TIME PREFERENCE AND THE "EQUITY PREMIUM PUZZLE"

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ABSTRACT

We re-examine the Mehra and Prescott (1985) model. Allowing the time preference factor to be greater than one resolves the "equity premium puzzle." We show that this solution is consistent with finite expected utility and a positive risk-free rate of interest. For somewhat higher values of consumption variance, it is possible to get time preference factors less than one.

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I. INTRODUCTION

In a recent and influential paper, Mehra and Prescott (1985) show that a class of equilibrium models commonly used in the theoretical and empirical literature cannot replicate the historic (1889-1978) averages of certain relevant financial market and consumption variables for the U. S. economy. In this paper we reconsider the Mehra and Prescott "equity premium puzzle." We show that when the time preference factor is allowed to be greater than unity, the simple representative consumer model with a time-additive, constant relative risk aversion (RRA) utility function does replicate these historic averages. The usual intuition about the time preference factor is that it should be less than unity, both to give positive real interest rates and to ensure that expected utility is finite. We show, however, that in Mehra-Prescott-type models both of these conditions can hold for time preference factors greater than one.

Mehra and Prescott simulate the equilibrium of a binomial model with a single, representative consumer. The model is a variation of Lucas's (1978) pure exchange asset pricing model. Endowments follow a Markov process. Mehra and Prescott attempt to match the average values of 6 variables (roughly matching historical values):
average growth in consumption: 1.83%
standard deviation of the growth rate of per capita consumption: 3.57%
first-order serial correlation of the growth rate of per capital consumption: -0.14
risk-free rate of interest: 0.80%
average real return on the stock market: 6.98%
standard deviation of stock market return: 16.54%

The puzzle which Mehra and Prescott report is that they cannot match these six values in their simulations for reasonable values of the representative consumer's relative risk aversion.

Several explanations for the Mehra-Prescott conundrum have been offered. Rietz (1988) relies on the existence of other, low-probability and unobserved, states of nature, although Mehra and Prescott (1988) discount this explanation. Constantinides (1988) shows that an equilibrium system with a time-dependent utility function can match the historic averages of the economic variables in question. Abel (1988) shows that agents' heterogeneous beliefs can help explain the puzzle.¹

In our view the importance of the "equity premium puzzle" that Mehra and Prescott posed is whether it makes any sense to use this type of model to try to understand asset pricing in actual economies. If sensible parameter values produce data that are very far away from what we observe, then this type of model is unlikely to help us understand the mechanism of asset pricing. We show that for empirically reasonable values of τ and values of

¹. Two other papers on this topic are Mankiw (1986) and Cecchetti, Lam, and Mark (1988).
δ that do not violate the ordinary concepts of equilibrium, this model can reproduce the historic averages of the six variables.

The explanation we offer is not only very simple, but it is attractive on scientific grounds, because it shows that a very minor and simple modification to a well-understood and widely-used general equilibrium model renders it capable of replicating the U. S. historic averages of the six variables.

In the following section we introduce the model and derive the analytical results. In Section III we show some simulation results, and in Section IV we discuss their economic significance, especially the meaning of the discount factor greater than unity. We show that the actual size of the discount factor is sensitive to the specification of the target parameter for consumption variance. A concluding section follows.

II. MODEL SPECIFICATION AND SOLUTION

In this section we derive the dynamic path of the model, and we solve for the values of the six variables that Mehra and Prescott try to match in their paper.

II.a. Model specification

We use a binomial, state-preference, infinite-horizon model, with a single representative consumer and a single consumption commodity. Uncertainty in this model is generated by the random consumption endowments. The representative consumer maximizes
the expected discounted utility of lifetime consumption. Thus the consumer's expected utility of a lifetime consumption stream $c$ is given by:

$$
(1) \quad EU(c) = \sum_{t=0}^{\infty} \delta^t EU(c_t).
$$

We follow Mehra and Prescott and assume that the utility function is exponential:

$$
(2) \quad U(x) = \frac{x^{1-\tau}}{1-\tau}, \quad \tau \neq 1, \quad U(x) = \log(x), \quad \tau = 1.
$$

Call the current state (a neutral starting point) 0. At date 1 consumption growth plus 1 is either $\alpha$ or $\beta$, with equal probabilities. The successor states to date 1 are (see Figure 1):

If date 1 growth is $\alpha$:
- growth of $\alpha$ with probability $\pi$
- growth of $\beta$ with probability $(1-\pi)$.

If date 1 growth is $\beta$:
- growth of $\alpha$ with probability $(1-\pi)$
- growth of $\beta$ with probability $\pi$.

Arbitrarily we assume that $\alpha < \beta$, i.e., $\alpha$ is the less desirable, "bad," state. It is easy to show that this transition probability scheme allows for any first-order correlation of consumption growth. In particular, consumption is positively
correlated if $\pi > 0.5$, and it is negatively correlated if $\pi < 0.5$. Since the historic averages cited by Mehra and Prescott show a small negative correlation, we shall find, when we simulate this correlation, that $\pi < 0.5$. Thus if the economy is in a state whose growth was $\alpha$ (a bad state), then the probability of a good state is higher than that of another bad state, and vice versa.

At date 1, the first-order conditions for this problem give the state prices:

If date 1 is an $\alpha$ state,

\[
\frac{\delta \pi U'(c_{\alpha\alpha})}{U'(c_\alpha)} = \frac{\delta \pi}{\alpha} = q_{\alpha\alpha},
\]

\[
\frac{\delta (1-\pi)U'(c_{\alpha\beta})}{U'(c_\alpha)} = \frac{\delta (1-\pi)}{\beta} = q_{\alpha\beta}.
\]

If date 1 is a $\beta$ state,

\[
\frac{\delta (1-\pi)U'(c_{\beta\alpha})}{U'(c_\beta)} = \frac{\delta (1-\pi)}{\alpha} = q_{\beta\alpha},
\]

\[
\frac{\delta (1-\pi)U'(c_{\beta\beta})}{U'(c_\beta)} = \frac{\delta \pi}{\beta} = q_{\beta\beta}.
\]

where $c_{\alpha\alpha}$ and $c_{\alpha\beta}$ are the consumptions in $\alpha$ and $\beta$ states respectively that follow an $\alpha$-state. Similarly $c_{\beta\alpha}$ and $c_{\beta\beta}$ are the consumptions in $\alpha$- and $\beta$-states that follow a $\beta$-state. The $q_{ij}$, $i,j=\alpha,\beta$, are the four possible one-period state prices.
II.b. Calculation of the equilibrium

In this sub-section we calculate the equilibrium values of the variables we wish to simulate. Since the consumption pattern is stationary, we can characterize completely all the state variables according to whether the economy is in an $\alpha$-state or a $\beta$-state.

II.b.1. Growth rate, standard deviation, and autocorrelation of consumption

The growth rate of consumption is a function only of $\alpha$, $\beta$, and $\pi$. The consumption growth rate is given by,

\begin{align}
(5a) & \quad \pi\alpha + (1-\pi)\beta - 1 \text{ starting from an } \alpha \text{ state}, \\
(5b) & \quad (1-\pi)\alpha + \pi\beta - 1 \text{ starting from a } \beta \text{ state}.
\end{align}

Thus the relevant average consumption growth rate is

\begin{equation}
(6) \quad g_C = (\alpha+\beta)/2 - 1.
\end{equation}

The variance of consumption growth, starting from an $\alpha$-state, is given by:

\begin{equation}
(7a) \quad \sigma_C^2(\alpha) = \pi\alpha^2 + (1-\pi)\beta^2 - [\pi\alpha + (1-\pi)\beta]^2 = \pi(1-\pi)(\alpha - \beta)^2,
\end{equation}

and starting from a $\beta$-state,

\begin{equation}
(7b) \quad \sigma_C^2(\beta) = (1-\pi)\alpha^2 + \pi\beta^2 - [(1-\pi)\alpha + \pi\beta]^2 = \pi(1-\pi)(\alpha - \beta)^2.
\end{equation}

Thus the average standard deviation of consumption growth is,

\begin{equation}
(8) \quad \sigma_C = (\beta-\alpha)[\pi(1-\pi)]^{1/2}.
\end{equation}
The autocorrelation of consumption can be calculated as follows: Suppose we start from a neutral state. With probability 0.5 consumption grows by $\alpha - 1$, followed by growth of $\alpha - 1$ with probability $\pi$ and growth of $\beta - 1$ with probability $1 - \pi$. With probability 0.5 consumption grows by $\beta - 1$, followed by growth of $\alpha - 1$ with probability $1 - \pi$ and growth of $\beta - 1$ with probability $\pi$.

The average date 1 consumption growth is $(\alpha + \beta)/2 - 1$, and the average date 2 consumption growth is 

$$[\pi \alpha + (1-\pi) \beta + \pi \beta + (1-\pi) \alpha]/2 - 1 = (\alpha + \beta)/2 - 1.$$ 

Therefore, the auto-covariance of consumption growth is

$$\text{covariance} = \frac{(2\pi - 1)(\alpha - \beta)^2}{4}.$$ 

The variance of consumption growth at date 1 and date 2 is $\pi(1-\pi)(\alpha - \beta)^2$. It follows that the correlation coefficient of consumption growth is

$$\text{(9) \quad \text{serial correlation} = \frac{-(1-2\pi)}{4\pi(1-\pi)} < 0, \text{ for } \pi < 0.5.}$$

II.b.2. The risk-free rate

The risk-free rate is a function of $\alpha$, $\beta$, $\pi$, $\delta$, and $\tau$. In an $\alpha$-state at date 1 this rate is given by

$$\text{(10a) \quad \frac{1}{1+r_{f\alpha}} = q_{\alpha \alpha} + q_{\alpha \beta} = \frac{\delta \pi}{\alpha \tau} + \frac{\delta(1-\pi)}{\beta \tau}},$$

or
\[(10b) \quad r_f\alpha = \frac{(\alpha \beta)^T}{\delta[\pi \beta^T + (1-\pi)\alpha^T]} - 1,\]

and in a \(\beta\)-state at date 1:

\[(10c) \quad r_f\beta = \frac{(\alpha \beta)^T}{\delta[(1-\pi)\beta^T + \pi \alpha^T]} - 1.\]

II.b.3. The value of the market

First we derive the unlevered market value. It is easy to calculate the market value for any leverage from this unlevered market value.

A market share gives the holder the right to a proportion of all uncertain future endowments, since this is an endowment economy. The value of the market then is just the present value of all future consumption.

At date 1, let the value of the market be \(M_\alpha\) if the economy is in the \(\alpha\)-state, and \(M_\beta\) if the economy is in the \(\beta\)-state. Starting from the \(\alpha\)-state, suppose the economy arrives at the next \(\alpha\)-state, the \(\alpha\alpha\)-state. All the possible consumption outcomes from the \(\alpha\alpha\)-state are identical to those from the \(\alpha\)-state, except that they are all multiplied by the growth relative \(\alpha\). Similarly, if the economy arrives at the \(\beta\)-state instead, the \(\alpha\beta\)-state, all possible consumption outcomes from the \(\alpha\beta\)-state are identical to those from the \(\beta\)-state, except that they are all multiplied by the growth relative \(\alpha\) (see Figure 1). The argument
is symmetric starting from the \( \beta \)-state, and it is a direct
consequence of the stationarity of the transition probabilities
and constancy of the consumption growth rates. This argument
implies,

\[
M_{\alpha \alpha} = \alpha M_{\alpha}, \quad M_{\alpha \beta} = \alpha M_{\beta}, \quad M_{\beta \alpha} = \beta M_{\alpha}, \quad M_{\beta \beta} = \beta M_{\beta}.
\]

The dividends in this endowment model are the consumptions
in each state. Given the constant growth structure, if
consumption in the initial state is \( c_0 \), consumption in the
following \( \alpha \)- and \( \beta \)-states is \( c_\alpha = \alpha c_0 \), \( c_\beta = \beta c_0 \). By the same
logic,

\[
c_{\alpha \alpha} = \alpha^2 c_0, \quad c_{\alpha \beta} = \alpha \beta c_0, \quad c_{\beta \alpha} = \beta \alpha c_0, \quad c_{\beta \beta} = \beta^2 c_0.
\]

The ex-dividend market value at date 1 in the \( \alpha \)- or the \( \beta \)-
state is the present value of next period's market price plus the
present value of next period's dividends. This gives the
difference equations,

\[
M_\alpha = q_{\alpha \alpha} [M_{\alpha \alpha} + c_{\alpha \alpha}] + q_{\alpha \beta} [M_{\alpha \beta} + c_{\alpha \beta}]
\]

\[
M_\beta = q_{\beta \alpha} [M_{\beta \alpha} + c_{\beta \alpha}] + q_{\beta \beta} [M_{\beta \beta} + c_{\beta \beta}].
\]

Substituting equations (11) and (12) reduces (13a) and (13b) to a
pair of simple equations in \( M_\alpha \) and \( M_\beta \), the market values at the
date-1 \( \alpha \)- and \( \beta \)-states:

\[
M_\alpha = q_{\alpha \alpha} [\alpha M_\alpha + \alpha^2 c_0] + q_{\alpha \beta} [\alpha M_\beta + \alpha \beta c_0]
\]

\[
M_\beta = q_{\beta \alpha} [\beta M_\alpha + \beta \alpha c_0] + q_{\beta \beta} [\beta M_\beta + \beta^2 c_0].
\]

We substitute the values for the \( q_{ij} \)'s from equations (3) and
(4) and solve this system to get
(15a) \[ M_\alpha = \frac{x_{13}x_{22} - x_{23}x_{12}}{x_{11}x_{22} - x_{21}x_{12}} \]

(15b) \[ M_\beta = \frac{x_{11}x_{23} - x_{21}x_{13}}{x_{11}x_{22} - x_{21}x_{12}} \]

where,

\[
x_{11} = 1 - \frac{\delta \pi}{\alpha^{\tau-1}}, \quad x_{12} = -\frac{\delta (1-\pi) \alpha}{\beta^\tau}, \quad x_{13} = \frac{\delta \pi}{\alpha^{\tau-2}} + \frac{\delta (1-\pi) \alpha}{\beta^{\tau-1}}
\]

\[
x_{21} = -\frac{\delta (1-\pi) \beta}{\alpha^\tau}, \quad x_{22} = 1 - \frac{\delta \pi}{\beta^{\tau-1}}, \quad x_{23} = \frac{\delta \pi}{\beta^{\tau-2}} + \frac{\delta (1-\pi) \beta}{\alpha^{\tau-1}}.
\]

The solution is given by,

(16a) \[ M_\alpha = \frac{\alpha \delta c_0}{\Gamma} \{\pi \beta^{\tau-1} + (1-\pi) \alpha^{\tau-1} + \delta (1-2\pi)\} \]

(16b) \[ M_\beta = \frac{\beta \delta c_0}{\Gamma} \{(1-\pi) \beta^{\tau-1} + \pi \alpha^{\tau-1} + \delta (1-2\pi)\}, \]

where

(17) \[ \Gamma = 1 - \delta \pi (\alpha^{1-\tau} + \beta^{1-\tau}) - \delta^2 (1-2\pi) (\alpha \beta)^{1-\tau}. \]

For equilibrium to exist, the market value must be finite and positive. For \( \pi < 1/2, 0 < M_\alpha, M_\beta < \infty \) if and only if \( \Gamma > 0 \). Thus a necessary condition for the existence of equilibrium in this model is \( \Gamma > 0 \).

Adjusting the market value and the market returns for leverage is a relatively simple matter. Suppose proportion \( L \) of
the market value is financed with debt. Then the value of equity in the \( \alpha \)-state is given by \( M_\alpha (1-L) \), and the value of equity in the \( \alpha \alpha \)-state is given by \( aM_\alpha + a^2 c_0 - LM_\alpha (1+r_{f\alpha}) \). The value of equity in the \( \alpha \beta \)-state is given by \( aM_\beta + \alpha \beta c_0 - LM_\alpha (1+r_{f\alpha}) \).

This gives the returns to levered equity in states \( \alpha \alpha \) and \( \alpha \beta \) as

\[
R_{\alpha \beta} = \frac{aM_\alpha + a^2 c_0 - LM_\alpha (1+r_{f\alpha})}{M_\alpha (1-L)},
\]

\[
R_{\alpha \beta} = \frac{aM_\beta + \alpha \beta c_0 - LM_\alpha (1+r_{f\alpha})}{M_\alpha (1-L)}.
\]

Similar calculations show that in states \( \beta \alpha \) and \( \beta \beta \), the returns to levered equity will be:

\[
R_{\beta \alpha} = \frac{BM_\alpha + \beta a c_0 - LM_\beta (1+r_{f\beta})}{M_\beta (1-L)},
\]

\[
R_{\beta \beta} = \frac{BM_\beta + \beta^2 c_0 - LM_\beta (1+r_{f\beta})}{M_\beta (1-L)}.
\]

The expected return and the standard deviation of levered equity returns are calculated from these relations.

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2. The risk-free interest rate is appropriate as long as the leveraged firm does not default. The results reported in subsequent sections are all for simulations in which bankruptcy does not occur.
II.b.4. **Lifetime expected utility**

In order to insure that the model has an economically meaningful solution, we must determine the conditions under which expected lifetime utility is finite and well-behaved. The argument we presented for the market valuation applies to expected utility as well, because of the stationary nature of the model. At date 1, let expected utility (after current consumption) be $EU_\alpha$ if the economy is in the $\alpha$-state, and $EU_\beta$ if the economy is in the $\beta$-state. Recall that all the possible consumption outcomes from the $\alpha\alpha$-state are identical to those from the $\alpha$-state, but multiplied by the growth relative $\alpha$. Similarly, all possible consumption outcomes from the other possible state, the $\alpha\beta$-state, are identical to those from the $\beta$-state, but multiplied by the growth relative $\alpha$. A similar argument holds starting from the date 1 $\beta$-state. Given the functional form of the utility function, this argument implies,

$$EU_{\alpha\alpha} = \alpha^{1-\tau}EU_\alpha, \; EU_{\alpha\beta} = \alpha^{1-\tau}EU_\beta,$$
$$EU_{\beta\alpha} = \beta^{1-\tau}EU_\alpha, \; EU_{\beta\beta} = \beta^{1-\tau}EU_\beta.$$

Expected utility at date 1 in the $\alpha$- or the $\beta$-state is next period's expected utilities discounted by the probability-adjusted subjective discount factor $\delta$, plus the utility of consumption in the two states, also discounted by the probability adjusted $\delta$. This gives the difference equations,

$$EU_\alpha = \pi\delta(EU_{\alpha\alpha} + U(c_{\alpha\alpha})) + (1-\pi)\delta(EU_{\alpha\beta} + U(c_{\alpha\beta}))$$
$$EU_\beta = (1-\pi)\delta(EU_{\beta\alpha} + U(c_{\beta\alpha})) + \pi\delta(EU_{\beta\beta} + U(c_{\beta\beta})).$$
Substituting equations (19) and (12) into (20) reduces these to a pair of simple equations in \( EU_\alpha \) and \( EU_\beta \), the expected utilities at the date-1 \( \alpha \)- and \( \beta \)-states:

\[
(21a) \quad (1-\pi \delta \alpha^{1-\tau}) EU_\alpha - (1-\pi) \delta \alpha^{1-\tau} EU_\beta = \frac{\delta [\alpha C_0]^{1-\tau}}{1-\tau} \left[ \pi \alpha^{1-\tau} + (1-\pi) \beta^{1-\tau} \right],
\]

\[
(21b) \quad -(1-\pi) \delta \beta^{1-\tau} EU_\alpha - (1-\pi \delta \beta^{1-\tau}) EU_\beta = \frac{\delta [\beta C_0]^{1-\tau}}{1-\tau} \left[ \pi \alpha^{1-\tau} + \pi \beta^{1-\tau} \right].
\]

The solution to this system of equations is given by:

\[
(22a) \quad EU_\alpha = \frac{\delta [\alpha \beta C_0]^{1-\tau}}{1-\tau} \left\{ 1 - \pi [1-(\alpha/\beta)^{1-\tau}] + \delta (1-2\pi) \alpha^{1-\tau} \right\},
\]

\[
(22b) \quad EU_\beta = \frac{\delta [\alpha \beta C_0]^{1-\tau}}{1-\tau} \left\{ 1 - \pi [1-(\beta/\alpha)^{1-\tau}] + \delta (1-2\pi) \beta^{1-\tau} \right\},
\]

where,

\[
(23) \quad \phi = 1 - \pi (\beta^{1-\tau} + \alpha^{1-\tau}) - \delta^2 (1-2\pi) (\alpha \beta)^{1-\tau}.
\]

Expected lifetime utility must have the same sign as instantaneous utility. Therefore,

\[
(24) \quad \infty > \frac{(1 - \pi [1-(\beta/\alpha)^{1-\tau}] + \delta (1-2\pi) \beta^{1-\tau})}{\phi} > 0.
\]
For \( \pi < 1/2 \), the numerator of (24) is positive. Therefore expected utility is defined only when \( \Phi > 0 \).

It is interesting to explore the connection between the necessary condition for the existence of a market price \( \Gamma > 0 \), and the necessary condition for the existence of lifetime expected utility, \( \Phi > 0 \). \( \Phi - \Gamma = \pi(\delta-1)(\alpha^{1-\gamma} + \beta^{1-\gamma}) \). This implies that \( \Phi > \Gamma \) if and only if \( \delta > 1 \). Thus, if \( \delta < 1, \Phi > 0 \) is the binding condition for the existence of equilibrium, and if \( \delta > 1, \Gamma > 0 \) is the binding condition.

III. Simulation results

Our simulation strategy is as follows: We take \( r \) as given. Using the Mehra and Prescott target values, we calculate the remaining five model parameters—\( \alpha, \beta, \pi, \delta \), and \( L \)—as follows:

1. The state probability \( \pi \) is set to achieve the desired serial correlation of the growth rate. For the Mehra-Prescott target of serial correlation of -0.14, this gives \( \pi \approx 0.43 \).

2. The consumption growth and standard deviation determine \( \alpha \) and \( \beta \). For \( g_C = 1.83\% \) and \( \sigma_C = 3.57\% \), this gives \( \alpha \approx 0.98 \), and \( \beta \approx 1.05 \).

3. The \( \delta \) is determined by the average risk-free rate.

4. The leverage ratio \( L \) is set to give the expected return on the market.

These parameters then give us the standard deviation of the levered market returns, \( \sigma_M \).
Table 1 gives simulation results that match the Mehra-Prescott targets of \( E(R_M) = 6.98\% \), risk-free rate = 0.8\%, expected growth in consumption = 1.83\%, and standard deviation of consumption growth of 3.57\%. The 1889-1978 average of the standard deviation of stock market returns 16.54\%, and it is to this value that the simulation results should be compared.

Table 1 shows that our model produces equilibria that match the U. S. historic average values of the 6 key variables targeted by Mehra and Prescott, with empirically accepted ranges of values for leverage (L) and relative risk aversion (\( \tau \)).

An interesting feature of the results is that as the relative risk aversion \( \tau \) increases, the leverage ratio L needed to achieve the target results decreases, and the resulting standard deviation of the market return decreases. When the standard deviation of the market return is in the range indicated by Mehra and Prescott (i.e. 15-20\%), the corresponding leverage ratios of 45-65\% seem to accord reasonably well with market data.

The reason for the decrease in the leverage is that, when the relative risk aversion increases, the consumer's risk premium for the unlevered market return increases. Thus the amount of leverage needed to bring the levered market return up to the target of 16.54\% decreases with increasing \( \tau \).

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3. Masulis (1988), for example, states that the annual average debt to total asset ratio of all U. S. corporations between 1937 and 1984 ranged between 53\% and 75\% (these figures are for market values).
IV. ANALYSIS OF THE RESULTS

It is clear from Table 1 that it is not hard to find equilibria that meet the target values of the six variables singled out by Mehra and Prescott, while keeping the leverage ratio (L) and the RRA (r) within their empirical consensus ranges. There are two interesting features of these equilibria that need to be discussed. One is the value of the time discount factor (\( \delta \)). The other is the sensitivity of the results to the target values.

IV.a. The importance of \( \delta > 1 \)

A feature of our results is that \( \delta > 1 \). This may be disconcerting and counter-intuitive to many economists. The common intuition and indeed an almost universal assumption in general equilibrium models is that \( \delta < 1 \). This intuition comes primarily from two sources. First, since \( \delta \) is the discount factor of future expected utility, \( \delta < 1 \) is necessary to insure that the discounted value of the utility of many plausible consumption paths (e.g., constant consumption) is finite for general utility functions. Second, in the steady-state of infinite horizon certainty models, \( \delta \) is the inverse of one plus
the risk-free rate. Since most economists believe that the long run risk-free real interest rate is positive, or equivalently, that the marginal product of capital is positive, it is natural to think of the time preference factor as smaller than unity.

At the purely technical level, the condition for a well-defined lifetime expected utility is $\Phi > 0$ in our model. This condition, as well as the existence condition for market value, $\Gamma > 0$, are not violated by $\delta > 1$, at least for some values of $\alpha$, $\beta$, and $\pi$. Indeed, we do not report any results in Table 1 that violate these conditions. Nonetheless, it is important to explore why $\delta > 1$ is consistent with equilibrium, and why the successful simulations require such high values of $\delta$.

It is possible for lifetime expected utility to be finite for $\delta > 1$ because of the peculiar nature of the constant RRA utility function. For $\tau > 1$, utility is negative. As consumption rises, its utility rises as well, meaning that the utility can get arbitrarily close to zero. This implies that if utility is rising towards zero fast enough it will overcome the exponential increase in the subjective discounting factor, and the sum of the utilities will be finite.

The intuitive reason that many equilibria have large $\delta$'s follows from the first-order conditions for the risk-free rate. For example, starting from an initial state $c_0$ in which consumption growth was $\alpha$, we have:

$$\frac{1}{1+r_f} = \frac{\delta \pi U'(c_\alpha)}{U'(c_0)} + \frac{\delta (1-\pi) U'(c_\beta)}{U'(c_0)} = \delta [\pi \alpha^{-\tau} + (1-\pi) \beta^{-\tau}].$$
Only consumption growth and its variance determine $\alpha$ and $\beta$, and the autocorrelation of consumption fixes $\pi$. The target values for these variables produce—using the Mehra and Prescott historical averages—parameter values of $\alpha \approx 0.98$ and $\beta \approx 1.05$, and $\pi \approx 0.43$. For these parameter values of $\alpha$, $\beta$, and $\pi$, the expression $[\pi \alpha^{-\gamma} + (1-\pi)\beta^{-\gamma}]$ is an everywhere decreasing function of $\gamma$, and it is less than unity for the range of $\gamma$s for which a solution exists. The low interest rate then forces $\delta > 1$.\(^4\)

The appropriate way to think about $\delta$ is that $\gamma$ and $\delta$ are the only two parameters for this utility function and we have to use both of them to match the average values of the six variables. There is some empirical evidence on the value of RRA which acts as a constraint on the simulation values of $\gamma$. However, there is no comparable direct evidence on $\delta$. The evidence in favor of positive discounting of future consumption is circumstantial. Furthermore, the values we get for $\delta$ are specific to this particular utility function and the target values for the simulation, and they need not generalize to other utility functions or to different target values.

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\(^4\) If we raise the target standard deviation of consumption growth, $[\pi \alpha^{-\gamma} + (1-\pi)\beta^{-\gamma}]$ may be parabolic in shape, resulting in a different pattern of $\delta$s. See the next subsection for examples.
IV.b. Sensitivity of the results to the target values

In assessing the value of using this type of general equilibrium model, it is important to understand how the model parameters adjust to changes of the target values. This is particularly important because the target values of the six variables are measured with different degrees of precision. The stock market and the interest rate data are not subject to serious measurement errors. By contrast, the variance and the autocorrelation of consumption are difficult to measure, and their target values may be very far from the values that would correspond to the theoretical constructs of the model.

In Table 2 we show the effect of increasing the target value of the standard deviation of consumption. For higher target standard deviation of consumption growth, $\delta$ declines with increasing $\tau$, unlike the case for the Mehra and Prescott target values. As can be seen from the table, even moderate increases in the estimates of consumption standard deviation (increases well within the realm of plausibility), lead to $\delta < 1$. However, it does not appear that plausible leverage ratios result from the simulations for these cases.

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Insert Table 2
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V. Conclusion

In this paper we have re-examined the model simulated by Mehra and Prescott (1985). In that paper, Mehra and Prescott reported that their model, an adaptation of Lucas's (1978) asset pricing model, was unable to replicate the historical averages of six major economic variables: return and standard deviation of the equity market, consumption growth and standard deviation, negative autocorrelation of consumption growth, and the risk-free rate.

Our major finding is that allowing the pure time discount factor $\delta$ to exceed unity enables us to replicate all of the historical averages. Furthermore we show that--while the pure time discount factor is usually assumed to be less than one--there is nothing in the Mehra and Prescott model which prevents it from being greater than one: A pure time discount factor greater than one is consistent with finite discounted expected utility, a positive market value, and a positive risk-free rate of interest.

Finally, we show that $\delta > 1$ is not an inevitable outcome of the model. Rather, $\delta > 1$ results from the large difference between the simulation targets for consumption and market variance. If this difference is narrowed somewhat, we can obtain simulations in which $\delta < 1$, although in these simulations the market leverage ratio appears to be too low.

The importance of the Mehra and Prescott paper is that if the Mehra and Prescott results are correct and general, then one
must doubt the explanatory power of a standard economic paradigm of equilibrium. In this paper we have shown that the Mehra and Prescott results are of more limited generality than at first thought, and to this extent, one may view our paper as a vindication of a powerful theoretical model.
REFERENCES

Abel, Andrew (1988). "Asset Prices under Heterogeneous Beliefs: Towards the Resolution of the Equity Premium Puzzle," mimeo, The Wharton School of the University of Pennsylvania and NBER.


TABLE 1
SIMULATION RESULTS

This table reports simulations that fit the historic values given by Mehra and Prescott (1985). The table shows the relation between $\tau$, $L$, $\delta$, and the standard deviation of the market return. Every reported simulation is characterized by expected market return $E(R_M) = 6.98\%$, a risk-free rate of return of $0.8\%$, and expected consumption growth of $1.83\%$, and standard deviation of consumption growth of $3.57\%$. These consumption parameters combine to give $\alpha = 0.9822$ and $\beta = 1.0544$.

As $\tau$ increases, the leverage ratio needed to match the expected return on the market declines. For $\tau < 6$, the model does not converge, and for $\tau > 18$, no positive leverage ratio matches the expected return on the market. Hence only results are reported for which $6 \leq \tau \leq 18$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\delta$</th>
<th>Leverage</th>
<th>Market $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1.0784</td>
<td>0.8068</td>
<td>0.2976</td>
</tr>
<tr>
<td>7</td>
<td>1.0891</td>
<td>0.7585</td>
<td>0.2563</td>
</tr>
<tr>
<td>8</td>
<td>1.0985</td>
<td>0.7062</td>
<td>0.2255</td>
</tr>
<tr>
<td>9</td>
<td>1.1068</td>
<td>0.6501</td>
<td>0.2017</td>
</tr>
<tr>
<td>10</td>
<td>1.1139</td>
<td>0.5904</td>
<td>0.1828</td>
</tr>
<tr>
<td>11</td>
<td>1.1199</td>
<td>0.5275</td>
<td>0.1674</td>
</tr>
<tr>
<td>12</td>
<td>1.1247</td>
<td>0.4617</td>
<td>0.1547</td>
</tr>
<tr>
<td>13</td>
<td>1.1284</td>
<td>0.3933</td>
<td>0.1440</td>
</tr>
<tr>
<td>14</td>
<td>1.1309</td>
<td>0.3226</td>
<td>0.1350</td>
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<tr>
<td>15</td>
<td>1.1324</td>
<td>0.2501</td>
<td>0.1272</td>
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<tr>
<td>16</td>
<td>1.1327</td>
<td>0.1760</td>
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</tr>
<tr>
<td>17</td>
<td>1.1321</td>
<td>0.1008</td>
<td>0.1147</td>
</tr>
<tr>
<td>18</td>
<td>1.1304</td>
<td>0.0246</td>
<td>0.1095</td>
</tr>
</tbody>
</table>
Table 2

Panel A

In this panel we simulate the Mehra and Prescott model, setting the standard deviation of consumption growth equal to 6%. The model converges for $9 \geq \tau \geq 3$. Note that, compared to Table 1, values of the pure time preference factor $\delta$ are smaller given the same values of the RRA $\tau$. Furthermore, note that $\delta$ is a declining function of $\tau$ for these target values. Given consumption growth of 1.83% and standard deviation of consumption growth of 6%, $\alpha = 0.9577$, and $\beta = 1.0789$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\delta$</th>
<th>leverage</th>
<th>market $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.0262</td>
<td>0.7885</td>
<td>0.3526</td>
</tr>
<tr>
<td>4</td>
<td>1.0309</td>
<td>0.6919</td>
<td>0.2664</td>
</tr>
<tr>
<td>5</td>
<td>1.0322</td>
<td>0.5839</td>
<td>0.2151</td>
</tr>
<tr>
<td>6</td>
<td>1.0303</td>
<td>0.4656</td>
<td>0.1813</td>
</tr>
<tr>
<td>7</td>
<td>1.0253</td>
<td>0.3386</td>
<td>0.1574</td>
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<tr>
<td>8</td>
<td>1.0173</td>
<td>0.2044</td>
<td>0.1398</td>
</tr>
<tr>
<td>9</td>
<td>1.0066</td>
<td>0.0647</td>
<td>0.1263</td>
</tr>
</tbody>
</table>

Panel B

In this panel we set the standard deviation of consumption growth equal to 8%. The model converges for $6 \geq \tau \geq 2$. For these targets, $\alpha = 0.9375$ and $\beta = 1.0991$. For $\tau = 5, 6$, the pure time discount factor is less than unity.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\delta$</th>
<th>leverage</th>
<th>market $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.0099</td>
<td>0.7727</td>
<td>0.3955</td>
</tr>
<tr>
<td>3</td>
<td>1.0099</td>
<td>0.6237</td>
<td>0.2662</td>
</tr>
<tr>
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<td>0.4541</td>
<td>0.2022</td>
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<td>0.9927</td>
<td>0.2671</td>
<td>0.1644</td>
</tr>
<tr>
<td>6</td>
<td>0.9763</td>
<td>0.0665</td>
<td>0.1397</td>
</tr>
</tbody>
</table>
This figure illustrates the time-state structure for the first four dates of the model.