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Author
Young, Jonathan D.

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University of California

Ernest O. Lawrence
Radiation Laboratory

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NUMERICAL APPLICATIONS OF HYPERBOLIC SPLINE FUNCTIONS

Jonathan D. Young

May 1968
This article describes the application of hyperbolic spline fitting to a set of points:

\[(t_i, x_i); \ i = 1, I \text{ with } I \leq 3\]

to obtain:

1. a computational definition of a smooth curve, \(x(t)\);
2. estimates of the first derivative \(x'(t)\) at each \(t_i\);
3. estimates of the second derivative \(x''(t)\) at each \(t_i\);
4. interpolated values for \(x\) at any \(t\), \(t_1 \leq t \leq t_I\).

The \(t_i\) must be distinct and increasing with \(i\), but need not be uniformly spaced.

Introduction

A \(t\)-dependent quantity, \(x(t)\), frequently is known (from observation, table etc) only in the discrete form as a set of points

\[(t_i, x_i); \ i = 1, I\]
with the $t_i$ distinct and increasing with $i$. Reference (1) constructs and gives the properties of a cubic spline function which fits such a set of points. For reasonably large $I$, the cubic spline fit will in general have fewer inflection points than the polynomial of degree $I-1$ which fits the data. However, we may find even among the inflection points of the cubic spline fit, some which we consider undesirable. If for example in the double subinterval $[t_{i-1}, t_{i+1}]$ for some $i; i = 2, I-1$ we have

$$x_{i-1} \leq x_i \geq x_{i+1}$$

we should expect that a proper fitting function have everywhere in the double interval, a non-positive second derivative, or no inflections. In cubic spline fitting inflection points can occur here or in the parallel case

$$x_{i-1} \geq x_i \leq x_{i+1}$$

In either case such inflection points are said to be extraneous.

Reference (2) considers fitting the set $(t_i, x_i)$ so that in each subinterval the fitting function represents a simply supported spline under uniform tension, $p^2$. It is shown that in the limiting case $p^2 = 0$, the fitting function is in fact a cubic spline fit and that for sufficiently large $p^2$, the fit will have no extraneous inflection points. Unfortunately, although totally adequate for the theoretical development, the formulation given in Reference (2) is for a table $(t_i, x_i)$ with unit uniform steps in $t$: 
In what follows, we provide formulation for non-uniform steps in $t$ to define the fitting function and determine its first and second derivatives at table points. The fitting function (as in reference (2)) is required to have the following properties:

1. over any subinterval $[t_i, t_{i+1}]$; $i = 1, I - 1$ the fitting function $f(t)$ coincides with the hyperbolic spline function

$$g_i(\tau) = x_i + a_i \tau + b_i \left[ \frac{(1 - \tau) \sinh p - \sinh p(1 - \tau)}{\sinh p - p} \right]$$

$$+ c_i \left[ \frac{\tau \sinh p - \sinh p}{\sinh p - p} \right]$$

where $\tau = (t - t_i)/(t_{i+1} - t_i)$

2. $f(t_i) = x_i$; $i = 1, I$ (exact fit)

3. over the whole interval $[t_1, t_I]$ $f$ has continuous first and second derivatives, $f'(t)$ and $f''(t)$.

4. in the limiting case $p \to 0$, $f$ is the cubic spline fit but for sufficiently large $p$, $f$ has no extraneous inflection points.

Provided two additional conditions such as the values:

(a) $x_1'$ and $x_I'$

or

(b) $x_1''$ and $x_I''$

are specified, the matching at $t_i$ required by (2) and (3) is sufficient to determine $a_i; b_i$ and $c_i$; $i = 1, I - 1$ and thereby determine $f$. 
However under the same conditions $f$ is completely determined by the known set

$$(t_i, x_i) \ i = 1, I$$

and the computable

$$f_i' = f'(t_i)$$

Computational processes on the values

$$(t_i, f_i, f_i') \ i = 1, I$$

provide for second order differentiation and interpolation.

Hyperbolic Spline Fit, First derivative.

The problem of defining $f$ is logically equivalent to finding $f_i' \ i = 1, I$ since for any $i; i = 1, I - 1$ the hyperbolic spline segment, $g_i$, of $f$ is determined by $(t_i, x_i, f_i')$ and $(t_{i+1}, x_{i+1}, f_{i+1}')$.

The computation for the $f_i'$ proceeds as follows.

Let

$$\alpha = \frac{(p \cosh \tau - \sinh \tau)}{(\sinh \tau - p)}$$

$$\beta = \frac{(p^2 \sinh \tau)}{(\sinh \tau - p)}$$

Matching $f_i, f_i', f_i''$ at $i = 2, I - 1$, we have

$$f_i = g_i(0) = g_{i-1}(1) = x_i$$

$$f_i' = \left[ g_i'(0) \right] \left[ \frac{1}{t_{i+1} - t_i} \right] = \left[ g_{i-1}'(1) \right] \left[ \frac{1}{t_i - t_{i-1}} \right]$$

$$f_i'' = \left[ g_i''(0) \right] \left[ \frac{1}{t_{i+1} - t_i} \right]^2 = \left[ g_{i-1}''(1) \right] \left[ \frac{1}{t_i - t_{i-1}} \right]^2$$

where derivatives of $g$ are with respect to $\tau$. From which we obtain after elimination of $f_i''$ the following:
\[(t_{i+1} - t_i)f_{i-1}' + \alpha(t_{i+1} - t_{i-1})f_i' + (t_i - t_{i-1})f_{i+1}' = (\alpha + 1) \frac{(t_{i+1} - t_i)(x_i - x_{i-1})}{t_i - t_{i-1}} + \frac{(t_i - t_{i-1})(x_{i+1} - x_i)}{t_{i+1} - t_i}
\]

\[i = 2, I - 1 \quad (1.0)\]

This system of \(I-2\) equations involves \(I\) unknowns, \(f_i', \ i = 1, I\), consequently two additional equations must be provided.

If as in case (a) above we specify values for \(x_i'\) and \(x_i''\), we then have

\[f_1' = x_1' \quad (1.1)\]

\[f_I' = x_I' \quad (1.2)\]

and the system is determinate.

We may however specify \(x_1''\) and \(x_I''\) (case (b)), then

\[\alpha f_1' + f_2' = (\alpha + 1)(x_2' - x_1') \quad \frac{(\alpha^2 - 1)(t_2' - t_1')f_1''}{t_2' - t_1'} \quad (1.3)\]

\[f_{I-1}' + f_I' = (\alpha + 1)(x_I' - x_{I-1}') \quad \frac{(\alpha^2 - 1)(t_I' - t_{I-1}')f_I''}{t_I' - t_{I-1}'} \quad (1.4)\]

with equations (1.0) provide a determinate system.

Second derivative

The values of the second derivative

\[f_i'' = f''(t_i)\]
can be readily computed from the set:

\[(t_i, x_i, f'_i) \quad i = 1, I\]

by

\[
f''_i = \frac{\beta}{\alpha^2 - 1} \left[ \frac{(1 + \alpha)(x_k - x_i)}{(t_k - t_i)^2} \frac{a_i f'_i + f'_k}{t_k - t_i} \right]
\]

where \(t_k\) is adjacent to \(t_i\). For \(i = 2, I - 1\), the \(k\) may be either \(i - 1\) or \(i + 1\). For \(i = 1, k\) must equal 2 and for \(i = I, k\) must be \(I - 1\).

**Interpolation**

Interpolation for \(x(t^*)\) for

\[t_1 < t^* < t_I\]

is accomplished by computing \(f(t^*)\) from the set

\[(t_i, x_i, f'_i) \quad i = 1, I.\]

For some \(i, i = 1, I - 1\) we have

\[t_i \leq t^* < t_{i+1}\]

and

\[f(t^*) = g_1(\tau^*)\]

where \(\tau^* = (t^* - t_i)/(t_{i+1} - t_i)\).

\[g_1(\tau^*) = x_i + a_i \tau^* + b_i \left[ \frac{(1 - \tau^*) \sinh p - \sinh(1 - \tau^*)}{\sinh p - p} + c_i \frac{\tau^* \sinh p - \sinh p \tau^*}{\sinh p - p} \right] \]
where \( a_i = (x_{i+1} - x_i) \)

\[
b_i = -\frac{1}{\alpha^2} \left[ (t_{i+1} - t_i)(f_i' + \alpha f_i') - (a + 1)a_i \right]
\]

\[
c_i = (t_{i+1} - t_i)f_i' - a_i - \alpha b_i
\]

Consequently \( g_i(T^*) \) is readily computable from \( (t_i, x_i, f_i') \) and \( (t_{i+1}, x_{i+1}, f_{i+1}') \).

**Conclusion**

While the cubic spline fit is computationally very convenient, it may for some data have extraneous inflection points. When this does occur it may be desirable to find the hyperbolic-spline fit which has no such inflection points. The question arises as to the magnitude of \( p \) to effect this. In reference (2) a method is given for estimating the value, denoted by \( p^* \), in the case of uniform steps in \( t \). However, no formal method exists when the steps are non-uniform.

The quantity \( \alpha \) which appears in the equation relating first derivatives to values of \( x \), may provide some clue as to the effect of \( p \) on the fit. Thus for \( p = 0 \) (cubic spline fit) we find \( \alpha = 2 \). Near zero \( p \) the value of \( \alpha \) does not change greatly. Starting with \( p = 3 \), the following table pertains.
and as $p$ increases, $\alpha$ approaches $p - 1$ from above.

In fitting the set 
$$(t_i, x_i); i = 1, I$$

it is suggested that the cubic spline fit be made first and tested for extraneous inflection points. If none exist the fit is accepted, otherwise introduce values $p = 3, 4, 5, 6, 7, 8, 9$, accepting any fit which has no extraneous inflection points. If extraneous inflection points still appear, higher values for example $p = 10, 20, 30, 40, 50, 60, 70, 80, 90$ may be tried.

References


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