enough values of the \( J_1 \)'s. This will be done in the next section.
VI. APPROXIMATE SOLUTIONS FOR LATTICE QCD$_4$

As for Abelian theories, the leading term, all $J_1 = 0$, of Equation (V.9) is just the leading term of the strong coupling expansion

$$<P> = \frac{I_2(\beta)}{I_1(\beta)}$$  \hspace{1cm} (VI.1)

This is plotted as curve 1 in Figure 4. As is well known, this is accurate only in the very large coupling limit.

If we include the first nonleading term, one of the $J_1 = \frac{1}{2}$, we obtain the polynomial\(^*\)

$$<P> - \frac{I_2(\beta)}{I_1(\beta)} - 4 <P>^5 (1 + \frac{3I_3(\beta)}{I_1(\beta)}) + 16 \frac{I_2(\beta)}{I_1(\beta)} <P>^6 = 0$$  \hspace{1cm} (VI.2)

Note the similarity of (VI.2) to (III.6). This similarity disappears when higher order terms are included. Equation (VI.2) is plotted as curve 2 in Figure 4. That figure shows that agreement with Monte Carlo improved markedly upon the inclusion of the first nonleading term. Moreover, already at this very low order, curve 2 exhibits a (mild) crossover region at about the same coupling where the Monte Carlo data exhibits such a behaviour.

\(^*\) Actually, for $\beta$ less than about 1, the contribution from the $(1,0,0,0)$ configuration of $J_1$ is larger than that from $(\frac{1}{2},0,0,0)$, but both are small enough to be unimportant. Nonleading terms, however, become important for $\beta \geq 1$, and in this region the $(\frac{1}{2},0,0,0)$ term is the larger. That is why it appears in (VI.2) as the first nonleading term.
The importance of higher order terms is estimated in the same way as for the Abelian theories. Using that method, we find that at $\beta \approx 2$, the values of $J_1$ that give the dominant contributions are $(0,0,0,0)$, $(\frac{1}{2},0,0,0)$, $(\frac{1}{2}, \frac{1}{2}, 0,0)$, $(\frac{3}{2},0,0,0)$, and $(\frac{3}{2}, \frac{1}{2},0,0)$. Contributions from other configurations of $J_1$ may be ignored up to $\beta \approx 2$. Notice that we are not attempting to find all the roots of (V.9) but are verifying that (V.9) has a solution that agrees very well with Monte Carlo data.

Solving Equation (V.9) keeping only the above mentioned terms yields curve 3 in Figure 4. This curve is in excellent agreement with the Monte Carlo data points. It tracks the crossover with very high accuracy, and even up to $\beta = 5$, the discrepancy between the mean plaquette and the Monte Carlo curves is only 5%.

One of the most gratifying results of this calculation is the obvious absence of a phase transition.

The fact that we kept only a few terms from the expansions of the lattice Bianchi identities, and the excellent agreement between the mean-plaquette curve and Monte Carlo results, tell us that the correlation length is quite short even up to $\beta = 5$.

Moreover, the excellent agreement with Monte Carlo indicates that we were justified in dropping the terms that we dropped. To improve agreement in the $\beta > 2.8$ region, one must keep some more higher order terms. This is straightforward but requires the use of a computer.
VII. MEAN-PLAQUETTE FOR SO(3) LATTICE GAUGE THEORY

The plaquette energy for SO(3) lattice gauge theory is

$$\frac{\langle x_1(P_{\alpha\beta}(r_1)) \rangle}{3} = z^{-1} \int_{\mathcal{D}_{\mu\nu}} \left[ \prod_{c=1}^{5} \delta(P_c - 1) \right] \frac{x_1(P_{\alpha\beta}(r_1))}{3} e^{\frac{\beta}{3} \sum_{\mu < \nu} x_1(P_{\mu\nu}(r))}$$

(VII.1)

where $x_1(P)$ is the trace of $P$ in the three dimensional representation.

Going through the same steps as for QCD$_4$, with the mean-plaquette substitution being

$$\frac{1}{3} x_1(P_{\mu\nu}(r)) \rightarrow \langle P \rangle$$

(VII.2)

gives

$$\langle P \rangle = z^{-1} \sum_{J_1=1}^{4} \left[ \prod_{i=1}^{2j_1} (2j_1 + 1)^{-4} u_2^{5} U_2(\langle P \rangle), U_0(\langle P \rangle) \right] \int dP_{\alpha\beta}(r_1)$$

$$\times \left[ -\prod_{2j_1} \left( \frac{\text{Tr} P_{\alpha\beta}(r_1)}{2} \right) \right] \frac{U_2(\frac{\text{Tr} P_{\alpha\beta}(r_1)}{2})}{3} e^{\frac{\beta}{3} x_1(P_{\alpha\beta}(r_1))}$$

(VII.3)

$$J_1 = 0, 1, 2, 3, \ldots$$

where $U_2^{j_1} \{ U_2(\langle P \rangle), U_0(\langle P \rangle) \}$ means that we must express $U_2^{j_1}$ in terms of $U_2$ and $U_0$ (which can always be done because $J_1$ takes only integer values), then put $U_2(\langle P \rangle) = 3 \langle P \rangle$ (by Equations (V.7), (VII.2)).

Unlike QCD$_4$, the $J_1$ here take only integer values because the SO(3) action is even under $P_{\mu\nu}(r) \rightarrow -P_{\mu\nu}(r)$ while $x_{\frac{2n+1}{2}}(P_{\mu\nu}(r))$ is odd and thus the integrals vanish.

Again, by explicit calculation we find that for $\beta < 2.5$, the dominant $J_1$ configurations and $(0,0,0,0), (1,0,0,0), (1,1,0,0), (3,0,0,0), (1,1,1,0), (1,3,0,0)$. Taking only these into account, Equation (VII.3) is solved
numerically yielding Figure 5. That figure shows that the mean-plaquette calculation predicts a first order phase transition at $\beta_c \approx 2.5$, in excellent agreement with Monte Carlo simulations.

Agreement between mean-plaquette and Monte Carlo is not so good in the weak coupling phase. The reason is that the terms that were dropped for $\beta < 2.5$, because they were small, become important for $\beta > 2.5$ and must therefore be kept. Therefore, to improve agreement between the mean-plaquette calculation and Monte Carlo, for SO(3) and QCD$_4$ at weak coupling, more terms should be kept when solving (V.9) or (VII.3). These terms are of interest in the strong coupling region as well where, although their effect is small, it is not zero. They may change the results by about 3%. We may also need to keep a few more Bianchi identities, according to the same prescription as for lattice QED$_4$, in order to get better agreement in the weak coupling region especially for SO(3).

The fact that keeping only a few terms from the expansion of the lattice Bianchi identity did give the plaquette energy for the weak coupling phase of SO(3) (although not accurately, in the order we worked with), means that the fluctuations of the dual potential ($J_1$) are still not very large. In other words the plaquettes are still quite disordered and the correlation length rather short.

An interesting question is then why does QCD$_4$ have a crossover while SO(3) has a phase transition? The only difference between Equations (V.9) and (VII.3), other than the value of the integrals, is that in (V.9) all values of $J_1$ contribute whereas in (VII.3) only integer values do. As mentioned before, the absence of half odd
integer values of $J_\perp$ for $SO(3)$ is due to the inability of this action to distinguish between $P_{\alpha\beta}(r)$ and $-P_{\alpha\beta}(r)$. On the other hand, the QCD$_4$ action can distinguish these two cases and therefore has an extra $Z(2)$ degree of freedom that $SO(3)$ lacks. The effect of this $Z(2)$ degree of freedom in QCD$_4$ is to allow half odd integer values of $J_\perp$ to contribute in (V.9). Moreover, as pointed out in Section VI, the nonzero values of $J_\perp$ that give the dominant contributions to $\langle P \rangle$ for $\beta \approx 2$ (i.e. near the crossover) are half odd integers. This indicates that the $Z(2)$ degree of freedom is playing an important role at the crossover. This seems to agree with the idea that the $Z(2)$ degree of freedom smooths out the phase transition and makes it a continuous crossover.
VIII. CONCLUSIONS

Finally we present our physical interpretation of this method and the reasons it seems to work better than previous mean-field schemes.

It is clear that topological excitations play an important role in the phase structure of lattice gauge theories. Usual mean-field methods are insensitive to fluctuations of these excitations and it is, perhaps, for this reason that, in a low number of dimensions, these methods give the wrong critical behaviour.

The mean-plaquette method presented here, automatically takes into account the existence of the topological excitations and the fluctuations in their density. This is easiest to illustrate with the Abelian theories. As mentioned in Section (III), the strong coupling ansatz (III.1) says that the topological excitations form a condensate with small fluctuations in its density. As we keep contributions of larger values of $n_1$ in (III.3), we are in fact calculating, perturbatively, the effect of larger fluctuations in the density of this condensate - i.e. longer correlation lengths. And since the difference in the density of the topological excitations, between the strong and weak coupling phases, is nonperturbative, the strong coupling ansatz (III.1) cannot give the weak coupling results. The same interpretation applies to non-Abelian theories, only here the fluctuations in the density of the topological excitations seem to be always perturbative.

This is the extent of our understanding of why the method works. It is not complete, and more work is needed to understand it more fully.
Equivalently, summing over the configurations of the dual potential $n$ in (II.7) means that we are effectively keeping the lattice Bianchi identity and thus correlations among the plaquettes. This is so because as discussed in chapter (I), the dual potential is the Fourier conjugate to the lattice Bianchi identity.

A very interesting point to note is the following. Previous mean field methods become exact as the dimensionality, $d$, goes to infinity because corrections are of order $1/d$. Our mean-plaquette method, however, is exact in two dimensions (which is the lowest dimensionality where a gauge theory can be defined). And since a dimensionality of four is much closer to two than it is to infinity, this may have something to do with why this method works better than mean field in four dimensions.

Finally, everything we said in this chapter can be applied to abelian and non-abelian spin models. There we would have a mean-link method which is exact in one dimension.
REFERENCES

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2. G. G. Batrouni, Lawrence Berkeley Laboratory preprint, LBL-13991 (February 1982), and chapter (I).
FIGURE CAPTIONS

Figure 1. QED$_4$ plaquette energy. The dots are Monte Carlo data points$^5,6$. Curve 1 is the leading term of the strong coupling expansion. Curve 2 is the solution to Equation (III.6) and is indistinguishable from the solution to the polynomial of order 21 generated by (III.3). $\beta_c$(mean-plaquette) = 0.91, $\beta_c$(Monte Carlo) = 1.

Figure 2. QED$_5$ plaquette energy. The points are Monte Carlo data points,$^5$ and the solid curve is the mean-plaquette result. $\beta_c$(mean plaquette) = 0.79 and $\beta_c$(Monte Carlo) = 0.74.

Figure 3. Plaquette energy of Z(2) in 4 dimensions. The dots are Monte Carlo data,$^7$ and the solid curve is the mean plaquette result. $\beta_c$(mean plaquette) $\approx$ 0.4247, $\beta_c$(exact) = 0.4407.

Figure 4. QCD$_4$ plaquette energy. The dots are Monte Carlo data.$^6$ Curve 1 is the leading term of the strong coupling expansion (VI.1). Curve 2 is the solution of (VI.2) and curve 3 is explained below Equation (VI.2).

Figure 5. Plaquette energy for SO(3). The dots are Monte Carlo data.$^6$ The solid lines are the mean plaquette results. $\beta_c$(mean-plaquette) = 2.5, $\beta_c$(Monte Carlo) = 2.48.
Figure 3

\[ \langle P \rangle \]

\[ \beta \]

- Cooling
- Heating
This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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