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The Phase-Space-Lagrangian Action Principle
and The Generalized K-x Theorem*

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THE PHASE-SPACE-LAGRANGIAN ACTION PRINCIPLE
AND THE GENERALIZED $K\chi$ THEOREM

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Abstract

The covariant coupled equations for plasma dynamics and the Maxwell field are expressed as a phase-space-Lagrangian action principle. The linear interaction is transformed to the bilinear beat Hamiltonian by a gauge-invariant Lagrangian Lie transform. The result yields the generalized linear susceptibility directly.
The fundamental relation between nonlinear ponderomotive effects and linear plasma response has come to be known as the K-\(\chi\) theorem. Ponderomotive effects are embodied in the oscillation-center Hamiltonian \(K(z)\), introduced by Dewar.\(^1\) It describes the (oscillation-averaged) orbit of a single particle in an oscillatory field, with the dominant effects quadratic in the wave amplitude. On the other hand, the linear susceptibility \(\chi\), a functional of the unperturbed particle distribution, describes the oscillatory current density linear in the wave amplitude.

This surprising relation between the quadratic Hamiltonian \(K_2\) and the susceptibility \(\chi\), for the case of a single wave, was observed\(^2\) some years ago, and then proved by Johnston\(^3\) and by Cary.\(^4\) The underlying reason for the relation, however, became clear only with the recent development\(^5\) of phase-space Lagrangian action principles, and the realization that the plasma action term quadratic in the wave amplitude \([-\text{ift}(z)K_2(z)]\) was simultaneously both the oscillation-center energy and the plasma part of the wave Lagrangian. (To be sure, this fact was at least implicit in the earlier work of Dewar\(^1\) and of Johnston.\(^3\)) The importance of this realization, with its embodiment in the action principle, is best exemplified in Similon's recent study\(^6\) of self-consistency in the stabilization of a confined plasma by the ponderomotive effects of an electromagnetic wave.

The ponderomotive beat Hamiltonian, introduced by Johnston\(^7\) for the scattering of two waves, and now of especial use for the theory of free-electron lasers and beat-wave accelerators, is a conceptually simple extension of oscillation-center ideas to particles that resonant with the beat of two primary waves. Its utility led Grebogi\(^8\) to the conjecture that it too is related to the linear susceptibility. This paper presents a simple proof of that desired relation, and then illustrates it by an explicit calculation.
That calculation, in turn, is based on the use of a powerful new perturbation technique, invented by Littlejohn\(^9\) for a system governed by a phase-space Lagrangian. Whereas the standard Hamiltonian perturbation theories (such as the Hamiltonian Lie transform\(^10\)) preserve the Poisson structure, the new method enables one to perform the desired averaging directly on the Poisson (or symplectic) structure. As a result, the generator of the transform can be made gauge invariant and physically meaningful.

The calculation is here outlined for a field-free background. The extension to the case of a strong background field is conceptually easy, but of course algebraically complex, and will be published later. We begin with the definition of the two-point linear susceptibility tensor,\(^11\),\(^5\) as a functional derivative:

\[
\chi^{\mu\nu}(x_1,x_2) = \delta j^\mu(x_1)/\delta A^\nu(x_2). \tag{1a}
\]

It is convenient to use covariant notation, with metric \((1, 1, 1, -1)\) and \(c = 1\). Thus \((x = x, t), j^\mu = (j, \rho), A^\nu = (A, -\phi)\). In terms of the Fourier transforms [e.g., \(j^\mu(k) = \int d^4x \ j^\mu(x) \exp(-ik \cdot x); k^\mu = (k, -\omega)\)], the susceptibility reads

\[
\chi^{\mu\nu}(k_1,k_2) = \delta j^\mu(k_1)/\delta A^\nu(k_2). \tag{1b}
\]

In Eq. (1), \(j\) is the linear current response to a perturbing electromagnetic potential \(A\). Since \(j\) must be invariant under gauge transformations of \(A\), the susceptibility must satisfy \(\chi^{\mu\nu}(k, k') k'_\nu = 0\). In addition, charge conservation (\(\partial j^\mu/\partial x^\mu = 0\)) implies that

\[
k^\mu \chi^{\mu\nu}(k,k') = 0.
\]

Because each particle responds to the perturbing field independently, the current density is additive in the particles; hence the susceptibility is a linear functional of the unperturbed distribution.

The ponderomotive Hamiltonian \(K_2(z)\) is (by definition) that term, of the oscillation-center Hamiltonian \(K(z)\), which is quadratic in the perturbing potential. Its most general form is thus
\[ K_2(z) = \frac{i}{\hbar} \int d^4 x_1 \int d^4 x_2 A_\mu (x_1) A_\nu (x_2) K^{\mu \nu} (z; x_1, x_2) \]  
\[ = \frac{i}{\hbar} \int d^4 k_1 \int d^4 k_2 A_\mu^* (k_1) A_\nu (k_2) K^{\mu \nu} (z; k_1, k_2), \]  
\( (2a) \) \( (2b) \)

(We absorb \((2\pi)^{-4}\) into the element \(d^4k\).)

We may interpret the integrand of (2b) as the contribution, to the oscillation-center Hamiltonian, of the nonlinear beat between two plane waves with wave-vectors \(k_1\) and \(k_2\). The relation we wish to prove, the "generalized K-\(\chi\) theorem," is

\[ K^{\mu \nu} (z; k_1, k_2) = - \delta \chi^{\mu \nu} (k_1, k_2)/\delta f(z). \]  
\( (3) \)

That this relation has not heretofore been observed is probably due to the fact that almost all calculations of \(\chi\) make specific assumptions on the form of \(f\). However, the functional derivative in (3) requires the susceptibility for completely general \(f\).

One restriction which we do make is that \(f\) include only those particles which are \textit{non-resonant} with the \textit{primary} waves \(k_1, k_2\). Hence \(\chi\) is Hermitian, and \(K_2\) is real. The proper treatment of primary resonances is a large subject in itself, with important contributions especially by Dewar and his co-workers.\(^{12}\)

The system action \(S\) is a functional of the potential field \(A(x)\), and of the particle orbits in 8-dimensional phase space, denoted \(z^{\alpha} (\tau) = [r^H (\tau), \pi_\mu (\tau)]\), with \(\pi_\mu = (\pi, -h)\) the \textit{kinetic} 4-momentum, \(h\) the kinetic energy, and \(\tau\) an arbitrary orbit parameter. The \textit{single}-particle action is \(S_1 = \int (\pi \cdot dr + eA(r) \cdot dr)\). We demand that \(\delta S_1 = 0\) for variation of orbits constrained to the 7-dimensional mass surface \(0 = H(z) = (\pi^2 + m^2)/2m\). With a Lagrangian multiplier \(\lambda (\tau)\), we have

\[ 0 = \delta \left[ \pi \cdot dr + eA(r) \cdot dr - \lambda (\tau) d\tau H(z) \right]. \]  
\( (4) \)
Variation with respect to (wrt) $r(t)$ yields $dr_\mu = eF_{\mu\nu} dr^\nu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, while variation wrt $\pi(t)$ yields $d\pi^\mu = \lambda(t) d\tau \pi^\mu / m$. The mass constraint determines $\lambda^2(t) = -(dr(t)/dt)^2$; if one wishes $d\tau$ to represent the particle's proper-time interval, then $\lambda = 1$.

The total action is $S = S_i S_i + S_m$, where $S_i$ is the action of particle $i$, and $S_m = \int d^4x F_{\mu\nu} F^{\mu\nu}/16\pi$ is the Maxwell action. The interaction part of $S$ can be expressed as $\int d^4x J^\mu(x) A_\mu(x)$, with $[u_\mu = dr^\mu/d\tau = \pi^\mu / m]$

$$j^\mu(x) = \sum_i e \int d\tau u^\mu(\tau) \delta^4(x-r(\tau)) = e \int d^8 z f(z) u^\mu \delta^4(x-r).$$

We have introduced the particle phase-space density (for each species)

$$f(z) = \sum_i \int d\tau \delta^4(r-r_i(\tau)) \delta^4(\pi-\pi_i(\tau)).$$  \hspace{1cm} (5)

Variation of $S$ wrt $A(x)$ yields the Maxwell equation $\partial_\mu F^{\mu\nu}(x) = -4\pi j^\nu(x)$.

The distribution $f$ satisfies the Vlasov equation \(^5\) \{f, H\} = 0, in terms of the noncanonical Poisson bracket (PB) \{\(g_1, g_2\)\} = $J^{\alpha\beta}(z) (\partial g_1/\partial z^\alpha)(\partial g_2/\partial z^\beta)$. The Poisson tensor $J(z)$ is the reciprocal of the Lagrange tensor (or symplectic 2-form), \(^9\) $\omega_{rr} = eF(r)$, $\omega_{r\pi} = -\omega_{\pi r} = 1$, $\omega_{\pi\pi} = 0$. Thus $J^{rr} = 0$, $J^{r\pi} = -J^{\pi r} = 1$, $J^{\pi\pi} = eF(r)$, and the PB is expressed in the physical variables $r$, $\pi$, $F$:

$$\{g_1, g_2\} = (\partial g_1/\partial r) \cdot (\partial g_2/\partial \pi) - (\partial g_1/\partial \pi) \cdot (\partial g_2/\partial r)$$

$$+ e(\partial g_1/\partial \pi) \cdot F(r) \cdot (\partial g_2/\partial \pi).$$  \hspace{1cm} (6)

For a wave field $F_{\mu\nu}(x)$, oscillations occur in the PB (6) for nonresonant particles. Our aim is to transform away this term, linear in $F$, by a change of variables from particle coordinates $z^\alpha$ to oscillation-center (OC) coordinates $\tilde{z}^\alpha(z;F)$. The linear oscillation induced by $F$ is denoted $\tilde{z} = z - \dot{z}$. We see that $\tilde{z}^\alpha(z)$ is a physically meaningful vector field; it is the generator of the Lagrangian Lie transform.

In terms of the Fourier transform $F_{\mu\nu}(k)$, the linearized particle equations yield the oscillation:
\[
\begin{align*}
\vec{r}(\dot{z};F) &= e \int d^4k F(k) \cdot \dot{u} \cdot (ik \cdot \dot{u})^{-1} \exp ik \cdot \vec{r}, \\
\vec{\pi}(\dot{z};F) &= -(e/m) \int d^4k F(k) \cdot \dot{u} \cdot (k \cdot \dot{u})^{-2} \exp ik \cdot \vec{r},
\end{align*}
\] (7)

as a vector field on OC phase space. In order that (7) be well-defined, we consider only that portion of phase space which has no primary resonances; i.e., \( k \cdot \dot{u} \neq 0 \) for all \( k \) such that \( F(k) \neq 0 \).

Our aim is to make the PB canonical, when expressed in OC variables:

\[
\{g_1, g_2\} = (\partial g_1/\partial \pi)(\partial g_2/\partial \pi) - (\partial g_1/\partial \pi) \cdot (\partial g_2/\partial \pi).
\] (8)

Space limitations permit us only to quote the result of using the Lagrangian Lie transform, which is based on differential-geometric methods. We obtain the OC Hamiltonian \( K(\dot{z}) = H(\dot{z}) + K_2(\dot{z}) \), with the ponderomotive term given by the virial:

\[
K_2(\dot{z};F) = -\frac{1}{2} \vec{r}(\dot{z};F) \cdot [eF(\dot{r}) \cdot \dot{u}].
\] (9)

The canonical Hamiltonian equations then yield

\[
\frac{d\pi}{d\tau} = -\partial K/\partial \dot{r} = -\partial K_2/\partial \dot{r}
\] (10a)

for the ponderomotive force, and

\[
\frac{d\dot{r}}{d\tau} = \partial K/\partial \pi = \pi/m + \partial K_2/\partial \pi,
\] (10b)

a gauge-invariant expression for the canonical OC momentum \( \dot{r} \), in terms to the OC velocity \( d\dot{r}/d\tau \) and the quadratic term (related to wave monentum).

(The mass constraint now reads \( 0 = H(z) = K(\dot{z}); \) i.e., the Hamiltonian transforms as a scalar under the coordinate change.)

The one-particle action is now, in the OC representation, including the Hamiltonian constraint:
\[ S_1 = \int (\pi \cdot \text{d}r - K(\tilde{z}; F) \text{d}t). \] (11)

The terms of \( \Sigma S_i \) quadratic in \( F \) are thus

\[ S^{(2)} = - \sum_i \int \text{d}t_i K_2(z_i(\tau_i); F) = - \int d^4 z f(z) K_2(\tilde{z}; F). \] (12)

Noting from (9) and (7) that \( K_2 \) and \( \tilde{z} \) are manifestly gauge-invariant, we proceed to express \( K_2 \) in the desired form (2b), using \( F_{\mu \nu}(k) = i(k_\mu A_\nu - k_\nu A_\mu) \); we obtain

\[ K^{\mu \nu}(z; k_1, k_2) = (e^2/m) \left[ (k_1 \cdot u)^2 + (k_2 \cdot u)^2 \right] \mathcal{K}^{\mu \nu}(u; k_1, k_2) \exp[i (k_2 - k_1) \cdot r]; \] (13)

with \( \mathcal{K}^{\mu \nu}(u; k_1, k_2) = k_1 \cdot u \ k_2^{\mu} u^{\nu} + k_2 \cdot u \ u^{\mu} k_1^\nu - k_1 \cdot k_2 \ u^{\mu} u^{\nu} - k_1 \cdot u \ k_2 \cdot u \ u^{\mu} u^{\nu}. \)

On substituting (2b) into (12), we obtain

\[ S^{(2)} = - \int d^4 z f(z) \int d^4 k_1 \int d^4 k_2 A_\mu (k_1) A_\nu (k_2) K^{\mu \nu}(\tilde{z}; k_1, k_2) \] (14)

Recalling that \( j^\mu(x) = \delta \sum_i S_i / \delta A_\mu(x) \), we see from (1a) that

\[ \chi^{\mu \nu}(x_1, x_2) = \delta^2 S / \delta A_\mu (x_1) \delta A_\nu (x_2), \] (15)

or \( \chi^{\mu \nu}(k_1, k_2) = \delta^2 S / \delta A^{*}_\mu (k_1) \delta A_\nu (k_2). \)

Applying this to (14), we obtain

\[ \chi^{\mu \nu}(k_1, k_2) = - \int d^4 z f(z) K^{\mu \nu}(\tilde{z}; k_1, k_2), \] (16)

which is equivalent to the desired theorem (3).

If we set \( k_2 = k_1 \) in (13) and (16), we obtain the covariant form of the single-wave K-\( \chi \) theorem.5,15
In summary, we have indicated that a phase-space transformation from particle to oscillation-center coordinates, using the oscillation vector field as the generator of a Lagrangian Lie transform, converts the Poisson Bracket to canonical but gauge-invariant form, and converts the linear interaction to a bilinear form, which simultaneously is the beat Hamiltonian and expresses the generalized linear susceptibility.

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