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A theory of breakers and breaking waves

Cointe, Raymond, Ph.D.
University of California, Santa Barbara, 1987
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Santa Barbara

A THEORY OF BREAKERS AND BREAKING WAVES

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mechanical and Environmental Engineering

by

Raymond Cointe

Committee in charge:

Professor Marshall P. Tulin, Chairman
Professor Jean-Louis Armand
Professor Harold Lewis
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August 1987
The thesis of Raymond Cointe is approved by

[Signatures]

Committee Chairperson

August 1987
August 4, 1987

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Finally, I wish to insist on the fact that it would have been very difficult to carry out this thesis without the help of my Macintosh personal computer which has been used for the word processing, graphics and numerical computations.
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ABSTRACT

A THEORY OF BREAKERS AND BREAKING WAVES

by

Raymond Cointe

Breakers are ubiquitous on the surface of large water bodies in nature and almost always accompany the flow past bodies immersed in flowing water, such as ships and bridge piers. Despite this fact and their considerable practical importance, they have been not well understood and a variety of questions arises in connection with their occurrence, onset, equilibrium configuration, stability and consequences. From a scientific point of view, the subject is made more difficult by the fact that breakers in nature are often unsteady and involve nonlinear free surface phenomena and turbulent dissipative processes.

A physical and mathematical model for steady spilling breakers is proposed which is shown to be in very good agreement with careful systematic measurements of Duncan, based on breaking waves produced by a towed hydrofoil. The steady breaker
is modeled as an almost stagnant eddy riding on the front of an underlying gravity wave. The equilibrium of the breaker is the result of balance between the hydrostatic pressures due to the weight of the eddy and friction at the dividing streamline between the eddy and the underlying flow. The breaker causes a pressure to act on the underlying wave, determining its ultimate shape and leading to its suppression. The analysis tends to explain the appearance of a threshold steepness for breaking in experiments and the existence of a marginal stability zone where both breaking and non-breaking solutions can exist.

This model for steady breakers is extended to include studies of breaker stability and natural modes and a nonlinear theory of non-steady breakers is developed. The unsteady breaker is visualized as turbulent gravity current which rides down the forward slope of the underlying wave and grows due to turbulent entrainment from the flow below. Small oscillations around equilibrium are studied analytically. This analysis allows to assess the stability of the breaker and tends to explain Duncan's observations of natural oscillations in the length of the breaker. A numerical scheme is developed to study the dynamics of the breaker under conditions when the underlying wave is not steady.

Conclusions are drawn on the implications of the model for some engineering applications.
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"La mer jusqu'à l'approche de ses limites est une chose simple qui se répète flot par flot. Mais les choses les plus simples dans la nature ne s'abordent pas sans y mettre beaucoup de formes, faire beaucoup de façons, les choses les plus épaisses sans subir quelque aménusisement."

Francis Ponge,
*Le Parti Pris des Choses.*

1/ INTRODUCTION

Anybody visiting a coastline or flying over the ocean has observed the phenomenon of wave breaking. Plunging breakers on a beach or whitecaps offshore can be seen almost every day. However, even after thousands of years of observations, our scientific understanding of phenomena related to breaking and breaking waves is still surprisingly scanty. Longuet-Higgins (1980a) has characterized breaking as one of the most important basic unsolved problems in hydrodynamics. Breaking is not only a frontier scientific problem which plays a great role in ocean physics, but its is also an important engineering problem. At present, theoretical information on breaking waves is still inadequate for ocean, coastal and naval engineers.

Considering the dramatic changes waves undergo in the surf zone, it is easy to imagine that wave breaking is of major importance there. Breaking waves exert major loadings on coastal structures. They are responsible for the dissipation of wave energy on the beach and for the transfer of momentum to longshore currents. They also play a major role in the erosion of beaches and the transport of sediment.
Even offshore, the surface of the ocean is unbroken only in the lightest winds, with much of the mass, momentum and energy transfer occurring during periods of high wind. Wind wave breaking is therefore essential for the air-sea exchanges and global climatology. For the same reason, its understanding is needed for the full development of satellite remote sensing of the ocean, both thermal and radar.

Breaking is the major mechanism for the dissipation of wave energy, and therefore plays a vital role in determining sea surface spectra. Breaking waves are generally associated with steep waves, and consideration of breaking cannot be ignored in the statistical prediction of wave height which is of great importance for ocean engineers. Moreover, the impact of breaking waves against ships or offshore structures may cause serious safety problems and structural damage of the systems. Understanding of breaking waves is therefore also needed for the safe design of marine structures.

The motion of a ship is almost always accompanied by breakers in the surrounding water. The white water thus produced is one of the most easily observed feature of the ship disturbance. Despite this fact, the importance of wave breaking in the determination of the wave resistance of ships has been pointed out only in recent times. Breaking is not only important for resistance, but also in connection with air mixing and its appearance around sonar domes and the ship's screw; and possibly in the remote sensing (SAR) of narrow V wakes behind surface ships.
Despite its considerable importance, very little is really understood about wave breaking. Laboratory or field measurements do exist, especially for breaking waves on beaches, but our knowledge of the phenomenon is still largely empirical. The subject is made more difficult by the fact that from a scientific point of view the breakers in nature involve turbulent dissipative processes and are often unsteady.

In chapter 2, the previous work concerning breaking waves is reviewed. It is only in the past 15 years that important steps toward the physical and mathematical modeling of breaking waves have been made. Most of them concern the study of the instability mechanisms which lead to breaking and have been conducted within the framework of potential flow theory. In particular, impressive results have been obtained for the mathematical modeling of plunging breakers.

However, in nature, steep waves seem readily to break down, resulting in the formation of dissipative breakers. Very few studies concern the study of waves which are really breaking (or broken), for which due account should be taken of viscous processes, notably turbulent shear. Our theoretical understanding of breakers is still imperfect, and many mysteries remain to be explained about their formation and consequences.

A major advance in the understanding of "steady" spilling breakers was made through the systematic measurements of Duncan (1981 & 1983) where breaking was produced by a towed hydrofoil - see figure 1.1. These experiments provided for the first time not only careful qualitative observations, but also a set of measurements
relating breaker and wave dimensions. These data allow a comparison with theory and have stimulated us to carry out the work reported in this thesis.

Figure 1.1 - The Breaking Wave Created by a Towed Hydrofoil (Schematic)

The Duncan observations give rise to a variety of questions. Some of those which have most intrigued us are:

- What is the physical nature of a steady spilling breaker? What is an appropriate physical model? What is an appropriate mathematical model?
• How is the size of the breaker related to the resistance exerted on the hydrofoil?
• How is the onset of breaking determined?
• What is the relation between the resistance exerted on the hydrofoil and the amplitude of the following waves?
• How does the breaker suppress the following wave?
• Why does a fully formed breaker appear at incipient breaking?

In chapter 3, we propose a physical and mathematical model for steady spilling breakers. In this model, the breaker is modeled explicitly as a low energy, essentially stagnant, recirculating eddy riding on the face of the breaking wave and kept in place by the turbulent friction between the underlying flow and the eddy. The breaker is trailed by a viscous momentum wake corresponding to the frictional force acting on the breaker mass. The breaker bears in this model a strong similarity to attached separated regions behind bluff bodies, as modeled by Tulin & Hsu (1980). The requirements for force equilibrium and for head loss lead to the determination of breaker height in terms of the crest height of the underlying wave. The resistance associated with the breaking wave is determined in two ways: by a momentum balance and by direct potential wave calculation. The effect of the trailing wake is considered and shown to be noticeable.

It is shown that the existence of a steady broken wave arises from the fact that it allows the resistance of the incoming wave to be balanced by the resistance of the breaker plus that of the following waves. These results tend to explain the appearance of a threshold steepness for breaking in the experiments. For wave steepness beyond
threshold, but below Stokes' limiting wave steepness, two breaking solutions are, in general, found to exist: a strong solution (with a large breaker) and a weak solution. The results are shown to be in very good qualitative and quantitative agreement with the laboratory experiments of Duncan in the strong breaking regime.

In chapter 4, the physical model is broadened to take into account non-steady breakers. The breaker is modeled as a dynamic density current, in a way similar to that of Longuet-Higgins & Turner (1974). This current rides down the forward face of the underlying wave and grows due to turbulent entrainment from the flow below. The current is assumed to be thin in comparison to its length so that its dynamics are governed by differential equations of the "shallow water" type, taking into account of stress and entrainment at the bottom. Equations for the height of the current and its speed at every point are derived; they form a hyperbolic nonlinear system.

Utilizing these equations and assuming that the underlying wave evolves in a quasi-steady way, small amplitude oscillations around equilibrium are analyzed. It is shown that only the strong breakers are stable, as observed. For stable breakers, the motion is shown to be damped-oscillating with a Bessel-like mode shape. This result tends to explain Duncan's observation of natural oscillations in the length of the breaker. In the present theory, the frequency of oscillations is shown to depend critically on the boundary conditions imposed at the leading edge of the breaker. An approximate leading edge boundary condition is derived and good quantitative agreement with Duncan's experiments is found.
A numerical scheme is proposed to solve the nonlinear unsteady equations governing the breaker's dynamics. This scheme is used to study small amplitude oscillations around equilibrium and compared to the approximate analytic solution. Large evolutions of the breaker are also studied under conditions when the underlying wave is not steady.

Finally, in chapter 5, conclusions are drawn and the implications of the model for practical engineering applications are discussed. Directions for extension of the theory and its applications are indicated.

In order to focus on the most important physical aspects of our modeling of breakers and breaking waves in the main body of this thesis, lengthy mathematical developments are given in appendices. Note that, in order to avoid cumbersome notations, non-dimensional variables can be introduced in the appendices and special notations used. However, variables in the main body of the thesis are always dimensional.
2/ PREVIOUS WORK

2.1/ INTRODUCTION

In this chapter, we review the previous work concerning breaking waves and related phenomena. An extensive review would probably be impossible, and we focus here on the status of physical and mathematical modeling of breakers and breaking waves. Of interest to us is the understanding of the fluid dynamics of wave breaking near the free surface. Further references can be found in the reviews of Cokelet (1977a), Tulin (1979), Yuen & Lake (1980), Schwartz & Fenton (1982) and Peregrine (1983).

In the first section, we describe the status of measurements, for breakers on beaches or in deep water. The different types of breakers are rapidly defined. Then, in the second section, we review our present theoretical knowledge of breaking waves. Finally, in a third section, we describe some engineering problems for which a better understanding of breaking is needed.

2.2/ A DESCRIPTION OF BREAKING AND THE STATUS OF MEASUREMENT

2.2.1/ BREAKING WAVES ON BEACHES

The classification of the different types of breakers that may appear on a beach was apparently first made by Mason (1952) - see figure 2.1:

* a \textit{plunging} breaker is associated with a prominent jet falling near the base of the wave;
* a spilling breaker is characterized by white water spilling down the front face, starting from the crest.

Less frequent breakers are **collapsing** and **surging** breakers (e.g. Galvin, 1968).

![Diagram of Plunging Breaker](image1)

Plunging Breaker

![Diagram of Spilling Breaker](image2)

Spilling Breaker

**Figure 2.1 - Various Breaking Waves on a Beach (Schematic)**

Experiments concerning shore breaking waves are generally conducted with variable depth flumes in order to model a wave train incident on a plane beach of constant slope. The two primary factors are found to be the deep water wave steepness.
and the slope of the beach (Cokelet, 1977a; Peregrine, 1983; Wiegel, 1982; Mei, 1983; Dean & Dalrymple, 1984). The qualitative time-dependent description of breakers in shallow water is rather satisfactory, essentially because observations and experiments are relatively easy. As a train of waves propagates into shallower water, the height of the wave increases. At some depth, a wave of given characteristics will become unstable and break, dissipating energy in the form of turbulence. This line of first breaking is called the breaker line. As stated above, its position is related to the deep water wave steepness and the slope of the beach.

For a plunging breaker, the wave steepens, becomes asymmetric and then a portion of its surface overturns, projects forward, and forms a jet of water. Then, vortices appear (Miller, 1976). The first vortex is created when the jet closes on front of the wave. The closing of the jet also results in a splash-up. The mechanism of this phenomenon and the origin of the water in the splash-up are not clear (Peregrine, 1983). The splash-up creates a second vortex in front of the first one which eventually forms a surface eddy. The process has been observed to repeat, but each successive vortex is weaker than the preceding one. Since the first vortex translates horizontally backward, Basco (1985) claimed that this may create a new surface wave that propagates forward with new wave kinematics.

The initial breaking seems to be similar for spilling breakers, but with vortices generated on a much smaller scale and confined to the near region of the free surface. Miller (1976) in particular illustrated this with a photograph in which overturning was occurring on a scale of 5 mm. Jansen (1986) developed a technique to visualize the
flow in the aerated upper part of breaking waves. He observed jet-splash motions qualitatively similar for spilling and plunging breakers. The jet motion consists of particles moving down along the forward face of the wave. Subsequently a splash motion is formed by the impact of the jet and its penetration through the free surface.

After the plunge point and any ensuing plunge-splash-up cycle, most shallow water waves settle into a quasi-steady state, spilling-like or bore-like. The main feature of the flow in both cases seems to be that the wave carries along a volume of water (Svendsen, 1984). This mass of water will be referred to here as the "breaker." Behind the breaker, the turbulence continues to spread in a way similar to a two-dimensional turbulent wake behind a body (Peregrine & Svendsen, 1978). Detailed measurements for laboratory shore waves (Stive, 1980) confirm this picture of the flow.

2.2.2/ BREAKING WAVES IN DEEP WATER

Only the plunging and spilling breakers seem to be observed at sea (Cokelet, 1977a). It is thought that the spilling breaker in the most common of deep water ocean breakers (Longuet-Higgins, 1973; Banner & Phillips, 1974). Most of the experiments concerning breaking waves have been conducted for shallow water waves and, as that was stated before, the bottom plays an essential role in this case. It is only recently that some carefully controlled experiments have been conducted for deep water waves. Because the breaking of ocean waves is "sporadic and fugitive, developing fairly abruptly, persisting for a time and then subsiding as the wave crest passes on" (Banner & Phillips, 1974), the lack of specific datas concerning unsteady breakers as they occur in nature is not surprising.
Figure 2.2 - A Breaking Ship Bow Wave (Schematic)

*Quasi-Steady Spilling Breakers*

Quasi-steady breakers are commonly observed around ships (see figure 2.2), but Longuet-Higgins (1974) was apparently among the first to recognize that "the time-dependence inherent in a breaking wave can be partly eliminated by studying the pattern of waves generated by a body in steady motion through the water." He reported laboratory measurement of the velocities in a breaking bow-wave where a wave similar to a spilling breaker appeared.

Extensive studies of the contribution of wave breaking to the wave resistance of ships have been made in Japan (see Inui, 1981). Actually, Inui described the phenomenon by the appearance of "free-surface shock waves" (FSSW). "FSSWs are classified as nonlinear waves that do not have the property of dispersion, and it is explained by the solution of a nonlinear partial differential equation" (Inui, 1981).

Banner & Phillips (1974) generated a standing spilling breaker in a small flume
by a small bar placed laterally across the flume a few centimeters above the floor. The
water depth was sufficiently great that the influence of the bottom was unimportant for
the breaking region. They observed that a turbulent wake extended behind the breaking
region. They suggested that a rolling eddy existed in the vicinity of the crest.

Duncan (1981 & 1983), Battjes and Sakai (1981), and Mori (1986) used a
submerged hydrofoil to induce breaking, see figure 1.1. The resulting quasi-steady
spilling breaker is also characterized by the turbulent breaker carried along at the phase
velocity and the turbulent wake. It is therefore very similar to a quasi-steady spilling
breaker on a beach. Duncan's systematic measurements, which for the first time
provided not only careful qualitative observations but a set of measurements relating the
geometry of the breaking region to the wave dimensions, have been most revealing.
Duncan's observations give rise to a variety of questions, which were already
discussed in introduction. One of the most striking conclusions of his study is that, in
some circumstances, both breaking and non-breaking waves can exist.

Even when a quasi-steady breaker is created, variations of the mean quantities
can be observed. Longuet-Higgins & Turner (1974) analyzed a ciné film by Kjeldsen &
Olsen (1971) of laboratory experiments on breaking waves. They measured the
horizontal position of the wave crest, the inclination to the horizontal of the line joining
the crest to the toe of the whitecap, the length of the whitecap and its maximum
thickness. They observed a marked intermittency in the flow. For the steady spilling
breaker created by a towed hydrofoil, Duncan (1981) observed oscillations of the
breaker with a period approximately equal to 4.4 times the wave period. Banner &
Fooks (1985) reported in detail the wavenumber-frequency spectral properties of a steady spilling breaker. They observed a very marked peak in the surface disturbance frequency spectrum at a much higher frequency, corresponding to a period approximatively equal to one fifth of the wave period.

Concerning the breaker itself, it is composed of an air-water mixture. Longuet-Higgins & Turner (1974) discussed the process of aeration and estimated the fluid density, $\rho'$, of the air water mixture in a breaker to be as low as 0.7 $\rho$. Duncan (1981) obtained a value of 0.6 $\rho$. However, as this is discussed in this thesis, this estimate has been obtained indirectly. Bezzabotnov (1985) reported some measurements of the characteristics of the breaking crests of wind waves which were in the Caspian and Baltic seas using a remote system for macrophotography. The average bubble diameter and concentration, as well as the density of the two-phase mixture in the breaker were estimated. Most direct experiments seem to show that a reasonable range for the ratio $\rho'/\rho$ is [0.7-1.0], but care should be exercised concerning the definition of the mean density $\rho'$.

*Deep Water Laboratory Waves*

Unsteady deep water breakers can also be generated in the laboratory. Longuet-Higgins (1974) suggested that the dispersive properties of deep water waves be used to converge the wave energy in the neighborhood of a single point of the tank. This method has been used, both experimentally and numerically, at MIT to produce a deep water plunging breaker (see Dommermuth & Yue, 1986). Kjeldsen & Myrhaug (1979) used both this method and a convergent channel. They observed plunging breakers,
deep water bores and spilling breakers - see figure 2.3.

Figure 2.3 - Various Breaking Waves in Deep Water (Schematic)

Van Dorn & Pazan (1975) used a convergent channel at the Scripps Institution's Hydraulic Facility. The growth toward breaking was found to be characterized by three successive, steepness dependent phases (H is the height of the wave, L its length):

- a young phase (0 < H/L < 0.10), wherein surface profiles and internal flow fields are symmetric and are roughly in accord with those predicted by Stokes steady wave theory to 5th order;
- a pre-breaking phase (0.10 < H/L < 0.125), wherein there is no obvious
surface manifestation of crest instability, but where crest profiles are markedly unsymmetric, flow isolines are concave upwards and tilted in the direction of wave propagation, and strong velocity gradients develop just beneath the crest;

- a breaking phase ($0.125 < H/L < 0.14$), involving verticality of the forward crest face, obvious crest instability, and increased tilting and curvature of sub-crest fluid velocity isolines so as to form a plunging jet, within which flow velocities equal or exceed the local crest speed.

Similar observations were made at the Institut de Mécanique Statistique de la Turbulence (IMST) in Marseille. In particular, Bonmarin (1984) noticed very rapid variations of the geometric parameters characterizing the wave profile near breaking. In the laboratory, the growth toward breaking seems therefore to be very similar for deep water waves and shallow water waves.

Ochi & Tsai (1983) established a steepness criterion for irregular waves in a laboratory flume. They however obtained a limiting steepness much lower than Stokes' limiting value ($H/L = 0.14$). Most laboratory experiments (Ochi & Tsai, 1983; Koga, 1984; Schultz, Griffin & Ramberg, 1986) confirm that the mean value at which breaking occurs is in the range $H/L = 0.10 - 0.11$, a value remarkably close from the one obtained by Duncan (1983) for steady breaking.

**Deep Water Ocean Waves**

Experimental data on deep water breakers as they appear on the ocean are much scarcer. Deep water breaking of ocean waves is fundamentally more complex than
shallow water or quasi-steady breaking: deep water waves are essentially three dimensional and transient (or unsteady). They are affected by wind, currents, and wave-wave energy transfer (Cokelet, 1977a). More recently, it has even been observed in the laboratory that a uniform train of plane deep water waves may evolve to two or three dimensional breaking through intrinsic hydrodynamic instabilities without external forcing (Melville, 1982; Su & Green, 1984 & 1985; Su, 1986). In this domain, theoretical studies seem to have preceded experiments and this phenomenon will be described in more details in the following section.

The scarcity of detailed data is made clear by the recent statement of Longuet-Higgins & Smith (1983) that "the frequency of occurrence of breaking waves and also their intensity have been little studied." Observations of deep water breakers as they appear in nature are essentially qualitative. For instance, Donelan, Longuet-Higgins & Turner (1972) observed that whitecaps are formed sporadically, each making its appearance down-wave of the following one. They estimated the period of breaking to be about double the wave period in a laboratory tank. This has been explained by the difference between the phase and group velocity in deep water.

It is only very recently that some quantitative data became available. Due to the development of remote sensing of the ocean, some estimates of the variation of whitecap coverage with wind speed have been made (e.g. Monahan, 1971; Wu, 1979; Monahan & Muircheartaigh, 1981). They lead to a power-law expression for the whitecap coverage. Through a detailed theoretical study, measures of sea spectra should also be able to yield valuable informations on the global phenomena (Phillips,
Longuet-Higgins & Smith (1983) used a surface jump meter to estimate the frequency of breaking events in a given wave field. At a wind speed of 14 m/s, they evaluated the number of either steep or breaking wave to be about 1 every 100 wave period. Weissman, Ataktürk & Katsaros (1984) and Holthuijsen & Herbers (1986) obtained a much higher ratio, of order 1 breaking wave for 10 waves. However, this quantitative difference may plausibly be ascribed to the difference in the definition of a breaking wave and the related observation technique.

As a result, even simple geometric data on deep water breaking waves is rare. Observations of whitecaps in the open sea seem to lead to much lower values of the steepness, of order 0.03 - 0.04 (Weissman, Ataktürk & Katsaros, 1984; Holthuijsen & Herbers, 1986)\(^1\) than laboratory experiments. Moreover, statistical wave-by-wave analysis indicates that wave height, steepness or skewness of a breaking wave are not always larger than those of a non-breaking wave (Koga, 1984; Holthuijsen & Herbers, 1986).

\(^1\) The interpretation of this result should however be made carefully. In the laboratory, it is the steepness just prior to breaking which is measured (e.g. Ochi & Tsai, 1983), and this is made possible by the reproducibility of the experiments. At sea, one of the problems is to distinguish between breaking and non-breaking waves. As a result, it seems that it is the steepness of broken waves which is measured. It is very likely that breaking induces a rapid decrease in the amplitude of the wave, and this might account, at least partly, for this important difference. In the steady case, Duncan's experiments (1981, 83) have shown that, as breaking occurs, there is a jump in the steepness.
1986): all pairs of probability density functions largely overlap.

2.3/ THE STATUS OF PHYSICAL AND MATHEMATICAL MODELING

2.3.1/ POTENTIAL FLOW APPROACH

As that was mentioned before, it seems that breaking is generally associated, at least in its initial stage, with the creation of a jet which then creates itself vorticity and turbulence. Therefore, as pointed out by Peregrine (1983), the account of when and how waves break should entirely be in terms of inviscid, initially irrotational flow. This - and the fact that most hydrodynamicists are used to deal with potential flow theory - probably explains why much attention has been given to the study of steep waves, both steady and unsteady, just before breaking. It should however be noticed that experiments (Duncan, 1981 & 83) show that a steady breaking wave can be created even in circumstances where stable potential waves exist. Yet, mathematical studies of waves which are actually broken are much scarcer.

Steep Periodic Waves

It is remarkable that the subject of steep, periodic gravity waves which has its origin in the last century, has seen such advances in both complexity and understanding during the last 15 years. Nonlinear finite amplitude effects lead to the concept of a limiting wave. In 1847, Stokes showed that the flow inside a corner of 120° exactly satisfies the free surface boundary condition, i.e. a constant pressure on the wall. This solution has been regarded as corresponding to the local flow near the crest of the steepest wave. The steepest gravity wave has been found approximately using different methods (see the review of Schwartz & Fenton, 1982). In deep water, the steepness of
the limiting wave (height over length - H/L) is around 0.14. In finite depth, H/L = 0.14 tanh (2πD/L), where D is the water depth. These values are often taken as the critical value above which breaking occurs, for instance by coastal engineers. Note, however, that experiments lead to a lower critical value, even for steady waves, around 0.11 in deep water (Duncan, 1983).

Attempts at computing the limiting wave have shown that it is not possible to use the classical Stokes' expansion, because the first Fourier coefficient of the expansion, which is usually taken as the independent parameter, is not a monotonic function of the wave height (Schwartz, 1974). A very interesting related result, found by Longuet-Higgins (1975), is that the phase speed, the momentum, and the kinetic and potential energy reach their maximum for a wave less steep than the steepest wave. This behavior, characteristic of a nonlinear system, has very important implications:

- the steepest wave can only be reached with some dissipation (for instance by breaking) and some hysteresis-related phenomena might occur, since any decrease in the energy of the steepest wave implies a jump. This has in particular be thought to be relevant to the intermittency observed for spilling breakers (Longuet-Higgins & Turner, 1974);

- it is very likely that some instability appears before the steepest wave.

Similar results exist for the solitary wave (Longuet-Higgins & Fenton, 1974) and waves at intermediate depth (Cokelet, 1977b). For deep water waves, simplified computations have been proposed (Longuet-Higgins & Fox, 1977 & 1978; Longuet-Higgins, 1979a).
As that was discussed in the foregoing, as waves steepen they do not retain the symmetry about their crests that a periodic wave train on uniform depth has. Limiting steepness waves provide, therefore, only a starting point for theoretical studies of wave breaking. But this development of computations of nonlinear waves have revealed a number of unexpected physical phenomena and stimulated further research. As stated by Schwartz & Fenton (1982), "the non-linearity of the describing equations produces a complexity of solution structure that is only now being to be appreciated."

**Instabilities of Finite Amplitude Waves**

In particular, much attention has been given to the problems of instability of waves of finite steepness (see the review by Yuen & Lake, 1980). Benjamin & Feir (1967) exhibited a side-band instability for weakly nonlinear waves. Longuet-Higgins (1978) showed that this instability disappears for a large enough steepness \(H/L = 0.11\), but that another type of instability occurs for an even greater steepness \(H/L = 0.13\). McLean (1982) designated these two types of instability as type I and II. He moreover showed that there are some three-dimensional instabilities of type II. Type II instability is predominantly three dimensional up to the first maximum of the wave energy (Kharif, 1987).

The existence of these instabilities indicates that the wave train is normally in a modulated state. Because the phase and group velocities are not the same in deep water, this implies that the individual waves are unsteady. Experiments demonstrate that these instabilities lead to the formation of strongly modulated wave groups (Lake et al., 1977:...
Melville, 1982; Su et al., 1982). Theoretical studies of wave modulation have been made with approximate equations and/or numerical methods (e.g. Lake et al., 1977; Stiassnie & Kroszynski, 1982; Lo & Mei, 1985; Dold & Peregrine, 1986). These phenomena, which are not directly related to the fluid dynamics of wave breaking, will not be further discussed here. It is important, however, to notice that in some circumstances these instabilities can lead to breaking. This has in particular been shown experimentally (Melville, 1982; Su & Green, 1984 & 1985; Su, 1986).

**Plunging Breakers**

The connection between these instabilities of steep waves and overturning with the production of a forward plunging jet has also been studied recently numerically. Numerical solutions showing overturning were first given by Longuet-Higgins & Cokelet (1976) in a pioneering paper. These authors used an integral equation on the free surface and a Lagrangian time-stepping procedure to solve the full nonlinear problem. In this first paper, overturning was produced by application of a pressure to the rear face of the wave and occurred before the formation of a sharp crest. Later on, Longuet-Higgins & Cokelet (1978) obtained overturning without applying any pressure, by letting unstable modes grow freely.

Peregrine (1983) gave a detailed description of the overturning motion. He found that, even before the face of the wave has a vertical tangent:

- water-particle velocities exceed the wave velocity (this is actually often considered as a criterion for wave breaking);
- water accelerations exceed the acceleration of gravity in a thin region on the
front of the wave;

- low water accelerations exist in an extensive region on and beneath the back slope of the wave.

New, McIver, and Peregrine (1985) extended the method of Longuet-Higgins & Cokelet (1976) to finite (but constant) depth, using the Green's function corresponding to this case. They obtained a large variety of overturning motions, ranging from the projection of a small-scale jet at the crest (which is likely to generate a spilling breaker) to large-scale plunging breakers, and noticed a remarkable similarity in the overturning regions even for differently induced breakers (i.e. using a pressure distribution or a variation of the depth). If the origin of overturning remains unexplained, the fact that the wave crest is vulnerable to perturbations, even for a wave lower than the highest, may eventually give a clue. Since this is a local instability, that might also explain the similarity between the overturnings, despite their initial conditions.

Following the work of Longuet-Higgins & Cokelet (1976), a numerical method for simulation of nonlinear two dimensional free surface problems was also developed (Vinje & Brevig, 1981; Vinje, Maogang & Brevig, 1983) which includes floating bodies. Similar methods have been further developed and applied at MIT (Lin, 1984; Lin, Newman & Yue, 1984; Dommermuth & Yue, 1986; Dommermuth & Yue, 1987), at Bristol (New, McIver & Peregrine, 1985; Dold & Peregrine, 1984 & 1986) and in other places (see for instance the proceedings of the International Workshops on Water Waves and Floating Bodies, 1986 & 1987). If some problems still exist (especially
when free surface piercing bodies are present), these methods are promising and some impressive results have been produced. However, these methods are strictly limited to potential flow theory.

Besides numerical techniques, analytical insight into the problem can greatly contribute to a better understanding of the fluid dynamics of the phenomenon. Longuet-Higgins (1980a & b, 1983) gave a model for the jet of a plunging breaker having the shape of a rotating hyperbola. Longuet-Higgins (1982) also presented a cubic model for the loop of a plunging breaker which fits remarkably well experimental data. New (1983) fitted ellipses of ratio of the axes equal to the square root of 3 to large portions of the face of the wave. Greenhow & Lin (1983) also fitted these ellipses to the splash created by the entry of a wedge or a cylinder into water. It seems particularly promising that such a general shape exists in all these cases. Greenhow (1983) showed that Longuet-Higgins' jet model and New's ellipse model are, for large time, complementary solutions of the same unsteady free surface equation (linear and semi-Lagrangian in form) derived by John (1953).

**A First Conclusion**

In most of the studies of breaking waves we have seen so far, the procedure to induce breaking is the same. Some energy is given to an individual wave by applying a pressure distribution on the back of the wave (Longuet-Higgins & Cokelet, 1976; New, McIver & Peregrine, 1985), by modifying the water depth (New, McIver & Peregrine, 1985), or by focusing the energy locally (Dommermuth & Yue, 1986). A generally accepted conclusion of these results was that waves break because they are
forced to contain more energy that would be contained in the most energetic steady
wave with the same wavelength. However, experimental evidences have shown that
breaking occurs before the most energetic wave, at a steepness of about 0.10 - 0.11 for
laboratory waves, both steady and unsteady (Duncan, 1983, Ochi & Tsai, 1983; Mori,
1986; Schultz, Griffin & Ramberg, 1986). The reason for this breaking before the
limiting wave has been sought in the instability mechanisms. As a matter of fact, these
instabilities have been shown to lead to breaking, both experimentally (Melville, 1982;
Su & Green, 1984 & 1985; Su, 1986) and numerically (Longuet-Higgins & Cokelet,
1978). But it seems that these instabilities only lead to a local steepening of the wave
(via a modulation of the wave train) which in turn induces breaking.

Since Duncan's experiments show that in some cases a steady breaking wave
solution or a potential wave solution can be obtained, depending upon the initial
conditions, it might well be that breaking is not a consequence of the limiting wave.
Another way to look at the problem for steady waves seems therefore to consider the
possibility of the bifurcation towards another steady wave incorporating a breaker
which is in some sense more stable than the potential wave. The subject is made more
difficult by the fact that broken waves involve turbulent dissipative processes but
simplifications arise because the flow is steady.

2.3.2/ QUASI-STeady BREAKERS

As that was mentioned before, the main characteristic of a quasi-steady breaker
is the presence of a recirculating region trailed by a wake. Thus the modeling of a
quasi-steady breakers demands that due account be taken of viscous processes.
Considering the case of shallow water, it is yet tempting to model a bore as a classical hydraulic jump where both the upstream and downstream flows are assumed potential and where the boundary shear stress on the bed is neglected. In this case, a detailed modeling of the region where dissipation takes place is not necessary. Benjamin and Lighthill (1954) used an energy-momentum flux diagram to relate the upstream and downstream flows. They assumed (and showed for long waves) that the volume flow rate, the momentum flow rate, and the energy totally determine a train of stationary gravity waves. Therefore, a bore is entirely described by the energy loss it undergoes (since, by assumption, no momentum loss occurs). If moreover the pressure is assumed to be hydrostatic downstream, the ratio of the depths is given by Belanger's momentum equation (Rajaratnam, 1967) and the energy loss is determined.

Yet, it has been pointed out (Banner & Phillips, 1974) in the case of a spilling breaker that a viscous wake existed behind the breaking region. There is a priori a momentum deficit associated with this wake which is due to the shear stress applied by the breaker on the wave (Duncan, 1981). The situation seems therefore quite similar to the cavity flow theory (Tulin & Hsu, 1980) where cavity drag is associated with a momentum deficit in the wake. The case of the breaking wave behind an hydrofoil (Duncan, 1981 & 1983; Battjes & Sakai, 1981; Mori, 1986) is particularly interesting because this momentum deficit is directly related to the drag on the hydrofoil. The estimation of this drag therefore demands a proper modeling of the breaker.

Such a modeling for a spilling breaker was first proposed by Longuet-Higgins & Turner (1974) who regarded the breaker as a turbulent gravity current riding down
the forward face of the wave. They used results given by Ellison & Turner (1959) to obtain a self-similar solution where they took into account the velocity of the entrained fluid, the pressure force on the layer and the accelerating gravity to write the momentum equation. However, for a steady spilling breaker (such as the one created by a towed hydrofoil), the eddy is essentially stagnant. Therefore, Duncan (1981) related the weight of the breaker to the shearing force it exerts on the forward face of the wave and to the momentum deficit in the wake. He obtained good agreement with his experimental data.

A local solution of the flow near the forward edge of a spilling breaker was first proposed by Longuet-Higgins (1973). He represented the turbulence by a constant eddy viscosity and the tangential stress across the interface between the laminar and turbulent zones by a drag coefficient. He found the inclination of the free surface as a function of the drag coefficient and the densities in the potential and turbulent regions. However, experiments show that the toe is not a stagnation point (Banner & Phillips, 1974; Svendsen and Madsen, 1984) and, as in the case of cavity flow theory, a mixing layer model is probably more appropriate. This model was suggested by Peregrine & Svendsen (1978) and thereafter used by Madsen & Svendsen (1983) and Svendsen & Madsen (1984). These authors developed a numerical model for turbulent bores on beaches where no distinction was made between the breaker and the wake. They integrated the equations over the vertical, with some profile assumed allowing for a recirculating region, assumed the pressure hydrostatic, and made a turbulent closure.

Even for quasi-steady breakers, some fluctuations have been observed in the
length of the breaker (Duncan, 1981). The period of oscillation of the breaking region was found to be approximatively 4.4 times the period of a linear wave of phase speed the velocity of the breaker. Longuet-Higgins (1980a) noticed that this period is not far from the period of the fastest-growing instability of a uniform train of wave (Longuet-Higgins, 1978); while Duncan (1981 & 1983) explained this phenomenon by transient disturbances created by the motion of the hydrofoil when it is started from rest. The higher frequency oscillations observed by Banner & Fooks (1985) have not been explained.

If Duncan's experiments (1981 & 1983) provided an important set of measurements concerning quasi-steady breakers and gave rise to a variety of questions, most of them have not been answered. These unanswered questions motivated the work reported in this thesis.

2.3.3/ OCEAN WAVES: UNSTEADY BREAKERS

Despite the apparent complexity of the phenomenon (or may be because of it), a "threshold" variable which controls the stability of the gravity wave flow has been sought. Because there exists a wave of limiting steepness in the steady case, the first such threshold variable to be considered was the wave steepness. However, as that was stated before, recent experiments have shown that:

- in the laboratory, breaking occurs well before Stokes' limiting wave, even in the steady case (Duncan, 1983; Ochi & Tsai, 1983; Koga, 1984; Schultz, Griffin & Ramberg, 1986);
- for ocean waves, breaking might occur even at a smaller steepness
(Weissman, Ataktürk & Katsaro, 1984; Holthuijsen & Herbers, 1986);

- the steepness of a breaking wave is not always larger than the steepness of a non-breaking wave (Koga, 1984; Holthuijsen & Herbers, 1986).

Another threshold variable involves the vertical acceleration in the wave. Phillips (1958) introduced the idea that, if the local surface acceleration at a wave crest reaches some fraction of the gravitational acceleration g, the wave would then break. Snyder & Kennedy (1983) have used the value of -1/2 g, the acceleration in a Stokes' corner flow. In reinterpreting the experimental results of Ochi & Tsai (1983), Srokosz (1986) proposed a limiting acceleration of - 0.4 g. In a recent paper, Longuet-Higgins (1985) studied the accelerations in steep gravity waves and reached the conclusion that "if some criterion for wave breaking is to be based on some upper bound for the magnitude of theoretical acceleration it should, paradoxically, be the upwards acceleration in the trough that is restricted, not the downward acceleration at the crest." It is likely that no simple criterion for breaking of ocean waves can be expected. As stated by Phillips (1985), "it seems that the recent time history of the surface configuration is much more pertinent to the matter than a single local threshold variable." The non-linearity of the governing equations (even in the absence of viscous processes) and Duncan's experiments (1981 & 1983), where non-unique solutions were observed, support this statement.

Nevertheless, the fact that ocean waves break at a steepness much less than expected has still to be explained. A number of factors might be important:

- the first one is related to unsteadiness and three-dimensional effects, which, as
that was stated before, are certain to develop due to intrinsic instabilities. Studies of instabilities (e.g. Su, 1986) do predict breaking for an initial steepness lower than Stokes' limiting value. However, these values are still larger than the values observed at sea. Moreover, since they concern the initial steepness of the unstable wave, they do not seem to be correlated to the measured value of the steepness at breaking. Yet, experiments confirm that breaking waves are essentially unsteady (Van Dorn & Pazan, 1975). Yuen & Lake (1980) claimed that unsteady waves could break even when their energy is much less than the corresponding limiting wave energy. Similarly, Longuet-Higgins (1985) suggested that unsteady waves may achieve higher acceleration at lower overall steepness, and this might account for an early breaking. If this was confirmed, this would indicate that the breaking of deep water waves could be very dependent upon the extent to which they are unsteady. However, no quantitative estimate of such effects has been given;

* a second possibility is the modulation of the short waves by longer waves. As a result of the straining associated with long waves, the short wavelength (and so the steepness) varies somewhat with respect to the phase of the long wave (Longuet-Higgins & Steward, 1960; Phillips & Banner, 1974). However, this modulation of the steepness (considered in a quasi-steady approach) cannot explain why waves break before they reach their maximum steepness;

* a third possibility is the effect of the wind, which creates a thin, sheared layer of water at the surface, the wind drift layer. The magnitude of the wind drift is of the order of 4% of the wind velocity at 10 meters. This drift augments the fluid particle velocity at the wave crest, forcing the wave to break at a height below the maximum (Banner & Phillips, 1974). Phillips & Banner (1974) also showed that this wave height
might be further reduced when a short breaking wave rides over a longer wave. However, Weissman, Atatürk & Katsaros (1984) evaluated this effect and argued that it could not explain the observed low steepness.

- finally, it must be pointed out again that the problem could be linked to a poor definition of the steepness of a "breaking" wave. The fact that the steepnesses of pre-breaking and of breaking (or broken) waves could be quite different does not seem to have been regarded as a possible explanation.

If carefully controlled laboratory and in-situ experiments are, therefore, still needed to clarify the problem, it seems fruitful to develop theory at least to a certain point, as such developments lead to emphasis on certain aspects of the flows which might later guide experimentations.

2.4/ THE UNDERSTANDING OF BREAKING WAVES: AN ENGINEERING PROBLEM

If surprisingly little is known about breaking waves, the phenomenon of breaking is important for a variety of reasons. As stated before, breaking waves are important for the transfer of momentum to surface currents, the mixing of the upper layer of the ocean and the enhancement of the air-sea exchanges. They, therefore, play an essential role in oceanography and climatology. But breaking waves are also of great importance for coastal and ocean engineers and some of the recent advances in this field were motivated by engineering problems. However, practical information relevant to marine engineers is still very scarce.
Breaking waves have essentially been of interest for coastal engineers. Clearly, one of the major features of coastal waters is the breaking of waves within the surf zone. The importance of the phenomenon in connection with hydrodynamic loads on coastal structures, with sediment transport and with the dissipation of wave energy has, of course, been noticed. Yet many questions about breaking waves remain to be answered. This is made clear by some recent statements:

- "the astonishing fact remains that our quantitative knowledge of the velocity field in breaking waves is very meagre" (Longuet-Higgins, 1980a);
- "to predict the loadings exerted on marine structures, it is necessary to understand finely the phenomenon of breaking. This is a very long term research theme" (Civil Engineering state-of-the-art report, Ministère de l'Equipement, France, 1982);
- "at present, theoretical information on breaking waves on a sloping beach is still inadequate" (Mei, 1983).

Also most of the rules applied in this domain are empirical. For instance, experiments have constantly shown that breaking waves exert much higher loadings on coastal structures than non-breaking waves (e.g. Miller et al., 1974; Wiegel, 1982; Tanimoto et al., 1986). However, even the relation between the breaker type and the magnitude of the impact force has still to be clarified. Not surprisingly, the estimate of breaking-induced loads on coastal structures is generally made by introducing a "curling factor" which allows to relate empirically the phase velocity of the wave, the height of its crest above the mean water level and the breaking-induced loadings (e.g. Wiegel, 1982). Clearly, a better knowledge of the geometry and kinematics of a
breaking wave, as well as a proper description of the impact phenomenon, would be of great practical importance.

In deep water, breaking waves are even less studied and are almost not understood at all. Even though they are recognized as a being the principal loss mechanism regulating the growth of the sea state toward spectral equilibrium (e.g. Phillips, 1985), as being responsible for the transfer of momentum from wind to mean surface flow, and, last but not least, as being able to cause serious safety problems and structural damages to fixed or floating marine structures, they are generally ignored in practice. Even in deep water, forces due to breaking waves have been measured to be an order of magnitude greater than those due to large non-breaking waves (Ochi & Tsai, 1984; Easson & Greated, 1985). Over the last two decades, many large ocean waves have been encountered by ships and offshore structures and some have caused much damage (Su, 1986). Again, recent statements enlighten the need for a better understanding of these phenomena:

- "the literature search in the area of green-sea loadings was disappointing and in some contrast to the importance as a damage factor. The committee hopes that researchers will devote their efforts to this admittedly very difficult problem area" (Committe II.3, ISSC 1985);

- "the primary objective of the Ocean Engineering Program is to provide understanding of the marine environment as it affects deep and shallow water ocean engineering works and operations. In many cases this involves developing theoretical and fundamental understanding to replace the many empirical relations being used by ocean engineers today" (Broad Agency Announcement University Research Initiative
Program, ONR & DARPA, 1986, which quotes breaking as an area of current interest).

The importance of wave breaking in the determination of the wave resistance of ships has only been pointed out in recent times (Baba, 1969; Dagan & Tulin, 1970 & 1972; Tulin, 1979, Inui, 1981). This subject has been studied in detail in Japan (see the review of Inui (1981)) and measurements confirm the potential importance of resistance due to breaking for wide beamed ships. For instance, as much as 15\% of the total resistance of a full-form ship can be associated with the breaking bow wave. Breaking might also be important in connection with the remote sensing of narrow V wakes behind surface ships.

Finally, the importance of wave breaking in connection with satellite remote sensing of the ocean has been noticed recently. In particular, it is thought that breaking might be of major importance for microwave backscattering (Kwoh & Lake, 1984; Banner & Fooks, 1985, Gjessing & Hjelmstad, 1986), since experiments have shown that microwave backscattering occurs in discrete bursts which are highly correlated with "gentle" breaking of the waves (Kwoh & Lake, 1984).
3/ A THEORY OF STEADY SPILLING BREAKERS

3.1/ INTRODUCTION

3.1.1/ THE MODELING OF STEADY SPILLING BREAKERS

The physical and mathematical modeling of a "steady" spilling breaker in deep water (as produced by a towed hydrofoil - see figure 1.1) has been approached in steps within the last fourteen years. This prior work was reviewed in the previous chapter. A major advance was made through the systematic measurements of Duncan (1981 & 1983) which for the first time provided not only careful qualitative observations but a set of measurements relating breaker and wave dimensions. These data allow a comparison with theory and stimulated the work reported in this thesis.

In our theory, which has already been presented partly in Cointe & Tulin (1985) and Tulin & Cointe (1986), the breaker is modeled as a low energy recirculating eddy riding on the forward face of the breaking wave:

- the breaker is kept in place by the friction between the underlying flow and the eddy. It is trailed by a viscous momentum wake corresponding to the frictional force acting on the breaker mass;
- the breaker shape is determined by the balance between hydrostatic pressures and frictional stresses acting on its bounding surface with the underlying wave;
- the turbulent region between the breaker and the underlying wave is modeled as a classical mixing region;
- the crest height is determined as a linear declining function of the eddy vertical height;
the relation between the incoming (unbroken) wave and the following (broken) wave is determined through the application of momentum balance or, alternatively with the same result, by a direct computation of the wave created by the breaker and its subtractive effect on the incoming wave.

The qualitative and quantitative results are compared with Duncan's observations (1981 & 1983) and are found to be in good agreement. The present model of the spilling breaker would, therefore, seem well validated in its overall predictions of the actual physical processes which have been observed. It would thus seem to provide an appropriate basis for understanding of these breakers.

3.1.2/ GENERAL DESCRIPTION OF THE FLOW

Following the observations of Duncan, the breaker is regarded as a closed recirculating region of aerated water in contact with the underlying wave from the crest at point (b), downward over the wave face to its leading edge at point (a), see figure 3.1. The flow is of course turbulent and exchanges take place from instant to instant between the breaker and the underlying wave, resulting in shear stresses. But we hereafter refer to the mean steady flow (i.e. time averaged).

The flow is studied in a reference frame fixed in the breaker (i.e. progressing with the wave at its phase velocity, c). In this reference frame, the underlying flow moves aft under the breaker, with a speed of order of magnitude c. We assume that the flow speeds in the body of the breaker are small compared with those in the wave (this assumption is verified in the analysis).
Figure 3.1 - Physical Model of a Steady Spilling Breaker

As a result of the speed contrast between the fluid in the eddy and beneath it, a strong shear exists along the dividing streamline originating at the leading edge (a) and extending to the end of the breaker which coincides with the crest (b). This mixing region is regarded local to (a) as resembling the free turbulent mixing layer between a
uniform flow and a stagnant region, see figure 3.1. Inside the breaker, the geometry suggests that the flow speed slows and must approach zero as point (b) is approached. The assumption that the crest is a stagnation point is justified by the results and comparison with observations. This assumption is identical to that made by Banner & Phillips (1974) in their analysis of the effects of surface drift on breaking and corresponds of course to the situation in case of the Stokes' limiting potential wave.

We also assume that the fluid within the breaker is aerated due to air mixing, resulting in a substantially lower mean density, $\rho_e$, than in the underlying wave, $\rho$. The degree of aeration is not determined from first principles but has to be estimated from experimental observations and appears to be a very important parameter.

The shear acting on the free streamline between (a) and (b) results in momentum losses in the underlying flow and as a result there exists just beneath the free surface aft of (b) a turbulent wake which thickens and weakens as it flows aft, as observed and measured by Duncan.

Finally, the breaker weight causes a positive pressure to exist on the underlying wave and this causes a modification of the wave (as do the shear stresses and the following wake to a slighter extent), comprising a strong repression of the incoming wave. This was observed and measured by Duncan and it has been one of the major goals of the present theory to explain this effect.
3.2/ THE MODEL OF THE BREAKER

3.2.1/ THE MIXING LAYER AND THE DIVIDING STREAMLINE

Since the crest of the breaking wave is assumed to be a stagnation point, the height of the wave under the dividing streamline should depend upon the head loss experienced by the flow along the dividing streamline, which is a continuation of the free surface. In our model, this flow experiences sudden losses at the leading edge of the breaker (a) where strong initial mixing with the overlying eddy occurs. We visualize this flow as resembling its mixing layer idealization, see figure 3.2. In this flow the total head on the dividing streamline experiences a jump at (a) and is constant thereafter.
The same assumption will be made here, and can be further justified by writing the
conservation of momentum along the dividing streamline. Neglecting deviatoric
stresses which are not shear stresses, this gives using streamline coordinates:

\[ \rho_\sigma q_\sigma \frac{dq_\sigma}{d\sigma} + \rho_\sigma \frac{dp}{d\sigma} + \rho_\sigma g \sin \theta = \frac{d\tau}{d\eta} + 2 K_\sigma \tau \]  

(3.2.1)

where \( q_\sigma \) is the velocity, \( \rho_\sigma \) the density of the fluid and \( \tau \) the shear stress along the
dividing streamline; and \( K_\sigma \) is the curvature of the streamline\(^1\). Note that we expect \( \rho_\sigma \)
to be smaller than \( \rho \) due to the entrainment of air into the breaker (aeration), but greater
than the mean density \( \rho_e \). In the vicinity downstream of (a), we neglect the curvature of
the streamline and we assume that the shear stress peaks on the dividing streamline, as
in the idealized mixing flow. In the vicinity of (b), we assume that the shear stresses are
everywhere small. We therefore get, after integration along the dividing streamline (see
figure 3.3 for nomenclature),

\[ \frac{1}{2} \rho_\sigma q_\sigma^2 + p_\sigma = \rho_\sigma g (\zeta_0 - \zeta) \]  

(3.2.2)

valid at any point between (a) and (b). The integration constant was found using the
fact that both the pressure and the velocity are equal to zero at the crest (b).

\(^1\) See appendix 1 for the definitions of the curvilinear abscissas \( \sigma \) and \( \eta \) and the derivation of this
equation.
Figure 3.3 - Geometric Definitions
Before (a), we assume that the flow is steady and potential and we therefore apply Bernoulli's equation which yields:

$$\frac{q_0^2}{2} + g \zeta = \frac{c^2}{2}$$  \hspace{1cm} (3.2.3)

where $c$ is the phase velocity of the wave and $q_0$ the velocity at the free surface in the potential flow. This choice of the constant in Bernoulli's equation specifies the elevation of the horizontal axis. In deep water, if the flow is uniform far upstream, the horizontal axis coincides with the free surface far upstream and if an incoming wave is present, the horizontal axis coincides with the mean water level far upstream (see Lamb (1932), art. 250).

Across (a), we allow for a jump in head or velocity, and we therefore write:

$$\rho \sigma q_{\sigma}^2 (a) = \beta^2 \rho q_0^2 (a)$$ \hspace{1cm} (3.2.4)

where $\beta$ is smaller than 1.

A consequence of (3.2.2), (3.2.3) and (3.2.4) is that the crest height of the breaking wave, $\zeta_{br}$, becomes determined in terms of the height of the breaker itself, $h^*$:
\[ 2 g \frac{\zeta_b}{c^2} = 1 + 2 \left( 1 - \frac{\rho_\sigma}{\rho \beta^2} \right) g \frac{h^*}{c^2} \]  

(3.2.5)

Furthermore, the elevations above the mean water level of the crest of the breaking wave and of the following crests are essentially identical (this will be confirmed later using potential flow wave calculations). If the steepness of the following waves is small, linear theory can be used and we therefore have \( \zeta_b = a_r/2 \), where \( a_r \) is the height of the following waves. As a result, the prediction (3.3.5) can be compared with Duncan's data. The agreement is excellent for a value

\[ \beta^2 = \frac{\rho_\sigma}{2 \rho} \]  

(3.2.6)

Using numerical results for finite amplitude waves (Longuet-Higgins, 1975; Cokelet, 1977b), it is possible to relate the height of the following waves and the height of their crest, see appendix 4. If we still assume that the height of the crest of the breaking wave is the same as the height of the following crests, this allows to relate \( a_r \) and \( h^* \) from equation (3.2.5) taking into account finite amplitude effects. Both the nonlinear and linear results are plotted on figure 3.4. Note that finite amplitude effects (for the following waves) are not important in the range of Duncan's experiments. This suggests that, in this range, a linear wave theory can be used for the following waves.
Figure 3.4 - Breaker Height vs. Following Waves Height

This analysis can be extended to take into account the effect of surface drift, as due to wind effects, in the same way as in Banner & Phillips (1974). If \( q_d \) is the drift velocity at the mean water level, \( \zeta = 0 \), and co-current with the wave velocity, then the
velocity just upstream of (a) is reduced and equation (3.2.5) becomes (small amplitude waves):

\[
g \frac{a_f}{c^2} = (1 - \frac{q_d}{c})^2 + 2 \left(1 - \frac{\rho_\sigma}{\rho \beta^2}\right) g \frac{h^*}{c^2} \tag{3.2.7}
\]

Clearly, the effect of a drift in the direction of wave propagation is to reduce the maximum height which can be reached by a potential wave (for which \(h^* = 0\)) and to induce early breaking. This was demonstrated by Duncan (1983) where it was shown that early breaking could be simulated by dragging a sheet on the free surface before the hydrofoil wave system.

This result (3.2.7) highlights the fact that breaking is self-induced, because the breaker induces its own drift current which allows stagnation to be reached at the crest.

3.2.2/ FORCE EQUILIBRIA ON THE STEADY BREAKER

These force equilibria lead to several important conclusions (see figure 3.3 for nomenclature). First, we write the vertical equilibrium:

\[
\int_\zeta^{b} \int_{a}^{b} g \, d\mathcal{V} - \int_\zeta^{b} p \, n_\gamma \, d\sigma + \int_\zeta^{b} \tau n_\lambda \, d\sigma = 0 \tag{3.2.8}
\]

Since the hydrostatic pressure balances gravity forces, this equation reduces to
\[ \int_{a}^{b} p_{\text{dyn}} \, dx + \int_{a}^{b} \tau \, dy = 0 \quad (3.2.9) \]

where \( p_{\text{dyn}} \) is the dynamic pressure due to the circulatory motion within the eddy. An estimate of \( p_{\text{dyn}} \) follows from equation (3.2.9) upon assuming that the face of the wave is flat (constant slope \( \theta \)), as suggested by Duncan's observations:

\[ \left| \frac{p_{\text{dyn}}}{\rho c^2} \right| = \frac{\tau}{\rho c^2} \tan \theta \approx 0.007 \frac{\rho c}{\rho} \quad (3.2.10) \]

\( c \) is the wave celerity and the overbars refer to averages over the boundary (a)-(b). The estimate of the averaged value of the stress is based on other data, as that will be discussed later. The estimate (3.2.10) leads to the conclusion that the circulation within the eddy is very weak, just as in a stable separated zone behind a bluff body. On the other hand, the maximum hydrostatic pressure, similarly non-dimensionalized is observed to be approximatively \( 0.35 \rho c / \rho \).

The horizontal equilibrium leads to:

\[ - \int_{a}^{b} p n_x \, d\sigma - \int_{a}^{b} \tau n_y \, d\sigma = 0 \quad (3.2.11) \]
which can also be written, introducing the hydrostatic pressure and assuming the
density to be constant within the breaker:

\[
\int_{a}^{b} \{ (\rho_e gh + p_{\text{dyn}}) \tan \theta - \tau \} \, dx = 0
\]  

(3.2.12)

If we assume \( \theta \) to be a constant and use equation (3.2.9) to eliminate the dynamic
pressure, integration yields

\[
A = \frac{\tau d}{\rho_e g \tan \theta} \frac{1}{\cos^2 \theta}
\]  

(3.2.13)

This equation was first derived by Duncan (1981). The \( \cos^2 \theta \) term corresponds to the
contribution of the dynamic pressure. It is approximatively equal to 0.95 (\( \theta = 13^\circ \)). The
effect of the dynamic pressure is therefore estimated to be 5%. This further justifies the
hydrostatic theory which will be derived in 3.2.3.

Using (3.3.2), (3.3.4) and (3.3.6), this leads to an estimate for the friction
coefficient acting on the dividing streamline near the point (a) in the mixing zone,

\[
\bar{c}_f = \frac{\tau}{\rho \, q_0^2 (a)} = \frac{A}{4 \, d^2} \frac{\rho_e}{\rho} \cos^2 \theta
\]  

(3.2.14)
Duncan (1981) found experimentally $A/d^2 = 0.11$ (constant). The friction coefficient varies therefore linearly with the density ratio and reaches a maximum value of approximately 0.025 when the two densities are equal. This estimate is close to that found experimentally in the mixing zone behind a bluff body (vertical flat plate) by
Arie & Rouse (1956), 0.013; this value would correspond to $\rho_e/\rho = 0.5$.

Equation (3.2.14) also leads to an estimate of the "shape factor" of the breaker, $h^*d/2A$:

$$\frac{h^*d}{2A} = \frac{\rho_e}{8\rho} \frac{\sin\theta}{c_f} \quad (3.2.15)$$

This result is compared in figure 3.5 with data taken again from Duncan (1981). It would seem that the shape factor is, overall, close to unity in value and that the breaker is therefore approximately triangular in shape. This remarkable property of the breaker will thereafter be used to obtain simplified equations.

3.2.3/ HYDROSTATIC THEORY OF THE BREAKER

In view of equation (3.2.10) and the result (3.2.13), the dynamic pressure can be neglected in comparison with the hydrostatic pressure as their mean ratio is equal to $\sin^2\theta$, i.e. approximatively of order $5 \times 10^{-2}$. This leads to a very simple theory in which the shape of the breaker is determined by the horizontal balance between the horizontal pressure and the turbulent shear stresses acting on the side and bottom boundary of a vertical slice of the eddy. As a result, the height of the breaker above the dividing streamline is given by:

$$\rho_e g h \frac{dh}{dx} = \tau - \rho_e g h \tan\theta \quad (3.2.16)$$
or,

\[ \rho_e g h \frac{d(\zeta + h)}{dx} = \tau \quad (3.2.17) \]

Given that \( \tau(a) \) is non-zero, as in the mixing layer, then immediately in the vicinity of (a), the front shape of the hydrostatic eddy is parabolic, rising rapidly as the square root of the horizontal distance from (a),

\[ h = \left\{ 2 \tau(a) \frac{(x - x(a))}{\rho_e g} \right\}^2 \quad (3.2.18) \]

Equation (3.2.18) shows moreover that the breaker height, \( h \), has to remain smaller than \( \zeta_b - \zeta \) as long as the shear stress remains positive\(^2\). The distribution of \( \tau \) along the dividing streamline is unknown, but likely falls rapidly with the speed from a maximum value \( \tau(a) \) near (a) to zero at the stagnation point (b), remaining positive between. From (3.2.2) and the assumption that the pressure in the breaker is hydrostatic, we obtain

\[ q_{\sigma}^2 = 2 g \left\{ (\zeta_b - \zeta) - \frac{\rho_e}{\rho_\sigma} h \right\} \quad (3.2.19) \]

---

\(^2\) As a result, the shape factor \( h^*d/2A \) should always be greater than 1 (if \( \theta \) is constant). Note however that in Duncan's experiments values of \( h^*d/2A \) smaller than 1 have been measured (see figure 3.5). This might be explained by the fact that the density is not constant on a vertical.
As the breaker height, $h$, rises and approaches its limiting value, $(\zeta_b - \zeta)$, the velocity decreases rapidly. This decrease will be especially important for a ratio $\rho_e/\rho_o$ of order 1. As a result, and even if a more detailed modeling of the flow near the leading edge of the breaker might eventually be useful, we expect the shear stresses to be concentrated in a narrow zone near the leading edge. This leads to a very simple model which will be especially useful for the extension of the hydrostatic theory to unsteady evolutions of the breaker.

3.2.4/ THE HEAD DRAG MODEL

We now assume that a finite drag can exist at the leading edge of the breaker, so that

$$\tau^* = \lim_{t \to a} \int_a^t \tau \, dx \neq 0$$  \hspace{1cm} (3.2.20)

Simple equilibrium considerations (or direct integration of equation (3.2.17)) lead to

$$\tau^* = \frac{1}{2} \rho_e g h^*^2$$  \hspace{1cm} (3.2.21)

where $h^*$ is by definition the jump in height at the leading edge (or, more physically, the increase in height across the region of high shear). If we assume that the shear stresses are equal to zero, except near the leading edge, $\tau = 0$ and $\tau^* \neq 0$, we obtain the head drag model where the breaker is flat-topped and its total height is equal to $h^*$ (as
in figure 3.7) and is simply related to the head drag, $\tau^*$, by equation (3.2.21).

Local similarity suggests that $\tau^*$ scales with the height of the breaker, $h^*$, and the speed in the underlying wave just upstream of the leading edge of the breaker, $q_0(a)$. We therefore define a drag coefficient, $c_D^*$, by

$$
\tau^* \equiv \rho \ c_D^* \ h^* \ q_0^2(a)
$$

(3.2.22)

Using equations (3.2.3) and (3.2.21) and assuming again that $2\xi_b = a_f$, equation (3.2.22) yields

$$
\frac{g a_f}{c^2} = 1 + (2 \cdot \frac{\rho_e}{2\rho} \cdot \frac{1}{c_D^*} \cdot \frac{gh^*}{c^2})
$$

(3.2.23)

Duncan’s experimental data (1981) show a linear relationship between $a_f$ and $h^*$ (see figure 3.4) and the agreement is excellent for a value of the head drag,

$$
c_D^* = \frac{1}{8} \cdot \frac{\rho_e}{\rho}
$$

(3.2.24)

The mixing layer model with a constant value of $\beta$ or the head drag model with a constant value of the head drag, $c_D^*$, both lead to the same linear relationship between the total height of the breaker, $h^*$, and the height of the breaking crest, $\xi_b$. 

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3.3/ THE EFFECT OF THE BREAKER ON THE FOLLOWING WAVES

3.3.1/ GENERAL

The breaker is riding on the forward face of the breaking wave and kept in place by shear stresses acting at the dividing streamline. In the previous section, it has been shown how this determines the shape of the breaker. The breaker has been found essentially flat-topped and its total height has been related to the height of the crest of the breaking wave. Reciprocally, the breaker causes a modification of the wave, comprising a strong repression of the incoming wave. This was observed and measured by Duncan and it has been one of the major goals of this theory to explain this effect.

The breaker causes a pressure to act on the underlying wave as well as shear stresses to act on the free streamline (a) - (b), resulting in the following wake. In order to find the free surface elevation downstream of point (b), the action of the shear and its wake can be interpreted as corresponding to a secondary pressure distribution additional to the primary hydrostatic pressure caused by the weight of the breaker eddy. Its effect is estimated here to be noticeable, but small. The detailed analysis for the study of the wake and of its effect on the wave is given in appendices 1 and 2. The main idea in this analysis is that the real viscous flow can be modeled by an appropriate potential flow, the *pseudo flow*.

At the breaking crest, energy is extracted from the underlying wave through the action on the forward face of the wave, resulting in the following wave repression. The
extracted energy is dissipated in the breaker turbulence.

In the following analysis, the effect is calculated in deep water in two separate ways with agreement:

- by a momentum balance taken across the breaking wave, a kind of shock analysis. The balance is phrased in terms of the total resistance resulting in the entire wave system, the resistance associated with the following waves, and the resistance due to the breaker. The detailed analysis for the computation of the wave resistance is given in appendix 3;

- by explicitly calculating the counterwave due to the breaker induced pressure and its subtractive effect on the advancing wave. The detailed analysis for this computation is given in appendix 4.

The first method is quite general but only leads to global properties, essentially the height of the breaker, $h^*$, and the height of the following waves, $a_f$. The only parameter is the incoming resistance (or momentum flux) $R_0$. The resistance source (submerged or free surface piercing body, incoming wave) has no influence. It yields very simple results when a linear wave theory is used to compute the wave resistance and the effect of the wake is neglected. It can also be extended to take into account the effect of the secondary pressure distribution created by the wake and finite amplitude effects using integral properties of waves of arbitrary steepness which have been computed numerically in the literature.

The second analysis gives more information concerning the precise geometry of
the flow, but it requires the resolution of the complete boundary value problem, comprising the source of resistance and the free surface pressure due to the breaker. The procedure will be described on an example, and comparison with the preceding analysis will be made.

3.3.2/ THE WAKE: BOUNDARY LAYER ANALYSIS

General

The trailing wake behind the spilling breaker seems to behave very much like the turbulent wake behind a two-dimensional body, according to the measurements of Duncan (1981) and Battjes & Sakai (1981). We will, therefore, use here arguments similar to those of the classical boundary layer theory along a fully submerged body. Upon the assumption that the wake is suitably thin, its effect may be calculated by finding the outer potential flow, referred to here as the pseudo flow, corresponding to the actual physical wave. This problem is stated and solved mathematically using the method of matched asymptotic expansions in appendix 2. Some results needed in this appendix are derived in appendix 1. Since some algebra is involved, the discussion which follows will only use the results of this mathematical study and focus on their physical interpretation. This analysis is closely related to that of Longuet-Higgins (1969) who investigated the effect of wind shear on wave propagation.

The Displacement and Momentum Thicknesses of the Wake

The major effect is that the pseudo flow free surface is displaced downward by a distance corresponding to the wake displacement thickness, \( \delta_1 \). As the wake flows along the wavy free streamline of the pseudo flow, the pseudo flow speed fluctuates
accordingly (increased in the through and decreased at the crest), and so therefore does the displacement thickness.

In the case of the wake, we assume that the displacement and momentum thicknesses are proportional. It is shown in appendix 2 that Von Kármán's boundary layer momentum equation applies and leads to

$$\delta_1 = \left(\frac{c}{q_\Pi}\right)^{H+2} \Delta$$  \hspace{1cm} (3.3.1)

where $\delta_1$ is the displacement thickness, $c$ the celerity of the wave, $q_\Pi$ the velocity at the free surface of the pseudo flow, $\Delta$ a constant (the momentum thickness where $q_\Pi = c$) and $H$ the shape factor of the wake ($\delta_1/\delta_2$), equal to 1 far downstream. This modulation of the boundary layer thickness is shown schematically on figure 3.7.

The question arises to know how to relate $\Delta$ and the shear applied along the dividing streamline below the breaker. Von Kármán's boundary layer momentum equation can be written as

$$\frac{d\delta_2}{d\sigma} + \frac{1}{q_\Pi} \frac{\partial q_\Pi}{\partial \sigma} (H+2) \delta_2 = \frac{\tau}{\rho q_\Pi^2}$$  \hspace{1cm} (3.3.2)

where $\delta_2$ is the momentum thickness of the boundary layer.
If we assume that \( q_{\Pi} = c \) everywhere along the free surface of the pseudo flow aft of the leading edge of the breaker, point (a), we get (we assume \( H = 1 \) far downstream)

\[
\Delta = \int_{a}^{b} \frac{x}{\rho c^2} \, d\sigma \quad \text{(3.3.3)}
\]

Equation (3.3.3) gives the classical relation between the wake momentum thickness and the integral of the shear along the boundary of the wake. In the case studied here, we however see that this expression is only approximate because the variations of the velocity of the pseudo flow are neglected. Duncan (1981) postulated this relation, equation (3.3.3).

Combined with our estimate of the shear along the dividing streamline, Duncan's estimate (3.3.3) of the wake momentum thickness yields a relation between several experimental observables:

\[
\Delta = \frac{\rho c}{\rho} \sin \theta \frac{gA}{c^2} \quad \text{(3.3.4)}
\]

This relation was first given by Duncan (1981) who measured \( \Delta \) using a pitot traverse in the following wave field and otherwise measured \( \theta \) and \( A \) (1981 & 1983). \( \Delta \) is plotted as a function of \( \sin \theta \frac{gA}{c^2} \) in figure 3.6. A linear relationship appears and,
according to equation (3.3.4), leads to an estimate of $\rho_e/\rho$ around 0.5 (0.6 if only the
data of the first Duncan's paper are used). This estimate appears to be rather small
compared to direct experimental observations, but the probable range quoted by
Longuet-Higgins & Turner (1974) ($\rho'/\rho$ between 0.7 and 1) might be more relevant to
the density at the dividing streamline, $\rho_\sigma$.

However, this estimate is only indirect and the assumption $q_\Pi = c$ cannot be
justified in the vicinity of the breaker. A scaling of $\Delta$ with $q_0(a)$ rather than $c$ in
equation (3.3.4) could, for instance, be more consistent with our head drag model and
would also agree with the few available experimental data for $\Delta$. Considering the
analogy with the drag on a body (see for instance Schlichting, 1968, chapter 25), we
see that relating the momentum thickness, $\delta_2$, and the skin friction (the integral of the
shear stresses along (a)-(b)) is equivalent to determining the form drag. This is a very
difficult problem because it is necessary to have a relation between the value of $q_\Pi$ and
the velocity profile in the boundary layer (which determines $H$). More measurements
would probably be necessary in order to solve this problem.

We, therefore, conclude that care should be exercised in evaluating the density
ratio, $\rho_e/\rho$. This ratio will be taken here as parameter and its effect on the results will be
evaluated (see figures 3.10 and 3.14). Note however that we will always assume that
equations (3.2.6) or (3.2.24) hold.
Figure 3.6 - Skin Friction vs. Momentum Thickness of the Wake
(Experiments)

We will however use Duncan's measurements (1981) for the thickness of the
wake which we write under the alternative form, consistent with the head drag model,
\[
\frac{g \Delta}{c^2} = \gamma \left( \frac{c h}{c^2} \right)^2
\]  \hspace{1cm} (3.3.5)

with \( \gamma \) being a constant. Duncan's experiments (1981 & 1983) lead to assume \( \gamma \) to be in the range \([0.2-0.3] \). The effect of \( \gamma \) on the results will also be studied.

**The Effect of the Wake on the Wave Length of the Following Waves**

Another effect of the boundary layer is that its weight acts upon the pseudo flow free surface causing an additional secondary pressure. Similarly, the curvature of the streamline of the pseudo flow induces an additional secondary pressure. Along the streamline of the pseudo flow, the pressure is therefore not constant but varying with the thickness of the boundary layer. If the pressure at the free surface of the real, viscous flow is constant, equal to zero, the pressure acting along the free surface of the pseudo flow is (see appendix 2):

\[
\frac{p_{\Pi}}{\rho} = g \cos \theta_{\Pi} \delta_1 + q_{\Pi} \frac{\partial q_{\Pi}}{\partial \eta} \delta_2
\]  \hspace{1cm} (3.3.6)

For the wake aft of the crest, point (b), this additional pressure causes a modification of the dispersion relation. It is shown in appendix 1 that the wavenumber \( \kappa \) corresponding to a small amplitude wave for celerity \( c \) is (we assume \( H = 1 \)):

---

3 Equation (3.3.3) would imply \( \gamma = 1/2 \rho e / \rho \).
Figure 3.7 - Effect of Breaker Wake on Following Wavelength
\[ \kappa = (1 + 4 \frac{gA}{c^2}) \frac{g}{c^2} \quad (3.3.7) \]

Combining equations (3.3.5) and (3.3.7), we obtain finally

\[ \kappa = (1 + 4 \gamma \left[ \frac{gh^+}{c^2} \right]^2 ) \frac{g}{c^2} \quad (3.3.8) \]

This relation is plotted on figure 3.7 for different values of \( \gamma \) in the range [0.1-0.3]. It compares well with Duncan's (1981) measurements of the shortened following wavelength. This effect has not been otherwise explained. In particular, the steepness of the following waves is too small for finite amplitude effects to explain the observations (the non-dimensional height of the following waves, \( ga_p/c^2 \), is less than 0.35 - see figure 3.4).

3.3.3/ MOMENTUM BALANCE: SHOCK CALCULATION

In the previous section, we have shown how the flow can be divided in two regions: an outer domain, where the flow is potential, and an inner domain, where boundary layer equations hold. One of the main implication of this model is that, provided that the displacement and momentum thicknesses of the boundary layer are known, the free surface elevation can be computed using potential flow theory only. This means that the real, viscous flow can be modeled by a potential flow, the pseudo flow. This potential flow can be computed using techniques which are now well...
developed.

In this section, we wish to calculate the repressive effect of the breaker upon the advancing wave indirectly, by applying a momentum balance. A direct calculation is possible (and will be described in the next section) but is much more involved and less general.

\[ R_0 = R_b + R_f \]

Figure 3.8 - Control Volume for the Momentum Balance

We will describe successive approximations taking into account different phenomena. First, we will consider the simplest approximation: only the primary
hydrostatic pressure due to the breaker will be considered and a linear wave theory for infinite depth will be used. Then wake effects, finite amplitude effects, and finite depth effects will be discussed.

In order to perform the momentum balance, we consider the pseudo potential flow and the control volume shown in figure 3.8, which includes the free surface of the pseudo flow. Note that the control volumes also includes the bottom (assumed to be horizontal). Results for infinite depth are obtained by letting the depth (and, therefore, the bottom of the control volume) go to infinity. We will assume that aft of the crest (b) the flow is established and, therefore, periodic in x. Using the results of appendix 3, we will then write

$$R_0 = R_b + R_f$$  \hspace{1cm} (3.3.9)

where $R_0$ is the total resistance associated to the disturbance, $R_b$ the resistance associated with breaking, and $R_f$ the resistance associated with the following waves. By definition, $R_b$ is equal to the integral of the pressure along the free surface of the pseudo flow below the breaker. Note that the distinction between the resistance associated with breaking and the resistance associated with the following waves is somehow arbitrary. Coupling between the two exist because of the wake, and the distinction is only clear in the simplest approximation given.

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4 Figure 3.8 corresponds to the case of an incoming wave (described in 3.3.4), but any other momentum source could be considered.
Simplest Approximation

We first consider only the breaker hydrostatic pressure and neglect the secondary pressure due to the boundary layer and curvature effects. For a flat-topped breaker, the integral of the pressure along the dividing streamline (which coincides in this approximation with the free surface of the pseudo potential flow) between the leading edge and the crest is just equal to

\[ R_b = \rho_e g \frac{h^*}{2} = \tau^* \] (3.3.10)

This integral is not a function of the shape of the dividing streamline and can be computed explicitly. It is just equal to the integral of the shear stress along the dividing streamline (a)-(b), i.e. the head drag.

According to linear theory, and for infinite depth, the resistance associated with the following waves, \( R_f \), is given by

\[ R_f = \rho g \frac{a_f^2}{16} \] (3.3.11)

According to this approximation, the total resistance resulting in the entire wave train is, therefore, the sum of the resistance associated with the following wave train - the wave resistance - and of the resistance associated with breaking - just equal to the
head drag (i.e. the skin friction). That is

\[ R_0 = \rho e g \frac{h^*}{2} + \rho g \frac{a_r^2}{16} \]  

(3.3.12)

(3.3.12) clearly shows the contributions of the skin friction due to the breaker and of the momentum flux associated with the following waves. For this first approximation, the resistance associated with breaking (or with the momentum thickness of the wake - see equation (3.3.5)) and the resistance associated with the following waves are not coupled. For a naval architect, this result is very reminiscent of Froude's division of the total resistance into two components and of the wake survey method (see Tulin, 1978). Note that it has been derived here theoretically by modeling the real viscous flow by a potential pseudo flow and by neglecting the effect of the wake on the outer pseudo flow. It is, however, only a first approximation and coupling effects will be discussed below.

Because of the relation existing between \( a_r \) and \( h^* \) (equation (3.2.23)), we note that \( R_b \) decreases with increasing \( a_r \) while the opposite is true for \( R_f \). Their sum can be found in terms of \( a_r \) by eliminating \( h^* \), leading to

\[ \frac{g R_0}{\rho c^4} = \frac{1}{16} \left[ \frac{g a_r}{c^2} \right]^2 + \frac{\rho e}{8 \rho} \left\{ 1 - \frac{2g a_r}{c^2} + \left[ \frac{g a_r}{c^2} \right] ^2 \right\} \]  

(3.3.13)
R_b, R_f and R_0 are plotted on figure 3.9 for \( \rho_e/\rho = 0.8 \). This figure reveals that the existence of the broken wave arises from the fact that it allows the resistance resulting in the entire wave system to be balanced by its own breaker resistance plus the residual resistance of the following waves.

From this figure we note that R_0 reaches a minimum for some value of \( ga_f/c^2 \) between 0 and 1. From equation (3.3.13), this value of the height of the following waves at minimum total wave resistance can be evaluated as

\[
\frac{ga_f}{c^2} = \frac{2\rho_e}{\rho + 2\rho_e}
\]  

(3.3.14)

The corresponding minimum value of the resistance, \( R_{\text{min}} \), is given by

\[
\frac{g R_{\text{min}}}{\rho c^4} = \frac{1}{8} \frac{\rho_e}{\rho + 2\rho_e}
\]  

(3.3.15)

The maximum value of the resistance corresponding to the present theory, \( R_{\text{max}} \), is obtained for \( a_f = 0 \), that is when the breaker has totally suppressed the following wave train. It is given by

\[
\frac{g R_{\text{max}}}{\rho c^4} = \frac{1}{8} \frac{\rho_e}{\rho}
\]  

(3.3.16)
Figure 3.9 - Breaking Resistance, Wave Resistance and Total Resistance
Finally, we note that according to linear wave theory a non-breaking potential wave train of resistance $R_0$ has an height $a_0$ given by

$$\frac{gR_0}{\rho c^4} = \frac{1}{16} \left[ \frac{ga_0}{c^2} \right]^2$$  \hspace{1cm} (3.3.17)

Bernoulli's equation implies that the resistance carried out by a non-breaking potential wave train has a maximum value, $R^*$. From equation (3.3.17), this maximum value can be evaluated to be

$$\frac{gR^*}{\rho c^4} = \frac{1}{16}$$ \hspace{1cm} (3.3.18)

Note, however, that this result (as well as other results of this section) was obtained using a linear wave theory to compute the wave resistance and the nonlinear Bernoulli's equation to express that stagnation is reached at the crest. Nonlinear finite amplitude effects will be discussed later.

The minimum and maximum resistance of a steady breaking wave are plotted in figure 3.10 as a function of the density of the breaker together with the maximum wave resistance which can be carried out by a non-breaking potential wave train. Breaking solutions exist for $R_0$ between $R_{\text{min}}$ and $R_{\text{max}}$ and non-breaking solutions for $R_0$ lower than $R^*$. From figure 3.10 we note that for $\rho_{\text{e}}/\rho \geq 0.5$, $R^*$ is greater than $R_{\text{min}}$ and
lower than $R_{\text{max}}$.

Figure 3.10 - Effect of the Density of the Breaker

_The Meaning of the Results_

This simple model yields excellent qualitative insight into the breaking phenomenon. The total resistance resulting in the entire wave train is plotted as a
function of height of the following waves on figure 3.11 for $\rho_e/\rho = 0.8$. This figure reveals that:

- steady breaking solutions exist above a critical minimum incident resistance $R_{\text{min}}$ and below a critical maximum resistance $R_{\text{max}}$;

- according to calculations using linear wave theory in part (and therefore subject to revision), $R_{\text{min}}$ is lower than $R^*$, the maximum resistance which can be carried out by a non-breaking potential wave train, for $\rho_e/\rho \geq 0.5$. Breaking is not a consequence of the limiting wave.

A wave train can therefore be characterized by the resistance resulting in the entire wave system, $R_0$. Possible solutions are as follow:

- for $R_0 < R_{\text{min}}$, no balance is possible, no breaking wave solution exists, and breaking cannot therefore occur. Only a non-breaking (potential) solution is possible;

- for $R_{\text{min}} \leq R_0 \leq R^*$, two breaking solutions exist - a weak solution with a smaller breaker and larger following waves and a strong solution with a larger breaker and smaller following waves - concurrently with the non-breaking potential solution. Presumably (as this is shown in the next chapter), the weak breaker is unstable and only the strong breaker is observed;

- for $R^* < R_0 \leq R_{\text{max}}$, only the strong breaking solution is possible and the residual resistance associated with the following waves is negligible;

- for $R_{\text{max}} < R_0$, the present model predicts that no steady breaking solution (and, of course, no steady non-breaking solution) can exist.
Figure 3.11 - Shock Relations - Simplest Model
These predictions strongly resemble Duncan's observations (1981 & 1983) which have shown that:

- breaking first occurs for a steepness less than that of the limiting waves;
- for a certain range of total resistance, breaking and non-breaking states are possible;
- at the first appearance of breaking, a fully formed breaker appears;
- thereafter, the breaker grows with increasing hydrofoil resistance and the height of the following waves decreases.

The present simple model of the spilling breaker would seem well validated in its overall predictions of the actual physical processes which have been observed. However, this analysis is incomplete. First of all, we have neglected the effect of the boundary layer in the evaluation of the resistance. Moreover, we have only used a linear wave theory to compute the residual wave resistance, an approach which is certainly inconsistent when \( g \alpha c^2 \) is large (i.e. in the weak breaking regime). These approximations are now going to be discussed and quantitative estimates of \( R_{\text{min}} \), \( R_{\text{max}} \) and \( R^* \) sought.

*The Effect of the Wake*

As that has been shown previously, a first effect of the wake is to modify the wavelength of the following waves. Here, we will consider the effect of the wake on the shock relations. However, we will always use a linear wave theory to compute the wave resistance. The asymptotic expansion corresponding to this approximation is detailed in appendix 3. If \( \alpha \) is a non-dimensional quantity scaling the steepness of the
following waves and $\delta$ a non-dimensional quantity scaling the boundary layer thickness, this expansion is valid in the limit where $\alpha$ and $\delta$ go to zero up to terms of order $\alpha$, $\delta$ and $\alpha\delta$.

$R_b$ is equal to the integral of the pressure at the free surface of the pseudo flow between (a) and (b). This pressure is the sum of the primary pressure due to the breaker and of the secondary pressure due to the wake. If we assume that:

- the pressure within the breaker and the boundary layer are hydrostatic (for the wake, this means that we neglect the term related to the curvature of the streamline);
- the density within the breaker and the boundary layer is $\rho_e$;
- the breaker is flat-topped;

the integral of the pressure can be expressed explicitly as

$$R_b = \rho_e g \frac{h^* - \delta_1(b)^2}{2}$$  \hspace{1cm} (3.3.19)

where $\delta_1(b)$ is the displacement thickness of the boundary layer at the crest (b). Note again that this expression is not a function of the shape of the free streamline: with the assumptions made, the growth of the boundary layer has no effect.

The contribution to $R_b$ of the boundary layer is therefore estimated to be second order in $\delta$ (which scales the boundary layer thickness). Thus it can be neglected in our approximation and
\[ \frac{g R_b}{\rho c^4} \approx \frac{\rho c}{2\rho} \left[ \frac{gh^*}{c^2} \right]^2 \]  

(3.3.20)

Assuming that the flow is established (and periodic) aft of the crest (b) and that the density within the wake is \( \rho \), the resistance associated with the following wave can be estimated to be (see appendix 3)

\[ \frac{g R_f}{\rho c^4} \approx \frac{1}{16} \left[ \frac{ga_f}{c^2} \right]^2 - \frac{g \Delta}{\rho c^2} \frac{\zeta_b}{c^2} \]  

(3.3.21)

where \( \Delta \) is the displacement thickness of the wake at a point where it crosses the mean water level and \( \zeta_b \) is the elevation of the crest above the mean water level. Note that at first order in \( \delta \), it is not necessary to distinguish between the elevation of the free surface of the pseudo flow and the elevation of the real flow.

We finally get from equation (3.3.5) an estimate of the total resistance in presence of the wake, \( R_\Delta \),

\[ \frac{g R_\Delta}{\rho c^4} \approx \frac{g R_0}{\rho c^4} - \frac{\gamma}{2} \left[ \frac{gh^*}{c^2} \right]^2 \frac{ga_f}{c^2} \]  

(3.3.22)

where \( \gamma \) is estimated to be in the range [0.2-0.3] and \( R_0 \) is the total resistance in the
absence of the wake (equation (3.3.13)). This effect of the wake on the shock relations is shown on figure 3.11 where a value of $\gamma = 0.3$ is used (and $\rho_u/\rho = 0.8$).

![Diagram showing total resistance $R_0$ with and without the wake](image)

**Figure 3.11 - Shock Relations - Effect of the Wake**
This figure is very similar to figure 3.9. The effect of the wake is small but noticeable. The major effect is to reduce the minimum resistance which, in presence of the wake, is approximatively given by

\[
\frac{g R_{\Delta \text{min}}}{\rho c^4} = \frac{1}{8} \frac{\rho_e}{\rho + 2\rho_e} - \frac{\gamma}{4} \frac{\rho_e \rho^2}{(\rho + 2\rho_e)^3}
\] (3.3.23)

According to this theory, the maximum resistance is not affected by the presence of the boundary layer.

From this analysis, we therefore conclude that:

* the effect of the boundary layer appears as a coupled term which is first order in the boundary layer thickness and first order in the following wave steepness;

* the presence of the wake decreases the minimum resistance at which breaking can occur;

* there is no effect, at this order, of the modulation of the thickness of the wake by the orbital velocity of the wave.

Note, however, that this study is a priori only valid for following waves of small steepness (\( \alpha \ll 1 \)) and therefore is incorrect for \( ga/c^2 = 1 \), that is in the weak breaking regime. Finite amplitude effects will now be discussed. Difficulties are anticipated because of the coupling between the two parameters of the expansion, \( \alpha \) and \( \delta \).
**Finite Amplitude Effects**

The analysis is carried out in appendix 3. Finite amplitude effects have a tremendous influence on the maximum wave resistance which can be carried out by a non-breaking wave train. This maximum wave resistance is estimated in appendix 3 to be equal in deep water to

\[
\frac{g R_{NL}^*}{\rho c^4} = 0.02
\]  

(3.3.24)

a result already obtained by Duncan (1982). Note that \( R_{NL}^* \) is much smaller than the previously estimated value \( R^* \) (their ratio \( R_{NL}^*/R^* = 0.32 \)) and that this maximum wave resistance is reached for a wave less steep than Stokes' limiting wave (\( ga/c^2 = 0.69 \) vs. \( ga/c^2 = 0.74 \)) - see figure 3.13.

If the effect of the wake is neglected, or if the thickness of the wake is assumed to be constant, finite amplitude effects can be evaluated and are shown on figure 3.13 for \( \rho_w/\rho = 0.8 \). When the breaker is large and the amplitude of the following waves is small, this analysis agrees with the preceding results.

However, finite amplitude effects lead to a very important modification of the shock relations for following waves of large steepness. The total resistance of a breaking wave train, considered as a function of \( a_p \) has no minimum, except for a small density ratio. Such a behavior of the resistance curve would imply that there is only one
breaking regime and that there is a smooth transition between breaking and non-breaking waves (except for some strange phenomena occurring in the vicinity of the wave of maximum resistance), in contradiction with Duncan’s observations.

![Diagram](image)

Figure 3.13 - Shock Relations - Finite Amplitude Effects (Small Aeration)

Note, however, that our estimate of the secondary pressure due to the wake is incorrect for very small breakers and very steep following waves. In particular:
• the modulation of the thickness of the boundary layer should be very important because of the low velocities reached at the crests (see equation 3.3.1);
  • nonlinear effects should be considered to evaluate $p_{II}$ (see equation (3.3.6)).

Yet, the largest following waves observed by Duncan are not very steep and his estimate of $R_{min}$ is as low as 0.83 $R_{NL}^*$. Presumably, finite amplitude effects are not very important for the following waves in the strong breaking regime up to incipient breaking. From equation (3.3.23) with $\gamma = 1/2 \, \rho_e/\rho$, Duncan's estimate of the minimum resistance leads to a value of the density ratio $\rho_e/\rho$ of order 0.2, a value which seems rather small. This low density estimate could be due to the fact that the breaker does not remain flat-topped as the underlying wave steepens - a tendency which seems to be confirmed by figure 3.5. Note also that for small breakers slope of the dividing streamline is greater, so that aeration might be higher. It might therefore well be that the "apparent" density $\rho_e$ - the mass of the breaker divided by the area of the flat-topped surface which contains the breaker - is not constant but varies with the size of the breaker.

Since we estimate the maximum breaking resistance, $R_{max}$, to be given by equation (3.3.16) and the maximum non-breaking wave resistance, $R_{NL}^*$, by equation (3.3.24), we obtain:

$$\frac{R_{max}}{R_{NL}^*} = 6.25 \frac{\rho_e}{\rho}$$

(3.3.25)
Duncan (1983) measured a total resistance as high as $3.1 \frac{R_{NL}}{R_{NL}^*}$, which seems to imply $\rho_e / \rho \geq 0.5$ for large breakers.

![Graph showing $R_{fo1}/R_{NL}^*$ vs $R_0/R_{NL}^*$ with points indicating experiments and present theory.]

**Figure 3.14 - Wave Resistance vs. Total Resistance**

This phenomenon is illustrated on figure 3.14 where the apparent resistance associated with the following waves, $R_{fo1}$, is plotted as a function of the total resistance, $R_0$. Both experimental data from Duncan (1983) and our theoretical results

---

$R_{fo1}$ is the resistance that would be carried out by a non-breaking wave train of height $a_f$. Note that $R_{fo1}$ is not, in general, equal to $R_f$. 

---
(for finite amplitude waves and with $\gamma = 1/2 \rho e/p$) for different values of the density ratio are plotted. Apparently, values of the apparent density ratio as low as 0.2 are needed near incipient breaking and values as high as 0.6 for large breakers.

\textit{Finite Depth Effects}

The preceding computations have also been carried out in finite depth. Actually, the deep water case was obtained in appendix 3 by considering the limit of the finite depth result. For finite amplitude waves, this limiting process is far from being obvious, and significant differences with Duncan's analysis (1982) are shown. Results are found to be qualitatively very similar in intermediate depth and deep water. For small amplitude waves, the shock relations are obtained by substituting to equation (3.3.11) the corresponding expression for the resistance in finite depth, see equation (A3.30). For finite amplitude waves, the numerical results of Cokelet (1977b) are used.

The most interesting phenomenon which appears in finite depth is the creation of an horizontal current in the direction of wave propagation. This current is due to the wake and, therefore, is related to viscous (turbulent) phenomena accompanying breaking. It might be of major importance for coastal engineers interested in sediment transport. The velocity of this current, $c_{br}$, is related to the height of the breaker, $h^*$, (for a depth much greater than $c^2/g$) by

$$c_{br} = \gamma \frac{g}{c} \frac{h^*}{d}$$

(3.3.26)
where d is the water depth and c the celerity of the wave (the constant γ is in the range [0.2-0.3]).

3.3.4/ DIRECT WAVE CALCULATION

In this section, we wish to calculate directly the repressive effect of the breaker upon the advancing wave. Here, we take the point of view that this may be done within the framework of linear potential flow theory and we neglect the effect of the wake on the pseudo flow. This computation will therefore correspond to the first of the approximations discussed in the foregoing. In lieu of the hydrofoil wave system it is simpler to consider the more fundamental problem of a non-breaking advancing wave, height \( a_0 \), for which breaking is somehow stimulated on a particular crest, resulting in its repression and a diminished following wave. For comparison with the foregoing, the advancing wave may be thought of as generated by a source of resistance \( R_0 = \rho g a_0^2/16 \) (according to linear wave theory).

This computation is performed here in order to illustrate how a direct wave calculation can be done. A similar calculation could be done numerically using higher order or fully nonlinear wave theories and/or any disturbance as source of resistance. This would, of course, demand an increased computational effort (and heavy numerical methods). It would, however, only require a potential flow computation for which known methods are available.

The detailed computation using the assumptions described is given in appendix 4. The main difficulty in this computation is that the pressure distribution to be applied
at the free surface of the pseudo flow is not known a priori. The position of the leading edge of the breaker and of the crest of the wave, as well as the height of the breaker have to be found by iteration.

Figure 3.15 - Height of the Breaker vs. Total Resistance
On figure 3.15, the height of the breaker is plotted as a function of the total resistance for $\rho_e/\rho = 0.8$. The plain curves correspond to the shock relations described in the previous section. The squares are results of the numerical computation. The agreement between the two computations is excellent. Computed wave profiles corresponding to the strong breaking regime (shown by a thick line on figure 3.15) are shown on figure 3.16 for $\rho_e/\rho = 0.8$. The same scale is used for the horizontal and vertical axes in order to show the aspect of the wave train. Note that at incipient breaking a fully formed breaker exists and that the steeper the advancing wave (or the greater the total resistance), the larger the breaker and the weaker the following wave. Note also that these computations validate the assumption made in the shock calculation that the elevation of the breaking and following crests are identical.

3.4/ CONCLUSIONS

In this chapter, we have derived a physical and mathematical model for steady spilling breakers. This model, based on Duncan's observations (1981 & 1983), shows that a steady spilling breakers consists of an essentially stagnant eddy (the breaker itself) held in place on the forward face of the breaking wave by the turbulent shear stresses acting on the streamline which separates the breaker and the underlying flow.

This model allows to describe the geometry of the breaker and the effect of the breaker on the breaking and following waves. Its qualitative predictions are in excellent agreement with Duncan's experiments. They show that the existence of the broken wave arises from the fact that breaking allows the resistance resulting in the entire wave system to be balanced by the breaker resistance plus that of the following waves. This
balance is only possible for a sufficiently large total resistance and this analysis explains the appearance of a threshold resistance (and therefore steepness) for steady breaking in the experiments. This threshold resistance is a priori smaller, but not necessarily equal, to that the maximum non-breaking wave resistance. Breaking is, therefore, not a direct consequence of the wave of limiting steepness and this result tends to explain the existence of a marginal stability zone where both breaking and non-breaking solutions can exist.

\[ g \alpha/c^2 = 0.785 \] (incipient breaking)

\[ g \alpha/c^2 = 0.85 \]

\[ g \alpha/c^2 = 0.90 \]

Figure 3.16 - Computed Wave Profiles  
(the same scale is used on the x and y axes)
For a resistance beyond threshold but less than the maximum non-breaking wave resistance, two breaking regimes are, in general, found to exist, strong and weak. Only the strong regime is observed (Duncan, 1981 & 1983) so that at threshold a fully formed breaker appears. For an even higher resistance, only the strong regime exist. The maximum steady breaking resistance is limited by the size of the breaker.

Good quantitative comparisons between theory and experiments have also been found, see figures 3.4, 3.5, 3.7, 3.14 and our estimate of the maximum breaking resistance. However, large variations of the density ratio are needed in order to account for all of Duncan's observations (1981 & 1983), suggesting that the shape of the breaker might vary and the degree of aeration change with the size of the breaker.

The present model for steady spilling breakers would therefore seem provide an appropriate basis for understanding of these breakers. It will be broadened in the next chapter to take into account unsteady effects. In particular, this will allow to study the stability of the breaking solutions and the oscillations in the length of the breaker observed by Duncan (1981).
4/ A THEORY OF UNSTEADY SPILLING BREAKERS

4.1/ INTRODUCTION

In order to take into account unsteady effects and to model unsteady breaking waves, the first step is to extend the hydrostatic theory of the breaker to unsteady evolutions. In this chapter, we therefore focus on the modeling of the dynamics of the breaker and the underlying wave is assumed to evolve in a quasi-steady way. Some indications are given for the full unsteady modeling of the underlying wave in chapter 5. Longuet-Higgins & Turner (1974) were the first to model the breaker as a turbulent gravity current riding down the forward slope of the wave. As stated by Longuet-Higgins & Turner, "the basic assumption in [this] theoretical model is that the whitecap can be regarded as a distinct turbulent flow, which is driven down the slope by the component of gravity in that direction in just the same way as a turbulent gravity current on a solid sloping boundary." The distinction between the flows in and below the breaker is therefore stressed in the present model, and this makes it very different from the model used by Madsen & Svendsen (1983) and Svendsen & Madsen (1984) for turbulent bores in shallow water. Note however that the present model will take into account the motion of the sloping boundary.

Under the assumptions that the flow was steady in time and that the underlying wave remained undisturbed, Longuet-Higgins & Turner obtained a similarity solution. Here, we shall rather take into account acceleration terms and show their importance. We shall also model the advance of the front of the breaker and show the importance of
the coupling with the underlying wave. In this chapter,

- a general theory for unsteady breakers, inspired from that developed by Tulin (1973) for gravity currents, is derived. It generalizes our hydrostatic theory. In this theory, the breaker is assumed to be thin in comparison to its length so that its dynamics are therefore governed by differential equations of the "shallow water" type;

- linearized equations corresponding to a perturbation of the hydrostatic solution found in chapter 3 are formulated. The eigen modes of this set of hyperbolic equations are found analytically (see appendix 5). The importance of an appropriate modeling of the leading edge of the breaker is shown;

- a simplified leading edge boundary condition is derived. It leads to the conclusion that weak breakers are unstable and strong breakers are stable and allows comparison with Duncan's observations (1981) of natural oscillations in the length of the breaker. Good qualitative and quantitative agreements are found;

- however, because the breaker spills on the front of the wave, a complete model for large evolutions of the breaker should take into account nonlinear terms and eventually incorporate a more refined description of the coupling between the two evolutions. A numerical scheme is derived to solve the nonlinear equations governing the dynamics of the breaker (see appendix 6). The results are compared with the analytic estimate of the period of oscillation of the breaker. The model is used to study breaker stability and the onset of breaking under conditions when the underlying wave is non-steady.
Figure 4.1 - Gravity Current Model for the Breaker (Schematic)
4.2/ GENERAL THEORY

Following the work of Longuet-Higgins & Turner (1974), the breaker is modeled as a turbulent gravity current riding on the front face of the wave. In order to derive equations to describe the dynamics of the breaker, we will take into account both shear stresses acting at the bottom of the breaker and entrainment of mass and momentum from the underlying wave. We will also model the flow in the vicinity of the leading edge of the breaker by considering the existence of shear stresses and entrainment concentrated in the vicinity of the leading edge. This physical model for the unsteady breaker is sketched on figure 4.1.

We will consider in this chapter a coordinate system $(X,Z)$ having for origin the crest $b$, oriented as shown in figure 4.2. Note that this convention is not the same as in the other chapters. The velocity of the crest in a fixed coordinate system is $(u_b, v_b)$. We call $(u_e, v_e)$ the velocity in the breaker in the moving $(X,Z)$ coordinate system, $h$ the local height of the breaker, $h^*$ the value of $h$ at the leading edge (which can be non-zero) and $d$ its length. The line $a-b$ is defined by $Z = \xi(X,t)$. Note that because of this convention, $\xi$ is the elevation of the line $a-b$ relative to the crest. It is therefore different from $\zeta$ introduce before in the steady case (in fact $\xi = \zeta_b - \zeta$). The basic assumption here is that $h/d << 1$, so that the pressure can be assumed hydrostatic as in the shallow water theory.

We will write the continuity equation and the conservation of horizontal momentum for a slice $(r,s,t,u)$ of the breaker, see figure 4.2. The boundary of the slice is fixed at a given instant. The equations are written first on such a volume element in
order to be able to derive the equations at the extremities of the breaker (where we will assume that concentrated forces can act) in an elementary manner.

Figure 4.2 - Geometric Definitions
We first write the continuity equation. We assume that particles at the free surface remain at the free surface, so that the vertical velocity there is \( \partial(h-\xi)/\partial t + u_e \partial(h-\xi)/\partial X \). If we call \( E \, dX \) the flux entering the breaker through the bottom of a slice of thickness \( dX \) (equal to zero if there is no exchange between the breaker and the underneath flow), we get:

\[
- \int_{r}^{u} u_e \, dZ + \int_{t}^{s} u_e \, dZ + \int_{t}^{u} (E - \frac{\partial \xi}{\partial t}) \, dX = \int_{s}^{r} \frac{\partial(h-\xi)}{\partial t} \, dX \quad (4.2.1)
\]

As \( t \to u \) and \( s \to r \), equation (4.2.1) leads to the local equation at a point between the leading edge and the crest:

\[
E = \frac{\partial h}{\partial t} + \frac{\partial}{\partial X} \int_{\xi-h}^{\xi} u_e \, dZ \quad (4.2.2)
\]

At the leading edge itself, special care must be taken. We will assume that a finite volume can be entrained over a very small distance and define

\[
E^* = \lim_{t \to a} \int_{t}^{a} E \, dX \quad (4.2.3)
\]

Equation (4.2.1) can then be written between \( t \) and \( a \). If we allow the height of the
breaker to grow from 0 to $h^*$ there, we finally get in the limit where $t \to a$:

$$h^* \frac{dd}{dt} = \int_{\xi^*-h^*}^{\xi^*} u_e \, dZ + E^* \tag{4.2.4}$$

We now write the conservation of horizontal momentum, i.e. Euler's theorem, for the same slice. The pressure is assumed to be hydrostatic and is denoted $p_{\text{stat}}$.

$$\int \frac{\partial (\rho_e \{u_e + u_b\})}{\partial t} \, d\mathcal{V} = \int_{t}^{u} \rho_e v (E - \frac{\partial \xi}{\partial t}) \, dX + \int_{s}^{r} \rho_e u_e \frac{\partial (h - \xi)}{\partial t} \, dX + \int_{r}^{u} \rho_e u_e^2 \, dZ - \int_{s}^{t} \rho_e u_e^2 \, dZ = \int_{t}^{u} (p_{\text{stat}} - \tau) \, dX - \int_{r}^{t} p_{\text{stat}} \, dZ + \int_{s}^{t} p_{\text{stat}} \, dZ \tag{4.2.5}$$

where $v$ is the horizontal velocity of the entrained fluid. For a point between the leading edge and the crest, this yields:

$$\int_{\xi-h}^{\xi} \frac{\partial (\rho_e \{u_e + u_b\})}{\partial t} \, dZ - \rho_e (E - \frac{\partial \xi}{\partial t}) v + \rho_e u_e \frac{\partial (h - \xi)}{\partial t} + \frac{\partial}{\partial X} \int_{\xi-h}^{\xi} \rho_e u_e^2 \, dZ =$$

$$= g \tan \theta \int_{\xi-h}^{\xi} \rho_e \, dZ - \tau - g \frac{\partial}{\partial X} \int_{\xi-h}^{\xi} \rho_e (k - \xi + h) \, dK \tag{4.2.6}$$

Again, special care must be taken at the leading edge. Equation (4.2.5) has to be
modified by taking $u_e = dd/dt$ there, so that we finally obtain

$$
-\rho_e E^* u + \int_{\xi^*-h^*}^{\xi^*} \rho_e \left( \frac{dd^2}{dt^2} - u_e^2 \right) + \tau^* - \delta \int dZ \int_{\xi^*-h^*}^{\xi^*} \rho_e (\xi - \xi^* + h^*) \, d\xi = 0 \quad (4.2.7)
$$

In order to handle equations (4.2.2), (4.2.4), (4.2.6) and (4.2.7) we need to model the velocity and concentration profiles. As a first approximation, we will assume that $\rho_e$ is a constant and that $u_e$ is only a function of $X$ and $t$. To allow for arbitrary, but similar, profiles, we would need to introduce profile constants (as did Tulin, 1973 and Longuet-Higgins & Turner, 1974). This is not expected, however, to modify the qualitative results obtained here. We therefore finally get for a point between the leading edge and the crest two equations involving $u_e (X, t)$ and $h (X, t)$:

$$
E = \frac{\partial h}{\partial t} + \frac{\partial (hu_e)}{\partial X} \quad (4.2.8)
$$

$$
\frac{h}{u_e} \frac{\partial (u_e + u_b)}{\partial t} + u_e \frac{\partial (h - \xi)}{\partial t} - (E - \frac{\partial \xi}{\partial t})u + \frac{\partial (hu_e^2)}{\partial X} =
$$

$$
= gh \tan \theta - \frac{x}{\rho_e} - gh \frac{\partial h}{\partial X} \quad (4.2.9)
$$

At the leading edge, we have
\[ E^* = h^* \left( \frac{dd}{dt} - u_e \right) \]  \hspace{1cm} (4.2.10)

\[ -E^* \nu + h^* \left( \frac{dd}{dt} - u_e^2 \right) = \frac{\tau^*}{\rho_e} + g \frac{h^*}{2} \]  \hspace{1cm} (4.2.11)

In the steady case, these equations are similar to those derived by Longuet-Higgins & Turner (1974). It should also be noticed that the hydrostatic solution is included in these equations and is obtained by taking the right hand sides of equations (4.2.9) and (4.2.11) equal to zero - see equation (3.2.16).

In order to solve equations (4.2.9) to (4.2.11), we first need some model for the entrainment and the shear stress distribution so that we can solve for the velocity in the breaker and its height. Since the hydrostatic solution derived in chapter 3 appears here as an equilibrium solution, we first look at the solutions in the vicinity of the hydrostatic solution. This should give useful informations concerning the stability of these solutions and the time scale of the evolution of the breaker.

4.3/ AN ANALYTIC MODEL FOR SMALL OSCILLATIONS

4.3.1/ LINEARIZED EQUATIONS FOR THE GRAVITY CURRENT

In this section, we shall derive linearized equations for the evolution of the breaker near equilibrium. A wave equation is obtained and solved in terms of Bessel functions. The effect of entrainment of mass and momentum is studied. The importance of the boundary condition at the leading edge and of the coupling with the evolution of
the underlying wave is shown.

As in the steady case, we will assume that the shear stress $\tau$ is concentrated near the leading edge of the breaker, so that $\tau = 0$, $\tau^* > 0$. In the steady case, the entrainment is equal to zero when $a-b$ is taken as the dividing streamline. We will therefore assume

$$E = \Sigma u_e, \Sigma \text{ being a constant} \quad (4.3.1)$$

Experimental data support this relation and lead to a value of $\Sigma$ (which in fact decreases with the Richardson number) of order 0.1 (see Longuet-Higgins & Turner, 1974\(^1\)).

At equilibrium, the breaker is flat-topped and therefore (see equation (4.2.9))

$$h (X,t) = \xi (X,t) \quad (4.3.2)$$

We will assume in this section that the breaker remains almost flat-topped (even when the underlying wave evolves). We, therefore, call $\phi = h - \xi$ and linearize the equations in the vicinity of the hydrostatic solution ($h=\xi$, $u_e=0$, $\partial \xi/\partial t=0$, $du_e/\partial t=0$). This yields:

\(^1\) Longuet-Higgins & Turner wrote $E = \Sigma (u_e - v)$. However, the term $\Sigma v$, which is non-zero when $u_e$ is equal to zero, is included here in the shear stress $\tau$. 

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\[
\frac{\partial \theta}{\partial t} = - \frac{\partial (\xi u_e)}{\partial X} + \Sigma u_e - \frac{\partial \xi}{\partial t} \tag{4.3.3}
\]

\[
g \xi \frac{\partial \theta}{\partial X} = - \frac{\partial (\xi u_e)}{\partial t} + \Sigma u_e \nu - \xi \frac{\partial \nu}{\partial t} + \xi \frac{\partial u_b}{\partial t} \tag{4.3.4}
\]

In order to model the entrainment of momentum in a simple way, we will assume that the velocity of the entrained fluid varies linearly with \(\xi\) between the leading edge of the breaker and the crest of the wave. We, therefore, write

\[
- \frac{\nu}{\xi} = \frac{q_a}{\xi_a} \equiv \frac{2 \omega_1}{\Sigma} \tag{4.3.5}
\]

where \(\omega_1\) is homogeneous to a frequency. This modeling of the distribution of the entrainment of momentum is rather crude, but we will check that its effect on the frequency of oscillation is small. Using equation (4.3.5), we finally get by eliminating \(u_e\) from (4.3.3) and (4.3.4):

\[
\frac{\partial^2 \theta}{\partial t^2} - g \frac{\partial}{\partial X} \left( \xi \frac{\partial \theta}{\partial X} \right) + 2 \omega_1 \frac{\partial \theta}{\partial t} + \Sigma g \frac{\partial \theta}{\partial X} =
\]

\[
= (\tan \theta - \Sigma) \frac{\partial u_b}{\partial t} - \frac{2 \omega_1}{\Sigma} \frac{\partial}{\partial X} \left( \xi \frac{\partial \xi}{\partial t} \right) - \frac{\partial^2 \xi}{\partial t^2} \tag{4.3.6}
\]
Clearly, this equation is hyperbolic. It is a linear wave equation for the height of the breaker. Free modes, solution of the corresponding homogeneous equations, are studied in appendix 5. It is shown that there exists two families of separable solutions parametrized by the frequency \( \omega_0 \) (plus an exponentially decaying mode which will not be discussed here).

The first family is given when the solution is not overdamped \( (\omega_1 < \omega_0) \) by \( (\alpha, \beta \text{ and } \psi \text{ are arbitrary constants}) \)

\[
\vartheta (X,t) = \cos (\Omega t + \psi) \exp (-\omega_1 t) \frac{\nu}{2} \left\{ \alpha J_\nu \left( \sqrt{\kappa X} \right) + \beta Y_\nu \left( \sqrt{\kappa X} \right) \right\}
\]

with \( \Omega = \sqrt{\frac{\omega_0^2 - \omega_1^2}{\tan \theta}} \), \( \nu = \frac{\Sigma}{\tan \theta} \), \( \kappa = \frac{4 \omega_0^2}{g \tan \theta} \) \hspace{1cm} (4.3.7)

These modes have Bessel-like shape functions in space and are oscillating and decaying in time. They will be referred to later as oscillatory modes and correspond to stable evolutions of the breaker. Note that entrainment of momentum (modeled by the parameter \( \omega_1 \)) has for effect to damp the oscillations of the breaker, but, as long as \( \omega_1 \) is small compared to \( \omega_0 \), has almost no influence on the frequency of oscillation.

The second family is given by \( (a, b, \alpha \text{ and } \beta \text{ are arbitrary constants}) \)

\[
\vartheta (X,t) = \{ a \exp((\Omega - \omega_1)t) + b \exp(-(\Omega + \omega_1)t) \} .
\]
\[ X^2 \{ \alpha I_\nu (\sqrt{\kappa X}) + \beta K_\nu (\sqrt{\kappa X}) \}, \quad \text{with} \quad \Omega = \sqrt{\omega_0^2 + \omega_1^2} \quad (4.3.8) \]

These modes also have Bessel-like shape functions in space, but their amplitude is exponentially increasing with time (if \( a \) is not equal to zero). They will be referred to later as exponential modes and correspond to unstable evolutions of the breaker. Note that for these mode, the effect of entrainment is very small as long as \( \omega_1 \) is smaller than \( \omega_0 \).

This study of the modes of the linearized wave equation describing the evolutions of the breaker in the vicinity of equilibrium shows that both oscillatory (stable) and exponential (unstable) modes can exist. In order to assess the stability of the breaker, it appears therefore crucial to determine the appropriate boundary conditions.

### 4.3.2 Boundary Conditions

At the crest, we will assume that \( h \) goes to zero and therefore we have \( \psi(0) = 0 \). This implies that the solution (4.3.7) (or (4.3.8)) has to be expressed in terms of \( I_\nu \) (or \( I_\nu \)) only.

Critical to the stability analysis and to the determination of the natural frequency is therefore the boundary condition at the leading edge of the breaker. There we have (equations (4.2.10) and (4.2.11)):
\[
\frac{dd}{dt} = \frac{E^*}{h^*} + u_e (d) \tag{4.3.9}
\]

\[
\tau^* = \rho_e \frac{gh^*^2}{2} - \rho_e E^* \left( \frac{E^*}{h^*} + 2u_e (d) - \nu (d) \right) \tag{4.3.10}
\]

Equation (4.3.9) states that if fluid is entrained at the leading edge of the breaker, the advance of the front will be greater than the local velocity at the front. Equation (4.3.10) states that the entrainment of momentum modifies the effective drag at the leading edge.

We now assume that

\[
E^* = \Sigma^* h^* u_e (d) \tag{4.3.11}
\]

\[
\tau^* = \rho h^* c_D^* q_0^2 (a) \tag{4.3.12}
\]

\[
\nu (d) = - \Lambda^* q_0 (a) \tag{4.3.13}
\]

Equation (4.3.11) states that the bulk entrainment at the leading edge is proportional to the velocity at the leading edge and to the height of the breaker there. Equation (4.3.12) states that the drag at the leading edge is proportional to the square of the velocity in the underlying flow and to the height of the breaker. Finally, equation (4.3.13) states that the entrained fluid has a velocity proportional to the velocity of the underlying flow just upstream of the leading edge of the breaker.
We define $\delta \equiv d - D$ ($D$ is the length of the breaker at equilibrium) and, linearizing the preceding equations in the vicinity of equilibrium, we obtain

$$\frac{d\delta}{dt} = u_e(D) \{1 + \Sigma^*\} \quad (4.3.14)$$

$$\dot{\theta}(d) = \frac{2\rho}{\rho_e} \frac{c_b^*}{g} q_0^2(a) + \frac{2 \Sigma^* \Lambda^*}{g} q_0(a) u_e(d) - \xi_a(d) \quad (4.3.15)$$

$\xi_a$ is the elevation of the leading edge of the breaker relative to the crest or, using a fixed system of reference,

$$\xi_a = \zeta_b - \zeta_a \quad (4.3.16)$$

where $\zeta_b$ is the elevation of the crest and $\zeta_a$ is the elevation of the leading edge of the breaker.

$q_0(a)$ is the velocity in the potential flow just upstream of $(a)$, and we will assume it is given by Bernoulli’s equation (i.e. that the incident wave is not perturbed upstream of $(a)$ and/or evolve in a quasi-steady way).

$$q_0^2(a) = c^2 - 2g \zeta_a \quad (4.3.17)$$
The Linearized Head Condition without Concentrated Entrainment of Momentum

In the absence of concentrated entrainment of momentum \((\Lambda^* = 0)\), the boundary condition at the leading edge take a very simple expression which allows an analytical resolution of the problem. We first write equation (4.3.15) as

\[
\vartheta (d) = \frac{2 \rho}{\rho_e} \frac{c_D^*}{g} c^2 + \left(1 - \frac{4 \rho}{\rho_e} c_D^* \right) \zeta_a - \zeta_b
\]  

(4.3.18)

Let us now assume that the underlying wave responds instantaneously to the evolution of the breaker (an hypothesis which will be discussed later) and that the breaker can be parametrized by its length \(d\). With these assumptions, \(\zeta_a\) and \(\zeta_b\) are only functions of \(d\), the instantaneous length of the breaker. Equation (4.3.18) can then be written (we linearize around the equilibrium configuration)

\[
\frac{\partial}{\partial t} \vartheta (D) = \left\{ \left(1 - \frac{4 \rho}{\rho_e} c_D^* \right) \frac{d\zeta_a}{dd} - \frac{d\zeta_b}{dd} \right\} \frac{dd}{dt}
\]  

(4.3.19)

We define

\[
\Gamma \equiv \frac{d}{dd} \left\{ \left(1 - \frac{4 \rho}{\rho_e} c_D^* \right) \zeta_a - \zeta_b \right\} = \frac{d}{dd} \left\{ h^* \left( \zeta_b - \zeta_a \right) \right\}
\]  

(4.3.20)
and, using equation (4.3.14), we finally write the boundary condition at the leading edge as

$$\forall \, t, \quad \frac{\partial \phi}{\partial t}(D, t) = (1+\Sigma^*) \Gamma \, u_e(D, t) \quad (4.3.21)$$

This equation appears to be crucial for the understanding of the behavior of the solution. It allows to study analytically the stability of the breaker and its natural frequencies of oscillations. It appears (see appendix 5) that

- if $\Gamma$ is lower than 0, exponential eigen modes exist and the breaker is unstable;
- if $\Gamma$ is greater than 0, only oscillatory eigen modes exist and the breaker is stable.

Note that concentrated entrainment of mass at the leading edge of the breaker has no influence on the stability. It has only for effect to increase the magnitude of $\Gamma$.

It is possible to give a very simple interpretation of this result. From equation (4.3.20), it appears that $\Gamma$ is positive if the top of the breaker at its leading edge is above the crest when the breaker length increases. Similarly, $\Gamma$ is negative if the top of the breaker at its leading edge falls below the crest when the breaker length increases, see figure 4.3.
\[ \Gamma \text{ can be estimated by assuming that the underlying wave evolves in a quasi-steady way. This hypothesis is well justified if the time scale of the evolution of the breaker is greater than } T_D = d/c_g, \text{ where } d \text{ is the length of the breaker and } c_g \text{ the group velocity of the wave. } T_D \text{ corresponds to the time for a disturbance to travel a distance equal to the length of the breaker. Numerical computations of the evolution of the free surface elevation of an initially undisturbed wave on which the pressure distribution corresponding to the fully formed breaker was applied impulsively have been performed at IFP (Nays, 1987). They confirm that a steady state is reached below the breaker on a time scale } T_D. \text{ However, small oscillations around the steady state are expected at a period } T_{cr} = 8\pi c/g \text{ (Maruo, 1957; Duncan, 1981).}

We only consider the deep water case. At each instant, we will assume that}
equation (3.3.12) holds (the effect of the boundary layer is neglected and linear wave theory is used), so that

$$\frac{a_b^2}{16} - \frac{\zeta_b^2}{4} - \frac{\rho_e}{\rho} \frac{h^2}{2} = 0 \quad (4.3.22)$$

From this equation and equation (4.3.10), we obtain (all quantities are evaluated at equilibrium, i.e. for $d = D$, and we take $c_D^* = \rho_e/(8\rho)$)

$$\frac{d\zeta_b}{dd} = -2 \frac{\rho_e}{\rho} \frac{h^*}{\zeta_b} \frac{dh^*}{dd} = \frac{\rho_e}{\rho} \frac{h^*}{\zeta_b} \frac{d\zeta_a}{dd} \quad (4.3.23)$$

We, therefore, obtain for $\Gamma$

$$\Gamma = \left(1 - \frac{\rho_e}{\rho} \frac{h^*}{\zeta_b}\right) \frac{d\zeta_a}{dd} \quad (4.3.24)$$

Using the relation between $h^*$ and $\zeta_b$ at equilibrium and assuming that $d\zeta_a/dd$ is negative (note, however, that this is the only hypothesis we have to make on the variation of the elevation of the leading edge of the breaker with its length), we finally obtain the stability criterion:
・ the breaker is stable if
\[ h^* > \frac{1}{2 \left( 1 + \frac{\rho_e}{\rho} \right)} \] (4.3.25)

・ the breaker is unstable if
\[ h^* < \frac{1}{2 \left( 1 + \frac{\rho_e}{\rho} \right)} \] (4.3.26)

Figure 4.4. - Stability Domain for the Breaker
The stable and unstable domains are shown on figure 4.4. It can be verified (see appendix 4) that equation (4.3.25) exactly corresponds to the strong breaking regime and equation (4.3.26) to the weak breaking regime. This analysis seems therefore to show that strong breakers are stable and weak breakers unstable, a result which is, as that was discussed in the foregoing, in agreement with Duncan's experimental observations (1981 & 1983). In particular, this result explains why a fully formed breaker appears at incipient breaking. It is rather remarkable that this stability analysis, which is very much independent from the computation of the equilibrium configuration of the underlying wave, exactly predicts that weak breakers are unstable and strong breakers stable.

This estimate of $\Gamma$ also allows to give a first quantitative estimate of the first natural period of oscillation of the breaker. We use the simplified model which is described in appendix 4 to compute the geometry of the breaker at equilibrium and therefore $\zeta_0$. Then the equation giving $\omega_0$ in terms of $\Gamma$ (see appendix 5) is solved numerically. If we assume that $\omega_1$ is "much" smaller than $\omega_0$, the frequency of oscillation $\Omega$ is almost equal to $\omega_0$. The first (greatest) natural period $T_{br} = 1/\omega_0$ predicted by this theory is plotted on figure 4.5 as a function of the height of the breaker (we took the values $\rho_c/\rho = 0.8$ and $\Sigma = 0.1$ and neglected concentrated entrainment of mass at the leading edge, $\Sigma^* = 0$). Duncan's experimental results concerning the period of oscillation of the breaker is also plotted on the same figure.
Figure 4.5 - Period of Oscillation vs. Height of the Breaker
From this figure, we observe that

- as incipient breaking is approached \((gh^*c^2 = 0.192\) according to the theory used), the computed period of oscillation goes to infinity;

- as the breaker height increases, the computed period decreases and approaches the value \(gT_{br}/c = 4.5\). The effect of a concentrated entrainment of momentum at the leading edge of the breaker would be to decrease this value;

- the results of the present model are in very good agreement with Duncan's measurements which yield \(gT_{br}/c = 4.4\);

- the hypothesis made that the wave responds instantaneously to the breaker deformation is rather well justified since \(T_D/T_{br} = 0.2\). Yet, \(T_{cr}\) and \(T_{br}\) are almost equal and the possibility of a resonance cannot be excluded. Note in particular that Duncan (1981) explained the oscillation of the breaker by a forcing phenomenon due to oscillations of the underlying wave at a period \(T_{cr}\).

However, the present theory is limited to the vicinity of the equilibrium configuration and needs further justification and/or extensions. In particular:

- we had to assume that \(\tan \theta\) was not a function of time in order to obtain an analytic (separable) solution. However, this hypothesis inconsistent with the estimate of \(\Gamma\) which has been made and which implies that \(d, \zeta_a\) and \(\zeta_b\) are varying with time;

- we have assumed a quasi-steady evolution for the underlying wave. In particular, the hypothesis that \(\zeta_a\) and \(\zeta_b\) are functions of \(d\) only is crucial for the preceding analysis;

- we have used a linear wave theory to compute the quasi-steady underlying wave;
we have neglected any possible concentrated entrainment of momentum at the leading edge. This hypothesis is also crucial in order to obtain an analytical (separable) solution;

we, of course, have neglected any non-linearity.

In order to assess the validity of these assumptions and/or to be able to study more general situations, it seems therefore necessary to build a more general theory. Due to the non-linearities which are involved in both the large scale evolutions of the breaker and the coupling with the underlying wave, this cannot be done analytically and we will therefore introduce a numerical scheme.

4.4/ A NUMERICAL MODEL FOR THE BREAKER DYNAMICS

4.4.1/ GENERAL

In this section, we develop a numerical model to predict the dynamic evolution of the breaker. Difference equations to predict the evolution of the height of the breaker and the velocity within it will be derived assuming known the evolution of the underlying flow, i.e. the slope of the line a-b, the length of the breaker, d, and the position of the crest. These equations are quite general and could be used with any model for the evolution of the underlying wave. Here, we will limit ourselves to simple coupling equations using a quasi-steady model for the underlying wave and a linear wave theory. The resulting model will be used to study the stability of the hydrostatic solutions and to describe large motions of the breaker under conditions when the underlying wave is non-steady. Possible extensions of the model to include a full transient modeling of the underlying wave are indicated in chapter 5.
4.4.2/ DISCRETIZED EQUATIONS FOR THE GRAVITY CURRENT

We shall now attempt to solve numerically the system (4.2.8) - (4.2.9) with the boundary conditions (4.2.10) - (4.2.11). In order to solve these equations over a fixed domain, we first define a new variable

\[ x = \frac{X}{d} \]  

(4.4.1)

where \( d \) is the length of the breaker (which varies with time). Guided by the analytical solution obtained above, we assume that the system is mainly hyperbolic and we write equations (4.2.8) and (4.2.9) as:

\[ \frac{\partial h}{\partial t} + \frac{1}{d} \frac{\partial h u_e}{\partial x} = \alpha \]  

(4.4.2)

\[ \frac{\partial u_e}{\partial t} + \frac{g}{d} \frac{\partial h}{\partial x} + \frac{u_e}{d} \frac{\partial u_e}{\partial x} = \beta \]  

(4.4.3)

where \( \alpha \) and \( \beta \) are given by:

\[ \alpha = E + \frac{x}{d} \frac{\partial h}{\partial x} \frac{dd}{dt} \]  

(4.4.4)

\[ \beta = g \tan \theta - \frac{x}{\rho_e h} - \frac{1}{h} (E - \frac{d \xi}{dt}) (u_e - u) - \frac{du_e}{dt} \]
\[ -\frac{1}{h} \frac{x}{d} \frac{dd}{dt} \tan \theta (u_e - v) + \frac{x}{d} \frac{dd}{dt} \frac{\partial u_e}{\partial x} \] (4.4.5)

A finite differences scheme can be obtained to approximate equations (4.4.2) and (4.4.3) by use of the integral method, as this is described in appendix 6. The hyperbolic character of the system is used explicitly in deriving these equations. As boundary conditions, the height of the breaker is prescribed at both ends \(x=0\) and \(x=1\). The scheme obtained is explicit and second order accurate as long as \(d\) is assumed to be constant. Comparison of the numerical results and of known analytical solutions (for appropriate choices of \(\alpha\) and \(\beta\)) are shown in appendix 6.

In order to solve these equations in general, we need to prescribe the value of \(d\), to define the left hand sides \(\alpha\) and \(\beta\), and to specify more precisely the boundary conditions. Clearly this is not a trivial task, since it demands that entrainment be properly modeled and that the evolution of the underlying wave be characterized in terms of the evolution of the breaker. Extending the developments made in the foregoing and assuming that the underlying wave is linear and evolves in a quasi-steady way (i.e. is given at each instant by the simplified model derived in appendix 4), the following simple closure can be proposed:

* length of the breaker

\[ \frac{dd}{dt} = u_e (d) (1+\Sigma^*) \] (4.4.6)
* **modeling of the entrainment**

- mass entrainment: left hand side $\alpha$

\[
\alpha = \Sigma u_e + \frac{x}{d} \frac{\partial h}{\partial x} \frac{dd}{dt}
\]  

(4.4.7)

- momentum entrainment: left hand side $\beta$

\[
\beta = g \tan \theta - 2\omega_1 u_e - \frac{x}{d} \frac{dd}{dt} \frac{\partial u_e}{\partial x}
\]  

(4.4.8)

* **boundary conditions**

- at the crest of the breaking wave

\[
h(0) = 0
\]  

(4.4.9)

- at the leading edge of the breaker

\[
h(d) = \frac{1}{4} \left( \frac{c^2}{g} - 2\zeta_a \right) + 2 \frac{\Sigma \Lambda^*}{g} \sqrt{\frac{c^2}{g} - 2g\zeta_a} u_e(d)
\]  

(4.4.10)

* **parametrization of the underlying wave**

- elevation of the leading edge

\[
\zeta_a = \frac{a_0}{2} \cos \left( \frac{gd}{c^2} \right)
\]  

(4.4.11)

- elevation of the crest

\[
\frac{a_0^2}{16} = \frac{\zeta_b^2}{4} - \frac{\rho_e h^2}{\rho} - \frac{\Sigma \Lambda^*}{g} \sqrt{\frac{c^2}{g} - 2g\zeta_a} u_e(d)
\]  

(4.4.12)

- slope of the bottom of the breaker

\[
\tan \theta = \frac{\zeta_b - \zeta_a}{d}
\]  

(4.4.13)
These coupling equations have been implemented in the numerical model describing the dynamics of the breaker, providing a first, but hopefully meaningful, model for the evolution of the system breaker-breaking wave. Sample applications of this model are presented in the next sections.

### 4.4.3 SMALL OSCILLATIONS AROUND EQUILIBRIUM

A comparison between the numerical solution of the nonlinear equations and the simplified analytic solution derived in the foregoing has first been sought. In order to perform this comparison, we computed the evolution of the breaker starting from an equilibrium configuration perturbed by the first oscillating mode of the simplified solution. Therefore, using equations (4.3.14) and (A5.15), we defined the initial conditions as

\[
h_{0}(\chi) = h_{eq}(\chi) + \varepsilon \frac{c^2}{g} (\kappa \chi) \frac{\nu}{v} I_v(\sqrt{\kappa \chi}) \tag{4.4.14}
\]

\[
u_{c0}(\chi) = 0 \tag{4.4.15}
\]

\[
\phi_{0} = \phi_{eq} + \varepsilon \frac{c^2}{\omega_0^2} \{1+\Sigma^*\} \frac{d}{d\chi} \{ (\kappa \chi)^{\frac{\nu}{v}} I_v(\sqrt{\kappa \chi}) \} \tag{4.4.16}
\]

where \(\kappa\) and \(\omega_0\) correspond to the first oscillating mode of the breaker and \(\varepsilon\) is a small
perturbation parameter.

According to the simplified analytic theory, the breaker should therefore oscillate at the first natural frequency. Note however that, for weak breakers, instabilities are expected to develop. In order to illustrate the results, we have taken $a_0 = 0.8 \, c^2/g$ and $\rho_e/\rho = 0.8$ and plotted the time evolution of the length of the breaker for both strong and weak breakers. In order to perform the numerical computation, the breaker is divided into 6 segments and the time step is equal to $c/6g$.

Figure 4.6 corresponds to the strong breaking regime. We took $\Sigma = 0.1$, $\Lambda^* = 0$ and $\omega_1 = 0.03 \, g/c$. The perturbation parameter, $\varepsilon$, is equal to 0.0001. The numerical computation confirms that the breaker is stable. The computed period of oscillation agrees rather well with the one predicted by the analytic model. However, there is a small phase shift and the period appears to be slightly higher in the numerical computation. Yet, the differences for both the period of oscillation and the rate of damping are small.

Figure 4.7 corresponds to the weak breaking regime. We took $\Sigma = 0.1$, $\Lambda^* = 0$ and $\omega_1 = 0.1 \, g/c$. The perturbation parameter, $\varepsilon$, is equal to 0.0001 for the top curve and to -0.0001 for the bottom curve. If the initial motion of the breaker follows well the first oscillating mode, instabilities develop rapidly. Depending upon the sign of the perturbation, these instabilities seem to lead to an exponential growth or decay. Presumably, the weak breaker will either disappear (and the wave unbreak) or grow and reach the strong breaking equilibrium.
These computations seem therefore to justify our analytic developments and the simplifying assumptions associated with them. On the other hand, they also seem to validate our numerical scheme. Of course, the numerical scheme is not limited to the cases where the analytic solution exists. It allows, for instance, to assess the effect of concentrated entrainment of momentum at the leading edge. Figure 4.8 corresponds to figure 4.6, but with $\omega_1 = 0$ and $\Lambda^* \Sigma^* = 0.1$. It appears that the effect of entrainment
of momentum is essentially the same at the leading edge and along the bottom of the breaker. It does not affect much the period of oscillation, but damps the oscillations. This justifies our crude distribution of momentum entrainment along the bottom of the breaker. Note, however, that the magnitude of the damping is difficult to assess analytically and that experiments would be very useful to determine the corresponding time scale.

Figure 4.7 - Small Oscillations - Weak (Unstable) Breaking
4.4.4/ LARGE EVOLUTIONS OF THE BREAKER

In this section, we use the numerical scheme to compute large evolutions of the breaker in situations when the underlying wave is not steady, but evolving. Keeping in mind the analogy with Duncan's experiments, this would correspond to situations when the motion of the hydrofoil is not steady. Because of the lack of experimental
data corresponding to this situation, we will give here a rather qualitative description of the phenomenon. Note however that the present model seems to have the capabilities to produce quantitative results.

Because of our coupling equations, this model is limited to cases where the underlying wave evolves in a quasi-steady way, that is when the time scale of the evolution of the foil (or, more generally, of the external forcing), $T_F$, is large compared to $\lambda/c_g$, where $\lambda$ is the wave length (which scales the distance between the foil or the disturbance and the breaking wave) and $c_g$ is the group velocity. We will therefore assume that $T_F$ is greater than $4\pi c/g$. As that as been discussed before, the time scale associated with breaking, $T_{br}$, should also be greater than $T_D = d/c_g$, where $d$ is the length of the breaker.

In order to perform the simulation, $a_0$ is no more a constant but a slowly varying function of time. $a_0$ parametrizes the momentum flux given to the wave and corresponds to the total height that would be reached by the wave if it were not breaking. Since the present model is limited to cases where the underlying wave evolves in a quasi-steady way, the threshold steepness for breaking is expected to be the same as in the steady case. The importance of the unsteadiness of the underlying wave on the inception of breaking cannot therefore be evaluated. Because we are using a linear wave theory to compute the shape of the wave, we expect breaking to occur for $ga_0/c^2 = 1$ in the absence of disturbance and for $ga_0/c^2$ as low as
\[ a_{0\text{ min}} = \sqrt{\frac{1 - \frac{\rho}{\rho + 2\rho_e}}{\rho}} \] (4.4.17)

if the wave is disturbed.

Figure 4.9 - Growth and Decay of a Breaker on an Unsteady Wave
An example is shown on figure 4.9 for which, again, \( \rho_e/\rho = 0.8 \). The elevation of the crest of the wave (breaking or non-breaking), \( \zeta_b \), is plotted as a function of time for a given "history" of \( a_0 \). \( ga_0/c^2 \) grows from 0.6 to 0.9 and then decays to reach 0.6 again, a period of the cycle being equal to 30 wave periods. When the wave height reaches a value of \( ga_0/c^2 = 0.8 \), the wave is disturbed. This is done numerically by putting a small breaker corresponding to the weak solution on the wave. Then the system breaker-wave evolves without any external forcing. The breaker is divided into 6 segments and the time step is equal to \( c/6g \). \( \Sigma \) is equal to 0.1, \( \Lambda^* \Sigma^* \) is equal to 0.1 and \( \omega_1 \) is given at each instant by equation (4.3.5).

When the disturbance is applied, the instability takes some time to develop. The wave breaks first gently and then collapses rapidly. After this breaking phase, a quasi-steady broken state is reached, corresponding to the strong breaking regime. Then, as the momentum flux given to the wave decreases and falls below threshold, the wave unbreaks: the wave height increases and the breaker finally disappears. The qualitative implications of the model are therefore as follows:

- as a wave beyond threshold is disturbed, the breaker will have a tendency to grow to reach the strong breaking regime;
- if the wave stays beyond threshold only a "small" time (compared to the time scale of the growth of the breaker), the wave will only break gently and a fully formed strong breaker will not have sufficient time to develop;
- if the wave stays beyond threshold a long time, a fully formed quasi-steady breaker will appear, corresponding to the strong breaking regime;
- similarly, as the momentum flux given to the wave decreases and falls below
threshold during a sufficiently large time, the breaker will disappear.

These predictions are characteristic of a nonlinear system for which hysteresis-related phenomena can occur. They stem from the fact that, in a certain domain, two quasi-steady stable solutions exist (the unbroken wave and the broken wave in the strong breaking regime) while in other domains only one stable quasi-steady solution exists. Therefore, a real solution has to "jump" from one of the solutions to the other and the time history of the evolution is essential. This jump is essentially unsteady, and is related to the growth or decay of the breaker.

This modeling of the system breaker-breaking wave seems to be in good qualitative agreement with observations of wave breaking. However, it is clear that in order to estimate properly the different time scales involved more experiments would have be done, for instance by giving an unsteady motion to the hydrofoil in Duncan-like experiments. It is hoped that such experiments might be guided by the present work.

4.5/ CONCLUSIONS

In this chapter, the physical model for steady spilling breaker has been extended to account for unsteady effects. The breaker has been modeled as an unsteady gravity current riding the forward slope of the wave and general equations describing its dynamics derived. The crucial importance of the coupling between the evolutions of the breaker and the waves has been shown, but the present study has been limited to quasi-steady underlying waves.
Small amplitude motions of the breaker near equilibrium have been studied both analytically and numerically, with a good agreement. A simple modeling of the entrainment of mass and momentum have been proposed and the analytic solution shows the effect of these phenomena. The present model of the breaker tends to account for the breaker stability and experimental observations of oscillations in the length of the breaker. Good quantitative agreement is found with Duncan's experiments (1981).

The numerical model developed seems to be able to describe large evolutions of the breaker. Results of unsteady computations confirm the existence of two stable quasi-steady states: a non-breaking state, which exists only below a critical incoming momentum flux, $R_{\text{NL}*}$; and a broken state, which exists only above a critical incoming momentum flux, $R_{\text{min}}$. A real solution has to jump from one solution to another, explaining the importance of the recent time history of the phenomenon. Unfortunately, the lack of experiments concerning unsteady breakers does not allow a detailed quantitative comparison. More experiments would be extremely useful in order to evaluate the different time scales involved in the phenomenon. It is hoped that they might be guided by this study.
5/ CONCLUSION

In this thesis, a physical and mathematical model for steady and unsteady spilling breakers has been proposed.

Our theory of steady spilling breakers is based on the measurements of Duncan (1981 & 1983) which have been most revealing. Most of the questions rose by these experiments, and stated in introduction, have been answered:

- the breaker is modeled as a mass of aerated water which rides the forward slope of the wave. Its shape is determined from conservation principles;
- the breaker imparts both pressures and shear stresses along the boundary with underlying wave. The breaking resistance is directly related to the size of the breaker;
- breaking first appears when the total resistance resulting in the entire wave system can be balanced by the following wave resistance plus the breaking resistance. This balance is not a direct consequence of Stokes' limiting wave;
  - as the total resistance increases, the breaking resistance becomes predominant and the amplitude of the following waves decreases;
  - near incipient breaking, two breaking solutions are, in general, possible: a weak solution with a smaller breaker and a strong solution with a larger breaker.

This theory is found to be in good agreement with Duncan's observation and seems to provide an appropriate basis for the understanding of spilling breakers. It has therefore been broadened to account for unsteady effects. The unsteady theory allows to assess the stability of the breaker and, since only strong breakers are found to be
stable, explains why a fully formed breaker appears at incipient breaking. It also accounts for the oscillations in the length of the breaker observed by Duncan.

Unfortunately, very few other data exist for unsteady breakers and the theory can only be validated qualitatively. Computations showing the growth and decay of a breaker on an unsteady underlying wave have been performed. They show the importance of the time history of the phenomenon - a result characteristic of a nonlinear system admitting several steady state solutions. More experiments would be needed to determine the different time scales involved, but these computations lead to a first speculative description of inception of breaking and of the breaking cycle. As the wave energy grows, it reaches a critical value above which breaking can occur. Some local disturbance (such as the creation of a jet at the tip) can then lead to the creation of a weak breaker. This breaker is unstable and grows further to form a strong breaker, if it has sufficient time to do so. The strong breaker is stable, except that the underlying wave is losing energy to dissipation in the breaker. As the wave collapses, the breaker finally disappears.

This model for breakers and breaking waves has already several important implications for engineering applications. These applications suggest some directions for further work:

- the model allows an estimate of the breaking resistance in terms of the total incoming resistance. Coupling of this model with a steady potential flow computation (an extension of the computation described in chapter 4) would allow to compute directly the breaking resistance for any submerged resistance source. Extension of the
model to the case of a breaking ship bow wave would also allow to estimate the breaking resistance for ships;

- the fact that the breaking wave carries along a mass of water at its phase velocity explains why impact forces are much greater for breaking waves than non-breaking waves, even for a spilling breaker. Preliminary estimates of the impact forces due to a spilling breaker have been made using the present theory and a slamming-type analysis (Cointe, 1987a & b). They seem to justify the use of a "curling factor" but would need a better estimate of the density ratio $\rho_c/\rho$ and/or of the shape of the breaker to lead to theoretical results which could be used by marine engineers. Experiments relating the geometry of the breaker and the loads exerted would be especially useful;

- the model accounts for the creation of a current in the direction of wave propagation and relates it to the size of the breaker. Applications of the model to the description of breaking-induced currents and sediment transport suggest themselves;

- preliminary computations of unsteady breakers are encouraging, and extensions of the model to describe more precisely the underlying wave would be needed for a better understanding of ocean breakers and their probability of occurrence - which itself would be useful for the applications quoted above. This might include the modeling of the energy input (for instance through wave-wave or wave-current interaction) and/or a full transient modeling of the wave. This would also demand further experimental work. Eventually, the importance of the unsteadiness of the underlying wave might be assessed.

Some of these extensions are ongoing. Experiments concerning steady and unsteady breakers are being conducted at the University of California, Santa Barbara
(UCSB). Work is being done at the French Petroleum Institute (IFP) to couple the model for the breaker and a potential flow numerical model for the underlying wave using the mixed Eulerian-Lagrangian approach introduced by Longuet-Higgins & Cokelet (1976). The final (but ambitious) objective is to include breaking effects in the nonlinear simulation of two-dimensional unsteady flows.
A1/ CONSERVATION LAWS IN ORTHOGONAL CURVILINEAR COORDINATES

A1.1/ INTRODUCTION

In this appendix, we will derive the equations expressing the conservation laws in two-dimensional orthogonal curvilinear coordinates. These derivations are straightforward but rather involved. We will therefore limit ourselves here to the results used in this thesis, especially in appendix 2. More general results concerning general three-dimensional curvilinear coordinates can be found, for instance, in Richmond et al. (1986).

General definitions will first be given, and the notion of scale factors will be introduced. In particular, the curvilinear abscissa along a given coordinate line will be defined. Conservation laws will then be written in orthogonal curvilinear coordinates, the results being derived from the vectorial (intrinsic) form of these laws. Finally, the special case of streamline coordinates will be studied and the steady Navier-Stokes equations will be written in the form used in appendix 2.

A1.2/ GENERAL EQUATIONS

A1.2.1/ DEFINITIONS

We consider a two-dimensional Euclidean space with Cartesian coordinates \((x_1, x_2)\). Curvilinear coordinates are defined by specifying two one-to-one functions of \((x_1, x_2)\), \(s = s(x_1, x_2)\) and \(n = n(x_1, x_2)\). We will only consider orthogonal curvilinear coordinates, i.e. curvilinear coordinates such that, at any point \(P\), the curves \(s = \ldots\)
constant and \( n = \text{constant} \) are orthogonal.

It is useful to introduce the scale factors which are non-negative functions defined by (sum on \( i \)):

\[
h_s^2 = \frac{\partial x_i}{\partial s} \frac{\partial x_i}{\partial s}, \quad h_n^2 = \frac{\partial x_i}{\partial n} \frac{\partial x_i}{\partial n}
\]  \hspace{1cm} (A1.1)

If we define \( s' = s'(s) \) and \( n' = n'(n) \), where \( s' \) and \( n' \) are two one-to-one functions, we see that \( h_{s'} = |ds/ds'| h_s \) and \( h_{n'} = |dn/dn'| h_n \). In other words, it is possible to multiply \( h_s \) and \( h_n \) by a function of \( s \) and \( n \), respectively, by an appropriate scaling of the coordinates without modifying the pattern of curvilinear coordinates.

Along the line \( n = n_0 \), if there exists \( \sigma \) such that \( d\sigma/ds = h_\sigma(s, n_0) \), we see that \( h_\sigma = 1 \). Similarly, along the line \( s = s_0 \), \( h_\eta = 1 \) for \( \eta \) such that \( d\eta/dn = h_\eta(s_0, n) \). \( \sigma \) and \( \eta \) are the *curvilinear abscissas* along the lines \( n = n_0 \) and \( s = s_0 \). We will assume that they can be defined, at least locally, along any coordinate line. This means that \( h_s \) and \( h_n \) can be taken equal to 1 at a given point. We will denote everywhere in this thesis \((s, n)\) arbitrary curvilinear coordinates and \((\sigma, \eta)\) curvilinear abscissas. Note, however, that the derivatives of the scale factors \( \partial h_\sigma/\partial n \) and \( \partial h_\eta/\partial s \) are intrinsic properties of the pattern of curvilinear coordinates and cannot be chosen arbitrarily.

As a matter of fact, the curvatures of the curvilinear coordinates \( n = \text{constant} \) and \( s = \text{constant} \) are given respectively by
\[ K_s = \frac{1}{h_s h_n} \frac{\partial h_s}{\partial n}, \quad K_n \equiv \frac{1}{h_s h_n} \frac{\partial h_n}{\partial s} \]  \hspace{1cm} (A1.2)

The equations for the conservation laws in the coordinates (s,n) will be derived from their vectorial (intrinsic) form by use of the expressions of the divergence, gradient and curl in orthogonal curvilinear coordinates (e.g. Malvern, 1969).

A1.2.2/ CONSERVATION OF MASS

From the vectorial form

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho U) = 0 \]  \hspace{1cm} (A1.3)

we get

\[ \frac{\partial \rho}{\partial t} + \rho K_n u_s + \rho K_s u_n + \frac{1}{h_s} \frac{\partial}{\partial s} (\rho u_s) + \frac{1}{h_n} \frac{\partial}{\partial n} (\rho u_n) = 0 \]  \hspace{1cm} (A1.4)

For an incompressible fluid, we have \( \text{div}(U) = 0 \) which can be written

\[ K_n u_s + K_s u_n + \frac{1}{h_s} \frac{\partial u_s}{\partial s} + \frac{1}{h_n} \frac{\partial u_n}{\partial n} = 0 \]  \hspace{1cm} (A1.5)
In this case, equation (A1.4) reduces to

\[
\frac{\partial \rho}{\partial t} + \frac{u_s}{h_s} \frac{\partial \rho}{\partial s} + \frac{u_n}{h_n} \frac{\partial \rho}{\partial n} = 0
\]  

(A1.6)

A1.2.3/ CONSERVATION OF MOMENTUM

The equations for the conservation of momentum are obtained from the vectorial form

\[
\frac{\partial}{\partial t} (\rho U) + \text{div} (\rho U \otimes U) + \text{grad} p - f = \text{div} \tau
\]  

(A1.7)

where \( f \) represents the volumic forces (such as gravity) and \( \tau \) the deviatoric stress tensor. The Navier-Stokes equations can be obtained with the appropriate closure, but (A1.7) is more general and also applies for the averaged quantities of a turbulent flow modeled with Reynolds' stresses.

This equation yields:

\[
\frac{\partial}{\partial t} (\rho u_s) + \frac{1}{h_s} \frac{\partial}{\partial s} (\rho u_s u_s) + \frac{1}{h_n} \frac{\partial}{\partial n} (\rho u_s u_n) + \rho K_n (u_s^2 - u_n^2) + 2 \rho K_s u_s u_n +
\]

\[
+ \frac{1}{h_s} \frac{\partial p}{\partial s} - f_s = \frac{1}{h_s} \frac{\partial}{\partial s} (\tau_{ss}) + \frac{1}{h_n} \frac{\partial}{\partial n} (\tau_{sn}) + K_n (\tau_{ss} - \tau_{nn}) + 2 K_s \tau_{ss}
\]  

(A1.8)
\[
\frac{\partial}{\partial t} (\rho u_n) + \frac{1}{h_s} \frac{\partial}{\partial s} (\rho u_s u_n) + \frac{1}{h_n} \frac{\partial}{\partial n} (\rho u_n u_n) + \rho K_s (u_n^2 - u_s^2) + 2pK_n u_s u_n + \\
+ \frac{1}{h_n} \frac{\partial p}{\partial n} - f_n = \frac{1}{h_s} \frac{\partial}{\partial s} (\tau_{sn}) + \frac{1}{h_n} \frac{\partial}{\partial n} (\tau_{nn}) + K_s (\tau_{nn} - \tau_{ss}) + 2K_n \tau_{sn} \quad (A.1.9)
\]

These equations can be simplified using the conservation of mass (equation (A1.4), yielding

\[
\frac{\partial u_s}{\partial t} + \frac{u_s}{h_s} \frac{\partial u_s}{\partial s} + \frac{u_n}{h_n} \frac{\partial u_s}{\partial n} - K_n u_n u_n + K_s u_s u_s + \frac{1}{\rho h_s} \frac{\partial p}{\partial s} - \frac{1}{\rho} f_s = \\
= \frac{1}{\rho} \left\{ \frac{1}{h_s} \frac{\partial}{\partial s} (\tau_{sn}) + \frac{1}{h_n} \frac{\partial}{\partial n} (\tau_{nn}) + K_n (\tau_{nn} - \tau_{ss}) + 2K_s \tau_{sn} \right\} \quad (A.1.10)
\]

\[
\frac{\partial u_n}{\partial t} + \frac{u_n}{h_s} \frac{\partial u_n}{\partial s} + \frac{u_n}{h_n} \frac{\partial u_n}{\partial n} - K_n u_n u_s + K_s u_s u_s + \frac{1}{\rho h_n} \frac{\partial p}{\partial n} - \frac{1}{\rho} f_n = \\
= \frac{1}{\rho} \left\{ \frac{1}{h_s} \frac{\partial}{\partial s} (\tau_{sn}) + \frac{1}{h_n} \frac{\partial}{\partial n} (\tau_{nn}) + K_s (\tau_{nn} - \tau_{ss}) + 2K_n \tau_{sn} \right\} \quad (A.1.11)
\]

A1.2.4/ NAVIER-STOKES EQUATIONS

The Navier-Stokes equations are readily obtained from their vectorial form:

\[
\frac{\partial}{\partial t} (\rho U) + \text{div}(\rho U \otimes U) + \text{grad} p - f = -\mu \text{curl} \omega \quad (A.1.12)
\]

where the rotation, \(\omega\), is a vector normal to the plane of the flow, of component
\[
\omega = \frac{1}{h_s} \frac{\partial u_n}{\partial s} - \frac{1}{h_n} \frac{\partial u_s}{\partial n} + K_n u_n - K_s u_s \tag{A1.13}
\]

so that we finally obtain

\[
\frac{\partial u_s}{\partial t} + \frac{u_s}{h_s} \frac{\partial u_s}{\partial s} + \frac{u_n}{h_n} \frac{\partial u_s}{\partial n} - K_n u_n u_n + K_s u_s u_s + \frac{1}{\rho h_s} \frac{\partial p}{\partial s} - \frac{1}{\rho} f_s = \]

\[
= -v \frac{1}{h_n} \frac{\partial \omega}{\partial n} \tag{A1.14}
\]

\[
\frac{\partial u_n}{\partial t} + \frac{u_s}{h_s} \frac{\partial u_n}{\partial s} + \frac{u_n}{h_n} \frac{\partial u_n}{\partial n} + K_n u_n u_s - K_s u_s u_s + \frac{1}{\rho h_n} \frac{\partial p}{\partial n} - \frac{1}{\rho} f_n = \]

\[
= v \frac{1}{h_s} \frac{\partial \omega}{\partial s} \tag{A1.15}
\]

A1.3/ EQUATIONS IN STREAMLINE COORDINATES

A1.3.1/ INTRODUCTION

In this section, we will derive expressions for the scale factors when the orthogonal curvilinear coordinates are chosen such that the lines \( n = \) constant are the streamlines of an incompressible irrotational flow at a given instant. Then we will write the steady Navier-Stokes equations in such coordinates. These results will be useful for the study of the action of a stress at the free surface (appendix 2) were the conservation laws will have to be written in such coordinates.
A1.3.2/ SCALE FACTORS FOR AN INCOMPRESSIBLE, IRROTATIONAL FLOW

We assume that at a given time \( t = t_0 \) it is possible to choose (at least locally) the curvilinear coordinates such that

\[
u_s = q, \quad u_n = 0
\]  \hspace{1cm} (A1.16)

Note that if the flow is not steady, (A1.16) will generally only be true at \( t = t_0 \). We will assume that the flow is incompressible (\( \text{div} \, (U) = 0 \)) and irrotational (\( \omega = 0 \)).

The continuity equation, equation (A1.5) yields

\[
K_n q + \frac{1}{h_s} \frac{\partial q}{\partial s} = \frac{1}{h_s h_n} \frac{\partial}{\partial s} (h_n q) = 0 \Rightarrow h_n = \frac{q_0(n)}{q}
\]  \hspace{1cm} (A1.17)

Similarly, the equation for the rotation (equation (A1.13)) yields

\[
\omega = - \frac{1}{h_n} \frac{\partial q}{\partial n} - q K_s = - \frac{1}{h_s h_n} \frac{\partial}{\partial n} (h_s q) = 0 \Rightarrow h_s = \frac{q_0(s)}{q}
\]  \hspace{1cm} (A1.18)

We will choose here \( s \) and \( n \) so that \( q_0(n) \) and \( q_0'(s) \) be constant and equal to \( q_0 \) (as that has been discussed above, such a choice is assumed to be possible). Then we see that the scale factors \( h_s \) and \( h_n \) are equal and inversely proportional to the magnitude of the velocity.
With this choice of the scale factors, it is easy to check that \( s \) and \( n \) are proportional to \( \phi \), the velocity potential, and \( \psi \), the stream function. The preceding results just express that the pair \((\phi, \psi)\) defines orthogonal curvilinear coordinates. This is, of course, a classical result. Note moreover that the curvature of the streamlines is proportional to the normal gradient of the velocity (see equations (A1.2) and (A1.18)).

**A1.3.3/ STEADY NAVIER-STOKES EQUATIONS (LAMINAR FLOW)**

We will now write the equations of motion for a *rotational* flow in the curvilinear coordinates defined above and corresponding to the streamlines of an incompressible and irrotational flow at a given instant \( t_0 \). We scale the velocities of the rotational flow by \( q \) and we define new variables by

\[
\begin{align*}
w_s & \equiv \frac{u_s}{q}, & w_n & \equiv \frac{u_n}{q}
\end{align*}
\]  

(A1.19)

The equations of motion (continuity, \( s \) and \( n \) momentum) are then easily obtained from the equations derived in A1.1. We will only write here the Navier-Stokes equations for a *steady, laminar* flow:

\[
\begin{align*}
\frac{\partial w_s}{\partial s} + \frac{\partial w_n}{\partial n} &= 0 \\
w_s \frac{\partial w_s}{\partial s} + w_n \frac{\partial w_s}{\partial n} + (w_s^2 + w_n^2) \frac{\partial}{\partial s} (\ln q) + \frac{1}{\rho q^2} \left( \frac{\partial p}{\partial s} - F_s \right) &= -\frac{\nu}{q^2} \frac{\partial \omega}{\partial n}
\end{align*}
\]  

(A1.20)
\[ w_s \frac{\partial w_n}{\partial s} + w_n \frac{\partial w_n}{\partial n} + (w_s^2 + w_n^2) \frac{\partial}{\partial n} (\ln q) + \frac{1}{\rho q} (\rho \frac{\partial p}{\partial n} - F_n) = \frac{v}{q} \frac{\partial \omega}{\partial s} \quad (A1.22) \]

where the rotation \( \omega \) is given by

\[ \omega = \frac{q}{q_0} \left( \frac{\partial w_n}{\partial s} - \frac{\partial w_s}{\partial n} \right) \quad (A1.23) \]

and the volumic forces \( F_s \) and \( F_n \) are defined by \( F \equiv q_0/q \, f \).

Finally, we give the expression for the shear stress \( \tau_{sn} \) which will be useful in appendix 3.

\[ \tau_{sn} = \frac{1}{q_0} \left\{ \frac{\partial}{\partial s} (q^2 w_n) + \frac{\partial}{\partial n} (q^2 w_s) \right\} \quad (A1.24) \]
A2/ THE ACTION OF A STRESS AT THE SURFACE OF WATER WAVES

A2.1/ INTRODUCTION

A2.1.1/ GENERAL

In this appendix, we will use the method of matched asymptotic expansions to study the action of a stress at the surface of water waves. For suitably small shear stresses applied along the free surface, the flow far from the free surface (outer flow) will be represented by a potential flow whereas the flow in the vicinity of the free surface (inner flow) will be described by classical boundary layer equations. The question we wish to answer in this appendix is what potential flow to choose in order to represent the real viscous flow. This potential flow is referred to in this thesis as the pseudo flow.

It is not possible to proceed directly and to expand the outer solution starting from the solution where no shear is applied along the free surface. The procedure fails because the wavelength of the waves is not the same for the flows with or without shear, as that will be demonstrated. A possible way to overcome this difficulty would be to use multiple scales in the asymptotic expansion, allowing the wavelength to vary with the small parameter. However, this would limit the scope of this study since the procedure would be limited to cases where the dispersion relation can be explicitied analytically - i.e. to waves of small steepness - and increase the computational difficulty.
We will therefore use here a less direct but more general (and simpler) approach. We will assume that the real viscous problem - referred to as the \((\Sigma)\) problem - has a solution where both the pressure and the shear are prescribed along the free surface (which is a streamline), \(S_\Sigma\). We will write the equations for the \((\Sigma)\) problem in streamline coordinates corresponding to an arbitrary potential flow \((\Pi)\). Then we will show that for a particular choice of the \((\Pi)\) flow, an outer solution to the \((\Sigma)\) problem, valid up to second order, has a very simple expression in these coordinates. This method, therefore, yields conditions to be satisfied by a second order outer approximation. However, it is not possible in general to solve the resulting boundary value problem, or even to assess the existence of its solution. Nevertheless, the solution can be constructed in the case of a wake at the surface of waves of small steepness. In this case, it will be shown how the dispersion relation is modified for deep water waves. The effect of the wake will be further discussed in appendix 3 in connection with the wave resistance.

The physical picture of the flow we obtain here is closely related to that of Longuet-Higgins (1969) even if this author did not consider the modification of the dispersion relation. In the model described here:

- the flow far from the free surface is represented by a potential flow, the pseudo flow;
- the flow near the free surface is represented by boundary layer equations;
- the major effect of the boundary layer is that the pseudo flow free surface is displaced downward by a distance corresponding to the boundary layer displacement thickness;
• the weight of the fluid within the displaced layer acts upon the pseudo flow free surface causing an additional secondary pressure distribution. An additional pressure arises due to the curvature of the streamline of the pseudo flow;

• Because the thickness of the boundary layer is related to the velocity of the pseudo flow at the free surface, this effect induces a modification of the dispersion relation.

The computations will be performed here using the Navier Stokes equations for a laminar flow, but the results can be generalized to turbulent flows by use of the appropriate closure for the Reynolds' stresses.

A2.1.2/ CONVENTIONS

For convenience, we will denote by a star the dimensional variables in this appendix, so that \{variable\}^* is dimensional and \{variable\} is not. Note that this convention is not used in the main body of this thesis where only dimensional variables are considered. Important results will be given under both their non-dimensional and dimensional expressions.

A2.2/ FORMULATION OF THE (Σ) AND (Π) PROBLEMS

A2.2.1/ THE (Σ) PROBLEM

We assume that there exists a flow (Σ) which satisfies the following conditions (see figure A2.1):

• \(S_\Sigma\) is a streamline

• pressure \(p_\Sigma^*\) along \(S_\Sigma\)
• shear $\tau_{\Sigma}^*$ along $S_{\Sigma}$.
• uniform flow upstream (velocity $c^*$, $p_{\Sigma}^* = \tau_{\Sigma}^* = 0$, $S_{\Sigma}$ coincides with the $x^*$-axis)
• other conditions (bottom boundary conditions, disturbance (such as a submerged hydrofoil), etc ...)

![Diagram showing no pressure and no shear along the streamline $S_{\Sigma}$, with $p_{\Sigma}^*$ and $\tau_{\Sigma}^*$ indicated.]

Figure A2.1 - The ($\Sigma$) Problem

A2.2.2/ THE ($\Pi$) FLOW

We now consider a potential flow ($\Pi$) which satisfies the following conditions (see figure A2.2):
• pressure $p_{\Pi}^*$ along $S_{\Sigma}$.
• normal velocity $w_n^*$ along $S_{\Sigma}$.
• uniform flow upstream (velocity $c^*$, $p_{\Pi}^* = 0$ and $w_n^* = 0$ along $S_{\Sigma}$ which
coincides with the x*-axis)

- other conditions (bottom boundary conditions, disturbance (such as a submerged hydrofoil), etc ...)

Figure A2.2 - The (Π) Problem

We will not consider any source of vorticity in the "other" boundary conditions since we only consider the effect of the stress at the free surface. The (Π) flow is only introduced at this stage in order to provide an appropriate coordinate system to study the (Σ) problem. It can be chosen arbitrarily.
Figure A2.3 - Curvilinear Coordinates for the ($\Sigma$) Problem

A2.2.3/ COORDINATE SYSTEM

The pressure in the ($\Pi$) problem is $p_{\Pi^*}$, the modulus of the velocity $q_{\Pi^*}$. We define orthogonal curvilinear coordinates $(s^*, n^*)$ such that the lines $n^* = \text{constant}$ are the streamlines of the potential flow ($\Pi$). The line $n^* = n_{\Sigma^*} (s^*)$ (which is not supposed to be a streamline of the pseudo flow) is the line $S_{\Sigma}$. Using the results of appendix 1, we will write the Navier-Stokes equations for a laminar flow in these coordinates. We assume that the only volumic forces acting are due to gravity ($f_x^* = 0$, $f_z^* = -g$). The constant appearing in the definition of the scale factors in appendix 1 (i.e. $\phi_0^*$) is chosen equal to $c^*$. The curvilinear coordinates are sketched on figure A2.3.

Non-dimensional variables are built taking as length scale $L^*$ (as yet
unspecified), as pressure scale \( \rho c^2 \) and as velocity scale \( c^* \) (therefore, for instance, \( \eta = \eta^*/c^* \)). The Reynolds number is defined by \( \text{Re} \equiv c^*L^*/v \). The Froude number is defined by \( \text{Fr}^2 \equiv c^*/gL^* \).

\( w_s \) and \( w_n \) being defined as \( u_s^*/\eta \) and \( u_n^*/\eta \), where \( u_s^* \) and \( u_n^* \) are the components of the velocity of the real viscous flow (\( \Sigma \)) tangential and normal to the streamline of the (\( \Pi \)) flow, the Navier-Stokes equations are expressed as (this choice of the coordinates is similar to that made by Kevorkian & Cole (1981) for the classical case of the boundary layer along a body):

\[
\frac{\partial w_s}{\partial s} + \frac{\partial w_n}{\partial n} = 0 \tag{A2.1}
\]

\[
w_s \frac{\partial w_s}{\partial s} + w_n \frac{\partial w_n}{\partial n} + (w_s^2 + w_n^2) \frac{\partial}{\partial s} (\ln \eta) + \frac{1}{\eta^2} \frac{\partial p}{\partial s} + \frac{1}{\text{Fr}^2 \eta^3} \sin \theta_{\Pi} = \]

\[
= \frac{1}{\text{Re}} \left\{ \frac{\partial^2 w_s}{\partial s^2} + \frac{\partial^2 w_s}{\partial n^2} - 2 \frac{\partial}{\partial n} \ln(\eta) \left[ \frac{\partial w_n}{\partial s} - \frac{\partial w_n}{\partial n} \right] \right\} \tag{A2.2}
\]

\[
= \frac{1}{\text{Re}} \left\{ \frac{\partial^2 w_n}{\partial s^2} + \frac{\partial^2 w_n}{\partial n^2} - 2 \frac{\partial}{\partial s} \ln(\eta) \left[ \frac{\partial w_n}{\partial s} - \frac{\partial w_n}{\partial n} \right] \right\} \tag{A2.3}
\]

where \( \theta_{\Pi} \) defines the inclination of the streamline of the (\( \Pi \)) flow with respect to the x-axis at the point considered.
The boundary conditions for the \((\Sigma)\) problem in these coordinates are given along \(S_\Sigma\) by:

- \(S_\Sigma\) is a streamline, so that the velocity is tangential along \(S_\Sigma\).  
- the shear stress along \(S_\Sigma\) is given, equal to \(\tau_\Sigma^* = \tau^* \tau_\Sigma(s)\), where \(\tau^*\) scales the prescribed shear, yielding from equation (A1.24):

\[
\frac{\partial}{\partial n} (q_{\Pi}^2 w_\Sigma) = \frac{Re \tau^*}{\rho c_s^*} \tau_\Sigma(s) \tag{A2.4}
\]

- the pressure along \(S_\Sigma\) is given, equal to \(p_\Sigma^*\) (diagonal deviatoric stresses appear at higher order and are neglected).

The other boundary conditions will not be written here since we are only interested by the development of the boundary layer at the free surface.

**A2.3/ RESOLUTION OF THE \((\Sigma)\) PROBLEM**

**A2.3.1/ OUTER EXPANSION**

We define \(\varepsilon = 1/Re\) and we assume that \(\varepsilon\) is much smaller than 1. We will solve the \((\Sigma)\) problem using asymptotic methods. The \((\Pi)\) flow can be chosen arbitrarily. Here, we will impose that \((\Pi)\) be a valid first order approximation of the outer flow and the streamline \(n = 0\) be a first order approximation of \(S_\Sigma\). Thus we will assume the following expansions for the outer solution of \((\Sigma)\):
\[ w_s = 1 + \alpha (\varepsilon) w_{s1} + o (\alpha) \]  
(A2.5)

\[ w_n = \beta (\varepsilon) w_{n1} + o (\beta) \]  
(A2.6)

\[ p = p_\Pi + \gamma (\varepsilon) p_1 + o (\gamma) \]  
(A2.7)

\[ n_{\Sigma} = \eta (\varepsilon) n_{\Sigma 1} + o (\eta) \]  
(A2.8)

where \( \alpha, \beta, \gamma \) and \( \eta \) are functions of \( \varepsilon \) much smaller than 1. Moreover, we will need to have \( |p_\Pi - p_\Sigma| \) of order \( \varepsilon \) along \( S_\Sigma \).

It is not obvious a priori that such a choice for the \( (\Pi) \) flow is possible. Actually, we will show that the potential flow solution of all the equations satisfied by the \( (\Sigma) \) flow, except the free surface shear boundary condition, does not, in general, satisfies these conditions. This stems from the fact the boundary layer induces a modification of the dispersion relation. We will, therefore, first assume that such a choice is possible and then construct the \( (\Pi) \) flow. This should appear less artificial than defining a priori a first order solution.

**A2.3.2/ FIRST ORDER INNER SOLUTION**

The first order outer solution defined above is not uniformly valid because it does not satisfy the free surface shear boundary condition. We therefore define an inner problem in the vicinity of the line \( S_\Sigma \). The inner variables are given by
\[ s = s, \quad n = \frac{n - n_{\Sigma}}{\delta(\varepsilon)} \quad (A2.9) \]

and the inner unknowns are expanded as (only the leading order terms are retained):

\[ w_s = w_s, \quad w_n = \frac{w_n}{\delta(\varepsilon)}, \quad p = \frac{p - p_{\Sigma}}{\delta(\varepsilon)} \quad (A2.10) \]

where \( \delta(\varepsilon) \) is a function of \( \varepsilon \) much smaller than 1. \( \delta \) scales the thickness of the inner domain (i.e. of the boundary layer).

The principle of least degeneracy leads to choose \( \eta = \delta \). We then define:

\[ W_n = w_n - \frac{d n_{\Sigma}}{d s} w_s \quad (A2.11) \]

This leads to the first order inner problem

\[ \frac{\partial w_s}{\partial s} + \frac{\partial W_n}{\partial \eta} = 0 \quad (A2.12) \]

\[ w_s \frac{\partial w_s}{\partial s} + W_n \frac{\partial w_s}{\partial \eta} + w_s^2 \frac{\partial}{\partial s} (\ln q_{\Pi}) + \frac{1}{q_{\Pi}^2} \frac{\partial p_{\Pi}}{\partial s} + \frac{1}{Fr^2 q_{\Pi}^3} \sin \theta_{\Pi} = \]
\[ w_s = \frac{1}{\delta^2 \text{Re}} \frac{\partial w_s}{\partial \eta^2} \]  \hspace{1cm} (A2.13)

\[ \frac{w_s^2}{\partial \eta} \left( \ln q_{II} \right) + \frac{1}{2} \frac{\partial p_{II}}{\partial \eta} + \frac{1}{2} \frac{\partial p}{\partial \eta} - \frac{1}{F \eta^2 q_{II}^3} \cos \theta_{II} = 0 \]  \hspace{1cm} (A2.14)

It is important to notice that in all these equations, the variables corresponding to the (II) problem (denoted by a subscript II) are only involved through their trace on \( S_{\Sigma} \). They are functions of \( s \) only.

These equations are classical boundary layer equations. Only the outer pressure field is involved in equations (A2.12)-(A2.13) so that these two equations can be solved independently of equation (A2.14). Then equation (A2.14) can be used to find the inner pressure field. The principle of least degeneracy leads to choose \( \delta^2 \text{Re} = 1 \) so that the thickness of the boundary layer is inversely proportional to the square root of the Reynolds' number, a classical result.

Of course, these equations have to be completed by boundary conditions. On the streamline \( S_{\Sigma} \) we have

\[ w_n = 0 , \quad \frac{\partial w}{\partial \eta} = \frac{1}{\delta^2 q_{II}} \frac{\delta \text{Re} \tau^*}{\rho c^*^2} \tau_{\Sigma} (s) \]  \hspace{1cm} (A2.15)
Therefore the principle of least degeneracy imposes the choice of the length scale $L^*$ and leads to choose finally

$$\epsilon \equiv \frac{1}{Re} = \left[ \frac{\tau^*}{\rho \, c^*} \right]^2, \quad \delta = \frac{\tau^*}{\rho \, c^*}$$  \hspace{1cm} (A2.16)

Equation (A2.16) implies that the present asymptotic theory is valid when the imposed shear stress is much smaller than $\rho c^*^2$, where $c^*$ scales the celerity of the wave. There is no condition on the pressure.

The behavior at infinity of the inner solution is given by the matching with the outer solution. This yields

$$w_s \rightarrow 1 \quad \text{as} \quad n \rightarrow +\infty$$  \hspace{1cm} (A2.17)

A2.3.3/ THE BOUNDARY LAYER MOMENTUM EQUATION

Since $p_{\Pi}$ is the pressure for the $(\Pi)$ flow, we have

$$\frac{\partial}{\partial s} \left( \ln q_{\Pi} \right) + \frac{1}{q_{\Pi}^2} \frac{\partial p_{\Pi}}{\partial s} + \frac{1}{Fr^2 q_{\Pi}^3} \sin \theta_{\Pi} = 0$$  \hspace{1cm} (A2.18)
\[
\frac{\partial}{\partial n} (\ln q_\Pi) + \frac{1}{q_\Pi^2} \frac{\partial p_\Pi}{\partial n} - \frac{1}{Fr^2 q_\Pi^3} \cos \theta_\Pi = 0 \quad (A2.19)
\]

Using equations (A2.18) and (A2.12), equation (A2.13) can be rewritten:

\[
\frac{\partial w_s^2}{\partial x} + \frac{\partial w_s W_0}{\partial \Pi} + \frac{w_s^2 - 1}{q_\Pi} \frac{\partial q_\Pi}{\partial s} = \frac{\partial^2 w_s}{\partial \Pi^2} \quad (A2.20)
\]

We now define the displacement thickness, \( \delta_1 \), and the momentum thickness, \( \delta_2 \), by

\[
q_\Pi \delta_1 = \int_0^\infty (1 - w_s) \, d\eta \quad (A2.21)
\]

\[
q_\Pi \delta_2 = \int_0^\infty (w_s - w_s^2) \, d\eta \quad (A2.22)
\]

The factor \( q_\Pi \) in these equations stems from the fact that the integration is performed with respect to \( \eta \), the stream function, and not with respect to the normal distance (see the definition of the scale factors). They are introduced here in order to define \( \delta_1 \) and \( \delta_2 \) according to usage. Using dimensional variables and the curvilinear abscissa along the normal, \( \eta^* \), we actually have:
\[
\delta_1^* \equiv L^* \delta_1 = \int_{\Sigma}^\infty (1 - \frac{u_s^*}{q_{\Pi^*}}) \, d\eta^*
\] (A2.23)

\[
\delta_2^* \equiv L^* \delta_2 = \int_{\Sigma}^\infty \left( \frac{u_s^*}{q_{\Pi^*}} \right)^2 \, d\eta^*
\] (A2.24)

Integration of (A2.20) making use of (A2.15) and (A2.17) yields the momentum equation:

\[
q_{\Pi^*} \frac{d\delta_2^*}{ds} + q_{\Pi^*} \frac{\partial q_{\Pi^*}}{\partial s} (\delta_1^* + 2\delta_2^*) = \frac{1}{q_{\Pi^*}} \tau_{\Sigma^*}
\] (A2.25)

Again, the \(1/q_{\Pi^*}\) factor on the right hand should be understood as a scale factor. Using dimensional variables, equation (A2.25) can be rewritten:

\[
q_{\Pi^*} \frac{d\delta_2^*}{d\sigma^*} + q_{\Pi^*} \frac{\partial q_{\Pi^*}}{\partial \sigma^*} (\delta_1^* + 2\delta_2^*) = \frac{\tau_{\Sigma^*}}{\rho}
\] (A2.26)

Equation (A2.26) is just classical Von Kármán's momentum equation (e.g. Schlichting (1968), p. 146). It has been derived here for a laminar flow satisfying the Navier Stokes equations but, as long as no assumption is made concerning the shear stress \(\tau\), it also applies to turbulent flows (but, of course, a different scaling would be necessary for the asymptotic expansion). Note that this equation takes into account the curvature
of the curvilinear coordinates used.

Similarly, integration of equation (A2.14) (taking into account the fact that the pressure is \( p_\Sigma (s,0) \) on \( S_{\Pi} \)) yields

\[
\lim_{n \to \infty} p (s, \Pi) = q_{\Pi}^2 \frac{\partial q_{\Pi}}{\partial s} (\delta_1 + \delta_2) + p_\Sigma (s,0)
\]  
(A2.27)

A2.3.4/ Next Order Matching

We now match the outer solution with the inner solution at the next order. This is made demanding that the behavior of the outer solution in the vicinity of \( n = 0 \) be the same as the behavior of the inner solution in the vicinity of \( \Pi = \infty \). This implies \( \beta = \gamma = \delta \). Matching of the normal velocities yields

\[
w_{n1} = \frac{\partial}{\partial s} (q_{\Pi} \delta_1) + \frac{dn_{\Sigma 1}}{ds} \text{ along } S_{\Sigma}
\]  
(A2.28)

Matching of the pressures yields (using equation (A2.27))

\[
p_1 = q_{\Pi}^2 \frac{\partial q_{\Pi}}{\partial n} (\delta_1 + \delta_2) + \frac{p_\Sigma - p_{\Pi}}{\delta (e)} \text{ along } S_{\Sigma}
\]  
(A2.29)

We now remark that if it is possible to choose \( p_{\Pi} \) such that
\[ p_{\Pi} = p_{\Sigma} + \delta \left( q_{\Pi}^2 \frac{\partial q_{\Pi}}{\partial n} (\delta_1 + \delta_2) + q_{\Pi} \delta_1 \frac{\partial p_{\Pi}}{\partial n} \right) \]  \hspace{1cm} (A2.30)

along the line \( n = 0 \) (a streamline of the potential flow (\( \Pi \))) then the potential flow (\( \Pi \)) will be a valid second order outer approximation of the real flow for which a pressure \( p_{\Sigma} \) an a shear stress \( \tau_{\Sigma} \) are applied along \( S_{\Sigma} \) (a streamline of the real flow (\( \Sigma \))) given by

\[ n_{\Sigma} (s) = - \delta \left( q_{\Pi} \delta_1 \right) \]  \hspace{1cm} (A2.31)

Using equation (A2.19), equation (A2.30) can be expressed under the alternative and simpler form

\[ p_{\Pi} = p_{\Sigma} + \delta \left( \frac{\cos \theta_{\Pi}}{Fr^2} \delta_1 + q_{\Pi}^2 \frac{\partial q_{\Pi}}{\partial n} \delta_2 \right) \]  \hspace{1cm} (A2.32)

Using dimensional variables, equation (A2.32) can be rewritten:

\[ \frac{p_{\Pi}^*}{\rho} = \frac{p_{\Sigma}^*}{\rho} + g\delta_1^* \cos \theta_{\Pi} + q_{\Pi}^* \frac{\partial q_{\Pi}^*}{\partial y^*} \delta_2^* \]  \hspace{1cm} (A2.33)

where \( p_{\Pi}^* \) is the pressure applied at the free surface of the pseudo flow; \( p_{\Sigma}^* \) is the pressure applied at the free surface of the real, viscous flow; \( \delta_1^* \) and \( \delta_2^* \) the displacement and momentum thicknesses; \( \theta_{\Pi} \) the inclination of the free surface of the pseudo potential flow with respect to the x axis; and \( q_{\Pi}^* \) the velocity at the free surface
of the pseudo potential flow.

Equation (A2.31), which relates the position of the streamline of the real flow and the position of the streamline of the pseudo flow, and equation (A2.33), which relates the pressure at the surface of the streamline of the real flow and the pressure at the surface of the streamline of the pseudo flow, constitute the main results of this appendix. If equation (A2.31) is quite classical and is the same as for the boundary layer along a body, equation (A2.33) is (at our knowledge) new and implies a modification of the free surface boundary condition. The pseudo flow is sketched in figure A2.4.

\[ S_\Sigma: \text{pressure } p_\Sigma^* \text{ & shear } \tau_\Sigma^* \]

\[ S_\Pi: \text{pressure } p_\Pi^* \]

Figure A2.4 - The Real and Pseudo Flows
A2.4/ INTERPRETATION OF THE RESULT

A2.4.1/ GENERAL

We have shown that for a suitably small shear stress $\tau_2^*$ (much smaller than $\rho c^*^2$, where $c^*$ scales the celerity of the waves) applied at the free surface of water waves, the real viscous flow can be modeled by a potential flow, the pseudo flow. The free surface of this pseudo flow is displaced downward by a distance corresponding to the boundary layer displacement thickness (equation (A2.31)). An additional pressure has to be applied at the free surface of the pseudo flow (equation (A2.33)). Two terms appear in this additional pressure. The first term is proportional to the displacement thickness, $\delta_1$, and corresponds exactly to the weight of the displaced layer. The effect of this term was considered by Longuet-Higgins (1969). The second term is proportional to the momentum thickness, $\delta_2$, and is related to the curvature of the free streamline of the pseudo flow (the curvature $K_s$ is equal to $-\partial q_\Pi/\partial n$).

It is not possible, in general, to solve the resulting boundary value problem for the pseudo flow, since Von Kármán's momentum equation has to be solved in order to express the boundary condition (A2.33) in closed form and since the velocity at the free streamline of the pseudo flow has to be known in order to solve this boundary layer equation. If the solution can be found, it will therefore require, in general, an iterative process.

In the special case of the far wake over small amplitude waves, its effect can however be found explicitly. This will be done in the next section. The results are important because they show how the dispersion relation is modified and they allow a
first comparison with experiments.

A2.4.2 THE WAKE OVER WAVES OF SMALL STEEPNESS

We will here make two assumptions in order to show, in a particular case, the effect the additional pressure related to the boundary layer. We will first assume that we are in the wake far downstream of the shear disturbance, so that

- there is no shear applied at the free surface;
- the displacement and momentum thicknesses, \( \delta_1 \) and \( \delta_2 \) are proportional.

The boundary layer shape parameter is defined classically as \( H = \delta_1 / \delta_2 \) (see for instance Schlichting (1968), p. 714). When there is no shear applied at the boundary, Von Kármán’s momentum equation can be written in terms of \( H \) and \( \delta_2 \) as

\[
\frac{1}{\delta_2} \frac{d \delta_2}{ds} + (H + 2) \frac{1}{q_{\Pi}} \frac{dq_{\Pi}}{ds} = 0 \tag{A2.34}
\]

Since \( H \) is assumed here to be a constant, integration yields

\[
\delta_1 = H \delta_2 = \frac{\Delta}{q_{\Pi}} \tag{A2.35}
\]

where \( \Delta \) is a constant (the displacement thickness for \( q_{\Pi} = 1 \)).
Moreover, we will assume that the pseudo flow is periodic in x (with a
cwavenumber \kappa). In this case (see Lamb (1932), Art. 250), \Delta is the displacement
thickness at the mean level of the free surface of the pseudo flow. We will also assume
that the steepness of the waves of the pseudo flow is small. For deep water waves, we
will, therefore, write that:

\[ q_{\Pi} = 1 + \alpha \sin \kappa s \ e^{-\kappa n} \]  \hspace{1cm} (A2.36)

where \alpha is the amplitude of the waves of the pseudo flow. We expand \( x(s,n) \) and
\( z(s,n) \) in terms of \( \alpha \):

\[ x(s,n) = s + \alpha x_1(s,n) \] \hspace{1cm} (A2.37)

\[ z(s,n) = -n + \alpha z_1(s,n) \] \hspace{1cm} (A2.38)

Since \( h_n \) is equal to \( 1/q_{\Pi} \), the relation between \( z_1 \) and \( q_{\Pi} \) is easily found and leads to
(the constant term is omitted):

\[ z_1(s,n) = -\frac{\sin \kappa s}{\kappa} \ e^{-\kappa n} \] \hspace{1cm} (A2.39)

On a free streamline \( n = 0 \) along which the pressure is constant, Bernoulli's
equation implies \( q_{\Pi}^2 + 2/\text{Fr}^2 \ z = \text{constant} \), where Fr is the Froude number, \( \text{Fr}^2 = \)
c*2/(gL*). In this case, equations (A2.36) and (A2.39) leads to the classical dispersion relation for the wavenumber, \( \kappa_0 \), of gravity waves of small steepness:

\[
\kappa_0^* = \frac{g}{c^*^2}
\]  
(A2.40)

On the streamline \( n = 0 \) of the pseudo flow, we have from equations (A2.32) (up to first order in \( \alpha \)):

\[
p_{II}(s, 0) = p_\Sigma + \frac{\delta \Delta}{Fr^2} - \alpha \sin \kappa \left \{ \left( \frac{2+H}{Fr^2} + \frac{\kappa}{H} \right) \delta \Delta \right \}
\]  
(A2.41)

If we assume that \( p_\Sigma \) is a constant, Bernoulli's equation leads to a new dispersion relation for the waves of the pseudo flow:

\[
\kappa^2 \frac{\delta \Delta}{H} + \left( 1 - \frac{2+H}{Fr^2} \delta \Delta \right) \kappa - \frac{1}{Fr^2} = 0
\]  
(A2.42)

Expanding the real solution of this equation in terms of \( \delta \), we get

\[
\kappa = \kappa_0 \left( 1 + \frac{\delta \Delta}{Fr^2} \left( 2+H+\frac{1}{H} \right) \right)
\]  
(A2.43)

This equation shows that the effect of the modulation of wake on the outer
pseudo flow is to increase the wavenumber or to decrease the wavelength. This result is confirmed by Duncan's experiments (1981), as this is discuss in the main part of this thesis.

It should be noticed that the effect of the modulation of wake on the pseudo flow is exactly equivalent to a modification of the gravity according to

\[ g' = g \left( 1 - \frac{\Delta^*}{c^2} \left\{ 2 + H + \frac{1}{H} \right\} \right) \]  
(A2.44)

This result (A2.44) is, however, only valid for waves of small steepness. Further discussion of the effect of the wake (in finite and infinite depth) is given in appendix 3.
A3/ ON THE EVALUATION OF THE WAVE PROFILE AFT OF A DISTURBANCE

A3.1/ INTRODUCTION

A3.1.1/ GENERAL

We consider the following problem: a steady stream (velocity $c_0$, depth $d_0$ and surface pressure $p_0 = 0$) is submitted to a disturbance applied on a bounded domain (length $L$). We assume that there exists a steady potential solution to the problem and that at infinity aft of the disturbance, the flow is periodic in the horizontal direction. The mean water level is $d$ and the surface pressure is $p$ (a periodic function of $x$) far downstream. Finally we assume that the bottom is flat. The flow is sketched on figure A3.1.

In this appendix, we wish to characterize the flow downstream as precisely as possible in terms of global considerations concerning the disturbance. This calculation is necessary in order to perform the "shock" analysis for the pseudo flow. For this shock analysis, the disturbance consists of a submerged body (the hydrofoil) and a surface pressure distribution (due to the breaker). Due to the presence of the wake, a periodic pressure is also applied at the free surface of the pseudo flow far downstream. The calculation is performed here for an arbitrary finite depth. The limiting case of infinite depth is considered and significant differences with Duncan (1982 & 1983) are shown in the analysis of the results.
In this analysis:

- an exact expression is derived for the resistance in terms of the wave profile far downstream taking into account the pressure applied at the free surface of the pseudo flow to model the wake;

- an expansion of this relation is given in the limiting case of waves of small steepness and of a wake of small steepness;

- finite amplitude effects are considered using the numerical results of Longuet-Higgins (1975) and Cokelet (1977b).

A3.1.2/ CONVENTIONS

The solution far aft of the disturbance being assumed to be spatially periodic in
the horizontal direction, we define $\kappa$ such that $2\pi/\kappa$ is the wave length. In order to keep
the same notations as Duncan (1981 & 1983) we denote here a the height of the waves
(distance crest to trough), $\alpha$ their amplitude ($a/2$). Non-dimensional variables are built
taken $1/\kappa$ as length scale and $(g/\kappa)^{1/2}$ as velocity scale ($g$ is the acceleration of gravity).

For convenience (and because we will only use one adimensionalization), we
will not distinguish dimensional and non-dimensional variables in this appendix, i.e.
we will define units such that $\kappa = 1$ and $g = 1$. Note that these conventions (and
adimensionalisations) are not the same as in appendix 2.

A3.2/ THE BOUNDARY VALUE PROBLEM FAR DOWNSTREAM

A3.2.1/ FORMULATION OF THE PROBLEM

We formulate the boundary value problem far downstream in terms of the
velocity potential, $\phi$, and of the free surface elevation, $\eta$. The velocities are given by $u$
$= \phi_x$ and $v = \phi_y$. The flow is assumed to be steady in our coordinate system, so that $\phi$
is a function of $(x,y)$ and $\eta$ is a function of $x$. The flow is also assumed to be spacially
periodic, of period $2\pi$ (in non-dimensional variables), so that $\phi_x (x+2\pi,y) = \phi_x (x,y),$
$\phi_y (x+2\pi,y) = \phi_y (x,y)$ and $\eta (x+2\pi) = \eta (x)$. Note that this does not imply that the
velocity potential itself is periodic: only the velocities are. However, it is possible to
write $\phi = \phi + c x$, where $\phi$ is periodic and $c$ is a constant. $c$ has an important physical
meaning since it represents the mean horizontal velocity (at any depth) far downstream.
We will therefore formulate the boundary value problem for $\phi$ and $c$ rather than $\phi$.

The $x$-axis is chosen so that the mean water level far downstream is equal to
zero. We therefore have:

\[
\int_{x}^{x+2\pi} \eta(x) \, dx \equiv 0
\]  

(A3.1)

for any \( x \) much greater than \( L \). The origin of the \( x \)-axis will be taken just aft of the disturbance which, therefore, is applied between \(-L\) and 0.

The periodic velocity potential, \( \phi \), and the free surface elevation, \( \eta \), far downstream have moreover to satisfy the following equations:

\[
\phi_{xx} + \phi_{yy} = 0 \quad \text{in the fluid domain}
\]  

(A3.2)

\[
\begin{aligned}
&c\phi_x + \frac{1}{2} (\phi_x^2 + \phi_y^2) + \eta + \frac{p}{\rho} = K_1 \\
&c\eta_x + \phi_x \eta_x = \phi_y \\
\end{aligned}
\]

for \( y = \eta(x) \)  

(A3.3)

\[
\phi_y = 0 \quad \text{for } y = -d
\]  

(A3.4)

where \( K_1 \) is a constant.

**A3.2.2/ RELATIONS WITH THE FLOW UPSTREAM**

In order to relate the flows upstream and downstream, we will apply
Bernoulli's theorem, the conservation of mass and the conservation of horizontal momentum. The control volume shown on figure A3.2, which includes the disturbance, will be used.

Figure A3.2 - Control Volume

Because the flow is steady and potential, Bernoulli's theorem yields

\[ c \varphi_x + \frac{1}{2} (\varphi_x^2 + \varphi_y^2) + \eta + \frac{p}{\rho} = \frac{c_0^2 - c^2}{2} + (d_0 - d) \]  \hspace{2cm} (A3.5)

so that the constant \( K_1 \) is determined by the flow upstream.

If we assume that there is no gain or loss of mass in the region where the disturbance is applied, conservation of mass yields
\[ c_0 d_0 = \int_{-d} u(y) \, dy \]  
(A.3.6)

We then write the conservation of horizontal momentum (Euler's theorem) which yields

\[ R = \rho \frac{d_0^2}{2} + \rho c_0^2 d_0 - \int_{-d} (p + \rho u^2) \, dy + \int_{0}^{x} p \frac{\partial \eta}{\partial \chi} \, d\chi \]  
(A.3.7)

where \( R \) is the resistance associated to the disturbance. Of course, \( R \) has to be independent of \( x \) and so does the right hand side of the equation when \( x >> L \).

In the case where the disturbance consists of a submerged body and a pressure distribution applied on the free surface, one easily gets:

\[ R = R_0 - \int_{-L}^{0} p \tan \theta \, dx \]  
(A.3.8)

where \( R_0 \) is the horizontal force exerted on the submerged body (positive when directed against the flow) and \( p \) is the pressure acting at the free surface of the pseudo flow in the region where the disturbance is applied.
By use of Bernoulli's equation and of the conservation of mass it is possible to eliminate the pressure from equation (A3.7). The resistance can, therefore, be expressed under the alternative form

\[
\frac{R}{\rho} = \int_{-d}^{(x)} \left( \frac{\varphi_y^2}{2} - \frac{\varphi_x^2}{2} \right) dy + \left( \frac{c^2 - c_0^2}{2} + d - d_0 \right) (\eta + d) + \frac{\eta^2}{2} + \\
+ \frac{d_0^2 - d^2}{2} + c_0 d_0 (c_0 - c) + \int_0^x \frac{p}{\rho} \frac{\partial \eta}{\partial \chi} d\chi
\]  

(A3.9)

We have formulated the problem far aft of the disturbance in terms of 4 unknowns: the free surface elevation \( \eta (x) \), a periodic function of zero mean value; the velocity potential \( \varphi (x,y) \), a periodic function of \( x \); the mean water level, \( d \); and the mean horizontal velocity, \( c \). In what follows, we will try to solve this problem in terms of the velocity and the depth of the flow upstream, \( c_0 \) and \( d_0 \); the pressure applied at the free surface of the pseudo flow far downstream, \( p (x) \), a periodic function; and the resistance associated with the disturbance, \( R \).
A3.3/ ASYMPTOTIC EXPANSION

A3.3.1/ INTRODUCTION

In this section, we solve the problem far from the disturbance by asymptotic methods. Two (small) parameters appear in the expansion: the steepness of the following waves and the thickness of the wake aft of the disturbance. Here we solve at the leading order and obtain an expression for the resistance. This allows to express the shock relations in the strong breaking regime. The effect of the wake on the evaluation of the resistance appears as a coupled term in the expansion. This coupling lets expect difficulties in finding the effect of the wake for finite amplitude waves.

A3.3.2/ THE EXPRESSION FOR THE PRESSURE AFT OF THE DISTURBANCE

We first make some assumptions concerning the pressure applied at the free surface of the pseudo flow far downstream. We will assume that this pressure can be expanded in terms of a small parameter $\Delta$ as

$$p(x) = \Delta p_\Delta(x) + o(\Delta)$$  \hspace{1cm} (A3.10)

Of interest to us will be the case where $p$ is equal to $p_\Pi$ given in appendix 2 (see equations (A2.29) and (A2.31)). It will allow us to evaluate the wave resistance for the pseudo flow, and therefore to take into account the effect of the wake. Note that, because of the adimensionalisation (which is different here and in appendix 2), $\Delta$ will represent in this appendix the product of the displacement thickness of the wake for $d_\Pi = c$ by the wavenumber $\kappa$. The pressure will be given by
\[
\frac{p}{\rho} = \Delta \left( \frac{c}{q_{\Pi}} \right)^{2+H} \{ \cos \theta_{\Pi} + \frac{1}{H} q_{\Pi} \frac{\partial q_{\Pi}}{\partial \eta} \} + o(\Delta) \tag{A3.11}
\]

where \(q_{\Pi}\) is the velocity at the free surface of the pseudo flow, \(\theta_{\Pi}\) its inclination with respect to the \(x\) axis, \(H\) the wake shape factor and \(\eta\) the curvilinear abscissa along the normal to the free surface.

In order to solve this problem, we will assume that the steepness of the waves aft of the disturbance is small. Some caution should be taken in performing the expansion because there exists another small parameter, \(\Delta\), which scales the boundary layer thickness. As shown by equation (A3.10), this parameter is involved explicitly in the expansion of the surface pressure \(p\). We will adopt here the rather naive point of view that all terms of order \(\Delta^2\) or higher can be neglected. The validity of this approximation will be discussed later. We will therefore expand the unknowns in terms of the small parameter \(\alpha\) (the non-dimensional amplitude of the following waves) only, without indicating the dependency on \(\Delta\).

A3.3.3/ EXPANSION OF THE PROBLEM

For a given value of \(\Delta\), the different unknowns are expanded in terms of \(\alpha\) as follows

\[
\varphi = \alpha \varphi_1 + o(\alpha) \tag{A3.12}
\]
\[ \eta = \alpha \eta_1 + o(\alpha) \quad (A3.13) \]

\[ c = c_1 + \alpha c_2 + o(\alpha) \quad (A3.14) \]

\[ d = d_1 + \alpha d_2 + o(\alpha) \quad (A3.15) \]

This expansion is justified by the fact that, as \( \alpha \) goes to zero, we expect the periodic part of the velocity potential and the free surface elevation to go to zero. Note, however, that \( c_1 \) and \( d_1 \) have no reason a priori to be equal to \( c_0 \) and \( d_0 \) for an arbitrary pressure \( p(x) \).

From equation (A3.11), we expand \( p(x) \) as

\[ \frac{p}{\rho} = \Delta - \alpha \Delta \left\{ (2+H) \frac{\varphi_{1x}}{c_1} + \frac{c_1}{H} \varphi_{1xy} \right\} + o(\alpha) \quad (A3.16) \]

We will assume that this expression is valid aft of the disturbance, i.e. for \( x > 0 \). We then expand the equations giving \( \varphi \) and \( \eta \) (equations (A3.2) to (A3.4)) in terms of \( \alpha \). This yields:

\[ \alpha \left( \varphi_{1xx} + \varphi_{1yy} \right) = o(\alpha) \quad \text{in the fluid domain} \quad (A3.17) \]
\[
\begin{align*}
\Delta + \frac{c_1^2 c_0^2}{2} + d_1 - d_0 + \alpha \left\{ c_1 - \frac{(2 + H)\Delta}{c_1} \right\} \phi_{1x} - \\
- \frac{\Delta c_1}{H} \phi_{1xy} + \eta_1 + c_1 c_2 + d_2 \right\} = o(\alpha) \quad \text{for } y = 0 \\
\alpha \{ c_1 \eta_{1x} - \phi_{1y} \} = o(\alpha) \quad (A3.18)
\end{align*}
\]

\[\alpha \phi_{1y} = o(\alpha) \quad \text{for } y = -d_1 \quad (A3.19)\]

We also expand the equation expressing the conservation of mass, equation (A3.6), which yields

\[
c_0 d_0 = c_1 d_1 + \alpha \left\{ c_1 \eta_1 + \int_{-d_1}^{0} \phi_{1x} \, dy + c_1 d_2 + c_2 d_1 \right\} + o(\alpha) \quad (A3.20)
\]

**A3.3.4/ ZEROTH ORDER APPROXIMATION**

At zeroth order in \( \alpha \), equations (A3.18) and (A3.20) imply that

\[
\Delta = \frac{c_0^2 - c_1^2}{2} + d_0 - d_1 \quad (A3.21)
\]

\[
c_0 d_0 = c_1 d_1 \quad (A3.22)
\]
These two relations allow to find the mean horizontal velocity, $c_1$, and the mean depth, $d_1$, aft of the disturbance. The solution can be expanded in terms of $\Delta$ as

$$c_1 = c_0 (1 + \frac{\Delta}{d_0 - c_0^2}), \quad d_1 = d_0 (1 - \frac{\Delta}{d_0 - c_0^2}) \quad \text{(A3.23)}$$

Note that critical speed, $c_0 = d_0^{1/2}$, appears here as an important parameter. We will only consider here the case of large depth ($d_0 >> c_0^2$) for which the effect of the boundary layer is to increase the mean horizontal velocity. Note in particular that this might be relevant to the problem of sediment transport induced by breaking waves since it accounts for the creation of a breaking induced current in the direction of wave propagation.

**A3.3.5/ FIRST ORDER APPROXIMATION**

Since $\phi_{1x}$ and $\eta_1$ are, by hypothesis, periodic and have a zero mean value over a period, equations (A3.18) and (A3.20) imply that

$$c_1 c_2 + d_2 = 0 \quad \text{(A3.24)}$$

$$c_1 d_2 + c_2 d_1 = 0 \quad \text{(A3.25)}$$

Except at the critical speed ($d_1 = c_1^2$), this yields $c_2 = d_2 = 0$. This is a classical result: at first order in $\alpha$, the mean depth and the mean horizontal velocity are not a function of
the steepness of the wave. A correction only appears (in finite depth) at second order (for a zero free surface pressure, this expansion has been carried out at second order by Nays (1987)).

The free surface boundary conditions, equation (A3.20) can now be written as

\[
\begin{align*}
\left\{ c_1 \frac{(2+H)\Delta x}{c_1} \varphi_{1x} - \frac{\Delta c_1}{H} \varphi_{1xy} + \eta_1 &= 0 \quad \text{for } y = 0 \\
- c_1 \eta_{1x} - \varphi_{1y} &= 0
\end{align*}
\]

(A3.26)

A solution is, therefore, given by (an arbitrary phase \( \psi \) is introduced)

\[
\eta_1 = \cos (x+\psi) \\
\varphi_1 = - \frac{c_1}{\mbox{th} (d_1)} \sin (x+\psi) \frac{\mbox{ch} (y+d_1)}{\mbox{ch} (d_1)}
\]

(A3.27)

(A3.28)

with the dispersion relation

\[
\mbox{th} (d_1) = c_1^2 \left\{ 1 - \frac{\Delta}{c_1^2} (2 + H + \frac{c_1^2 \mbox{th} (d_1)}{H}) \right\}
\]

(A3.29)
Note that, for infinite depth, this expression is the same as the one derived in appendix 2.

A3.3.6/ ASYMPTOTIC EXPANSION FOR THE RESISTANCE

From the solution we have derived for the flow aft of the disturbance, we can now express the resistance. The solution for the flow is valid up to terms of order $\alpha$, $\Delta$ and $\alpha\Delta$. From equation (A3.9), we can express the resistance up to terms of order $\alpha^2$, $\Delta$ and $\alpha\Delta$. This stems from the fact that terms of order $\alpha$ in the expansions of $\varphi$ and $\eta$ give a contribution to the resistance of order $\alpha^2$. After simplification, equation (A3.9) yields

$$\frac{R}{\rho} = \frac{\alpha^2}{4} \left[ 1 - \frac{2d_1}{\text{sh}(2d_1)} \right] - \Delta \eta_1(0) + o(\alpha^2, \Delta, \alpha\Delta) \quad (A3.30)$$

The first term on the right hand side is the classical second order wave resistance (e.g. Lamb (1932), Art. 249). Note that the correction due to the presence of the wake appears at first order in $\alpha$. In the case of the wake following a breaking wave, the origin of the $x$-axis is taken at the crest of the breaking wave, so that $\eta_1(0)$ is equal to the height of the crest, $\zeta_b$, which is itself assumed to be equal to $\alpha/2$.

A3.4/ FINITE AMPLITUDE EFFECTS

A3.4.1/ GENERAL

In order to be able to predict the critical amplitude at which breaking can first occur, it seems important to take into account finite amplitude effects. Results obtained
from linear wave theory are valid in the strong breaking regime, but probably not in the weak breaking regime. In this section, we wish to study finite amplitude effects on the momentum balance. We will, therefore, not assume that the steepness of the waves aft of the disturbance is small. We will use integral properties of waves of arbitrary steepness to compute the mass and momentum flux. Theses properties have been computed numerically by Longuet-Higgins (1975) for infinite depth and by Cokelet (1977b) for intermediate depth. Unfortunately, they have only been computed for a constant pressure applied at the free surface and it is therefore impossible, at present, to take into account both finite amplitude effects and the modulation of the thickness of the wake. Limitations of the present theory, where the thickness of the wake is assumed to be constant, are discussed in chapter 3.

The computation will be performed for finite depth and the limit for infinite depth will be considered. This analysis is closely related to that of Duncan (1982). However, we believe that his limiting process is incorrect and we will show how his result can be justified.

A3.4.2/ FINITE DEPTH

c₀ and d₀ being the velocity and depth upstream; c and d the mean horizontal velocity and mean depth, η the free surface elevation and φ the velocity potential aft of the disturbance, conservation of mass and horizontal momentum yield:
\[ c_0 d_0 = c (d + \eta) + \int_{-d}^{\eta} \phi_x \, dy \]  
(A3.31)

\[ \frac{R}{\rho} = \frac{d_0^2}{2} + c_0^2 d_0 \int_{-d}^{\eta} \left( \frac{p}{\rho} + \phi_x^2 + 2c\phi_x + c^2 \right) \, dy - \Delta d \cdot \Delta \eta (0) \]  
(A3.32)

where we have assumed that \( p = \rho \Delta \) along the free surface for \( x > 0 \).

Taking the mean value of these expression over a wavelength, we get

\[ c_0 d_0 = c d - \frac{I}{\rho} \]  
(A3.33)

\[ \frac{R}{\rho} = \frac{d_0^2 \cdot d^2}{2} + c_0 d_0 \cdot c^2 d \cdot \frac{S_{xx}}{\rho} + 2 \frac{I}{\rho} c - \Delta d \cdot \Delta \eta (0) \]  
(A3.34)

Finally, we write Bernoulli's equation as

\[ -2 \Delta + c_0^2 + 2 (d_0 \cdot d) = K - 2 \frac{I}{\rho c} \]  
(A3.35)

where the mass flux \( I \) and the radiation stress \( S_{xx} \) are defined by (the overbar denotes the average over a wavelength).
\[ I \equiv - \int_{-d}^{\eta} \rho \varphi_x \, dy \]  
(A3.36)

\[ S_{xx} \equiv \int_{-d}^{\eta} \left( p - \rho \Delta + \rho \varphi_x^2 \right) \, dy - \frac{1}{2} \rho \, g \, d \]  
(A3.37)

I, S_{xx}, K and c are tabulated in Cokelet (1977b) in function of the amplitude of the following wave train (equal here to \( \alpha \)) for a given depth (note, however, that the depth parameter taken by Cokelet is, according to our notations, \( d_c = d - I/\rho c \)). Therefore, when \( d_c \), \( \Delta \) and \( \alpha \) are given, equations (A3.33) and (A3.35) allow to compute \( d_0 \) and \( c_0 \).

Then, equation (A3.34) can be used to compute \( R \), provided that there exists a relation between the elevation of the free surface of the pseudo flow at \( x = 0 \), \( \eta (0) \), and the height of the following waves. For the the wake aft of a breaking wave \( \eta (0) \) is equal the elevation of the crest of the breaking wave above the mean water level, \( \zeta_b \), and we will assume that \( \zeta_b \) is equal to the elevation of the crests of the following wave train, \( \zeta_c \). For finite amplitude wave, this elevation is not equal to half the total height of the wave, \( a_r \), but the numerical results of Longuet-Higgins (1975) and Cokelet (1977b) allow to relate \( \zeta_c \) and \( a_r \) (see Nays (1987) for a detailed derivation of this relation).

This totally solves our problem. Resistance curves for finite depth are given by Nays (1987).
A3.4.3/ INFINITE DEPTH

We now wish to find the limit of equation (A3.35) when the depth $d_0$ goes to infinity. Thus we take as small parameter $\epsilon = 1/d_0$ and we expand the mean velocity and depth aft of the disturbance in terms of $\epsilon$. Since we expect the mean velocity to have a finite limit when the depth upstream goes to infinity, but the mean depth to behave like $d_0$, we write:

$$c = c_1 + \epsilon c_2 + o(\epsilon) \quad (A3.38)$$

$$d = \frac{d_1}{\epsilon} + d_2 + \epsilon d_3 + o(\epsilon) \quad (A3.39)$$

Plugging these expansions in equations (A3.33) and (A3.35), we finally get

$$c_1 = c_0, \quad d_1 = 1 \quad (A3.40)$$

$$c_2 = \left[ \frac{1}{\rho} \right]_{\epsilon = 0} + c_0 \Delta, \quad d_2 = -\Delta \quad (A3.41)$$

$$d_3 = \frac{d_0}{2} \left[ \frac{\partial}{\partial \epsilon} \left( K - 2 \frac{I}{\rho c} \right) \right]_{\epsilon = 0} \quad (A3.42)$$
Then, the resistance $R$ is given by

$$\frac{R}{\rho} = -d_3 \left[ \frac{S_{xx}}{\rho} \right]_{\varepsilon=0} - c_0^2 \Delta \eta (0) + o (\varepsilon, \Delta) \quad (A3.43)$$

This result (A3.43) shows that the asymptotic expansion of the resistance for large depth is far from being trivial, even in the case where $\Delta$ is equal to zero. It is not sufficient to know the limit as the depth goes to infinity of $S_{xx}$, $I$ and $K$. The difficulty in the limiting process arises from the fact that the mean horizontal velocities and that the mean water depths upstream and downstream differ by a term of order $1/d$ (when $\Delta = 0$). These terms have to be evaluated in order to obtain the proper asymptotic expansion because the depth of the control volume is of order $d$. This difficulty was not recognized by Duncan (1982) who did not consider these higher order terms and had to introduce a non-zero vertical mass flux at infinite depth to maintain mass conservation - a rather unphysical phenomenon.

The evaluation of $d_3$ in equation (A3.42) is somewhat difficult. Conservation of vertical momentum yields

$$K - 2 \frac{1}{\rho} c = c^2 + \left[ \bar{\phi}_x^2 \right]_{y=-d} \quad (A3.44)$$

where the overbar denotes the average value over a wavelength. If we assume (a result
which appears reasonable but which we were not able to prove, except for the first terms of Stokes' expansion) that the periodic part of the velocity potential at the bottom decrease exponentially with the depth, we can relate \( d_3 \) and \( c_2 \), yielding

\[
d_3 = - c_0 \left( \frac{I}{\rho} + c_0 \Delta \right)
\] (A3.45)

This finally gives in infinite depth

\[
\frac{R}{\rho} = - \frac{S_{xx}}{\rho} + \frac{I}{\rho} c_0 - \Delta \eta (0)
\] (A3.46)

When \( \Delta \) is equal to zero, this result (A3.46) is the same as the one obtained by Duncan (1982), but by a completely different method. We give in table A3.1 the value of the wave resistance associated with the following waves for \( \Delta \) equal to zero, \( R' \), and the value of the elevation of a crest of a following wave above the mean water level, \( \zeta_c \), corresponding to a given value of the total height of the following waves, \( a_p \), as computed from the numerical results of Longuet-Higgins (1975). These values have been used for figures 3.4, 3.13 and 3.14.
\begin{tabular}{ccc}
\hline
$g\alpha/c^2$ & $gR'/\rho c^4$ & $g\zeta/c^2$ \\
\hline
0.278722 & 0.004705 & 0.149547 \\
0.388134 & 0.008809 & 0.214848 \\
0.467686 & 0.012305 & 0.265677 \\
0.530809 & 0.015170 & 0.308823 \\
0.582698 & 0.017391 & 0.346966 \\
0.605302 & 0.018250 & 0.364630 \\
0.625979 & 0.018944 & 0.381529 \\
0.644883 & 0.019468 & 0.397762 \\
0.662175 & 0.019818 & 0.413417 \\
0.677961 & 0.019995 & 0.428575 \\
0.692362 & 0.020001 & 0.443314 \\
0.705545 & 0.019845 & 0.457719 \\
0.717743 & 0.019550 & 0.471883 \\
0.729614 & 0.019195 & 0.485929 \\
0.7431 & 0.0191 & 0.5000 \\
\hline
\end{tabular}

Table A3.1 - Wave Height, Wave Resistance and Elevation of the Crest for Finite Amplitude Waves (from the Results of Longuet-Higgins, 1975)
A4/ DIRECT COMPUTATION OF A STEADY SPILLING BREAKER

A4.1/ INTRODUCTION

A4.1.1/ GENERAL

In this appendix, we show how a direct computation of the breaking wave and the following wave train is possible. We wish to calculate directly the repressive effect of the breaker upon the advancing wave.

We will first describe a simplified model based on the shock analysis. Simple geometrical considerations allow to determine approximatively the wave profile. Then a numerical procedure to calculate the breaking wave and the following wave train is described. The main purpose here is to illustrate how a direct wave calculation is possible. Some simplifying assumptions, which do not limit the generality of the method, are, therefore, introduced. In particular:

- we take the point of view that the calculation of the pseudo potential flow can be made according to linear theory;
- we only consider the case of deep water waves;
- we assume that breaking is somehow stimulated on a particular crest of an advancing wave train, the elevation of which is given by \(a_0/2 \cos (x)\);
- we neglect the effect of the boundary layer and only consider the effect of the primary pressure distribution due to the weight of the breaker.

Finally, we will compare the results of the simplified model and of the full
calculation (performed with the same hypotheses described in the foregoing). This will validate this simplified model which will be used in chapter 4 to study the stability of the equilibrium configurations of the breaker.

A4.1.2/ CONVENTIONS

As in appendix 3, the adimensionalisation is such that $\kappa = 1$ and $g = 1$. Note that, because of the hypotheses made in the foregoing, the dispersion relation yields here $c = 1$. We consider a coordinate system in which the flow is steady. The horizontal $x$-axis is directed in the direction of the current (i.e. opposite to the direction of wave propagation) and its origin is taken at the crest of the breaking wave.

A4.2/ SIMPLIFIED MODEL

We first describe a simplified method to compute the shape of the breaking wave and that of the following waves. Given $a_0$, the height of the incident waves, the shock relation allows us to find the height of the breaker and the height of the following waves. Two solutions are actually possible, corresponding to the strong and weak breaking regime. If linear theory is used and the boundary layer effect neglected, they are given by

- strong breaking:
\[ h^* = \frac{1}{2 (1 + 2 \frac{\rho_c}{\rho})} \sqrt{1 - (1 + 2 \frac{\rho_c}{\rho}) (1 - a_v^2)} \]  \hspace{1cm} (A4.1)

\[ h^* = \frac{1 - \sqrt{1 - (1 + 2 \frac{\rho_c}{\rho}) (1 - a_v^2)}}{2 (1 + 2 \frac{\rho_c}{\rho})} \]  \hspace{1cm} (A4.2)

* weak breaking:

These results (phrased in terms of the wave resistance \( ga_0/c^2 \)) are shown on figure 3.14.

We have assumed here that the crest of the breaking wave and the crest of the following waves have the same height. \( a_f \) is related to \( h^* \) by

\[ a_f = 1 - 2 h^* \]  \hspace{1cm} (A4.3)

An approximate solution for the wave profile is found as follows. We assume that the crest of the wave does not move horizontally, i.e. \( x_b = 0 \). We also assume that the flow is undisturbed upstream of point (a), so that
\[ \zeta(x) = \frac{a_0}{2} \cos(x) \quad \text{for } x < -d \] (A4.4)

We also assume that the flow is established downstream of point (b), so that

\[ \zeta(x) = \frac{a_f}{2} \cos(x) \quad \text{for } x > 0 \] (A4.5)

Since \( h^* \) is equal to \( \zeta_b - \zeta_a \), the length of the breaker, \( d \), is given by

\[ \frac{a_f}{2} - \frac{a_0}{2} \cos(d) = h^* \] (A4.6)

Assuming that the dividing streamline is flat between point (a) and point (b), this totally determines an approximate solution for the free surface elevation of the breaking wave and the following waves.

A4.3/ NUMERICAL SOLUTION

A4.3.1/ THE WAVE CREATED BY AN ADVANCING QUADRATIC PRESSURE DISTRIBUTION

According to linear theory, the wave \( \zeta^* \) due to a pressure distribution moving with constant speed \( c \) is given in deep water by (Lamb, 1932)
\[ \rho \zeta^* = -2 \int_{-\infty}^{x} p(u) \sin(x-u) \, du + \]
\[ + \int_{-\infty}^{\infty} p(u) \int_{0}^{\infty} \frac{m}{m^2 + 1} \exp[-m|x-u|] \, dm \, du \]
\[ (A4.7) \]

We consider here the case where the pressure \( p \) is defined on the interval \([x_a, x_b]\) (i.e. it is equal to zero outside this interval) and is quadratic on this interval. Thus we write:

\[ \frac{p(x)}{\rho} = a_1 + a_2 x + a_3 x^2 \quad \text{for} \quad x \in [x_a, x_b] \]
\[ (A4.8) \]

In this section, we wish to compute the free surface elevation (A4.7) for a quadratic pressure distribution (A4.8).

We first compute the first term on the right hand side of (A4.7), which we denote \( \zeta_p^* \). This yields directly

* for \( x < x_a \):
\[ \zeta_p^*(x) = 0 \]
\[ (A4.9) \]

* for \( x_a < x < x_b \):
\[ \zeta_p^* (x) = - 2 \left\{ (-x_a a_2 - a_1 - (x_a^2 - 2) a_3) \cos (x-x_a) + 
\right. \\
\left. + (a_1 + x a_2 + (x^2 - 2) a_3) - (a_2 + 2 a_3 x_a) \sin (x-x_a) \right\} \quad (A4.10) \]

* for \( x_b < x \):

\[ \zeta_p^* (x) = - 2 \left\{ (-x_a a_2 - a_1 - (x_a^2 - 2) a_3) \cos (x-x_a) + 
\right. \\
\left. + (a_1 + x_b a_2 + (x_b^2 - 2) a_3) \cos (x-x_b) - (a_2 + 2 a_3 x_b) \sin (x-x_b) \right\} \quad (A4.11) \]

We now compute the second term on the right hand side, which we denote \( \zeta_t^* \).

We define two functions \( f \) and \( g \) by

\[ f (x) = \int_0^{\infty} \frac{e^{- x t}}{t^2 + 1} \, dt \quad (x > 0) \quad (A4.12) \]

\[ g (x) = \int_0^{\infty} \frac{t e^{- x t}}{t^2 + 1} \, dt \quad (x > 0) \quad (A4.13) \]

\( f \) and \( g \) are closely related to the Sine and Cosine integrals (see Abramowitz & Stegun (1972), p. 232) and can easily be evaluated numerically.

Using integrations by parts, \( \zeta_t^* \) can be expressed in terms of \( f \) and \( g \). We denote
\[ f_a = f(|x-x_a|), \quad f_b = f(|x-x_b|), \quad g_a = g(|x-x_a|), \quad g_b = g(|x-x_b|) \]
\[ l_a = \ln(|x-x_a|), \quad l_b = \ln(|x-x_b|) \]

\[ \pi \zeta_t^* (x) = a_1 \{ f_a - f_b \} + a_2 \{ x f_a - x f_b - l_a + l_b - g_a + g_b \} + \]
\[ + a_3 \{(2-x_b) f_b - (2-x_a) f_a + 2 x_b g_b - 2 x_a g_a + 2 x (l_b - l_a) + 2 (x_b-x_a)\} \]

\[ \pi \zeta_l^* (x) = a_1 \{ \pi f_a - f_b \} + a_2 \{ \pi x - x f_a - x f_b - l_a - g_a + g_b \} + \]
\[ + a_3 \{(2-x_b) f_b + (2-x_a) f_a + \pi (x^2 - 2) + 2 x_b g_b - 2 x_a g_a + \]
\[ + 2 x (l_b - l_a) + 2 (x_b-x_a)\} \]

\[ \pi \zeta_b^* (x) = a_1 \{ f_b - f_a \} + a_2 \{ x_b f_b - x_a f_a + l_b - l_a + g_b - g_a \} + \]
\[ + a_3 \{(2-x_a) f_a - (2-x_b) f_b + 2 x_b g_b - 2 x_a g_a + 2 x (l_b - l_a) + 2 (x_b-x_a)\} \]

It can be checked that the elevation \( \zeta^* (x) \) is continuous at \( x_a \) and \( x_b \).
A4.3.2/ Determination of the Pressure Distribution

We will consider here only the primary pressure distribution due to the breaker. This pressure distribution has to satisfy the following conditions:

- the breaker is flat-topped;
- the height of the breaker is related to the height of the crest by equation (3.2.5);
- point (b) is a crest of the resulting wave.

In order to solve numerically these equations, we assume that the pressure distribution is quadratic. The density of the water within the breaker is assumed to be known ($\rho_w/\rho$ is taken as a parameter). If the pressure distribution is known, the free surface elevation is given by $\zeta = \zeta_p + \zeta_l + a_0/2 \cos(x)$.

$x_a$ and $x_b$ being given, the conditions

\[
p(x_a) = \rho_w h^* \quad (A4.18)
\]

\[
p(x_b) = 0 \quad (A4.19)
\]

allow to express $a_1$ and $a_2$ in terms of $x_a$, $x_b$, $h^*$, and $a_3$. Then $h^*$ and $a_3$ are found by iterations so that they satisfy

\[
h^* = \zeta(x_b) - \zeta(x_a) \quad (A4.20)
\]
\[ \zeta \left( \frac{x_a + x_b}{2} \right) = \zeta (x_b) \]  
\hspace{1cm} \text{(A4.21)}

the last equation corresponding numerically to the condition for the breaker to be flat-topped (of course, better approximations could be made using higher order polynomials to approximate the pressure).

Then \( x_a \) and \( x_b \) have to be found so that the two following conditions are satisfied:

\[ h^* + \zeta (x_b) - 0.5 = 0 \]  
\hspace{1cm} \text{(A4.22)}

\[ \left[ \frac{d\zeta}{dx} \right]_{x = x_b} = 0 \]  
\hspace{1cm} \text{(A4.23)}

Equation (A4.23) expresses that the point (b) is a crest of the resulting breaking wave. \( x_a \) and \( x_b \) are found numerically by minimizing (in the least square sense) the magnitude of the left hand sides of equations (A4.22) and (A4.23). Initial values of \( x_a \) and \( x_b \) are given by the simplified model described above. The conjugate-direction method of Fletcher & Reeves is used in order to find a direction search. Then the Golden Ratio method is used for the one-dimensional optimization (for a more detailed description of these methods, see for instance Vanderplaats (1984)). A solution is found within a few
iterations.

A4.4/ COMPARISON OF THE SIMPLIFIED MODEL AND THE NUMERICAL CALCULATION

The simplified described in A4.2 and the direct numerical calculation of the wave described in A4.3 have been compared and this comparison is summarized in table A4.1. The two computations agree very well for the height of the breaker (the maximum difference is approximatively equal to 3%). Note that the height of the breaker is computed in the simplified analysis using the shock relations derived in chapter 3 and that, therefore, the shock relations agree very well with a direct computation of the wave.

The agreement is also very good for the abscissa of the leading edge of the breaker, $x_a$. The difference is less than 4% of the length of the breaker. This justifies well the assumption that the flow upstream of the leading edge of the breaker is almost not disturbed. Not surprisingly, the agreement is less good for the abscissa of the crest of the breaking wave, $x_b$, which was crudely estimated to be equal to zero in the simplified analysis. Note however that the difference is less than 20% of the length of the breaker, so that the approximation is rather fair.
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<th>Solution</th>
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<th>$g_{xq}/c^2$</th>
<th>$g_{xq}/c^2$</th>
<th>$gh^*/c^2$</th>
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<td>-0.22</td>
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Table A4.1 - Comparison of the Simplified Model and the Numerical Calculation
A5/ FREE MODES OF THE BREAKER

A5.1/ INTRODUCTION

In this appendix, we determine analytically the free modes of the wave equation derived to model the breaker near equilibrium in chapter 4. We use separation of variables to express the modes as the product of a function of time and a Bessel-like shape function in space. Two different families of modes are found which are distinguished by their behavior in time: stable (oscillatory) modes and unstable (exponential) modes. These two families correspond respectively to stable and unstable gravity currents. Since boundary conditions have to be prescribed to specify the possible modes in a specific case, it appears that the boundary condition at the leading edge of the breaker is crucial for the stability of the breaker and the determination of its natural frequency of oscillation. The simplified boundary condition derived in chapter 4 is used here and the corresponding solutions are studied. According to this analysis, the stability of the breaker is determined by the sign of the parameter $\Gamma$ introduced in chapter 4. The equation giving the natural frequencies of stable breakers is derived.

A5.2/ SOLUTION OF THE FIELD EQUATION

A5.2.1/ FORMULATION OF THE PROBLEM

The equation we wish to solve here is the homogenous equation corresponding to equation (4.3.4):
\[
\frac{\partial^2 \phi}{\partial t^2} - g \frac{\partial}{\partial X} \left( \xi \frac{\partial \phi}{\partial X} \right) + 2\omega_1 \frac{\partial \phi}{\partial t} + \Sigma g \frac{\partial \phi}{\partial X} = 0 \quad (A.5.1)
\]

We will moreover assume here that the line (a)-(b) is straight so that

\[
\xi = X \tan \theta \quad (A.5.2)
\]

Finally, in order to obtain a separable solution, we will assume that the slope \( \theta \) is not a function of time. In this case, equation (A.5.1) is equivalent to the system

\[
\frac{\partial \phi}{\partial t} + \frac{\partial (\xi u_e)}{\partial X} = \Sigma u_e \quad (A.5.3)
\]

\[
g \frac{\partial \phi}{\partial X} + \frac{\partial u_e}{\partial t} = -2\omega_1 u_e \quad (A.5.4)
\]

In this system, \( \phi \) represents the disturbed height of the breaker and \( u_e \) the horizontal velocity within the breaker.

In order to solve equation (A.5.1) (or, equivalently, the system (A.5.3)-(A.5.4)), we use separation of variables and we look for a solution expressed as

\[
\phi(X,t) = \bar{\phi}(X) \hat{\phi}(t) \quad (A.5.5)
\]
Under this assumption, equation (A5.1) yields, with \( \chi \) being as yet an arbitrary constant,

\[
\frac{d^2 \hat{\phi}}{dt^2} + 2 \omega_1 \frac{d \hat{\phi}}{dt} = -\chi \hat{\phi}
\]

(A5.6)

\[
X \tan \theta \frac{d^2 \hat{\phi}}{dX^2} + (\tan \theta - \Sigma) \frac{d \hat{\phi}}{dX} = -\frac{\chi}{g} \hat{\phi}
\]

(A5.7)

Solutions of these equations can be computed explicitly. Two cases appear, depending upon the sign of the constant \( \chi \).

### A5.2.2/ STABLE MODES

We first consider the case \( \chi > 0 \) and we define \( \omega_0 = \chi^{1/2} \) and \( \Omega^2 = \omega_0^2 - \omega_1^2 \).

The solutions of equation (A5.6) are then given for \( \Omega^2 \) positive (i.e. when the solution is not overdamped) by

\[
\hat{\phi}_1 = \cos (\Omega t) \exp (-\omega_1 t) \quad ; \quad \hat{\phi}_2 = \sin (\Omega t) \exp (-\omega_1 t)
\]

(A5.8)

This equation shows that, in this case, the breaker will oscillate at the frequency \( \Omega \) and that the amplitude of the oscillations will decay in time. The effect of the entrainment of momentum, modeled by the parameter \( \omega_1 \), is to "stabilize" the breaker.

With the change of variable \( Y^2 = X \), equation (A5.7) becomes
\[ \frac{d^2 \overline{\theta}}{dY^2} - (2 \frac{\Sigma}{\tan \theta} - 1) \frac{d \overline{\theta}}{dY} + \frac{4 \omega_0^2}{g \tan \theta} \overline{\theta} = 0 \]  

(A5.9)

The solutions of (A5.9) are classically given in terms of Bessel functions (see for instance Abramowitz & Stegun, 1972, p. 362) and are expressed in terms of the two parameters

\[ \nu \equiv \frac{\Sigma}{\tan \theta} \geq 0 ; \quad \kappa \equiv \frac{4 \omega_0^2}{g \tan \theta} \]  

(A5.10)

With these notations, the solutions of (A5.7) are finally given by

\[ \overline{\theta}_1 = X^2 J_\nu \left( \sqrt{\kappa X} \right) ; \quad \overline{\theta}_2 = X^2 Y_\nu \left( \sqrt{\kappa X} \right) \]  

(A5.11)

\( J_\nu \) and \( Y_\nu \) are Bessel functions of the first and second kind, respectively. They can easily be evaluated numerically. The case where \( \nu \) is equal to zero corresponds to shallow water waves in a canal of depth increasing linearly from the origin. This is not surprising, since in this case both tangential stress on the bottom and entrainment are neglected. This problem is described in details by Lamb (1945), Art. 186, and will not be rediscussed here. We will limit ourselves to the case where \( \nu > 0 \).
In the case where $\chi = 0$, it is easy to check that one also obtain a stable 
(exponentially decaying) mode which is proportional to 

$$\hat{\vartheta}_0 = X^v \exp(-2\omega_1 t) \quad \text{(A5.12)}$$

This mode, which is neither oscillatory nor unstable, will not be further discussed here.

**A5.2.3/ UNSTABLE MODES**

We now consider the case $\chi < 0$. We define $\omega_0 = (-\chi)^{1/2}$ and $\Omega^2 = \omega_0^2 + \omega_1^2$.

The solutions of equation (A5.6) are now given by

$$\hat{\vartheta}_1 = \exp((\Omega - \omega_1) t) \quad \hat{\vartheta}_2 = \exp(-(\Omega + \omega_1) t) \quad \text{(A5.13)}$$

This equation shows that in this case the breaker is not oscillating but growing or decaying exponentially. The existence of such a solution will therefore imply that the breaker is unstable.

Solutions of equation (A5.7) are now given by

$$\hat{\vartheta}_1 = X^v I_v (\sqrt{\kappa X}) \quad \hat{\vartheta}_2 = X^v K_v (\sqrt{\kappa X}) \quad \text{(A5.14)}$$

where $v$ and $\kappa$ are given by equation (A5.10). $I_v$ and $K_v$ are modified Bessel functions.
of the first and second kind, respectively.

A5.2.4/ CONCLUSION

In this section, we have found separable solution of equation (A5.1). For a separable mode given by equation (A5.5), different from \( \tilde{\theta}_0 \), the velocity \( u_e \) is given by

\[
u_e = \frac{g}{\chi} \frac{d\hat{\theta}}{dt} \frac{d\hat{\theta}}{dX}
\]  

(A5.15)

so that the solutions found are also solutions of the system (A5.3)-(A5.4). Both stable (oscillatory) and unstable (exponential) modes have been found. The existence of stable and/or unstable solutions and the determination of the natural frequencies of the system require now that the boundary conditions be specified.

A5.3/ BOUNDARY CONDITIONS AND NATURAL FREQUENCIES

A5.3.1/ BOUNDARY CONDITIONS

At the crest of the breaking wave \( X = 0 \), we assume that the height of the breaker is equal to zero and we, therefore, write

\[
\tilde{\theta} (0, t) = 0
\]  

(A5.16)

The behavior near the origin of the modes found in the foregoing can easily be obtained using an asymptotic expansion of the Bessel functions for small argument. For both oscillatory and exponential modes, we obtain
\[ \bar{\vartheta}_1 \sim X^v \text{ as } X \to 0^+ ; \quad \bar{\vartheta}_2 \sim \text{constant as } X \to 0^+ \quad (A5.17) \]

Since \( v \) is assumed to be greater than 0, it appears that only the first mode satisfies this boundary condition (the mode \( \vartheta_0 \) will not be considered).

In chapter 4, a simplified boundary condition was derived, valid when there is no concentrated entrainment of momentum at the leading edge of the breaker. This boundary condition is expressed as (for simplicity, we take \( \Sigma^* = 0 \))

\[ \forall t, \quad \frac{\partial \varrho}{\partial t}(D, t) = \Gamma u_e(D, t) \quad (A5.18) \]

where \( D \) is the length of the breaker at equilibrium and \( \Gamma \) is a constant which can be computed from the equilibrium configuration of the breaker.

We, therefore, finally get the leading edge boundary condition for a separable mode

\[ \frac{\bar{\vartheta}}{\chi} \frac{d \bar{\vartheta}}{dX}(D) = 0 \quad (A5.19) \]

Note that it was possible to derive this boundary condition because equation (A5.18) is compatible with the assumption that there exists a separable solution. This would not have been the case if a concentrated entrainment of momentum had existed at the
leading edge.

In order to perform the stability analysis and to find the natural frequencies of the breaker, we will assume (without any proof) that the set of separable solutions of equation (A5.1) determined in the foregoing satisfying the boundary conditions (A5.16) and (A5.19) is complete in the appropriate space of the solutions of the boundary value problem (A5.1), (A5.16) & (A5.19). In other words, we will assume that any regular enough solution of this boundary value problem can be expressed as the sum of the separable solutions we have found.

A5.3.2/ Stability Analysis

An unstable mode (A5.14) will be solution of the field equation (A5.1) with the boundary conditions (A5.16) and (A5.19) if there exists at least one $\omega_0$ positive such that

$$-\ddot{\vartheta}(D) + \frac{g}{\omega_0^2} \Gamma \frac{d\dot{\vartheta}}{dx}(D) = 0$$

(A5.20)

with

$$\ddot{\vartheta} = X^2 \lambda_v (\sqrt{\kappa X}) ; \quad \kappa = \frac{4 \omega_0^2}{g \tan \theta}$$

(A5.21)

Clearly, both the shape function and its derivative are positive. Therefore

* if $\Gamma$ is positive, equation (A5.20) cannot have a solution, the unstable modes
are not solution of the boundary value problem and the breaker is stable;

On the other hand, we were not able to show analytically but we verified numerically that

- if $\Gamma$ is negative, equation (A5.20) has at least a solution, there exists at least an unstable mode solution of the boundary value problem and the breaker is unstable.

A5.3.3/ NATURAL FREQUENCIES

Similarly, a stable mode (A5.11) will be solution of the field equation (A5.1) with the boundary conditions (A5.16) and (A5.19) if there exists $\omega_0$ positive such that

$$ -\bar{\theta} (D) - \frac{g}{\omega_0^2} \Gamma \frac{d\bar{\theta}}{dX} (D) = 0 $$  \hspace{1cm} (A5.22)

with

$$ -\bar{\theta} = X^2 J_\nu (\sqrt{\kappa X}) ; \quad \kappa = \frac{4 \omega_0^2}{g \tan \theta} $$  \hspace{1cm} (A5.23)

This equation always admit solutions. The first mode is associated with the smallest value of $\omega_0$. The corresponding natural frequency, $\Omega$, is given by (we assume that the solution is not overdamped)

$$ \Omega = \sqrt{\omega_0^2 - \omega_1^2} $$  \hspace{1cm} (A5.24)

Equations (A5.22) to (A5.24) therefore allow to find the natural frequencies of
the breaker. They can easily be solved numerically using, for instance, Newton's method.
A6/ NUMERICAL SCHEME FOR THE BREAKER DYNAMICS

A6.1/ INTRODUCTION

A6.1.1/ GENERAL

In this appendix, we derive discrete equations to approximate the hyperbolic nonlinear system

\[ \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = \alpha \]  \hspace{1cm} (A6.1)

\[ \frac{\partial u}{\partial t} + \frac{\partial h}{\partial x} + u \frac{\partial u}{\partial x} = \beta \]  \hspace{1cm} (A6.2)

with the boundary conditions

\[ h(0) = h_0 \]  \hspace{1cm} (A6.3)

\[ h(d) = h_d \]  \hspace{1cm} (A6.4)

The equations which govern the dynamics of the breaker are derived in chapter 4. With a proper choice of \( \alpha \) and \( \beta \) and an appropriate scaling of the \( x \)-coordinate, they can be expressed under the form (A6.1)-(A6.4). In these equations, \( x \) corresponds to the horizontal distance from the crest of the wave, \( t \) to the time, \( h(x, t) \) to the height of
the breaker and \( u(x, t) \) to the horizontal velocity within the breaker (for convenience, the subscript \( e \) used in chapter 4 for this velocity is omitted in this appendix).

A6.1.2/ CONVENTIONS

Non-dimensional variables are defined so that \( g = 1, c = 1 \) and (therefore) \( \kappa = 1 \). The two-dimensional \( x-t \) space is discretized and constant spacing is used on the \( x \) and \( t \) axes (note, however, that only the constant \( x \)-spacing is essential to the calculation). The space and time steps are denoted by \( \Delta x \) and \( \Delta t \). The following notations will be used for the values of the unknowns at the nodes of the grid:

\[
h_i^n = h(i \Delta x, n \Delta t) \quad (A6.5)
\]

The same notation will also be used for points located between the nodes.

Finite difference equations for the unknowns at the node will be derived so that the governing equations are satisfied on the average over control volumes - see for instance Lick & Gaskins (1984).

A6.2/ DISCRETE EQUATIONS

A6.2.1/ FIELD EQUATIONS

In order to obtain discretized equations corresponding to (A6.1) and (A6.2), we consider the control volume \([i-1/2, i+1/2] \otimes [n, n+1]\). The choice of this control volume stems from the hyperbolic nature of the system and the fact that an explicit
scheme is sought. The procedure is somehow lengthy but straightforward.

We first integrate equation (A6.1) over the control volume. This yields the exact expression

$$\int_{i-1/2}^{i+1/2} [h^{n+1} - h^n] \, dx + \int_n^{n+1} [(hu)_{i+1/2}^{-} - (hu)_{i-1/2}^{-}] \, dt = \iint \alpha \, dx \, dt$$  \hspace{1cm} (A6.6)

In order to obtain an equation involving the values of the unknowns at the nodes, the three integrals appearing in equation (A6.6) have now to approximated. Approximations valid up to second order are sought.

The first integral on the LHS is estimated as

$$\int_{i-1/2}^{i+1/2} [h^{n+1} - h^n] \, dx = (h_i^{n+1} - h_i^n) \Delta x$$  \hspace{1cm} (A6.7)

Note that this expression is valid up to second order because of the constant x-spacing of the nodes (i.e. $x_i$ is at the middle of $[x_{i-1/2}, x_{i+1/2}]$).

Similarly, the second integral on the LHS is written as
\[
\int_{n}^{n+1} ([hu]_{i+1/2} - [hu]_{i-1/2}) \, dt = \left[(hu)_{i+1}^{n} - (hu)_{i-1}^{n}\right] \frac{\Delta t}{2} + \\
+ \left[\frac{\partial hu}{\partial t} \right]_{i+1/2}^{n} - \left[\frac{\partial hu}{\partial t} \right]_{i-1/2}^{n} \frac{\Delta t^2}{2} \tag{A6.8}
\]

In order to evaluate the time derivative of \( hu \), we use equations (A6.1) and (A6.2) which yield

\[
\frac{\partial hu}{\partial t} = \alpha u + \beta h - \frac{\partial hu^2}{\partial x} - \frac{1}{2} \frac{\partial h^2}{\partial x} \tag{A6.9}
\]

Equation (A6.9) allows to express equation (A6.8) as

\[
\int_{n}^{n+1} ([hu]_{i+1/2} - [hu]_{i-1/2}) \, dt = \left[(hu)_{i+1}^{n} - (hu)_{i-1}^{n}\right] \frac{\Delta t}{2} + \\
+ \left[\left(\alpha u\right)_{i+1}^{n} - \left(\alpha u\right)_{i-1}^{n} + \left(\beta h\right)_{i+1}^{n} - \left(\beta h\right)_{i-1}^{n}\right] \frac{\Delta t^2}{4} - \\
- \left[(hu)_{i+1}^{2n} + (hu)_{i-1}^{2n} + \frac{1}{2} \left\{ (h^2)_{i+1}^{2n} - 2(h^2)_{i}^{2n} + (h^2)_{i-1}^{2n} \right\}\right] \frac{\Delta t^2}{2\Delta x} \tag{A6.10}
\]

Note that by using equation (A6.2) to evaluate the time derivative of \( hu \), we were able to obtain an explicit second-order estimate of the integral (A6.8).
Finally, the integral on the RHS of equation (A6.6) is evaluated as

\[
\int_{i-1/2}^{i+1/2} \int_{n}^{n+1} \alpha_i \, dx \, dt = \alpha_i^n \, \Delta x \, \Delta t + \left( \frac{\partial \alpha_i^n}{\partial t} \right)_i \, \Delta x \, \frac{\Delta t^2}{2}
\]

(A6.11)

Equations (A6.7), (A6.10) and (A6.11) can be plugged into equation (A6.6) to obtain the discrete approximation of (A6.1). The same method is used to approximate equation (A6.2). The algebra being carried out, discretized forms of equations (A6.1) and (A6.2) can finally be expressed as

\[
h_i^{n+1} = h_i^n - \left( (hu)_{i+1}^n - (hu)_{i-1}^n \right) \frac{\Delta t}{2\Delta x} + \left[ (h^2)_{i+1}^n - 2(h^2)_{i}^n + (h^2)_{i-1}^n \right] \frac{\Delta t^2}{4\Delta x} +
\]

\[
+ \left[ (hu^2)_{i+1}^n - 2(hu^2)_{i}^n + (hu^2)_{i-1}^n \right] \frac{\Delta t^2}{2\Delta x^2} - \left[ (\alpha u)^n_{i+1} - (\alpha u)^n_{i-1} \right] \frac{\Delta t^2}{4\Delta x} - \left[ (\beta h)_{i+1}^n - (\beta h)_{i-1}^n \right] \frac{\Delta t^2}{4\Delta x} + \alpha_i^n \, \Delta t + \left( \frac{\partial \alpha_i^n}{\partial t} \right)_i^n \frac{\Delta t^2}{2}
\]

(A6.12)

\[
u_i^{n+1} = u_i^n - \left( h_i^n - h_i^{n-1} \right) \frac{\Delta t}{2\Delta x} - \left[ (u^2)_{i+1}^n - (u^2)_{i-1}^n \right] \frac{\Delta t}{4\Delta x} +
\]

\[
+ \left[ 3h_{i+1}^n u_{i+1}^n + h_{i+1}^n u_{i+1}^n - h_i^n u_{i+1}^n - 6h_{i+1}^n u_{i+1}^n + h_{i+1}^n u_{i+1}^n - h_i^n u_{i+1}^n + 3h_{i+1}^n u_{i+1}^n \right] \frac{\Delta t^2}{4\Delta x^2} +
\]

\[
+ \left[ (u^3)_{i+1}^n - 2(u^3)_{i}^n + (u^3)_{i-1}^n \right] \frac{\Delta t^2}{6\Delta x^2} - \left[ (\alpha u_i^n)_{i+1} - (\alpha u_i^n)_{i-1} \right] \frac{\Delta t^2}{4\Delta x} -
\]
\[ - [(\beta u)^n_{i+1} - (\beta u)^n_{i-1}] \frac{\Delta t^2}{4\Delta x} + \beta^n_i \Delta t + \left( \frac{\partial \beta}{\partial t} \right)^n_i \frac{\Delta t^2}{2} \]  

(A6.13)

**A6.2.2/ BOUNDARY CONDITIONS**

In order to derive discretized boundary conditions, the same method is used, but with a special control volume. The calculation is performed here for \( x = d \) or \( i = i_{\text{max}} = I \). Of course, the boundary condition for \( x = 0 \) can be derived in a similar way. The value of \( h \) at \( x = d \) is assumed known, and given by \( h(d) = h_I = h_d \).

Integration of equation (A6.2) over the control volume \([I-1/2, I] \otimes [n, n+1]\) yields:

\[
\int_{I-1/2}^{I} [u^{n+1} - u^n] \, dx + \int_{n}^{n+1} [h_I - h_{I-1/2}] \, dt + \frac{1}{2} \int_{n}^{n+1} [(u^2)^{I-1} - (u^2)_{I-1/2}] \, dt = \\
\int_{I-1/2}^{I} \int_{n}^{n+1} \beta \, dx \, dt \quad \text{(A6.14)}
\]

In order to evaluate the first integral on the LHS, we expand \( u \) as

\[
u (x) = u_I + \left( \frac{\partial u}{\partial x} \right)_{I-1/2} (x-x_I) + o (\Delta x) \quad \text{(A6.15)}
\]

This yields
\[
\int_{1-1/2}^{1} [u^{n+1} - u^n] \, dx = [u_t^{n+1} - u_t^n] \frac{\Delta x}{2} - [u_t^{n+1} - u_{t,1-1/2}^{n+1} - u_t^n + u_{t,1-1/2}^n] \frac{\Delta x}{8} \tag{A6.16}
\]

Similarly, the second integral on the LHS is estimated using Taylor's series expansions as

\[
\int_{n}^{n+1} [h_t - h_{t,1/2}] \, dt = (\frac{\partial h}{\partial x})_{t,1-1/2}^n \frac{\Delta x}{2} \Delta t + (\frac{\partial^2 h}{\partial x^2})_{t,1-1/2}^n \frac{\Delta x^2}{8} \Delta t + \frac{\partial^2 h}{\partial x \partial t}_{t,1-1/2}^n \frac{\Delta x}{4} \Delta t^2 \tag{A6.17}
\]

The third integral on the LHS is given by a similar expression

\[
\frac{1}{2} \int_{n}^{n+1} [(u_t^2)_{t,1-1/2} - (u_t^n)_{t,1-1/2}] \, dx = (\frac{\partial u^2}{\partial x})_{t,1-1/2}^n \frac{\Delta x}{4} \Delta t + (\frac{\partial^2 u^2}{\partial x^2})_{t,1-1/2}^n \frac{\Delta x^2}{16} \Delta t + \frac{\partial^2 u}{\partial x \partial t}_{t,1-1/2}^n \frac{\Delta x}{8} \Delta t^2 \tag{A6.18}
\]

By using equation (A6.2), we get

\[
\frac{\partial^2 h}{\partial x^2} = \frac{\partial \beta}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \tag{A6.19}
\]
Equation (A6.19) allows to eliminate the $\partial/\partial x^2$ terms from the LHS of equation (A6.14). The other terms ($\partial/\partial x$ and $\partial^2/\partial x \partial t$) are evaluated by the difference equations

\[
\left( \frac{\partial {\cdot}}{\partial x} \right)^n_{I-1/2} = \left[ \cdot \right]_I^n - \left[ \cdot \right]_{I-1}^n \frac{1}{\Delta x} \tag{A6.20}
\]

\[
\left( \frac{\partial^2 {\cdot}}{\partial x \partial t} \right)^n_{I-1/2} = \left[ \cdot \right]_{I}^{n+1} - \left[ \cdot \right]_{I-1}^{n+1} - \left[ \cdot \right]_I^n - \left[ \cdot \right]_{I-1}^n \frac{1}{\Delta x \Delta t} \tag{A6.21}
\]

Note that equation (A6.21) is not implicit, because the values at $n+1$ are given by the boundary equations or the solution of the field equations.

The integral of the RHS is also evaluated using Taylor's series expansion, yielding

\[
\int_{I-1/2}^{I} \int_{n}^{n+1} \beta \ dx \ dt = \beta^n_{I-1/2} \Delta x \Delta t + \frac{\partial \beta^n_{I-1/2}}{\partial x} \frac{\Delta x^2}{8} \Delta t + \frac{\partial \beta^n_{I-1/2}}{\partial t} \Delta x \frac{\Delta t^2}{4} \tag{A6.22}
\]

When all the algebra is carried out, the boundary condition can finally be expressed as
\( \begin{align*}
\{1 + \frac{\Delta t}{2\Delta x} u^n_i \} u^{n+1}_i &= u^n_i - \{1 - \frac{\Delta t}{2\Delta x} u^n_{i-1} \} u^{n+1}_{i-1} + u^n_{i-1} - \\
- [h^n_i - h^n_{i-1} + h^{n+1}_{i-1} - h^{n+1}_{i-1}] \frac{\Delta t}{\Delta x} + 2 \beta^n_{i-1/2} \Delta t + \left(\frac{\partial \beta}{\partial t}\right)_{i-1/2} \Delta t^2
\end{align*} \) (A6.23)

\[ \vartheta_0 = 3.3 \times 10^{-4} \frac{d \tan \theta}{2} \]

Figure A6.1 - Nonlinear Computation vs. Linear Analytic Solution

A6.3/ IMPLEMENTATION

This numerical scheme, as well as the other numerical algorithms described in this thesis, have been written in Pascal and implemented on a Macintosh microcomputer using the Macintosh toolbox and the "user friendly" interface.
In order to check the algorithm described in the foregoing, we took $\alpha = 0$ and $\beta = \tan \theta$ (a constant). The boundary conditions were taken as $h(0) = 0$ and $h(d) = dt \tan \theta$. This special choice allows a comparison with the analytic solution derived in appendix 5 (with $\Gamma = 0$). An example is shown on figure A6.1, where $d = 1.33$ and $\tan \theta = 0.17$. 20 segments are used to discretize the breaker and the time step is equal to 0.05 c/g. The initial velocity is equal to zero and the initial height of the breaker is given by $xt \tan \theta$ plus a perturbation corresponding to the first mode of the analytic solution. The height of the breaker at $x = d/2$ is plotted as a function of time. As can be seen, the numerical solution is slightly damped, but the estimate of the period of oscillation is very good.
BIBLIOGRAPHY


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