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On the Existence and Optimality of Competitive Equilibrium for a Slave Economy

T. BERGSTROM
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With their fascinating paper, *Economics of Slavery in the Ante-bellum South*, Alfred Conrad and John Meyer [2] have stimulated considerable interest in the question of the institutional stability of slavery. Prior to their work, a conventional view of historians was that slave prices in the antebellum South were inflated well above the capitalized value of slave services. Overcapitalization, it was argued, might have led to the demise of slavery as an institution even in the absence of harsh political action.

Conrad and Meyer undermine the premise of this argument by offering evidence that the prices of slaves in the decade before the Civil War were approximately equal to the present value of their services. The Conrad-Meyer results suggest that there were well functioning competitive markets for slaves during this period.

As the authors acknowledge, their evidence does not provide "a sufficient guarantee of the continuity of southern slavery". In this paper, a general equilibrium model will be constructed to characterize a competitive economy in which the institution of slavery is present. The existence and welfare properties of an equilibrium for such an economy are investigated. The construction of the model and the theorems derived suggest additional empirical questions which are germane to the issue of the efficiency and institutional stability of slavery.

Data from the 1850 and 1860 censuses are used to lend additional credence to the hypothesis that slavery in the South would not have disappeared due to purely economic forces.

A plausible general equilibrium model of a slave economy requires substantial modification of some assumptions crucial to the traditional theorems on the existence and optimality of competitive equilibrium. In particular, the theorems in this paper extend the results of Arrow, Debreu, and McKenzie [1], [4], [5], [8], [9], to a class of economies where initial wealth allocations are somewhat less rich and diversified than is required for the traditional theorems.

In the construction of the model, the following definitions will be used:

- $I$ the set of nonslaves;
- $J$ the set of slaves;
- $M = I \cup J$, the set of all consumers;
- $K$ the set of firms;
- $a_{ij}$ the fraction of the present value of the $j$th slave which is owned by the $i$th nonslave. It is assumed that $\sum_{i \in I} a_{ij} = 1$. That is, the entire present value of the $j$th slave, as a slave, is owned by members of $I$;

1 First version received April 1969, final version received March 1970 (Eds).
the fraction of the profits of the kth firm which accrue to i. We assume
\[ \sum_{i \in I} \theta_{ik} = 1. \]

All shares in profits of firms are owned by nonslaves.

\( T_m \) the feasible trade set of the mth consumer for \( m \in M \), \( T_m \) is assumed to be a
subset of Euclidean \( n \) space. \( x_m \in T_m \) is a vector of net flows to consumer \( m \).
Flows received or emitted at different times are represented by different com-
ponents of \( x_m \). The restriction to \( E^x \) requires that time is divided into discrete
periods and that there is a finite (possibly very distant) time horizon. If, for
example, consumer \( m \) in period \( t \) is a net recipient of food and surrenders labour
services to others, then the component of \( x_m \) corresponding to food in time \( t \)
has a positive sign while the component corresponding to his own labour in
time \( t \) has a negative sign.

\( R_m \) the preference ordering of consumer \( m \);

\( P_m \) the strict preference relation corresponding to \( R_m \);

\( Y_k \) the production possibility set of the kth firm.

In addition, for \( j \in J \) we define \( T_j^* \subseteq T_j \) as the trade set which is feasible for \( j \) when \( j \)
is owned by others. \( T_j^* \) need not be identical to \( T_j \) if the incentives which can be enforced
on \( j \) when he is enslaved are insufficient to elicit from him some trades which he could
make if he were free.

At any price \( p \in E^x \), if the trade vector assigned the jth slave is such as to elicit
the highest present value which can be extracted from him so long as he is a slave, then the
present value of the jth slave is \( \max_{x \in T_j^*} (-px) \).\(^1\) (This is true since by our convention on
the signs of trade vectors, negative elements of the vector \( x_j \) represent flows from \( j \) and
positive elements represent flows to \( j \).)

The main formal difference between a slavery equilibrium, as defined in this paper,
and a competitive equilibrium as usually defined is that the pattern of ownership of
resources assigns budgets of a somewhat different kind. In particular, since it will be
assumed that all consumers are locally nonsatiated, it is in the interest of a slaveowner
(as well as his legal right) to extract the maximum possible present value from his slaves.
A slave is obliged to surrender to his master a vector of commodities with this maximum
value. Since this is the case, the formal treatment of a slave in this model is somewhat
similar to that of a firm which is owned by nonslaves. There are, however, important
differences.

One difference is that while the production possibility sets of firms are assumed to be
convex and to contain the origin, this assumption, as will be shown, seems quite un-
reasonable for the sets, \( T_j^* \), for \( j \in J \).

Furthermore, since the set, \( T_j^* \) of trades which can be elicited from slave \( j \) is not
necessarily equal to the set, \( T_j \), of trades which he can make as a free man, the model will
be constructed so as to allow the possibility that a slave may purchase his freedom for a
price equal to his present value. For this reason, the preferences of a slave as well as his
productivity play a role in the model.

A slave economy will be defined so that at prices, \( p \), any slave, \( j \), is allowed to choose
any trade \( x_j \) in \( T_j \) for which the net cost, \( px_j \), of the trade plus his value to his owner does

\(^1\) We can easily interpret the vector of trades made by a slave to include those of his offspring. In
particular, the child of a female slave in the American South was owned by the owner of the mother. (See
Meyer and Conrad [2] and Hollander [7]). We could then interpret the trade set of a female slave to include
the commodities made available and the commodities consumed by her offspring (with appropriate time
designations). The present value of a female slave would thus include the present value of her reproductive
capabilities.
not exceed zero. If, for example, $\bar{x}_j$ maximizes $-px$ on $T^*_j$ then $-p\bar{x}_j$ is the value of slave
j to his owner. If there is another trade $\bar{x}_j$ in $T_j$ such that $p\bar{x}_j - p\bar{x}_j \leq 0$ and $\bar{x}_j P J \bar{x}_j$ then
the slave may purchase his freedom at the price, $-p\bar{x}_j$, and make the preferred trade, $\bar{x}_j$.
If, on the other hand, there is no way in which $j$ as a free man could offer a trade which
cost him no more than $-p\bar{x}_j$ and which he preferred to $\bar{x}_j$, then there is no way in which
he could purchase his freedom and make himself better off than he would be as a slave.

The competitive budgets, $b(p)$, are defined as follows:

$$
\begin{align*}
\text{If } j \in J, \quad b_j(p) & \equiv - \max \{ -px \} \equiv \min \{ px \}. \\
\text{If } i \in I, \quad b_i(p) & \equiv \sum_{k \in K} \theta_{ik} \max \{ py \} + \sum_{j \in J} a_{ij} b_j(p).
\end{align*}
$$

A slavery equilibrium is a point, $(\bar{p}, \bar{x}, \bar{y}) \in (E^n, \prod M T_m, \prod K Y_k)$ such that:

1. For all $k \in K$, $\bar{p} \bar{y}_k \geq \bar{p}y$ for all $y \in Y_k$.
2. For all $m \in M$, $\bar{x}_m$ maximizes $R_m$ on $T_m \cap \{ x \mid \bar{p}x \leq b_m(\bar{p}) \}$.
3. $\sum_M \bar{x}_m - \sum_K \bar{y}_k = 0$.

It is also of interest to consider economies in which slaves are not allowed to purchase
their freedom and are simply assigned present value maximizing tasks by their masters.¹

A slave exploitation equilibrium is a point $(\bar{p}, \bar{x}, \bar{y}) \in (E^n, \prod I T_i, \prod J T^*_j, \prod K Y_k)$ such
that conditions (1) and (3) above apply as well as

1. For $i \in I$, $\bar{x}_i$ is $R_i$ maximal on $T_i \cap \{ x \mid \bar{p}x \leq b_i(\bar{p}) \}$.
2. For $j \in J$, $\bar{p}\bar{x}_j = \min \{ \bar{p}x \}$.

The proof of existence offered here, unlike the proofs of McKenzie [9] and Debreu [5]
do not employ the assumption that $0 \in T_m$ or that $T_m$ intersects the asymptotic cone of
$\sum_k Y_k$. Such an assumption would seem economically unreasonable since if consumer $m$
owns no land or stocks of food and if the production of food requires land then it would
be the case that $T_m \cap \sum_k Y_k = \phi$.

One could argue that the origin should be contained in the trade possibility set of any
consumer since he could offer the zero trade vector simply by dying immediately. But
if for this reason the origin were included in his trade possibility set, it would no longer
seem reasonable to assume that this set is convex. Consider a consumer who cannot
produce food without trade and who receives barely enough food to survive, while offering
some labour. If his trade possibility set were convex and contained the origin, he could
offer some labour while receiving only half as much food. This does not seem plausible.

The strategy pursued in this paper will be to assume that $T_m$ is a convex set for all
$m \in M$ but not to assume that $T_m$ contains the origin. An economic interpretation of the
sets, $T_m$, can be constructed as follows. Each consumer has a natural lifetime of $n_m$ years.
The set, $T_m$, consists of those trades which he can offer and which enable him to survive
for his entire natural lifetime.²

¹ This appears to have been the case in several states in the American South in the 19th century. See Table 1.
² It would, of course, be more satisfactory if the model allowed a natural interpretation for the possibility
that the length of life could take values between 0 and $n_m$: with an individual's longevity being determined by
the trades which he makes. If this is done, it becomes less reasonable to assume that the sets, $T_m$, are convex.
An adequate treatment of this problem would seem to require the use of continuous rather than discrete
time.
It will be useful in the ensuing discussion to define the set $\overline{T}_m$ as the convex hull of the set $\{0\} \cup T_m$. That is, $\overline{T}_m \equiv \{x \mid x = \lambda x' \text{ for some } x' \in T_m \text{ and some } \lambda \text{ such that } 0 \leq \lambda \leq 1\}.$

A set, $S$, of generally useful bundles is defined so that $S$ has the following characteristic. If an allocation $x \in \bigcap_m T_m$ is feasible in the sense that $\sum M \in \sum K Y_k$ then the addition of the resources in a vector $s \in S$ to the resources of the economy would make feasible an allocation $z$ where for all $i \in I$:

(a) If $x_i \in T_i$, then $z_i P_i x_i$;

(b) if $x_i \in \overline{T}_i$ but $x_i \notin T_i$, then either $z_i \in T_i$ or $z_i = x_i$.

Formally, $S = \{s \mid \forall x \in \bigcap_m T_m \text{ such that } \sum M x_m = y \text{ for some } y \in \sum K Y_k \text{ there exists a } z \in \bigcap_m \overline{T}_m \text{ and a } y' \in \bigcap K Y_k \text{ such that } \sum M z_m = y' + s \text{ and}

(a) For $i \in I$ such that $x_i \in T_i$, $z_i P_i x_i$,

(b) for $i \in I$ such that $x_i \notin T_i$, either $z_i = x_i$ or $z_i \in T_i$.}$

It may be helpful in interpreting the definition of $S$ to observe that if $0 \in T_i$ for all $i \in I$ then condition (b) becomes superfluous and condition (a) could be written "For all $i \in I$, $z_i P_i x_i". In this case, $S$ consists of all vectors, $s$, such that if $x$ is a feasible allocation then the addition of the vector $s$, to the aggregate resources of the economy would make it possible with the use of feasible production processes to produce an aggregate output which could be divided in such a way as to achieve an allocation preferred by every non-slave to the feasible allocation, $x$.

If, for example, $s$ is a vector of goods such that for all $i \in I$ and all $x_i \in T_i$, $x_i + s P_i x_i$ (and if $\sum M T_m \cap \sum K Y_k \neq \emptyset$) then $s \in S$. The set $S$ also includes vectors, $s$, such that the resources in $s$ together with existing resources can be transformed by possible production processes into desirable goods.

It may be helpful to consider a simple Ricardian economy. There are three commodities, land, labour, and food. Land is used only for the production of food. There are positive but non-increasing returns to labour in the production of food. Each consumer has a positive minimal food requirement for survival. If he receives at least this minimal requirement he can offer any amount up to some limit, $L_m$, of labour. It is not difficult to show that if all the land is owned by persons who could survive without trade, then $s \in S$ if the following is true. The vector $s$ has a negative component, $-F$, for food, a zero component for land and a positive component, $+L$, for labour such that when $L$ is used together with the total supply of land and the total potential supply of labour, the incremental output due to the application of $L$ exceeds $F$.

A key assumption used to prove the existence of a slavery equilibrium is that for all $m \in M$, $-S \cap T_m \neq \emptyset$. This means that each consumer can offer a vector in $S$ to the rest of the economy and yet survive.

If in the Ricardian economy discussed above, the landholders could add the labour $L_m$ of consumer $m$ to the labour of all other consumers and produce an increment of food greater than the amount, $F_m$, that he requires to survive, then the vector $(-F_m, L_m, 0) \in S$ and $(+F_m, -L_m, 0) \in T_m$ so that $-S \cap T_m \neq \emptyset$.

If the model is interpreted so that commodity bundles are distinguished by location as well as by time, the interpretation of the assumption, $-S \cap T_m \neq \emptyset$, remains fairly reasonable. Observe that it is not required that consumer $m$ be able to offer a vector of commodities which itself is desired by every other consumer. It is only assumed that each consumer, $m$, can offer a trade vector which with the aid of possible production methods (including transportation) and some redistribution of consumption bundles
among individuals makes possible an allocation which would be preferred by any consumer other than \( m \) to any allocation possible when consumer \( m \) does not make any trades.\(^1\)

Theorem 1 can now be stated.

**Theorem 1.** (a) A slavery equilibrium exists if:

1. \( \sum_{k} Y_k \) is closed and convex and \( \{0\} \in Y_k \) for all \( k \in K \);
2. For all \( i \in I \), \( T_i \) is closed and convex. For all \( j \in J \), \( T_j \) and \( T_j^* \) are closed and convex;
3. For all \( m \in M \), \( R_m \) is a continuous, weakly convex, and locally non-satiated quasi-ordering defined on \( T_m \);
4. For all \( m \in M \), \( T_m \cap \{ \sum_{k} Y_k - \sum_{i \in M \setminus m} T_i \} \) is bounded and \( \{ \sum_{i \in I} T_i + \sum_{j \in J} T_j^* \} \cap \sum Y_k \) has a nonempty interior;
5. There exists a nonzero vector \( v \in \mathbb{R}^n \) such that for all \( m \in M \) and for all \( x_m \in T_m \), \( x_m + \lambda v R_m x_m \) for all \( \lambda \geq 0 \);
6. For all \( m \in M \), \( -S \cap T_m \neq \emptyset \);
7. If \( j \in J \), \( x_j \in \text{Boundary} T_j \) and \( x_j^* P_j x_j \), then \( x_j^* - x_j \in S \).

(b) If assumptions 1-6 hold, and if \( T_j = T_j^* \) for all \( j \in J \), then a slave exploitation equilibrium exists.

Assumptions 1-4 are familiar from other general equilibrium models (Debreu [4] and [5]). Assumption 5 is satisfied if there is some good which is always harmless. Since this good could also be totally useless, the assumption seems to involve no substantial loss of generality.\(^2\) Assumption 6 is discussed above. Assumption 7 requires that if the level of preference of a slave is to be raised, he must absorb a bundle of resources desirable to the community. A similar assumption is introduced by Rader (11) under the name of quasi-transferability. Since the proof of Theorem 1 is lengthy and rather intricate, it is confined to the appendix to this paper.

The remaining theorems deal with the optimality properties of slavery equilibrium and slave exploitation equilibrium.

For any partition \( \{ I, J \} \) of the set, \( M \), of consumers, we say that \( x \) is \( I \) superior to \( x' \) if \( x \) and \( x' \) are in the set \( \prod_{m \in M} T_m \) and if \( x_i R_i x_i' \) for all \( i \in I \), and \( x_i P_i x_i' \) for some \( i \in I \). We say that \( x \) is \( I \) optimal if \( \sum_{i \in I} x_i \in \{ \sum_{k} Y_k - \sum_{j \in J} T_j \} \) and if for any \( x' \) which is \( I \) superior to \( x \), \( \sum_{j \in J} x_j' \notin \{ \sum_{k} Y_k - \sum_{j \in J} T_j \} \). A feasible allocation, \( x \), is \( I \) optimal if there is no allocation feasible for the economy as a whole which is like at least as well by all members of \( I \) and better by some member of \( I \) than the allocation \( x \).

**Theorem 2.** If preferences are locally nonsatiated and there are no externalities:

1. A slavery equilibrium is Pareto optimal;
2. If \( T_j = T_j^* \) for all \( j \in J \), then a slave exploitation equilibrium is \( I \) optimal where \( I \) is the set of nonslaves and \( J \) is the set of slaves.

\(^1\) For example, if an additional bushel of potatoes were made available in Maine, it might not be possible to make everyone in the U.S. economy better off by shipping a fraction of a potato to everyone in the U.S. On the other hand, there might be subtle chains of adjustment in production and distribution which could enable everyone to benefit from the availability of a few additional potatoes in Maine. A nice example of this phenomenon is discussed in H. Working’s statistical study of potato prices in Minneapolis-St. Paul [16]. See also, Sigler [15].

\(^2\) In fact, an artificial commodity could be introduced into the commodity space such that the commodity existed in fixed supply and was useless and harmless.
Proof. (1) Consider any allocation, $x$, Pareto superior to a slavery equilibrium, $(\bar{p}, \bar{x}, \bar{y})$. From the definition of a slavery equilibrium and the assumption of local non-satiation, it follows that $\bar{p}x_m \geq b_m(\bar{p})$ for all $m \in M$ and that $\bar{p}x_m > b_m(\bar{p})$ for some $m \in M$. Therefore $\bar{p} \sum_M x_m > \sum_M b_m(\bar{p})$. But $\sum_M b_m(\bar{p}) = \bar{p} \sum_K \bar{y}_k$. Profit maximization implies that $\bar{p} \sum_K \bar{y}_k \geq \bar{p} \sum_K y_k$ for all $\sum_K y_k \in \sum_K \bar{y}_k$. Therefore $\bar{p} \sum_M x_m > \bar{p} \sum_K y_k$ for all $\sum_K y_k \in \sum_K \bar{y}_k$. This cannot be the case if $\sum_M x_m \in \sum_K \bar{y}_k$. It follows that a slavery equilibrium is Pareto optimal.

(2) If the allocation $x$ is $I$ superior to the slave exploitation equilibrium, $(\bar{p}, \bar{x}, \bar{y})$, then $\bar{p} \sum_I x_i > \sum_I b_i(\bar{p})$. In a slave exploitation equilibrium, $\sum_I b_i(\bar{p}) = \max_{K \in \mathcal{Y}_k} \bar{p} y + \max_{J \in \mathcal{T}_j} (\sum_I x_j)$. Since, by assumption $T_j = T_j^*$ for all $j \in J$, it must be that $\bar{p} \sum_I x_i > \sum_I b_i(\bar{p}) \geq (\sum_K y_k - \sum_I x_j)$ for all $\sum_K y_k \in \sum_K \bar{y}_k$. It follows that a slave exploitation equilibrium is $I$ optimal.

The next result is a counterpart to the theorem that a competitive equilibrium is in the core of an exchange economy (Debreu and Scarf [6]). As noted by Nikaido [11] and Rader [12], since the production possibility sets of firms are convex and contain the origin, there is no loss of generality in assuming that each consumer, $i$, can choose any production activity in the (closed, convex) set $\sum_{K \in \mathcal{K}} \theta_{ik} Y_k$. For all $i \in I$, define $\tilde{Y}_I = \sum_{K \in \mathcal{K}} \theta_{ik} Y_k$.

An allocation $(\bar{x}, \bar{y}) \in (\prod_M T_m, \prod_I \tilde{Y}_I)$ is said to be blocked by the coalition, $C \subseteq M$, if there is an allocation $x \in \prod_C T_c$ such that (a) for all $c \in C$, $x_c R_c \bar{x}_c$; (b) for some $c \in C$, $x_c P_c \bar{x}_c$ and (c) $\sum_C x_c \in \sum_C \tilde{Y}_c$.

This means, roughly, that the allocation $(\bar{x}, \bar{y})$ is blocked by the coalition, $C$, if the resources which it is physically possible for members of $C$ to offer in trade can be transformed by means of production facilities owned by members of $C$ in such a way as to produce an allocation of resources which each member of $C$ likes at least as well as $(\bar{x}, \bar{y})$ and some member likes better.

In the following discussion it will be assumed that each slave has only one owner. So long as the property claims of slaveholders are honoured, a slave will not be allowed to enter economic coalitions without the consent of his owner.

A coalition, $C$, is said to be admissible under slavery obligations if for every slave, $j$, in the set $C \cap J$, the owner of $j$ also belongs to $C$.

Theorem 3. If the assumptions of Theorem 2 are true and if every slave has only one owner, then a slavery equilibrium in which every slave has a non-negative present value cannot be blocked by any coalition which is admissible under slave obligations.

Proof. Consider a coalition, $C$, which is admissible under slave obligations. By methods similar to those employed in the proof of theorem 2 it can be shown that if $(\bar{p}, \bar{x}, \bar{y})$ is a slavery equilibrium and if $x_c R_c \bar{x}_c$ for all $c \in C$ and $x_P \bar{x}$ for some $c \in C$, then $\bar{p} \sum_C x_c > \sum_C b_c(\bar{p})$. But $\sum_C b_c(\bar{p}) = \sum_{j \in I \cap C} b_j(\bar{p}) + \bar{p} \sum_C \bar{y}_c - \sum_{i \in I \cap C} \sum_{j \notin C} a_{ij} b_j(\bar{p})$. Since $C$ is admissible under slave obligations this reduces to $\sum_C b_c(\bar{p}) = \bar{p} \sum_C \bar{y}_c - \sum_{i \in I \cap C} \sum_{j \notin C} a_{ij} b_j(\bar{p})$. 

\[ \sum_C b_c(\bar{p}) = \bar{p} \sum_C \bar{y}_c - \sum_{i \in I \cap C} \sum_{j \notin C} a_{ij} b_j(\bar{p}). \]
By assumption, the present value, \(-b_j(\bar{p})\), of each slave, \(j\), is nonnegative. Hence
\[ \bar{p} \sum_c x_c \geq \sum_c b_c(\bar{p}) \geq \bar{p} \sum_c y_c. \]
Profit maximization implies that for all \(\bar{c} \in \mathcal{C}_c\), \(\bar{p} \sum \bar{x}_c \geq \bar{p} \sum \bar{y}_c\). Therefore
\[ \sum_c x_c \notin \sum_c \bar{y}_c. \]
This proves Theorem 3.

Another way of stating the result of Theorem 3 is to say that if the economy is at a slavery equilibrium, a slave can enter no coalition which with only the resources available to members of the coalition could make the slave better off without either making other members of the coalition worse off or making his owner worse off.

In general, of course, it would be possible if slavery obligations were abolished for a slave to enter a coalition which does not contain his owner and which blocks a slavery equilibrium.\(^1\)

Theorems 2 and 3 suggest that if the economy is at a slavery equilibrium, any improvement in the welfare of slaves could take place only at the cost of a loss of welfare to their owners.

It is nevertheless possible that slavery might disappear as an observable institution even if all slavery contracts were honoured. Our definition of a slavery equilibrium does not require that in equilibrium every member of \(J\) remains a slave. All that is required is that if he is to gain his freedom, he must pay his owner a price equal to his present value as a slave.

If the incentive system which can be enforced on a slave is not sufficient to induce him to make some trades which he could make if he were free, it might be that the slave could improve his lot by self-purchase.

However, it appears that instances of manumission were rare in the antebellum South. The 1850 and 1860 censuses indicate that fewer than 1/10 of 1 cent of the slaves received their freedom in each year.\(^2\) This small percentage could be explained in at least three different ways: (1) Manumission of slaves was outlawed in several states during this period. (2) There were not adequate capital markets or other contractual institutions to allow slaves to finance self-purchase. (3) A slave if freed would be unable to pursue activities which he preferred to slavery and still pay the cost of self-purchase.

Manumission was forbidden in several southern states. Enforcement of these laws was lenient, however. [See Stampp [14] and Matison [10]]. In fact, as indicated by Table I, the rates of manumission were little different whether state law forbade manumission or not. Thus explanation (1) seems inadequate.

Matison cites several interesting examples of methods whereby slaves financed self-purchase. These include illicitly accumulated savings, winnings in a lottery, loans from non-slaves and incentive plans offered by owners or employers. The natural way for a slave to finance his self-purchase would seem to have been to contract with his owner to apply that part of his output beyond some stipulated amount to his self-purchase until he earned his freedom. Slaves, however, had no property rights and in most states, no contract between a slave and his master was binding on the master.\(^3\) (Stampp [14],

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\(^1\) There are conceivable slavery equilibria for which this is not true. If, for example, a slave belongs to an owner who holds all the land in the economy and if any consumption which he likes at least as well as what he receives as a slave must include some products of the land, then even if he were freed he could enter no coalition which blocks the slavery equilibrium.

\(^2\) Nor was the percentage of slaves who escaped bondage by flight ever very substantial. The 1850 and 1860 censuses indicate that less than 1/20 of 1% of the slave population escaped in each year. Source: 8th U.S. Census, Vol. 4, p. 338.

\(^3\) For example, the Arkansas supreme court ruled that, "If the master contract . . . that the slave shall be emancipated upon his paying to his master a sum of money, or rendering him some stipulated amount of labor, although the slave may pay the money, . . . or perform the labor, yet he cannot compel his master to execute the contract, because both the money and the labor of the slave belong to the master and could constitute no legal consideration for the contract." (Stampp [14], p. 197.)
Three states, Delaware, Louisiana, and Tennessee, enforced written contracts for manumission between master and slave. With the exception of Delaware, which had only a very small number of slaves, these states were little different from the others in their rates of manumission (see Table I).

It thus appears plausible that the majority of slaves could, in fact, be forced under slavery to achieve at least as much productivity as they would if free. This being the case, they simply could not afford to purchase themselves.

It is assumed in theorem 1 that for all consumers, $m \in M$, $S \cap T_m \neq \emptyset$. One implication of this assumption is that in equilibrium slaves have a positive present value.

### TABLE I

**Rates of manumission and legality of manumission in Southern states**

<table>
<thead>
<tr>
<th>States</th>
<th>1850 Census</th>
<th>1860 Census</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of slaves</td>
<td>Per cent manumitted</td>
</tr>
<tr>
<td>Alabama</td>
<td>343,000</td>
<td>0.0046</td>
</tr>
<tr>
<td>Arkansas</td>
<td>47,000</td>
<td>0.0021</td>
</tr>
<tr>
<td>Delaware</td>
<td>2,200</td>
<td>12.0960</td>
</tr>
<tr>
<td>Florida</td>
<td>39,000</td>
<td>0.0359</td>
</tr>
<tr>
<td>Georgia</td>
<td>382,000</td>
<td>0.0049</td>
</tr>
<tr>
<td>Kentucky</td>
<td>211,000</td>
<td>0.0720</td>
</tr>
<tr>
<td>Louisiana</td>
<td>245,000</td>
<td>0.0649</td>
</tr>
<tr>
<td>Maryland</td>
<td>90,000</td>
<td>0.5455</td>
</tr>
<tr>
<td>Mississippi</td>
<td>310,000</td>
<td>0.0019</td>
</tr>
<tr>
<td>Missouri</td>
<td>87,000</td>
<td>0.0571</td>
</tr>
<tr>
<td>North Carolina</td>
<td>289,000</td>
<td>0.0006</td>
</tr>
<tr>
<td>South Carolina</td>
<td>385,000</td>
<td>0.0005</td>
</tr>
<tr>
<td>Tennessee</td>
<td>239,000</td>
<td>0.0187</td>
</tr>
<tr>
<td>Texas</td>
<td>58,000</td>
<td>0.0385</td>
</tr>
<tr>
<td>Virginia</td>
<td>473,000</td>
<td>0.0461</td>
</tr>
<tr>
<td>Dist. of Columbia</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Total</td>
<td>3,200,000</td>
<td>0.0458</td>
</tr>
</tbody>
</table>

Sources: Rates of manumission: 8th U.S. Census, Vol. 4, p. 337. Year of prohibition of manumission, Matison (op. cit.).

Available data indicates that adult slaves sold for prices well above zero. Meyer and Conrad have verified that these prices are reasonably accurate reflections of present value. Our assumption, however, would have to be rejected even if infants had non-positive present values.

The institution of slavery might have gradually vanished if infant slaves had been of negative present value. Had this been the case, slave owners would probably have discouraged reproduction or perhaps have engaged in infanticide.

The price of infants can rarely be directly observed since very young children were seldom sold apart from their mothers. The present value of an infant slave can be found indirectly by use of the following formula:

$$ V = \sum_{K=1}^{18} P(K) \sum_{T=1}^{K} \frac{R_T}{(1+r)^T} + \left(1 - \sum_{K=1}^{18} P(K)\right) \left[\sum_{T=1}^{18} \frac{R_T}{(1+r)^T} + \frac{S_{18}}{(1+r)^{18}}\right] $$

where $R_T$ is the net return from a slave of age $T$, $P(K)$ is the probability that he dies in his $K$th year, $S_{18}$ is the expected sale price of an eighteen-year-old slave, and $r$ is the interest
rate. $V$ is the sum which an entrepreneur who attempts to maximize the expected present value of his income stream would be willing to pay for an infant slave.

Meyer and Conrad [2] have estimated the prices in 1850 of 18-year-old male and female slaves as $925 and $825, respectively. They have also estimated the net returns from slave children of each age. These estimates will be used here for $S_{18}$ and $R_T$. An important cost in the rearing of young slaves is that a substantial proportion die before they reach their productive years. Death rates for slaves less than ten years of age are not available from the census. We use as a proxy the death rates of Negro children in the District of Columbia in 1901. These are probably not sufficiently different from the rates we seek to substantially alter our conclusions.\(^1\) Table II illustrates the estimated present values of infant slaves calculated for various interest rates.

If one accepts the argument of Meyer and Conrad that the interest rate available to planters was from 6 to 8 per cent in the period considered, it appears that infant slaves of both sexes did have positive present values. At rates of 12 per cent, females take on a negative present value. At slightly higher interest rates this would be the case for both sexes.

**TABLE II**

*Present value of infant slaves of each sex at alternative interest rates*

<table>
<thead>
<tr>
<th>Sex</th>
<th>Interest rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5 per cent</td>
</tr>
<tr>
<td>Male</td>
<td>$220</td>
</tr>
<tr>
<td>Female</td>
<td>$115</td>
</tr>
</tbody>
</table>

It thus appears that reproduction of slaves was economically profitable. Corroborating evidence is given by the fact that the slave population grew steadily and substantially during the years after 1808 when the slave trade was eliminated until the Civil War. This is shown by Table III.

In addition to being an efficient instrument of exploitation, the institution of slavery is also very flexible. In fact, any subgroup, $I$, of the economy could employ the institution of slavery to sustain nearly any $I$ optimal pattern of exploitation of the other members of the economy. This result is stated formally in theorem 4.

\(^1\) The percentage of the slave population which died in its second decade of life was reported in the 1860 census to be about 11 per cent. The corresponding percentage for the District of Columbia in 1901 was 10 per cent. Sources: 8th U.S. Census (1860), Vol. 4, p. 337. Bureau of the Census: United States Life, Tables: 1900-1931. pp. 120-123.
Theorem 4. If

(1) For all \( m \in M \),
   (a) \( R_m \) is convex, continuous and locally nonsatiated,
   (b) \( T_m \) is convex and \( T_m \cap -S \neq \phi \),
   (c) \( T_m = T_m^* \),

(2) \( \sum_k Y_k \) is convex;

(3) \( \sum M T_m \cap \sum_k Y_k \) is bounded, closed and has a nonempty interior;

then for any partition \((I, J)\) of the population, any \( I \) optimal allocation \((\bar{x}, \bar{y})\) such that \( \bar{x}_i - s_i \in T_i \) for some \( s_i \in S \) for all \( i \in I \) can be sustained as a slavery equilibrium with some price vector \( \bar{p} \) and some assignment of claims on profits and the trades made by group \( J \).

The proof of this theorem, which is a consequence of Minkowski's theorem on the separation of convex sets, will be found in the appendix.

SUMMARY

Theorem 1 displays plausible conditions under which equilibrium exists for a slave economy. The results of Conrad and Meyer indicate that slave prices were not widely divergent from competitive prices. Theorems 2 and 3 suggest that if the economy were at a slavery equilibrium, slavery obligations would be unlikely to disappear simply because of the inefficiency of the system.

In the antebellum South, rates of manumission were low both in states which imposed legal barriers to self-purchase by slaves and in states which did not. The present values of infant slaves were probably non-negative. These results suggest that so long as the ownership of slaves was upheld by the law, the number of persons enslaved would not have substantially diminished either because of self-purchase or because of economically motivated birth control.

The slavery system would appear to be an efficient means of achieving any particular distribution of wealth favourable to the slave owning class. The case against slavery, it seems, must be made not primarily on grounds of inefficiency but on grounds of the morality of the resultant distribution of wealth.

APPENDIX

(A) Proof of Theorem 1

It turns out to be necessary to consider only prices on a half sphere \( P \subset E^n \) where

\[
P = \{ p \mid pv \geq 0 \} \cap \left\{ p \mid \sum_{i=1}^{n} p_i^2 = 1 \right\}
\]

where \( v \) is the vector whose existence is postulated in assumption 5.

As in Debreu (5), attention will first be confined to choice restricted to bounded subsets of \( E^n \). In particular, choose \( B^1 \) a closed bounded cube in \( E^n \) such that for all \( a \in M \), \( T_a \cap \sum \{ \sum_k Y_k \cap \sum_{m \in M \setminus a} T_m \} \subset \text{Int} \cdot B^1 \). Assumption 4 ensures that this can be done.

Let \( \{ B^q \} \) be a sequence of closed bounded cubes such that if \( q' \geq q \) then \( B^q \subset B^{q'} \) and such that \( \lim_{q \to \infty} B^q = E^n \).
For \( m \in M \), let \( T^n_m = T_m \cap \mathbb{R}^g \) and \( \overline{T}^n_m = \text{convex hull} \{ 0 \} \cup T^n_m \). For \( j \in J \), let \( T^*_j = T_j \cap \mathbb{R}^g \) and \( \overline{T}^*_j = \text{convex hull} \{ 0 \} \cup T^*_j \). For \( k \in K \), let \( Y^q_k = Y_k \cap \mathbb{R}^g \).

Define the budget constraints so that:

For \( j \in J \), \( \bar{b}_j(p) = \min_{x \in \overline{T}^*_j} (px) \);

for \( i \in I \), \( \bar{b}_i(p) = \sum_{k \in K} \theta_{ik} \max_{y \in Y^q_k} (py) - \sum_{j \in J} a_{ij} \bar{b}_j(p) \).

For all \( q \geq 1 \) and all \( k \in K \) define the correspondence \( F^q_k : P \rightarrow Y^q_k \) by \( F^q_k(p) = \{ y \mid y \in Y^q_k \text{ and } py \geq py' \text{ for all } y' \in Y^q_k \} \).

For all \( q \geq 1 \) and \( m \in M \), define \( E^q_m : P \rightarrow \overline{T}^q_m \) as follows:

If \( px < \bar{b}_m^q(p) \) for some \( x \in \overline{T}^q_m \), let \( E^q_m(p) = \{ x \mid x \text{ maximizes } R_m \text{ on } \overline{T}^q_m \cap \{ x \mid px \leq \bar{b}_m^q(p) \} \} \).

If \( px \geq \bar{b}_m^q(p) \) for all \( x \in \overline{T}^q_m \), let \( E^q_m(p) = \{ x \mid x \in \overline{T}^q_m \text{ and } px = \bar{b}_m^q(p) \} \).

Define \( G^q : P \rightarrow \mathbb{R}^{q \times B} \) so that \( G^q(p) = \sum_M E^q_m(p) - \sum_K F^q_k(p) \).

**Lemma 1.** For all \( p \in P \) and all \( q \geq 1 \), (a) \( G^q(p) \) is a nonempty convex set and \( G^q \) is upper semi-continuous; (b) if \( x \in G^q(p) \), then \( px \leq 0 \).

Debreu [4], shows that for all \( k \in K \) and all \( p \in P \), \( F^q_k(p) \) is nonempty and convex and that \( F^q_k \) is u.s.c.

It is easily verified that \( \bar{b}_m^q(p) = 0 \) for all \( m \in M \) and all \( p \in P \). Since \( \overline{T}^q_m \) is compact and contains the origin, it follows that \( \overline{T}^q_m \cap \{ x \mid px \leq \bar{b}_m^q(p) \} \) is nonempty and compact for any \( q \geq 1 \). Since \( R_m \) is continuous, \( E^q_m(p) \) is nonempty for all \( m \in M \) and all \( p \in P \). Convexity of \( E^q_m(p) \) follows from the weak convexity of preferences.

For all \( m \in M \), the function \( \bar{b}_m^q(p) \) is the sum of maxima of continuous functions on compact sets and is therefore continuous. If \( px < \bar{b}_m^q(p) \) for some \( x \in \overline{T}^q_m \) the proof that \( E^q_m(p) \) is u.s.c. at \( p \) is the same as that given by Debreu [6]. It is also easily verified that \( px \geq \bar{b}_m^q(p) \) for all \( x \in \overline{T}^q_m \), \( E^q_m(p) \) is u.s.c. at \( p \). Thus \( E^q_m(p) \) is u.s.c. at \( p \) for all \( p \in P \).

Since, by definition, \( G^q(p) = \sum_M E^q_m(p) - \sum_K F^q_k(p) \), part (a) of Lemma 1 follows immediately.

If \( x_m \in E^q_m(p) \) for all \( m \in M \), then \( \sum_M px_m \leq \sum_M \bar{b}_m^q(p) \). But \( \sum_M \bar{b}_m^q(p) = \sum_k py_k \) where \( y_k \in F^q_k(p) \). It follows that if \( \sum_M x_m \in \sum_M E^q_m(p) \) and \( \sum_K y_k \in \sum_K F^q_k(p) \), then \( p(\sum_M x_m - \sum_K y_k) \leq 0 \). This proves part (b) of Lemma 1.

**Lemma 2.** For all \( q \geq 1 \), there exists a \( \bar{p}(q) \) such that \( 0 \in G^q(\bar{p}(q)) \).

A theorem of Debreu [4], states that if \( P \) is the intersection of the unit sphere

\[ \left\{ p \mid \sum_{i=1}^n p_i^2 = 1 \right\} \]

and a closed convex cone which is not a linear manifold and if the correspondence, \( G \), maps \( P \) into a bounded subset of Euclidean \( n \) space and satisfies the conditions of Lemma 1, above, then there is a \( \bar{p} \in P \) such that \( G(\bar{p}) \cap \mathbb{R}^n \neq \phi \) where \( \mathbb{R}^n = \{ -x \mid px \geq 0 \text{ for all } p \in P \} \).

Since \( \left\{ p \mid px \geq 0 \right\} \) is a closed half space, and for all \( q \geq 1 \), \( G^q \) satisfies Lemma 1, there is a \( \bar{p}(q) \) such that \( G^q(\bar{p}(q)) \cap \mathbb{R}^n \neq \phi \). From the duality theorem for closed convex cones (Nikaido [11]) it follows that \( \mathbb{R}^n = \left\{ -\lambda v \mid \lambda \geq 0 \right\} \). Therefore for some \( \bar{p}(q) \) and some \( \lambda \geq 0 \), \( -\lambda v \in G^q(\bar{p}(q)) \).

It will now be shown that for \( \bar{p} = \bar{p}(q) \), \( 0 \in G^q(\bar{p}) \).
Consider a point \( \bar{x} \in \prod M T_m^a \) such that \( \bar{x}_m \in E_m^a(\bar{p}) \) for all \( m \in M \) and such that for some \( \bar{y} \in \sum K F_k^a(\bar{p}), \sum M \bar{x}_m - \bar{y} = -\lambda \bar{v} \) where \( \lambda \geq 0 \). It is not hard to show that for any \( a \in M \), \( \bar{x}_a + \lambda \bar{v} \in T_a \). Since \( \bar{x}_a + \lambda \bar{v} + \sum \bar{x}_m = \bar{y} \), it follows from the definition of \( B^a \) that \( \bar{x}_m \in \text{Int } B^a \) for all \( m \in M \) and that \( \bar{x}_m + \lambda \bar{v} \in \text{Int } B^a \) for all \( m \in M \). Since this is the case, the assumption of local nonsatiation can be used to show that \( \bar{p} \bar{x}_m = b_m^a(\bar{p}) \) for all \( m \in M \). Therefore \( \bar{p} \sum M \bar{x}_m = \sum M b_m^a(\bar{p}) = \bar{p} \bar{y} \). But \( \sum M \bar{x}_m + \lambda \bar{v} = \bar{y} \). It must be that \( \bar{p} \lambda \bar{v} = 0 \).

For some \( a \in M \), let \( \bar{x}_a = \bar{x}_a + \lambda \bar{v} \). Then \( \bar{p} \bar{x}_a = \bar{p} \bar{x}_a \). From the definition of \( v \) it follows that \( \bar{x}_a \in E^a_m(\bar{p}) \). But \( \bar{x}_a + \sum \bar{x}_m = \bar{y} \). Therefore \( 0 \in G^a(\bar{p}) \).

**Lemma 3.** There exists a point \((\bar{p}, \bar{x}, \bar{y}) \in (P, \prod M T_m, \prod K Y_k)\) such that:

1. \( \bar{p} \bar{y}_k \geq \bar{p} y \) for all \( y \in Y_k \);
2. For all \( m \in M \),
   
   a. \( \bar{x}_m \) maximizes \( R_m \) on \( \{ x \mid \bar{p} x \leq b_m(\bar{p}) \} \cap T_m \) if \( \min x \in T_m \bar{p} x < b_m(\bar{p}) \),
   
   b. \( \bar{p} \bar{x}_m = b_m(\bar{p}) \) if \( \min x \in T_m \bar{p} x \geq b_m(\bar{p}) \);
3. \( \sum M \bar{x}_m = \sum K \bar{y}_k \).

**Proof.** Lemma 2 guarantees that there is a sequence \( \{ p(q), x(q), y(q) \} \) such that \( (p(q), x(q), y(q)) \in (P, \prod M T_m, \prod K Y_k) \) and the following conditions are satisfied for all \( q \geq 1 \).

1. For all \( k \in K \), \( y_k(q) \in F^a_k(p(q)) \);
2. For all \( m \in M \), \( x_m(q) \in E^a_m(p(q)) \);
3. \( \sum M x_m(q) = \sum K y_k(q) \).

Condition (3), together with Assumption 4 of Theorem 1 guarantees that for all \( q \geq 1 \), \( (q(q), x(q), y(q)) \) belongs to the compact set \( \{ P, \prod M T_m, \prod K Y_k \} \).

Therefore, for some \( (\bar{p}, \bar{x}, \bar{y}) \in \{ P, \prod M T_m, \prod K Y_k \}, (p(q), x(q), y(q)) \rightarrow (\bar{p}, \bar{x}, \bar{y}) \). It is a straightforward albeit somewhat tedious exercise to show that \( (\bar{p}, \bar{x}, \bar{y}) \) satisfies the conditions of Lemma 3. Q.E.D.

**Lemma 4.** If \( s \in S \), and \((\bar{p}, \bar{x}, \bar{y}) \) satisfies Lemma 3, then \( \bar{p} s > 0 \).

**Proof.** The definition of \( S \) states that if \( s \in S \), there exists a \( z \in \prod M T_m \) such that \( \sum M z_m = \sum K \bar{y}_k + s \) and if \( \bar{x}_m \in T_m \) then \( z_m P_a \bar{x}_m \), if \( \bar{x}_m \notin T_m \) then \( z_m = \bar{x}_m \). It is readily verified that in either case \( \bar{p} z_m \geq \bar{p} \bar{x}_m \).

Assumption 4 of Theorem 1 implies that \( \sum M T_m \cap \sum K Y_k \) has a nonempty interior. It follows that for all \( p \in P \), there is some \( a \in M \) such that \( \min x \in T_a \bar{p} x < b_a(\bar{p}) \). Since this is the case and since \( z_a P_a \bar{x}_a \), it must be that \( \bar{p} z_a > \bar{p} \bar{x}_a \). (See Debreu [4]). Therefore,

\[ \sum M \bar{p} z_m > \sum M \bar{p} \bar{x}_m. \]

But \( \sum M \bar{p} z_m = \bar{p} \sum M z_m = \bar{p} \sum M \bar{x}_m + \bar{p} s \). Hence \( \bar{p} s > 0 \). Q.E.D.
Since \( 0 \in Y_k \) and \( 0 \in T_i \) for all \( k \in K \) and all \( j \in J \), \( b_j(p) \geq 0 \) for all \( i \in I \) and all \( p \in P \). According to Lemma 4 and Assumption 6, \( \bar{p}s > 0 \) for all \( s \in S \) and \( -S \cap T_i \neq \emptyset \). Therefore, \( \min_{x \in T_i} \bar{p}x < 0 \leq b_j(p) \) for all \( i \in I \). The fact that \( \bar{x}_i \in E_j(\bar{p}) \) implies that \( \bar{x}_i \) maximizes \( R_i \) on \( T_i \cap \{x \mid \bar{p}x \leq b_j(p)\} \) for all \( i \in I \).

Part (a) of Theorem 1 will follow if it can be shown that at \( \bar{x} \), preferences of slaves are also maximized subject to \( \bar{p}x_j \leq b_j(p) \).

If for \( j \in J \), \( \min_{x \in T_j} \bar{p}x < b_j(\bar{p}) \), it is immediate that \( \bar{x}_j \) maximizes \( R_j \) on \( T_j \cap \{x \mid \bar{p}x \leq b_j(\bar{p})\} \).

If \( \min_{x \in T_j} \bar{p}x \geq b_j(\bar{p}) \), it will be shown that \( \bar{x}_j \in \text{Boundary } T_j \).

Since \( T_j^* \cap -S \neq \emptyset \), \( b_j(\bar{p}) \equiv \min_{x \in T_j} (\bar{p}x) < 0 \). Since \( \bar{x}_j \in E_j(\bar{p}) \), \( \bar{p}\bar{x}_j = b_j(\bar{p}) \), and \( \bar{x}_j \neq 0 \).

Suppose \( \bar{x}_j \in T_j \). Since \( \bar{x}_j \in T_j \), \( \lambda \bar{x}_j \in T_j \) for some \( \lambda > 1 \). But \( \bar{p}\bar{x}_j = b_j(\bar{p}) < 0 \). Therefore, \( \bar{p}(\lambda \bar{x}_j) < b_j(\bar{p}) \). Hence \( \min_{x \in T_j} \bar{p}x < b_j(\bar{p}) \). It follows that if \( \min_{x \in T_j} \bar{p}x \geq b_j(\bar{p}) \), then \( \bar{x}_j \in T_j \) and that \( \bar{x}_j \in \text{Boundary } T_j \).

According to Assumption 7, if \( x_j P_i \bar{x}_j \), then \( x_j = \bar{x}_j + s \) for some \( s \in S \). But \( \bar{p}s > 0 \). Therefore, \( x_j P_i \bar{x}_j \), then \( \bar{p}x_j > \bar{p}\bar{x}_j \).

It follows that for all \( j \in J \), \( \bar{x}_j \) maximizes \( R_j \) on \( T_j \cap \{x \mid \bar{p}x \leq b_j(\bar{p})\} \). This proves part (a) of Theorem 1.

It was shown, using only assumptions 1-6, that for all \( i \in I \), \( \bar{x}_i \) maximizes \( R_i \) on \( T_i \cap \{x \mid \bar{p}x \leq b_j(\bar{p})\} \), and that for all \( j \in J \), \( \bar{p}\bar{x}_j = \min_{x \in T_j} \bar{p}x \) and \( \bar{x}_j \in T_j \). If for all \( j \in J \), \( T_j = T_j^* \), then \( \bar{p}\bar{x}_j = \min_{x \in T_j} \bar{p}x \) and \( \bar{x}_j \in T_j \). This proves part (b) of Theorem 1.

Q.E.D.

B. PROOF OF THEOREM 4

Let \( \bar{P}_f(\bar{x}) = \{x_i \mid x_i \in T_i \} \) and \( x_i R_i \bar{x}_i \) for all \( i \in I \) and \( x_i P_i \bar{x}_i \) for some \( i \in I \). If \( (\bar{x}, \bar{y}) \) is \( I \) optimal, then \( \bar{P}_f(\bar{x}) \cap \{\sum_{i} T_i - \sum_{j} Y_j\} = \emptyset \). Since preferences are convex, \( \bar{P}_f(\bar{x}) \) is convex. Also \( \{\sum_{k} Y_k - \sum_{j} T_j\} \) is the sum of the convex sets and is therefore convex.

According to Minkowski's theorem, there exists a vector, \( \bar{p} \), such that \( \sum_{i} x_i \in \bar{P}_f(\bar{x}) \) and implies that \( \bar{p} \sum_{i} x_i \leq \bar{p} \left( \sum_{k} y_k - \sum_{j} T_j \right) \). Since \( \sum_{i} \bar{x}_i = \sum_{k} \bar{y}_k - \sum_{j} \bar{x}_j \), it follows that at prices \( \bar{p}, \bar{y}_k \) is a profit maximizing output for \( k \in K \) and \( x_j \) maximizes \( -\bar{p}x_j \) on the set \( T_j^* \) (by assumption, \( T_j = T_j^* \)).

Lemma 4, above, can be employed to show that for all \( s \in S \), \( \bar{p}s > 0 \). Since, by assumption, for all \( i \in I \), \( \bar{x}_i - s_i \in T_i \) for some \( s_i \in S \) and since \( \bar{p}(\bar{x}_i - s_i) = \bar{p}\bar{x}_i - \bar{p}s_i < \bar{p}\bar{x}_i \), \( \bar{x}_i \) is not a cost minimizing trade in \( T_i \). Therefore for all \( i \in I \), \( x_i P_i \bar{x}_i \) implies \( \bar{p}x_i > \bar{p}\bar{x}_i \). (See Debreu [4]).

Collecting results, we have:

1. \( y \in Y_j \) implies \( \bar{y}y \leq \bar{p}\bar{y}j \);

2. For all \( i \in I \), \( \bar{x}_i \) maximizes \( R_i \) on \( \{x \mid \bar{p}x \leq \bar{p}\bar{x}_i\} \),

   for \( j \in J \), \( \bar{p}\bar{x}_j = \min_{x \in T_j} \bar{p}x < 0 \).

Since \( (\bar{x}, \bar{y}) \) is \( I \) optimal,

3. \( \sum_{M} \bar{x}_m = \sum_{k} \bar{y}_k \).
If all members of \( I \) are given budgets, \( b_i(\bar{p}) = \bar{p} \bar{x}_i \), then according to (1), (2), and (3), \((\bar{p}, \bar{x}, \bar{y})\) is a slave exploitation equilibrium. Since \( \bar{p} \sum_{i} \bar{x}_i = \bar{p} \sum_{k} \bar{y}_k - \bar{p} \sum_{j} \bar{x}_j \), one can assign claims on profits and the present value of slaves in such a way that \( b_i(\bar{p}) = \bar{p} \bar{x}_i \) and \( \sum_{T} b_i(\bar{p}) = \bar{p} \sum_{T} \bar{x}_i = \bar{p} \sum_{k} \bar{y}_k - \bar{p} \sum_{j} \bar{x}_j \). This establishes Theorem 4.1

REFERENCES


1 It might be useful to change the definition of I optimal to allow the possibility that members of J be allowed to starve. If this is the case, one would replace \( T_j \) by \( T_j \cup \{0\} \) everywhere in the definition of I optimality. If \( (0) \notin X_1 \) then these sets are non-convex. With the assumption that \( -S \cap T_j \neq \emptyset \), however, it is easy to show that if \((\bar{x}, \bar{y})\) is I optimal, then for all \( j \in J, \bar{x}_j \neq 0 \). The same hyperplane \( p \) used in Theorem 4 will still sustain \((\bar{x}, \bar{y})\) as a slave exploitation equilibrium.