A theoretical understanding of circular polarization memory in random media

by

Julia Dark

A technical report submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics

2016

Committee Members:
Professor Arnold Kim, Chair
Professor Harish Bhat
Professor Boaz Ilan
Copyright
Julia Dark, 2017
All rights reserved
The Dissertation of Julia Paige Dark is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

__________________________________________________________________________
Harish Bhat

__________________________________________________________________________
Boaz Ilan

__________________________________________________________________________
Arnold Kim, Graduate Chair

University of California, Merced
2017
# Contents

Signature Page ................................................................. iii  
List of Symbols ............................................................... vi  
Acknowledgments ............................................................. ix  
Abstract ............................................................................. xiii

## 1 Introduction 1

1.1 Background ................................................................. 1  
1.2 Polarized light in forward scattering media ......................... 2  
1.3 Polarized light in strong scattering and weakly absorbing media 2

## 2 Background 4

2.1 The Stokes vector ......................................................... 4  
2.2 Scattering events ......................................................... 6  
2.3 The radiative transport equation ........................................ 7  
2.3.1 Spectrum of the scalar scattering operator ...................... 8  
2.3.2 Spectrum of the vector scattering operator ..................... 10  
2.4 One-dimensional radiative transport theory ......................... 12  
2.4.1 Direct and diffuse radiance ......................................... 13  
2.4.2 Fourier expansion in azimuth ..................................... 13  
2.4.3 Half-range expansion ................................................ 15  
2.4.4 Expansion of the scattering matrix ............................... 15  
2.4.5 The discrete ordinate method ...................................... 17

## 3 Polarized light in forward-peaked scattering media 24

3.1 Introduction ................................................................. 24  
3.2 The scalar Fokker-Planck approximation ............................. 25  
3.3 Derivation of the vector Fokker-Planck approximation ............. 25  
3.3.1 Expansion of the scattering operator in differential operators 26  
3.3.2 The vector Fokker-Planck approximation ....................... 28  
3.4 Results ........................................................................... 30

## 4 Polarized light in strongly scattering, weakly absorbing media 32

4.1 Introduction ................................................................. 32  
4.2 Derivation of the Polarized Diffusion Approximation ............ 33  
4.2.1 Solution outside the boundary layer ............................. 35  
4.2.2 Complementary solution .......................................... 38  
4.3 Polarized Diffusion Discrete Ordinate Method ..................... 41  
4.3.1 Fourier expansion of the complementary solution .......... 43
4.3.2 Computing the complementary solution ........................................... 47
4.3.3 Computing the outer solution ...................................................... 48
4.3.4 Validation of the asymptotic model for zeroth Fourier ....................... 50
4.4 Results ......................................................................................... 51
  4.4.1 Scattering models ................................................................. 51
  4.4.2 Reflectance of a normally incident beam on a halfspace ................. 52
  4.4.3 Quantifying circular polarization memory with the discrete spectrum .. 55

5 Discussion and Future Work .......................................................... 59
  5.1 Insight into circular polarization memory ........................................ 59
  5.2 Polarized propagation in tissues with forward-peaked and large-angle scattering .... 60
  5.3 Parameter inference with the polarized diffusion approximation .......... 60
  5.4 Light scattering by anisotropic media .......................................... 61

A Rotations ......................................................................................... 63

B The auxiliary functions ................................................................... 63
  B.1 Definitions ................................................................................. 64
  B.2 Orthogonality ........................................................................... 65
  B.3 Completeness ............................................................................ 65
  B.4 Recurrence relation ................................................................. 66

C Expansion of the scattering matrix ................................................. 67
## List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>radiance/intensity</td>
</tr>
<tr>
<td>$Q, U, V$</td>
<td>Stokes parameters</td>
</tr>
<tr>
<td>$I$</td>
<td>Stokes vector</td>
</tr>
<tr>
<td>$\kappa_a$</td>
<td>absorption coefficient</td>
</tr>
<tr>
<td>$\kappa_s$</td>
<td>scattering coefficient</td>
</tr>
<tr>
<td>$\kappa_t$</td>
<td>attenuation factor</td>
</tr>
<tr>
<td>$\varpi$</td>
<td>single scattering albedo</td>
</tr>
<tr>
<td>$H(\alpha)$</td>
<td>change of basis matrix</td>
</tr>
<tr>
<td>$x$</td>
<td>position</td>
</tr>
<tr>
<td>$\hat{s}$</td>
<td>direction of propagation</td>
</tr>
<tr>
<td>$\mu$</td>
<td>polar cosine</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>azimuth angle</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>scattering angle</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$f(\hat{s} \cdot \hat{s}')$</td>
<td>phase function</td>
</tr>
<tr>
<td>$F(\hat{s} \cdot \hat{s}')$</td>
<td>phase matrix</td>
</tr>
<tr>
<td>$Z(\hat{s}, \hat{s}')$</td>
<td>scattering matrix</td>
</tr>
<tr>
<td>$E$</td>
<td>identity matrix</td>
</tr>
<tr>
<td>$\hat{i}_0$</td>
<td>unpolarized, isotropic Stokes vector [1, 0, 0, 0]</td>
</tr>
<tr>
<td>$\mathbb{S}^2$</td>
<td>the unit sphere</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>scattering operator</td>
</tr>
<tr>
<td>$P_\ell(x)$</td>
<td>Legendre polynomial</td>
</tr>
<tr>
<td>$Y_{\ell,k}(\hat{s})$</td>
<td>spherical harmonics</td>
</tr>
<tr>
<td>$E(\alpha, \beta, \gamma)$</td>
<td>rotation given by Euler angles in the zyz-convention</td>
</tr>
<tr>
<td>$\mathcal{R}(\omega, \hat{n})$</td>
<td>rotation of angle $\omega$ about direction $\hat{n}$</td>
</tr>
<tr>
<td>$d_{m,n}(\Theta)$</td>
<td>Wigner $d$-functions</td>
</tr>
<tr>
<td>$\Phi_{m,k}(\phi)$</td>
<td>diagonal matrix of trigonometric functions</td>
</tr>
<tr>
<td>$\mathcal{P}_{\ell,k}(x)$</td>
<td>matrix of auxiliary rotation functions</td>
</tr>
<tr>
<td>$\mathcal{D}_{\ell,m,k}(\hat{s})$</td>
<td>diagonal matrix diag(1, 1, -1, -1)</td>
</tr>
<tr>
<td>$\mathcal{D}_{34}$</td>
<td>diagonal matrix diag(1, 1, -1, -1)</td>
</tr>
<tr>
<td>$Q$</td>
<td>source term</td>
</tr>
<tr>
<td>$\delta_{m,n}$</td>
<td>Dirac delta</td>
</tr>
<tr>
<td>$\Delta_3$</td>
<td>spherical Laplacian</td>
</tr>
<tr>
<td>$J$</td>
<td>total angular momentum operator</td>
</tr>
</tbody>
</table>
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Experimental reference frame for light traveling in direction $\hat{s}$.</td>
<td>6</td>
</tr>
<tr>
<td>4.1</td>
<td>Plot of the max error of the reflectance for the 1D PDDOM compared to the vector radiative transport equation.</td>
<td>50</td>
</tr>
<tr>
<td>4.2</td>
<td>The profile of the incident beam $f(x, y)$.</td>
<td>52</td>
</tr>
<tr>
<td>4.3</td>
<td>Reflected flux computed with PDDOM due to a right-handed circular polarized beam on a halfspace.</td>
<td>53</td>
</tr>
<tr>
<td>4.4</td>
<td>Polar cosine dependence of the dominant planewave solution for a Rayleigh scattering model.</td>
<td>56</td>
</tr>
<tr>
<td>4.5</td>
<td>The eigenvector of the discretized radiative transport equation associated with the smallest eigenvalue with a non-zero contribution to the circular polarization state for a Rayleigh model, Venutian atmosphere model, aerosols model, and cloud model.</td>
<td>57</td>
</tr>
</tbody>
</table>
# List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Table of the depolarization length scales of the dominant planewave solutions to discrete radiative transport equation.</td>
<td>56</td>
</tr>
</tbody>
</table>
Acknowledgments

This dissertation could not have been completed with the support, advice, and guidance of a number of people. First and foremost, I would like to thank my advisor, Professor Arnold D. Kim, who first inspired my interest in applied mathematics when I was a freshmen at UC Merced. His guidance and encouragement has been felt throughout my academic career. Next, I would like to thank my committee members Professor Harish Bhat and Professor Boaz Ilan for their feedback on my research. I greatly appreciate the support I have received from both of them. I would like to thank the School of Natural Science and Department of Mathematics at the University of California, Merced for financial support. Finally, I would like to thank the graduate students in my department, especially Michael Stobb and Jason Dark, for the numerous conversations that lead to insights into my research, debugging of compilation errors, encouragement to try new technologies and techniques, and all the little tips that accumulate into a large help over the years.
Julia Dark
Curriculum Vitae

Education

**PhD, Applied Mathematics.** University of California, Merced. Expected December 2016

Dissertation title: A theoretical understanding of circular polarization memory in random media

Advisor: Arnold D. Kim

**BS, Mathematics.** University of California, Los Angeles. June 2010

Publications and Presentations

**Journal Publications**


**Conference Proceedings**


**Invited Talks**


**Local Presentations**


Workshops


Leadership and Service

**President of the SIAM Student Chapter.** Summer 2015 - Spring 2016

The UC Merced student chapter of the Society for Industrial and Applied Mathematics (SIAM) is a group dedicated to encouraging and creating development opportunities for all students interested in applied mathematics.

- Planning and coordinating the first annual California Central Valley SIAM Student Conference. (April 29, 2016).

- Organizing and recruiting speakers for the “SAMPLE” seminar series.

- Collaborating with the University of California, Davis to create an exchange of graduate student speakers.
WSTEM Steering Committee Member. Spring 2015 - Spring 2016
Women in Science, Technology, Engineering, and Mathematics (WSTEM) is a group centered around mentoring and encouraging women in STEM fields.
   ▶ Co-organizing WSTEM discussion on jobs outside of academia. (Nov. 19, 2015).
   ▶ Co-organized WSTEM Kick-off Mixer. (Sept. 8, 2015).
   ▶ Co-organized and chaired panel for WSTEM Graduating Student Celebration and Panel. (May 8, 2015).

Quantitative, Analytic, and Computational Consulting. Spring 2015 - Fall 2015
QACC is an interdisciplinary group of graduate researchers who aim to increase the productivity and overall quality of research by serving as a local source of knowledge and experience.
   ▶ Founding member of the Quantitative, Analytic, and Computational Consulting (QACC) group.

Graduate Student Peer Mentor Program. Fall 2011 - Spring 2016
The Graduate Student Peer Mentoring Program is a group of peer mentors and peer mentees who meet once a month to discuss a topic relevant to graduate students, generally with an invited speaker.
   ▶ Active involvement in encouraging and advising other graduate students.
   ▶ Participated in a panel on finding a graduate committee. (May 16, 2015).

The Scientific Computing boot camp is a one to two day program that introduces incoming graduate students to basic scientific computing tools. The emphasis is on useful tools that are not general taught in standard courses.
   ▶ Collaborated with a small group of continuing applied mathematics graduate students to develop and implement the Scientific Computing Boot Camp.
   ▶ Ran sessions on version control and work flow management tools.

Getting the Most of Your Research Experience Workshop. Summer 2015
   ▶ Participated as a panelist in a workshop for undergraduate students involved in summer research programs.

Teaching Experience

Teaching Fellow. Spring 2013 - Current
School of Natural Science, University of California, Merced.

Teaching Assistant. Summer 2010 - Fall 2012
School of Natural Science, University of California, Merced.

Fellowships

SNS Dean’s Distinguished Scholar Award. Spring 2016.
School of Natural Science. University of California, Merced.

Applied Mathematics Distinguished Student Award. Summer 2015.
Department of Applied Mathematics. University of California, Merced.

Applied Mathematics Summer Fellowship. Summer 2015.
Department of Applied Mathematics. University of California, Merced.

Applied Mathematics Summer Fellowship.  
Applied Mathematics Graduate Group. University of California, Merced.  
Summer 2014.

Applied Mathematics Summer Fellowship.  
Applied Mathematics Graduate Group. University of California, Merced.  
Summer 2013.

Graduate Student Summer Fellowship.  
Graduate Division. University of California, Merced.  
Summer 2012.
A theoretical understanding of circular polarization memory

by

Julia Dark

Doctorate in Applied Mathematics
Arnold Kim, Chair
University of California, Merced
2016

Abstract

Radiative transport theory describes the propagation of light in random media that absorb, scatter, and emit radiation. To describe the propagation of light, the full polarization state is quantified using the Stokes parameters. For the sake of mathematical convenience, the polarization state of light is often neglected leading to the scalar radiative transport equation for the intensity only. For scalar transport theory, there is a well-established body of literature on numerical and analytic approximations to the radiative transport equation. We extend the scalar theory to the vector radiative transport equation (vRTE). In particular, we are interested in the theoretical basis for a phenomena called circular polarization memory.

Circular polarization memory is the physical phenomena whereby circular polarization retains its ellipticity and handedness when propagating in random media. This is in contrast to the propagation of linear polarization in random media, which depolarizes at a faster rate, and specular reflection of circular polarization, whereby the circular polarization handedness flips. We investigate two limits that are of known interest in the phenomena of circular polarization memory. The first limit we investigate is that of forward-peaked scattering, i.e. the limit where most scattering events occur in the forward or near-forward directions. The second limit we consider is that of strong scattering and weak absorption.

In the forward-peaked scattering limit we approximate the vRTE by a system of partial differential equations motivated by the scalar Fokker-Planck approximation. We call the leading order approximation the vector Fokker-Planck approximation. The vector Fokker Planck approximation predicts that strongly forward-peaked media exhibit circular polarization memory where the strength of the effect can be calculated from the expansion of the scattering matrix in special functions. In addition, we find in this limit that total intensity, linear polarization, and circular polarization decouple. From this result we conclude, that in the Fokker-Planck limit the scalar approximation is an appropriate leading order approximation.

In the strong scattering and weak absorbing limit the vector radiative transport equation can be analyzed using boundary layer theory. In this case, the problem of light scattering in an optically thick medium is reduced to a 1D vRTE near the boundary and a 3D diffusion equation in the interior. We develop and implement a numerical solver for the boundary layer problem by using a discrete ordinate solver in the boundary layer and a spectral method to solve the diffusion approximation in the interior. We implement the method in Fortran 95 with external dependencies on BLAS, LAPACK, and FFTW. By analyzing the spectrum of the discretized vRTE in the boundary layer, we are able to predict the presence of circular polarization memory in a given medium.
Chapter 1

Introduction

1.1 Background

Radiative transport theory describes the propagation of light through a medium of random scatterers [8, 27]. To fully describe a general radiative field we need to specify both the radiance, also called intensity, as well as the polarization state of the field; however, the polarization state is often neglected leading to scalar transport theory for the radiance alone. Although the scalar transport equation is mathematically simpler, the full polarization state is needed to fully describe scattering events. In this dissertation we seek to investigate the propagation of polarized light in random media. In particular, we are interested in understanding the phenomenon of circular polarization memory. Circular polarization memory is the physical phenomenon whereby circular polarization retains its helicity and handedness when propagating in anisotropic random media. In comparison, the linear polarization state is randomized at a faster rate. This is also in contrast to circular polarized light that undergoes specular reflection, in which case circular polarization flips handedness.

The phenomena of circular polarization memory is useful for enhancing signals in strongly scattering media [40], improving clarity in long-distance imaging in fog [16], as a signal in underwater wireless communications[28], and for signal enhancement and path discrimination in biomedical imaging [21, 39, 50]. Circular polarization memory occurs in anisotropic media. It is hypothesized that the preservation of handedness occurs because light undergoes many forward scattering events, rather than backscattering directly [32]. Although it has been observed that circular polarization memory correlates with the degree of anisotropy of the medium, bulk scattering properties such as the scattering coefficient, the absorption coefficient, and the anisotropy factor have done a poor job of predicting circular polarization memory [1, 18, 19].

We seek to gain a deeper theoretical understanding of circular polarization by investigating it in two limits. First, we investigate the effects of anisotropy on circular polarization memory by studying media where scattering is peaked in the forward direction. Next, we seek to study circular polarization as light scattering becomes diffuse. This corresponds to light propagation in strongly scattering and weakly absorbing media. In both of these limits, there is a well-established body of literature on the scalar transport theory. We seek to extend scalar theory to the full vector radiative transport equation to investigate polarization memory in these limits.
1.2 Polarized light in forward scattering media

The first limit we consider is that of forward-peaked scattering. We say a medium is forward-peaked if most scattering events occur in the forward and near-forward directions. It is known that circular polarization memory is especially prominent in forward-peaked scattering media [32, 39, 58]. Moreover, many commonly occurring scattering media, such as clouds, ocean, and biological tissue, media are forward peaked. The forward-peaked limit is a well-studied limit of the scalar radiative transport equation, both because it is both common in nature and because it is difficult to numerically solve the radiative transport equation in this case [27].

Among the approximations commonly used to study forward-peaked scattering in the scalar case are the Fokker-Planck approximation [7, 47], the Fermi pencil beam approximation [3], the small-angle approximation [27], and the $\delta$-Eddington approximation [29]. There has been some work studying forward-peaked scattering of polarized light. Liu and Weng [38] look to expand the $\delta$-Eddington method. Budak et al. [4] present a numerical method for handling forward-peaked scattering using a modified spherical harmonic method. However, much of the theory developed to study the scalar transport theory either has not been investigated in the polarized case or does not easily generalize to include polarization effects.

We are interested in studying the Fokker-Planck approximation. The scalar Fokker-Planck approximation is obtained by approximating the scattering operator with the spherical Laplacian. This approximation is commonly used in nuclear physics. While the Fokker-Planck approximation itself only holds in very strong limit where there is little to no large angle scattering [47], it can be generalized for higher order expansions [37, 48], and it can be used in combination with other operators to account for large angle scattering [20]. Moreover, it can be used to derive the Fermi-Pencil beam approximation [3]. In Chapter 3 we study forward-peaked scattering of polarized light by deriving a generalization of the Fokker-Planck approximation that from the vector radiative transport equation.

1.3 Polarized light in strong scattering and weakly absorbing media

The second regime we consider is that of a strongly scattering and weakly absorbing medium. In this regime we seek to investigate the depolarization of light as it propagates into the diffusion limit. Sufficiently deep in the scattering medium, light becomes depolarized and isotropic in the sense that the radiance is independent of direction. We seek to study the transition of polarized light into the diffusion limit. The scattering and absorption coefficients of a medium are frequency dependent quantities. In a strongly scattering and weakly absorbing medium, it is possible to interrogate at a greater depth. However, there is a loss of resolution due to scattering events. In such media, polarization information can be used to enhance signals [40], discriminate between short- and long-path photons [50], and improve contrast for long-range imaging [16]. In biomedical tissue, light in the near-infrared spectrum is strongly scattered and weakly absorbed, with the main absorbers being hemoglobin/myoglobin, water, and lipids [14]. One of the advantages of imaging with infrared light is that it is low energy and does not ionize or photo-damage.

Although polarization has found to be useful in the strong scattering, weakly absorbing media, it is computationally expensive to solve the radiative transport equation in optically thick media. Notable exceptions are the “eigenmatrix” discrete ordinate method [52] which constructs a general solution of the vector radiative transport equation that can be solved at any optical depth and the bidirectional reflectance method for computing the reflectance from a halfspace [42], which was recently generalized to be applicable to the vector radiative transport equation [43]. However, both
of these methods only address scattering in one-spatial dimension.

Recently, Kim and Moscoso [33] derived the diffusion approximation from the vector radiative transport theory using asymptotic boundary layer theory. This theory allows the problem of light scattering in an optically thick medium to be reduced to the one-dimensional vector radiative transport equation near the boundary, and the three-dimensional scalar diffusion equation in the interior of the medium away from the boundary.

In Chapter 4 we seek to further simplify the boundary layer theory in order to construct a framework for studying the depolarization of light into the diffusion limit. We use this framework to develop and implement a novel deterministic numerical solver for polarized diffusion approximation using the discrete ordinate method. We use the numerical solver to study reflectance measurements due to a beam normally incident on a halfspace. Finally, since the discrete ordinate method constructs the general solution of the vector radiative transport equation using planewave solutions constructed from the eigenvalues and eigenvectors of the discrete radiative transport equation, we show you can use the spectrum of the discrete radiative transport equation to investigate the existence of circular polarization in a given medium without solving the full transport equation.

1.4 Summary

In the following work we study the vector radiative transport equation. In Chapter 2 we introduce background to radiative transport theory. In Chapter 3 we seek to generalize the scalar Fokker-Planck approximation to include polarization effects. In Chapter 4 we use the polarization diffusion approximation to investigate the depolarization of circular polarized light in strongly scattering and weakly absorbing media. Finally, in Chapter 5 we discuss the insight we have gained from this research, and we discuss future work.
Chapter 2

Background

Radiative transport theory describes the propagation of polarized light through a medium of random scatterers [8, 27]. This includes a plethora of different media such as atmospheres, oceans, biological tissues, tree canopies, and paper. To describe the propagation of light, the full polarization state is quantified using the Stokes parameters. In this chapter we will introduce radiative transport theory, emphasizing the differences between unpolarized (scalar) transport theory and polarized (vector) transport theory. First, we use the concept of an electromagnetic planewave to define the Stokes vector. Next, we discuss the redistribution of light during a scattering event. Then we discuss the radiative transport equation for both vector and scalar theory. Finally, we discuss studying and solving the one-dimensional steady state radiative transport equation.

2.1 The Stokes vector

To describe a general radiation field we need to specify four quantities: one to describe the total radiance and three to specify the polarization state. The polarization state is determined by the ellipticity, the orientation, and the degree of polarization. To represent all four quantities, we use the Stokes vector. This is an especially convenient choice of representation as the Stokes vector is additive for incoherent light [8]. In scalar transport theory, the polarization state of light is assumed to be negligible. This reduces the dimensions of the problem to one quantity: the radiance. In this section we will define the radinace and polarization state. We will then define the Stokes vector and show that it represents all four quantities.

Throughout this section we will use the concept of a monochromatic planewave. A monochromatic planewave wave traveling in the direction $\hat{s}$ with angular frequency $\omega$ is a traverse wave of the form

$$E = E_r^{(0)} \sin(\omega t - \phi_1)\hat{r} + E_\ell^{(0)} \sin(\omega t - \phi_2)\hat{\ell}$$

where $\hat{r}$ and $\hat{\ell}$ are unit vectors such that

$$\hat{r} \times \hat{\ell} = \hat{s},$$

and the the amplitudes $E_r^{(0)}, E_\ell^{(0)}$ and the phases $\phi_1, \phi_2$ are constants.

The radiance, also called the intensity, $I_\nu$, at a position $\mathbf{x}$ traveling in the direction $\hat{s}$ at time $t$ is defined as the amount of radiant power $dP_\nu$ in a frequency interval $(\nu, \nu + d\nu)$ flowing within a solid angle $d\Omega$ through a volume element $d\sigma$ at an angle $\theta$ from the normal of the volume element, [8]

$$dP_\nu = I_\nu(\hat{s}, \mathbf{x}, t) \cos \theta d\sigma d\Omega d\nu.$$
For a monochromatic planewave given by equation (2.1), the radiance is

\[ I_\nu = |E| = E_\ell E_\ell^* + E_r E_r^*. \] (2.4)

In what follows we will assume light is monochromatic, and drop the subscript \( \nu \) for ease of notation.

In general, light consists of a random mixture of waves with varying polarization. First, we consider the polarization state defined by a single monochromatic planewave as described in (2.1). Over time this wave traces out an ellipse. The ellipticity of the polarization is defined to be the ratio of the major-axis to minor-axis of this ellipse. The orientation is the angle between the axis \( \hat{r} \) and the major-axis. For incoherent light the polarization state can be understood as a combination of uncorrelated waves. The degree of polarization gives the ratio of the polarized portion of the light to the radiance.

To track the polarization state, we introduce the Stokes vector. The Stokes vector fully describes the polarization state of light [8]. Consider again a monochromatic planewave traveling in direction \( \hat{s} \) given by (2.1). The Stokes vector is defined to be the vector \( I = [I, Q, U, V]^T \) where the Stokes parameters are given by

\[
\begin{align*}
I &= E_\ell E_\ell^* + E_r E_r^*, \\
Q &= E_\ell E_\ell^* - E_r E_r^*, \\
U &= E_\ell E_r^* + E_r E_\ell^*, \\
V &= i(E_\ell E_r^* - E_r E_\ell^*).
\end{align*}
\] (2.5-2.8)

Here \( I \) is exactly the radiance. Define \( \chi \) to be the orientation angle and \( \beta \) to be the angle such that \( \tan \beta \) is the ellipticity. The Stokes parameters \( Q, U, \) and \( V \) are of the form,

\[
\begin{align*}
Q &= I \cos 2\beta \cos 2\chi, \\
U &= I \cos 2\beta \sin 2\chi, \\
V &= I \sin 2\beta.
\end{align*}
\] (2.9-2.11)

For an arbitrary mixture of light, the Stokes vector is the sum of the Stokes vectors of the components. If the polarization states of the components are completely incoherent, the light is said to be unpolarized. In general, the degree of polarization, \( p \), is given by

\[ p = \frac{I}{\sqrt{Q^2 + U^2 + V^2}}. \] (2.12)

The Stokes parameters \( Q \) and \( U \) are dependent on the choice of reference frame. A rotation of the reference frame counter-clockwise by an angle \( \alpha \) correspond to the transformation

\[ I \mapsto H(\alpha)I \] (2.13)

where

\[
H(\alpha) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\alpha & \sin 2\alpha & 0 \\
0 & -\sin 2\alpha & \cos 2\alpha & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\] (2.14)

We draw emphasis to the fact that we have defined the rotation using what is considered to be the standard definition for a passive rotation. This is opposite of the definition set forth by Chandrasekhar [8] who defines the change-of-basis matrix for a clockwise rotation of the axis.
Given a experimental coordinate system \( \{ \hat{x}, \hat{y}, \hat{z} \} \), it is conventional to pick \( \hat{r} \) normal to the meridian plane containing (see Figure 2.1). In fact, the choice of letters ‘r’ and ‘l’ follow from this definition: ‘r’ stands for letter of perpendicular and ‘l’ stands for the last letter of parallel. Since the representation of the Stokes vector is invariant under a 180-degree rotation of then reference frame, the choice of normal direction is arbitrary. Here we choose to define

\[
\hat{r} = \hat{z} \times \hat{s}. \tag{2.15}
\]

By definition, we have

\[
\hat{l} = \hat{s} \times \hat{r}. \tag{2.16}
\]

### 2.2 Scattering events

Consider a light beam measured in a fixed experimental coordinate system, \( \{ \hat{x}, \hat{y}, \hat{z} \} \), that is scattered from incident direction \( \hat{s}' \) into the scattered direction \( \hat{s} \). As discussed above, it is convention to define the Stokes vector with respect to a reference plane defined by the direction of propagation and the normal vector of the meridian plane containing the direction propagation. With this in mind, we define the experimental reference frame for ingoing radiance, \( \{ \hat{r}', \hat{l}', \hat{s}' \} \), where

\[
\hat{r}' = \frac{\hat{z} \times \hat{s}'}{||\hat{z} \times \hat{s}'||}, \quad \hat{l}' = \frac{\hat{s}' \times \hat{r}'}{||\hat{s}' \times \hat{r}'||}, \tag{2.17}
\]

and we define the experimental reference frame for the outgoing light, \( \{ \hat{r}, \hat{l}, \hat{s} \} \), where

\[
\hat{r} = \frac{\hat{z} \times \hat{s}}{||\hat{z} \times \hat{s}||}, \quad \hat{l} = \frac{\hat{s} \times \hat{r}}{||\hat{s} \times \hat{r}||}. \tag{2.18}
\]

For many types of scatterers (e.g. spheres, cylinders, ellipsoids), the redistribution of the radiance field can be computed using Maxwell’s equations. However, in order to transform the Stokes vector, it must first be expressed with respect to a reference frame local to the scattering event. We define the scattering normal, which we denotes as \( \hat{r}_s \), to be the unit vector orthogonal to both \( \hat{s}' \) and \( \hat{s} \). We define the scattering reference frame for the incident light to be \( \{ \hat{r}_s, \hat{l}_s, \hat{s}' \} \) where

\[
\hat{r}_s = \frac{\hat{s}' \times \hat{s}}{||\hat{s}' \times \hat{s}||}, \quad \hat{l}_s = \frac{\hat{s}' \times \hat{r}_s}{||\hat{s}' \times \hat{r}_s||}, \tag{2.19}
\]
and we define the scattering reference frame for the outgoing light to be \( \{ \hat{r}_s, \hat{\ell}_s, \hat{s} \} \) where

\[
\hat{r}_s = \frac{\hat{s}' \times \hat{s}}{\|\hat{s}' \times \hat{s}\|}, \quad \hat{\ell}_s = \frac{\hat{s} \times \hat{r}_s}{\|\hat{s} \times \hat{r}_s\|}.
\]

Let \( 0 \leq \Theta \leq \pi \) denote the angle between the incident and scattered directions, \( \hat{s} \cdot \hat{s}' = \cos \Theta \). The redistribution of radiation that occurs during a scattering event is be describe by the phase matrix, \( F(\cos \Theta) \), where the Stokes vector for the ingoing radiation is expressed with respect to the reference frame \( \{ \hat{r}_s, \hat{\ell}_s, \hat{s}' \} \) and the outgoing radiation is expressed with respect to the reference frame \( \{ \hat{r}_s, \hat{\ell}_s, \hat{s} \} \).

For a medium consisting of randomly oriented particles and their mirror particles, the phase matrix has the general form \([54]\)

\[
F(\cos \Theta) = \begin{bmatrix}
a_1 & b_1 & 0 & 0 \\
b_1 & a_2 & 0 & 0 \\
0 & 0 & a_3 & b_2 \\
0 & 0 & -b_2 & a_4
\end{bmatrix}.
\]

(2.21)

Here the component \( a_1 \), called the phase function, satisfies the properties

\[
0 \leq a_1 \leq 1,
\]

(2.22)

and

\[
\frac{1}{4\pi} \int_{4\pi} a_1(\hat{s} \cdot \hat{s}') \hat{s} = 1.
\]

(2.23)

The magnitude of the remaining components of the phase matrix are bounded by \( a_1 \),

\[
|a_j| \leq a_1, \; j = 2, 3, 4,
\]

(2.24)

and

\[
|b_j| \leq a_1, \; j = 1, 2.
\]

(2.25)

We now define the scattering matrix, \( Z(\hat{s}, \hat{s}') \), to be the matrix which describes the redistribution of the Stokes vector during a scattering event from the incident direction to the scattered in the experimental reference frames. Let \( 0 \leq \chi < 2\pi \) describe the rotation from the experimental reference frame for the ingoing radiation to the scattering reference frame for the ingoing radiation. Let \( 0 \leq \Phi < 2\pi \) describe the rotation from the experimental reference frame for the outgoing radiation to the scattering reference frame for the outgoing radiation. The scattering matrix can be computed by pre- and post- multiplying the phase matrix by the change of basis matrices,

\[
Z(\hat{s}, \hat{s}') = H^T(\Phi) F(\cos \Theta) H(\chi).
\]

(2.26)

### 2.3 The radiative transport equation

As a radiation field propagates through a random medium energy is lost due to absorption, gained due to emission, and redistributed by scattering. The radiative transport equation balances the change in the radiation field over time as radiant energy is absorbed, emitted, and scattered in random media. Although the radiative transport theory was originally derived heuristically from conservation laws, it was recently shown to be a rigorous limit of the wave equation \([49]\).
Let \( \kappa_a \) denote the absorption coefficient, \( \kappa_s \) denote the scattering coefficient, \( \mathbf{Q} \) denote any sources illuminating the medium, and \( c \) be the speed of light in the medium. The vector radiative transport equation is given by

\[
\frac{1}{c} \frac{\partial}{\partial t} \mathbf{I} + \mathbf{s} \cdot \nabla \mathbf{I} + \kappa_a \mathbf{I} + \kappa_s \mathbf{L} \mathbf{I} = \mathbf{Q}
\]  
(2.27)

where the vector scattering operator is

\[
\mathbf{L} \mathbf{I} = \mathbf{I} - \frac{1}{4\pi} \int_{4\pi} Z(\mathbf{s}, \mathbf{s'}) \mathbf{I}(\mathbf{s'}) d\mathbf{s'}.
\]  
(2.28)

with \( Z \) denoting the scattering matrix defined in the previous section. We can parametrize the integral in the scattering operator using the polar cosine \( \mu \) and azimuth \( \varphi \),

\[
\int_{4\pi} d\mathbf{s'} = \int_0^{2\pi} \int_{-1}^{1} d\mu d\varphi.
\]  
(2.29)

The scalar radiative transport equation is given by

\[
\frac{1}{c} \frac{\partial}{\partial t} I + \mathbf{s} \cdot \nabla I + \mu_a I + \mu_s \mathbf{L} I = S
\]  
(2.30)

where the scattering operator is

\[
\mathbf{L} I = I - \frac{1}{4\pi} \int_{4\pi} f(\mathbf{s} \cdot \mathbf{s'}) I(\mathbf{s'}) d\mathbf{s'}.
\]  
(2.31)

The scalar radiative transport equation differs from the vector equation in that the Stokes vector is replaced with the radiance, and the scattering matrix is replaced with phase function \( f \). The phase function corresponds the first element in the scattering matrix, \( f(\mathbf{s} \cdot \mathbf{s'}) = a_1(\mathbf{s} \cdot \mathbf{s'}) \).

### 2.3.1 Spectrum of the scalar scattering operator

In this section we will briefly review results from scalar transport theory. The scattering operator for the radiance is given by

\[
\mathbf{L} I = I - \frac{1}{4\pi} \int_{4\pi} f(\mathbf{s} \cdot \mathbf{s'}) I(\mathbf{s'}) d\mathbf{s'}
\]  
(2.32)

where \( I \) is the radiance and \( f = a_1 \) is the phase function. The phase function satisfies the properties of a probability distribution function. That is,

\[
0 \leq f(\mathbf{s} \cdot \mathbf{s'}) \leq 1,
\]  
(2.33)

and the average of the phase function over all directions is 1,

\[
\frac{1}{4\pi} \int_{4\pi} f(\mathbf{s} \cdot \mathbf{s'}) d\mathbf{s'} = 1,
\]  
(2.34)

or,

\[
\frac{1}{2} \int_{-1}^{1} f(x) dx = 1.
\]  
(2.35)
The eigenfunctions of the scalar scattering operator are spherical harmonics. To see this we expand the phase function as a series of Legendre polynomials,

\[ f(\hat{s} \cdot \hat{s}') = \sum_{\ell=0}^{\infty} f_{\ell} P_{\ell}(\hat{s} \cdot \hat{s}'), \]  

(2.36)

with

\[ f_{\ell} = \frac{2\ell + 1}{2} \int_{-1}^{1} f(x) P_{\ell}(x) \, dx. \]  

(2.37)

The Legendre polynomials are defined as normal,

\[ P_{\ell}(x) = \frac{1}{2^{\ell} \ell!} \frac{d^\ell}{dx^\ell} [(x - 1)^\ell]. \]  

(2.38)

The Legendre polynomials satisfy the addition theorem

\[ P_{\ell}(\hat{s} \cdot \hat{s}') = \frac{4\pi}{2\ell + 1} \sum_{k=-\ell}^{\ell} Y_{\ell k}(\hat{s}) Y_{\ell k}^*(\hat{s}'). \]  

(2.39)

where \( Y_{\ell k}(\hat{s}) \) are the spherical harmonics and \( * \) denotes complex conjugation. Here we use the convention

\[ Y_{\ell k}(\hat{s}) = \sqrt{(2\ell + 1)(\ell - k)!} \frac{P_{\ell}^{k}(\mu)}{4\pi(\ell + k)!} e^{ik\phi} \]  

(2.40)

where \( P_{\ell}^{k}(\mu) \) denotes the associated Legendre polynomials,

\[ P_{\ell}^{k}(x) = (1 - \mu^2)^{k/2} \frac{d^k}{dx^k} P_{\ell}(x). \]  

(2.41)

The spherical harmonics are defined here with orthogonality property

\[ \int_{4\pi} Y_{\ell k}(\hat{s}) Y_{\ell' k'}(\hat{s}) \, d\hat{s} = \delta_{\ell \ell'} \delta_{kk'}. \]  

(2.42)

We use the addition theorem in the expansion of the phase function given in equation (2.36),

\[ f(\hat{s} \cdot \hat{s}') = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell + 1} f_{\ell} \sum_{k=-\ell}^{\ell} Y_{\ell k}(\hat{s}) Y_{\ell k}^*(\hat{s}'). \]  

(2.43)

We substitute the expansion of the phase function into the scattering operator to find

\[ \mathcal{L} I = I - \sum_{\ell=0}^{\infty} \frac{1}{2\ell + 1} f_{\ell} Y_{\ell k}(\hat{s}) \int_{4\pi} Y_{\ell k}^*(\hat{s}') I(\hat{s}') \, d\hat{s}'. \]  

(2.44)

Using the orthogonality property (2.42) it is straight-forward to compute

\[ \mathcal{L} Y_{\ell k}(\hat{s}) = \left( 1 - \frac{1}{2\ell + 1} f_{\ell} \right) Y_{\ell k}(\hat{s}). \]  

(2.45)
2.3.2 Spectrum of the vector scattering operator

We can compute the spectrum of the vector scattering operator in an analogous way to the computation of the scalar transport equation. The phase matrix is expanded in a basis of special functions, and an addition theorem is used to transform that basis.

The vector scattering operator is given by

\[ \mathcal{L} I = I - \frac{1}{4\pi} \int \! Z(\hat{s}, \hat{s}') I(\hat{s}') \, d\hat{s}' \]  

(2.46)

where \( Z(\hat{s}, \hat{s}') \) is the scattering matrix. Let the angles \( \chi \) and \( \Phi \) be the angles such that \( R(\chi, \hat{s}') \) and \( R(\Phi, \hat{s}) \) are the rotations from the experimental frame to the scattering frame. Using the standard Stokes vector we can write the scattering matrix as

\[ Z(\hat{s}, \hat{s}') = H^T(\Phi) F(\cos \Theta) H(\chi) \]  

(2.47)

where the phase function for a medium consisting of randomly oriented scatterers has the general form

\[ F(\cos \Theta) = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{bmatrix} \]  

(2.48)

Kuščer and Ribarič [36] used special functions called generalized spherical functions to expand the scattering matrix. Here we summarize the results using the related Wigner \( d \)-functions \( d_{m,n}^\ell(\theta) \). See Appendix B for a discussion on the properties of these functions. We expand the components of the phase matrix in Wigner \( d \)-functions,

\[ a_1(\cos \Theta) = \sum_{\ell=0}^\infty \alpha_1^\ell d_{0,0}^\ell(\Theta), \]  

(2.49)

\[ a_2(\cos \Theta) + a_3(\cos \Theta) = \sum_{\ell=0}^\infty (\alpha_2^\ell + \alpha_3^\ell) d_{2,2}^\ell(\Theta), \]  

(2.50)

\[ a_2(\cos \Theta) - a_3(\cos \Theta) = \sum_{\ell=0}^\infty (\alpha_2^\ell - \alpha_3^\ell) d_{2,2}^\ell(\Theta), \]  

(2.51)

\[ a_4(\cos \Theta) = \sum_{\ell=0}^\infty \alpha_4^\ell d_{0,0}^\ell(\Theta), \]  

(2.52)

\[ b_1(\cos \Theta) = \sum_{\ell=0}^\infty \beta_1^\ell d_{2,0}^\ell(\Theta), \]  

(2.53)

\[ b_2(\cos \Theta) = \sum_{\ell=0}^\infty \beta_2^\ell d_{2,0}^\ell(\Theta), \]  

(2.54)
where the expansion coefficients are defined by

\[
\alpha_\ell = \frac{2\ell + 1}{2} \int_0^\pi a_1(\cos \Theta) d\ell_{0,0}(\Theta) \sin \Theta d\Theta,
\]

\[
\alpha_\ell^2 + \alpha_\ell^3 = \frac{2\ell + 1}{2} \int_{-1}^1 \left( a_2(\cos \Theta) + a_3(\cos \Theta) \right) d\ell_{2,2}(\Theta) \sin \Theta d\Theta,
\]

\[
\alpha_\ell^2 - \alpha_\ell^3 = \frac{2\ell + 1}{2} \int_{-1}^1 \left( a_2(\cos \Theta) + a_3(\cos \Theta) \right) d\ell_{2,-2}(\Theta) \sin \Theta d\Theta,
\]

\[
\alpha_\ell^4 = \frac{2\ell + 1}{2} \int_{-1}^1 a_4(\cos \Theta) d\ell_{0,0}(\Theta) \sin \Theta d\Theta,
\]

\[
\beta_\ell = \frac{2\ell + 1}{2} \int_{-1}^1 b_1(\cos \Theta) d\ell_{2,0}(\Theta) \sin \Theta d\Theta,
\]

\[
\beta_\ell^2 = \frac{2\ell + 1}{2} \int_{-1}^1 b_2(\cos \Theta) d\ell_{2,0}(\Theta) \sin \Theta d\Theta.
\]

Using the work of [36], it can be shown that

\[
Z(\hat{s}, \hat{s}') = \sum_{\ell=0}^\infty \sum_{u=1}^2 \sum_{u'=-\ell}^\ell \mathcal{D}_{\ell, m, k} \hat{s}) F_\ell \mathcal{D}_{\ell, m, k} \hat{s} \right)
\]

where the coefficient matrix \( F_\ell \) is given by

\[
F_\ell = \begin{bmatrix}
\alpha_\ell & \beta_\ell & 0 & 0 \\
\beta_\ell & \alpha_\ell & 0 & 0 \\
0 & 0 & \alpha_\ell & \beta_\ell \\
0 & 0 & -\beta_\ell & \alpha_\ell
\end{bmatrix},
\]

and \( \mathcal{D}_{\ell, m, k} \hat{s} \) is defined as

\[
\mathcal{D}_{\ell, m, k} \hat{s} = \frac{2 - \delta_{k,0}}{2} \Phi_{m, k}(\phi) \mathcal{P}_{\ell, k}(\mu)
\]

where

\[
\Phi_{1, k}(\phi) = \text{diag}(\cos k\phi, \cos k\phi, \sin k\phi, \sin k\phi),
\]

\[
\Phi_{2, k}(\phi) = \text{diag}(-\sin k\phi, -\sin k\phi, \cos k\phi, \cos k\phi),
\]

and

\[
\mathcal{P}_{\ell, k}(x) = \begin{bmatrix}
P_{\ell, k}(x) & 0 & 0 & 0 \\
0 & R_{\ell, k}(x) & -T_{\ell, k}(x) & 0 \\
0 & -T_{\ell, k}(x) & R_{\ell, k}(x) & 0 \\
0 & 0 & 0 & P_{\ell, k}(x)
\end{bmatrix}.
\]

Here the auxiliary rotation functions \( P_{\ell, k}(x) \), \( R_{\ell, k}(x) \), and \( T_{\ell, k}(x) \) are as defined in [52].

\[
P_{\ell, k}(\cos \theta) = d_{k,0}^\ell(\theta),
\]

\[
R_{\ell, k}(\cos \theta) = \frac{1}{2} \left( d_{k,2}^\ell(\theta) + d_{k,-2}^\ell(\theta) \right),
\]
We substitute this expansion into the scattering matrix to find,

\[
\mathcal{L} \mathbf{I} = \mathbf{I} - \sum_{\ell=0}^{\infty} \sum_{m=1}^{2} \sum_{u\ell=-\ell}^{2} \mathcal{D}_{\ell,m,k}(\mathbf{s}) F_{\ell} \left( \frac{1}{4\pi} \int_{S^2} \mathcal{D}_{\ell,m,k}(\mathbf{s}') \mathbf{I}(\mathbf{s}') d\mathbf{s}' \right)
\]

(2.70)

The matrices \(\mathcal{D}_{\ell,m,k}(\mathbf{s})\) satisfy the orthogonality property

\[
\int_{S^2} \mathcal{D}_{\ell,m,k}(\mathbf{s}) \mathcal{D}_{\ell',m',k'}(\mathbf{s}) d\mathbf{s} = \frac{4\pi}{2\ell + 1} \delta_{\ell,\ell'} \delta_{m,m'} \delta_{k,k'}.
\]

(2.71)

Let \(\mathbf{E}\) denote the 4 × 4 identity matrix. Using this we find

\[
\mathcal{L} \mathcal{D}_{\ell,m,k}(\mathbf{s}) \mathbf{v} = \mathcal{D}_{\ell,m,k}(\mathbf{s}) \left( \mathbf{E} - \frac{1}{2\ell + 1} F_{\ell} \right) \mathbf{v}.
\]

(2.72)

Hence, we can construct the eigenvalues and eigenfunctions of the scattering operator given the eigenvalues and eigenvectors

\[
\left( \mathbf{E} - \frac{1}{2\ell + 1} F_{\ell} \right) \mathbf{v}^{(i)}_{\ell} = \gamma^{(i)}_{\ell} \mathbf{v}^{(i)}_{\ell}.
\]

(2.73)

This can be done analytically as \(F_{\ell}\) is a 4 × 4 matrix that is block-diagonal. In particular, we draw attention to the fact an isotropic, unpolarized light source is an eigenvector with eigenvalues 0,

\[
\mathcal{L} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

(2.74)

### 2.4 One-dimensional radiative transport theory

In this section we study the steady-state radiative transport equation in one spatial dimension. Specifically, we consider light scattering through a one-dimensional slab. We will use some standard techniques to simplify the problem, and then introduce the discrete ordinate method.

We write the vector radiative transport equation in the non-dimensionalized form

\[
\mu \frac{\partial}{\partial \tau} \mathbf{I} + \frac{\bar{\omega}}{4\pi} \int_{\theta_0}^{\theta_1} \int_{-1}^{1} Z(\mu, \mu', \varphi, \varphi') \mathbf{I}(\tau, \mu', \varphi') d\mu' d\varphi' = 0
\]

(2.75)

where \(\tau = (\kappa_{\alpha} + \kappa_{\sigma}) z\) is the optical depth and \(\bar{\omega} = \kappa_{\sigma}/(\kappa_{\alpha} + \kappa_{\sigma})\) is the single scattering albedo.

We consider scattering on a slab \(0 < \tau \leq \tau_0\) with a beam incident on the lower boundary. The boundary conditions are given by

\[
\mathbf{I}(0, \mu, \varphi) = \pi \delta(\mu - \mu_0) \delta(\varphi - \varphi_0) \mathbf{I}_{inc}
\]

(2.76)

where \(\mathbf{I}_{inc} = [I_{inc}, Q_{inc}, U_{inc}, V_{inc}]^T\) is the Stokes vector of the incident beam, and

\[
\mathbf{I}(\tau_0, -\mu, \varphi) = 0
\]

(2.77)

for \(0 < \mu \leq 1\) and \(0 \leq \varphi < 2\pi\). As a subcase, we implement this method for a halfspace where the upper boundary condition becomes

\[
\lim_{\tau \to \infty} \mathbf{I}(\tau, \mu, \varphi) = 0
\]

(2.78)

for \(-1 \leq \mu \leq 1\) and \(0 \leq \varphi < 2\pi\).
2.4.1 Direct and diffuse radiance

The solution to scattering on the slab can be split into $I_{\text{dir}}(\tau; \mu, \varphi)$, the unscattered or direct radiance, and $I_{\text{diff}}(\tau; \mu, \varphi)$, the reduced or diffuse radiance,

$$I(\tau, \mu, \varphi) = I_{\text{dir}}(\tau, \mu, \varphi) + I_{\text{diff}}(\tau, \mu, \varphi). \quad (2.79)$$

The direct radiance solves the initial value problem

$$\mu \frac{\partial}{\partial \tau} I_{\text{dir}}(\tau, \mu, \varphi) = -I_{\text{dir}}(\tau, \mu, \varphi) \quad (2.80)$$

with initial condition

$$I(0, \mu, \varphi) = \pi \delta(\mu - \mu_0) \delta(\varphi - \varphi_0) I_{\text{inc}} \quad (2.81)$$

on the interval $0 < \mu \leq 1$. We use the method of characteristics to solve this directly,

$$I_{\text{dir}}(\tau, \mu, \varphi) = \pi \delta(\mu - \mu_0) \delta(\varphi - \varphi_0) I_{\text{inc}} e^{-\tau/\mu}. \quad (2.82)$$

We substitute $I_{\text{dir}}(\tau, \mu, \varphi)$ into the boundary value problem for the vector radiative transport equation to obtain the new boundary value problem

$$\mu \frac{\partial}{\partial \tau} I_{\text{diff}}(\tau, \mu, \varphi) = -I_{\text{diff}}(\tau, \mu, \varphi)$$

$$+ \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} Z(\mu, \mu', \varphi - \varphi') I_{\text{diff}}(\tau, \mu', \varphi') d\varphi' d\mu' + Q(\tau, \mu, \varphi) \quad (2.83)$$

where $Q(\tau, \mu, \varphi)$ is a source term given by

$$Q(\tau, \mu, \varphi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} Z(\mu, \mu', \varphi - \varphi') I_{\text{diff}}(\tau, \mu, \varphi) d\varphi' d\mu'$$

$$= \frac{\omega}{4} Z(\mu, \mu_0, \varphi - \varphi_0) I_{\text{inc}} e^{-\tau/\mu_0}. \quad (2.84)$$

The boundary conditions for the diffuse Stokes vector

$$I_{\text{diff}}(0, \mu, \varphi) = 0, \quad (2.85)$$

and

$$I_{\text{diff}}(\tau_0, -\mu, \varphi) = 0. \quad (2.86)$$

If we consider a semi-infinite halfspace, (2.86) is replaced by

$$\lim_{\tau \to \infty} I_{\text{diff}}(\tau, \mu, \varphi) = 0. \quad (2.87)$$

2.4.2 Fourier expansion in azimuth

In order to solve for diffuse Stokes vector we follow [52] and expand the diffuse radiance in a Fourier series,

$$I_{\text{diff}}(\tau, \mu, \varphi) = \sum_{k=0}^{\infty} \left( \frac{2 - \delta_{k,0}}{2} \right) \sum_{\alpha=1}^{2} \Phi_{\alpha,k}(\varphi - \varphi_0) I_{\alpha,k}(\tau, \mu) \quad (2.88)$$
where we define
\[ \Phi_{1,k}(\phi) = \text{diag}(\cos k\phi, \cos k\phi, \sin k\phi, \sin k\phi), \]
\[ \Phi_{2,k}(\phi) = \text{diag}(-\sin k\phi, -\sin k\phi, \cos k\phi, \cos k\phi). \]

The scattering matrix can be expanded in the Fourier series [26, §3.4.1]
\[ Z(\mu, \mu', \varphi - \varphi') = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{2} (2 - \delta_{k,0}) \Phi_{\alpha,k}(\varphi) Z_k(\mu, \mu') \Phi_{\alpha,k}(\varphi') \]
(2.91)
or, equivalently, as
\[ Z(\mu, \mu', \varphi - \varphi') = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{2} (2 - \delta_{k,0}) \Phi_{\alpha,k}(\varphi - \varphi') Z_k(\mu, \mu') D_{\alpha} \]
(2.92)
with the Fourier modes
\[ Z_k(\mu, \mu') = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \Phi_{1,k}(\phi) + \Phi_{2,k}(\phi) \right] Z(\mu, \mu', \phi) d\phi \]
(2.93)
and diagonal matrices
\[ D_1 = \text{diag}(1, 1, 0, 0) \]
(2.94)
and
\[ D_2 = \text{diag}(0, 0, 1, 1). \]
(2.95)

We use the expansion of the scattering matrix and basic trigonometric identities to compute
\[ \frac{\overline{\sigma}}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} Z(\mu, \mu', \varphi - \varphi') \Phi_{\alpha,k}(\varphi - \varphi_0) I_{\alpha,k}(\tau, \mu) d\varphi' d\mu' \]
\[ = \Phi_{\alpha,k}(\varphi - \varphi_0) \frac{\overline{\sigma}}{2} \int_{-1}^{1} Z_k(\mu, \mu') I_{\alpha,k}(\tau, \mu') d\mu'. \]
(2.96)
Moreover, by substituting (2.92) into (2.84),
\[ Q(\tau, \mu, \varphi) = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{2} \left( \frac{2 - \delta_{k,0}}{2} \right) \Phi_{\alpha,k}(\varphi - \varphi_0) Q_{\alpha,k}(\tau, \mu) \]
(2.97)
where
\[ Q_{\alpha,k}(\tau, \mu) = \frac{\overline{\sigma}}{2} Z_k(\mu, \mu_0) D_{\alpha} I_{\alpha,k}(\tau, \mu) e^{-\tau/\mu_0}. \]
(2.98)

We substitute (2.96) and (2.97) back into (2.83) to find
\[ \mu \frac{\partial}{\partial \tau} I_{\alpha,k}(\tau, \mu) = -I_{\alpha,k}(\tau, \mu) + \frac{\overline{\sigma}}{2} \int_{-1}^{1} Z_k(\mu, \mu') I_{\alpha,k}(\tau, \mu') d\mu' + Q_{\alpha,k}(\tau, \mu) \]
(2.99)

together with boundary conditions for the Fourier modes of the diffuse Stokes vector given by
\[ I_{\alpha,k}(0, \mu) = 0, \]
(2.100)
and
\[ I_{\alpha,k}(\tau_0, -\mu) = 0, \]
(2.101)
on a slab of thickness \( \tau_0 \), or given by
\[ I_{\alpha,k}(0, \mu) = 0, \]
(2.102)
and
\[ \lim_{\tau \to \infty} I_{\alpha,k}(\tau, \mu) = 0, \]
(2.103)
on a semi-infinite halfspace.
2.4.3 Half-range expansion

We split the radiative transport equation into two coupled equations on the half-range \(0 < \mu \leq 1\). We define the Stokes vectors propagating in the upward and downward directions by

\[
I^{(\pm)}_{\alpha,k}(\tau, \mu) = I^{(\pm)}_{\alpha,k}(\tau, \pm \mu)
\]

on the half-range \(0 < \mu \leq 1\). Here we take advantage of the symmetry property (see Hovenier et al. [26, p.85] for example)

\[
Z_k(\mu, -\mu') = D_{34} Z_k(\mu, \mu') D_{34}
\]

with

\[
D_{34} = \text{diag}(1, 1, -1, -1).
\]

We write the radiative transport equation on \(0 < \mu \leq 1\) as

\[
\mu \frac{\partial}{\partial \tau} I^{(+)}_{\alpha,k}(\tau, \mu) + I^{(+)}_{\alpha,k}(\tau, \mu) = \frac{\alpha_{\omega}}{2} \int_0^1 Z_k(\mu, \mu') I^{(+)}_{\alpha,k}(\tau, \mu') d\mu' + \frac{\alpha_{\omega}}{2} \int_0^1 D_{34} Z_k(-\mu, \mu') D_{34} I^{(-)}_{\alpha,k}(\tau, \mu') d\mu' + Q_{\alpha,k}(\tau, \mu).
\]

We substitute \(\mu \mapsto -\mu\) in equation (2.107) and left-multiply by \(D_{34}\) to get

\[
-\mu \frac{\partial}{\partial \tau} D_{34} I^{(-)}_{\alpha,k}(\tau, \mu) + D_{34} I^{(-)}_{\alpha,k}(\tau, \mu) = \frac{\alpha_{\omega}}{2} \int_0^1 D_{34} Z_k(-\mu, \mu') I^{(+)}_{\alpha,k}(\tau, \mu') d\mu' + \frac{\alpha_{\omega}}{2} \int_0^1 Z_k(\mu, \mu') D_{34} I^{(-)}_{\alpha,k}(\tau, \mu') d\mu' + D_{34} Q_{\alpha,k}(\tau, -\mu).
\]

We have reduced the problem from the range \(\mu \in [-1, 1]\) to the range \(\mu \in (0, 1]\). The boundary conditions are given by

\[
I^{(+)}_{\alpha,k}(0, \mu) = 0,
\]

and

\[
I^{(-)}_{\alpha,k}(\tau_0, \mu) = 0.
\]

on a slab of thickness \(\tau_0\), or given by

\[
I^{(+)}_{\alpha,k}(0, \mu) = 0.
\]

and

\[
\lim_{\tau \to \infty} I^{(\pm)}_{\alpha,k}(\tau, \mu) = 0
\]

on a semi-infinite halfspace.

2.4.4 Expansion of the scattering matrix

In order to construct the Fourier modes \(Z_k(\mu, \mu')\) of the scattering matrix we use the expansion of the scattering matrix in special functions. As discussed above, the scattering matrix can expanded as

\[
Z_k(\mu, \mu') = \sum_{\ell = k}^L \mathcal{P}_{\ell,k}(\mu) F_{\ell,k} \mathcal{P}_{\ell,k}(\mu')
\]
where $F_\ell$ is the coefficient matrix

$$F_\ell = \begin{bmatrix}
\alpha_1^\ell & \beta_1^\ell & 0 & 0 \\
\beta_1^\ell & \alpha_2^\ell & 0 & 0 \\
0 & 0 & \alpha_3^\ell & \beta_2^\ell \\
0 & 0 & -\beta_2^\ell & \alpha_4^\ell
\end{bmatrix}. \quad (2.114)$$

The matrix $\mathcal{P}_{\ell,k}(\mu)$ is a matrix of special functions,

$$\mathcal{P}_{\ell,k}(\mu) = \begin{bmatrix}
P_{\ell,k}(\mu) & 0 & 0 & 0 \\
0 & R_{\ell,k}(\mu) & -T_{\ell,k}(\mu) & 0 \\
0 & -T_{\ell,k}(\mu) & R_{\ell,k}(\mu) & 0 \\
0 & 0 & 0 & P_{\ell,k}(\mu)
\end{bmatrix}. \quad (2.115)$$

We call the special functions $P_{\ell,k}(\mu)$, $R_{\ell,k}(\mu)$, and $T_{\ell,k}(\mu)$ auxiliary rotation function, or just auxiliary functions. These functions are discussed in detail in the Appendix B. In theory, the expansion may be infinite; however, we assume the scattering matrix is approximately of finite order $L$.

Thus we can write the integrals over the half-range as

$$\int_0^1 Z_k(\mu, \mu') I^{(+)}_{\alpha,k}(\tau, \mu') d\mu' = \sum_{\ell=k}^{L} \mathcal{P}_{\ell,k}(\mu) F_\ell \left( \frac{\omega}{2} \int_0^1 \mathcal{P}_{\ell,k}(\mu') I^{(+)}_{\alpha,k}(\tau, \mu') d\mu' \right) \quad (2.116)$$

and

$$\int_0^1 Z_k(\mu, \mu') D_{34} I^{(-)}_{\alpha,k}(\tau, \mu') d\mu' = \sum_{\ell=k}^{L} \mathcal{P}_{\ell,k}(\mu) F_\ell \left( \frac{\omega}{2} \int_0^1 \mathcal{P}_{\ell,k}(\mu') D_{34} I^{(-)}_{\alpha,k}(\tau, \mu') d\mu' \right). \quad (2.117)$$

Hence, the split vector radiative transport equations are

$$\mu \frac{\partial}{\partial \tau} I^{(+)}_{\alpha,k}(\tau, \mu) + I^{(+)}_{\alpha,k}(\tau, \mu) = \sum_{\ell=k}^{L} \mathcal{P}_{\ell,k}(\mu) F_\ell \left( \frac{\omega}{2} \int_0^1 \mathcal{P}_{\ell,k}(\mu') I^{(+)}_{\alpha,k}(\tau, \mu') d\mu' \right)$$

$$+ \sum_{\ell=k}^{L} D_{34} \mathcal{P}_{\ell,k}(-\mu) F_\ell \left( \frac{\omega}{2} \int_0^1 \mathcal{P}_{\ell,k}(\mu') D_{34} I^{(-)}_{\alpha,k}(\tau, \mu') d\mu' \right) + Q_{\alpha,k}(\tau, \mu) \quad (2.118)$$

and

$$-\mu \frac{\partial}{\partial \tau} D_{34} I^{(-)}_{\alpha,k}(\tau, \mu) + D_{34} I^{(-)}_{\alpha,k}(\tau, \mu) = \sum_{\ell=k}^{L} D_{34} \mathcal{P}_{\ell,k}(-\mu) F_\ell \left( \frac{\omega}{2} \int_0^1 \mathcal{P}_{\ell,k}(\mu') I^{(+)}_{\alpha,k}(\tau, \mu') d\mu' \right)$$

$$+ \sum_{\ell=k}^{L} \mathcal{P}_{\ell,k}(\mu) F_\ell \left( \frac{\omega}{2} \int_0^1 \mathcal{P}_{\ell,k}(\mu') D_{34} I^{(-)}_{\alpha,k}(\tau, \mu') d\mu' \right) + D_{34} Q_{\alpha,k}(\tau, -\mu). \quad (2.119)$$
2.4.5 The discrete ordinate method

The discrete ordinate method is a numerical technique for solving the radiative transport equation developed by Chandrasekhar [8]. The “ordinates”, or polar cosine, are discretized to reduce the integral equation to a linear first-order system of differential equations. In the one-dimensional case the Fourier modes are decoupled, and so the resulting system of equations can be solved independently.

In our work we use the Gauss-Legendre quadrature with $2N$ gridpoints. However, since we split the transport equation on the half-range, we only use the $N$ positive quadrature points in our computations. In principle any quadrature on the half-range $(0, 1]$ can be used. It is important that the origin, $\mu = 0$, is not included in the quadrature as the radiative transport equation is singular at the origin.

The discrete transport equations is given by

$$
M \frac{\partial}{\partial \tau} \hat{I}^{(+)}_{\alpha,k} = \left( -E + \sum_{\ell=k}^{L} \mathcal{J}^{(+)}_{\ell,k} \right) \hat{I}^{(+)}_{\alpha,k} + \left( \sum_{\ell=k}^{L} \mathcal{J}^{(-)}_{\ell,k} \right) \Delta_{34} \hat{I}^{(-)}_{\alpha,k} + Q^{(+)}_{\alpha,k} e^{-\tau/\mu_0},
$$

(2.120)

and

$$
- M \frac{\partial}{\partial \tau} \Delta_{34} \hat{I}^{(-)}_{\alpha,k} = \left( \sum_{\ell=k}^{L} \mathcal{J}^{(-)}_{\ell,k} \right) \hat{I}^{(+)}_{\alpha,k} + \left( -E + \sum_{\ell=k}^{L} \mathcal{J}^{(+)}_{\ell,k} \right) \Delta_{34} \hat{I}^{(-)}_{\alpha,k} + \Delta_{34} Q^{(-)}_{\alpha,k} e^{-\tau/\mu_0},
$$

(2.121)

with boundary conditions given by

$$
\hat{I}^{(+)}_{\alpha,k}(0) = 0,
$$

(2.122)

and

$$
\hat{I}^{(-)}_{\alpha,k}(\tau_0) = 0,
$$

(2.123)

on a slab of thickness $\tau_0$, or given by

$$
\hat{I}^{(+)}_{\alpha,k}(0) = 0,
$$

(2.124)

and

$$
\lim_{\tau \to \infty} \hat{I}^{(\pm)}_{\alpha,k}(\tau, \mu) = 0,
$$

(2.125)

on a semi-infinite halfspace, where we define the following.

Let $\mu_i$ denote the positive quadrature points for $i = 1, 2, \ldots, N$, and let $w_i$ denote the associated weights. We define the discretize Stokes vector over the positive and negative half-ranges as the vectors $\hat{I}^{(+)}_{\alpha,k}$ and $\hat{I}^{(-)}_{\alpha,k}$ of length $4N$ given by

$$
\hat{I}^{(\pm)}_{\alpha,k}(\tau) = \begin{bmatrix}
\hat{I}^{(\pm)}_{\alpha,k}(\tau, \mu_1) \\
\hat{I}^{(\pm)}_{\alpha,k}(\tau, \mu_2) \\
\vdots \\
\hat{I}^{(\pm)}_{\alpha,k}(\tau, \mu_N)
\end{bmatrix}.
$$

(2.126)

Let $M$ and $W$ be the $4N \times 4N$ diagonal matrices

$$
M = \text{diag}(\mu_1 E_{4 \times 4}, \mu_2 E_{4 \times 4}, \ldots, \mu_N E_{4 \times 4}),
$$

(2.127)
and
\[ W = \text{diag} \left( w_1 E_{4 \times 4}, w_2 E_{4 \times 4}, \ldots, w_N E_{4 \times 4} \right). \] (2.128)

We let \( P_{\ell,k} \) be the \( 4N \times 4 \) matrix
\[ P_{\ell,k} = \begin{bmatrix} P_{\ell,k}(\mu_1) \\ P_{\ell,k}(\mu_2) \\ \vdots \\ P_{\ell,k}(\mu_N) \end{bmatrix}. \] (2.129)

Let source terms \( \hat{Q}_{\alpha,k}^{(+)} \) be defined as
\[ \hat{Q}_{\alpha,k}^{(+)} = \sum_{\ell=k}^{L} \frac{\partial}{\partial x} F_{\ell,k} P_{\ell,k}(\mu_0) D_{\alpha} I_{\text{inc}}, \] (2.130)
and let \( \hat{Q}_{\alpha,k}^{(-)} \) be defined as
\[ \Delta_{34} \hat{Q}_{\alpha,k}^{(-)} = \sum_{\ell=k}^{L} (-1)^{\ell-k} P_{\ell,k} D_{34} \frac{\partial}{\partial x} F_{\ell,k}(\mu_0) D_{\alpha} I_{\text{inc}}, \] (2.131)
where \( \Delta_{34} \) is the \( 4N \times 4N \) diagonal matrix
\[ \Delta_{34} = \text{diag}(D_{34}, D_{34}, \ldots, D_{34}), \] (2.132)

with \( D_{34} = \text{diag}(1, 1, -1, -1) \) as defined above. Finally, we define the \( 4N \times 4N \) matrices
\[ F_{\ell,k}^{(+)} = \frac{\partial}{\partial x} P_{\ell,k} F_{\ell,k} P_{\ell,k}^T W, \] (2.133)
and
\[ F_{\ell,k}^{(-)} = \frac{\partial}{\partial x} (-1)^{\ell-k} P_{\ell,k} D_{34} F_{\ell,k} P_{\ell,k}^T W. \] (2.134)

We can write equations (2.120) and (2.121) in matrix form as
\[ \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} I_{\alpha,k}^{(+)}(\tau) \\ \Delta_{34} I_{\alpha,k}^{(-)}(\tau) \end{bmatrix} = \left( -\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} + \sum_{\ell=k}^{L} \begin{bmatrix} F_{\ell,k}^{(+)} & F_{\ell,k}^{(-)} \\ F_{\ell,k}^{(-)} & F_{\ell,k}^{(+)} \end{bmatrix} \right) \begin{bmatrix} I_{\alpha,k}^{(+)}(\tau) \\ \Delta_{34} I_{\alpha,k}^{(-)}(\tau) \end{bmatrix} + \begin{bmatrix} \hat{Q}_{\alpha,k}^{(+)} \\ \Delta_{34} \hat{Q}_{\alpha,k}^{(-)} \end{bmatrix} e^{-\tau/\mu_0}, \] (2.135)

with boundary conditions
\[ \begin{bmatrix} I_{\alpha,k}^{(+)}(0) \\ I_{\alpha,k}^{(-)}(\tau_0) \end{bmatrix} = 0. \] (2.136)

This is simply a first order system of equations, and can be solved using standard techniques.

Instead of directly solving the \( 8N \times 8N \) first order system of differential equations given in equation (2.135), we use the symmetry of the system to convert into an \( 4N \times 4N \) second-order differential system of equations. We introduce the vectors
\[ \hat{\Psi}_{\alpha,k}(\tau) = \frac{1}{2} M \left( I_{\alpha,k}^{(+)}(\tau) + \Delta_{34} I_{\alpha,k}^{(-)}(\tau) \right) \] (2.137)
and
\[
\hat{\Gamma}_{\alpha,k}(\tau) = \frac{1}{2} M \left( \hat{I}^{(1)}_{\alpha,k}(\tau) - \Delta_{34} \hat{I}^{(-)}_{\alpha,k}(\tau) \right).
\] (2.138)

Functions \( \hat{\Psi}(\tau) \) and \( \hat{\Gamma}(\tau) \) solve the differential system
\[
\hat{\Psi}'_{\alpha,k}(\tau) = A_k \hat{\Psi}_{\alpha,k}(\tau) + \sigma^{(1)}_{\alpha,k} e^{-\tau/\mu_0},
\] (2.139)
\[
\hat{\Gamma}'_{\alpha,k}(\tau) = B_k \hat{\Psi}_{\alpha,k}(\tau) + \sigma^{(2)}_{\alpha,k} e^{-\tau/\mu_0}.
\] (2.140)

The matrices \( A \) and \( B \) are
\[
A_k = \left( -E_{4N \times 4N} + \frac{\omega}{2} \sum_{\ell = k}^{L} P_{\ell,k} \left( E_{4 \times 4} - (-1)^{\ell-k} D_{34} \right) F_{\ell} P_{\ell,k}^T W \right) M^{-1},
\] (2.141)
and
\[
B_k = \left( -E_{4N \times 4N} + \frac{\omega}{2} \sum_{\ell = k}^{L} P_{\ell,k} \left( E_{4 \times 4} + (-1)^{\ell-k} D_{34} \right) F_{\ell} P_{\ell,k}^T W \right) M^{-1},
\] (2.142)
and the the source terms \( \sigma_1 \) and \( \sigma_2 \) are
\[
\sigma^{(1)}_{\alpha,k} = \frac{\omega}{4} \sum_{\ell = k}^{L} P_{\ell,k} \left( E_{4 \times 4} - (-1)^{\ell-k} D_{34} \right) F_{\ell} P_{\ell,k}(\mu_0) D_{\alpha} I_{\text{inc}},
\] (2.143)
and
\[
\sigma^{(2)}_{\alpha,k} = \frac{\omega}{4} \sum_{\ell = k}^{L} P_{\ell,k} \left( E_{4 \times 4} + (-1)^{\ell-k} D_{34} \right) F_{\ell} P_{\ell,k}(\mu_0) D_{\alpha} I_{\text{inc}}.
\] (2.144)

We reduce the dimensions of the system given in (2.139) and (2.140) by instead solving a second-order differential equation for \( \hat{\Psi}_{\alpha,k}(\tau) \) and using this solution to construct \( \hat{\Gamma}_{\alpha,k}(\tau) \). This method is called the double Gauss method, and it is implemented in the discrete ordinate method for the vector radiative transport equation described in [52]. We compute a second-order differential equation for \( \hat{\Psi}_{\alpha,k}(\tau) \) by differentiating (2.139) and substitute in (2.140),
\[
\hat{\Psi}''_{\alpha,k}(\tau) = A_k B_k \hat{\Psi}_{\alpha,k}(\tau) + \tilde{\sigma}_{\alpha,k} e^{-\tau/\mu_0},
\] (2.145)
where
\[
\tilde{\sigma}_{\alpha,k} = -\frac{1}{\mu_0} \sigma^{(1)}_{\alpha,k} + A_k \sigma^{(2)}_{\alpha,k}.
\] (2.146)
We use the spectrum of \( A_k B_k \) to solve the second-order differential system. The matrix \( A_k B_k \) is non-degenerate and all eigenvalues have non-negative real part. There is a eigenvalue with zero real part if and only if scattering is conservative,\( i.e. \omega = 1 \). We will discuss the solution for non-conservative scattering first, and then address the case of conservative scattering.

**Non-conservative scattering**

In the case where \( \omega < 1 \), the matrix \( A_k B_k \) is non-degenerate and all eigenvalues have a positive real part. We solve (2.145) by computing the eigenvalue decomposition
\[
A_k B_k = P_k A^2_k P^{-1}_k,
\] (2.147)
where \( P_k \) is some, possibly complex, invertible matrix,

\[
P_k = \begin{bmatrix} \hat{\Psi}_{1,k} & \hat{\Psi}_{2,k} & \cdots & \hat{\Psi}_{4N,k} \end{bmatrix},
\]

and \( \Lambda_k \) is a diagonal matrix,

\[
\Lambda_k = \text{diag}(\lambda_{1,k}, \lambda_{2,k}, \ldots, \lambda_{4N,k}),
\]

where \( \lambda^2_{j,k} \) are the \( 4N \) eigenvalues of \( A_k B_k \) and \( \text{Re} \left[ \lambda_{j,k} \right] > 0 \).

The homogeneous solution of (2.145) is

\[
\hat{\Psi}_{h,\alpha,k}(\tau) = \sum_{u=1}^{4N} c_{j,\alpha,k} \hat{\Psi}_{j,k} e^{-\lambda_{j,k}\tau} + \sum_{u=1}^{4N} c_{j,\alpha,k} \hat{\Psi}_{j,k} e^{-\lambda_{j,k}(\tau_0-\tau)}.
\]

The particular solution has the general form

\[
\hat{\Psi}_{p,\alpha,k}(\tau) = \left( \tau w^{(0)}_{\alpha,k} + w^{(1)}_{\alpha,k} \right) e^{-\tau/\mu_0},
\]

where \( v^{(0)}_{\alpha,k} \) is identically zero except in the unlikely case were \( 1/\mu_0^2 \) is an eigenvalue of \( A_k B_k \). We substitute this into ansatz into (2.145) to find

\[
\frac{1}{\mu_0} v^{(0)}_{\alpha,k} - A_k B_k v^{(0)}_{\alpha,k} = 0,
\]

and

\[
\left( \frac{1}{\mu_0^2} E - A_k B_k \right) v^{(1)}_{\alpha,k} - \frac{2}{\mu_0} v^{(0)}_{\alpha,k} = \tilde{\sigma}_{\alpha,k}.
\]

Next, we compute \( \hat{\Gamma}_{\alpha,k}(\tau) \) from \( \hat{\Psi}_{\alpha,k}(\tau) \) using (2.140). The particular solution is

\[
\hat{\Gamma}_{p,\alpha,k}(\tau) = \left( \tau w^{(0)}_{\alpha,k} + w^{(2)}_{\alpha,k} \right) e^{-\tau/\mu_0},
\]

where

\[
w^{(0)}_{\alpha,k} = -\mu_0 B v^{(0)}_{\alpha,k},
\]

and

\[
w^{(1)}_{\alpha,k} = -\mu_0 B v^{(1)}_{\alpha,k} - \mu_0^2 v^{(0)}_{\alpha,k} - \mu_0 \sigma^{(2)}_{\alpha,k}.
\]

The homogeneous solution is

\[
\hat{\Gamma}_{h,\alpha,k}(\tau) = \sum_{j=1}^{4N} c_{j,\alpha,k} \hat{\Gamma}_{j,k} e^{-\lambda_{j,k}\tau} - \sum_{j=1}^{4N} c_{j,\alpha,k} \hat{\Gamma}_{j,k} e^{-\lambda_{j,k}(\tau_0-\tau)},
\]

with

\[
\hat{\Gamma}_{j,k} = -\frac{1}{\lambda_{j,k}} B \hat{\Psi}_{j,k}.
\]

Using definitions (2.137) and (2.138) we can now construct the solution of the Stokes vector,

\[
\hat{I}^{(+)}_{\alpha,k}(\tau) = \hat{I}^{(+)}_{p,\alpha,k}(\tau) + \sum_{j=1}^{4N} c_{j} \hat{I}^{(+)}_{j,k} e^{-\lambda_{j,k}\tau} + \sum_{j=1}^{4N} c_{j} \Delta_{3,4} \hat{I}^{(+)}_{j,k} e^{-\lambda_{j,k}(\tau_0-\tau)},
\]

with

\[
\Delta_{3,4} = \frac{1}{\lambda_{3,4}} B \hat{\Psi}_{3,4}.
\]
\[
\tilde{I}_{\alpha,k}^{(-)}(\tau) = \tilde{I}_{p,\alpha,k}^{(-)}(\tau) + \sum_{j=1}^{4N} c_{j,\alpha,k} \tilde{I}_{j,k}^{(-)} e^{-\lambda_{j,k} \tau} + \sum_{j=1}^{4N} c_{-j,\Delta_{3,4},4} \tilde{I}_{j,k}^{(+)} e^{-\lambda_{j,k} (\tau_0 - \tau)},
\]  
(2.160)

where the particular solutions are given by

\[
\tilde{I}_{p,\alpha,k}^{(+)}(\tau) = \left[\tau \left( v_{\alpha,k}^{(0)} + w_{\alpha,k}^{(0)} \right) + \left( v_{\alpha,k}^{(1)} + w_{\alpha,k}^{(1)} \right) \right] e^{-\tau/\mu_0},
\]
(2.161)

\[
\tilde{I}_{p,\alpha,k}^{(-)}(\tau) = \left[\tau \Delta_{3,4} \left( v_{\alpha,k}^{(0)} - w_{\alpha,k}^{(0)} \right) + \Delta_{3,4} \left( v_{\alpha,k}^{(1)} - w_{\alpha,k}^{(1)} \right) \right] e^{-\tau/\mu_0},
\]
(2.162)

and the eigenvectors that form the homogeneous solution are given by

\[
\tilde{I}_{j,k}^{(+)} = \Psi_{j,k} + \Gamma_{j,k},
\]
(2.163)

\[
\tilde{I}_{j,k}^{(-)} = \Delta_{3,4} \left( \Psi_{j,k} - \Gamma_{j,k} \right).
\]
(2.164)

We solve for the expansion coefficients using the boundary conditions

\[
\sum_{j=1}^{4N} c_{j,\alpha,k} \tilde{I}_{j,k}^{(+)} + \sum_{j=1}^{4N} c_{-j,\alpha,k} \Delta_{3,4} \tilde{I}_{j,k}^{(-)} e^{-\lambda_{j,k} \tau_0} = -\tilde{I}_{p,\alpha,k}^{(+)}(0),
\]
(2.165)

and

\[
\sum_{j=1}^{4N} c_{j,\alpha,k} \tilde{I}_{j,k}^{(-)} e^{-\lambda_{j,k} \tau_0} + \sum_{j=1}^{4N} c_{-j,\alpha,k} \Delta_{3,4} \tilde{I}_{j,k}^{(+)} = -\tilde{I}_{p,\alpha,k}^{(-)}(\tau_0).
\]
(2.166)

If we are solving the problem on a halfspace, the boundary conditions reduced to

\[
\sum_{j=1}^{4N} c_{j,\alpha,k} \tilde{I}_{j,k}^{(+)} = -\tilde{I}_{p,\alpha,k}^{(+)}(0),
\]
(2.167)

and

\[
c_{-j,\alpha,k} = 0, j = 1, 2, ... 4N.
\]
(2.168)

**Conservative scattering**

In the case of conservative scattering, there is exactly one zero eigenvalue in the zeroth Fourier mode. In this case, we find

\[
D \hat{\Psi}_0 = 0,
\]
(2.169)

for

\[
\hat{\Psi}_0 = [[\mu_1, 0, 0, 0], [\mu_2, 0, 0, 0], \ldots, [\mu_N, 0, 0, 0]]^T.
\]
(2.170)

Hence,

\[
\begin{bmatrix}
\hat{\Psi}_0 \\
0
\end{bmatrix}
\]
(2.171)

is a solution. Next, we use the ansatz

\[
\begin{bmatrix}
\tau \hat{\Psi}_0 \\
\hat{\Gamma}_0
\end{bmatrix}
\]
(2.172)

to find a second solution that corresponds to the zero eigenvalue. This is satisfied when

\[
\hat{\Psi}_0 = A \hat{\Gamma}_0.
\]
(2.173)

21
We can solve this analytically to find

$$\hat{\Gamma}_0 = \frac{1}{1 - g} M \hat{\Psi}_0,$$  \hspace{1cm} (2.174)

where $g = \alpha_1^1/3$ is the anisotropy factor. The remaining of the analysis is the same as the non-conservative case, and the full solution is

$$\hat{I}_{\alpha,k}^{(+)}(\tau) = I_{p,\alpha,k}^{(+)}(\tau) + c_1 \hat{i}_0 + c_{-1} \left( \tau \hat{i}_0 + \frac{1}{1 - g} M \hat{i}_0 \right) + \sum_{j=2}^{4N} c_{j,\alpha,k} I_{j,k}^{(+)} e^{-\lambda_{j,k}\tau} + \sum_{j=1}^{4N} c_{-j,\alpha,k} \Delta_{3,4} I_{j,k}^{(-)} e^{-\lambda_{j,k}(\tau_0 - \tau)},  \hspace{1cm} (2.175)$$

and

$$\hat{I}_{\alpha,k}^{(-)}(\tau) = I_{p,\alpha,k}^{(-)}(\tau) + c_1 \hat{i}_0 + c_{-1} \left( \tau \hat{i}_0 + \frac{1}{1 - g} M \hat{i}_0 \right) e^{-\tau/\mu_0} + \sum_{j=2}^{4N} c_{j,\alpha,k} I_{j,k}^{(-)} e^{-\lambda_{j,k}\tau} + \sum_{j=2}^{4N} c_{-j,\alpha,k} \Delta_{3,4} I_{j,k}^{(+)} e^{-\lambda_{j,k}(\tau_0 - \tau)},  \hspace{1cm} (2.176)$$

where $\hat{i}_0$ is an unpolarized, isotropic source,

$$\hat{i}_0 = [[1, 0, 0, 0], [1, 0, 0, 0], ..., [1, 0, 0, 0]]^T,$$ \hspace{1cm} (2.177)

and the functions $\hat{I}_{p,\alpha,k}^{(+)}(\tau)$ and vectors $\hat{I}_{j,k}^{(\pm)}$ are as defined above.

In this case, the expansion coefficients are found using the boundary conditions. For a slab of thickness $\tau_0$, the boundary conditions are

$$c_1 \hat{i}_0 + c_{-1} \left( \frac{1}{1 - g} M \hat{i}_0 \right) + \sum_{j=2}^{4N} c_{j,\alpha,k} I_{j,k}^{(+)} + \sum_{j=2}^{4N} c_{-j,\alpha,k} \Delta_{3,4} I_{j,k}^{(-)} e^{-\lambda_{j,k}\tau_0} = -I_{p,\alpha,k}^{(+)}(0),  \hspace{1cm} (2.178)$$

and

$$c_1 \hat{i}_0 + c_{-1} \left( \tau \hat{i}_0 + \frac{1}{1 - g} M \hat{i}_0 \right) e^{-\tau/\mu_0} + \sum_{j=2}^{4N} c_{j,\alpha,k} I_{j,k}^{(-)} e^{-\lambda_{j,k}\tau_0} + \sum_{j=2}^{4N} c_{-j,\alpha,k} \Delta_{3,4} I_{j,k}^{(+)} = -I_{p,\alpha,k}^{(-)}(\tau_0).  \hspace{1cm} (2.179)$$

For a semi-infinite halfspace, the boundary conditions are

$$c_1 \hat{i}_0 + \sum_{j=2}^{4N} c_{j,\alpha,k} I_{j,k}^{(+)} = -I_{p,\alpha,k}^{(+)}(0),  \hspace{1cm} (2.180)$$

and

$$c_{-j,\alpha,k} = 0, \hspace{0.5cm} j = 1, 2, ... 4N.  \hspace{1cm} (2.181)$$
2.5 Summary

In this chapter we introduced radiative transport theory. First, we used the concept of an electromagnetic planewave to define the Stokes vector. We discussed the redistribution of light during a scattering event. Next we discussed the radiative transport equation and introduced a spectral decomposition of the scattering operator. Finally, we discussed studying and solving the one-dimensional steady state radiative transport equation using the discrete ordinate method. In Chapter 3 we will study the scattering operator when scattering is peaked in the forward direction, and in Chapter 4 we will use both the spectral decomposition of the scattering operator and the discrete ordinate method to approximate the radiative transport equation when scattering is strong and absorption is weak.
Chapter 3

Polarized light in forward-peaked scattering media

3.1 Introduction

For many multiple scattering media, such as clouds, oceans, and biological tissue, light scattering is peaked about the forward direction. Moreover, circular polarization memory is known to be especially prominent in these forward-peaked scattering media [32, 39, 58]. Recall that circular polarization memory is the physical phenomenon whereby circular polarization retains its ellipticity and handedness when propagating in anisotropic random media. To study circular polarization in forward-peaked media we seek to extend approximations of the scalar radiative transport equation to the vector radiative transport equation. Forward-peaked scattering has been studied extensively for the case in which polarization is neglected. Consider the scalar radiative transport equation,

\[ \tilde{s} \cdot \nabla I + \kappa_a I + \kappa_s \mathcal{L} I = Q, \]  

(3.1)

where \( I \) denotes the radiance or intensity, \( \kappa_a \) is the absorption coefficient, \( \kappa_s \) is the scattering coefficient, \( Q \) is a scalar source term, and \( \mathcal{L} \) is the scattering operator

\[ \mathcal{L} I = I - \frac{1}{4\pi} \int_{S^2} f(\tilde{s} \cdot \tilde{s}') I(x, \tilde{s}') d\tilde{s}'. \]  

(3.2)

Here \( f(\tilde{s} \cdot \tilde{s}') \) is the phase function which gives the amount of light scattered from incident direction \( \tilde{s}' \) into scattered direction \( \tilde{s}' \). We say that scattering is forward-peaked if scattering is dominated by scattering events in the forward direction. This corresponds to a a peak in the phase function about \( \tilde{s} = \tilde{s}' \). In the case of forward-peaked scattering the phase function becomes difficult to resolve, and computing the solution to the radiative transport equation becomes numerically expensive. A number of approximations have been proposed in order to simplify computations in this case. These approximations include the Fokker-Planck approximation [7, 47], the Fermi pencil beam approximation [3], the small-angle approximation [27], and the \( \delta \)-Eddington approximation [29].

The scalar Fokker-Planck approximation is obtained by approximation the scattering operator with the spherical Laplacian. While the Fokker-Planck approximation itself only holds in very strong limit where there is little to no large angle scattering [47], it can be generalized for higher order expansions [37, 48], and it can be used in combination with other operators to account for large angle scattering [20]. Moreover, it can be used to derive the Fermi-Pencil beam approximation [3]. In this chapter we will derive a generalization of the Fokker-Planck approximation for the vector
radiative transport equation. This work was previously published in [11]. First, we give a brief introduction to the scalar Fokker-Planck approximation. Next, we derive the vector Fokker-Planck approximation. Finally, we discuss the results from our analysis.

3.2 The scalar Fokker-Planck approximation

The scalar Fokker-Planck approximation consists of approximating the scattering operator according to

\[ \mathcal{L}I \approx -\frac{1}{2} (1 - g) \Delta_{\hat{s}} I, \] (3.3)

where \( g \) is the anisotropy factor, or average cosine,

\[ g = \frac{1}{2} \int_0^\pi \cos \Theta f(\Theta) \sin \Theta d\Theta, \] (3.4)

and \( \Delta_{\hat{s}} \) is the spherical Laplacian,

\[ \Delta_{\hat{s}} = \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial}{\partial \mu} \right] + \frac{1}{1 - \mu^2} \frac{\partial^2}{\partial \varphi^2}. \] (3.5)

In the classical derivation of the Fokker-Planck approximation, the intensity is expanded in a Taylor series in the incident direction \( \hat{s}' \) about the scattered direction \( \hat{s} \) in the scattering operator. The Fokker-Planck approximation is then derived by truncating the Taylor series after two terms. Later, it was shown the Fokker-Planck approximations can be derived as an asymptotic limit of the radiative transport equation [37, 47, 48]. In the asymptotic theory, it is noted that the scattering operator and the spherical Laplacian both have spherical harmonics as eigenfunctions where

\[ \mathcal{L}Y_{\ell,k}(\hat{s}) = (1 - f_{\ell})Y_{\ell,k}(\hat{s}), \] (3.6)

and

\[ -\frac{1}{2} (1 - g) \Delta_{\hat{s}} Y_{\ell,k}(\hat{s}) = \frac{1}{2} \ell(\ell + 1)(1 - g) Y_{\ell,k}(\hat{s}). \] (3.7)

In particular, it is worth noting

\[ (1 - f_0) = 0, \] (3.8)

and

\[ (1 - f_1) = (1 - g), \] (3.9)

hence the first two eigenvalues of the scattering operator match those of \( -\frac{1}{2} (1 - g) \Delta_{\hat{s}} \) exactly.

3.3 Derivation of the vector Fokker-Planck approximation

Here we seek to generalize the Fokker-Planck approximation to include the polarization state. The vector radiative transport equation is

\[ \hat{s} \cdot \nabla I + \kappa_\alpha I + \kappa_s \mathcal{L} I = Q, \] (3.10)

where \( Q \) is a source term, \( \kappa_\alpha \) is the absorption coefficient, \( \kappa_s \) is the scattering coefficient, and the scattering operator \( \mathcal{L} \) is given by

\[ \mathcal{L} I = I - \frac{1}{4\pi} \int_{S^2} Z(\hat{s}, \hat{s}') I(\hat{s}') d\hat{s}'. \] (3.11)
Recall from Section 2.3.2, the scattering matrix \( Z(\hat{s}, \hat{s}') \) can be written as the product
\[
Z(\hat{s}, \hat{s}') = H^T(\Phi)F(\Theta)H(\chi),
\]
where the angles \( \chi \) and \( \Phi \) are the angles such that \( R(\chi, \hat{s}') \) and \( R(\Phi, \hat{s}) \) are the rotations from the experimental frame to the scattering frame. The rotations of the Stokes vector is given by
\[
H(\sigma) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\sigma & -\sin 2\sigma & 0 \\
0 & \sin 2\sigma & \cos 2\sigma & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]
and the phase matrix \( F \) has the general form
\[
F(\cos \Theta) = \begin{bmatrix}
a_1 & b_1 & 0 & 0 \\
b_1 & a_2 & 0 & 0 \\
0 & 0 & a_3 & b_2 \\
0 & 0 & -b_2 & a_4
\end{bmatrix}.
\]

The component \( 0 \leq a_1 \leq 1 \) is exactly the scattering phase function, \( a_1 = f \), and is normalized according to
\[
\frac{1}{2} \int_0^\pi a_1(\cos \Theta) \sin \Theta d\Theta = 1.
\]
In addition, \( a_1 \geq |b_i| \) for \( i = 1, 2 \) and \( a_1 \geq |a_j| \) for \( j = 2, 3, 4 \).

### 3.3.1 Expansion of the scattering operator in differential operators

Rather than Taylor expanding the Stokes vector \( I(\hat{s}') \) about \( \hat{s} \), we use that \( H(\chi)I(\hat{s}') \) is related to \( H(\Phi)I(\hat{s}) \) by a rotation of \( \Theta \) in the scattering plane,
\[
H(\chi)I(\hat{s}') = R(\Theta, \hat{n}_s)H(\Phi)I(\hat{s}),
\]
where \( \hat{n}_s \) is a unit vector normal to both \( \hat{s} \) and \( \hat{s}' \). Let \( J \) denote the total angular momentum operator, and define \( J_s \) to be the component of the total angular momentum operator in the direction of the \( \hat{n}_s \),
\[
J_s = \hat{n}_s \cdot J.
\]
Since the operator \( J_s \) generates infinitesimal rotations in the scatter plane, \( H(\chi)I(\hat{s}') \) can be expanded with respect to \( H(\Phi)I(\hat{s}) \) about \( \Theta = 0 \) as
\[
H(\chi)I(\hat{s}') = e^{-i\Theta J_s}H(\Phi)I(\hat{s}),
\]
or,
\[
H(\chi)I(\hat{s}') = \sum_{k=0}^\infty \frac{(-i\Theta)^k}{k!} J_s^k [H(\Phi)I(\hat{s})].
\]

We substitute (3.19) into the scattering operator and assume the integrals and series may be exchanged,
\[
\mathcal{L} = I - \sum_{k=0}^\infty \frac{1}{2\pi} \int_0^{2\pi} H^T(\Phi) \left( \frac{1}{2} \int_0^\pi (-i\Theta)^k \left[ F(\Theta) \sin \Theta d\Theta \right] J_s^k [H(\Phi)I(\hat{s})] \right) d\Phi.
\]
The integrals in (3.20) are identically zero for odd $k$. To see this we use the contravariant spherical components of the total angular momentum operator. Let $\mathbf{\hat{s}} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, the contravariant spherical components of the total angular momentum operator are given by Varshalovich et al. [57, p.75] in terms of the Euler angles $\varphi, \theta,$ and $\Phi$ by

$$J_{\pm 1}' = \frac{i}{\sqrt{2}} e^{\mp i\Phi} \left( \pm \cot \theta \frac{\partial}{\partial \Phi} + i \frac{\partial}{\partial \theta} \mp \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right), \quad (3.21)$$

$$J_0' = -i \frac{\partial}{\partial \Phi}. \quad (3.22)$$

The normal to the scattering plane, $\mathbf{\hat{n}}$, can be written in terms of contravariant spherical components as

$$\mathbf{\hat{n}} = \frac{1}{\sqrt{2}} (\mathbf{\hat{e}}_{+1}' - \mathbf{\hat{e}}_{-1}'). \quad (3.23)$$

Hence,

$$J_s = \frac{1}{\sqrt{2}} (J_{+1}' - J_{-1}'). \quad (3.24)$$

Now taking the $k$th power we find

$$J_s^k = \left( \frac{1}{2} \right)^{k/2} (J_{+1}' - J_{-1}')^k. \quad (3.25)$$

The operators $J_{\pm 1}'$ act as raising and lowering operators on the Wigner $d$-functions. Using basic expansion properties of the Wigner $d$-functions and the orthogonality properties of the complex exponential it follows that for any smooth function $\psi$,

$$\frac{1}{2\pi} \int_0^{2\pi} H_T(\Phi) J_s^{2k+1} [H(\Phi) \psi(\theta, \varphi)] d\Phi = 0, \quad (3.26)$$

Therefore, the scattering operator is given by

$$\mathbf{\mathcal{L}} \mathbf{I} = \sum_{k=0}^{\infty} L_k \mathbf{I}, \quad (3.27)$$

where we define

$$L_0 \mathbf{I} = \mathbf{I} - \left( \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi H_T(\Phi) F(\Theta) H(\Phi) \sin \Theta d\Theta \right) \mathbf{I}(\mathbf{\hat{s}}) d\Phi, \quad (3.28)$$

and

$$L_k \mathbf{I} = \frac{1}{2\pi} \int_0^{2\pi} H_T(\Phi) \left( \frac{1}{2} \int_0^\pi (\frac{-1}{(2k)!}) \Theta^{2k} F(\Theta) \sin \Theta d\Theta \right) J_s^{2k} [H(\Phi) \mathbf{I}(\mathbf{\hat{s}})] d\Phi. \quad (3.29)$$

Notice that although the operators in (3.28) and (3.29) are defined using integrals, these integrals do not act directly on the Stokes vector $\mathbf{I}$. In fact, the $k$th operator $L_k$ is $2k$-th order differential operator. In writing the operators in this form, we make strong assumptions on the differentiability of $\mathbf{I}$.
3.3.2 The vector Fokker-Planck approximation

Here we introduce the notation
\[
\langle f(\Theta) \rangle_u = \frac{1}{2} \int_0^\pi \Theta^k f(\Theta) \sin \Theta d\Theta.
\] (3.30)

We follow Leakeas and Larsen [37] and quantify forward-peakedness to correspond to the ordering
\[
\langle a_1(\cos \Theta) \rangle_0 \gg \langle a_1(\cos \Theta) \rangle_1 \gg \langle a_1(\cos \Theta) \rangle_2 \gg \cdots.
\] (3.31)

Since \(|a_j|, |b_i| < a_1\) for \(j = 2, 3, 4\) and \(i = 1, 2\),
\[
\|\langle F(\cos \Theta) \rangle_{2k} \|_\infty = \langle a_1(\cos \Theta) \rangle_{2k},
\] (3.32)
and so we find
\[
\|\langle F(\cos \Theta) \rangle_0 \|_\infty \gg \|\langle F(\cos \Theta) \rangle_2 \|_\infty \gg \|\langle F(\cos \Theta) \rangle_4 \|_\infty \gg \cdots.\] (3.33)

Moreover, in the case of forward scattering the phase matrix is given by [24]
\[
F(1) = \begin{bmatrix}
a_1(1) & 0 & 0 & 0 \\
0 & a_2(1) & 0 & 0 \\
0 & 0 & a_2(1) & 0 \\
0 & 0 & 0 & a_4(1)
\end{bmatrix}.
\] (3.34)

Hence, when scattering is forward-peaked we make the further assumptions that
\[
\langle a_2(\cos \Theta) \rangle_k \sim \langle a_3(\cos \Theta) \rangle_k,
\] (3.35)
and \(\langle b_i(\cos \Theta) \rangle_k\) are negligible compared to \(\langle a_j(\cos \Theta) \rangle_k\) for \(i = 1, 2\) and \(j = 1, 2, 3, 4\). We define
\[
c_1(\cos \Theta) = \frac{1}{2} (a_2 + a_3).
\] (3.36)

Given these assumptions, we approximate the scattering operator by truncating (3.27),
\[
\mathcal{L}I \approx L_0 I + L_1 I.
\] (3.37)

In addition, we make the approximation
\[
\langle F \rangle_0 \approx \text{diag} \left(1, c_1^{(0)}, c_1^{(0)}, a_4^{(0)}\right),
\] (3.38)
where we let
\[
\bar{a}_j^{(0)} = \frac{1}{2} \int_0^\pi \bar{a}_j(\cos \Theta) \sin \Theta d\Theta,
\] (3.39)
and
\[
c_1^{(0)} = \frac{1}{2} (a_2^{(0)} + a_3^{(0)}),
\] (3.40)
so that
\[
L_0 I \approx \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 - c_1^{(0)} & 0 & 0 \\
0 & 0 & 1 - c_1^{(0)} & 0 \\
0 & 0 & 0 & 1 - a_4^{(0)}
\end{bmatrix} I.
\] (3.41)
Since we are expanding about $\Theta = 0$, we replace $\Theta^2/2 \approx (1 - \cos \Theta)$, and approximate

$$
\frac{1}{2} \left< F(\cos \Theta) \right>_2 \approx \begin{bmatrix}
1 - g_1 & 0 & 0 & 0 \\
0 & c_1^{(0)} - h_1 & 0 & 0 \\
0 & 0 & c_1^{(0)} - h_1 & 0 \\
0 & 0 & 0 & \tilde{a}_4^{(0)} - g_4
\end{bmatrix},
$$

(3.42)

where

$$
g_j = \frac{1}{2} \int_0^{\pi} \cos \Theta a_j(\cos \Theta) \sin \Theta d\Theta,
$$

(3.43)

and

$$
h = \frac{1}{2} (g_2 + g_3).
$$

(3.44)

Let $\theta$ denote the polar angle and $\varphi$ denote the azimuthal angle in the experimental reference frame. Using

$$
J_s = i \left( \cos \Phi \cot \theta \partial_\Phi + \sin \Phi \partial_\theta - \cos \Phi \csc \theta \partial_\varphi \right),
$$

(3.45)

we find that

$$
\frac{1}{2\pi} \int_0^{2\pi} H^T(\Phi) J_s^2 [H(\Phi) I(\hat{s})] d\Phi = -\frac{1}{2} \Delta_s I - 2 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 - \cot^2 \theta & \frac{\cos \theta}{\sin^2 \theta} \partial_\varphi & 0 \\
0 & -\frac{\cos \theta}{\sin^2 \theta} \partial_\varphi & 1 - \cot^2 \theta & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} I,
$$

(3.46)

with

$$
\Delta_s = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial^2_\varphi,
$$

(3.47)

denoting the orbital angular momentum operator, or spherical Laplacian. We put this together to get the approximation

$$
\mathcal{L} I \approx \text{diag} \left( 0, 1 - c_1^{(0)}, 1 - c_1^{(0)}, 1 - a_4^{(0)} \right) I
$$

$$
- \text{diag} \left( 1 - g_1, c_1^{(0)} - h, c_1^{(0)} - h, a_4^{(0)} - g_4 \right) \Delta_s I
$$

$$
- 2(c_1^{(0)} - h) \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 - \cot^2 \theta & \frac{\cos \theta}{\sin^2 \theta} \partial_\varphi & 0 \\
0 & -\frac{\cos \theta}{\sin^2 \theta} \partial_\varphi & 1 - \cot^2 \theta & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} I.
$$

(3.48)

Replacing $\mathcal{L}$ in the vector radiative transport equation with (3.48), we obtain

$$
\hat{s} \cdot \nabla I + \kappa_a I - \kappa_s L_0 I - \kappa_s L_2 I = 0,
$$

(3.49)

which we call the vector Fokker-Planck approximation. We write the four equations contained in equation (3.49) for each of the Stokes parameters below. The radiance $I$ satisfies

$$
\hat{s} \cdot \nabla I + \kappa_a I - \frac{1}{2} \kappa_s (1 - g_1) \Delta_s I = 0,
$$

(3.50)
which is the Fokker-Planck approximation established for the scalar radiative transfer equation. The
Stokes parameters $Q$ and $U$ satisfy the system

$$
\hat{s} \cdot \nabla Q + \kappa_a Q + \kappa_s (1 - c_1^{(0)})Q - \frac{1}{2} \kappa_s (c_1^{(0)} - h_1) \Delta_\hat{s} Q - 2(c_1^{(0)} - h_1)(1 - \cot^2 \theta)Q \\
- 2(c_1^{(0)} - h_1) \frac{\cos \theta}{\sin^2 \theta} \partial_\varphi U = 0, \quad (3.51)
$$

and

$$
\hat{s} \cdot \nabla U + \kappa_a U + \kappa_s (1 - c_1^{(0)})U - \frac{1}{2} \kappa_s (c_1^{(0)} - h_1) \Delta_\hat{s} U + 2(c_1^{(0)} - h_1)(1 - \cot^2 \theta)U \\
- 2(c_1^{(0)} - h_1) \frac{\cos \theta}{\sin^2 \theta} \partial_\varphi Q = 0, \quad (3.52)
$$

respectively with $\gamma^{(2)} = (\alpha_2^{(2)} + \alpha_3^{(2)})/2$. Equations (3.51) and (3.52) are coupled due to the rotations
of references frames manifested by forward-peaked scattering that, in turn, modify $Q$ and $U$. The
Stokes parameter $V$ satisfies

$$
\hat{s} \cdot \nabla V + \kappa_a V + \kappa_s (1 - \alpha_4^{(0)})V - \frac{1}{2} \kappa_s (\bar{a}_4^{(0)} - g_4) \Delta_\hat{s} V = 0.
\quad (3.53)
$$

### 3.4 Results

The vector Fokker-Planck equation, defined by (3.49)–(3.52), governs the Stokes parameters in
a forward-peaked multiple scattering medium. For arbitrarily polarized light incident on a forward-
peaked scattering medium, the vector Fokker-Planck approximation yields decoupled equations for
the radiance, $I$, circular polarization, $V$, and linear polarization, $Q$ and $U$. These results are physically intuitive since forward-peaked scattering is mostly restricted to a narrow cone of directions
about the forward direction.

Consider the following special case as an example. Suppose we seek to solve the vector Fokker-
Planck approximation in a plane-parallel slab $0 < z < z_0$ due to circular polarized light incident
normally on the boundary $z = 0$, and no other source of light. Because $Q$ and $U$ are decoupled
from $I$ and $V$, we would only need to solve (3.50) and (3.53). The linear polarization is identically
zero inside the scattering media $Q = U = 0$. We note that as the radiance $I$ decouples from the
polarization state in strongly forward-peaked media, the scalar transport theory is accurate in this
regime. The radiance satisfies the equation

$$
\mu \frac{\partial}{\partial z} I + \kappa_a I - \frac{1}{2} \kappa_s (1 - g_1) \Delta_\hat{s} I = 0, \quad (3.54)
$$

with boundary conditions

$$
I|_{z=0} = \delta(1 - \mu)I_0, \quad 0 < \mu \leq 1, \quad (3.55)
$$

and

$$
I|_{z=z_0} = 0, \quad -1 \leq \mu < 0. \quad (3.56)
$$

The $V$ component of the Stokes vector satisfies the equation

$$
\mu \frac{\partial}{\partial z} V + \left( \kappa_a + \kappa_s \left(1 - \alpha_4^{(0)} \right) \right) V - \frac{1}{2} \kappa_s (\bar{a}_4^{(0)} - g_4) \Delta_\hat{s} V = 0, \quad (3.57)
$$
with boundary conditions
\[ V|_{z=0} = \delta(1 - \mu)V_0, \quad 0 < \mu \leq 1, \quad (3.58) \]
and
\[ V|_{z=z_0} = 0, \quad -1 \leq \mu < 0. \quad (3.59) \]

The solutions to boundary value problems for \( I \) and \( V \) are both well-posed, and have unique sign-definite solutions [6, 51]. Hence, the handedness of the light does not change all inside this medium. Moreover, since \( V \) is decoupled from \( Q \) and \( U \), an incident circularly polarized beam retains its ellipticity. This result shows that strong forward-peaked media also exhibit strong circular polarization memory. This is consistent with the fact that circular polarization memory has been correlated with the anisotropy factor \( g \).

### 3.5 Conclusion

In this chapter we have derived a generalization to the Fokker-Planck approximation from the vector radiative transport equation that takes into account polarization effects. The vector Fokker-Planck approximation shows that media that exhibit strongly forward-peaked scattering, will exhibit circular polarization memory. This is consistent with the fact that circular polarization memory has been correlated with the anisotropy factor \( g \). Moreover, we find that scalar theory is a good approximation in this regime.
Chapter 4

Polarized light in strongly scattering, weakly absorbing media

4.1 Introduction

In this chapter we study the transport of polarized light in the strongly scattering and weakly absorbing limit. In this regime we seek to investigate the depolarization of light as it propagates into the diffusion limit. Sufficiently deep in the scattering medium, light becomes depolarized and isotropic in the sense that the radiance is independent of direction. We seek to study the transition of polarized light into the diffusion limit. Because light is unpolarized in the diffusion limit, in general the diffusion approximation is treated as an asymptotic limit of the scalar radiative transport equation. On the other hand, research into the depolarization of light in this limit is done in the context of studying the full vector radiative transport equation. This has done both with analytic techniques [5, 32], and more frequently, Monte Carlo methods [1, 19, 21, 55, 58].

In a strongly scattering and weakly absorbing medium, it is possible to interrogate at a greater depth. However, there is a loss of resolution due to scattering events. In such media, polarization information can be used to enhance signals [40], discriminate between short- and long- path photons [50], and improve contrast for long-range imaging [16]. In biomedical tissue, light in the near-infrared spectrum is strongly scattered and weakly absorbed, with the main absorbers being hemoglobin/myoglobin, water, and lipids [14]. One of the advantages of imaging with infrared light is that it is low energy and does not ionize or photo-damage. Hence, diffuse imaging is a non-invasive imaging method.

In this limit, the radiative transport equation can be analyzed using boundary layer theory to derive the diffusion approximation. Kim and Moscoso [33] derive a polarized correction to the diffusion approximation in this limit, which we call the polarized diffusion approximation. In this chapter we derive the polarized diffusion approximation as a uniformly valid solution to the vector radiative transport equation in the limit of strong scattering and weak absorption. We use this method to fully express the outer solution and the boundary layer solution as the solution to scalar diffusion approximation and a one-dimensional radiative transport equation that are fully decoupled. We propose and implement a numerical method to solve the polarized diffusion approximation that uses a discrete ordinate method to solve the boundary layer, and a spectral method to compute the outer solution. We call the method the polarized diffusion discrete ordinate method (PDDOM). First, we use the PDDOM to study the light reflected from a halfspace due to a normally incident beam. We then use the PDDOM to investigate the presence of circular polarization memory in a given medium by analyzing the spectrum of the discretized radiative transport equation.
4.2 Derivation of the Polarized Diffusion Approximation

We seek to approximate the radiative transport equation in the case where scattering is strong and absorption is weak. Kim and Moscoso [33] derive the polarized diffusion approximation equation by non-dimensionalizing the radiative transport equation. We write the vector radiative transport equation in the form

\[ \hat{s} \cdot \nabla I + \kappa_a I + \kappa_s \mathcal{L} I = 0, \]  

(4.1)

where \( \mathcal{L} \) is the scattering operator

\[ \mathcal{L} I = I - \frac{1}{4\pi} \int_{S^2} Z(\hat{s}, \hat{s}') I(\hat{s}') \, d\hat{s}'. \]  

(4.2)

Here \( \kappa_a \) and \( \kappa_s \) are the absorbing and scattering coefficients, respectively, \( I = [I, Q, U, V] \) is the Stokes vector, and \( Z \) is the scattering matrix. We let \( x \) and \( \hat{s} \) denote the position and direction. We parameterize direction with the polar cosine \( \mu \) and azimuth \( \varphi \) so that

\[ \hat{s} = \sqrt{1 - \mu^2} \cos \varphi \hat{x} + \sqrt{1 - \mu^2} \sin \varphi \hat{y} + \mu \hat{z}. \]  

(4.3)

In this chapter we will use the additional notation

\[ p = x \hat{x} + y \hat{y}, \]  

(4.4)

\[ \nabla \perp = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}, \]  

(4.5)

and

\[ \hat{s}_\perp = \sqrt{1 - \mu^2} \cos \varphi \hat{x} + \sqrt{1 - \mu^2} \sin \varphi \hat{y}. \]  

(4.6)

We restrict our attention to a collimated beam incident on a halfspace. Hence, we prescribe the boundary conditions

\[ I(0, p, \hat{s}) = \pi \delta(\hat{s} - \hat{s}_0) f(p) I_{inc}, \quad \hat{z} \cdot \hat{s} > 0, \]  

(4.7)

and

\[ \lim_{z \to \infty} I(z, p, \hat{s}) = 0. \]  

(4.8)

Here we assume the boundary is non-reflective, i.e. the refractive index inside the halfspace matches that on the outside.

Before we continue, we recall the solution to the vector radiative transport equation can be split into a direct solution, \( I_{dir} \), and a diffuse or scattered solution, \( I_{diff} \). The direct solution satisfies

\[ \hat{s} \cdot \nabla I_{dir} + \kappa_a I_{dir} + \kappa_s I_{dir} = 0, \]  

(4.9)

with the boundary condition

\[ I_{dir}(0, p, \hat{s}) = \pi \delta(\hat{s} - \hat{s}_0) f(p) I_{inc}, \quad \hat{z} \cdot \hat{s} > 0. \]  

(4.10)

We can solve this analytically using the method of characteristics to find

\[ I_{dir} = \pi \delta(\hat{s} - \hat{s}_0) f(p - z\hat{s}_\perp / \mu) e^{-(\kappa_a + \kappa_s)z / \mu}. \]  

(4.11)

We define the source term as

\[ Q(x, \hat{s}) = \frac{1}{4} f(p - z\hat{s}_0 \perp / \mu_0) Z(\hat{s}, \hat{s}_0) e^{-(\kappa_a + \kappa_s)z / \mu_0}. \]  

(4.12)
We substitute the direction solution back into the radiative transport equation to find the diffuse solution satisfies

$$\hat{s} \cdot \nabla I_{\text{dif}} + \kappa_a I_{\text{dif}} + \mathcal{L} I_{\text{dif}} = \kappa_s Q,$$

(4.13)

with homogeneous boundary condition

$$I_{\text{dif}}|_{z=0} = 0, \quad \hat{z} \cdot \hat{s} > 0,$$

(4.14)

and

$$\lim_{z \to \infty} I_{\text{dif}}(z, \mathbf{p}, \hat{s}) = 0.$$

(4.15)

We will apply our analysis to (4.13) with boundary condition (4.14) and radiation condition (4.15). In what follows, we do not use the subscript $\text{dif}$.

We begin our analysis by introducing the small parameter $0 < \epsilon \ll 1$ such that

$$\kappa_s = \frac{1}{\epsilon \sigma},$$

(4.16)

and

$$\kappa_a = \epsilon \alpha,$$

(4.17)

where $\sigma, \alpha = O(1)$ as $\epsilon \to \infty$. The radiative transport equation becomes

$$\epsilon \hat{s} \cdot \nabla I + \epsilon^2 \alpha I + \sigma \mathcal{L} I = \sigma Q$$

(4.18)

with boundary conditions

$$I|_{z=0} = 0, \quad \hat{z} \cdot \hat{s} > 0,$$

(4.19)

and

$$\lim_{z \to \infty} I|_{z=z_0} = 0.$$

(4.20)

We make the additional assumption that the beam profile is not small with respect to other scales in the problem. To make this explicit, we assume all derivatives of $f$ are $O(1)$ as $\epsilon \to 0^+$. Equation (4.18) with boundary conditions (4.19) and (4.20) is a singular perturbation problem. There is a boundary layer of thickness $\epsilon$ about $z = 0$. We use the method of successive complementary solutions to asymptotically approximate (4.18) in the limit as $\epsilon \to 0^+$. We introduce the stretched variable $\zeta$, defined as

$$\zeta = \kappa_s z = \sigma(\mathbf{x}) z / \epsilon.$$

(4.21)

The uniformly valid asymptotic solution takes the form

$$I^{(N)} \sim \sum_{i=0}^{N} \epsilon^i \left[ I^{(i)}_{\text{out}}(z, \mathbf{p}, \hat{s}) + I^{(i)}_{\text{bl}}(\zeta, \mathbf{p}, \hat{s}) \right], \quad \epsilon \to 0^+,$$

(4.22)

where $I^{(i)}_{\text{out}}$ are solutions to equation (4.18) in the outer region that are obtained from a regular perturbation series and $I^{(i)}_{\text{bl}}$ are the complementary solutions that exist in the boundary layer. We construct $I^{(i)}_{\text{bl}}$ such that the boundary conditions

$$I^{(i)}_{\text{out}}|_{z=0} + I^{(i)}_{\text{bl}}|_{\zeta=0} = 0, \quad \hat{z} \cdot \hat{s} > 0,$$

(4.23)

are solved exactly at each order. The polarized diffusion approximation is the uniformly valid approximation with $N = 1$. 

34
4.2.1 Solution outside the boundary layer

We begin by considering the outer solution. We define the regular outer solution as

\[ I_{r,\text{out}} \sim \sum_{n=1}^{\infty} \epsilon^n I_{\text{out}}^{(n)}, \quad \epsilon \to 0^{+}. \]  

(4.24)

Since the source term \( Q \) given in (4.12) is exponentially small outside of the boundary layer, the outer solution satisfies

\[ \sigma \mathcal{L} I_{\text{out}} = -\epsilon \mathbf{s} \cdot \nabla I_{\text{out}} - \epsilon^2 \alpha I_{\text{out}}. \]  

(4.25)

We substitute the expansion (4.24) into the radiative transport equation (4.25) and collect like powers of \( \epsilon \) to obtain

\[ \mathcal{L} I_{\text{out}}^{(0)} = 0, \]  

(4.26)

\[ \sigma \mathcal{L} I_{\text{out}}^{(1)} = -\mathbf{s} \cdot \nabla I_{\text{out}}^{(0)}, \]  

(4.27)

and

\[ \sigma \mathcal{L} I_{\text{out}}^{(n)} = -\mathbf{s} \cdot \nabla I_{\text{out}}^{(n-1)} - \alpha I_{\text{out}}^{(n-2)}, \]  

(4.28)

for \( n \geq 2 \).

We will find the general solution of (4.28) by using the expansion of the scattering matrix. Recall the scattering matrix has expansion

\[ Z(\mathbf{s}, \mathbf{s}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} 2 \sum_{u=0}^{2} \sum_{u=-1}^{u} \mathcal{D}_{\ell,m,k}(\mathbf{s}) F_{\ell} \mathcal{D}_{\ell,m,k}(\mathbf{s}'), \]  

(4.29)

where \( \mathcal{D}_{\ell,m,k}(\mathbf{s}) \) and \( F_{\ell} \) are as defined in Section 2.3.

**Theorem 4.2.1.** Let \( Q = [q_1, q_2, q_3, q_4] \) be a smooth, bounded, vector-valued function defined on \( \mathbb{S}^2 \). Then the equation

\[ \mathcal{L} \Psi(\mathbf{s}) = Q(\mathbf{s}) \]  

(4.30)

is solvable only if

\[ \int_{\mathbb{S}^2} q_1(\mathbf{s}) d\mathbf{s} = 0, \]  

(4.31)

and the general solution is given by

\[ \Psi(\mathbf{s}) = c_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2(1 - \alpha_4^{(0)})} \int_{\mathbb{S}^2} q_4(\mathbf{s}) d\mathbf{s} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sum_{\ell=1}^{\infty} \sum_{k=0}^{\ell} \sum_{m=1}^{2} \mathcal{D}_{\ell,m,k}(\mathbf{s}) \Psi_{\ell,m,k} \]  

(4.32)

where \( c_0 \) is an arbitrary constant and for \( \ell > 0, |k| \leq \ell, \) and \( m = 1, 2, \)

\[ \Psi_{\ell,m,k} = \left( E - \frac{1}{2\ell + 1} F_{\ell} \right)^{-1} \left( \frac{2\ell + 1}{4\pi} \int_{\mathbb{S}^2} \mathcal{D}_{\ell,m,k}(\mathbf{s}) Q(\mathbf{s}) d\mathbf{s} \right). \]  

(4.33)
Proof. We begin by expanding the solution $\Psi$ as

$$\Psi(\hat{s}) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{2} \sum_{k=0}^{\infty} \mathcal{D}_{\ell,m,k}(\hat{s}) \Psi_{\ell,m,k},$$

(4.34)

where

$$\Psi_{\ell,m,k} = \frac{2\ell + 1}{4\pi} \int_{S^2} \mathcal{D}^T_{\ell,m,k}(\hat{s}) \Psi(\hat{s}) d\hat{s}.$$  

(4.35)

We substitute the expansion into the scattering operator, and use the expansion of the scattering operator from Section 2.3.2 to find

$$\mathcal{L} = \sum_{\ell=0}^{\infty} \sum_{m=1}^{2} \sum_{k=-\ell}^{\ell} \mathcal{D}_{\ell,m,k}(\hat{s}) \left( E - \frac{1}{2\ell + 1} F_{\ell} \right) \Psi_{\ell,m,k}.$$  

(4.36)

Hence, (4.30) is given by

$$\sum_{\ell=0}^{\infty} \sum_{m=1}^{2} \sum_{k=-\ell}^{\ell} \mathcal{D}_{\ell,m,k}(\hat{s}) \left( E - \frac{1}{2\ell + 1} F_{\ell} \right) \Psi_{\ell,m,k} = Q(\hat{s})$$  

(4.37)

Using the orthogonality property

$$\int_{S^2} \mathcal{D}_{\ell,m,k}(\hat{s}) \mathcal{D}^T_{\ell',m',k'}(\hat{s}) d\hat{s} = \frac{4\pi}{2\ell + 1} E \delta_{\ell,\ell'} \delta_{m,m'} \delta_{k,k'}$$

(4.38)

yields

$$\left( E - \frac{1}{2\ell + 1} F_{\ell} \right) \Psi_{\ell,m,k} = \frac{2\ell + 1}{4\pi} \int_{S^2} \mathcal{D}_{\ell,m,k}(\hat{s}) Q(\hat{s}) d\hat{s}.$$  

(4.39)

Now, the matrix

$$\left( E - \frac{1}{2\ell + 1} F_{\ell} \right)$$

(4.40)

is non-invertible if and only if either or both

$$\left( 2\ell + 1 - \alpha_1^{(\ell)} \right) \left( 2\ell + 1 - \alpha_2^{(\ell)} \right) - \left( \beta_1^{(\ell)} \right)^2 = 0,$$

(4.41)

or

$$\left( 2\ell + 1 - \alpha_3^{(\ell)} \right) \left( 2\ell + 1 - \alpha_4^{(\ell)} \right) + \left( \beta_2^{(\ell)} \right)^2 = 0.$$  

(4.42)

From [56], (4.41) is only satisfied for $\ell = 0$, in which case the zero eigenvalue has associated eigenvector $\hat{i}_0 = [1, 0, 0, 0]^T$. Also from [56], (4.42) is only satisfied if $\ell = 0$ and $a_4 \equiv a_1$. However, $a_4$ is not identically equal to $a_1$ for any physical scattering matrix, and so (4.42) is never satisfied. Thus, for $\ell \neq 0$, we can simply invert the matrix to find

$$\Psi_{\ell,m,k} = \left( E - \frac{1}{2\ell + 1} F_{\ell} \right)^{-1} Q_{\ell,m,k}.$$  

(4.43)

For $\ell = 0$,

$$E - \frac{1}{2} F_0 = \text{diag}(0, 1, 1, 1 - \alpha_4^{(0)})$$

(4.44)

$$\mathcal{D}_{0,1,0}(\hat{s}) = \frac{1}{2} \text{diag}(1, 0, 0, 0).$$

(4.45)
and
\[ \mathcal{D}_{0,2,0}(\mathbf{s}) = \frac{1}{2} \text{diag}(0, 0, 0, 1). \] (4.46)

Hence, we seek to solve
\[ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 - \alpha_4^{(0)} \end{bmatrix} (\Psi_{0,1,0} + \Psi_{0,2,0}) = \frac{1}{2} \int_{S^2} q_1(\mathbf{s})d\mathbf{s} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \int_{S^2} q_4(\mathbf{s})d\mathbf{s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \] (4.47)

This is only solvable if
\[ \frac{1}{2} \int_{S^2} q_1(\mathbf{s})d\mathbf{s} = 0, \] (4.48)
in which case
\[ \Psi_{0,1,0} + \Psi_{0,2,0} = c_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2(1 - \alpha_4^{(0)})} \int_{S^2} q_4(\mathbf{s})d\mathbf{s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \] (4.49)

where the coefficient \( c_0 \) is arbitrary.

We use Theorem 4.2.1 to compute the solution to (4.26) and (4.27). In particular, we find
\[ I_{out}^{(0)} = u^{(0)}(x)\hat{i}_0, \] (4.50)
and
\[ I_{out}^{(1)} = u^{(1)}(x)\hat{i}_0 - \frac{1}{(1 - g)\sigma} \mathbf{s} \cdot \nabla u^{(0)}(x)\hat{i}_0, \] (4.51)

where \( u^{(0)}, u^{(1)} \) are isotropic scalar functions and
\[ g = \frac{1}{3} \alpha_4^{\frac{1}{3}} = \frac{1}{2} \int_{-1}^{1} \cos \theta a_1(\cos \theta) d \cos \theta \] (4.52)
is the anisotropy factor. In order to find \( u^{(0)} \), we substitute \( I_{out}^{(0)} \) and \( I_{out}^{(1)} \) into equation (4.28) for \( n = 2 \),
\[ \sigma \mathcal{L} I_{out}^{(2)} = -\mathbf{s} \cdot \nabla u^{(1)}(x)\hat{i}_0 + \mathbf{s} \cdot \nabla \left[ \frac{1}{\sigma(1 - g)} u^{(0)}(x) \right] \hat{i}_0 - \alpha u^{(0)}(x)\hat{i}_0. \] (4.53)

For this to be solvable, by Property 4.2.1, we must have that
\[ \int_{S^2} \left( \mathbf{s} \cdot \nabla u^{(1)}(x) - \mathbf{s} \cdot \nabla \left[ \frac{1}{\sigma(1 - g)} u^{(0)}(x) \right] + \alpha u^{(0)}(x) \right) d\mathbf{s} = 0. \] (4.54)

Integrating by parts yields
\[ \nabla \cdot \left[ \frac{1}{3\sigma(1 - g)} \nabla u^{(0)}(x) \right] - \alpha u^{(0)}(x) = 0. \] (4.55)

Similarly, for the equation for \( I_{out}^{(3)} \) to be solvable, we must have
\[ \nabla \cdot \left[ \frac{1}{3\sigma(1 - g)} \nabla u^{(1)}(x) \right] - \alpha u^{(1)}(x) = 0. \] (4.56)
Notice, by multiplying both equations by $\epsilon$ we can write them in terms of the scattering coefficient and absorption coefficient

\[
\nabla \cdot \left[ \frac{1}{3\kappa_s(1-g)} \nabla u^{(j)}(x) \right] - \kappa_a u^{(j)}(x) = 0, \quad j = 0, 1. \tag{4.57}
\]

The leading order solution to the boundary value problem is

\[
I_{r,\text{out}}(x, \hat{s}) = u^{(0)}(x) \hat{i}_0 + \epsilon u^{(1)}(x) \hat{i}_0 - \frac{1}{\kappa_s(1-g)} \hat{s} \cdot \nabla u^{(0)}(x) \hat{i}_0 + O(\epsilon^2). \tag{4.58}
\]

### 4.2.2 Complementary solution

We begin by expressing the radiative transport equation in the boundary layer using the stretched variable $\zeta = z/\kappa_s$, or equivalently, $\zeta = \sigma(x)z/\epsilon$. In terms of $\zeta$,

\[
\sigma \frac{\partial}{\partial \zeta} I_{bl} + \epsilon \sigma \hat{s}_\perp \cdot \nabla I_{bl} + \epsilon \left( \frac{\hat{s} \cdot \nabla \sigma}{\sigma} \right) \zeta \frac{\partial}{\partial \zeta} I_{bl} + \epsilon^2 \alpha I_{bl} + \sigma \mathcal{L} I_{bl} = \sigma Q(\zeta, p, \hat{s}), \tag{4.59}
\]

where

\[
Q(\zeta, p, \hat{s}) = \frac{1}{4} f \left( p - \frac{\epsilon \zeta}{\sigma \mu_0} \hat{s}_0, \perp \right) Z(\hat{s}, \hat{s}_0) I_{\text{inc}} e^{-\zeta/\mu_0} e^{-\epsilon^2 \alpha z/(\sigma \mu_0)}. \tag{4.60}
\]

We want to find the leading order solution in the boundary layer,

\[
I = I^{(0)}_{bl} + \epsilon I^{(1)}_{bl} + O(\epsilon^2), \tag{4.61}
\]

such that $I^{(j)}_{bl}$ satisfy the boundary condition

\[
I^{(0)}_{\text{out}}|_{z=0} + I^{(0)}_{bl}|_{\zeta=0} = 0, \quad \hat{z} \cdot \hat{s} > 0, \tag{4.62}
\]

and

\[
\epsilon I^{(1)}_{\text{out}}|_{z=0} + \epsilon I^{(1)}_{bl}|_{\zeta=0} = 0, \quad \hat{z} \cdot \hat{s} > 0. \tag{4.63}
\]

In addition, we require the boundary layer solution decays outside of the boundary layer,

\[
\lim_{\zeta \to 0^+} I^{(j)}_{bl} = 0, \tag{4.64}
\]

for $j = 0, 1$.

To leading order, the boundary layer solution satisfies

\[
\mu \frac{\partial}{\partial \zeta} I^{(0)}_{bl}(\zeta, p, \hat{s}) + \mathcal{L} I^{(0)}_{bl}(\zeta, p, \hat{s}) = \frac{1}{4} f(p) Z(\hat{s}, \hat{s}_0) I_{\text{inc}} e^{-\zeta/\mu_0}, \tag{4.65}
\]

with boundary conditions

\[
u^{(0)}|_{z=0} + I^{(0)}_{bl}|_{\zeta=0} = 0, \quad \hat{s} \cdot \hat{z} > 0, \tag{4.66}
\]

and

\[
\lim_{\zeta \to \infty} I^{(0)}_{bl} = 0. \tag{4.67}
\]
We notice that the operator on the left-hand side of (4.65) does not depend on $x$ or $y$, which motivates us to write
\[ I_{bd}^{(0)}(\zeta, p, \hat{s}) = f(x, y) I_{bd,0}^{(0)}(\zeta, \hat{s}), \tag{4.68} \]
where $I_{bd}^{(0)}$ satisfies the one-dimensional radiative transport equation
\[ \mu \frac{\partial}{\partial \zeta} I_{bd}^{(0)}(\zeta, \hat{s}) + \mathcal{L} I_{bd}^{(0)}(\zeta, \hat{s}) = \frac{1}{4} Z(\hat{s}, \hat{s}_0) I_{bd,0}^{(0)} e^{-\zeta/\mu_0}. \tag{4.69} \]
The boundary condition (4.66) becomes
\[ u^{(0)}|_{z=0} \hat{s}_0 + f(p) I_{bd,0}^{(0)}(0, \hat{s}) = 0, \quad \hat{s} \cdot \hat{z} > 0, \tag{4.70} \]
and the radiation condition (4.67) yields
\[ \lim_{\zeta \to \infty} I_{bd,0}^{(0)} = 0. \tag{4.71} \]
Since $u^{(0)}|_{z=0}$ is independent of $\hat{s}$ and $I_{bd,0}^{(0)}$ is independent of $p$, we may separate the boundary condition into the two equations
\[ c_0^{(0)} \hat{i}_0 + I_{bd}^{(0)}|_{\zeta=0} = 0, \quad \hat{s} \cdot \hat{z} > 0, \tag{4.72} \]
and
\[ u^{(0)}|_{z=0} = c_0^{(0)} f(p). \tag{4.73} \]
This motivates us to define $u_0^{(0)}$ such that
\[ \nabla \cdot \left[ \frac{1}{3\kappa_s(1-g)} \nabla u_0^{(0)}(x) \right] - \kappa a u_0^{(0)}(x) = 0. \tag{4.74} \]
with boundary conditions
\[ u_0^{(0)}(x, y, 0) = f(x, y), \tag{4.75} \]
and
\[ \lim_{z \to \infty} u_0^{(0)} = 0. \tag{4.76} \]
Using this notation, we can now write the leading order outer and complementary solutions as
\[ I_{out}^{(0)}(x, \hat{s}) = c_0^{(0)} u_0^{(0)}(x), \tag{4.77} \]
and
\[ I_{bd}^{(0)}(x, \hat{s}) = f(x, y) I_{bd,0}^{(0)}(\kappa_s z, \hat{s}). \tag{4.78} \]
Notice that while $I_{out}^{(0)}$ depends on the solution to both the one-dimensional radiative transport equation and the scalar diffusion approximation, the equations themselves are completely decoupled.

Next, we consider the first order correction. If we assume the the scattering coefficient varies slowly, $\nabla \sigma = O(\epsilon)$, then the complementary solution to the first order correction satisfies the equation
\[ \mu \frac{\partial}{\partial \zeta} I_b^{(1)}(\zeta, p, \hat{s}) + \mathcal{L} I_b^{(1)}(\zeta, p, \hat{s}) = -\frac{1}{4\mu_0} (\hat{s}_{0,\perp} \cdot \nabla \perp f(p)) \ Z(\hat{s}, \hat{s}_0) \ e^{-\zeta/\mu_0} \]
\[ - \hat{s}_\perp \cdot \nabla \perp f(p) I_{bd,0}^{(0)}(\zeta, \hat{s}), \tag{4.79} \]
with boundary condition
\[ \epsilon I_{\text{out}}^{(1)}|_{\zeta=0} + \epsilon I_{b|}^{(1)}|_{\zeta=0} = 0, \quad \hat{z} \cdot \hat{s} > 0, \] (4.80)
and radiation condition
\[ \lim_{\zeta \to \infty} I_{b|}^{(1)} = 0. \] (4.81)
We substitute the solution of \( I_{\text{out}}^{(1)} \) into the boundary condition to find
\[ \epsilon u^{(1)}|_{\zeta=0} = \frac{c_0^{(0)}}{\kappa_s(1-g)} \hat{s} \cdot \nabla u^{(0)}|_{\zeta=0} \hat{i}_0, \quad \hat{s} \cdot \hat{z} > 0. \] (4.82)
Proceeding as we did in the leading order approximation, we find
\[ \epsilon I_{bl}^{(1)}(\zeta, \hat{s}) = c_0^{(0)} \frac{\partial u^{(0)}}{\partial z} \bigg|_{\zeta=0} I_{bl,0}^{(1)}(\zeta, \hat{s}) + f_x(p) I_{bl,1}^{(1)}(\zeta, \hat{s}) + f_y(p) I_{bl,2}^{(1)}(\zeta, \hat{s}), \] (4.83)
where \( I_{bl,i}^{(1)} \) satisfy the one-dimensional radiative transport equation,
\[ \mu \frac{\partial}{\partial \zeta} I_{bl,i}^{(1)}(\zeta, \hat{s}) + \mathcal{L} I_{bl,i}^{(1)}(\zeta, \hat{s}) = Q_i^{(1)}(\zeta, \hat{s}), \quad i = 0, 1, 2, \] (4.84)
with source terms
\[ Q_0^{(1)}(\zeta, \hat{s}) = 0, \] (4.85)
\[ Q_1^{(1)}(\zeta, \hat{s}) = -\frac{\sqrt{1 - \mu_0^2} \cos \varphi_0}{4\mu_0} Z(\hat{s}, \hat{s}_0) \iota_{\text{inc}} e^{-\zeta/\mu_0} - \sqrt{1 - \mu^2} \cos \varphi I_{bl}^{(0)}, \] (4.86)
\[ Q_2^{(1)}(\zeta, \hat{s}) = -\frac{\sqrt{1 - \mu_0^2} \sin \varphi_0}{4\mu_0} Z(\hat{s}, \hat{s}_0) \iota_{\text{mid}} e^{-\zeta/\mu_0} - \sqrt{1 - \mu^2} \sin \varphi I_{bl}^{(0)}. \] (4.87)
The boundary conditions for \( I_{bl,i}^{(1)} \) are given by
\[ c_0^{(1)} \hat{i}_0 + I_{bl,0}^{(1)}|_{\zeta=0} = \frac{\mu}{\kappa_s(1-g)} \hat{i}_0, \quad \hat{s} \cdot \hat{z} > 0, \] (4.88)
\[ c_1^{(1)} \hat{i}_0 + I_{bl,1}^{(1)}|_{\zeta=0} = \frac{c_0^{(0)} \sqrt{1 - \mu_0^2} \cos \varphi_0}{\kappa_s(1-g)} \hat{i}_0, \quad \hat{s} \cdot \hat{z} > 0, \] (4.89)
\[ c_2^{(1)} \hat{i}_0 + I_{bl,2}^{(1)}|_{\zeta=0} = \frac{c_0^{(0)} \sqrt{1 - \mu_0^2} \sin \varphi_0}{\kappa_s(1-g)} \hat{i}_0, \quad \hat{s} \cdot \hat{z} > 0, \] (4.90)
and they satisfy the radiation condition
\[ \lim_{\zeta \to \infty} I_{bl,i}^{(1)} = 0, \quad i = 0, 1, 2. \] (4.91)
Here we have used that
\[ \hat{s} \cdot \nabla u^{(0)}|_{\zeta=0} = \sqrt{1 - \mu^2} \cos \varphi f_x(p) + \sqrt{1 - \mu^2} \sin \varphi f_y(p) + \mu \frac{\partial u^{(0)}}{\partial z} \bigg|_{\zeta=0}. \] (4.92)
Moreover, we can write \( u^{(1)} \) as
\[ \epsilon u^{(1)}(x) = c_0^{(1)} u_0^{(1)}(x) + c_1^{(1)} u_1^{(1)}(x) + c_2^{(1)} u_2^{(1)}(x), \] (4.93)
where we define \( u_0^{(1)}, u_1^{(1)}, \) and \( u_2^{(2)} \) such that they satisfy the scalar diffusion equation

\[
\nabla \cdot \left[ \frac{1}{3\kappa_s(1-g)} \nabla u_i^{(1)}(x) \right] - \kappa_a u_i^{(1)}(x) = 0, \quad i = 0, 1, 2, \tag{4.94}
\]

with boundary conditions

\[
u_0^{(1)}(x, y, 0) = \frac{\partial u_0^{(0)}}{\partial z} \bigg|_{z=0},
\]

\[
u_1^{(1)}(x, y, 0) = f_x(x, y),
\]

\[
u_2^{(1)}(x, y, 0) = f_y(x, y),
\]

and

\[
\lim_{z \to \infty} u_i^{(1)} = 0, \quad i = 0, 1, 2. \tag{4.98}
\]

We use these results to write

\[
\epsilon I_{out}^{(1)}(x, \hat{s}) = \left[-\frac{c_0^{(0)}}{\kappa_s(1-g)} \hat{s} \cdot \nabla u(x) + c_0^{(0)} c_1^{(1)} u_0^{(0)}(x) + c_1^{(1)} u_1^{(1)}(x) + c_2^{(1)} u_2^{(1)}(x) \right] \hat{i}_0, \tag{4.99}
\]

and

\[
\epsilon I_{bl}^{(1)}(x, \hat{s}) = c_0^{(0)} \frac{\partial u_0^{(0)}}{\partial z} \bigg|_{z=0} I_{bl,0}^{(1)}(\kappa_s z, \hat{s}) + f_x(p) I_{bl,1}^{(1)}(\kappa_s z, \hat{s}) + f_y(p) I_{bl,2}^{(1)}(\kappa_s z, \hat{s}). \tag{4.100}
\]

Notice here that, while \( I_{bl,1}^{(1)} \) and \( I_{bl,2}^{(1)} \) depend on \( I_{bl,0}^{(0)} \) and \( u_0^{(0)} \) depends on \( u_0^{(0)} \), the solutions to the scalar diffusion equations and the solution to the one-dimensional radiative transport equations are independent of each other.

### 4.3 Polarized Diffusion Discrete Ordinate Method

In the previous sections we derived the polarized diffusion approximation for light scattering on a halfspace with scattering coefficient \( \kappa_s \) and absorption coefficient \( \kappa_a \) due to a collimated beam with profile \( f(x, y) \) incident on the lower boundary in direction \( \hat{s}_0 = (\mu_0, \varphi_0) \). We summarize the results here.

The polarized diffusion approximation is given by

\[
I = I_{out}^{(0)}(x, \hat{s}) + I_{bl}^{(0)}(x, \hat{s}) + \epsilon I_{out}^{(1)}(x, \hat{s}) + \epsilon I_{bl}^{(1)}(x, \hat{s}), \tag{4.101}
\]

where

\[
I_{out}^{(0)}(x, \hat{s}) = c_0^{(0)} u^{(0)}(x) \hat{i}_0, \tag{4.102}
\]

\[
I_{bl}^{(0)}(x, \hat{s}) = f(x, y) I_{bl,0}^{(0)}(\kappa_s z, \hat{s}), \tag{4.103}
\]

\[
\epsilon I_{out}^{(1)}(x, \hat{s}) = \left[-\frac{c_0^{(0)}}{\kappa_s(1-g)} \hat{s} \cdot \nabla u(x) + c_0^{(0)} c_1^{(1)} u_0^{(0)}(x) + c_1^{(1)} u_1^{(1)}(x) + c_2^{(1)} u_2^{(1)}(x) \right] \hat{i}_0. \tag{4.104}
\]
and
\[ e^{(1)}_{i}(x, \mathbf{\hat{s}}) = c_{0}^{(0)} \frac{\partial u_{0}^{(0)}}{\partial z} \bigg|_{z=0} I_{bl,0}^{(1)}(\kappa_{s} z, \mathbf{\hat{s}}) + f_{x}(p) I_{bl,1}^{(1)}(\kappa_{s} z, \mathbf{\hat{s}}) + f_{y}(p) I_{bl,2}^{(1)}(\kappa_{s} z, \mathbf{\hat{s}}). \]  

(4.105)

The terms \( u_{i}^{(j)} \) satisfy the diffusion equation
\[ \nabla \cdot \left[ \frac{1}{3 \kappa_{s} (1 - g)} \nabla u_{i}^{(j)}(x) \right] - \kappa_{a} u_{i}^{(j)}(x) = 0, \]

for \( i = 0, 1, 2 \) and \( j = 0, 1 \), with boundary conditions
\[ u_{0}^{(0)} \big|_{z=0} = f(p), \]
\[ u_{0}^{(1)} \big|_{z=0} = \frac{\partial u_{0}^{(0)}}{\partial z} \bigg|_{z=0}, \]
\[ u_{1}^{(1)} \big|_{z=0} = f_{x}(p), \]
\[ u_{2}^{(1)} \big|_{z=0} = f_{y}(p), \]

and
\[ \lim_{z \to \infty} u_{i}^{(j)} = 0. \]

(4.111)

The terms \( I_{bl,i}^{(j)} \) satisfy the one-dimensional radiative transport equation
\[ \mu \frac{\partial}{\partial \zeta} I_{bl,i}^{(j)}(\zeta, \mathbf{\hat{s}}) + \mathcal{L} I_{bl,i}^{(j)}(\zeta, \mathbf{\hat{s}}) = Q_{i}^{(j)}(\zeta, \mathbf{\hat{s}}), \]

(4.112)

for \( i = 0, 1, 2 \) and \( j = 0, 1 \), where the source terms are given by
\[ Q_{0}^{(0)}(\zeta, \mathbf{\hat{s}}) = \frac{1}{4} Z(\mathbf{\hat{s}}, \mathbf{\hat{s}}_{0}) I_{n0} e^{-\zeta/\mu_{0}}, \]
\[ Q_{0}^{(1)}(\zeta, \mathbf{\hat{s}}) = 0, \]
\[ Q_{1}^{(1)}(\zeta, \mathbf{\hat{s}}) = \frac{\sqrt{1 - \mu_{0}^{2} \cos \varphi_{0}}}{4 \mu_{0}} Z(\mathbf{\hat{s}}, \mathbf{\hat{s}}_{0}) I_{n0} e^{-\zeta/\mu_{0}} - \sqrt{1 - \mu_{0}^{2}} \cos \varphi I_{bl}^{(0)}(\zeta, \mathbf{\hat{s}}), \]
\[ Q_{2}^{(1)}(\zeta, \mathbf{\hat{s}}) = - \frac{\sqrt{1 - \mu_{0}^{2} \sin \varphi_{0}}}{4 \mu_{0}} Z(\mathbf{\hat{s}}, \mathbf{\hat{s}}_{0}) I_{n0} e^{-\zeta/\mu_{0}} - \sqrt{1 - \mu_{0}^{2}} \sin \varphi I_{bl}^{(0)}(\zeta, \mathbf{\hat{s}}), \]

(4.113) - (4.116)

with boundary conditions
\[ c_{0}^{(0)} \mathbf{\hat{i}}_{0} + I_{bl,0}^{(0)}(\zeta=0) = 0, \]
\[ c_{0}^{(1)} \mathbf{\hat{i}}_{0} + I_{bl,0}^{(1)}(\zeta=0) = \frac{\mu}{\kappa_{s} (1 - g)} \mathbf{\hat{i}}_{0}, \quad \mathbf{\hat{s}} \cdot \mathbf{\hat{z}} > 0, \]
\[ c_{1}^{(1)} \mathbf{\hat{i}}_{0} + I_{bl,1}^{(1)}(\zeta=0) = \frac{c_{0}^{(0)} \sqrt{1 - \mu_{0}^{2} \cos \varphi_{0}}}{\kappa_{s} (1 - g)} \mathbf{\hat{i}}_{0}, \quad \mathbf{\hat{s}} \cdot \mathbf{\hat{z}} > 0, \]
\[ c_{2}^{(1)} \mathbf{\hat{i}}_{0} + I_{bl,2}^{(1)}(\zeta=0) = \frac{c_{0}^{(0)} \sqrt{1 - \mu_{0}^{2} \sin \varphi_{0}}}{\kappa_{s} (1 - g)} \mathbf{\hat{i}}_{0}, \quad \mathbf{\hat{s}} \cdot \mathbf{\hat{z}} > 0, \]

(4.117) - (4.120)

and the radiation conditions
\[ \lim_{\zeta \to \infty} I_{bl,i}^{(j)} = 0. \]

(4.121)
4.3.1 Fourier expansion of the complementary solution

As discussed in Section 2.4.2, if we expand the Stokes vector in an appropriate Fourier series, then the Fourier modes of the one-dimensional radiative transport equation solve uncoupled equations. We expand the complementary solutions in the Fourier series

\[
I_{u\xi,\xi}(\zeta, \hat{s}) = \sum_{m=1}^{2} \sum_{k=0}^{\infty} \left( \frac{2 - \delta_{k,0}}{2} \right) \Phi_{m,k}(\varphi - \varphi_0) I_{m,k}^{(j)}(\zeta, \mu),
\]

for \( i = 0, 1, 2 \) and \( j = 0, 1 \). Here we define the diagonal matrices \( \Phi_{m,k}(\phi) \) as before,

\[
\Phi_{1,k}(\varphi) = \text{diag}(\cos \varphi, \cos \varphi, \sin \varphi, \sin \varphi),
\]

and

\[
\Phi_{2,k}(\varphi) = \text{diag}(-\sin \varphi, -\sin \varphi, \cos \varphi, \cos \varphi).
\]

As shown in Section 2.4.2, the Fourier components of the Stokes vector satisfy the equations

\[
\mu \frac{\partial}{\partial \zeta} I_{i,m,k}^{(j)}(\zeta, \mu) + I_{i,m,k}^{(j)}(\zeta, \mu) - \frac{1}{2} \int_{-1}^{1} Z_k(\mu, \mu') I_{i,m,k}^{(j)}(\zeta, \mu') d\mu' = Q_{i,m,k}^{(j)}(\zeta, \mu),
\]

where

\[
Z_k(\mu, \mu') = \frac{1}{2\pi} \int_{0}^{2\pi} [\Phi_{1,k}(\phi) + \Phi_{2,k}(\phi)] Z(\mu, \mu', \phi) d\phi,
\]

and

\[
Q_{i,m,k}^{(j)} = \frac{1}{\pi} \int_{0}^{2\pi} \Phi_{m,k}(\varphi - \varphi_0) Q_{i,m,k}^{(j)}(\zeta, \mu, \varphi) d\varphi.
\]

First, we consider the source term and boundary conditions for the Fourier modes of the leading order complementary solution \( I_{0,m,k}^{(0)} \). The source term has Fourier modes

\[
Q_{0,m,k}^{(0)}(\zeta, \mu) = \frac{1}{2} Z_k(\mu, \mu_0) D_{\mu}^{m} e^{-\zeta/\mu_0}
\]

where \( D_1 = \text{diag}(1, 1, 0, 0) \) and \( D_2 = \text{diag}(0, 0, 1, 1) \). The boundary conditions for the Fourier components of the leading order solution is given by

\[
c_0^{(0)} I_{0,1,0}^{(0)} |_{\zeta=0} = 0, \quad 0 < \mu \leq 1,
\]

and

\[
\lim_{\zeta \to \infty} I_{0,1,0}^{(0)} = 0
\]

when \( m = 1, k = 0 \). For all other Fourier modes,

\[
I_{0,m,k}^{(0)} |_{\zeta=0} = 0, \quad 0 < \mu \leq 1,
\]

and

\[
\lim_{\zeta \to \infty} I_{0,m,k}^{(0)} = 0.
\]

Next we consider the source term and boundary conditions for \( I_{1,m,k}^{(1)} \). Since the source term is identically zero and

\[
\frac{\mu}{\kappa_s (1 - g)} \hat{i}_0 = \Phi_{1,0}(\varphi - \varphi_0) \frac{\mu}{\kappa_s (1 - g)} \hat{i}_0,
\]
the only non-zero Fourier mode corresponds to \( \Phi_{1,0}(\varphi - \varphi_0) \). Hence,

\[
I^{(1)}_{b,l,0}(\zeta, \mathbf{s}) = \Phi_{1,0}(\varphi - \varphi_0)I^{(1)}_{0,1,0}(\zeta, \mu)
\]  \hspace{1cm} (4.134)

where

\[
I^{(1)}_{0,1,0}(0, \mu) = \frac{\mu}{\kappa_0 (1 - g)} \hat{I}_0, \quad 0 < \mu \leq 1,
\]  \hspace{1cm} (4.135)

and

\[
\lim_{\zeta \to \infty} I^{(1)}_{0,1,0} = 0.
\]  \hspace{1cm} (4.136)

Next we consider the source term and boundary conditions for \( I^{(1)}_{b,l,1} \). The source term is given by

\[
Q^{(1)}_1(\zeta, \mathbf{s}) = -\sqrt{1 - \mu^2} \cos \varphi \cos \varphi_0 Z(\mathbf{s}, \mathbf{s}_0) e^{-\zeta/\mu_0}
\]

\[
- \sqrt{1 - \mu^2} \sin \varphi \sin \varphi_0 Z(\mathbf{s}, \mathbf{s}_0) e^{-\zeta/\mu_0}
\]

\[
- \frac{2 - \delta_{k,0}}{2} \Phi_{m,k}(\varphi - \varphi_0) I^{(0)}_{0,m,k}(\zeta, \mu).
\]  \hspace{1cm} (4.137)

We use trigonometric identities and the expansion of the scattering matrix to compute the following. For \( k \neq 0 \), the Fourier components of \( Q^{(1)}_1 \) are given by

\[
Q^{(1)}_{1,1,k}(\zeta, \mu) = -\sqrt{1 - \mu^2} \cos \varphi \cos \varphi_0 Z_k(\mu, \mu_0) e^{-\zeta/\mu_0}
\]

\[
- \frac{1}{2} \sqrt{1 - \mu^2} \cos \varphi \left( I^{(0)}_{0,1,k-1}(\zeta, \mu) + I^{(0)}_{0,1,k+1}(\zeta, \mu) \right)
\]

\[
- \sin \varphi \left( -I^{(0)}_{0,2,k-1}(\zeta, \mu) + I^{(0)}_{0,2,k+1}(\zeta, \mu) \right),
\]  \hspace{1cm} (4.138)

and

\[
Q^{(1)}_{1,2,k}(\zeta, \mu) = -\sqrt{1 - \mu^2} \cos \varphi \sin \varphi_0 Z_k(\mu, \mu_0) e^{-\zeta/\mu_0}
\]

\[
- \frac{1}{2} \sqrt{1 - \mu^2} \cos \varphi \left( I^{(0)}_{0,2,k-1}(\zeta, \mu) + I^{(0)}_{0,2,k+1}(\zeta, \mu) \right)
\]

\[
- \sin \varphi \left( I^{(0)}_{0,1,k-1}(\zeta, \mu) - I^{(0)}_{0,1,k+1}(\zeta, \mu) \right) \right),
\]  \hspace{1cm} (4.139)

For \( k = 0 \), the Fourier components of \( Q^{(1)}_1 \) are

\[
Q^{(1)}_{1,1,0}(\zeta, \mu) = -\sqrt{1 - \mu^2} \cos \varphi \cos \varphi_0 Z_0(\mu, \mu_0) e^{-\zeta/\mu_0}
\]

\[
- \sqrt{1 - \mu^2} \left( \cos \varphi_0 I^{(0)}_{0,1,1}(\zeta, \mu) - \sin \varphi_0 I^{(0)}_{0,1,1}(\zeta, \mu) \right),
\]  \hspace{1cm} (4.140)

and

\[
Q^{(1)}_{1,2,0}(\zeta, \mu) = -\sqrt{1 - \mu^2} \cos \varphi \sin \varphi_0 Z_0(\mu, \mu_0) e^{-\zeta/\mu_0}
\]

\[
- \sqrt{1 - \mu^2} \left( \cos \varphi_0 I^{(0)}_{0,2,1}(\zeta, \mu) + \sin \varphi_0 I^{(0)}_{0,2,1}(\zeta, \mu) \right).
\]  \hspace{1cm} (4.141)
To compute the boundary conditions we use that
\[ \frac{\sqrt{1 - \mu^2 \cos \varphi}}{\kappa_s(1 - g)} i_0 = \Phi_{1,1}(\varphi - \varphi_0) \frac{\cos \varphi_0 \sqrt{1 - \mu^2}}{\kappa_s(1 - g)} i_0 + \Phi_{2,1}(\varphi - \varphi_0) \frac{\sin \varphi_0 \sqrt{1 - \mu^2}}{\kappa_s(1 - g)} i_0. \quad (4.142) \]

Hence, the boundary conditions for the Fourier modes of \( I_{1,1}^{(1)} \) are given by
\[ c_1^{(1)} i_0 + I_{1,1,0}^{(1)} \zeta = 0, \quad k = 0, m = 1, \]
\[ I_{1,1,1}^{(0)} \zeta = \frac{\cos \varphi_0 \sqrt{1 - \mu^2}}{\kappa_s(1 - g)} i_0, \quad k = 1, m = 1, \]
\[ I_{1,2,1}^{(1)} \zeta = \frac{\sin \varphi_0 \sqrt{1 - \mu^2}}{\kappa_s(1 - g)} i_0, \quad k = 1, m = 2, \]
\[ I_{1,m,k}^{(0)} \zeta = 0, \quad \text{otherwise}. \quad (4.146) \]
and
\[ \lim_{\zeta \to \infty} I_{1,m,k}^{(1)} = 0. \quad (4.147) \]

Finally, we consider \( I_{1,2}^{(1)} \). The source term is given by
\[ Q_{1,2}^{(1)}(\zeta, \hat{s}) = -\frac{\sqrt{1 - \mu_0 \cos \varphi_0}}{4 \mu_0} Z(\hat{s}, \hat{s}_0) \zeta e^{-\zeta/\mu_0} \]
\[ + \sqrt{1 - \mu^2} \sin \varphi \sum_{m=1, k=0}^{\infty} \left( \frac{2 - \delta_{k,0}}{2} \right) \Phi_{m,k}(\varphi - \varphi_0) I_{0,m,k}^{(0)}(\zeta, \mu). \quad (4.148) \]

For \( k \neq 0 \), the Fourier components of \( Q_{1,2}^{(1)} \) are given by
\[ Q_{1,2,k}^{(1)}(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} \left( \sin \varphi_0 \left( I_{0,1,k-1}^{(0)}(\zeta, \mu) + I_{0,1,k+1}^{(0)}(\zeta, \mu) \right) \right. \]
\[ - \cos \varphi_0 \left( -I_{0,2,k-1}^{(0)}(\zeta, \mu) + I_{0,2,k+1}^{(0)}(\zeta, \mu) \right), \quad (4.149) \]
and
\[ Q_{1,2,k}^{(1)}(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} \left( \sin \varphi_0 \left( I_{0,2,k-1}^{(0)}(\zeta, \mu) + I_{0,2,k+1}^{(0)}(\zeta, \mu) \right) \right. \]
\[ - \cos \varphi_0 \left( I_{0,1,k-1}^{(0)}(\zeta, \mu) - I_{0,1,k+1}^{(0)}(\zeta, \mu) \right) \). \quad (4.150) \]

For \( k = 0 \), the Fourier components of \( Q_{1,2}^{(1)} \) are
\[ Q_{1,2,0}^{(1)}(\zeta, \mu) = \sqrt{1 - \mu^2} \left( \sin \varphi_0 I_{0,1,1}^{(0)}(\zeta, \mu) - \cos \varphi_0 I_{0,2,1}^{(0)}(\zeta, \mu) \right), \quad (4.151) \]
and
\[ Q_{2,2,0}^{(1)}(\zeta, \mu) = \sqrt{1 - \mu^2} \left( \sin \varphi_0 I_{0,2,1}^{(0)}(\zeta, \mu) + \cos \varphi_0 I_{0,1,1}^{(0)}(\zeta, \mu) \right). \quad (4.152) \]
The boundary conditions are given by

\[ c_2^{(1)\gamma} I_{0}^{(1)} + I_{2,m,k}^{(1)}|_{\zeta=0} = 0, \quad m = 1, k = 0, \]  
\[ I_{2,m,k}^{(1)}|_{\zeta=0} = \frac{\sin \varphi_0 \sqrt{1 - \mu^2}}{\kappa_s(1 - g)} I_0^{(1)}, \quad m = 1, k = 1, \]  
\[ I_{2,m,k}^{(1)}|_{\zeta=0} = -\frac{\cos \varphi_0 \sqrt{1 - \mu^2}}{\kappa_s(1 - g)} I_0^{(1)}, \quad m = 2, k = 1, \]  
\[ I_{2,m,k}^{(1)}|_{\zeta=0} = 0, \quad \text{otherwise.} \]

and

\[ \lim_{\zeta \to \infty} I_{2,m,k}^{(1)} = 0. \]

**Normally incident beam**

In the case of a normally incident beam, \( \hat{s}_0 = \hat{z} \), the above results simplify. For a normally incident beam, the choice of azimuth is arbitrary. Without loss of generality, we let \( \varphi_0 = 0 \).

First, we consider the source term and boundary conditions for the Fourier modes of the leading order complementary solution \( I_0^{(0)} \). The scattering matrix \( Z \) satisfies (see [9]),

\[ Z(\hat{s}, \hat{z}) = \Phi_{1,0}(\varphi)Z_0(\mu, 1) + \Phi_{2,0}(\varphi)Z_0(\mu, 1) + 2\Phi_{1,2}(\varphi)Z_2(\mu, 1) + 2\Phi_{2,2}(\varphi)Z_2(\mu, 1). \]  

Hence, the Fourier modes of the source term are given by

\[ Q_{0,m,k}^{(0)}(\zeta, \mu) = \begin{cases} \frac{1}{2} Z_k(\mu, 1) D_m I_{inc} e^{-\zeta}, & k = 0, 2, \\ 0, & \text{otherwise.} \end{cases} \]  

The leading order solution has the finite Fourier expansion

\[ I_{bl,0}^{(0)} = \frac{1}{2} I_{0,1,0}^{(0)}(\zeta, \mu) + \frac{1}{2} I_{0,2,0}^{(0)}(\zeta, \mu) + \Phi_{1,2}(\varphi) I_{0,2,1}^{(0)}(\zeta, \mu) + \Phi_{2,2}(\varphi) I_{0,2,2}^{(0)}(\zeta, \mu). \]  

The expansion to the first order correction \( I_{bl,1}^{(1)} \) is unchanged. However, \( I_{bl,1}^{(1)} \) and \( I_{bl,2}^{(1)} \) have the truncated expansion

\[ I_{bl,1}^{(1)}(\zeta, \hat{s}) = \sum_{m=1,2,k=1,3} \Phi_{m,k}(\varphi) I_{1,m,k}^{(1)}(\zeta, \mu), \]  

and

\[ I_{bl,2}^{(1)}(\zeta, \hat{s}) = \sum_{m=1,2,k=1,3} \Phi_{m,k}(\varphi) I_{2,m,k}^{(1)}(\zeta, \mu). \]  

The non-zero source terms for \( Q_{1}^{(1)} \) are given by

\[ Q_{1,1,1}^{(1)}(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} \left( I_{0,1,0}^{(0)}(\zeta, \mu) + I_{0,1,2}^{(0)}(\zeta, \mu) \right), \]  
\[ Q_{1,2,1}^{(1)}(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} \left( I_{0,2,0}^{(0)}(\zeta, \mu) + I_{0,2,2}^{(0)}(\zeta, \mu) \right), \]  
\[ Q_{1,1,3}^{(1)}(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} I_{0,1,2}^{(0)}(\zeta, \mu), \]  
\[ Q_{1,2,3}^{(1)}(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} I_{0,2,2}^{(0)}(\zeta, \mu). \]
and the non-zero source terms for $Q_2^{(1)}$ are

\[
Q_2,1,1(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} \left( I_{0,2,0}^{(1)}(\zeta, \mu) - I_{0,2,2}^{(0)}(\zeta, \mu) \right),
\]

\[
Q_2,2,1(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} \left( -I_{0,1,0}^{(0)}(\zeta, \mu) + I_{0,1,2}^{(0)}(\zeta, \mu) \right),
\]

\[
Q_2,1,3(\zeta, \mu) = \frac{1}{2} \sqrt{1 - \mu^2} I_{0,1,2}^{(0)}(\zeta, \mu),
\]

\[
Q_2,2,3(\zeta, \mu) = -\frac{1}{2} \sqrt{1 - \mu^2} I_{0,2,2}^{(0)}(\zeta, \mu).
\]

The boundary conditions become

\[
I_{1,1,1}^{(0)}|_{\zeta=0} = \frac{c_0^{(1)}}{\kappa_s(1 - g)} \hat{i}_0, \quad i = 1, m = 1, k = 1,
\]

\[
I_{2,2,2}^{(0)}|_{\zeta=0} = -\frac{c_0^{(1)}}{\kappa_s(1 - g)} \hat{i}_0, \quad i = 2, m = 2, k = 1,
\]

\[
I_{1,m,k}^{(0)}|_{\zeta=0} = 0, \quad \text{otherwise.}
\]

and

\[
\lim_{\zeta \to \infty} I_{i,m,k}^{(1)} = 0,
\]

for $i = 1, 2, m = 1, 2, k = 1, 3$. Notice for a normally incident beam

\[
c_1^{(1)} = c_2^{(1)} = 0.
\]

### 4.3.2 Computing the complementary solution

Because of the form of the boundary conditions (4.118)-(4.120), it is convenient to use the eigen-matrix discrete ordinate method as outlined in Section 2.4.5. Recall, the discrete ordinate method is obtained by discretizing the Fourier decomposed radiative transport equation as given in equation (4.125) in the polar cosine $\mu$ (also known as the ordinate directions) and approximating the integral operator with a numerical quadrature with $2N$ gridpoints $\mu_n$ with associated weights $w_n$ on $[-1, 1]$. The solutions is approximated at the ordinate directions by solving the resulting system of linear first-order differential equations. In Section (2.4.5) we show equations of the form (4.125) on a halfspace have the general solution

\[
I_{1,0}^{(j)}(\zeta, \mu_n) \approx \hat{I}_{p,1,0}^{(j)}(\zeta, \mu_n) + c_i^{(j)} \hat{i}_0 + \sum_{r=2}^{4N} c_i^{(j)} \hat{r}_{r,1,0}^{(j)}(\mu_n) e^{-\lambda_r(1-g)\zeta},
\]

and, for not both $m = 1$ and $k = 0$,

\[
I_{i,m,k}^{(j)}(\zeta, \mu_n) \approx \hat{I}_{i,m,k}^{(j)}(\zeta, \mu_n) + \sum_{r=1}^{4N} c_i^{(j)} \hat{r}_{r,m,k}^{(j)}(\mu_n) e^{-\lambda_r(1-g)\zeta}.
\]

Here we let $\hat{I}_{r,k}$ and $\lambda_{r,k}$ denote the eigenvectors and eigenvalues that solve the eigenvalue problem

\[
-\lambda_{r,k}\mu_n \hat{I}_{r,k}(\mu_n) + \hat{I}_{r,k}(\mu_n) - \frac{1}{2} \sum_{n'=1}^{N} w_{n'} Z_k(\mu_n, \mu_{n'}) \hat{I}_{r,k}(\mu_{n'}) = 0,
\]

47
and we let \( \tilde{I}_{p,i,m,k}^{(j)}(\zeta, \mu_n) \) be the particular solution to

\[
\mu_n \frac{\partial}{\partial \zeta} \tilde{I}_{p,i,m,k}^{(j)}(\zeta, \mu_n) + \tilde{I}_{p,i,m,k}^{(j)}(\zeta, \mu_n) - \frac{1}{2} \sum_{n' = 1}^N w_{n'} Z_k(\mu_n, \mu_{n'}) \tilde{I}_{p,i,m,k}^{(j)}(\mu_n') = Q_{i,m,k}^{(j)}(\zeta, \mu_n). \tag{4.179}
\]

We solve for the coefficients \( c_i^{(j)} \) and \( c_{i,r,m,k}^{(j)} \) using the boundary conditions.

The particular solution can be computed by using the method of variation of parameters. For \( i = 0 \), we follow the method outlined in Section 2.4.5 to find

\[
\tilde{I}_{p,0,m,k}^{(0)}(\zeta, \mu) = \left( \zeta v_{p,0,m,k}^{(0)} + v_{p,0,m,k}^{(0)} \right) e^{-\zeta/\mu_0}. \tag{4.180}
\]

For \( i = 1 \), we use the approximation of \( I_{0,m,k}^{(0)} \) at the quadrature points to approximate the source terms \( Q_{i,m,k}^{(1)} \). The particular solution of \( \tilde{I}_{p,i,m,k}^{(1)} \) has the general form

\[
\tilde{I}_{p,i,m,k}^{(1)}(\zeta, \mu) = \left( \zeta^2 v_{p,0,m,k}^{(1)} + \zeta v_{p,0,m,k}^{(1)} + v_{p,0,m,k}^{(1)} \right) e^{-\zeta/\mu_0} + \sum_{r=1}^{4N} \left( \zeta w_{1,r,m,k}^{(1)} + w_{1,r,m,k}^{(1)} \right) e^{-\lambda_{r,k-1} \zeta} + \sum_{r=1}^{4N} \left( \zeta y_{1,r,m,k}^{(1)} + y_{0,r,m,k}^{(1)} \right) e^{-\lambda_{r,k-1} \zeta}. \tag{4.181}
\]

### 4.3.3 Computing the outer solution

The diffusion approximation (4.106) can be solved using standard numerical methods for solving partial differential equations. Here we will address two special cases below. First, we given a closed form solution in the case of a uniform scattering and absorbing medium. Next, we discuss computing the reflectance using a spectral method.

#### 1D outer solution

Consider a uniform scattering and absorbing medium in one spatial dimension. In this case we have that

\[
u_1^{(1)}(z) = u_2^{(2)}(z) = 0. \tag{4.182}
\]

The equation for \( u_0^{(0)} \) is given by

\[
\frac{\partial^2 u_0^{(0)}(z)}{\partial z^2} - 3\kappa_a \kappa_s (1 - g) u_0^{(0)}(z) = 0, \tag{4.183}
\]

with boundary condition

\[
u_0^{(0)}(0) = 1, \tag{4.184}
\]

and radiation condition

\[
\lim_{z \to \infty} \nu_0^{(j)}(z) = 0. \tag{4.185}
\]

The equation \( u_0^{(1)} \) is given by

\[
\frac{\partial^2 u_0^{(1)}(z)}{\partial z^2} - 3\kappa_a \kappa_s (1 - g) u_0^{(1)}(z) = 0, \tag{4.186}
\]
with boundary condition

\[ u_0^{(1)}(0) = \frac{\partial u_0^{(1)}(0)}{\partial z}, \] (4.187)

and radiation condition

\[ \lim_{z \to \infty} u_0^{(j)} = 0. \] (4.188)

We can solve these boundary value problems analytically to find,

\[ I_{out}^{(0)}(z, \hat{s}) = c_0^{(0)} e^{-\sqrt{3} \kappa_a \kappa_s (1-g) z}, \] (4.189)

and

\[ e I_{out}^{(1)}(z, \hat{s}) = -c_0^{(0)} \left( c_0^{(1)} - \mu \right) \sqrt{3} \kappa_a \kappa_s (1-g) e^{-\sqrt{3} \kappa_a \kappa_s (1-g) z}, \] (4.190)

where \( c_0^{(0)} \) and \( c_0^{(1)} \) are computed from zeroth Fourier mode of the radiative transport equation discussed above.

The outer solution at \( z = 0 \)

There are many applications, such as the remote sensing of clouds and diffuse optical tomography, where we are interested in computing the reflectance. By the definition (4.102) and the boundary conditions (4.107),

\[ I_{out}^{(0)} \big|_{z=0} = c_0^{(0)} f(p) \hat{i}_0. \] (4.191)

Similarly, by (4.104) and boundary condition (4.108),

\[ e I_{out}^{(1)} \big|_{z=0} = c_0^{(0)} \left( c_0^{(1)} - \mu \right) \frac{\partial u_0^{(0)}}{\partial z} \big|_{z=0} \hat{i}_0 + \left( c_1^{(1)} - c_0^{(0)} \frac{1 - \mu^2 \cos \varphi}{\kappa_s (1-g)} \right) f_x(p) \hat{i}_0 + \left( c_2^{(1)} - c_0^{(0)} \frac{1 - \mu^2 \cos \varphi}{\kappa_s (1-g)} \right) f_y(p) \hat{i}_0. \] (4.192)

It remains to compute the derivative of \( u_0^{(0)} \) with respect to \( z \) at the boundary. This can be done by taking a Fourier transform of \( u \) in the \( x \) and \( y \) spatial directions, and solving the resulting equation for \( z \).

We define the Fourier transformed functions

\[ \hat{u}_0^{(j)}(k_x, k_y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_0^{(j)}(x, y, z) e^{-ixk_x} e^{-iyk_y} \, dx \, dy, \quad j = 0, 1, \] (4.193)

and

\[ \hat{f}(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{ixk_x} e^{iyk_y} \, dx \, dy. \] (4.194)

For a uniform medium, taking the Fourier transform of equation (4.106) yields

\[ \frac{\partial \hat{u}_0^{(0)}}{\partial z} - (3\kappa_a \kappa_s (1-g) + k_x^2 + k_y^2) \hat{u}_0^{(0)} = 0, \] (4.195)

with boundary condition

\[ \hat{u}_0^{(0)}(x, y, 0) = \hat{f}(k_x, k_y), \] (4.196)
and radiation condition
\[
\lim_{z \to \infty} \hat{u}_0^{(0)}(k_x, k_y, z) = 0.
\] (4.197)

This has solution
\[
\hat{u}_0^{(0)}(k_x, k_y, z) = e^{-\sqrt{3\kappa_a\kappa_s(1-g)+k_x^2+k_y^2} f(k_x, k_y)}.
\] (4.198)

We take the derivative with respect to \( z \) and evaluate at \( z = 0 \) to find
\[
\frac{\partial \hat{u}_0^{(0)}}{\partial z} \bigg|_{z=0} = -\sqrt{3\kappa_a\kappa_s(1-g)+k_x^2+k_y^2} f(k_x, k_y).
\] (4.199)

Finally, we take the inverse transform to obtain
\[
\frac{\partial u_0^{(0)}}{\partial z} \bigg|_{z=0} = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{3\kappa_a\kappa_s(1-g)+k_x^2+k_y^2} f(k_x, k_y) e^{ixk_x} e^{iyk_y} dk_x dk_y.
\] (4.200)

### 4.3.4 Validation of the asymptotic model for zeroth Fourier

Figure 4.1: Errors of the reflectance integrated over \( \varphi \) for \( I \) (upper left), \( Q \) (upper right), \( U \) (lower left), and \( V \) (lower right) incurred by the 1D PDDOM compared to 1D radiative transport equation. The domain is a halfspace with \( \kappa_a = \epsilon \text{ cm}^{-1} \) and \( \kappa_s = \epsilon^{-1} \text{ cm}^{-1} \). The scattering matrix used is the Venutian model [43]. Regression results show the \( O(\epsilon^2) \) error of the PDDOM.
Consider the one-dimensional vector radiative transport equation on a halfspace,

$$\mu \frac{\partial}{\partial z} \mathbf{I} + \kappa_a \mathbf{I} + \kappa_s \mathbf{C} \mathbf{I} = 0,$$  \hspace{1cm} (4.201)

subject to boundary conditions a normally incident circularly polarized beam

$$\mathbf{I}|_{z=0} = \pi \delta(\mu - \mu_0) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$  \hspace{1cm} (4.202)

and

$$\lim_{z \to \infty} \mathbf{I} = 0.$$  \hspace{1cm} (4.203)

We validate the PDDOM by comparing the solution to the zeroth Fourier mode of the diffuse Stokes vector computed at $z = 0$ using the PDDOM with the solution to the vector radiative transport equation using the discrete ordinate method for a halfspace with absorption coefficient $\kappa_a = \epsilon \text{cm}^{-1}$ and $\kappa_s = \epsilon^{-1} \text{cm}^{-1}$ as we vary the value of $\epsilon$. In Figure (4.1) we plot the max error of the PDDOM compared to the vector radiative transport equation at $z = 0$ as we vary the value of $\epsilon$ on a log-log scale. Doing a linear regression on the log-log scale shows the $O(\epsilon^2)$ error of the PDDOM. These results match the expected convergence of the polarized diffusion approximation.

### 4.4 Results

In this section we use results from the PDDOM to investigate the phenomena of circular polarization memory. First, we investigate the flux reflected from a halfspace due to a normally incident beam. Next, we discuss using the spectrum of the discrete radiative transport equation to quantify the depolarization of light as it travels into the diffusion limit.

#### 4.4.1 Scattering models

In this section we will use four scattering models: a Rayleigh scattering model, a model for the Venutian atmosphere, an aerosols model, and a cloud model. The Rayleigh model is the standard Rayleigh model without depolarization,

$$F(\cos \Theta) = \frac{3}{4} \begin{bmatrix} (1 + \cos^2 \Theta) & -(1 - \cos \Theta^2) & 0 & 0 \\ -(1 - \cos^2 \Theta) & (1 + \cos^2 \Theta) & 0 & 0 \\ 0 & 0 & 2 \cos \Theta & 0 \\ 0 & 0 & 0 & 2 \cos \Theta \end{bmatrix}.$$  \hspace{1cm} (4.204)

The anisotropy factor for the Rayleigh phase matrix is $g = 0$. The model for the Venutian atmosphere is the model provided by Hansen and Hovenier [22] and used in the bireflectance code by Mishchenko et al. [43]. This model consist of polydisperse, homogeneous, spherical particles with a relative refractive index of $m = 1.44$ and a size parameters described by the gamma size distribution

$$n(r) \propto r^{(1-3b)/b} \exp \left( -\frac{r}{ab} \right).$$  \hspace{1cm} (4.205)
with \( a = 1.05 \mu m, b = 0.07, \) and \( 0 \mu m \leq r \leq 5 \mu m. \) The distribution is normalized so that

\[
\int_0^5 n(r)dr = 1. \tag{4.206}
\]

The anisotropy factor of the Venutian atmosphere model is \( g = 0.71890. \) The aerosols and cloud models are models presented in Kokhanovsky et al. [34]. Both models are polydisperse media with lognormal distributions

\[
f(a) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\ln^2(a/a_0)}{2s^2} \right) \tag{4.207}
\]

with normalization

\[
\int_0^{\infty} f(a)da = 1. \tag{4.208}
\]

The upper limit of integration of the Mie optical cross section with respect to the radius of the particle \( a \) is bounded above by \( a_{end}. \) The aerosols model uses \( a_0 = 0.3 \mu m, s = 0.92, \) and \( a_{end} = 30 \mu m. \) The cloud model uses \( a_0 = 5 \mu m, s = 0.4, \) and \( a_{end} = 100 \mu m. \) The refractive index of the aerosols particles is \( m = 1.385, \) and the refractive index of the cloud model is \( m = 1.339. \) The anisotropy factor for the aerosols model is \( g = 0.79275. \) The anisotropy factor for the cloud model is \( g = 0.86114. \)

### 4.4.2 Reflectance of a normally incident beam on a halfspace

Reflectance measurements are used in a number of applications from diffusion optical tomography to finding targets in turbid media to remote sensing in clouds. We seek to study the light backscattered by an optically thick medium through the boundary measurements,

\[
F^-(x, y) = - \int_{\hat{s} \cdot \hat{z} < 0} \mathbf{I}(x, y, 0, \hat{s}) \hat{s} \cdot \hat{z} d\hat{s}. \tag{4.209}
\]

We write \( F^- = [F^-_I, F^-_Q, F^-_U, F^-_V]^T. \) The first component of \( F^-_I \) given in (4.209) is the flux leaving the boundary at \( z = 0. \) The other three components give the fractions of this flux corresponding to the \( Q, U, \) and \( V \) Stokes parameters, respectively.

![Beam profile](image)

**Figure 4.2:** The profile of the incident beam \( f(x, y) = \exp(-2(x^2 + y^2)). \)
We use the PDDOM to study the reflectance due to a Gaussian beam normally incident on a halfspace. In Section 4.3 we describe the PDDOM for a halfspace with boundary conditions of the general form
\[ I \bigg|_{z=0} = \delta(\hat{s} - \hat{z}) f(x, y) I_{inc}, \quad \hat{s} \cdot \hat{z} > 0. \] (4.210)
Here we will compute the reflectance due to a circularly polarized beam. In particular, we let \( f \) be a Gaussian beam (see Figure 4.2),
\[ f(x, y) = \exp(-2(x^2 + y^2)) \] (4.211)
and we let the incident polarization state be fully right-hand circularly polarized,
\[ I_{inc} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \] (4.212)

First, we write
\[ F_i(x, y) = -\int_{\hat{s} \cdot \hat{z} < 0} I^{(0)}_{out}(x, y, 0, \hat{s}) \hat{s} \cdot \hat{z} d\hat{s} - \int_{\hat{s} \cdot \hat{z} < 0} e I^{(1)}_{out}(x, y, 0, \hat{s}) \hat{s} \cdot \hat{z} d\hat{s} \]
\[ -\int_{\hat{s} \cdot \hat{z} < 0} I^{(0)}_{bl}(x, y, 0, \hat{s}) \hat{s} \cdot \hat{z} d\hat{s} - \int_{\hat{s} \cdot \hat{z} < 0} e I^{(1)}_{bl}(x, y, 0, \hat{s}) \hat{s} \cdot \hat{z} d\hat{s}. \] (4.213)
From the definition (4.102) and boundary condition (4.107), the contribution to the flux from the leading order outer solution is given by

$$ - \int_{\mathbf{s} \cdot \mathbf{z} < 0} I_{out}^{(0)}(x, y, 0, \mathbf{s}) \mathbf{z} \mathbf{d} \mathbf{s} = \frac{1}{2} c_0^{(0)} f(x, y). \quad (4.214) $$

Similarly, by (4.104) and (4.108),

$$ - \int_{\mathbf{s} \cdot \mathbf{z} < 0} I_{out}^{(1)}(x, y, 0, \mathbf{s}) \mathbf{z} \mathbf{d} \mathbf{s} = c_0^{(0)} \left( \frac{1}{2} c_0^{(1)} - \frac{1}{3} \kappa_s (1 - g) \right) \frac{\partial}{\partial z} u_0^{(0)}(x, y, 0). \quad (4.215) $$

For the leading-order complementary solution

$$ - \int_{\mathbf{s} \cdot \mathbf{z} < 0} I_{bl}^{(0)}(x, y, 0, \mathbf{s}) \mathbf{z} \mathbf{d} \mathbf{s} = \left( -\frac{1}{2} \int_{-1}^{0} I_{0,1,0}^{(0)}(0, \mu) \mu \mathbf{d} \mu \right) f(x, y) + \left( -\frac{1}{2} \int_{-1}^{0} I_{0,2,0}^{(0)}(0, \mu) \mu \mathbf{d} \mu \right) \frac{\partial}{\partial z} u_0^{(0)}(x, y, 0). \quad (4.216) $$

and for the first-order correction

$$ - \int_{\mathbf{s} \cdot \mathbf{z} < 0} I_{bl}^{(1)}(x, y, 0, \mathbf{s}) \mathbf{z} \mathbf{d} \mathbf{s} = c_0^{(0)} \left( -\frac{1}{2} \int_{-1}^{0} I_{0,1,0}^{(1)}(0, \mu) \mu \mathbf{d} \mu \right) \frac{\partial}{\partial z} u_0^{(0)}(x, y, 0). \quad (4.217) $$

We compute the reflected flux for the four models described above, a Rayleigh scattering model, a Venutian atmosphere model, an aerosols model, and a cloud model, using the PDDOM. We compute the complementary solution $I_{0,1,0}^{(0)}$, $I_{0,2,0}^{(0)}$, and $I_{0,1,0}^{(1)}$ and the coefficients $c_0^{(0)}$ and $c_0^{(1)}$ using the discrete ordinate method as outlined in 4.3.2. We compute $\frac{\partial}{\partial z} u_0^{(0)}$ using the spectral method discussed in 4.3.3. Here we numerically approximate the Fourier transform using FFTW. In Figure (4.3) we compare the computed reflected flux for the leading order solution to the asymptotic model and the first order correction when $\kappa_a = 0.01 \text{cm}^{-1}$ and $\kappa_s = 100 \text{cm}^{-1}$.

Notice to leading order the reflected flux $F_T^{-u}$, shown in the top left corner of Figure (4.3), is identical to spectral reflection of the beam. However, the leading order correction of $F_T^{-u}$, given in the top right corner of Figure (4.3), shows the effects of absorption. Since the absorption coefficient is constant in all four models, $F_T^{-u}$ serves as proxy for the path length of the photons (the more scattering events that occur, the more light is absorbed). Our results show that Rayleigh scattering has the shortest average path lengths, followed by the Venutian atmosphere model, then the aerosols model, and finally the cloud model. This is consistent with the relative anisotropy of the four models: we expect forward-peaked scattering to correspond to a greater average path length.

Next consider the polarization state of the four models shown in Figure (4.3). The components $F_U^{-u}$ and $F_U^{-v}$ are completely determined by $I_{0,2,0}^{(0)}$. Since the components $F_U^{-u}$ and $F_U^{-v}$ are determined entirely by the solution of the leading order boundary layer equation, the results are unaffected by small changes in the absorption and scattering coefficient. In particular, this means the characteristic of the $F_U^{-u}$ and $F_U^{-v}$ provide information about the phase matrix independent of the absorption and scattering coefficients. This is potentially useful for interrogating the shape of scatterers.

We see that the $U$ and $V$ components of the reflect flux of the four models studies in Figure (4.3) are in fact distinct. As expected the three anisotropic models exhibit circular polarization memory whereas the handedness of the circular polarization state of the Rayleigh model has flipped handedness. We see that the Venutian atmosphere has the greatest amount of mixing between the $F_Q^{-u}$ and the $F_Q^{-v}$ components, followed by the aerosols model, and then the cloud model. The Rayleigh model does not exhibit any mixing between the $F_Q^{-u}$ and $F_Q^{-v}$ components as $b_2 = 0$. 

54
4.4.3 Quantifying circular polarization memory with the discrete spectrum

In this section we seek to quantify the existence of circular polarization memory in a given medium. Here we will use the polarized diffusion approximation to study the propagation of circular polarized light into the diffusion limit by investigating the spectrum of the discrete transport equation. We will apply our analysis to the models presented above. This work is based on results previously published in [10].

Previous work has investigated circular polarization memory by investigating the effect of anisotropy, refractive index, and size parameters. When MacKintosh et al. [39] first investigated circular polarization memory, they noted that it is a phenomena that is observed in anisotropic medium. Kim and Moscoso [32] investigated the time-dependence of polarization memory, and posited that the occurrence of polarization memory corresponded to many forward-peaked scattering events. This time-dependent theory was validated experimentally by Ni and Alfano [45] and extended to detect targets in turbid media. Xu and Alfano [58] further investigated the role of the size and relatively refractive index of dielectric spherical scatterers. Bicout et al. [2] studied it as a function of the size of scatterers. However, as noted in previous works [1, 18, 19], bulk scattering properties such as the scattering coefficient, the absorption coefficient, and the anisotropy factor have done a poor job of predicting circular polarization memory. Our method provides a means of quantitatively investigating the strength of polarization memory at a low computational cost.

The polarized diffusion approximation shows that in the limit of strong scattering and weak absorption, light is only polarized near the boundary. We use the PDDOM discussed in Section 4.3 to find the solution to the diffusion approximation as an expansion of planewave solutions, that is, solutions of the form

\[ I_{r,m,k}(\zeta, \varphi, \mu_i) = \Phi_{m,k}(\varphi) \tilde{I}_{r,k}(\mu_i) e^{-\lambda_{r,k} \kappa z}, \]  

(4.218)

where \([\lambda_{r,k}, \tilde{I}_{r,k}]\) are the eigenpairs of the \(k\)th Fourier mode of the discrete radiative transport equation, and \(\Phi_{m,k}(\varphi)\) are given by

\[ \Phi_{1,k}(\varphi) = \text{diag}(\cos k\varphi, \cos k\varphi, \sin k\varphi, \sin k\varphi), \]  

(4.219)

and

\[ \Phi_{2,k}(\varphi) = \text{diag}(-\sin k\varphi, -\sin k\varphi, \cos k\varphi, \cos k\varphi). \]  

(4.220)

Here we will define the depolarization length scale \(\eta_{r,k}\) by

\[ \eta_{r,k} = \text{Re} \left[ \frac{1}{\lambda_{r,k}} \right]. \]  

(4.221)

for \(|\lambda_{r,k}| > 0\) (i.e. for all eigenvalues except the zero eigenvalue corresponding the to isotropic, unpolarized eigenfunction \(\tilde{i}_0\)). By ranking \(\eta_{r,k}\), we can investigate the relative depolarization of planewave solutions as the light transitions into the diffusion limit. We call the planewave solutions with the longest depolarization length scales “dominant.”

As discussed in Section 4.3, we discretize the radiative transport equation by expanding the scattering matrix in \(L\)th order series expansion in special functions computed using the double-Gauss quadrature. If we discretize the radiative transport equation using \(N\) quadrature points and a maximum of \(L_{\text{max}} = 2N\) expansion coefficients, then to compute the entire spectrum we must solve an \(4N \times 4N\) eigenvalue problems for each Fourier mode. In general, their are \(L + 1\) Fourier modes; however, we find the dominant eigenvalues are from the first few Fourier modes. Moreover, in section 4.3.1 we show for a normally incident beam the boundary layer solution only has contributions
form the first 3 Fourier modes. In addition, we can use deflation to solve the eigenvalue problems of each Fourier mode for only the smallest eigenvalues and associated eigenvectors.

Table 4.1: This table records the depolarization length scales of the dominant planewave solutions to the discretize radiative transport equation using the four models described above. The radiative transport equation was discretized using $N = 128$ gridpoints. Here $n$ denotes the order of the eigenvalues when ranked from smallest to largest for all Fourier modes and $k$ denotes which Fourier mode the depolarization length scale is from.

<table>
<thead>
<tr>
<th>n</th>
<th>k</th>
<th>Rayleigh</th>
<th></th>
<th>Venus</th>
<th></th>
<th>Aerosols</th>
<th></th>
<th>Cloud</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\eta_{r,k}$</td>
<td></td>
<td>$\eta_{r,k}$</td>
<td></td>
<td>$\eta_{r,k}$</td>
<td></td>
<td>$\eta_{r,k}$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.349586797</td>
<td>0</td>
<td>3.553230559</td>
<td>0</td>
<td>5.246634463</td>
<td>0</td>
<td>8.133171552</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1.172669768</td>
<td>2</td>
<td>3.397550626</td>
<td>2</td>
<td>4.512945312</td>
<td>2</td>
<td>6.635222425</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.093115538</td>
<td>1</td>
<td>2.358607386</td>
<td>1</td>
<td>3.044426207</td>
<td>1</td>
<td>4.395080247</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.999956065</td>
<td>1</td>
<td>1.918605044</td>
<td>1</td>
<td>2.358377275</td>
<td>1</td>
<td>3.332081147</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0.999956065</td>
<td>1</td>
<td>1.900097000</td>
<td>1</td>
<td>2.358377275</td>
<td>1</td>
<td>3.304734452</td>
<td></td>
</tr>
</tbody>
</table>

We compute the depolarization length scales for the four models described in Section 4.4.1: a Rayleigh scattering model, a Venutian atmosphere model, an aerosols model, and a cloud model. In Table 4.1 we record the depolarization length scales for the first five dominant planewave solution where the radiative transport equation is discretized with the double-Gauss quadrature for $N = 128$ gridpoints. The largest depolarization length scale determines the polarization state that dominates as polarization propagates into the diffusion limit. The larger the length scale, the further it can propagate into the medium before light becomes completely depolarized. In addition, the separation between the depolarization lengths gives the degree to which the planewave solution dominates.

The behavior of the Rayleigh model is distinct from the other three models. First, the Rayleigh model depolarizes at a much faster rate than the Venutian atmosphere, aerosols, and cloud models. In addition, the dominant planewave solution contributes to the total intensity and linear polarization, but not the circular polarization. We plot the eigenvector in Figure 4.4. In comparison, the dominant planewave solution in the other three cases contributes to the circular polarization. This is consistent with expected results: Rayleigh scattering is isotropic whereas the Venutian atmosphere, aerosols,
and cloud models are all relative anisotropic.

We now turn our attention to the circular polarization state as light propagates into the diffusion limit. For the Venutian atmosphere, aerosols, and cloud models the dominant planewave solution contributes to the circular polarization state. However, for the Rayleigh model we need to look at the second most dominant planewave solution instead. In all four models, the dominant contribution to circular polarization is given by a solution to the zeroth Fourier mode. By the symmetry of scattering, the matrix $Z_0(\mu, \mu')$ is block diagonal. Hence, the Stokes parameters $I, Q$ and the Stokes parameters $U, V$ decouple, and $I = Q = 0$ for the leading eigenvector for all four models. In the Rayleigh model $b_2 = 0$, and so the $U$ Stokes parameter is also identically zero.

![Figure 4.5](image)

Figure 4.5: The eigenvector of the discretized radiative transport equation that gives the $\mu$ dependence of the planewave solution that gives the dominant contribution to circular polarization memory for Rayleigh model (top left), Venutian atmosphere model (top right), aerosol model (bottom left), and cloud model (bottom right) using a double-Gauss quadrature with $N = 128$. For all four models the planewave solution is from the $k = 0$ Fourier mode, and the $I$ and $Q$ components of the eigenvector are identically zero.

As expected, the three anisotropic models (the Venutian atmosphere model, aerosols model, and cloud model), all exhibit polarization memory whereas the Rayleigh scattering model does not. Figure 4.5 shows the planewave solution associated with the dominant contribution to the circular polarization state as light propagates towards the diffusion limit. We see that the Venutian atmosphere model, the aerosols model, and the cloud model are all sign definite in the $V$ parameter. Hence, these models retain the handedness in all directions as light propagates into the diffusion limit. In contrast, the Rayleigh model flips handedness in the backward directions. The three anisotropic models also demonstrate varying levels of mixing with the linear polarization $U$. Here, the Venutian atmosphere demonstrates the strongest mixing, followed by the aerosols model, and finally the cloud model.

In Figure 4.5, we compute this eigenvector for a fine discretization and high order expansion ($N = 128, L_{max} = 256$). Although this number of grid points and expansion coefficients may be needed to solve the vector radiative transport equation for an anisotropic scattering phase matrix, in general we can get good estimates for the dominant planewave solutions for a coarse grid. Here we find good agreement for the dominant planewave solution with even two-point double Gauss quadrature.
4.5 Conclusion

In this chapter we derived the polarized diffusion approximation as a uniformly valid approximation to the vector radiative transport equation in the asymptotic limit of strong scattering and weak absorption. In general the vector radiative transport equation is computationally expensive to compute in full three spatial dimensions, especially when studying optically thick media. This is usually done via Monte Carlo simulation. We construct a deterministic numerical solver, which we call the polarized diffusion discrete ordinate method (PDDOM) that allows us to compute the solution to the full three dimensional problem.

We use the PDDOM to compute the flux reflected from a uniform medium due a circularly polarized beam normally incident on the boundary at $z = 0$. Using this method we are able to compute the circular polarization separately from the total intensity. Moreover, the circular polarization is not sensitive to perturbation of the scattering and absorption coefficients, and instead gives information about the scattering matrix. We compute the flux for four test models, a Rayleigh scattering model, a Venutian atmosphere model, an aerosols model, and a cloud model. The polarization information from the four models is distinct and can be useful for providing addition information about the scattering in a medium that is distinct from that recovered using the intensity measurements. Notably, all models except the Rayleigh model preserve the handedness of the circular polarization.

Since the PDDOM uses a construction of the full solution in terms of planewave solutions, we are able to use the slowest decaying planewave solutions to investigate the depolarization of light as it propagates into the diffusion limit. We use this observation to develop a method that allows us to quickly determine if a medium exhibits circular polarization memory. We apply this method to the four models discussed above: Rayleigh scattering, a Venutian atmosphere model, an aerosol model, and a cloud model. We found that light for the Rayleigh model depolarizes at a much greater rate than for the remaining four models. Moreover, we found the dominant contributions to the circular polarization in the diffusion limit for the Venutian, aerosols, and cloud model preserved the handedness in all directions, whereas backscattered light for the Rayleigh model flipped handedness.
Chapter 5

Discussion and Future Work

5.1 Insight into circular polarization memory

The objective of this dissertation was to investigate the propagation of polarized light in random media with an interest in understanding circular polarization memory. Recall that circular polarization memory is the physical phenomenon whereby circular polarization retains its ellipticity and handedness when propagating in anisotropic random media. We study circular polarization memory by approximating the vector radiative transport equation in two limits. The first limit was that of forward-peaked scattering, i.e. the limit where most scattering events occur in the forward or near-forward directions. The second limit is that of strong scattering and weak absorption.

In Chapter 3 we study scattering in forward-peaked media by deriving a generalization to the Fokker-Planck approximation, which we call the vector Fokker-Planck approximation. The scalar Fokker-Planck approximation consists of approximation scalar radiative transport equation by replacing the scattering operator with the orbital angular momentum operator, and can be derived by expanding the radiance as a Taylor series about the scattered direction. In the vector radiative transport equation, a similar approximation can be derived by expanding the Stokes vector as series in the generators of rotations about the scattered direction. The vector Fokker-Planck approximation shows that media that exhibit strongly forward-peaked scattering, will exhibit circular polarization memory. This is consistent with the fact that circular polarization memory has been correlated with the anisotropy factor $g$. Moreover, we find that scalar theory is a good approximation in this regime.

In Chapter 4 we study light propagating strongly scattering and weakly absorbing media using the polarized diffusion approximation. The polarized diffusion approximation allows us to reduce the problem of light scattering in an optically thick medium to a one-dimensional vector radiative transport equation near the boundary and a three-dimensional scalar diffusion equation in the interior via boundary layer asymptotic analysis. We derived the polarized diffusion approximation as a uniformly valid approximation to the vector radiative transport equation in the asymptotic limit of strong scattering and weak absorption. In general the vector radiative transport equation is computationally expensive to compute in full three spatial dimensions, especially when studying optically thick media. This is usually done via Monte Carlo simulation. We construct a deterministic numerical solver, which we call the polarized diffusion discrete ordinate method (PDDOM) that allows us to compute the solution to the full three dimensional problem.

We use the PDDOM to compute the flux reflected from a uniform medium due a circularly polarized beam normally incident on the boundary at $z = 0$. Using this method we are able to compute the circular polarization separately from the total intensity. Moreover, the circular polarization is not sensitive to perturbation of the scattering and absorption coefficients, and instead gives information
about the scattering matrix. We compute the flux for four test models, a Rayleigh scattering model, a Venutian atmosphere model, an aerosols model, and a cloud model. The polarization information from the four models is distinct and can be useful for providing addition information about the scattering in a medium that is distinct from that recovered using the intensity measurements. Notably, all models except the Rayleigh model preserve the handedness of the circular polarization.

Since the PDDOM uses a construction of the full solution in terms of planewave solutions, we are able to use the slowest decaying planewave solutions to investigate the depolarization of light as it propagates into the diffusion limit. We use this observation to develop a method that allows us to quickly determine if a medium exhibits circular polarization memory. We apply this method to the four models discussed above: Rayleigh scattering, a Venutian atmosphere model, an aerosol model, and a cloud model. We found that light for the Rayleigh model depolarizes at a much greater rate than for the remaining four models. Moreover, we found the dominant contributions to the circular polarization in the diffusion limit for the Venutian, aerosols, and cloud model preserved the handedness in all directions, whereas backscattered light for the Rayleigh model flipped handedness. We found our results were well-approximated using a coarse discretization of the radiative transport equation and a low order truncation of the expansion of the scattering matrix in Wigner $d$-functions.

In what follows, we describe general extensions to what we have done here.

### 5.2 Polarized propagation in tissues with forward-peaked and large-angle scattering

Both the scalar and vector Fokker-Planck approximations are only valid in the case where large-angle scattering is negligible. However, it is often the case that forward-scattering media do in fact have non-negligible large-angle scattering. This is the case, for instance, in imaging biological tissue [46]. In the scalar case, the Fokker-Planck approximation has been generalized to include higher-order scattering [37] and combined with other approximations, such as the $\delta$-Eddington approximation, to capture large angle scattering [20]. These scalar approximations are often derived by combining approximations to the scalar scattering operator and matching the spectrum up to some order. The same approach can be used for the vector approximation.

In Section 4.4.3 we found we could accurately predict the qualitative behavior of circular polarization memory in the three moderately anisotropic models for the Venutian atmosphere, aerosols, and clouds by truncating the expansion of the scattering matrix in Wigner $d$-functions at the order $\ell = 3$. The approximation of the vector Fokker-Planck approximation, on the other hand, corresponds to a truncation at $\ell = 1$. In order to capture higher-order modes we could truncate the expansion of the scattering matrix as a series of differential operators at a higher order; however, this leads to a numerically unstable approximation. Alternatively, we can subtract the vector Fokker-Planck approximation directly from the scattering operator before truncating the scattering operator at an order $\ell = 3$. The vector Fokker-Planck approximation captures the forward-scattering contributions from the higher order approximation and the $\ell = 2$ and $\ell = 3$ remainder from the scattering matrix captures the large-angle scattering.

### 5.3 Parameter inference with the polarized diffusion approximation

Although it has been shown that polarimetry can provide information that is unavailable in the polarization insensitive case [13], computing the full vector radiative transport equation is numeri-
cally expensive. In Chapter 4 we develop and implement a numerical solver for the polarized diffusion approximation which we call the polarized diffusion discrete ordinate method (PDDOM). The PDDOM is a method that is well-suited to use in parameter inference.

The polarized diffusion approximation is a boundary layer method that allowed us to separate the problem of three-dimensional scattering in a strongly scattering and weakly absorbing medium by decomposing the solution into an one-dimensional vector radiative transport equation and a scalar diffusion approximation. Since the solution in the boundary layer computation is independent of the values of the scattering and absorption coefficients, given the scattering matrix we can pre-compute the planewave solutions that are used to compute the Stokes vector or pre-compute the polarization reflectance measurements.

### 5.4 Light scattering by anisotropic media

Spatial Frequency Domain Imaging (SFDI) is an optical imaging method for measuring absorption, scattering, and fluorescence properties of biological tissue over a large field-of-view [12]. SFDI works by projecting spatially-modulated illumination schemes with various spatial frequencies over a large area of tissue. The remitted diffuse light is demodulated to extract the diffuse reflectance at multiple spatial frequencies. Since absorption and scattering characteristics of tissue are frequency dependent, the demodulated reflectance can then be used to reconstruct the absorption and scattering characteristics. SFDI is a non-contact method that allows pixel-by-pixel reconstruction of absorption and scattering over a large field of view.

In general, it is assumed that scattering in biological tissues is isotropic at the meso-scale, i.e. the orientations of individual scatterers in the tissues are uniformly distributed. However, many biological tissues, such as collagen, muscle fiber, dentin, and axon tracts, have an oriented pattern. Moreover, a loss of this oriented pattern can be useful as a hallmark of disease. In recent years, work at the Beckman Laser Institute at the University of California, Irvine has investigated recovering the spatially varying scattering orientation over a large field of view [35, 44]. To image oriented patterns in tissue with SFDI, light transport is modeled with an anisotropic diffusion approximation to the radiative transport equation.

However, the radiative transport equation, and consequently the diffusion approximation, is formulated under the assumption that the scattering medium does not have an oriented pattern. Heino et al. [23] proposed an anisotropic diffusion approximation based on a modified RTE. While the anisotropic diffusion approximation out-performs the isotropic diffusion approximation when studying partially oriented media, it does not give an accurate estimates of the reduced scattering coefficient when compared to Monte Carlo simulations of a mixture of isotropic and anisotropic scatterers [30, 31].

Anisotropic scattering, like polarized scattering, exhibits a dependence on the scattering frame of reference. Given an model of the anisotropic radiative transport equation, a diffusion approximation for anisotropic scattering can be systematically derived for the anisotropic diffusion approximation using a method similar to that we employed for the polarized diffusion approximation in Section 4.2. Notably, in order to derive an anisotropic diffusion approximation we need to expand the phase function in Wigner $d$-functions, as we do with the vector radiative transport equation, rather than Legendre functions, as is done in scalar transport theory. By implementing this method we can address the following model assumptions in the anisotropic diffusion approximation: (1) the formulation of the anisotropic RTE, (2) the derivation of the diffusion approximation from the anisotropic RTE, and (3) the choice of boundary conditions for the diffusion approximation. A systematic derivation of
the anisotropic diffusion approximation is critically needed to disentangle the effects of these three distinct modeling choices.
Appendix A

Rotations

We will use two different representations to describe rotations. First, we describe a rotation as a rotation of an angle about a fixed axis. We let $\mathcal{R}(\omega, \mathbf{n})$ denote a rotation of angle $0 \leq \omega \leq \pi$ about the fixed axis $\mathbf{n}$. Note, that the axis $\mathbf{n}$ will have the same polar angle representation before and after applying this rotation.

The second representation we use is the Euler angle representation. Here we use $zyz$-convention for describing the Euler angles. In addition, we use passive rotations, i.e., we consider rotations of the axis rather than rotations of the object. We let $\mathcal{E}(\alpha, \beta, \gamma)$ for $0 \leq \alpha, \gamma < 2\pi$ and $0 \leq \beta \leq \pi$ correspond to the rotation described by sequence of rotations:

1. Rotate about the $\hat{z}$-axis by an angle of $\alpha$.
2. Rotate about the new $\hat{y}$-axis by an angle of $\beta$.
3. Rotate about the new $\hat{z}$-axis by an angle $\gamma$.

Appendix B

The auxiliary functions

An expansion for the scattering matrix was presented by Kuščer and Ribarič [36] using generalized spherical functions as described by Gel’fand et al. [17]. This work has been expanded on, and many different conventions are used. We follow Siewert [52], and use the “normalized auxiliary functions.” Rather than defining the functions with respect to generalized spherical functions, we use the related Wigner $d$-functions. For more on the expansion of the radiative transport equation in special functions see Hovenier et al. [26] or Mishchenko et al. [41], for example.
B.1 Definitions

We define the Wigner $d$-functions $d_{m,k}^{\ell}(\theta)$ by the differential representation [57, p.77]

\[
d_{m,k}^{\ell}(\theta) = \frac{(-1)^{\ell-k}}{2^\ell} \left[ \frac{(\ell + k)!}{(\ell + m)! (\ell - m)! (\ell - k)!} \right]^{1/2} (1 - \cos \theta)^{-\frac{k-m}{2}} (1 + \cos \theta)^{-\frac{k+m}{2}} \times \frac{d^{\ell-k}}{d\cos \theta^{\ell-k}} \left[ (1 - \cos \theta)^{\ell-m} (1 + \cos \theta)^{\ell+m} \right]. \tag{B.1}
\]

The Wigner $d$-functions satisfy the symmetry property

\[
d_{m,n}^{\ell}(\theta) = d_{-n,-m}^{\ell}(\theta) = (-1)^{m-n} d_{n,m}^{\ell}(\theta) = (-1)^{m-n} d_{-m,-n}^{\ell}(\theta). \tag{B.2}
\]

The Wigner $D$-functions,

\[
D_{m,n}^{\ell}(\alpha, \beta, \gamma) = e^{-im\alpha} d_{m,n}^{\ell}(\beta) e^{-in\gamma}, \tag{B.3}
\]

are the matrix elements of the rotation $E(\alpha, \beta, \gamma)$ in the angular momentum representation. We follow Siewert [52] and define the normalized auxiliary rotation functions $P_{\ell,k}(x)$, $R_{\ell,k}(x)$, and $T_{\ell,k}(x)$ as

\[
P_{\ell,k}(\cos \theta) = d_{k,0}^{\ell}(\theta), \tag{B.4}
\]

\[
R_{\ell,k}(\cos \theta) = \frac{1}{2} \left( d_{k,2}^{\ell}(\theta) + d_{k,-2}^{\ell}(\theta) \right), \tag{B.5}
\]

and

\[
T_{\ell,k}(\cos \theta) = \frac{1}{2} \left( d_{k,2}^{\ell}(\theta) - d_{k,-2}^{\ell}(\theta) \right). \tag{B.6}
\]

In Siewert [52], the auxiliary rotation functions are defined using generalized spherical functions instead. The generalized spherical functions $P_{m,n}^{\ell}(x)$ are related to the Wigner $d$-functions by the definition

\[
P_{m,n}^{\ell}(\cos \theta) = i^{m-n} d_{m,n}^{\ell}(\theta). \tag{B.7}
\]

In terms of the generalized spherical functions, the auxiliary functions are given by

\[
P_{\ell,k}(\mu) = P_{0,k}^{\ell}(\mu), \tag{B.8}
\]

\[
R_{\ell,k}(\mu) = -\frac{i^k}{2} [P_{k,2}^{\ell}(\mu) + P_{k,-2}^{\ell}(\mu)], \tag{B.9}
\]

and

\[
T_{\ell,k}(\mu) = -\frac{i^k}{2} [P_{k,2}^{\ell}(\mu) + P_{k,-2}^{\ell}(\mu)]. \tag{B.10}
\]

We now define the matrix of auxiliary functions

\[
P_{\ell,k}(\mu) = \begin{bmatrix}
P_{\ell,k}(\mu) & 0 & 0 & 0 \\
0 & R_{\ell,k}(\mu) & -T_{\ell,k}(\mu) & 0 \\
0 & -T_{\ell,k}(\mu) & R_{\ell,k}(\mu) & 0 \\
0 & 0 & 0 & P_{\ell,k}(\mu)
\end{bmatrix}. \tag{B.11}
\]
The matrix $\mathcal{P}_{\ell,k}(\mu)$ is used as a basis to expand the scattering matrix. In addition, we define

$$\mathcal{D}_{\ell,m}(\hat{s}) = \frac{2 - \delta_{k,0}}{2} \Phi_{m,k}(\varphi) \mathcal{P}_{\ell,k}(\mu), \quad (B.12)$$

where

$$\Phi_{1,k}(\phi) = \text{diag}(\cos k\phi, \cos k\phi, \sin k\phi, \sin k\phi), \quad (B.13)$$

and

$$\Phi_{2,k}(\phi) = \text{diag}(-\sin k\phi, -\sin k\phi, \cos k\phi, \cos k\phi). \quad (B.14)$$

### B.2 Orthogonality

The Wigner $d$-functions satisfy the orthogonality property [57, pp.95]

$$\int_{S^2} d_{m,n}(\theta) d_{m,n}'(\theta) \sin \theta \, d\theta = \frac{2}{2\ell + 1} \delta_{\ell\ell'} \cdot (B.15)$$

Using this and the definition of $\mathcal{P}_{\ell,k}(\mu)$, it is straightforward to show

$$\int_{-1}^{1} \mathcal{P}_{\ell,k}(\mu) \mathcal{P}_{\ell',k}(\mu) \, d\mu = \frac{2}{2\ell + 1} \mathbf{E} \delta_{\ell\ell'}, \quad (B.16)$$

where $\mathbf{E}$ is the $4 \times 4$ identity matrix. In addition, the matrices $\mathcal{D}_{\ell,m,k}(\hat{s})$ satisfy the orthogonality property

$$\int_{S^2} \mathcal{D}_{\ell,m,k}(\hat{s}) \mathcal{D}_{\ell',m',k'}(\hat{s}) \, d\hat{s} = \frac{4\pi}{2\ell + 1} \mathbf{E} \delta_{\ell\ell'} \delta_{m,m'} \delta_{k,k'}. \quad (B.17)$$

### B.3 Completeness

Let $\Psi(\hat{s}) = [\Psi_1, \Psi_2, \Psi_3, \Psi_4]^T$ be a function such that

$$\int_{S^2} |\Psi_j(\hat{s})|^2 \, d\hat{s} < \infty, \quad (B.18)$$

for $j = 1, 2, 3, 4$. Then we can expand the function as a series

$$\Psi(\hat{s}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathcal{D}_{\ell,m,k}(\hat{s}) \Psi_{\ell,m,k}, \quad (B.19)$$

where

$$\Psi_{\ell,m,k} = \frac{2\ell + 1}{4\pi} \int_{S^2} \mathcal{D}_{\ell,m,k}^T(\hat{s}) \Psi(\hat{s}) \, d\hat{s}. \quad (B.20)$$
B.4 Recurrence relation

In order to compute the first values in the recurrence relation we use the differential representations of the Wigner $d$-functions [57, pp.77]. We use the definition

$$d^\ell_{mk}(\theta) = \frac{(-1)^{\ell-k}}{2^\ell} \left[ \frac{(\ell + k)!}{(\ell + m)! (\ell - m)! (\ell - k)!} \right]^{1/2} (1 - \cos \theta)^{-\frac{k-m}{2}} (1 + \cos \theta)^{-\frac{k+m}{2}} \times \frac{d^{\ell-k}}{d \cos \theta^{\ell-k}} \left[ (1 - \cos \theta)^{\ell-m} (1 + \cos \theta)^{\ell+m} \right]. \quad (B.21)$$

For $|m| < k$, we simplify

$$d^k_{mk}(\theta) = \frac{1}{2^k} \left[ \frac{(2k)!}{(k + m)! (k - m)!} \right]^{1/2} (1 - \cos \theta)^{(k-m)/2} (1 + \cos \theta)^{(k+m)/2}. \quad (B.22)$$

Since the Wigner $d$-functions satisfy the symmetry property [57, p.79]

$$d^\ell_{-mk}(\theta) = (-1)^{\ell-k} d^\ell_{mk}(\pi - \theta), \quad (B.23)$$

we also have that

$$d^{k}_{-mk}(\theta) = \frac{1}{2^k} \left[ \frac{(2k)!}{(k + m)! (k - m)!} \right]^{1/2} (1 + \cos \theta)^{(k-m)/2} (1 - \cos \theta)^{(k+m)/2}. \quad (B.24)$$

We use this to compute

$$d^0_{0,k}(\theta) = \frac{\sqrt{(2k)!}}{2^k k!} (1 - \cos^2 \theta)^{k/2}, \quad (B.25)$$

$$d^k_{2k}(\theta) = \frac{1}{2^k} \left[ \frac{(2k)!}{(k + 2)! (k - 2)!} \right]^{1/2} (1 - \cos \theta)^{(k-2)/2} (1 + \cos \theta)^{(k+2)/2}, \quad (B.26)$$

and

$$d^{k}_{-2k}(\theta) = (-1)^{\ell-k} \frac{1}{2^k} \left[ \frac{(2k)!}{(k + 2)! (k - 2)!} \right]^{1/2} (1 + \cos \theta)^{(k-2)/2} (1 - \cos \theta)^{(k+2)/2}. \quad (B.27)$$

For $P_{\ell k}(x)$ the initial two values in the recurrence relation are given by

$$P_{k,k}(x) = \frac{\sqrt{(2k)!}}{2^k k!} (1 - x^2)^{k/2}, \quad (B.28)$$

and

$$P_{k+1,k}(x) = \sqrt{2k + 1} x P_{k,k}(x). \quad (B.29)$$

For $k \geq 2$ the first value of $R_{\ell k}(x)$ and $T_{\ell k}(x)$ in the recurrence relations are given by

$$R_{k,k}(x) = \frac{1}{2^k} \left[ \frac{(2k)!}{(k + 2)! (k - 2)!} \right]^{1/2} (1 + x^2) (1 - x^2)^{1+k/2}, \quad (B.30)$$

and

$$T_{k,k}(x) = \frac{1}{2^k} \left[ \frac{(2k)!}{(k + 2)! (k - 2)!} \right]^{1/2} (2x) (1 - x^2)^{1+k/2}. \quad (B.31)$$
The second value of \( R_{\ell k}(x) \) and \( T_{\ell k}(x) \) are given
\[
R_{k+1,k}(x) = \sqrt{\frac{2k+1}{k^2-4}} \left[ (k+1)R_{k,k}(x) - 2T_{k,k}(x) \right],
\]
and
\[
T_{k+1,k}(x) = \sqrt{\frac{2k+1}{k^2-4}} \left[ (k+1)T_{k,k}(x) - 2R_{k,k}(x) \right].
\]

The normalized auxiliary functions can be computed using a recurrence relationship. The auxiliary function \( P_{\ell k}(x) \) satisfies the recurrence relation
\[
\sqrt{(\ell + 1)^2 - k^2} P_{\ell+1,k}(x) = (2\ell + 1)xP_{\ell k}(x) - \sqrt{\ell^2 - k^2} P_{\ell-1,k}(x),
\]
and \( R_{\ell k}(x) \) and \( T_{\ell k}(x) \) satisfy the coupled recurrence relations
\[
\ell \sqrt{(\ell + 1)^2 - k^2} R_{\ell+1,k}(x) = \ell(\ell + 1)(2\ell + 1)xR_{\ell k}(x)
- (\ell + 1)\sqrt{\ell^2 - 4\sqrt{\ell^2 - k^2}} R_{\ell-1,k}(x) - 2k(2\ell + 1)T_{\ell k}(x),
\]
and
\[
\ell \sqrt{(\ell + 1)^2 - 4} \frac{\sqrt{(\ell + 1)^2 - k^2}}{\sqrt{\ell^2 - k^2}} d_{\ell+1}^{\ell+1}(\theta) = (2\ell + 1) \left[ \ell(\ell + 1) \cos \theta \mp mk \right] d_{\ell k}^{\ell}(\theta)
- (\ell + 1)\sqrt{\ell^2 - m^2} \sqrt{\ell^2 - k^2} d_{\ell-1}^{\ell-1}(\theta),
\]
for the Wigner \( d \)-functions [57, pp.90].

**Appendix C**

**Expansion of the scattering matrix**

In both the vector Fokker-Planck approximation and polarized diffusion approximation we make use of the eigenvalues of the scattering operator. In order to compute the spectrum of the scattering operator it is convenient to express the Stokes vector in an alternative basis. Kuščer and Ribarič [36] derive an expansion for the scattering matrix in an alternative basis called the circular polarization.
(CP) basis. These results have been used extensively [15, 25, 53]. We summarize the expansion here for the scattering matrix for the standard Stokes vector.

The components of the phase matrix are expanded as

$$a_1(\cos \Theta) = \sum_{\ell=0}^{\infty} \alpha_1^\ell d_{0,0}^\ell(\Theta), \quad (C.1)$$

$$a_2(\cos \Theta) + a_3(\cos \Theta) = \sum_{\ell=0}^{\infty} (\alpha_2^\ell + \alpha_3^\ell) d_{2,2}^\ell(\Theta) \quad (C.2)$$

$$a_2(\cos \Theta) - a_3(\cos \Theta) = \sum_{\ell=0}^{\infty} (\alpha_2^\ell - \alpha_3^\ell) d_{2,2}^\ell(\Theta) \quad (C.3)$$

$$a_4(\cos \Theta) = \sum_{\ell=0}^{\infty} \alpha_4^\ell d_{0,0}^\ell(\Theta), \quad (C.4)$$

$$b_1(\cos \Theta) = \sum_{\ell=0}^{\infty} \beta_1^\ell d_{2,0}^\ell(\Theta), \quad (C.5)$$

$$b_2(\cos \Theta) = \sum_{\ell=0}^{\infty} \beta_2^\ell d_{2,0}^\ell(\Theta), \quad (C.6)$$

where the expansion coefficients are defined by

$$\alpha_1^\ell = \frac{2\ell + 1}{2} \int_{0}^{\pi} a_1(\cos \Theta)d_{0,0}^\ell(\Theta) \sin \Theta \, d\Theta, \quad (C.7)$$

$$\alpha_2^\ell + \alpha_3^\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} (a_2(\cos \Theta) + a_3(\cos \Theta))d_{2,2}^\ell(\Theta) \sin \Theta \, d\Theta, \quad (C.8)$$

$$\alpha_2^\ell - \alpha_3^\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} (a_2(\cos \Theta) + a_3(\cos \Theta))d_{2,2}^{\ell-2}(\Theta) \sin \Theta \, d\Theta, \quad (C.9)$$

$$\alpha_4^\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} a_4(\cos \Theta)d_{0,0}^\ell(\Theta) \sin \Theta \, d\Theta, \quad (C.10)$$

$$\beta_1^\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} b_1(\cos \Theta)d_{2,0}^\ell(\Theta) \, d\Theta, \quad (C.11)$$

$$\beta_2^\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} b_2(\cos \Theta)d_{2,0}^\ell(\Theta) \, d\Theta. \quad (C.12)$$

Define the coefficient matrices

$$F_\ell = \begin{bmatrix} \alpha_1^\ell & \beta_1^\ell & 0 & 0 \\ \beta_1^\ell & \alpha_2^\ell & 0 & 0 \\ 0 & 0 & \alpha_3^\ell & \beta_2^\ell \\ 0 & 0 & -\beta_2^\ell & \alpha_4^\ell \end{bmatrix}. \quad (C.13)$$

The scattering matrix has expansion

$$Z(\hat{s}, \hat{s}') = \sum_{\ell=0}^{\infty} \sum_{m=1,2}^{\ell} \sum_{k=-\ell}^{\ell} D_{\ell,m,k}(\hat{s}) F_\ell D_{\ell,m,k}(\hat{s}'), \quad (C.14)$$
where $\mathcal{D}_{\ell,m,k}(\hat{s})$ is as defined above,

$$
\mathcal{D}_{\ell,m,k}(\hat{s}) = \frac{2 - \delta_{k,0}}{2} \Phi_{m,k}(\varphi) \mathcal{P}_{\ell,k}(\mu), \quad (C.15)
$$

where

$$
\Phi_{1,k}(\phi) = \text{diag} \left( \cos k\phi, \cos k\phi, \sin k\phi, \sin k\phi \right), \quad (C.16)
$$

$$
\Phi_{2,k}(\phi) = \text{diag} \left( -\sin k\phi, -\sin k\phi, \cos k\phi, \cos k\phi \right), \quad (C.17)
$$

and

$$
\mathcal{P}_{\ell,k}(x) = 
\begin{bmatrix}
P_{\ell,k}(x) & 0 & 0 & 0 \\
0 & R_{\ell,k}(x) & -T_{\ell,k}(x) & 0 \\
0 & -T_{\ell,k}(x) & R_{\ell,k}(x) & 0 \\
0 & 0 & 0 & P_{\ell,k}(x)
\end{bmatrix}. \quad (C.18)
$$

Alternatively, it can be useful to first expand the scattering matrix as a Fourier series. By rearranging the terms in (C.14), we find

$$
Z(\hat{s}, \hat{s}') = \sum_{k=0}^{\infty} (2 - \delta_{k,0}) \left[ \Phi_{1,k}(\varphi) Z_k(\mu, \mu') \Phi_{1,k}(\varphi') + \Phi_{2,k}(\varphi) Z_k(\mu, \mu') \Phi_{2,k}(\varphi') \right], \quad (C.19)
$$

where

$$
Z_k(\mu, \mu') = \sum_{\ell=k}^{\infty} \mathcal{P}_{\ell,k}(\mu) F_\ell \mathcal{P}_{\ell,k}(\mu'). \quad (C.20)
$$
Bibliography


