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ON THE RATE OF CONVERGENCE OF THE CORE

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Abstract

We relax the two restrictive assumptions in Anderson's theorem (1986) which proves that in a general sequence of finite economies with smooth preferences, the rate of convergence of the competitive gap with respect to the "gap-minimizing" prices is the inverse of the square of the number of agents. The assumptions are the boundary condition and the strict positiveness of individual endowments.

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On the Rate of Convergence of the Core

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1. Introduction

In a recent paper, Anderson (1986) showed that the rate of convergence of the average competitive gap to zero with respect to suitably chosen prices is the inverse of the square of the number of agents, in a general sequence of economies with smooth preferences. This improvement upon the previous results which had established the rate $O(1/n)$ could be obtained essentially by combining the flattening effect of increasing the number of agents with smooth preferences at the tangent points, with the fundamental result of Anderson (1978).

Like others in the literature (see Debreu (1975) and Grodal (1975)), Anderson makes two restrictive assumptions: (i) all agents have strictly positive endowments of all commodities, and (ii) the closure of each indifference curve is contained in $\mathbb{R}_+^k$ (boundary condition). With these conditions, the consumption set can be confined to a subset of $\mathbb{R}_+^k$, without encountering any

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1 The author is very grateful to Professor Robert Anderson who suggested the problem and gave encouragement and invaluable comments.
unpleasant boundary problems. For example, the possibility that core allocations lie on the boundary of the consumption set is excluded.

Cheng (1981) was able to relax one of the two assumptions, the boundary condition, to obtain the rate $O(1/n)$ of convergence of the maximum competitive gap to zero with respect to the supporting prices at the core allocations in a sequence of type economies, by introducing a rather mild assumption which he called the Indecomposability condition. The condition essentially says that we cannot partition the set of agents into two groups such that each group consumes different set of commodities.

Anderson conjectured that Cheng's condition or some variant of it could replace the two assumptions in his theorems as well. Indeed, this turns out to be true in a general sequence of economies converging to a well-defined continuum economy. The purpose of this paper is to show how Cheng's condition which we will call the linkedness condition following Mas-Colell (1985), can be used to replace the two assumptions in Anderson (1986)'s theorem.

The basic references are Anderson (1981, 1986) and Mas-Colell (1985). We follow their convention in definitions and notations and readers are referred to them for the details. Their theorems and lemmas will be freely cited whenever needed, sometimes without proof.
2. Definitions and Notations

We begin with the concept of preference preorder $\succeq$ on $\mathbb{R}^k_+$, which is defined as a reflexive and transitive binary relation on $\mathbb{R}^k_+$. Throughout the paper, we assume that $\succeq$ is complete, continuous, strictly monotone, and strictly convex, and that $\succeq$ is smooth, i.e., representable by a $C^2$ utility function with no critical points. $\mathcal{P}$ denotes the set of all smooth preferences satisfying the above assumptions. $\mathcal{P}$ is endowed with the topology of uniform $C^2$ convergence on compacta of associated admissible utility functions.

An exchange economy is a map $\mathcal{E}: \mathcal{A} \to \mathcal{P} \times \mathbb{R}^k_+$, where $\mathcal{A}$ is a finite set or $[0,1]$. $\mathcal{E}(a) = (\succeq_a, e(a))$ has a standard interpretation: an agent $a \in \mathcal{A}$ in an exchange economy $\mathcal{E}$ is characterized by a preference preorder $\succeq_a$ and an endowment vector $e(a) \in \mathbb{R}^k_+$. We assume that total endowments are strictly positive, i.e., $\sum_{a \in \mathcal{A}} e(a) > 0$ (for a continuum economy, $\sum$ is replaced by $\int$). For a continuum economy with $\mathcal{A} = [0,1]$, we require that $\mathcal{E}$ be Borel measurable and $\text{supp} \mathcal{E}$ compact in $\mathcal{P} \times \mathbb{R}^k_+$ with the product topology.

An allocation $f$ is a map $f: \mathcal{A} \to \mathbb{R}^k_+$. An attainable allocation is an allocation satisfying $\sum_{a \in \mathcal{A}} f(a) = \sum_{a \in \mathcal{A}} e(a)$. The set of core allocations $C(\mathcal{E})$, and the set of Walrasian allocations $W(\mathcal{E})$ are
defined as usual. A price \( p \) is an element of the set \( D = \{ p \in \mathbb{R}^k_+ \mid \Vert p \Vert = 1 \} \), where \( \Vert p \Vert \) denotes the Euclidean length of \( p \).

We consider a sequence of finite exchange economies \( \{ \xi_n \} \) which converges to a continuum economy \( \xi \). We say \( \xi_n \rightharpoonup \xi \), if \( \text{supp} \xi_n \rightharpoonup \text{supp} \xi \) and \( \nu_n \rightharpoonup \nu \) weakly, where \( \nu_n, \nu \) are the distributions of characteristics induced by \( \xi_n, \xi \). Note that \( Q = (U_n \text{supp} \xi_n) \cup \text{supp} \xi \) is compact.

An allocation \( f \) is linked, if there are no partitions of agents \( (A = A_1 \cup A_2) \) and commodities \( (L = L_1 \cup L_2) \) such that \( f_{ij}(a) = 0 \) whenever \( (a, j) \in A_1 \times L_2 \) or \( A_2 \times L_1 \). Linkedness is the same as the Indecomposability condition in Cheng(1981) or the No Isolated Community condition in Smale(1974). In their models, this guarantees that the supporting price at a core allocation is unique.

A \((k-1)\) collection of pairs of commodities \( s = \{J_1, \ldots, J_{k-1}\} \) is linked if \( \bigcup_{h=1}^{k-1} J_h = L \) and for any partition \( s_1 \) and \( s_2 \) of \( s \), \( \{i \in J_h \mid J_h \in s_1 \} \cap \{i \in J_h \mid J_h \in s_2 \} \neq \emptyset \).

For a sequence of finite exchange economies \( \xi_n : A_n \rightarrow \mathfrak{S} \times \mathbb{R}^k_+ \) with \( \#A_n = n \), a pair of a coalition \( S_n \) and an allocation \( f_n, (S_n, f_n) \), is \( \delta \)-uniformly linked, if there exists a \( \delta > 0 \) such that \( \#S_n \geq \delta n \) and for a linked \((k-1)\) collection \( s \), \( \#(a \in S_n \mid f_{ij}(a) > \delta) \) for each \( j \in J \}/n > \delta \), for all \( J \in s \). \( \delta \)-uniform linkedness is the same as \( \delta \)-balancedness in Mas-Colell(1985).

Given an economy \( \xi \), an allocation \( f \), and a price \( p \in D \), the competitive gap \( \phi \) for an agent \( a \in A \) is defined as:
\[ \phi(p, f, a) = |p \cdot (f(a) - e(a)) + \inf \{ p \cdot (y - e(a)) | y \succeq_a f(a) \} |. \]

3. The Result

Now we can state the main theorem.

**Theorem.** Let \( \epsilon_n : A_n \rightarrow \mathcal{P} \times \mathbb{R}^k \) with \( \#A_n = n \) be a sequence of finite exchange economies converging to a continuum economy \( \epsilon \). If all the Walrasian allocations of \( \epsilon \) are linked, then there is a price vector \( p_n \in \mathcal{D} \) and a constant \( M > 0 \) such that, if \( f_n \) is a core allocation for \( \epsilon_n \), then

\[ \frac{1}{n} \sum_{a \in A_n} \phi(p_n, f_n, a) \leq \frac{M}{n^2} \]

The Theorem says that under the almost same hypotheses as in Proposition 7.4.12. in Mas-Colell(1985), the average competitive gap with respect to the "gap-minimizing" prices, is the order of \( 1/n^2 \). The proof will closely follow those of Anderson(1986) and Mas-Colell(1985). We begin with the definitions: for an exchange economy \( \epsilon \) and a core allocation \( f \), define \( \gamma(a) = \{ x - e(a) | x \succeq_a f(a) \} \), \( \pi(a) = \gamma(a) \cup \{0\} \), and \( \Pi = \sum_{a \in A} \pi(a) \). \( \bar{p} \) which maximizes \( \inf p \cdot \Pi \) is called the "gap-minimizing" price. Define \( \alpha = \bar{p} \cdot \Pi \).

For \( \xi \in [0,1) \), let \( \Gamma = \bigcup_{n \geq 1} \sum_{a \in A} \gamma(a) \). \( \tilde{q} \) maximizes \( \inf q \cdot \Gamma \).

Define \( \beta = \inf \tilde{q} \cdot \Gamma \) and \( g(a) \in \arg \min \tilde{q} \cdot \gamma(a) \). Define \( g(S) = \)

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$\sum_{a \in S} g(a)$ and $\rho(S) = \tilde{q} \cdot g(S)$. Note that the subscript $n$ can be used for $\epsilon_n$ (for example, $\tilde{q}_n$).

The next two lemmas have been proved under the much weaker hypotheses by Anderson (1986) and therefore will be stated without proofs.

**Lemma 1.** For any finite economy $\epsilon$ and $\xi \in [0,1)$, (i) $\rho \leq (1 - \xi) \alpha$ and (ii) $\inf \tilde{q} \cdot M \geq - (1 - \xi)^{-1} \sum_{k=1}^{k} M_{\epsilon}$, where $M_{\epsilon} = \max \{|e(a_1) + \ldots + e(a_h)| : a_1, \ldots, a_h \text{ are distinct elements of } \Lambda, 1 \leq h \leq k + 1\}$.

**Lemma 2.** For any finite economy $\epsilon$ and $\xi \in [0,1)$, there exist a coalition $S$ with $\#S \geq \xi n - (k + 1)$, $x = g(S)$, and a constant $M_1 > 0$ such that $|x| \leq M_1$ and $\rho(S) \leq (1 - \xi) \alpha$.

The following is the central lemma in this paper.

**Lemma 3.** Let $S_n$ and $x_n = g_n(S_n)$ be defined for the economy $\epsilon_n$ as in Lemma 2. Then there exist $N > 0$, $\delta > 0$, and a constant $M_2 > 0$ such that for $n > N$ and $\xi \geq 1 - \delta$, $\rho(S_n) \geq -M_2/n$.

**Proof of Lemma 3.** Step 1. Lemma 7.4.13 in Mas-Colell (1985) proves that under the same hypothesis except strict positiveness of individual endowments, there exist $\delta > 0$ and $N_1 > 0$ such that for any economy $\epsilon_n$ with $n > N_1$, $(\Lambda_n, f_n)$ is $4\delta$-uniformly linked.
But we can get rid of the assumption of strictly positive individual endowments by using the result of Anderson(1981).

Theorem 3 in Anderson(1981) proves that the distance between the core allocation and the demand at price $\bar{p}$ converges to zero in measure. This can replace the proposition 7.4.9 in the proof of Lemma 7.4.13 in Mas-Colell(1985). Clearly, for $\xi \geq 1-\delta$ and $n > \max\{N_1, (k+1)/\delta\}$, $(S_n, f_n)$ is $2\delta$-uniform linked. Hereafter, we will drop subscript $n$ for $S$, $f$, $g$, $x$, $\bar{p}$, and $\bar{q}$ for simplicity of notation, unless it is particularly confusing.

**Step 2.** Define an allocation $h$ such that for all $a \in A$, $h(a) = g(a) + e(a)$, which, by definition of $g(a)$, minimizes $\bar{q} \cdot v$ over $\{v \in \mathbb{R}^k_+ \mid v \succeq_a f(a)\}$. We claim that for $\delta$ obtained above, there exists $N_2 > 0$ such that for $n > N_2$,

$$\#\{a \in S \mid \|f(a) - h(a)\| > \delta\}/n < \delta$$

But this is implied by the proof of Theorem 3 in Anderson(1981).

Under our hypotheses all the assumptions of Theorem 3 are satisfied. In particular, assumption (v) is satisfied because for the limit economy with $\int e(a)da > 0$, there exists a $\delta > 0$ such that $\lambda(\{a \in A \mid e(a) > \delta\}) > \delta$ and $\zeta_n$ converges to a limit economy. Moreover by Lemma 2 (ii) above, $\bar{p}$ can be replaced by $\bar{q}$ in theorem 3.

**Step 3.** Let $N_3 = \max\{N_1, N_2, (k+1)/\delta\}$. Then the consequence of Step 1 and 2 is that there exists $\delta > 0$ such that for $n > N_3$ and $\xi \geq 1 - \delta$, $(S, h)$ is $\delta$-uniformly linked. Now that the structure of the problem is quite similar but a little different.
from Lemma 7.4.14 in Mas-Colell(1985), we will virtually repeat his proof of Lemma 7.4.14. We assume \( n > N_\delta \) and \( \delta > 1 - \delta \).

1. Because of strict monotonicity of preferences, there is \( \rho > 0 \) such that for \((\succ, e)\in Q\) and \( y \preceq \xi \), \( \nabla_{\succ} y > \rho \xi \) and \( \bar{q} > \rho \xi \) for all \( n \).

2. From the result of Anderson(1978) and Lemma 2 (ii) above, there is a \( K > 0 \) such that \( |\bar{q} \cdot f(a) - \bar{q} \cdot e(a)| < K \). Since \( e(a) \) is in a compact set and \( \bar{q} \) is bounded away from zero, \( f \), and therefore \( h \), is uniformly bounded. So there is \( \kappa > 0 \) such that \( h(a) \preceq \kappa \xi \), where \( \xi = (1, 1, \ldots, 1) \).

3. \( Q \) is compact. Therefore, smoothness of preferences implies that there is \( \theta > 0 \) such that if \((\succ, e) \in Q \) and \( y \preceq \kappa \xi \), then \( \{v \in \mathbb{R}_+^k | v \cdot y \} \) contains the nonnegative vectors of the ball of radius \( \theta \) and center \( y + \theta \nabla_{\succ} y \), where \( \nabla_{\succ} y \) is the unit normal to the indifference manifold of \( \succ \) through \( y \).

4. Consider any \((k-1)\) linked collection \( \mathcal{J}_m = (J_1, \ldots, J_{k-1}) \) and \( p \in D \). Let \( T_p = \{v \in \mathbb{R}^k | v \cdot p = 0\} \). For every \( s \leq k-1 \), let \( v_s \in T_p \) be such that \( \|v_s\| = 1 \) and \( v_s \neq 0 \) only if \( j \in J_s \). Because \( \mathcal{J}_m \) is linked, \((v_1, \ldots, v_{k-1})\) constitute a basis of \( T_p \). Therefore, for any \( z \in \mathbb{R}^k \), there exist unique \((\alpha_1, \ldots, \alpha_{k-1})\) such that \( z = \alpha_1 v_1 + \ldots + \alpha_{k-1} v_{k-1} \) \( (p \cdot z)p \). Pick \( c > 0 \) such that \( |\alpha_s| \leq c \) uniformly on \( m \), \( s \), and \( p \).

5. We can write \( x = g(S) = \alpha_1 v_1 + \ldots + \alpha_{k-1} v_{k-1} + (\bar{q} \cdot x)\bar{q} \). Let \( x_s = \alpha_s v_s + \frac{1}{k-1}(\bar{q} \cdot x)\bar{q} \). Then \( x = \sum_{s=1}^{k-1} x_s, \|x\| \leq (c+1)\|x\| \leq (c+1)M_1 \). Since \((S, h)\) is \( \delta \)-uniformly linked, we can partition \( S \)
and \( \xi \geq 1 - \delta \), \( \alpha \geq -\frac{M_2}{(1-\xi)n} \). But it is easy to show that with the gap minimizing price \( \bar{p} \), \( \sum_{a \in A} \phi(\bar{p}, f, a) \leq 4\alpha \). The proof is completed by letting \( M = \frac{4M_2}{(1-\xi)} \). Q.E.D.
into $k-1$ coalitions $S_1, \ldots, S_{k-1}$ such that for every $s$, (i) $h^j(a) > \sigma$ for each $j \in J_s$ and $a \in S_s$, and (ii) $\#S_s \geq \sigma n/2k$. Note that for $a \in S_s$, $\nu_S(h(a)) \cdot v_s = 0$.

(6) Now let $M_3 = 2k^3(c+1)^2M_1^2/\sigma^4 \theta \rho$ and $n > N_4 = \max\{N_3, 2k[(c+1)M_1/\sigma^2 + 1]/\sigma\}$. Then, we can show by contradiction that $\beta(S) = \tilde{q} \cdot g(S) \geq -M_3/\#S$. Assume $\tilde{q} \cdot g(S) < -M_3/\#S$. Then it is fairly easy to show that $S$ is a blocking coalition with the allocation $h'(a) = h(a) - \frac{1}{\#S_s} x_s$ if $a \in S_s$:

(i) $\Sigma_{a \in S}(h'(a) - e(a)) = \Sigma_{s=1}^{k-1} \Sigma_{a \in S_s}(h(a) - \frac{1}{\#S_s} x_s - e(a))$

$= \Sigma_{a \in S}(h(a) - e(a)) - x_s \Sigma_{s=1}^{k-1} x_s$

$= x - x = 0$.

(ii) $h'(a) \in \mathbb{R}_+$, since $\frac{1}{\#S_s} x_s \geq \frac{2k(c+1)M_1}{\sigma^2 n}$

(iii) $h'(a) > a h(a) \Sigma a f(a)$, because, denoting $v = \nu_S(h(a))$,

$\|h(a) + \theta v - h'(a)\|^2 = \|\theta v + \frac{1}{\#S_s} x_s\|^2$

$= \theta^2 + 2\theta \Sigma \frac{1}{\#S_s} v \cdot x_s + (\frac{1}{\#S_s})^2 \|x_s\|^2$

$\leq \theta^2 - \frac{2\theta}{\#S_s} \Sigma \frac{1}{\#S(k-1)} M_3(v \cdot \tilde{q}) + (\frac{1}{\#S_s})^2 \|x_s\|^2$

$\leq \theta^2 - \frac{2\theta}{n^2} M_3 \rho + \frac{4k^2}{\sigma^2} (c+1)^2 \frac{M_1^2}{\sigma n}$

$\leq \theta^2$.

(7) $\beta(S) \geq -M_3/\#S \geq -M_3/((1-\sigma)n-k-1)$. Therefore, $\beta(S) \geq -M_2/n$, if we let $N = \max\{N_4, 2(k+1)/(1-2\sigma)\}$ and $M_2 = 2M_3$. Q.E.D.

Proof of the Theorem. Lemma 2 and 3 establish that for $n > N$
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