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Lawrence Radiation Laboratory
Berkeley, California
AEC Contract No. W-7405-eng-48

ABSORPTION OF ION CYCLOTRON WAVES

Gordon Wayne Hamilton
(Ph. D. Thesis)
September 1, 1964
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ABSTRACT

A torsional hydromagnetic wave is induced by application of an oscillating radial electric field to a cylindrical deuterium plasma in a magnetic mirror field. The wave propagates axially to the center of the chamber where a resonance occurs between the hydromagnetic-wave frequency and the ion cyclotron frequency. Ninety percent of the energy is absorbed from the wave and is transformed to kinetic energy of charged particles. Measurements of the wave magnetic field, the oscillator power, and the plasma-energy content indicate an overall efficiency of 60% to 80% energy transfer. A numerical model has been constructed to study the wave propagation and damping in a magnetic mirror field under spatially nonuniform plasma conditions, subject to boundary conditions closely modeled after the experiment. The model is based upon the macroscopic equation of motion, the generalized Ohm's law, the wave equation, and the effects of ion cyclotron damping and damping by collisions between deuterons and other particle species. The equations are solved numerically by a single-line block overrelaxation technique. The numerical results show amplitudes and phases of wave currents and wave electric and magnetic fields throughout the chamber. A torsional wave is excited between the coaxial electrodes, and evanescent compressional waves are induced in the volumes defined by flux lines intersecting the electrodes. Reflections from the resonance are found when ion cyclotron damping is inefficient; for example, when (because of low densities) the wavelength is too long relative to the magnetic-field gradient.
I. INTRODUCTION

A. Statement of Problem

Damping of plasma waves at cyclotron resonances has attracted considerable attention because of intrinsic interest in the mechanism and because of the occurrence of cyclotron damping in physical situations. The absorption mechanism results from the strong interaction between an electromagnetic wave and a charged particle gyrating in a magnetic field at a frequency near the wave frequency. The particle will travel in a spiral trajectory, accelerating or decelerating in accordance with the relative phase between the particle velocity and the wave electric field. On the average, energy will be absorbed from the wave since the power transfer per particle is proportional to the particle velocity, and after a short time the particles being accelerated will have greater velocities than the particles being decelerated. The wave energy absorbed by a particle will depend upon how rapidly the relative phase is changing and therefore upon how near the condition is to exact resonance.

The energy of coherent particle motion must be randomized if energy is to be extracted from the wave. In a collisionless plasma the randomizing mechanism is phase mixing, which results from differences in particle velocities in the direction of the magnetic field. Particles that are initially gyrating in phase with each other will later be spread out longitudinally due to the velocity distribution. The phase of gyro motion will thereby be randomized relative to the phase of neighboring particles traveling with different velocities.

If cyclotron damping occurs in a magnetized plasma at the ion cyclotron frequency, the energy absorbed from the wave will be transformed to ion thermal energy, thus raising the possibility of heating the ions to temperatures of thermonuclear interest. In order for an oscillator to excite the wave efficiently, the plasma
must present a load with a sufficiently high impedance. The wave must therefore be excited under nonresonant conditions with a magnetic field in the excitation region that is higher than the magnetic field at the ion cyclotron resonance. In an ion cyclotron heating experiment, the wave is normally made to propagate in a magnetic field that is decreasing from the excitation region to the resonance.

A review of such experiments has been published by Hooke and Rothman. In most of these experiments (references 2 through 7, for example) the ion cyclotron waves are excited by an induction coil (around the outside of the plasma chamber) that applies an rf magnetic field in the radial and axial directions of a cylindrical geometry. In our experiment the wave was excited by the direct application of a radial rf electric field to the plasma. A further difference is in the significantly steeper magnetic-field gradient employed in our experiment for the transition between the excitation region and the ion cyclotron resonance.

A large literature exists on the theory of plasma waves and ion cyclotron damping. Most pertinent to this thesis are the works of Stix, Allis, Buchsbaum and Bers, and DeSilva, each of which contains a large bibliography on these topics. More recent theoretical studies of ion cyclotron damping have been on aspects such as the effects of collisions, an approximation by the method of geometrical optics, and several studies of charged-particle motion near cyclotron resonances.

In general the various theories are concerned with certain complications separately but do not attempt to consider all the interrelationships between the effects. In order to clarify certain aspects of this experiment a theoretical model is required for the propagation and absorption of ion cyclotron waves in a bounded plasma in a nonuniform divergenceless magnetic field with provisions for collisions between species, propagation in various directions relative to the external magnetic field, and nonuniformities in temperatures and densities of ions, electrons, neutral particles, and
impurity ions. Such a model has been constructed numerically to solve the magnetohydrodynamic equations and thereby determine the nature of the propagation and absorption of ion cyclotron waves in arbitrarily assumed nonuniform conditions.

The numerical model is based on an impedance tensor derived from the macroscopic equation of motion and the generalized Ohm's law. Since these equations do not include the cyclotron damping mechanism, an equivalent resistivity tensor is derived and added to the impedance tensor to represent the effect of ion cyclotron damping. The resulting tensor equation is combined with a wave equation and solved numerically. The solution is determined by a set of physically realistic boundary conditions discussed in Appendix B.

B. Notation

Vectors, unit vectors, tensors, and matrices are indicated as in these examples: vector, \( \vec{E} \); unit vector \( \hat{r} \); tensor or matrix \( \mathbf{z} \). Vector and tensor products are always dot products unless the cross product is specified. The dot is usually omitted.

Time-dependent quantities associated with the wave are actually real functions of position and time, but are replaced by the real part of their complex Fourier transforms. For example,

\[
\overline{E(r, t)} = \text{Re} \left( \int_{-\infty}^{\infty} E(r, \omega') e^{-i\omega' t} d\omega' \right).
\]

When monochromatic time dependence is assumed, \( E(r, \omega') = E(r) \delta(\omega' - \omega) \), so the integration can be performed

\[
\overline{E(r, t)} = \text{Re} \left[ E(r) e^{-i\omega t} \right].
\]

In keeping with common practice, we freely replace the real vector \( \overline{E(r, t)} \) with the complex vector \( \overline{E(r)} \), which consists of three complex components each with an amplitude and a phase angle. The time derivative can be replaced with \(-i\omega\). The factor \( e^{-i\omega t} \) and the
notation Re are dropped since they occur in every term of the equations.

Except in Appendix C, superscripts denote the components of vectors. Subscripts denote either the particle species, the partial derivative, or the index of the mesh point. The four species—ions, electrons, neutral atoms, and heavy impurity ions—are indicated by the subscripts i, e, n, and I. When the species is not indicated it is assumed to be ions. In Appendix C the superscripts and subscripts denote contravariant and covariant vector components.

Since a nonuniform magnetic field is not everywhere parallel to the axis of symmetry, the terms "transverse" and "longitudinal" refer to the directions perpendicular and parallel to the external magnetic field. The term "axial" indicates the direction parallel to the axis of symmetry.

The symbols used in this thesis (except in Appendices C and D) are listed below alphabetically:

- \( b \) wave magnetic field
- \( c \) velocity of light
- \( B \) externally applied magnetic field
- \( e \) electron charge
- \( f(u^z') \) distribution function for longitudinal ion velocities
- \( E \) wave electric-field vector
- \( j \) wave current density
- \( k \) wave number
- \( L \) transformation matrix to transform a vector from the local Cartesian coordinate system to the nonorthogonal coordinate system
- \( m \) particle mass
- \( n \) particle density
- \( p \) pressure
- \( P_{ij} \) rate of momentum transfer per unit volume from species \( j \) to species \( i \)
\[ P = \nabla \times (\nabla \times E) \] defined by (2.20)

\[ P_{\text{cyc}} \] power absorption per unit volume by cyclotron damping

\[ \Xi_c \] cyclotron resistivity tensor

\[ t \] time

\[ T \] temperature

\[ u \] velocity of an individual particle

\[ v \] macroscopic velocity of a species

\[ V_A = \frac{B}{\sqrt{\mu_0 \rho_1}} \] complex Alfven speed

\[ \chi \] quasi-reactive tensor

\[ z \] impedance tensor

\[ z_{xx'}, z_{xy'}, z_{zz} \] matrix elements of \( z \), defined by (2.19)

\[ Z \] number of charges of an impurity ion

\[ a^+, \beta^+ \] defined by (2.11)

\[ \alpha \] angle between \( B \) and the axis of symmetry

\[ \gamma \] momentum transfer coefficient

\[ \varepsilon_0 \] permittivity of free space

\[ \lambda \] wave length

\[ \mu_0 \] permeability of free space

\[ \nu_{ni} = n_i \sigma_{ni} \bar{u} \] ion-neutral collision frequency

\[ \eta \] ohmic resistivity tensor

\[ \eta_t \] transverse ohmic resistivity

\[ \eta_l \] longitudinal ohmic resistivity

\[ \eta_c \] magnitude of cyclotron resistivity tensor

\[ \eta_n \] equivalent resistivity due to neutrals

\[ \eta_l \] equivalent resistivity due to impurity ions

\[ \rho, \theta, \zeta \] nonorthogonal coordinates defined in Appendix C

\[ \rho_i = n_i m_i \] ion-mass density

\[ \rho_1 \] complex mass density

\[ \rho_2 \] contribution of neutrals and impurities to \( \rho_1 \)

\[ \sigma_{ni} \] charge transfer cross section

\[ \omega \] wave frequency

\[ \omega_c = eB/m_i \] ion cyclotron frequency

\[ \Omega = \frac{\omega}{\omega_c \rho_1} \] (This parameter approaches unity at the ion cyclotron resonance).
II. THEORY

A. Magnetohydrodynamic Equations

The set of equations describing the propagation of nonresonant magnetohydrodynamic waves in a medium consisting of singly charged ions, electrons, neutral atoms, and heavy impurity ions consists of a macroscopic equation of motion, a generalized Ohm's law, and two linearized Maxwell equations:

\[ \frac{\partial \mathbf{v}_i}{\partial t} = \mathbf{j} \times \mathbf{B} + \mathbf{P}_{\text{in}} + \mathbf{P}_{\text{II}} \] (2.1a)

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{n}_i \cdot \mathbf{j} + \mathbf{j} \times \mathbf{B}/\varepsilon_0 \] (2.2)

\[ \mathbf{v} \times \mathbf{b} = \mu_0 \mathbf{j} \] (2.3)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{b}}{\partial t} \] (2.4)

DeSilva has derived this set of equations under the following approximations:

a. Linearity (i.e., \( \mathbf{b} \ll \mathbf{B} \))

b. Charge quasi-neutrality (i.e., \( n_i = n_e = n \))

c. Neglect of \( \nabla \mathbf{p} \).

d. Neglect of terms of the order \( m_e/m_i \).

e. Neglect of displacement current. This is valid when the plasma dielectric constant \( K = 1 + \frac{\varepsilon_0 m_i}{\varepsilon_0 B^2} \) is much larger than unity.

f. Neglect of \( \mathbf{v} \cdot \nabla \mathbf{v} \) terms.

The Hall-effect term in (2.2) \( (\mathbf{j} \times \mathbf{B}/\varepsilon_0) \) is the term through which the effect of the ion cyclotron resonance enters these equations. This term is often dropped by authors who restrict themselves to Alfvén waves of frequencies far below the ion cyclotron resonance. However, we have not yet inserted the ion cyclotron damping mechanism into these equations.
B. Collisional Damping Mechanisms

Three types of collisional damping mechanisms have been introduced by momentum-transfer terms, each of which is considered proportional to the relative macroscopic velocity of ions with respect to some other species:

\[ \mathbf{P}_{ei} = e n \mathbf{\hat{n}} \cdot \mathbf{j} = e^2 n^2 \mathbf{\hat{n}} \cdot (\mathbf{v}_i - \mathbf{v}_e) \]

\[ \mathbf{P}_{ni} = \gamma_n (\mathbf{v}_i - \mathbf{v}_n) = \frac{-i \omega \rho_n}{1 - i \omega / \nu_n} \mathbf{v}_i \]

\[ \mathbf{P}_{li} = \gamma_l (\mathbf{v}_i - \mathbf{v}_l) . \]

Transfer of momentum from ions to any other species is a viscosity effect that inhibits the transverse wave motion. Momentum transfer between electrons and other species is qualitatively similar, but is smaller in magnitude by the ratio \( m_e / m_i \) and is therefore neglected with the exception of the ion-electron momentum transfer. Since \( \mathbf{P}_{ie} = -\mathbf{P}_{ei} \), these terms cancel in the equation of motion (2.1a), and generate the ohmic resistivity term \( (\mathbf{\hat{n}} \cdot \mathbf{j}) \) in the generalized Ohm's law (2.2). The ohmic resistivity is a tensor of the form

\[ \mathbf{\eta} = \begin{pmatrix} \eta_{\perp} & 0 & 0 \\ 0 & \eta_{\perp} & 0 \\ 0 & 0 & \eta_{||} \end{pmatrix} , \]

where \( \eta_{\perp} \) and \( \eta_{||} \), the ohmic resistivities across and parallel to the magnetic field, have been computed by Spitzer.\(^{18}\)

Since the ion velocity perpendicular to the magnetic field can become large at the resonance, the rate of momentum transfer and consequent damping due to all three collision processes becomes large at the resonance. In an ion-cyclotron-heating experiment it is necessary to ascertain which process is primarily responsible for wave damping observed at the resonance.

In Appendix A it is shown that the effects of \( \mathbf{P}_{ni} \) and \( \mathbf{P}_{li} \) may be represented by a complex mass density. We can therefore rewrite the equation of motion as
where the complex mass density $\rho_1$ includes the effects of neutral atoms and heavy impurity ions

$$\rho_1 = n m_i + \frac{\rho_n}{1 - i \omega / \nu_{ni}} + \frac{i \gamma_i}{\omega}. \quad (2.5)$$

For conditions of interest in this experiment, the imaginary part of the complex mass density is small compared to the real part.

C. Derivation of Impedance Tensor from the MHD Equations

The set of equations (2.1 b), (2.2), (2.3), and (2.4) involve four complex-vector unknowns associated with the wave: $E$, $b$, $j$, and $\nu_i$. The four equations can be solved analytically if the parameters $\rho_1$, $\eta$, and $B$ are spatially homogeneous and if purely axial propagation is assumed (i.e., with solutions of the form $f(r) \exp [i(pz - \omega t)]$). In the present experiment these conditions do not pertain, and it is necessary to solve the equations under the more general assumptions that the parameters $\rho_1$, $\eta$, and $B$ are arbitrarily specified functions of position with axial symmetry, and that the propagation will be in unknown directions (i.e., with solutions of the form $f(r, z) e^{-i \omega t}$). These generalizations require numerical techniques for solution.

The set of four equations may be combined into one equation with one unknown. For the numerical technique it is convenient to eliminate all unknowns except $E$. This is accomplished first by combining equations (2.1 b) and (2.2) to eliminate $\nu_i$, thereby obtaining another form of generalized Ohm's law. We then combine equations (2.3) and (2.4) to eliminate $b$ and obtain the wave equation, replacing the time derivative with $-i \omega$. After some algebra to express the first result in tensor form, we have the two equations

$$E = \mathbf{z} \cdot \mathbf{j} \quad \text{(generalized Ohm's law)} \quad (2.6)$$

and

$$\nabla \times (\nabla \times E) = i \omega \mu_0 \mathbf{j} \quad \text{(wave equation)} \quad (2.7)$$
Here the impedance tensor $\mathbf{z}$ is defined by the new generalized Ohm's law (2.6); its inverse $\mathbf{z}^{-1}$ is the conductivity tensor used by some authors.\cite{10, 12} It is related to the plasma dielectric tensor by the relation $\mathbf{K} = 1 + \frac{i}{\omega_\infty} \mathbf{z}^{-1}$. The impedance tensor can be decomposed into its resistive part and its quasi-reactive part ($\mathbf{z} = \mathbf{\eta} + i \mathbf{\chi}$).

The resistive part is the ohmic resistivity tensor and the quasi-reactive part $\mathbf{\chi}$ is derived from the $\mathbf{v} \times \mathbf{B}$ and $\mathbf{j} \times \mathbf{B}$ terms of (2.3). (If collisions of ions with neutral atoms and impurities are neglected, the quasi-reactive tensor is purely reactive.)

In a Cartesian ($x' - y' - z'$) coordinate system such that the $z'$ axis is parallel to the magnetic field $\mathbf{B}$ at a given point, the quasi-reactive tensor is

$$
\mathbf{\chi} = \frac{B^2}{\nu_1} \begin{pmatrix}
1 & -i\Omega & 0 \\
i\Omega & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \text{ where } \Omega = \frac{\omega_\infty}{\omega c} \left( \frac{\nu_1}{\rho_0 m_i} \right) = \frac{\omega_\infty}{e n B} \rho_1, \quad (2.8)
$$

The impedance tensor is composed of a Hermitian part and an anti-Hermitian part. The Hermitian part includes all the damping mechanisms, whereas the anti-Hermitian part describes the purely reactive impedance. This decomposition has been shown in Appendix A equation (A.6) with the assumption that the contribution of neutral particles and impurities to the complex mass density is small compared to the ion mass density.

**D. Equivalent Cyclotron Resistivity**

Before proceeding to the solution of equations (2.6) and (2.7), we must insert the ion cyclotron damping mechanism into the impedance tensor. We do this by computing an estimate of the power absorption per unit volume by ion cyclotron damping following Stix, Section 8.2.\cite{9} We then express this power density in a form analogous to $P_{\text{cyc}} = \rho c \mathbf{j}^2$, thus defining a resistivity tensor $\mathbf{\rho}_c$ to introduce the effect of cyclotron damping. In this model we are considering only the resistive effect of phase mixing, ignoring the reactive effect.
Stix finds that the reactive correction due to finite ion temperatures may be ignored if \(a^2 \gg 1\), where \(a^{-1} = \frac{(\omega - \omega_c)}{k u z_i}\) and \(u z_i\) is the ion thermal velocity parallel to the magnetic field. Sachs finds an almost identical criterion that justifies the use of moments of the kinetic equations, which is essentially the method used here. We find that for conditions of this experiment this criterion is satisfied except for a region within about two centimeters of the exact resonance. Most of the cyclotron power absorption occurs before the wave propagates into this region.

The linearized equation of motion for an individual ion with longitudinal velocity \(u z_i\) in an ion cyclotron wave is

\[
m \frac{\partial u}{\partial t} + m u z_i \frac{\partial u}{\partial z} = e E + e (u \times B) + e u z_i (\hat{z}' \times b).
\]

In this model we assume that the wave fields \(E\) and \(b\) vary as \(\exp [i(kt - \omega t)]\) and that \(E z'\) is negligible due to the low ohmic resistivity in a hot plasma. We have assumed \(B\) is constant for the short distance within which an individual ion is in resonance. The terms involving \(u z_i\) have resulted from a Lorentz transformation from a coordinate frame in which \(u z_i = 0\) to the laboratory frame \(x'-y'-z'\). From Maxwell's induction equation (2.4) we can write

\[
b = \frac{k}{\omega} (\hat{z}' \times E).
\]

The transverse component of the equation of motion is decoupled from the axial component. We can use rotating coordinates for the transverse components of (2.9), and are interested only in the rotating components of \(u\) and \(E\) that are circularly polarized in the direction of ion gyration:

\[
u^+ = u x' + i u y' \quad \quad \quad E^+ = E x' + i E y'.
\]

With these substitutions, the equation of motion (2.9) can be decomposed into components rotating in the left and right directions. Only the left rotating component plays a part in ion cyclotron damping.
\[ \frac{\partial u'^+}{\partial t} + i(\kappa u'^z + \omega_c)u'^+ = \frac{e}{m} \left( 1 - \frac{\kappa u'^z}{\omega} \right) E'^+ \exp \left[ i(\kappa z - \omega t) \right]. \quad (2.9\ a) \]

Using the initial condition \( u'^+ = u'^+_0 \) at time \( t = 0 \), we find that the solution is

\[ u'^+ = u'^+_0 + \frac{ie E'^+ (\omega - \kappa u'^z)}{m\omega} \left[ 1 - \exp \left( i(\omega - \omega_c - \kappa u'^z) t \right) \right]. \quad (2.10) \]

The rotating component of the macroscopic ion velocity is now computed by averaging \( u'^+ \) over the initial transverse velocity \( u'^+_0 \) and over longitudinal velocities \( u'^z \). Averaging over \( u'^+_0 \) results merely in dropping the first term of the solution (2.10), with the assumption that the phase of \( u'^+ \) is initially random. With a velocity distribution function \( f(u'^z) \), the result is

\[ v'^+ = \int_{-\infty}^{\infty} u'^+ f(u'^z) du'^z \]
\[ = \frac{ie}{2m} (\alpha'^+-i\beta'^+) E'^+, \]

where

\[ \alpha'^+ = \int_{-\infty}^{\infty} du'^z f(u'^z) \left( 1 - \frac{\kappa u'^z}{\omega} \right) \left[ 1 - \cos \left( \omega - \omega_c - \kappa u'^z \right) t \right] \]

and

\[ \beta'^+ = \int_{-\infty}^{\infty} du'^z f(u'^z) \left( 1 - \frac{\kappa u'^z}{\omega} \right) \sin \left( \omega - \omega_c - \kappa u'^z \right) t \left( \omega - \omega_c - \kappa u'^z \right) \]

The power absorption per unit volume by ion cyclotron damping is therefore

\[ P_{\text{cyc}} = ne \left( \text{Re} \ E'^+ \right) \left( \text{Re} \ v'^+ \right) = \frac{ne^2}{4m} \beta'^+ E'^+ \cdot E'^* + \frac{ne^2}{4m} |E^x' + iE^y'|^2 \beta'^+. \quad (2.12) \]

The integral \( \beta'^+ \) is a function of the time for which an average ion remains in the conditions described by the equation of motion. In a nonuniform magnetic field this is usually the time required for a resonant ion to move into a significantly different magnetic field due to its axial velocity. It is impractical to determine this time accurately, but the integral can be evaluated when the time is...
sufficiently short or sufficiently long for the two limiting cases:

a. If \((\omega - \omega_c - ku_z')t << 1\), the sine can be replaced with its argument, and \(\beta^+\) is linearly proportional to time

\[
\beta^+ \rightarrow \int_{-\infty}^{\infty} du_z' \ f(u_z') \ (1 - ku_z'/\omega)t .
\]

(2.13)

In this limit the relative phase angle between the circularly polarized \(E^+\) component and the ion gyro velocity is constant.

b. If \((\omega - \omega_c - ku_z')t >> 2\pi\), we can consider that the factor \(f(u_z') (1 - ku_z'/\omega)\) is virtually constant in a velocity interval over which the sine function oscillates several times. The ratio

\[
\frac{\sin (\omega - \omega_c - ku_z')t}{(\omega - \omega_c - ku_z')}\]

behaves like a delta function, and the integral approaches the following result for asymptotically long times --

\[
\beta^+ \rightarrow \frac{\pi \omega_c}{k \omega} f(\frac{\omega - \omega_x}{k}) .
\]

(2.14)

For times between these two limits, the integral will oscillate for several cycles as illustrated qualitatively by Fig. 1. It is out of the question to compute details of the oscillation without knowledge of the time \(t\), so we simply extend the two limiting cases (2.13) and (2.14) until they intersect. The damping power is computed by each of these equations, and we use whichever is smaller, usually (2.14).

The time \(t\) in (2.13) is estimated by the criterion that applies to the limit \(t \approx 0.1/|\omega - \omega_c - ku_z'|\), subject to the condition that \(t\) not be greater than the time between ion-ion collisions. The propagation constant \(k\) is estimated by the phase velocity of the torsional mode

\[
k \approx \frac{\omega}{B} \sqrt{\frac{p_0 \sigma^4}{I - \sigma}} .
\]

(This estimate can be checked after solution of the equations.)

It has been pointed out that the \(z'\) component of the \(u \times b\) term should be included in the equation of motion even if \(b\) is much smaller in magnitude than \(B\). The effect of the \(u \times b\) term will be an axial force that will change \(u_z'\), and thereby destroy the resonance.
Fig. 1. Dependence of $\beta^+$ on time (not to scale).
(The rate of change of the relative phase angle \((\omega - \omega_c - k\mathbf{u})\) must be small if resonant damping is to occur.) This point is probably true, but there are other complications due to the nonuniform magnetic field that are of at least equal importance.

a. The ion experiences an additional axial force proportional to \(\partial B/\partial z\) due to its interaction with the magnetic mirror. The magnitude of the magnetic mirror force is about the same as the magnitude of the \(\mathbf{u} \times \mathbf{b}\) force.

b. As the ion travels axially, the magnetic field and therefore \(\omega_c\) are changing. This is probably the most important complication.

c. The nonuniform magnetic field has been treated by more sophisticated W-K-B approximations for slowly changing magnetic fields.\(^9\),\(^14\) This condition does not pertain to the rather steep magnetic-field gradient of this experiment.

Each of these three complications affects the time for which an individual ion remains in resonance. It is believed that the approximation made is as good as the situation warrants.

Returning to Cartesian coordinates, we find that equation (2.12) can be expressed as

\[ P_{\text{cyc}} = b \mathbf{E}^* (\mathbf{c} \cdot \mathbf{E}) \]

where, for abbreviation,

\[ b = \pi B^+ / 4m \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

We convert this to a function of \(\mathbf{j}\) using the generalized Ohm's law (2.6). We now drop other damping mechanisms from the impedance tensor since they have been treated separately, and write (2.6) in a form including the cyclotron resistivity:

\[ \mathbf{E} = \mathbf{z} \cdot \mathbf{j}, \quad \text{where} \quad \mathbf{z} = \mathbf{z}_c + i \mathbf{z}_x. \]

The cyclotron resistivity is expected to be Hermitian, since we have dropped the reactive effect of phase mixing. The reactance
tensor $x$ is given by (2.8) and is also Hermitian if collisions between ions and other species are neglected. By eliminating $E$ from (2.15) we have

$$P_{cyc} = b (z, j)^* C (z, j)$$

(2.16)

$$= b j^* (z t^* C z) j.$$

(The transpose of the matrix $z$, results from interchanging the positions of $z$ and $j$ and is denoted by the superscript t. Complex conjugates are indicated by $\ast$. We are now dropping the dots from the dot products.)

The definition of cyclotron resistivity is based on the equation.

$$P_{cyc} = (\text{Re } E) \cdot (\text{Re } j) \quad \text{(time averaged)}$$

$$= \frac{1}{4}(E^* \cdot j + E \cdot j^*) = \frac{1}{4}(z^* \cdot j + (z \cdot j) \cdot j^*)$$

$$= \frac{1}{4} j^* (z t^* + z) \cdot j. \quad (2.17)$$

Using the Hermiticity of $r_c$ and $x$, we may simplify this to

$$z = r_c + i x$$

$$z^* = r_c^* - i x^*$$

$$z^t = r_c - i x.$$

Equation (2.17) now becomes $P_{cyc} = \frac{1}{2} j^* \cdot r_c \cdot j$.

This defines the cyclotron resistivity tensor. By comparing this definition to (2.16) we find the following matrix equation:

$$r_c = 2b z t^* C z = 2b (r_c - i x) C (r_c + i x)$$

$$= 2b (r_c C r_c + i r_c C x - i x C r_c + x C x).$$

The second and third terms cancel since the matrices commute. We find that $2b r_c C r_c$ is much smaller in magnitude than $r_c$ because the coefficient $b$ is very small. The solution is therefore

$$r_c = 2b x C x,$$ which yields the result

$$r_c = \eta_c \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.18)$$
Examining this result, we first find that $\tau^c$ is Hermitian and that consequently the cyclotron power absorption computed by (2.17) and (2.18) is real. The factor $\beta^+$ causes the cyclotron resistivity to be large near the cyclotron resonance and vanishingly small elsewhere. The breadth of the resonant regime is determined by the ion thermal velocity through the distribution function $f(u^2)$. The off-diagonal matrix elements have the effect of applying the cyclotron damping only to the rotating component of the transverse wave that is left-circularly polarized; i.e., in the direction of ion gyration. It may appear that the factor $(\omega/\omega - 1)^2$ introduces the incorrect behavior when $\omega < \omega$ near the resonance, but this is merely due to the definition $P_{\text{cycl}} = 1/2 \int^* \omega_c \cdot \tau^c$. The factor $(\omega/\omega - 1)^2$ is cancelled by the great increase in the transverse current at the resonance. The tensor $\eta_c$ is shown in Fig. 2 for a typical set of conditions as a function of $B$. Near the resonance the cyclotron resistivity is one or two orders of magnitude larger than the ohmic resistivity for typical conditions.

E. Total Impedance Tensor

We now add the cyclotron resistivity to the impedance tensor previously derived, thus inserting the mechanism of ion cyclotron damping into the generalized Ohm's law (2.6). The total impedance tensor is now the sum of the ohmic resistivity tensor, the cyclotron resistivity tensor, and the quasi-reactive tensor multiplied by $i$.

$$z = \eta + \tau^c + i \chi = \begin{pmatrix} \eta_{xx} & \eta_{xy} & 0 \\ -\eta_{xy} & \eta_{xx} & 0 \\ 0 & 0 & \eta_{zz} \end{pmatrix}$$

where, for abbreviation,
Fig. 2. Equivalent cyclotron resistivity as a function of magnetic field. Ion density = \(3 \times 10^{19}\) m\(^{-3}\). Ion temperature = 100 eV.
\[ z_{xx} = \eta_x + \eta \frac{\beta^2}{\omega \rho_1} \]
\[ z_{xy} = i \eta + \frac{\beta}{en} \]
\[ z_{zz} = \eta_z \]

**F. Numerical Solution**

The numerical techniques are briefly outlined in this section and described in more detail in the appendices.

Having derived the impedance tensor \( z \), which is assumed to be a known function of position in an inhomogeneous plasma, we are ready to commence the numerical solution of the equation formed by combining (2.6) and (2.7):

\[ \nabla \times (\nabla \times E) = i \omega \mu_{0z}^{-1} E = P, \]

(Here we have defined the vector \( P \) as an abbreviation for \( \nabla \times (\nabla \times E) \).)

1. **Coordinate System**

   In a cylindrical geometry it is normal to use cylindrical polar coordinates. However, this results in a computational instability driven by the axial component of the wave, since the flux lines are not parallel to the \( z \) axis in a magnetic mirror. The instability can be eliminated by choosing a coordinate system such that the transverse wave has no axial component, i.e., such that the transverse coordinate lines are orthogonal to magnetic flux lines. The axial coordinate lines are specified parallel to the \( z \) axis to simplify the boundaries. This combination of transverse and axial coordinate lines defines the nonorthogonal curvilinear coordinate system illustrated by Fig. 3. The coordinates are designated as \( \rho \), \( \theta \), and \( \zeta \), in analogy to the cylindrical polar coordinates \( r \), \( \theta \), and \( z \).

2. **Mesh System**

   A mesh system illustrated by Fig. 4 is established in the \( \rho - \zeta \) plane along coordinate lines, in preparation for conversion of the differential equation (2.20) to difference equations. The mesh
Transverse coordinate lines

\( (\zeta = \text{constant}) \)

Axial coordinate line

\( (\rho = \text{constant}) \)

Flux line

Fig. 3. Nonorthogonal coordinate system.
Fig. 4. Nonorthogonal mesh system.
should be as fine as possible to improve the accuracy and the computational stability. The limitations are the computer's memory and the time requirement. These requirements impose a limit of about 1400 mesh points, which determines the mesh interval. The mesh system consists of 14 axial mesh lines at intervals ($\Delta r$) of 0.762 cm and 99 transverse mesh lines at intervals ($\Delta z$) of about 1 cm in the plasma and 0.5 cm in the end insulator. Mesh points are identified by two indices; the first index defines the radial coordinate and the second index defines the axial coordinate.

3. Transformation of Equation (2.20) to the Nonorthogonal Coordinate System

In Appendix C, $\mathbf{P}$ is computed in the nonorthogonal system and a transformation matrix $L$ is derived to transform a vector from the local Cartesian coordinate system to the nonorthogonal system. In order to transform (2.20) to the nonorthogonal system, we need only replace $z$ with $L z L^{-1}$, which from (2.19) and (C.7) is

$$
L z L^{-1} = \begin{pmatrix}
z_{xx} & z_{xy} & (z_{zz} - z_{xx}) \sin a \\
-z_{xy} & z_{xx} & z_{zz} \sin a \\
0 & 0 & z_{zz}
\end{pmatrix}, \quad (2.21)
$$

where $a$ is the angle between $\mathbf{B}$ and $\hat{z}$ at a given point and $\sin a = B^r/B$.

4. Conversion to Difference Equations

We now convert the differential equation (2.20) to a set of finite difference equations, where the derivatives involved in $\nabla \times (\nabla \times \mathbf{E})$ in equation (C.10) at the mesh point $ij$ are approximated by the differences in $\mathbf{E}$ at neighboring points. Denoting the partial derivatives by the subscripts $\rho$ and $z$, we use the following approximations:
\[ E_\rho = \frac{1}{2\Delta \rho} [E_{i+1,j} - E_{i-1,j}] \]
\[ E_z = \frac{1}{2\Delta z} [E_{i,j+1} - E_{i,j-1}] \]
\[ E_{\rho\rho} = \frac{1}{\Delta \rho^2} [E_{i+1,j} - 2E_{ij} + E_{i-1,j}] \tag{2.22} \]
\[ E_{zz} = \frac{1}{\Delta z^2} [E_{i,j+1} - 2E_{ij} + E_{i,j-1}] \]
\[ E_{\rho z} = \frac{1}{4\Delta \rho \Delta z} [E_{i+1,j+1} - E_{i+1,j-1} - E_{i-1,j+1} + E_{i-1,j-1}] \].

These approximations may be justified by reference to Fig. 5, which shows the "nine-point star." These approximations are not used at the boundaries, which are treated by the boundary conditions described in Appendix B.

A system of complex linear equations results from combining equations (2.20), (2.21), (2.22), and (C.10) at each mesh point. At each mesh point we have three complex linear difference equations that result from the three components of (2.20). We also have three unknowns (\(E^\rho\), \(E^\theta\), and \(E^z\)) at each interior mesh point. The system of about 3300 simultaneous nonsingular complex linear equations is (in principle) solvable. The solution is performed by an iterative method that converges to a solution within the desired accuracy, provided that the matrix representing the system of equations is computationally stable.
Fig. 5. Nine-point star used for difference approximations for partial derivatives.
III. EXPERIMENT

A. Apparatus

The plasma is produced from deuterium gas at $5.0 \times 10^{-3}$ torr pressure and is contained in a copper cylinder 19.8 cm in diameter and 94.0 cm long, as shown in Fig. 6. The external magnetic field is supplied by the coils shown and is normally adjusted to provide a mirror field of approximately 1.9 tesla at the ends and 1.0 tesla in the center. These fields correspond to $\Omega = 0.45$ and $\Omega = 1.1$, where $\Omega$ is the parameter that approaches 1.0 at the resonance. The axial and radial magnetic-field intensities are shown by Fig. 27. The copper cylinder is closed by insulating end plates in one of which is mounted a coaxial molybdenum electrode of 6.25-cm diameter and 7.5-cm length. Inside the copper cylinder at the driving end is an outer electrode, the radius of which is designed to intersect the magnetic-field lines that just miss the cylinder walls at the midplane.

The system for plasma preparation consists of a pulse line and ignitron switch connected to the coaxial electrodes to apply a radial electric field to the plasma. The radial field produces an ionizing front of the type previously studied. The front proceeds down the cylinder, leaving a highly ionized rotating plasma behind it. When the front arrives at the opposite end of the cylinder, the gross rotation of the plasma is stopped by short circuiting the electrodes with the crowbar ignitron shown in Fig. 6. To reduce the turbulence generated either by the braking action or by the later application of the wave-driving fields, a copper screen is mounted at the end of the cylinder opposite the electrodes.

Approximately 30 $\mu$sec after the electrodes have been short circuited, an 8.3-Mc electric field is applied for 1.0 msec. (The radio-frequency circuit is isolated from the plasma preparation system by a quarter-wavelength transmission line and an rf filter.)
Fig. 6. Experimental apparatus.
The 8.3-Mc oscillator is capable of 1.0-MW power output, but it was not operated at full power in the experiments described here. Plasma conditions normally change during the first 300 μsec of the rf pulse, and then reach a steady state in which the ion cyclotron wave propagation is sufficiently reproducible to measure meaningfully. Apparently the plasma is warming up and losing some of its density during the first 300 μsec.

**B. Wave Field Measurements**

The wave magnetic fields were measured by means of five 4.2-mm-diameter four-turn coils spaced at 1.27-cm intervals inside an 8-mm glass tube. The tube was positioned along a radius of the cylinder and moved axially and azimuthally to measure the $b^\theta$ component of the wave magnetic field throughout the most interesting portion of the chamber. The amplitude of $b^\theta$ was measured about 800 μsec after the beginning of the rf pulse by averaging the amplitude over a 100-μsec period during which the amplitude variation was about 20%. The amplitude measurements are shown by Figs. 7, 8, and 9. Before the resonance the general behavior is as expected in that the amplitude slowly increases as the wave propagates axially and is proportional to $1/r$ in the annulus between electrodes. The amplitude drops sharply to one fifth its former value at the resonance, showing that less than 5% of the power is propagated through the resonance.

The phase of $b^\theta$, shown by Figs. 9 and 10, was measured with the phase of the rf current at the connection to the electrode as a reference. In the absence of reflections or other complications, the phase of a wave is given by

$$\phi(z) = \int_0^z k(z) \, dz.$$  

When the analytic theory was used to compute $k(z)$, the best fit to the phase data of Fig. 10 was obtained by assuming a density
Fig. 7. Wave measurements, $b^\theta$ amplitude, at axial positions of $z = 25$ cm, 35 cm, 47.5 cm, and 52.5 cm.
Fig. 8. Wave measurements, $b^\theta$ amplitude at a radius of 5.7 cm.
Fig. 9. Wave measurements, $b^\theta$ phase and amplitude shown by contour plots (shot-to-shot differences are of the order of one contour interval before the resonance, two-contour intervals after the resonance.)
Fig. 10. Wave measurements, $b\theta$ phase.
of $6 \times 10^{18} \text{ m}^{-3}$, which yielded the solid curve shown on Fig. 10. Other possible interpretations are considered in Sec. IV. D by comparison with the numerical results.

C. Diagnostics

Most of the usual diagnostic techniques could not be applied to this experiment because of the density regime ($10^{19}$ to $10^{20} \text{ m}^{-3}$), the nonuniform conditions, and the probable lack of local thermal equilibrium. Measurements have been made of the thermal energy density and of the impurity temperature. Lower limits have been placed on the electron density and the power loss by vacuum ultraviolet radiation.

1. Diamagnetic Measurements

A probe designed to measure the diamagnetic effect of the heated plasma provides the clearest information available about plasma conditions in this experiment. The probe consists of a coil of 1200 turns wound around a longitudinal axis and located within an axial glass tube of 8-mm outside diameter. A simple theory for this probe can be constructed either by equating the change in thermal pressure to the change in magnetic pressure during the heating pulse or by considering the plasma as a diamagnetic medium. The magnetic flux is confined within the copper vacuum shell during the application of wave power. The result of this theory can be stated as:

$$\text{The integrated diamagnetic signal} = \Delta H = \frac{(nkT_1 + nkT_e)}{B}$$

where $(nkT_1 + nkT_e)$ is the average kinetic-energy density (equal to the thermal pressure) that results from the motion of charged particles in the direction perpendicular to the external field. (These quantities are averaged over the cross section of the chamber.)

A small portion of the diamagnetic signal is due to the energy of coherent charged-particle motion associated with the wave. The wave energy density is $b^2/2\mu_0$, which has been measured, thereby accounting for about 5% of the energy density indicated by the
diamagnetic signal. The remaining 95% must be thermal energy of charged particles.

It was ascertained that the observed signal conforms to expectations in the following respects:

(a) $\Delta H$ is in the same direction as $B$ with either polarity of $B$. This is expected because the probe is actually measuring the induced currents in the copper shell, which are confining the flux within the shell.

(b) $\Delta H$ is proportional to the oscillator power.

(c) A typical signal (Fig. 11) indicates that the plasma is warming up during the first 300 $\mu$sec of the oscillator pulse, after which time the signal remains constant or slowly decreases. The integrated signal returns to zero after oscillator crowbar and has a decay time of about 30 $\mu$sec.

(d) The signal $\Delta H$ is dependent upon the magnetic field, having its maximum value at the axial position where the plasma is being heated by absorption of the ion cyclotron wave. Figs. 7 and 12 may be compared to verify this result.

(e) When the probe is located at the midplane of the chamber and the magnetic field at this point is varied, a distinct resonance is observed (see Fig. 13). The maximum signal is obtained with a magnetic field on the centerline about 5% above the resonance magnetic field. This small discrepancy is evidently due to the radial gradient of $B$. Fig. 27 shows that the magnetic field is weaker at the midplane at off-axis points, where most of the cyclotron damping occurs.

(f) When impurities are allowed to accumulate in the gas between pulses, the diamagnetic signal decreases.

(g) Measurements made by this probe were later compared to those made by a more conventional diamagnetic loop surrounding the plasma. The two measurements were consistent.

The data in Fig. 12 can be integrated over the volume of the chamber to compute a total energy content of about 6 joules, with
Fig. 11. Diamagnetic signal as a function of time (typical).  
Horizontal scale: 50 \( \mu \)sec per division  
Vertical scale: 1 G per division  
Oscillator pulse extends from \( t = 60 \) \( \mu \)sec to \( t = 400 \) \( \mu \)sec.
Fig. 12. Diamagnetic signal as a function of axial probe position.
Fig. 13. Diamagnetic signal as a function of magnetic field with a fixed probe position.
the assumption of an isotropic velocity distribution. The downward slope of the diamagnetic signal immediately after oscillator crowbar indicates an energy-loss rate of about 300 kW, which is about equal to the steady-state cyclotron wave power.

The ratio of thermal pressure to magnetic pressure ($\beta$) was about $4 \times 10^{-4}$. By improving the plasma purity and the oscillator power, this was later increased by a factor of 7. These improvements do not pertain to the conditions under which the wave measurements were conducted.

2. Vacuum Ultraviolet Radiation

The ultraviolet-radiation intensity at wavelengths less than 1000 Å was measured by a simple tungsten photocathode with instrumentation for measuring the photoelectron current. Because of inadequate data on quantum efficiencies for tungsten at wavelengths shorter than 400 Å, this measurement can be interpreted only as an approximate lower limit of 30 kW for the power loss due to far-vacuum ultraviolet radiation. We believe that the actual loss due to ultraviolet-impurity radiation considerably exceeds 30 kW, because the diamagnetic signal is markedly affected by the impurity content.

3. Electron Density

A lower limit of $3 \times 10^{19}$ m$^{-3}$ for the electron density measured on a diameter across the midplane was established by cutoff of 8-mm microwaves. The polarization with the microwave electric-vector perpendicular to B was used to increase the cut-off density to this value. This lower limit applies to the maximum density on the diameter that presumably includes a dense cool boundary layer.

4. Doppler Broadening

There is little Doppler broadening of spectral lines in the deuterium Balmer series because the hot ions do not radiate Balmer light. A search was made for spectral lines from excited impurities, which might indicate temperatures in the hot portion of the chamber. The highest temperature observed by Doppler broadening
was indicated by the 4089-Å line of silicon IV (ionization potential = 33.3 eV), which had a width corresponding to 46 eV.

The time for equipartition of energy between deuterons and silicon IV ions is less than a microsecond, according to Spitzer's equation (5-31). It is reasonable to assume that equipartition is achieved during the lifetime of the silicon IV ion, and that its energy is therefore indicative of the deuteron temperature.

D. Power Transfer

The power transferred into the plasma through the ion cyclotron waves has been measured at three points in the energy cycle:

(a). By oscillator-load tests the oscillator output under conditions of the wave measurements was found to be 380 kW.

(b) The power in the torsional wave has been computed by integrating the wave energy flux over the cross-sectional area

\[ P = \frac{\omega}{k} \int_{r_1}^{r_2} \frac{b^2 \theta^2}{\nu \rho} \left[ 1 + \left( \frac{b_r}{b_\theta} \right)^2 \right] (2\pi r \, dr). \]

The power in the torsional wave is calculated as 250 kW by means of the wave amplitudes shown in Fig. 7, the phase velocity (\(\omega/k\)) indicated by the data of Fig. 10, and the ratio \(b_r/b_\theta = 0.17\) predicted by the analytic theory. (This measurement is the least accurate of the three, since the amplitude of \(b_\theta\) is subject to a 20% fluctuation.)

(c) From the decay rate of the diamagnetic signal immediately after the end of the oscillator pulse, we find an energy-loss rate of 300 kW from the plasma.

Obviously it is impossible for the energy-loss rate to be greater than the wave power. This indicated the order of accuracy of the power measurements. The conclusion is that the power transfer from the oscillator to the wave and from the wave to plasma heat has an overall efficiency of 60% to 80%. Losses such as thermal
conductivity, impurity radiation, and charge-transfer collisions cause energy transport and particle recycling during the steady-state portion of the oscillator pulse.

E. Time Scales

The orders of magnitudes for the times of interest in this experiment are recapitulated below:

- Wave period: 0.1 μsec
- Ion-ion collision time: 1.0 μsec
- Ion-silicon equipartition time: 1.0 μsec
- Ion-electron equipartition time: 10.0 μsec
- Energy-containment time: 10.0 μsec
- Electron-electron collision time: 0.01 μsec
IV. NUMERICAL RESULTS

A. Debugging and Convergence

The correctness of the numerical model was tested first by verifying that the numerical solution to the magnetohydrodynamic equations was consistent with the analytical solution under conditions in which the analytical solution is valid (uniform conditions without ion cyclotron damping). Various parameters were then varied to make sure that the solution behaves as expected under well-understood conditions. The solutions were checked for self-consistency regarding boundary conditions, wavelengths, attenuation lengths, and other points of clear agreement with the analytic solutions.

After the computational instabilities described in Appendix E were eliminated, convergence to a solution was usually obtained after 20 to 30 iterations. The test for convergence consisted of verifying that there were no significant changes in the solution during the last 10 iterations. A normal computer run lasted for 10 minutes, including 3 minutes for loading the program and performing the precomputation described in Appendix D. The remaining 7 minutes provided time for about 34 iterations and several output tables of intermediate results. Sometimes the output tape was resubmitted to continue the run for another 10 minutes, but this was not usually required.

B. Presentation of Results

By a second computer run of about 4 minutes, the solution for $E$ at each mesh point was converted into $\mathbf{b}$ and $\mathbf{j}$, the wave magnetic field and wave current. Equations (2.4), (2.7), and (2.24) were used for this conversion. The solutions were converted to polar form and presented in the form of contour plots produced by a plotter as part of the output of the second run. Each contour on Figs. 14 through 24 represents a line of equal phase or of equal amplitudes of $E$, $\mathbf{b}$, or $\mathbf{j}$. The amplitude plots provide information
Fig. 14. Solution of MHD wave equations for conditions of Section IV.C (p. 1 of 5). Transverse current amplitudes.
Fig. 15. Solution of MHD wave equations for conditions of Section IV. C (p. 2 of 5). Axial current and magnetic field amplitudes.
Fig. 16. Solution of MHD wave equations for conditions of Section IV. C (p. 3 of 5). Transverse magnetic field amplitudes.
Fig. 17. Solution of MHD wave equations for conditions of Section IV. C (p. 4 of 5). Transverse electric field amplitudes.
Fig. 18. Solution of MHD wave equations for conditions of Section IV. C (p. 5 of 5). Transverse magnetic field phases.
Fig. 19. Solution of MHD wave equations for conditions of Section IV. D (p. 1 of 5). Transverse current amplitudes.
Fig. 20. Solution of MHD wave equations for conditions of Section IV. D (p. 2 of 5). Axial current and magnetic field amplitudes.
Fig. 21. Solution of MHD wave equations for conditions of Section IV. D (p. 3 of 5). Transverse magnetic field amplitudes.
Fig. 22. Solution of MHD wave equations for conditions of Section IV.D (p. 4 of 5). Transverse electric field amplitudes.
Fig. 23. Solution of MHD wave equations for conditions of Section IV. D (p. 5 of 5). Transverse magnetic field phases.
Fig. 24. Solution of MHD wave equations with neutral damping, magnetic field phase, and amplitude.
such as radial distributions, attenuation lengths, and standing waves. The phase plots provide information such as polarization and phase velocity. The tabular output must also be consulted in order for one to understand some aspects of these plots. When there are large discontinuities of phase angle (for example at nodes between standing waves), the contour-plotting subroutine fails and blank spaces result. It is more meaningful in the more extreme cases to leave such spaces blank rather than to attempt to draw all the contours. Phase contours terminating in blank spaces in the interior of the chamber should be interpreted accordingly.

The mesh system on which the contour plots are based is shown by Fig. 4. The contours are composed of a series of line segments connecting points on mesh lines computed by linear interpolation between mesh points.

The contour interval for phase angles is 30°. In the absence of reflections, the wavelength can be determined by measuring the spacing between contours and multiplying by 12. The contour interval for amplitudes is 10% of the maximum amplitude found in the chamber, and is therefore different for each plot, as noted on the figures.

C. Solution of Equations for a Moderately High Density

Figures 14 through 18 show a solution that is fairly easy to understand because of the lack of complications that are discussed in Sec. IV, D. The plasma parameters pertaining to this solution are as follows:

- ion density $1.6 \times 10^{20} \text{ m}^{-3}$
- neutral-particle density 0
- impurities 0
- ion temperature 50 eV
- electron temperature 10 eV
- magnetic field (standard mirror field described in Appendix F)
- location of resonances $z = 46 \text{ cm and } z = 54 \text{ cm}$.
This solution is discussed in some detail before we proceed to the more complicated solutions. A complete set of graphical results showing phases and amplitudes of all components of $E$, $b$, and $j$ consists of 18 plots. However, some of this information is redundant, and we shall therefore not reproduce the plots showing the phases of $E$, $j$ and $b^\perp$, nor the amplitude plot for $E^\perp$. The phases of $E^\rho$, $b^\theta$, and $j^\rho$ are virtually identical, and the phases of $E^\theta$, $b^\theta$, and $j^\theta$ are also virtually identical. The phases of the three axial components at the center of the chamber are $180^\circ$ from these phases near the outer wall, so these phase plots show little other information. Since $E^\perp \approx n_j^\perp$ in this coordinate system, the $E^\perp$ plot is identical to the $j^\perp$ plot if the electron temperature is uniform. Figures 14 through 18 display the 10 contour plots that most meaningfully present this solution.

1. Wave Currents

Wave rf currents originate at the center electrode following an annular tube of flux defined by the electrode radius. A similar annular current sheet is defined by the inner radius of the outer electrode. Most of the current flows longitudinally to the vicinity of the ion cyclotron resonance before flowing transversely to complete the circuit between the two current sheets. The two current sheets are most clearly seen on the plot showing the amplitude of $j^\perp$ (Fig. 15). Integrating over area, we find that the total axial currents at the ends of the two electrodes are equal in magnitude (about 4000 A) but opposite in phase.

The transverse wave current (Fig. 14) consists of $\rho$ and $\theta$ components that are almost equal in magnitude but $90^\circ$ out of phase. This is because most of the transverse current consists of ions following circular or spiral orbits. The difference between $j^\rho$ and $j^\theta$ is due mostly to the transverse electron current, which is almost entirely in the $\hat{\theta}$ direction and cancels some of the $j^\theta$ ion current.

The effect of inductance is to minimize the volume enclosed between the annular rf current sheets. This is the reason the current
enters and leaves the electrodes at the outer edge of the inner electrode and the inner edge of the outer electrode.

The annular current sheets have been experimentally observed by discontinuities in measurements of $b^\theta$ (shown by Fig. 10) and by direct evidence of plasma bombardment at the edges of electrodes.

The transverse wave current becomes large at the resonance because of the large ion orbits and consequent low impedance. All types of wave damping are therefore enhanced at the resonance. By comparing the equivalent cyclotron resistivity with the ohmic resistivity and with the equivalent resistivities due to other species (see Appendix A), we can ascertain that ion cyclotron damping dominates for this set of conditions.

2. Wave Polarization

The relative phase of $b^\rho$ and $b^\theta$ may be deduced either from the tabular output or from the dashed phase contours of Fig. 18, which designate the contours of $0^\circ$ phase angle. We find a $90^\circ$ phase difference between the two transverse components of $\mathbf{E}$, $\mathbf{b}$, and $\mathbf{j}$ at all points between the two annular current sheets. The wave is elliptically polarized in the "left" direction (the direction of ion gyration) in this region. The major axis of the elliptical polarization is in the $\hat{\rho}$ direction for the $\mathbf{E}$ and $\mathbf{j}$ wave fields and in the $\hat{\theta}$ direction for the $\mathbf{b}$ wave field. As the wave approaches resonance, the polarization becomes nearly circular rather than elliptical. This is evident in the amplitude plots showing $b^\rho$ and $E^\theta$ increasing as resonance is approached. (The ratio $|b^\rho/b^\theta|$ increases from 0.3 to 0.9 as the wave propagates from $z = 20$ cm to $z = 40$ cm.)

All this is characteristic of the torsional mode, also called the T mode or the slow hydromagnetic mode, which is the mode this experiment was designed to excite.

From the contour phase plots of Fig. 18 we can see a large discontinuity in the phase of $b^\theta$ at the annular current sheets, but
no discontinuity in the phase of $b^\theta$. This shows that the relative phase of $b^\rho$ and $b^\theta$ undergoes a large change at the current sheet. The wave is elliptically polarized in the right direction within the magnetic images of the two electrodes, but in the left direction in the annulus between the electrodes. (The volumes defined by the flux lines intersecting the electrodes are called magnetic images.)

Next we notice from Fig. 15 that $b^\xi$ is large only within the electrode images. The combination of large $b^\xi$ and right elliptical polarization is characteristic of the compressional mode (also called the TLA mode or the fast hydromagnetic mode), which has been studied by Swanson$^{23}$ and by Spillman. $^{24}$ The compressional mode is cut off when the wavelength is too large in comparison with the wave-guide diameter. (The condition for cutoff is $k_c V_A/\omega \gg 1$, where $V_A = B/\sqrt{\mu_0 \rho}$ is the Alfven speed and $k_c$ is the radial wave number of index $c$ defined by the condition $k_c = x_c/r_0$; $r_0$ is the waveguide radius and $x_c$ is the $c$th root of the Bessel function $J_1(x_c) = 0$; $x_c$ is approximately equal to $c\pi$.)

Since the compressional mode is cut off for the densities of this experiment, the wave in the electrode images must be evanescent, not propagating. It is excited by the tangential currents ($j^\theta$) in the annular current sheets, as a consequence of the finite geometry. The energy in the compressional mode (estimated by integrating $b^2/2\mu_0$ over the volume) is 10% or less than the energy in the torsional mode, for this set of conditions. The compressional mode is an unexpected finding in the numerical model, but is consistent with the experimental measurements of $b^\theta$.

3. **Wave Amplitudes**

The radial variation of the wave amplitudes is determined by the boundary conditions and by the annular current sheets. Between the two current sheets the dominant wave components ($E^\rho$ and $b^\theta$) are proportional to $1/r$.

The wave is strongly damped near the resonance. For this set of conditions the damping mechanism is ion cyclotron damping.
since neutral particles and impurities are assumed to be absent and the electron temperature is high enough that ohmic damping is unimportant. The very small wave transmitted through the resonance is attenuated by two orders of magnitude and is too weak to appear on an amplitude plot of this contour interval, although it shows up on the phase contour plots.

Some very small reflections in the region $10 \text{ cm} < z < 40 \text{ cm}$ can be seen in the amplitude plots for transverse components of $E$ and $b$, Figs. 16 and 17. (The contour presentation is a very sensitive indicator for standing waves.) The small standing waves with wavelengths of about $9 \text{ cm}$ are generated by reflections from the resonance. The wavelength of the ion cyclotron wave in this region (determined from the spacing between $30^\circ$ contours) is about $18 \text{ cm}$. This is consistent with the wavelength computed from the analytic theory for the torsional mode.

The small wave that propagates through the resonance is reflected from the conducting end screen, setting up a standing wave pattern in the downstream end of the chamber. In contrast to the standing waves previously discussed, the incident and reflected waves are almost equal in amplitude. Consequently the phase undergoes a discontinuity of almost $180^\circ$ at the nodes. This standing wave pattern appears on the phase plots as closely grouped contours at the nodes in the downstream end of the chamber. The irregular contours near the outer wall at the downstream end are caused by the tendency of the wave to follow flux lines in the magnetic mirror field.

As the torsional wave propagates in the nonresonant region between the driving electrodes and the coordinate $z = 40 \text{ cm}$, we see the amplitudes of the transverse wave change as follows:

- Fig. 16 $b^\phi$ increasing $b^\theta$ increasing slightly
- Fig. 17 $E^\phi$ decreasing $E^\theta$ increasing
For these conditions there are effectively no damping mechanisms in the nonresonant portion of the chamber, so we must explain these amplitude changes by the nonuniform magnetic field. The amplitudes of $E^\theta$ and $b^\theta$ increase as resonance is approached because the wave polarization is changing from elliptical to circular. The amplitude of $b^\theta$ is increasing slightly because the phase velocity is decreasing in the decreasing magnetic field. (This is because the wave-energy density $b^2/2\mu_0$ is inversely proportional to phase velocity if there is no damping. However, the wave is spreading radially as it propagates. This competing effect reduces the wave-energy density, so the net increase of $b^\theta$ is small.)

The decrease in $E^\theta$ may be understood by replacing $\nabla$ with $i k \hat{z}$ in the Maxwell equation (2.4) and solving for $E$:

$$ E = \frac{\omega}{k} b \times \hat{z}. $$

This shows that the amplitude of $E$ is proportional to the phase velocity if the amplitude of $b$ does not change. Since the phase velocity is decreasing in the decreasing magnetic field, $E^\theta$ is also decreasing.

4. Phase Velocity

The phase velocity is proportional to the spacing between contours in the absence of reflections. In the downstream end of the chamber the phase velocity can be deduced from the half-wavelength between nodes. The phase velocity conforms to the analytic theory for the torsional mode where the magnetic field is above resonance. Near the midplane where the magnetic field is below resonance, the torsional mode is evanescent with a large phase velocity.

The compressional mode in the electrode images is excited by the $j^\theta$ currents in the annular current sheets and is not propagating due to the wave-guide cutoff. Its apparent phase velocity is therefore primarily determined by the phase of $j^\theta$ and is almost equal to the phase velocity of the torsional mode.
D. Solutions for Low Densities (1 to $3 \times 10^{19} \text{ m}^{-3}$)

Figures 19 through 23 show a solution with an assumed density of $3.0 \times 10^{19} \text{ m}^{-3}$, which is about one-fifth that of the previously discussed density. All other parameters are unchanged. This result is somewhat closer to the experimental wave measurements, and is complicated by larger reflections from the resonance.

The reflections are excited by an interesting mechanism. The transverse current (Fig. 19) is concentrated in a toroid near the ion cyclotron resonance. The $j^\theta$ component excites a fairly large axial rf magnetic field ($b^\varphi$) (Fig. 20) near the centerline within the $j^\theta$ toroid. The energy associated with this $b^\varphi$ magnetic field cannot be absorbed by ion cyclotron damping nor can it propagate in the compressional mode in this low density. This energy is reflected as a wave propagating across magnetic-field lines from the concentrated $b^\varphi$ source. The result is a diagonal pattern of standing waves most clearly shown by the amplitude of $b^\theta$, Fig. 21. This pattern is formed by the interaction of the incident wave propagating axially and the reflected wave propagating in a direction with a large transverse component. It should be noted that this mechanism would not be found by a theory that assumes either an unbounded medium or purely axial propagation.

The effect of reflections upon the phase can be large even when the reflected power is small, as illustrated by this example. By comparing the amplitudes of the nodes and antinodes in this standing-wave pattern, it is found that the amplitude of the reflected wave is about one-half the incident amplitude and that the reflected power is therefore about 25%. Similar solutions have been obtained for lower plasma densities, and it is found that reflections are larger and the results are therefore more complicated in appearance. For densities below $10^{19} \text{ m}^{-3}$, most of the energy is reflected from the resonance. This is basically because the wavelength of the torsional mode becomes too long relative to the dimensions of the magnetic mirror for efficient ion cyclotron damping.
The reflection pattern disappears when some other type of damping is introduced. Figure 24 shows the phase and amplitude of \( b^\theta \) for conditions identical to those of Figs. 21 and 22 except that a density of neutral particles equal to the ion density has been specified. (Such a pessimistic condition could conceivably arise if the experiment were dominated by charge-transfer collisions.) No reflections occur in this condition because the \( b^\theta \) wave is damped by neutral damping but not by cyclotron damping. Computing the equivalent resistivity for neutral damping by equation (A.6), we find that the damping near the resonance is almost entirely caused by neutral particles.

Most of the other features discussed in the previous section are qualitatively similar to the low-density solutions. Another difference between the two sets of results is that at the lower density the elliptical polarization in the nonresonant region becomes almost plane-polarized. For example, at a nonresonant point where \( \omega_c = 0.6 \), the ratio \( b^r/b^\theta \) is 0.32 for the higher density \((1.6 \times 10^{20} \text{ m}^{-3})\), but only 0.069 for the lower density \((0.3 \times 10^{20} \text{ m}^{-3})\).

E. Comparison of Experimental and Numerical Results

Experimentally measured wave amplitudes agree well with results of the numerical model provided reflections are not too large. The amplitude of \( b^\theta \) was numerically found to be insensitive to plasma density except for the effect of reflections. The experimental data contain suggestions of small reflections, but the reflected amplitude is at most 20% of the incident amplitude. (This is the extent of amplitude variations in the nonresonant portion of the experiment, which may have masked the standing waves.)

The high-phase velocity \((7.5 \times 10^6 \text{ m/sec})\) deduced from the experimental data of Fig. 10 implies a substantial reduction of plasma density. The density reduction is probably not entirely responsible for the phase velocity measurement, since the density
and temperature measurements by diamagnetic probe and Doppler broadening are inconsistent with densities below $10^{19}$ m$^{-3}$.

Possibly the phase measurements were affected by small (20%) reflections or by density nonuniformities that were not detected experimentally because of shot-to-shot differences. Some numerical solutions have been obtained with radial density gradients that indicate complicated effects on the phase that would have been impossible to measure in detail.

A self-consistent set of parameters that reasonably satisfies the findings of the wave measurements, the numerical results, and the diagnostics is listed below.

- **Ion density**: $2 \times 10^{19}$ m$^{-3}$
- **Neutrals and impurities**: 10%
- **Ion temperature**: 50 eV at the resonance, 25 eV elsewhere
- **Electron temperature**: 20 eV at the resonance, 10 eV elsewhere

The lack of large reflections at this density could be accounted for by the presence of neutrals and impurities. This set of parameters is not a unique set of values satisfying the results, but is reliable within a factor of about four.
V. CONCLUSIONS

From the experiment it is concluded that a substantial portion of the oscillator power can be transferred into torsional hydromagnetic waves in a highly ionized plasma by direct application of an oscillatory radial electric field. The waves propagate in a relatively steep magnetic-mirror geometry to an ion cyclotron resonance region at the mirror center, where at least 90% of the energy is lost from the waves. Most of the wave energy is converted to kinetic energy of charged particles. The energy containment time is about 20 µsec.

A lower limit for the electron temperature may be inferred from the time for transfer of energy from ions to electrons; this time must be not less than the energy-containment time. From Spitzer's equation (5.31), and with an equipartition time of at least 20 µsec used for transfer of energy from hot deuterons to cooler electrons, the electron temperature must be at least 2 eV if the density is $10^{19}$ m$^{-3}$ or 10 eV if the density is $10^{20}$ m$^{-3}$.

Some of the ion thermal energy may be lost by mechanisms not involving electrons, such as charge transfer, thermal conductivity, or ion diffusion to walls. This would result in an energy-containment time shorter than the ion-electron equipartition time.

From a combination of the experimental and numerical results, we may conclude that the dominant damping mechanism is ion cyclotron damping. A small portion of the wave energy may have been reflected, but large reflections would have been experimentally observed. Damping by neutral particles may be ruled out as a dominant mechanism since the energy absorption by neutrals would not have produced the diamagnetic signal. This implies a high ionization maintained by electron temperatures of about 10 eV. For such electron temperatures the ohmic resistivity is several orders of magnitude below the cyclotron resistivity. For the reasons given in Appendix A, we must believe the damping by impurity ions to be less than the neutral damping.
The points of agreement between the experiment and the numerical model indicate that the physical principles on which the model is based are generally correct. Some of the assumed conditions do not conform precisely to the experiment; and some of the experimental measurements may have missed some complications found by the numerical model. The model has improved the understanding of the excitation and propagation of the two hydromagnetic modes and of the damping and reflection mechanisms at the ion cyclotron resonance.
ACKNOWLEDGMENTS

The author wishes to thank Drs. Forrest I. Boley, Wulf B. Kunkel, and John M. Wilcox and Mr. William R. Baker for advice and encouragement during this work. The wave measurements were conducted by a group consisting of Drs. Boley, Wilcox, Alan W. DeSilva, Mr. Peter R. Forman, Professor C. N. Watson-Munro*, and the author. Advice on the matrix iteration technique was provided by Dr. Gene Golub of the Stanford Computation Center, Stanford University. Assistance was furnished by many individuals at the E. O. Lawrence Radiation Laboratory, especially in the Sherwood Physics Group and in the Mathematics and Computing Section of the Theoretical Physics Group.

This work was performed under the auspices of the U. S. Atomic Energy Commission. The author was supported by a National Science Foundation Graduate Fellowship.

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APPENDICES

A. Neutral Particles and Impurity Ions

1. Neutral Particles

Neutral particles are treated by the approach of DeSilva and Lehnert. The rate of momentum transfer per unit volume from deuterons to neutral deuterium atoms by charge-transfer collisions is equal to the average momentum transfer per collision \([m(v_i - v_n)]\) multiplied by the number of charge-transfer collisions per unit volume per unit time \((n_n v_{ni})\).

\[
P_{ni} = m(v_i - v_n) n_n v_{ni} = m n_n v_{ni}. \tag{A. 1}
\]

Here \(v_{ni} = n_i \sigma_{ni} \bar{v}\) is the frequency of collisions of a neutral with ions, \(\sigma_{ni}\) is the charge-transfer cross section (about \(5 \times 10^{-19} \text{ m}^2\)), and \(\bar{v}\) is of the order of the ion thermal velocity.

We can replace the time derivative with \(-i\omega\) and solve for \(\dot{v}_{ni}\).

\[
\dot{v}_{ni} = \frac{1}{1 - i\omega/v_{ni}} v_i \tag{A. 2}
\]

Now we eliminate \(v_{ni}\) by combining (A. 1) and (A. 2) and obtain the result for \(P_{ni}\).

\[
P_{ni} = \frac{-i\omega m n_n v_i}{1 - i\omega/v_{ni}} \approx v_n n_n m v_i \quad \text{if} \quad \frac{\omega}{v_{ni}} \gg 1. \tag{A. 3}
\]

For the frequency and densities of this experiment, \(\omega/v_{ni} \gg 1\), but it is unnecessary to assume this condition in the numerical model.

2. Heavy Impurity Ions

Spillman has computed the rate of momentum transfer from deuterium ions to impurity ions of infinite mass by averaging the momentum transfer per collision over collision angles and ion
velocities. Using the Coulomb cross section and assuming infinite-mass impurities with zero velocity, Spillman obtained

\[ \sigma = \frac{e^4 Z^2}{4m^2 u^4 \sin^4 \psi/2} \]

\[ P_{\text{li}} = n_i n_i \frac{2\pi e^4 Z^2}{m} \left[ \frac{1}{u^2} \ln \frac{m^2 u^4 kT_i}{4ne^6 Z^2} \right] \frac{v_i}{\omega} \] (cgs units).

We can simplify this by assuming a Maxwellian distribution of ion velocities and carrying out the two averages with the usual Debye cutoff of impact parameters.

\[ P_{\text{li}} = n_i n_i \left( \frac{2\pi}{kT_i} \right)^{3/2} m^{1/2} e^4 Z^2 \ln \Lambda \frac{v_i}{\omega}. \] (A. 4)

In the above equations, \( Z \) is the number of charges of the impurity ion, \( \psi \) is the angle by which the deuteron velocity is deflected in a deuteron-impurity Coulomb collision, and \( \Lambda = \frac{3}{2Z e^3} \left( \frac{k^3 T_i^3}{\pi n_i^3} \right)^{1/2} \).

The assumption of infinite mass impurities with zero velocity is probably valid since the lightest impurity (carbon) is six times heavier than deuterium and most of the impurities are much heavier.

The results (A. 3) and (A. 4) both show that the momentum transfer to other species is proportional to the macroscopic ion velocity. In both cases the effect is a viscous damping of the transverse wave. (If \( \omega/v_ni \approx 1 \) or if the finite mass of the impurity ions were considered, there would be additional effects due to the participation of the other species in the wave motion, thus affecting the phase velocity. These effects do not occur in this experiment for the reasons mentioned.)

We may compare the magnitudes of the two rates of momentum transfer by computing the ratio of (A. 4) to (A. 3).

\[ \frac{P_{\text{li}}}{P_{\text{nli}}} = \frac{n_i}{n_n} \left( \frac{\pi}{kT_i} \right)^2 e^4 Z^2 \ln \Lambda \approx 0.17 \frac{n_i}{n_n}. \]
Here we have inserted typical values of $T_i = 50 \text{ eV}$, $\sigma_{ni} = 5 \times 10^{-19} \text{ m}^2$, and $\ln \Lambda = 10$. This shows that the charge-transfer cross section for neutral atoms is about six times that of the effective Coulomb cross section for heavy impurity ions. In spite of the high impurity content at the time wave measurements were made, it is unlikely that the wave damping by impurities is more important than the wave damping by neutral particles. The two effects are qualitatively identical, and we could not expect to differentiate between neutral damping and impurity damping by measurements of the wave absorption.

We can define a coefficient of viscosity for each damping mechanism since the momentum transfer is proportional to the macroscopic ion velocity relative to the other species.

\[
\begin{align*}
\mathbf{P}_{ni} = \mathbf{P}_{in} &= \gamma_n (\mathbf{v}_i - \mathbf{v}_n) \\
\mathbf{P}_{ii} = \mathbf{P}_{ii} &= \gamma_I \mathbf{v}_i
\end{align*}
\]

where

\[
\gamma_n = \frac{m_n}{n_i} \mathbf{v}_n \quad \text{(A. 5)}
\]

and

\[
\gamma_I = n_i n_1 \left( \frac{2\pi}{k \mathbf{T}_i} \right)^{3/2} m^{1/2} e^4 \frac{Z^2}{\ln \Lambda}.
\]

We now substitute these results for the momentum transfer into the macroscopic equation of motion (2.1) and rewrite the equation in the form

\[
\rho_i \frac{\partial \mathbf{v}_i}{\partial t} = \mathbf{j} \times \mathbf{B}, \quad (2.1B)
\]

thus defining the complex mass density

\[
\rho_1 \equiv \rho_1 + \rho_2 \equiv n_i m_i + \frac{n_n m_n}{1 - i \omega / \nu_{ni}} + \frac{i \gamma_I}{\omega}
\]

(2.5)

to incorporate the effects of neutrals and impurities on the wave.

If $\rho_2$, the contribution of neutrals and impurities to the complex density, is small compared to the ion mass density ($\rho_1 \gg |\rho_2|$), we can
derive equivalent resistivities to represent these effects. We do
this by rewriting the impedance tensor of (2.19), using the first two
terms of the binomial expansion for $1/\rho_1$.

$$
\rho_1^{-1} = (\rho_1 + \rho_2)^{-1} = \rho_1^{-1} - \rho_1^{-2}\rho_2 + \ldots
$$

$$
z_{xx} \approx \eta_\perp + \eta_c + \frac{iB^2}{\omega} \left( \frac{1}{\rho_1} - \frac{\rho_2}{\rho_1^2} \right)
\approx \eta_\perp + \eta_c + \eta_n + \eta_I + \frac{iB^2}{\omega n m}
$$

where

$$
\eta_n + \eta_I = \frac{-iB^2\rho_2}{\omega \rho_1^2}
$$

$$
\eta_n \approx \frac{B^2\nu_n \eta_n m}{\omega^2 \rho_1^2} \quad \text{(if } \omega/\nu_n \gg 1)\]
$$

and

$$
\eta_I \approx \frac{B^2\gamma_I}{\omega \rho_1^2}
$$

Here we have expressed $z_{xx}$, the diagonal element of the
impedance matrix, in a form in which the first four terms are the
equivalent transverse resistivities for the four damping mechanisms.
This is convenient to use to compare the relative importance of the
damping mechanisms in a given situation. In the computer model
we use the more accurate treatment by the complex density.

B. Boundary Conditions

(a) Electrode surfaces were assumed to be in contact with
the plasma because of rf currents at these surfaces. The electric-
field components parallel to these surfaces are therefore constrained
to be zero.

(b) The end screen is intended to be in electrical contact
with the plasma and there is evidence that contact is actually
established. The electric field parallel to the end screen is also constrained to be zero.

(c) The side wall is evidently insulated from the plasma by a layer of cool gas perhaps a few millimeters thick, according to wave measurements near the side walls. DeSilva has shown in his Appendix F that the condition at the interface between the plasma and the insulating layer is \( E_r = E^\theta = E^z = 0 \), provided the layer thickness falls between certain limits.\(^{11}\) The lower limit (about \( 10^{-3} \) mm) is determined by the condition that the capacitive reactance of the insulating layer must be much larger than the resistance of an equally thick layer of plasma. The upper limit is determined by the condition that the argument of the radial wave function \( (k_c r) \) must not vary appreciably across the layer thickness. The latter condition requires that the layer thickness must be less than \( 30/c \) millimeters; this condition is satisfied for the first few radial modes of index \( c \).

(d) The axial center line must have conditions such that the divergences of \( E, b, j, \) and \( v \) not be infinite. This results in the condition \( E_r = E^\theta = E^z = 0 \).

(e) The quartz insulator and its boundaries were believed to have an effect on the wave excitation, and were therefore included in the computer model. The mesh system illustrated by Fig. 4 was extended into the insulator, where the mesh interval \( \Delta z \) was reduced to 0.5 cm because of the steep field gradients in the insulator. The wave equation in the insulator takes the form

\[
\nabla \times (\nabla \times E) = -\frac{\omega^2}{c^2} K E, \tag{B.1}
\]

where the displacement current is considered but the conduction current is not, and where \( K \) is the dielectric constant of the insulator. This wave equation was converted to difference equations and solved by the method of Appendix D, where equation (B.1) is used in place of (2.20) in the insulator.
(f) The boundaries around the insulator were treated appropriately. Tangential components of electric fields are required to be zero at conducting surfaces. An rf potential of 1000 volts was specified between the electrodes. The inner electrode is at the same potential as the conducting surface at the outer face of the insulator, and the outer electrode is at the same ground potential as the outer wall. It was required that $b$, $E_r$, $E_\theta$, and the axial component of the displacement be continuous across the interface between the insulator and the plasma.

C. Coordinate Transformation

In this appendix equation (2.20) is transformed to a nonorthogonal curvilinear coordinate system. The following systems, illustrated by Fig. 25, are used in the transformation:

(a) $r-\theta$-$z$ system. Cylindrical polar coordinates. The $z$ axis is the axis of symmetry.

(b) $x$-$y$-$z$ system. Cartesian coordinates with the $z$ axis on the axis of symmetry.

(c) $x'$-$y'$-$z'$ system. Local Cartesian coordinates with the origin at a point $(r, \theta, z)$. The $z'$ axis is in the direction of the external magnetic field $\mathbf{B}$; the $y'$ axis is in the $\theta$ direction of the $r$-$\theta$-$z$ system; and the $x'$ axis is orthogonal to the $y'$ and $z'$ axes. This is the system in which the impedance tensor (2.19) has been derived.

(d) $\rho-\theta-\zeta$ system. Nonorthogonal curvilinear coordinates defined by the conditions that the coordinate lines in the transverse directions are orthogonal to magnetic flux lines; the $\theta$ coordinate is the same as in the $r$-$\theta$-$z$ system; and the axial coordinate lines are parallel to the $z$ axis. Three sets of base vectors are used in this coordinate system:

1. Covariant base vectors $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, $\mathbf{e}_\zeta$

2. Contravariant base vectors $\mathbf{e}^\rho$, $\mathbf{e}^\theta$, $\mathbf{e}^\zeta$
Fig. 25. Coordinate systems used in transformation.
(3) Unit base vectors \( \hat{\rho}, \hat{\theta}, \hat{\zeta} \) in the direction of the covariant base vectors. This is the set of base vectors to which (2.20) is transformed. The covariant and contravariant base vectors are not necessarily unit vectors. A set of vector components will be associated with each set of base vectors.

The following pairs of unit vectors are identical:

\[
\begin{align*}
\hat{x}' &= \hat{\rho} \\
\hat{y}' &= \hat{\theta} \\
\hat{z}' &= \hat{\zeta} \\
\hat{z} &= \frac{B}{B} \\
\end{align*}
\]

1. Transformation Equations

The magnitude of the \( \rho \) coordinate of a point will be the same as the magnitude of the \( r \) coordinate. The axially symmetric magnetic field will have the components \((B^r, 0, B^z)\) at any point, and will have the components \((0, 0, B^z)\) on the center line. From the definition of the magnetic scalar potential \( \phi \), we can write expressions for \( B^z \) at two points on the same transverse coordinate line, one of which is also on the center line. (We define \( \xi = z \) on the center line.)

\[
B^z(\rho, \xi) = \frac{\partial \phi(\xi)}{\partial z}; \quad B^z(0, \xi) = \frac{\partial \phi(\xi)}{\partial z} = \frac{\partial \phi(\xi)}{\partial \xi}
\]

The transverse coordinate line is a line of constant scalar potential due to its orthogonality to the flux lines. Therefore \( \phi \) is a function of \( \xi \) but not of \( \rho \), and we can combine these two expressions to obtain the relation between \( z \) and \( \xi \).

\[
\frac{\partial z}{\partial \xi} = \frac{B^z}{B^z}
\]

We can now write a set of transformation equations and their partial derivatives, where we define \( \alpha \) as the angle between \( \hat{\rho}' \) and \( \hat{\zeta} \) (\( \tan \alpha = B^r/B^z \)).
\[ x = \rho \cos \theta \quad y = \rho \sin \theta \quad z = \int \frac{B_0}{B^2} \, d\zeta \]

\[
\begin{align*}
\frac{\partial x}{\partial \rho} &= \cos \theta & \frac{\partial x}{\partial \theta} &= -\rho \sin \theta & \frac{\partial x}{\partial \zeta} &= 0 \\
\frac{\partial y}{\partial \rho} &= \sin \theta & \frac{\partial y}{\partial \theta} &= \rho \cos \theta & \frac{\partial y}{\partial \zeta} &= 0 \\
\frac{\partial z}{\partial \rho} &= -\tan \alpha & \frac{\partial z}{\partial \theta} &= 0 & \frac{\partial z}{\partial \zeta} &= \frac{B^r_0}{B^z} 
\end{align*}
\]

(C. 2)

2. Metric Tensor, Base Vectors, and Vector Components

Here we follow Margenau and Murphy, Section 5.16. The metric tensor is computed by Margenau and Murphy's \((M&M)\) equation (5-4a).

\[
\begin{pmatrix}
\delta_{ij} = \frac{\partial x^m}{\partial q^i} \frac{\partial x^n}{\partial q^j} \\
\sec^2 \alpha & 0 & -\tan \alpha \begin{pmatrix} B^z_0 \\ B^z \end{pmatrix} \\
0 & \rho^2 & 0 \\
-\tan \alpha \begin{pmatrix} B^z_0 \\ B^z \end{pmatrix} & 0 & \begin{pmatrix} B^z_0 \\ B^z \end{pmatrix}^2
\end{pmatrix}
\]

(C. 3)

The covariant base vectors are defined by

\[
\hat{e}_i = \frac{\partial \hat{r}}{\partial q_i}
\]

\[
e_\rho = \hat{x} \cos \theta + \hat{y} \sin \theta - \hat{z} \tan \alpha
\]

\[
= \hat{r} - \hat{z} \tan \alpha = \hat{\rho} \sec \alpha
\]

\[
e_\theta = \hat{x}(-\rho \sin \theta) + \hat{y}(\rho \cos \theta) = \rho \hat{\theta}
\]

\[
e_\zeta = \hat{z} \begin{pmatrix} B^z_0 \\ B^z \end{pmatrix}.
\]

(C. 4)

These base vectors are not unit vectors. We now define three sets of components of a vector \( \mathbf{E} \), each of which is associated with a set of base vectors.
a. Contravariant compounds

\[ E = E^\rho \, e^\rho + E^\theta \, e^\theta + E^\phi \, e^\phi \]

b. Components using nonorthogonal unit base vectors

\[ E = E^\rho \, \hat{\rho} + E^\theta \, \hat{\theta} + E^\phi \, \hat{\phi} \]

c. Cylindrical polar components

\[ E = E^r \, \hat{r} + E^\theta \, \hat{\theta} + E^z \, \hat{z} \]

By comparison after using (C.4), we can write the relationships among these three sets of vector components.

\[ E^\rho = \cos\alpha \, E^\rho' = E^r \]
\[ E^\theta = \frac{1}{\rho} \, E^\theta' = \frac{1}{\rho} \, E^\theta_c \]
\[ E^\phi = \frac{B_z^\rho}{B_z} (E^z + E^r \tan\alpha) = \frac{B_z^\rho}{B_z} E^\phi' \]

Finally we write the covariant components of \( E \) by the relation:

\[ E_i = g_{ij} \, E^j \]

\[ E^\rho = \sec^2\alpha E^\rho - \tan\alpha \frac{B_o^\rho}{B_z} E^\phi = \sec^2\alpha E^\rho' - \tan\alpha E^\phi' \]
\[ E^\theta = \rho^2 E^\theta = \rho E^\theta' \]
\[ E^\phi = -\tan\alpha \frac{B_o^\phi}{B_z} E^\rho + \left( \frac{B_o^\phi}{B_z} \right)^2 E^\phi = \frac{B_o^\phi}{B_z} ( -\tan\alpha \sec\alpha E^\rho + E^\phi' ) \]

3. Transformation Matrix

We need a matrix to transform a vector or tensor from the \( x'-y'-z' \) system to the \( \rho-\theta-\phi \) system. Referring to Fig. 25, we can first write a matrix to rotate the \( x'-y'-z' \) system about the \( y' \) axis by an angle \( \alpha \), thereby transforming into the \( r-\theta-z \) system:
From (C.5) we find that the matrix transforming the \( r-\theta-z \) system into the \( \rho-\theta-\zeta \) system is

\[
\mathbf{R} = \begin{pmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-\sin \alpha & 0 & \cos \alpha
\end{pmatrix}
\]

By multiplying these two matrices we obtain the desired transformation matrix, which we can also derive directly from Fig. 25:

\[
\mathbf{L} = \begin{pmatrix}
1 & 0 & \tan \alpha \\
0 & 1 & 0 \\
0 & 0 & \sec \alpha
\end{pmatrix}
\quad \text{(C.7)}
\]

Equation (2.20) is transformed into the \( \rho-\theta-\zeta \) system by replacement of the impedance tensor \( Z \) with \( \mathbf{L} Z \mathbf{L}^{-1} \). We must next compute \( \nabla \times (\nabla \times E) \) in this system.

4. \( \nabla \times E \) and \( \nabla \times (\nabla \times E) \)

We now compute the curl of a vector in the nonorthogonal system by Margenau and Murphy's equation (5.22a).

\[
\nabla \times E = (\text{Det} g)^{-1/2} \begin{vmatrix}
e_\rho & e_\theta & e_\zeta \\
\frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \zeta} \\
e_\rho & e_\theta & e_\zeta
\end{vmatrix}
\]
Here we have replaced the covariant base vectors and the covariant components with the expressions (C. 4) and (C. 6) that involve unit base vectors and the corresponding components in the nonorthogonal system. We have used small-angle approximations for $\alpha$, and have set the $\theta$ derivative to zero because of the axial symmetry. We have used (C. 1) and will differentiate with respect to $z$ rather than $\xi$. We will henceforth use only the components with unit base vectors and will drop the primes. Using subscripts to denote partial differentiation, we expand the determinant.

\[
(V \times E)^{\rho} = -E_{z}^{\rho}
\]

\[
(V \times E)^{\theta} = E_{z}^{\rho} - \frac{\partial}{\partial \rho}(\alpha E_{\rho}^{\rho}) - \frac{\partial}{\partial z}(\alpha E_{z}^{\rho})
\]

\[
+ E_{\rho}^{\rho} \left| \frac{B_{\rho}^{z}}{B_{z}^{z}} \right| - \alpha E_{\rho}^{\rho} \left| \frac{B_{\rho}^{z}}{B_{z}^{z}} \right|
\]

\[
(V \times E)^{\xi} = E_{\rho}^{\rho} + \frac{1}{\rho} E_{\rho}^{\theta}
\]

It is convenient, though perhaps not necessary, to simplify $(V \times E)^{\theta}$ by discarding the last four terms. This can be justified under certain conditions by examining the characteristic lengths for variation of $(\alpha E_{\rho}^{\rho})$, $E_{\rho}^{\rho}$, and $B_{z}^{z}$:

(a) The ratio of the fifth term to the second term is equal in magnitude to the ratio of the characteristic length for transverse variation of $E_{\rho}^{\rho}$ ($\frac{E_{\rho}^{\rho}}{E_{\rho}^{\rho}} \leq 3$ cm) to the characteristic length for
transverse variation of $B^z$ ($\frac{B^z}{B^\rho} \gtrsim 200$ cm). Similarly the ratio of
the sixth term to the third term is equal to the ratio of the characteristic length for transverse variation of $(aE^\rho)$ to the characteristic length for transverse variation of $B^z$. Thus the fifth and sixth terms are both small compared to other terms.

(b) The neglect of the third and fourth terms is usually justified, but this approximation should be reexamined after a solution is obtained. The fourth term is small compared to the second term if $\lambda \gg 2\pi a \frac{E^\rho}{E^z}$; and the third term is small compared to the first term if $\lambda \ll \frac{2\pi (aE^\rho)}{a \frac{\partial}{\partial \rho} (aE^\rho)}$. The first criterion is easily satisfied for conditions of this experiment. The second criterion is satisfied in the outer portion of the chamber because $E^\rho$ varies inversely with the radius and $a$ varies directly with radius; therefore the radial variation of $(aE^\rho)$ is small. The second criterion is satisfied in the inner portion of the chamber because $a$ is small. In a more strongly nonuniform field it would be necessary to retain the third term for $(\nabla \times E)^\theta$ in equation (C.9). We must verify as well these approximations regarding $(\nabla \times b)^\theta$.

To compute $\nabla \times (\nabla \times E)$, we merely replace $E$ with $(\nabla \times E)$ in (C.9), after dropping the four terms. The results are:

$$\begin{align*}
\rho^\rho &= [\nabla \times (\nabla \times E)]^\rho = -E^\rho_{zz} + E^\rho_{pz} \\
E^\theta &= [\nabla \times (\nabla \times E)]^\theta = -E^\theta_{zz} - E^\theta_{\rho\rho} - \frac{1}{\rho} E^\theta_{\rho} + \frac{1}{\rho^2} E^\theta \\
P^\rho &= [\nabla \times (\nabla \times E)]^\rho = -E^\rho_{pz} - E^\rho_{\rho\rho} + \frac{1}{\rho} E^\rho_{\rho} - \frac{1}{\rho^2} P^\rho
\end{align*}$$  

(C.10)

D. Matrix Iteration Techniques

1. Iteration Methods

The most obvious approach for solving a large system of linear difference equations would be first to solve for each unknown
\( E^k_{ij} \) as a function of the \( E^k_{ij} \) at neighboring mesh points. (Here the subscripts are the indices of the mesh point. The superscript denotes the component of \( E \) in the nonorthogonal system.) Starting with initially assumed values, we would compute revised values of each \( E^k_{ij} \) as a function of the assumed values at neighboring mesh points. The revised values would replace the assumed values for use in the next iteration. This process (the point Jacobi method) would eventually converge to the correct solution if the matrix is stable. The convergence rate can be greatly improved by several modifications.

(a) Rather than waiting until an iteration is complete before substituting the revised values of \( E^k_{ij} \), one may substitute each revised value as soon as it is generated, thereby always using the best available approximation to the correct solution. This is the point Gauss-Seidel method.

(b) After a correction \( (\delta E^k_{ij}) \) is computed, we may increase the correction by a relaxation factor \( \omega \), and apply the larger correction. If the relaxation factor is optimized (usually about 1.8) it is possible to improve the convergence rate by an order of magnitude.

(c) It may be possible to partition the system of equations into blocks and to solve exactly each block of equations for all \( E_{ij} \) within the block as a function of the \( E_{ij} \) outside the block. The solutions \( (E^*_{ij}) \) of the block of equations are used to compute the corrections \( \delta E_{ij} \). By combining the block method with the overrelaxation method, the procedure is as follows:

(i) Solve the block of equations simultaneously for the values of \( E^*_{ij} \) within the block. These solutions are denoted \( E^*_{ij} \).

(ii) Compute the corrections \( \delta E_{ij} = E^*_{ij} - E_{ij} \).

(iii) Apply the overrelaxation factor to the correction.

\[
E_{ij}^{t+1} = E_{ij}^t + \omega (\delta E_{ij}) = \omega E^*_{ij} + (1 - \omega) E_{ij}^t \quad \text{(D. 1)}
\]
(The superscript * denotes the simultaneous solution to the block of equations. The superscript t indicates the values of $E_{ij}$ resulting from the $t$th iteration and the superscript $t+1$ indicates the values resulting from the next iteration.)

(iv) Replace $E_{ij}^t$ with $E_{ij}^{t+1}$ and go on to the next block. The block should be as large as possible to optimize the convergence rate. By means of a factorization technique it is possible to solve for all the unknowns on a single mesh line with no more arithmetical work than the point method would require. We shall therefore define the block as all the $E_{ij}^k$ with a given index $i$.

This method, which we use in the numerical model, is the Single Line Over-Relaxation Method. A large body of theory is available for using this technique with simpler problems. Most of this theory does not apply to complications such as complex equations that are asymmetric with respect to interchange of the independent variables $p$ and $z$.

2. Factorization

For the simultaneous solution for the $E_{ij}$ in the block $i$, we shall first separate the vector $P = \nabla \times (\nabla \times E)$. The vector $Q_{ij}$ consists of all terms of the difference equations containing $E_{ij}$ and $E_{i,j\pm 1}$. The vector $R$ consists of the terms in which the first index is $i\pm 1$.

Using the definition $P_{ij} = Q_{ij} + R_{ij}$ and equations (2.22) and (2.23), we can write (2.20) in a form in which the left side is a function of block $i$ and the right side is a function of blocks $i\pm 1$.

$$\frac{a_{ij}}{2} E_{i,j-1} + \frac{b_{ij}}{2} E_{i,j} + \frac{c_{ij}}{2} E_{i,j+1} = R_{ij} \quad (D.2)$$

where

$$a_{ij} = \begin{pmatrix} 1/\Delta z^2 & 0 & 0 \\ 0 & 1/\Delta z^2 & 0 \\ 1/2r\Delta z & 0 & 0 \end{pmatrix}; \quad b_{ij} = \begin{pmatrix} 1/\Delta z^2 & 0 & 0 \\ 0 & 1/\Delta z^2 & 0 \\ -1/2r\Delta z & 0 & 0 \end{pmatrix}; \quad c_{ij} = \begin{pmatrix} 1/\Delta z^2 & 0 & 0 \\ 0 & 1/\Delta z^2 & 0 \\ 1/2r\Delta z & 0 & 0 \end{pmatrix};$$
The equations of the form (D. 2) for the block $i$ is now written as a partitioned matrix. We now drop the subscript $i$ for the remainder of this appendix.

\[
\begin{pmatrix}
\frac{2}{\Delta z^2} & 0 & 0 \\
0 & 2/ \Delta r^2 + 2/ \Delta z^2 + 1/ r^2 & 0 \\
0 & 0 & 2/ \Delta r^2 \\
\end{pmatrix}
\]

and

\[
R_{ij}^\rho = \frac{1}{4 \Delta r \Delta z} (E_{i+1,j+1}^\rho - E_{i+1,j-1}^\rho - E_{i-1,j+1}^\rho + E_{i-1,j-1}^\rho)
\]

\[
R_{ij}^\theta = \left(\frac{1}{\Delta r^2} + \frac{1}{2r \Delta r} \right) E_{i+1,j}^\theta - \left(\frac{1}{\Delta r^2} - \frac{1}{2r \Delta r} \right) E_{i-1,j}^\theta
\]

\[
R_{ij}^\tau = \left(\frac{1}{\Delta r^2} + \frac{1}{2r \Delta r} \right) E_{i+1,j}^\tau - \left(\frac{1}{\Delta r^2} - \frac{1}{2r \Delta r} \right) E_{i-1,j}^\tau
\]

\[
+ \frac{1}{4 \Delta r \Delta z} (E_{i+1,j+1}^\rho - E_{i+1,j-1}^\rho - E_{i-1,j+1}^\rho + E_{i-1,j-1}^\rho)
\]

The symbols $\mathbf{B}$ and $\mathbf{C}$ are vectors with 297 complex components. The block matrix $\mathbf{M}$ is quasi-tridiagonal—i.e., all of its elements are

\[
\begin{pmatrix}
\mathbf{b}_1 & \mathbf{c}_1 \\
\vdots & \ddots & \vdots \\
\mathbf{a}_{j-1} & \mathbf{b}_{j-1} & \mathbf{c}_{j-1} \\
\mathbf{a}_j & \mathbf{b}_j & \mathbf{c}_j \\
\vdots & \ddots & \ddots & \ddots \\
\mathbf{a}_{99} & \mathbf{b}_{99} & \mathbf{c}_{99} \\
\end{pmatrix}
\begin{pmatrix}
\mathbf{E}_1 \\
\vdots \\
\mathbf{E}_j \\
\vdots \\
\mathbf{E}_{99} \\
\end{pmatrix} =
\begin{pmatrix}
\mathbf{R}_1 \\
\vdots \\
\mathbf{R}_j \\
\vdots \\
\mathbf{R}_{99} \\
\end{pmatrix}
\]

or, for abbreviation: \( M \mathbf{E} = \mathbf{R} \)

The symbols $\mathbf{E}$ and $\mathbf{R}$ are vectors with 297 complex components. The block matrix $\mathbf{M}$ is quasi-tridiagonal—i.e., all of its elements are
zero except for the three subblocks of each row centered about the diagonal. We now solve for $E$ and therefore redesignate this vector as $E^*$, for which the $^*$ was explained in Sec. D.1. It is impractical to invert $M$ directly since the off-diagonal zeros would then be lost. We shall instead employ factorization and stipulate that $M$ is the product of a lower triangular matrix $N$ and an upper triangular matrix $U$ of the following forms:

$$
\begin{pmatrix}
I \\
- \mathbf{n}_{j-1} I \\
\mathbf{n}_j I \\
\mathbf{n}_{j+1} I
\end{pmatrix}
\quad
\begin{pmatrix}
\mathbf{u}_1 & \mathbf{c}_1 \\
\mathbf{u}_{j-1} & \mathbf{c}_{j-1} \\
\mathbf{u}_j & \mathbf{c}_j \\
\mathbf{u}_{j+1} & \mathbf{c}_{j+1}
\end{pmatrix}
$$

We have defined the diagonal subblocks of $N$ to be identity matrices. Consequently the off-diagonal subblocks of $U$ must be the same as the corresponding elements of $M$ (i.e., $c_j$). By carrying out the matrix multiplication for subblocks $a_j$ and $b_j$ we obtain the relations

For $j = 1$: \[ b_1 = u_1 \]
For $j \geq 2$: \[ a_j = \frac{b_j \mathbf{u}_{j-1}}{\mathbf{n}_j} \]
\[ b_j = \frac{b_j \mathbf{c}_{j-1} + \mathbf{u}_j}{\mathbf{n}_j} \]

By solving (D.6) for $n_j$ and $u_j$ we can start with $j = 2$ and compute all the $n_j$ and $u_j$: \[ n_j = a_j \frac{u_{j-1}}{n_{j-1}} \]
\[ u_j = b_j - n_j c_{j-1} \]
Now we can rewrite (D. 4) and (D. 5) and define a vector 

\[ Z \equiv U E \]

\[ M E = N U E = N Z = R \]

By carrying out the indicated multiplication, we have a recursion relation that enables us to start with \( j = 1 \) and compute all the \( Z_j \):

\[ R_1 = Z_1 \]

\[ R_j = n_j Z_{j-1} + Z_j \] (D. 8)

\[ Z_j = R_j - n_j Z_{j-1} \]

Now, using the definition of \( Z \), we obtain a similar recursion that enables us to start with \( j = 99 \) and work backward to find the solution for each \( E_j^* \) in the block \( i \). The vector \( E_{99}^* \) will be known from boundary conditions.

\[ Z = U E^* \]

\[ \begin{pmatrix} Z_1 \\ Z_j \\ Z_{j+1} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_j \\ u_j+1 \end{pmatrix} \begin{pmatrix} E_1^* \\ E_j^* \\ E_{j+1}^* \end{pmatrix} \]

\[ Z_j = u_j E_j^* + c_j E_{j+1}^* \]

or

\[ E_j^* = u_j^{-1} Z_j - u_j^{-1} c_j E_{j+1}^* \] (D. 9)
It would be possible to use (D. 7), (D. 8), and (D. 9) for the computation, but we can save some multiplication in the iteration process by defining a three-component vector $X_j = u_j^{-1} Z_j$.

Equations (D. 7) through (D. 9) now become

$$u_j = b_j - a_j (u_{j-1}^{-1} c_{j-1})$$  \hspace{1cm} \text{(D. 7a)}

$$X_j = u_j^{-1} R_j - (u_j^{-1} a_j) X_{j-1}$$  \hspace{1cm} \text{(D. 8a)}

and $E_j^* = X_j - (u_j^{-1} c_j) E_{j+1}^*$.  \hspace{1cm} \text{(D. 9a)}

The $3 \times 3$ complex matrices $u^{-1}$, $(u^{-1} a)$, and $(u^{-1} c)$ must be precomputed for each mesh point, by means of equations (2.19), (2.21), (2.22), (D. 2), and (D. 7a). The equations used in the iteration consist of (D. 3), (D. 8a), (D. 9a), and (D. 1). All possible computations are performed in the precomputation rather than during the iteration.

E. Computational Instabilities

A matrix iteration method will converge to a solution if the spectral radius (the maximum absolute value of the eigenvalues) is less than unity. In other words, the interaction between the unknown quantities must not be too strong. If a change in the value of an unknown produces changes in other unknowns, which in turn produce a larger change in the first unknown, the interaction will cause the changes to increase in magnitude with each iteration rather than converge to a solution. Computational instabilities were the principal difficulty encountered in the numerical model. For each type of instability the interaction driving the instability was diagnosed and then weakened or eliminated to stabilize the matrix. The diagnosis was rather difficult in some cases since it was not obvious which of the many terms in the difference equations were driving the instabilities—especially when more than one mode of instability was present.
An instability at the ion cyclotron resonance was foreseeable since the determinant of the impedance tensor approaches zero at the resonance in the absence of damping mechanisms. Consequently, referring to equation (D. 2), we see that the matrix elements of \( z^{-1} \) become large at the resonance, partially cancelling the diagonal elements \( 2/\Delta r^2 \) and \( 2/\Delta z^2 \) in subblock \( b_{ij} \). The matrix then loses the property of diagonal dominance and instability results. (Diagonal dominance, defined by the condition \( |a_{ii}| \geq \Sigma |a_{ij}| \), is a sufficient condition for computational stability. When the diagonal matrix elements become small relative to the off-diagonal elements, the matrix is unstable.)

The resonance-driven instability can be stabilized by three approaches.

(a) The mesh interval can be decreased, within the limit of the computer memory.

(b) A damping mechanism of any sort can be specified, thereby preventing the determinant of \( z \) from approaching zero.

(c) The diagonal elements of the reactance tensor \( (B^2/\omega \rho_1) \) can be increased, for example by specifying a sufficiently low density.

A combination of these three approaches was used. For conditions of this experiment the matrix is stable at the resonance for densities below \( 2 \times 10^{20} \) m\(^{-3} \), which includes the regime of experimental interest.

Two types of instabilities due to the interaction between \( E^r \) and \( E^z \) were encountered. In cylindrical polar coordinates the axial component of \( E \) is generated both by ohmic resistivity and by a component of the transverse \( E \) field due to the fact that \( B \) is not parallel to \( z \) in a nonuniform magnetic field.

\[
E^z \approx \eta_{||} j^z - E^r \tan \alpha \tag{E. 1}
\]
The component $E^z$ also interacts with $E^r$ through the derivative $E^z_{rz}$, which appears in equation (C.10). If $E^z$ becomes too large because of either term in (E. 1), the interaction is unstable. The instabilities due to these two terms were eliminated separately by the following means:

(a) For a warm plasma ($T_e = 5$ eV), the resistivity is sufficiently low so that the matrix is stable.

(b) The second term in (E. 1) can be eliminated by redefining the coordinate system so that the transverse $E$ field has no component in the axial direction. This is the reason the coordinate transformation described by Appendix C was required. It is interesting to compare the transformation matrix $L$ (equation C.7) with the rotation matrix $R$ (above C.7), which was used in place of $L$ when the problem was attempted in the cylindrical coordinate system. Using small-angle approximations for comparison, we find that the only difference between $R$ and $L$ is the element in the lower left corner, which drove the instability in the cylindrical system with the nonuniform magnetic field. When this element was changed to zero the instability disappeared.

F. Magnetic-Field Computations and Calibrations

The solenoid producing the external magnetic field consisted of eight coils of 120 turns each, connected to four dc power supplies. The magnetic field was easily variable, but was standardized for the experiments and computations reported here. The magnetic field was measured by a search coil on the axis of symmetry and was computed at other points by means of Tables for a Semi-Infinite Current Sheet. Good agreement between the measurement and the computation was obtained.

Using an orthogonal $14 \times 95$ mesh system in the r-z plane, we computed the ratios $B^z/I$ and $B^r/I$ in tesla/ampere at each mesh point for each of the eight solenoid coils. We performed this
computation by schematically replacing the 120-turn coil with five current sheets as indicated by Fig. 26, and carrying out the procedure specified in the introduction to the tables. The necessary interpolation was performed by a two-way second order interpolation subroutine. To avoid the necessity of interpolating with each run, the ratios \( B_z/I \) and \( B_r/I \), each of which comprised an \( 8 \times 14 \times 95 \) array, were stored on a tape used as an input for subsequent computations.

The current through each of the eight coils was specified, and \( B_z \) and \( B_r \) were computed at each point with the precomputed ratios. This result is shown by Fig. 27. The axial field was about 1.9 tesla at the ends and 1.0 tesla at the midplane. The radial gradient of the axial field is positive at the ends but negative at the midplane. The radial magnetic field is typical of a magnetic mirror—positive at one end, negative at the other end, and increasing in magnitude radially. The irregularities in \( B_r \) (and to a lesser degree in \( B_z \)) are due to the small gaps (5 mm) between coils. These irregularities are a tiny fraction of the total field, although they appear large in the \( B_r \) graph at the outside wall. The maximum value of \( B_r \) is 0.17 tesla, at the outside wall.

After the mesh points were located for the nonorthogonal mesh system, it was necessary to compute \( B \) and \( \sin \alpha \) at the new mesh points for use in the impedance tensor, equations (2.19) and (2.21). This was accomplished by interpolating for \( B_z \) and \( B_r \) between the two nearest points of the orthogonal mesh system.
Actual coil
9" i.d., 19" o.d.
120 turns at 1/2" spacing

Coil used in computer model
five current sheets

---

r = 9"

---

r = 8"

---

r = 7"

---

r = 6"

---

r = 5"

---

Vacuum wall

4.5"

---

Centerline

---

Fig. 26. Coil detail.
Fig. 27. $B_z$ and $B_r$ as functions of position, for the standard mirror field. The three curves on each graph indicate the magnetic fields at three radii: 0.76 cm, 5.3 cm, and 9.9 cm.
REFERENCES


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