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The Gelfand-Zeitlin Algebra and Polarizations of Regular Adjoint Orbits for Classical Groups

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Mark Colarusso

Committee in charge:

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2007
The dissertation of Mark Colarusso is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

2007
To Vince E. Fazari (1982-2004)
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The main results of this thesis describe and construct polarizations of regular adjoint orbits for certain classical groups. The thesis generalizes recent work of Bertram Kostant and Nolan Wallach (see [KW]). Kostant and Wallach construct polarizations of regular adjoint orbits for $n \times n$ complex matrices $M(n)$. ($X \in M(n)$ is said to be regular if it has a cyclic vector in $\mathbb{C}^n$.) They accomplish this by defining an $\frac{n(n-1)}{2}$ dimensional abelian complex Lie group $A$ that acts on $M(n)$ and stabilizes adjoint orbits. We study the $A$ orbit structure on $M(n)$ and generalize the construction to complex orthogonal Lie algebras $\mathfrak{so}(n)$. Since the $A$ orbits of dimension $\frac{n(n-1)}{2}$ contained in a given regular adjoint orbit form the leaves of a polarization of an open submanifold of that orbit, we then obtain descriptions of these polarizations. For $\mathfrak{so}(n)$ we construct polarizations of certain regular semi-simple orbits.

In the case of $M(n)$, we study the action of $A$ on matrices whose principal $i \times i$ submatrices in the top left hand corner ($i \times i$ cutoffs) are regular. We determine the $A$ orbit structure of Zariski closed subvarieties of these matrices defined by each cutoff having a fixed characteristic polynomial. We relate the number of $A$ orbits of maximal dimension $\frac{n(n-1)}{2}$ in such closed sets to the number of eigenvalues shared by adjacent cutoffs. The number of such $A$ orbits always turns out to be a power of 2. We are also able to describe all of the $A$ orbits of maximal dimension as orbits
of Zariski connected algebraic groups acting on certain quasi-affine subvarieties of $M(n)$. This work gives an explicit description of the leaves of polarizations of all regular adjoint orbits in $M(n)$.

In the case of the orthogonal Lie algebras, we construct a completely integrable system of commuting Hamiltonian vector fields on $\mathfrak{so}(n)$ using a classical analogue of the Gelfand-Zeitlin algebra of the universal enveloping algebra of $\mathfrak{so}(n)$ in the algebra of polynomials on $\mathfrak{so}(n)$. Integrating these vector fields gives rise to an action of $\mathbb{C}^{d/2}$, where $d$ is the dimension of a regular adjoint orbit in an orthogonal Lie algebra. The action of $\mathbb{C}^{d/2}$ stabilizes adjoint orbits. We then use the orbits of $\mathbb{C}^{d/2}$ to construct polarizations of certain regular semi-simple adjoint orbits in $\mathfrak{so}(n)$. We realize the leaves of these polarizations as orbits of a Zariski connected algebraic group.
1 Background

1.1 Introduction

In recent work Bertram Kostant and Nolan Wallach have revisited Gelfand-Zeitlin theory from the perspective of classical mechanics see [KW], [KW1]. In doing so they constructed polarizations of all regular adjoint orbits in the $n \times n$ complex matrices, $M(n)$. They accomplished this by constructing an $\frac{n(n-1)}{2} = \binom{n}{2}$ dimensional abelian, simply-connected, complex Lie group, denoted by $A$, that acts on $M(n)$ and stabilizes adjoint orbits. The $A$ orbits of dimension $\frac{n(n-1)}{2}$ contained in a given regular adjoint orbit then form the leaves of a polarization of an open submanifold of that orbit.

The group $A$ is constructed by integrating certain commuting vector fields derived from Gelfand-Zeitlin theory. The associative commutator on the universal enveloping algebra of $M(n)$, $U(n)$, gives $GrU(n)$ the structure of a Poisson algebra. Using the trace form, the Poisson structure on $GrU(n)$ can be carried over to the polynomials on $M(n)$, $P(n)$, making $P(n)$ into a Poisson algebra. If $m \leq n$, we can think of $M(m)$ as Lie a subalgebra of $M(n)$ by embedding it in the $m \times m$ upper left hand corner of an $n \times n$ matrix. Using the fact that the restriction of the trace form to $M(m)$ is non-degenerate, we can think of the polynomials on $M(m)$, $P(m)$, as a Poisson subalgebra of $P(n)$. Given an $X \in M$, we refer to the $i \times i$ submatrix in the top left hand corner as the $i \times i$ cutoff and denote it by $X_i$. We have a classical analogue to the Gelfand-Zeitlin algebra of $U(n)$ in $P(n)$, namely

$$J(M(n)) = P(1)^{G_1} \otimes \cdots \otimes P(n)^{G_n}$$

where $G_i = GL(i)$. $J(M(n))$ is a Poisson commutative subalgebra of $P(n)$. If we
let $f_{i,j}(X) = \text{tr}(X^j_i)$, then using the Poisson structure on $P(n)$, we can define a Hamiltonian, algebraic vector field $\xi_{f_{i,j}}$ on $M(n)$. $\xi_{f_{i,j}}$ is tangent to the adjoint orbits in $M(n)$ and agrees with the usual symplectic Hamiltonian vector field generated by $f_{i,j}$ on the orbit. Kostant and Wallach then show that $\xi_{f_{i,j}}$ integrates to a global action of $\mathbb{C}$ on $M(n)$ given by a rather simple formula:

$$\text{Ad} \left( \begin{bmatrix} \exp(t \cdot X^j_i) & 0 \\ 0 & \text{Id}_{n-i} \end{bmatrix} \right) \cdot X$$

(1.1)

for $t \in \mathbb{C}$. Since $J(M(n))$ is Poisson commutative, the vector fields $\xi_{f_{i,j}}$ for $1 \leq i \leq n - 1, 1 \leq j \leq i$ all commute and thus their flows in (1.1) commute, giving rise to an action of a group $A \cong \mathbb{C}^{\binom{n}{2}}$ on $M(n)$. We note that the dimension of this group is exactly half of the dimension of a regular adjoint orbit of $GL(n)$ in $M(n)$. Kostant and Wallach show that the set of $A$ orbits of maximal dimension $\binom{n}{2}$ is a non-empty Zariski open subset of $M(n)$ and that a regular adjoint orbit always contains $A$ orbits of maximal dimension. This fact allows one to construct polarizations of regular adjoint orbits of $M(n)$.

In [KW], the authors study the $A$ action on a special set of regular semi-simple matrices, denoted by $\Omega_n$. $\Omega_n$ is defined to be the set of matrices for which each cutoff $X_i$ is regular semi-simple and with the property that two adjacent cutoffs do not share any eigenvalues.

The motivation for considering what appears to be a very strange set of matrices comes from the theory of orthogonal polynomials on $\mathbb{R}$. One can define a suitable measure $\nu$ on $\mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}, \nu)$. The measure $\nu$ is defined so that any polynomial function $\phi(t)$ is integrable with respect to $\nu$ (see [KW, pg 28]). We find an orthonormal sequence of polynomials $\{\phi_k(t)\}_{k \in \mathbb{N}}$ by applying the Gram-Schmidt process to the functions $\{t^k\}_{k \in \mathbb{N}}$. The zeroes of the polynomials $\{\phi_k(t)\}_{k \in \mathbb{N}}$ are an interesting object of study. The amazing fact is that they can be realized as the eigenvalues of cutoffs of certain real symmetric matrices in $\Omega_n$ (see Theorem 2.20 in [KW, pg 29]).

Kostant and Wallach show that for $X \in \Omega_n$, $\dim(A \cdot X) = \binom{n}{2}$ and describe the action of $A$ on $\Omega_n$ via the action of an $\binom{n}{2}$ dimensional complex torus $(\mathbb{C}^\times)^{\binom{n}{2}}$. They show that Zariski closed set of matrices in $\Omega_n$ consisting of matrices each
of whose cutoffs have a prescribed spectrum is exactly one $A$ orbit and that the action of the group $A$ on such matrices can be realized by an a simply transitive, algebraic action of the torus $(\mathbb{C}^\times)^{(2)}$ (see Theorems 3.23 and 3.28 in [KW, pgs 45, 49]).

1.2 Summary of current research

Our work took on two directions. The first was to generalize Theorems 3.23 and 3.28 in [KW, pgs 45, 49] about the orbit structure of $A$ to a larger classes of matrices. This work gives explicit descriptions of leaves of polarizations of all regular adjoint orbits in $M(n)$ as orbits of abelian, connected algebraic groups acting regularly on certain quasi-affine subvarieties of $M(n)$.

The second direction of our recent work was to generalize the theory of Kostant and Wallach to orthogonal Lie algebras denoted by $\mathfrak{so}(n)$. This involves using a classical analogue of the Gelfand-Zeitlin algebra of $U(\mathfrak{so}(n))$ in the polynomials on $\mathfrak{so}(n)$ to construct $d/2$ commuting Hamiltonian vector fields, where $d$ is the dimension of a regular adjoint orbit in an orthogonal Lie algebra. We will see that we can integrate such vector fields to an action of a complex analytic group $B = \mathbb{C}^{d/2}$ on $\mathfrak{so}(n)$. We study the action of $B$ on the analogous set of regular semi-simple elements $\Omega_n \subset \mathfrak{so}(n)$ and show that we can polarize the adjoint orbits of such elements.

1.2.1 Summary of current research for $M(n)$

We study a larger set of matrices $\Theta_n \supset \Omega_n$. For matrices in $\Theta_n$, we still insist that adjacent cutoffs $X_i$ and $X_{i+1}$ share no eigenvalues and that each cutoff $X_i$ is regular, however we relax the condition that $X_i$ is semi-simple. In this case, we find analogous results to those found by Kostant and Wallach in the case of $\Omega_n$. For $X \in \Theta_n$ we show that $\dim(A \cdot X) = \binom{n}{2}$. Matrices in $\Theta_n$ each of whose cutoffs have a prescribed characteristic polynomial form one orbit under the action of $A$. However, the action of $A$ on these Zariski closed subvarieties of $\Theta_n$ is no longer realized by an algebraic torus of dimension $\binom{n}{2}$, but by different
commutative, connected complex algebraic groups which often have non-trivial
unipotent subgroups. For a slightly more detailed summary see section 2.3 below.
For the full statement of the results, please see Chapter 6, section 6.1.

We also consider the $A$ orbit structure of matrices whose cutoffs $X_i$ are
regular for $1 \leq i \leq n - 1$, but we allow for degeneracies in the eigenvalues of
adjacent cutoffs. We give an algebraic parameterization of $A$ orbits of arbitrary
dimension for such matrices. In these cases, it turns out that the set of matrices
each of whose cutoffs have a prescribed spectrum no longer form one $A$ orbit. We
are able to relate the number of $A$ orbits of maximal dimension in this set to the
number of degeneracies in the spectrum at each of the levels. It turns out that
the number of such $A$ orbits is always a power of 2. For a slightly more detailed
summary see section 2.4 below. For the full statement of the results see Chapter
6.

In particular, we consider the action of $A$ on a special subset of matrices
each of whose cutoffs $X_i$ are principal nilpotent elements of $M(i)$. We show that
the closure of $A$ orbits of maximal dimension $\binom{n}{2}$ in this set are exactly nilradicals
of Borel subalgebras that contain the standard Cartan. We find that there are
$2^{n-1}$ such $A$ orbits and in this way we are choosing $2^{n-1}$ special elements of the
Weyl group of $M(n)$ (see Chapter 6, section 6.2).

1.2.2 Gelfand-Zeitlin theory for orthogonal Lie algebras

For our purposes, we think of $\mathfrak{so}(n)$ as $n \times n$ complex skew-symmetric
matrices. This allows us to embed easily $\mathfrak{so}(i) \hookrightarrow \mathfrak{so}(n)$ as a Lie subalgebra of
$\mathfrak{so}(n)$, by embedding an $i \times i$ skew-symmetric matrix in the top left hand corner
of an $n \times n$ skew-symmetric matrix. As in the case of $M(n)$, we again have
that $P(\mathfrak{so}(n))$ is a Poisson algebra and $P(\mathfrak{so}(i))$ can be considered as a Poisson
subalgebra. The key observation in the orthogonal case is that the Gelfand-Zeitlin
algebra of $U(\mathfrak{so}(n))$ has exactly the right number of generators to give rise to $d/2$
commuting Hamiltonian vector fields on $\mathfrak{so}(n)$, which on an open subset of $\mathfrak{so}(n)$
define a tangent distribution of dimension $d/2$. As in the case of $M(n)$, these $d/2$
vector fields are derived from the analogous Poisson commutative subalgebra of
We define Hamiltonian vector fields which are tangent to adjoint orbits $\xi_{f_{i,j}}$, where $f_{i,j}(X_i) \in P(\mathfrak{so}(n)(i))^{SO(i)}$ is a fundamental adjoint invariant. We then show that each of these fields $\xi_{f_{i,j}}$ integrate to a global action of $\mathbb{C}$ on $\mathfrak{so}(n)$ (see Chapter 7, section 7.2).

The vector fields $\xi_{f_{i,j}}$ for $2 \leq i \leq n - 1$, $1 \leq j \leq \text{rank } \mathfrak{so}(i)$ commute on account of the Poisson commutativity of the subalgebra $J(\mathfrak{so}(n))$, just as in the case of $M(n)$. Thus, we get an action of $B = \mathbb{C}^{d/2}$ on $\mathfrak{so}(n)$. However, unlike in the case of $M(n)$, it is not clear that there are elements whose $B$ orbits are of maximal dimension $d/2$. We show that there are in fact such elements (see Chapter 7, section 7.6). More specifically, we show that if we consider the set $\Omega_n \subset \mathfrak{so}(n)$ defined by $X \in \Omega_n$ if and only if $X_i$ is regular semi-simple for each $1 \leq i \leq n$ and $X_i$ and $X_{i+1}$ share no eigenvalues then elements of $\Omega_n$ have the property that their $B$ orbits are of maximal dimension. We also show the analogues of Theorems 3.23 and 3.28 in [KW, pgs 45, 48] hold in this case. i.e. We show that the closed subsets of matrices in $\Omega_n$ for which each cutoff $X_i$ evaluated on the fundamental invariants $f_{i,1}, \ldots, f_{i,l}$ ($l = \text{rk}(\mathfrak{so}(i))$) is constant form one $B$ orbit. We also show that such closed subsets are homogeneous spaces for a free, algebraic action of the torus $\text{SO}(2)^{d/2} \simeq (\mathbb{C}^\times)^{d/2}$.

1.3 Poisson algebras and the Poisson structure on $U(\mathfrak{g})$

Poisson algebras play an important role in symplectic geometry and in the theory of Lie algebras and the orbit method. In much of what follows, we will be using Poisson algebras to describe the geometry of regular adjoint orbits in the reductive Lie algebra of $n \times n$ matrices $M(n)$ over $\mathbb{C}$ and the $n \times n$ complex orthogonal Lie algebra $\mathfrak{so}(n)$.

A Poisson algebra can roughly be defined as an a commutative associative algebra that is equipped with a Lie bracket that is compatible with the underlying
Definition 1.3.1. A commutative algebra $P$ over $\mathbb{C}$ equipped with a bilinear binary operation $\{\cdot, \cdot\} : P \times P \rightarrow P$ is said to be a Poisson algebra if the following two conditions are satisfied:

1) $\{\cdot, \cdot\}$ makes $P$ into a Lie algebra.

2) Liebniz Rule: $\{a, bc\} = \{a, b\}c + b\{a, c\}.$

In other words, the bracket operation $\{\cdot, \cdot\}$ is not only a derivation of the Lie structure, but also is a derivation of the underlying associative structure.

As mentioned above, Poisson algebras naturally arise in geometry, but they can also be constructed in a purely algebraic setting. Our discussion will follow that in section 1.3 [CG, pg 26]. Let $B$ be a filtered, associative, but not necessarily commutative algebra over $\mathbb{C}$. Recall that this means that there is an increasing sequence of $\mathbb{C}$ vector spaces $\mathbb{C} = B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots$ such that $B_i \cdot B_j \subset B_{i+j}$ and $B = \bigcup_{i=0}^{\infty} B_i$.

Now, given any associative, filtered algebra, we can form the associated graded algebra $Gr(B)$ which is given by

$$Gr(B) = \bigoplus_{i=0}^{\infty} B_i/B_{i-1} = \bigoplus_{i=0}^{\infty} Gr(B)^i$$

where we set $Gr(B)^i = B_i/B_{i-1}$ and we take $B_{-1} = \{0\}$.

It is easy to check that we have a well-defined multiplication:

$$Gr(B)^i \times Gr(B)^j \rightarrow Gr(B)^{i+j}$$

given by

$$a \mod B_{i-1} \times b \mod B_{j-1} = ab \mod B_{i+j-1}$$
that makes \( \text{Gr}(B) \) into a graded algebra over \( \mathbb{C} \). Even though we may not have that \( B \) is commutative, it can be the case that \( \text{Gr}(B) \) is commutative. In this situation, we make the following definition.

**Definition 1.3.2.** The filtered associative algebra \( B \) is said to be almost commutative if \( \text{Gr}(B) \) is commutative.

*Remark 1.3.1.* It is easy to see that \( B \) is almost commutative if and only if we have \([B_i, B_j] \subset B_{i+j-1}\)

In this case we have a natural Poisson structure on \( \text{Gr}(B) \).

**Proposition 1.3.1.** If \( B \) is almost commutative, then \( \text{Gr}(B) \) is naturally a Poisson algebra.

*Proof:*

The idea is to note that in any associative algebra \( A \), the associative commutator makes \( A \) into a Lie algebra and is also a derivation of the underlying associative structure. In other words, the associative commutator on \( A \) satisfies both conditions 1) and 2) of definition 1.3.1. However, \( A \) is itself not commutative, so it is not a Poisson algebra. If \( A \) is filtered, conditions 1) and 2) in Definition 1.3.1 are not be affected when we move to \( \text{Gr}A \). In our case \( \text{Gr}(B) \) is commutative, so to use the associative commutator to get a non-trivial Poisson structure, we have to take its second order symbol. In other words, we define

\[
\{ \cdot, \cdot \} : \text{Gr}(B)^i \times \text{Gr}(B)^j \to \text{Gr}(B)^{i+j-1}
\]

given by

\[
\{a_1, a_2\} = a_1 a_2 - a_2 a_1 \mod B_{i+j-2}
\]

One can easily check that this operation is well-defined and can be bilinearly extended to define a skew-symmetric bilinear operation on \( \text{Gr}(B) \times \text{Gr}(B) \to \text{Gr}(B) \) which satisfies conditions 1) and 2) in Definition 1.3.1.

We now consider one of the main examples of this type of Poisson structure.
Example 1.3.1. Let \( g \) be a finite-dimensional Lie algebra over \( \mathbb{C} \). Then one knows that the universal enveloping algebra \( U(g) \) is a filtered algebra and that \( Gr(U(g)) \) is commutative, so that \( U(g) \) is almost commutative. Thus, by Proposition 1.3.1, \( Gr(U(g)) \) is a Poisson algebra.

We recall that using the Poincaré-Birkhoff-Witt (PBW) theorem, we have a canonical isomorphism as graded algebras

\[ \Psi : S(g) \simeq Gr U(g) \]  

with \( S(g) \) being the symmetric algebra on \( g \). Recall that this map can be described in terms of a basis for \( U(g) \) and \( S(g) \) as follows. Let \( g \) have basis \( \{X_1, \cdots, X_n\} \). Then by PBW, the monomials \( X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} \mod U_{n-1}(g) \) with \( \sum j_i = n \) and with \( i_1 < \cdots < i_k \) form a basis for \( Gr(U(g))^n \). The above isomorphism can be described in terms of the basis by noting that it sends a coset \( X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} \mod U_{n-1}(g) \) to the monomial \( X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} \) which forms part of a basis for \( S^n(g) \) (see [Knp, pg 222]).

Using this isomorphism, we can carry over the Poisson structure on \( Gr(U(g)) \) to \( S(g) \).

Remark 1.3.2. It is important to note that if \( x, y \in S^1(g) \), then \( \{x, y\} = [x, y] \) recovers the Lie bracket of \( x, y \). We can see this as follows. We note \( U_1(g) = \mathbb{C} \oplus g \), so that \( Gr(U(g))^1 \simeq g \) as vector spaces. Thus for \( x, y \in Gr(U(g))^1 \simeq g \), we have that \( \{x, y\} = x \cdot y - y \cdot x = [x, y] \) by the defining relations for \( U(g) \).

Now, we also note that \( S(g) \simeq P(g^*) \) where \( P(g^*) \) denotes the polynomials on \( g^* \). Recall that the homogenous polynomials of degree \( k \) on \( g^* \) are the linear span of the functions

\[ v^* \to \prod_{i=1}^{k} \langle v^*, x_i \rangle \]

with \( x_i \in g \) (see [GW, pg 622]). Thus, we have a canonical Poisson structure on \( P(g^*) \).

In our work, we will be concerned solely with finite-dimensional reductive Lie algebras (namely \( M(n) \) and \( \mathfrak{so}(n) \)). Such a Lie algebra \( g \) comes equipped with an invariant, non-degenerate, symmetric bilinear form. This means a bilinear form \( \beta \) which has the following invariance property: \( \beta([x, y], z) = \beta(x, [y, z]) \)
for all $x, z \in \mathfrak{g}$. (In the case of $M(n)$ and $\mathfrak{so}(n)$, we will use the trace form $\beta(x, y) = \text{tr}(xy)$. In the case of a general semi-simple Lie algebra, the Killing form can be used.) Using this form one can canonically identify $\mathfrak{g} \simeq \mathfrak{g}^*$. $\phi \in \mathfrak{g}^*$ is identified with unique $g_\phi$ in $\mathfrak{g}$ such that $\phi(x) = \beta(x, g_\phi)$ for all $x \in \mathfrak{g}$. Using this isomorphism, one may then identify $\mathfrak{g} \simeq \mathfrak{g}^*$. $\phi \in \mathfrak{g}^*$ is identified with unique $g_\phi$ in $\mathfrak{g}$ such that $\phi(x) = \beta(x, g_\phi)$ for all $x \in \mathfrak{g}$. Using this isomorphism, one may then identify $\mathfrak{g} \simeq \mathfrak{g}^*$.

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1.3.1 The Poisson structure on $P(\mathfrak{g})$

In this section $\mathfrak{g}$ is a reductive Lie algebra over $\mathbb{C}$ with non-degenerate, symmetric, invariant bilinear form $\beta(\cdot, \cdot)$. It will be useful to have a description of the Poisson bracket of two polynomials on $f, g \in P(\mathfrak{g})$. To do this we have to make a few preliminary definitions.

Let $\mathcal{H}(\mathfrak{g})$ denote the algebra of holomorphic functions on $\mathfrak{g}$. Then if we are given $\psi \in \mathcal{H}(\mathfrak{g})$ and $x \in \mathfrak{g}$, then $d\psi_x \in T_x(\mathfrak{g})^* \simeq \mathfrak{g}^*$. Thus, using the form $\beta$, we can naturally identify $d\psi_x$ with an element of $\mathfrak{g}$, denoted by $d\psi(x)$ defined via the rule

$$\beta(d\psi(x), z) = \frac{d}{dt}|_{t=0} \psi(x + tz) = (\partial^z f)_x$$

(1.3)

for all $z \in \mathfrak{g}$. ($\partial^z f)_x$ denotes the directional derivative of $f$ in the direction of $z$ evaluated at $x$).

We claim that we have the following simple description of the Poisson bracket of two polynomials.

**Proposition 1.3.2.** If $f, g \in P(\mathfrak{g})$ and if $\{f, g\}$ denotes their Poisson bracket as described in the previous section, then we have:

$$\{f, g\}(x) = \beta(x, [df(x), dg(x)])$$

(1.4)

with $df(x), dg(x) \in \mathfrak{g}$ are as defined in equation (1.3).

**Proof:**
This proof is adapted from [CG, pg 36]. We observe that since both sides of
equation (1.4) satisfy the Liebniz rule that it suffices to prove (1.4) for \( f, g \in g^* \), since \( g^* \) generates \( P(g) \) as an algebra. We also observe that both sides of (1.4) are bilinear, so that is suffices to prove if for a basis \( \{\alpha_1, \ldots, \alpha_n\} \) of \( g^* \). To that effect, let \( \{e_1, \ldots, e_n\} \) be the corresponding basis of \( g \). This means that for any \( y \in g \), \( \alpha_i(y) = \beta(y, e_i) \). We have structure constants \( c^k_{ij} \in \mathbb{C} \) defined via

\[
[e_i, e_j] = \sum_{k=1}^{n} c^k_{ij} e_k
\]

Now, we consider first the LHS of (1.4) \( \{\alpha_i, \alpha_j\} \). Now, let \( \gamma : P(g^*) \to P(g) \) be the isomorphism described in the last section. From the definition of the Poisson structure on \( P(g) \) as coming from the Poisson structure on \( S(g) = P(g^*) \), and by Remark 1.3.2 we have that

\[
\{\alpha_i, \alpha_j\} = \gamma(\{e_i, e_j\}) = \gamma\left( \sum_{k=1}^{n} c^k_{ij} e_k \right) = \sum_{k=1}^{n} c^k_{ij} \alpha_k
\]

Thus, we obtain

\[
\{\alpha_i, \alpha_j\}(x) = \sum_{k=1}^{n} c^k_{ij} \beta(e_k, x)
\]

(1.5)

for any \( x \in g \). Now, we consider the RHS of (1.4). We claim that \( d\alpha_i(x) \in g = e_i \) for any \( x \in g \). To see this, consider

\[
\frac{d}{dt}\big|_{t=0} \alpha_i(x + ty) = \frac{d}{dt}\big|_{t=0} \beta(e_i, x + ty) = \beta(e_i, y)
\]

Thus, the RHS of (1.4) becomes

\[
\beta(x, [d\alpha_i(x), d\alpha_j(x)]) = \beta(x, [e_i, e_j]) = \sum_{i=1}^{n} c^k_{ij} \beta(x, e_k)
\]

(1.6)

This is equal to (1.5), which completes the proof.

Q.E.D.

Remark 1.3.3. Although we will not need it in what follows, it is worthwhile to note that the RHS (1.4) can be used to define the Poisson bracket of any pair of holomorphic functions \( f, g \in \mathcal{H}(g) \). This is actually a workable definition, since the RHS of (1.4) can be computed in terms of polynomial functions. We just need
to observe that if we have \( f \in \mathcal{H}(\mathfrak{g}) \) and \( \{y_1, \cdots, y_n\} \) a basis of \( \mathfrak{g} \) and \( \{x_1, \cdots, x_n\} \) the dual basis in \( \mathfrak{g}^* \), then
\[
\text{df}_x = \sum_{i=1}^n (\partial^{y_i} f)_x (dx_i)_x
\]
where \((\partial^{y_i} f)_x = \frac{d}{dt}|_{t=0} f(x + ty_i)\) is the directional derivative in the direction of \( y_i \).

Thus, by the bilinearity of the formula in (1.4), we need only know the Poisson brackets \( \{x_i, x_j\} \) to compute \( \{f, g\} \) for \( f, g \in \mathcal{H}(\mathfrak{g}) \). As we showed above, we can compute \( \{x_i, x_j\} \) in terms of structural constant of the Lie algebra. More precisely, the Poisson bracket of two holomorphic functions \( f, g \in \mathcal{H}(\mathfrak{g}) \) is given by
\[
\{f, g\}(x) = \sum_{i,j} (\partial^{y_i} f)_x (\partial^{y_j} g)_x \{x_i, x_j\} = \sum_{i,j} (\partial^{y_i} f)_x (\partial^{y_j} g)_x \sum_k c_{ij}^k x_k
\]

### 1.3.2 Hamiltonian vector fields on \( \mathfrak{g} \)

Recall from definition 1.3.1 that the Poisson bracket on \( \mathcal{P}(\mathfrak{g}) \) is a derivation of the underlying associative structure. This allows us to define algebraic Hamiltonian vector fields \( \xi_f \) for \( f \in \mathcal{P}(\mathfrak{g}) \) by
\[
\xi_f(g) = \{f, g\}
\]  \hspace{1cm} (1.7)
(By Remark 1.3.3, we could also make the same definition for \( f, g \in \mathcal{H}(\mathfrak{g}) \)). We will justify the term Hamiltonian in section 1.5.

Using Proposition 1.3.2, we can see that the vector fields \( \xi_f \) have a particularly simple expression in coordinates.

\[
\xi_f(g)(x) = \{f, g\}(x) = \beta(x, [df(x), dg(x)])
\]  \hspace{1cm} (1.8)
\[
= \beta([x, df(x)], dg(x)) = \frac{d}{dt}|_{t=0} g(x + t([x, df(x)]))
\]

Thus, the tangent vector \((\xi_f)_x\) is just the directional derivative in the direction of \([x, df(x)] \in \mathfrak{g}\) evaluated at \( x \) and thus, in coordinates, it has the expression
\[
(\xi_f)_x = (\partial^{[x, df(x)]})_x
\]  \hspace{1cm} (1.9)
1.4 Some basic symplectic geometry

We now have a notion of a Poisson structure on $P(\mathfrak{g})$ which is purely algebraic in the sense that it comes from the Poisson structure on $U(\mathfrak{g})$ described in section 1.3. One also knows that the adjoint orbits in $\mathfrak{g}$ have a natural symplectic structure, and therefore holomorphic functions on the adjoint orbits form a Poisson algebra. We want to see that this is the same Poisson structure as the one coming from $U(\mathfrak{g})$. Before, we do that let us review a few of the basic facts from symplectic geometry. For an introduction to symplectic manifolds [see Lee, pgs 480-490] and for a much more detailed exposition see [CdS].

Recall that a complex (real differentiable) manifold $(M, \omega)$ is symplectic if $\omega$ is a holomorphic (smooth) two form on $M$, which is non-degenerate and closed (i.e. $d\omega = 0$). In particular, this means that at every point $x \in M$, the tangent space $T_x(M)$ comes equipped with a non-degenerate, anti-symmetric bilinear form $\omega_x$. This fact allows us to define a bundle isomorphism:

$$\tilde{\omega} : T(M) \rightarrow T^*(M)$$

Using this bundle isomorphism, for any holomorphic (smooth) function $f$ on $M$ we can define a holomorphic (smooth) vector field $X_f$, referred to as the Hamiltonian vector field of $f$ as follows,

$$\omega_x((X_f)_x, v) = -df_x(v) \quad (1.10)$$

for $v \in T_x(M)$. Using (1.10), we can define a Poisson structure on the algebra of holomorphic (smooth) functions on $M$. If $f, g \in \mathcal{H}(M)$ then we define

$$\{f, g\}(x) = \omega_x((X_f)_x, (X_g)_x) \quad (1.11)$$

See [Lee, pg 489] or [CG, pg 21] to see this definition actually makes $\mathcal{H}(M)$ into a Poisson algebra. One of the most important facts about this Poisson structure is contained in the following proposition.

**Proposition 1.4.1.** The assignment $f \rightarrow X_f$ intertwines the Poisson bracket with commutator of vector fields. i.e. we have that

$$[X_f, X_g] = X_{\{f, g\}} \quad (1.12)$$
for any \( f, g \in \mathcal{H}(M) \).

Two holomorphic functions \( f, g \in \mathcal{H}(M) \) are said to Poisson commute, if we have
\[
\{ f, g \} = \omega(X_f, X_g) = -X_g f = X_f g = 0
\]

**Remark 1.4.1.** Note, that \( f, g \) Poisson commute if and only if \( f \) is constant along the Hamiltonian flow of \( g \). (since \( f, g \) Poisson commute if and only if \( X_g f = 0 \)).

### 1.5 The symplectic structure on the adjoint orbits

Let \( G \) be a finite dimensional complex Lie group and let \( \mathfrak{g} \) be its Lie algebra of left invariant vector fields. Then \( G \) acts on \( \mathfrak{g} \) via the adjoint action, which in the case of the classical groups is just conjugation. i.e.
\[
\text{Ad}(g) \cdot X = g X g^{-1}
\]

\( g \in G \) and \( X \in \mathfrak{g} \). The contragradient action to the adjoint action is referred to as the coadjoint representation on \( \mathfrak{g}^* \) and is given by
\[
\text{Ad}^*(g)(\lambda)(X) = \lambda(\text{Ad}(g^{-1})(X))
\]

for \( g \in G, \lambda \in \mathfrak{g}^* \), and \( X \in \mathfrak{g} \). If \( \lambda \in \mathfrak{g}^* \), let \( O_\lambda \) be its coadjoint orbit. The beautiful fact is that \( O_\lambda \) is naturally a symplectic manifold.

**Theorem 1.5.1.** Let \( \lambda \in \mathfrak{g}^* \) and let \( O_\lambda \) be its coadjoint orbit. Then \( O_\lambda \) is a symplectic manifold with the symplectic form given by
\[
\omega_\lambda(x, y) = \lambda([x, y])
\]

for \( x, y \in \mathfrak{g} \).

For a proof see ([CG], pg 23). This symplectic structure is often referred to in the literature as the Kirillov-Kostant-Souriau structure. Some explanation is required here. Note that the tangent space \( T_\lambda(O) \subset \mathfrak{g}^* \) is spanned by vectors
of the form $\lambda \circ (-ad(X))$ for $X \in \mathfrak{g}$ and isomorphic as a vector space to $\mathfrak{g}/\mathfrak{g}^\lambda$ where $\mathfrak{g}^\lambda$ is the Lie algebra of the isotropy group of $\lambda$ and is given by $\mathfrak{g}^\lambda = \{ X \in \mathfrak{g} | \lambda \circ (-ad(X)) = 0 \}$. Thus to define a bilinear form on $T_\lambda(\mathcal{O}_\lambda) \times T_\lambda(\mathcal{O}_\lambda)$ one can define it on $\mathfrak{g} \times \mathfrak{g}$ as we did in equation (1.13) and then show that it pushes down to the quotient $\mathfrak{g}/\mathfrak{g}^\lambda$, which is clear from its definition. The same argument will also show that the form is non-degenerate. What takes work is to show that it is closed (see [CG]).

In the cases we will be dealing with $G$ will be a complex reductive Lie group with reductive Lie algebra $\mathfrak{g}$. We can use the non-degenerate, invariant, bilinear form $\beta$ to transfer the symplectic structure on the coadjoint orbits to the adjoint orbits, making the the adjoint orbits into symplectic manifolds. More precisely if $\lambda \in \mathfrak{g}^*$, then $\mathcal{O}_\lambda \cong \mathcal{O}_{x_\lambda}$ where $x_\lambda \in \mathfrak{g}$ is given by $\lambda(y) = \beta(x_\lambda, y)$ for all $y \in \mathfrak{g}$. (Where the isomorphism is given by the form $\beta$ as described in section 1.3.) Now, using this isomorphism, we can compute the tangent space to $\mathcal{O}_x$ at $x \in \mathfrak{g}$ to be:

$$T_x(\mathcal{O}_x) = \text{span}\{ [y, x] | y \in \mathfrak{g} \} \subset \mathfrak{g}$$

(1.14)

The symplectic form on $\mathcal{O}_x$ is then given by

$$\alpha_x([y, x], [z, x]) = \beta(x, [y, z])$$

for $x, y, z \in \mathfrak{g}$. The fact that $\alpha$ is a symplectic form on $\mathcal{O}_X$ follows from that fact that it is the pullback of the form $\omega$ on the coadjoint orbit $\mathcal{O}_\lambda$ given in equation (1.13) under the isomorphism induced by $\beta$. However, it is an easy computation to see that $\alpha$ is well-defined and non-degenerate. We first show $\alpha_x$ is well-defined. Suppose that $[y, x] = [z, x]$. Then we have:

$$\alpha_x([y, x], [w, x]) = \beta(x, [y, w]) = \beta([x, y], w)$$

$$= \beta([x, z], w) = \beta(x, [z, w]) = \alpha_x([z, x], [w, x])$$

To see that the form is non degenerate suppose that we have $\alpha_x([y, x], [z, x]) = 0$ for all $z \in \mathfrak{g}$. Then, we compute

$$\alpha_x([y, x], [z, x]) = 0 \Rightarrow \beta(x, [y, z]) = 0 \Rightarrow$$

$$\beta([x, y], z) = 0 \Rightarrow [x, y] = 0$$
by the non degeneracy of $\beta$. The fact $\alpha$ is closed follows from the fact that the form $\omega$ on the coadjoint orbits is closed. For the rest of our discussion, we always transfer the natural structure on the coadjoint orbits to the adjoint orbits.

Now, recall equation (1.9) in section 1.3.2

$$(\xi_f)_x = (\partial^{[x, df(x)]})_x$$

Using this fact, we have the following proposition.

**Proposition 1.5.1.** Let $x \in \mathfrak{g}$ and let $O_x$ be its adjoint orbit. For any $p \in P(\mathfrak{g})$ (or more generally for any $p \in \mathcal{H}(\mathfrak{g})$), we have that the vector field $\xi_p$ defined in section 1.3.2 is tangent to $O_x$. i.e. we have

$$(\xi_p)_x \in T_x(O_x)$$

Thus, we now potentially have two different notions of Hamiltonian vector field and of Poisson bracket on an adjoint orbit $O_x \subset \mathfrak{g}$ of $x \in \mathfrak{g}$, one coming from the “global” Poisson structure of $\mathcal{H}(\mathfrak{g})$ and the other from the symplectic structure on $O$. The next result says that these two different structures agree, so that we were justified in calling the vector fields $\xi_f$ Hamiltonian in section 1.3.2.

**Proposition 1.5.2.** For $\phi \in \mathcal{H}(\mathfrak{g})$, $x \in \mathfrak{g}$ we have that $(\xi_\phi)_x \in T_x(O_x)$ and if $V \subset O_x$ is open and $q = \phi|_{O_x}$, then

$$\xi_\phi|_V = X_q|_V$$

(where $X_q$ is the Hamiltonian vector field derived from the symplectic structure on $O$). Thus we have

$${}\phi, \psi\}|_{O_x} = {}\phi, \psi\}|_{O_x}$$

(1.15)

(where the Poisson bracket on the LHS is the restriction of the Poisson bracket on $\mathcal{H}(\mathfrak{g})$ to the orbit $O_x$, and the RHS is the Poisson bracket coming from the symplectic structure of $O_x$).

For a proof of this proposition in the case of $\mathfrak{g} = M(n)$, see Proposition 1.3 in [KW, pg 12]. The methods used there carry over to a general reductive Lie algebra. For a proof in a slightly more general setting, see [K1, pg 5].
Now, we have an immediate corollary, which will play a very important role in what follows.

**Corollary 1.5.1.** For \( f, g \in \mathcal{H}(\mathfrak{g}) \) we have

\[
\xi_{\{f,g\}} = [\xi_f, \xi_g]
\]

(1.16)

**Proof:**

The corollary follows immediately from Proposition 1.5.2 and Proposition 1.4.1.

### 1.6 A classical analogue of the Gelfand-Zeitlin algebra

We expand upon some of the facts about the universal enveloping algebra of a finite dimensional Lie algebra over \( \mathbb{C} \) in order to set the stage for the definition of the classical analogue of the Gelfand-Zeitlin algebra of \( U(\mathfrak{g}) \) in \( S(\mathfrak{g}) \). For any \( n \geq 1 \), we can define a natural symmetric \( n \)-multilinear map

\[
\sigma_n : \mathfrak{g} \times \cdots \times \mathfrak{g} \to U_n(\mathfrak{g})
\]

given by

\[
\sigma_n(X_1, \cdots, X_n) = \frac{1}{n!} \sum_{\tau \in S_n} X_{\tau(1)} X_{\tau(2)} \cdots X_{\tau(n)}
\]

Since this map is symmetric, it descends to a linear map

\[
\sigma_n : S^n(\mathfrak{g}) \to U_n(\mathfrak{g})
\]

Taking the direct sum of the maps \( \sigma_n \) we get a linear map

\[
\text{Sym} : S(\mathfrak{g}) \to U(\mathfrak{g})
\]

which we denote by \( \text{Sym} \) and refer to as symmetrization. It is easy to see that the map \( \text{Sym} \) is a \( G \)-module map with respect to the \( G \) action induced on \( S(\mathfrak{g}) \) and \( U(\mathfrak{g}) \) from the adjoint action of \( G \) on \( \mathfrak{g} \). We will now recall some facts about this map (see [Knp, pgs 225-226]).
Proposition 1.6.1. The map \( \sigma_n : S^n(g) \to U_n(g) \) maps injectively onto a vector space compliment of \( U_{n-1}(g) \) in \( U_n(g) \). i.e., we have that
\[
U_n(g) = \sigma_n(S^n(g)) \oplus U_{n-1}(g) \quad (1.17)
\]
Moreover, the map \( \text{Sym} \) is a filtered \( G \)-module isomorphism
\[
\text{Sym} : S(g) \simeq U(g)
\]
whose associated graded map
\[
\text{Gr Sym} : S(g) \to \text{Gr} U(g)
\]
is map \( \Psi \) in equation (1.2).

Now, let \( U(g)^G \) be the \( G \) invariants in \( U(g) \). \( U(g) \) is naturally filtered by \( U_n(g)^G = U(g)^G \cap U_n(g) \). Now, we claim that we have
\[
U_n(g)^G = \sigma_n(S^n(g)^G) \oplus U_{n-1}(g)^G \quad (1.18)
\]
We note that inclusion \(( \supseteq \)) is clear. To see the other inclusion, consider \( x \in U_n(g)^G \). By equation (1.17), we note that we have \( x = y + z \) with \( y \in \sigma_n(S^n(g)) \), \( z \in U_{n-1}(g) \). We let \( g \in G \) act on \( x \) to get \( g \cdot x = g \cdot y + g \cdot z \). We note that \( U_{n-1}(g) \) is \( G \)-stable and \( \sigma_n \) is a \( G \) map, so that we have \( g \cdot y \in \sigma_n(S^n(g)) \) and \( g \cdot z \in U_{n-1}(g) \). Thus, we have that
\[
g \cdot x = g \cdot y + g \cdot z = y + z = x
\]
Since the sum in (1.17) is direct, we must have that \( g \cdot y = y \) and \( g \cdot z = z \). Using the fact the \( \sigma_n \) is a \( G \)-map and is 1-1, we get the inclusion \( \subseteq \) and thus equation (1.18). Equation (1.18) tells us that \( \text{Gr} \) commutes with taking invariants. In other words, we have that
\[
\text{Gr}(U(g)^G) \simeq \text{Gr}(U(g))^G
\]
Now, the upshot of this discussion is that \( \text{Gr Sym} \) gives a map
\[
\text{Gr Sym} : S(g)^G \simeq \text{Gr}(U(g)^G) \quad (1.19)
\]
which is the restriction of the map \( \Psi \) in (1.2) to the invariants in \( S(g) \). Now, we recall from section 1.3 that we use the isomorphism \( \Psi \) to define the Poisson structure on \( S(g) \). We want to see that (1.19) gives us that \( S(g)^G \) is a commutative Poisson algebra. This follows immediately from the following lemma.
Lemma 1.6.1. Let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$, then we have:

$$U(\mathfrak{g})^G = Z(\mathfrak{g})$$

Thus, we have that (1.19) becomes

$$Gr \ Sym : S(\mathfrak{g})^G \simeq Gr(Z(\mathfrak{g}))$$  \hspace{1cm} (1.20)

and $Gr(Z(\mathfrak{g}))$ is clearly a Poisson commutative subalgebra of $Gr(U(\mathfrak{g}))$, since $Z(\mathfrak{g})$ is abelian.

Now, for the remainder of this section $\mathfrak{g} = M(n)$ or $\mathfrak{g} = \mathfrak{so}(n)$. For $\mathfrak{g} = M(n)$ we note the we have a natural embedding of Lie subalgebras $\mathfrak{g}_i = M(i) \hookrightarrow M(n)$, where we think of $M(i)$ as embedded via

$$X \hookrightarrow \begin{bmatrix} X & 0 \\ 0 & 0_{n-i} \end{bmatrix}$$

If we realize $\mathfrak{so}(n)$ as $n \times n$ complex skew-symmetric matrices, then we have an identical embedding of $\mathfrak{so}(i) \hookrightarrow \mathfrak{so}(n)$, by embedding $\mathfrak{so}(i)$ in the top left hand corner. Now, we note that we can also embedd the adjoint groups $G_i = GL(i)$ or $SO(i)$ into $GL(n)$ or $SO(n)$, via the embedding,

$$g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & Id_{n-i} \end{bmatrix}$$

for $g \in G_i$. In both cases, we get a chain of inclusions of Lie subalgebras and thus of universal enveloping algebras

$$\mathfrak{g}_{i_1} \subseteq \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n \Rightarrow$$

$$U(\mathfrak{g}_{i_1}) \subseteq U(\mathfrak{g}_2) \subset \cdots \subset U(\mathfrak{g}_n)$$

where $\mathfrak{g}_n = \mathfrak{g} = M(n)$ or $\mathfrak{so}(n)$ and $i_1 = 1$ for $\mathfrak{g} = M(n)$ and $i_1 = 2$ for $\mathfrak{g} = \mathfrak{so}(n)$.

Now, let $Z(\mathfrak{g}_i)$ be the centre of $U(\mathfrak{g}_i)$. We have that $S(\mathfrak{g}_i) \subset S(\mathfrak{g})$ and $U(\mathfrak{g}_i) \subset U(\mathfrak{g})$. We note that we have $G_i$ module isomorphisms

$$Sym_i : S(\mathfrak{g}_i) \simeq U(\mathfrak{g}_i)$$
for $1 \leq i \leq n$. We note that it is easy to see that by the PBW Theorem and the definition of symmetrization that $\text{Sym}|_{\mathfrak{g}_i} = \text{Sym}_i$. So in particular this yields that $S(\mathfrak{g}_i)$ is a Poisson subalgebra of $S(\mathfrak{g})$. And since $G_i \subset G$ is a subgroup, we have that $\text{Sym}$ is also a $G_i$ module map. Putting this together, we get

$$\text{GrSym} = \Psi : S(\mathfrak{g}_i)^{G_i} \simeq \text{Gr}(Z(\mathfrak{g}_i)) \subset \text{Gr}(U(\mathfrak{g}))$$

Thus, $S(\mathfrak{g}_i)^{G_i}$ is a commutative Poisson subalgebra of $S(\mathfrak{g})$.

Now, we consider the subalgebra of $S(\mathfrak{g})$ generated by $S(\mathfrak{g}_{i_1})^{G_{i_1}}, \ldots, S(\mathfrak{g}_n)^{G_n}$ i.e. we consider the algebra:

$$J(\mathfrak{g}) = S(\mathfrak{g}_{i_1})^{G_{i_1}} \cdots S(\mathfrak{g}_n)^{G_n} \simeq S(\mathfrak{g}_{i_1})^{G_{i_1}} \otimes \cdots \otimes S(\mathfrak{g}_n)^{G_n}$$

We claim that this is a Poisson commuting subalgebra of $S(\mathfrak{g})$. We argue this inductively. We note that since $\text{Gr Sym} : S(\mathfrak{g})^G \simeq \text{Gr}(Z(\mathfrak{g}))$ by equation (1.20), we have that $S(\mathfrak{g})^G$ is contained in the Poisson centre of $S(\mathfrak{g})$. Hence, it commutes with all of the subalgebras $S(\mathfrak{g}_i)^{G_i}$ for $i_1 \leq i \leq n$. Now, we consider $S(\mathfrak{g}_{n-1})^{G_{n-1}}$, which via $\text{Gr Sym}$ is isomorphic to $\text{Gr}(Z(\mathfrak{g}_{n-1}))$. Observe that since $U(\mathfrak{g}_i) \subset U(\mathfrak{g}_{n-1})$ for $1 \leq i \leq n - 2$, we have that $Z(\mathfrak{g}_{n-1})$ commutes with $U(\mathfrak{g}_i)$ for $1 \leq i \leq n - 1$. So that $S(\mathfrak{g}_{n-1})^{G_{n-1}}$ Poisson commutes with $S(\mathfrak{g}_i)$ for $1 \leq i \leq n - 1$. Proceeding in this fashion, we see that $J(\mathfrak{g})$ is a Poisson commutative subalgebra of $S(\mathfrak{g})$.

We are now ready to define the Gelfand-Zeitlin algebra of $U(\mathfrak{g})$ and to see that $J(\mathfrak{g})$ is a classical analogue to the Gelfand-Zeitlin algebra in $S(\mathfrak{g})$. We define the Gelfand-Zeitlin algebra to be the associative subalgebra of $U(\mathfrak{g})$ generated by the centres $Z(\mathfrak{g}_i) \subset U(\mathfrak{g}_i) \subset U(\mathfrak{g})$. Thus, we define

$$GZ(\mathfrak{g}) = Z(\mathfrak{g}_{i_1}) \cdots Z(\mathfrak{g}_n)$$

We observe that $GZ(\mathfrak{g})$ is an abelian subalgebra of $U(\mathfrak{g})$. One argues this inductively. The argument is completely analogous to the one we gave to see that $J(\mathfrak{g})$ is Poisson commuting. We will do the first few steps. Since $U(\mathfrak{g}_{i_1}) \subset U(\mathfrak{g}_2)$, we have the centre of $Z(\mathfrak{g}_2)$ of $U(\mathfrak{g}_2)$ commutes with all of $U(\mathfrak{g}_{i_1})$, thus it commutes with $Z(\mathfrak{g}_{i_1})$. This gives that the associative algebra generated by $Z(\mathfrak{g}_{i_1})$ and $Z(\mathfrak{g}_2)$
is abelian. Now, since $U(\mathfrak{g}_{} ) \subset U(\mathfrak{g}_2 ) \subset U(\mathfrak{g}_3 )$, $Z(\mathfrak{g}_{} )$ commutes with $Z(\mathfrak{g}_{} ) Z(\mathfrak{g}_2 )$ and since $Z(\mathfrak{g}_3 )$ is abelian, the associative algebra $Z(\mathfrak{g}_{} ) Z(\mathfrak{g}_2 ) Z(\mathfrak{g}_3 )$ is abelian, and so on.

Now, we have the following deep result (see Theorem 10.4.5 [Dix, 343]).

**Theorem 1.6.1.** Let $S(\mathfrak{g}^{} )^G$ denote the $G$ invariant polynomials on $\mathfrak{g}^*$, (here we are identifying $S(\mathfrak{g} )$ with $P(\mathfrak{g}^* )$), then there exists an algebra isomorphism

$$Z(\mathfrak{g} ) \simeq S(\mathfrak{g}^{} )^G$$

The formula for such an isomorphism involve twists of the symmetrization map described above (due to Harish-Chandra and simplified by Duflo). Thus, we can consider $J(\mathfrak{g} )$ as an analogue of the Gelfand-Zeitlin algebra. We will see later that $GZ$ and $J(\mathfrak{g} )$ are free commutative algebras on the same number of generators, so that they are abstractly isomorphic.

Now, recall that we have that $S(\mathfrak{g}_{} )$ is a Poisson subalgebra of $S(\mathfrak{g} )$. Now $\mathfrak{g} = M(n), \mathfrak{so}(n)$, being reductive, comes equipped with an invariant, non-degenerate form, namely the trace form. One can easily show that $\mathfrak{g} = \mathfrak{g}_{} \oplus \mathfrak{g}_{}^\perp$ where $\mathfrak{g}_{}^\perp$ denotes the orthogonal compliment of $\mathfrak{g}_{}$ with respect to the trace form. This in particular means that the restriction of the form to $\mathfrak{g}_{}$ must be non-degenerate, since it is non-degenerate on $\mathfrak{g}$. Thus, we can use the trace form to establish an isomorphism $\mathfrak{g}^*_{} \simeq \mathfrak{g}_{}$ as we did for $\mathfrak{g}$ in section 1.3. Thus, we have that the isomorphism discussed in section 1.3 carries the subalgebra $S(\mathfrak{g}_{} ) \subset S(\mathfrak{g} )$ to the subalgebra $P(\mathfrak{g}_{} ) \subset P(\mathfrak{g} )$. Thus, we can naturally think of polynomials on $\mathfrak{g}_{}$ as subalgebra of all polynomials on $\mathfrak{g}$. By constuction this is clearly a Poisson subalgebra, since $S(\mathfrak{g}_{} )$ is a Poisson subalgebra of $S(\mathfrak{g} )$. From now on, we identify $P(\mathfrak{g}_{} )$ with its image in $P(\mathfrak{g} )$.

Now, we recall that we also embedd the adjoint groups $G_i = GL(i)$ or $SO(i)$ into $GL(n)$ or $SO(n)$, via the embedding,

$$g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & Id_{n-i} \end{bmatrix}$$

for $g \in G_i$. Using this embedding, it is easy to see that the trace form on $\mathfrak{g}$ is $G_i$-equivariant. Thus, it naturally maps the invariants $S(\mathfrak{g}_{} )^{G_i}$ isomorphically onto
Thus, we have an isomorphism

\[ J(\mathfrak{g}) \rightarrow P(\mathfrak{g}_{i_1})^{G_{i_1}} \otimes \cdots \otimes P(\mathfrak{g}_n)^{G_n} \]  

(1.23)

So that \( P(\mathfrak{g}_{i_1})^{G_{i_1}} \otimes \cdots \otimes P(\mathfrak{g}_n)^{G_n} \) is a Poisson commuting subalgebra of \( P(\mathfrak{g}) \). From now, on we will refer to \( P(\mathfrak{g}_{i_1})^{G_{i_1}} \otimes \cdots \otimes P(\mathfrak{g}_n)^{G_n} \) as \( J(\mathfrak{g}) \).

### 1.7 Commuting vector fields derived from the classical analogue of the Gelfand-Zeitlin algebra

We observed in the previous section that

\[ S(\mathfrak{g}_{i_1})^{G_{i_1}} \otimes \cdots \otimes S(\mathfrak{g}_n)^{G_n} \simeq P(\mathfrak{g}_{i_1})^{G_{i_1}} \otimes \cdots \otimes P(\mathfrak{g}_n)^{G_n} = J(\mathfrak{g}) \]  

(1.24)

is Poisson commutative. (Recall that \( i_1 = 1 \) for \( \mathfrak{g} = M(n) \) and \( i_1 = 2 \) for \( \mathfrak{g} = \mathfrak{so}(n) \).) For \( f, g \in J(\mathfrak{g}) \), we notice that by Corollary 1.5.1 we have that \( [\xi_f, \xi_g] = \xi_{\{f, g\}} = 0 \).

**Remark 1.7.1.** Suppose \( f \in P(\mathfrak{g})^G \) then it is easy to see that the Hamiltonian vector field generated by \( f \), \( \xi_f \) is trivial. This follows from the fact that \( f \in P(\mathfrak{g})^G \) is constant on an adjoint orbit and therefore \( \xi_f|_{\mathfrak{g}_x} = 0 \), by Proposition 1.5.2.

Let \( \text{rk}\, G_i \) denote the rank of the group \( G_i \). Now, if we take \( f_{i,j} \in P(\mathfrak{g}_i)^{G_i} \) for \( i_1 \leq i \leq n - 1, 1 \leq j \leq \text{rk}\, G_i \) to be a choice of fundamental adjoint invariants, then we see that we have a commutative Lie algebra of vector fields

\[ L = \text{span}\{\xi_{f_{i,j}}|i_1 \leq i \leq n - 1, 1 \leq j \leq \text{rk}\, G_i\} \]  

(1.25)

Now, the upper bound on the dimension of \( L \) is the sum \( m = \sum_{i=i_1}^{n-1} \text{rk}\, \mathfrak{g}_i \).

If we let \( d \) be the dimension of a regular adjoint orbit of \( G \) in \( \mathfrak{g} \), then the amazing fact is that in the cases of \( \mathfrak{g} = M(n) \) or \( \mathfrak{g} = \mathfrak{so}(n) \) we have

\[ m = d/2 \]
Thus, we have potentially found a commuting system of symplectic vector fields on the adjoint orbits $\mathcal{O}_X$ in $\mathfrak{g}$ of dimension exactly half the dimension of a regular adjoint orbit. If we could integrate this Lie algebra, it should in theory allow us to construct polarizations of regular adjoint orbits in $\mathfrak{g}$.

**Remark 1.7.2.** We note that we could have done the same analysis with the Gelfand-Zeitlin algebra and the Poisson commuting Lie algebra $J(\mathfrak{g})$ for Lie algebras of Type $C_l$. (In this case, we would have $\mathfrak{g}_i = \mathfrak{sp}(2i, \mathbb{C})$ for $1 \leq i \leq n - 1$.) We would also have obtained that the Lie algebra $L$ in equation (1.25) was commutative. However, in this case $m = \sum_{i=1}^{n-1} \text{rk } \mathfrak{g}_i$ is less than $d/2$. So this commuting system would not allow to construct polarizations of regular adjoint orbits. For example with the Lie algebra $C_5$ (i.e. $\mathfrak{sp}(10)$), one computes that $m = 10$, but $d/2 = 25$. This indicates that there is no direct generalization to Lie algebras of Type $C_n$.

Moreover, no subalgebra of dimension $d/2$ of the commutative Lie algebra of vector fields $V = \{\xi_f| f \in J(\mathfrak{g})\}$ could be integrated to a group action which would have orbits of dimension $d/2$. This is because any subalgebra $V'$ of $V$ defines a collection of subspaces $V'_x \subset V_x = \{(\xi_f)_x| f \in J(\mathfrak{g})\} \subset T_x(\mathfrak{g})$ for $x \in \mathfrak{g}$, which are of dimension at most $m$. This follows from the fact that $V_x = \text{span}\{ (\xi_{q_i})_x | i \in I \}$ where $\{q_i\}_{i \in I}$ generate the algebra $J(\mathfrak{g})$. Thus, the orbits of group constructed from integrating the subalgebra $V'$ could be of at most dimension $m$.

Let us recall what we mean by a regular or regular adjoint orbit and a polarization in this context.

**Definition 1.7.1.** Let $\mathfrak{g}$ be a finite dimensional, reductive Lie algebra over $\mathbb{C}$. Then $X \in \mathfrak{g}$ is regular if and only if its adjoint orbit $\mathcal{O}_X$ is of maximal dimension $\dim \mathfrak{g} - \text{rank } \mathfrak{g}$. Equivalently, $X \in \mathfrak{g}$ is regular if and only if its centralizer is of minimal dimension $\text{rank } \mathfrak{g}$.

The following definition is adapted from ([Ki, pg 24]).

**Definition 1.7.2.** We let $(M, \omega)$ be a symplectic manifold. Let $TM$ be the tangent bundle on $M$. A polarization $P \subset TM$ is an integrable subbundle of the tangent bundle $TM$ such that at each point $m \in M$, $P(m)$ is a maximal isotropic subspace.
(i.e. \(P(m) = P(m)^\perp\)) of the symplectic vector space over \(\mathbb{C}\), \((T_m(M), \omega_m)\). In particular this means that \(\dim P = 1/2 \dim M\).

The integral submanifolds of \(P\) are called leaves of the polarization and are necessarily Lagrangian submanifolds of \(M\).

Thus, to be able to use the Lie algebra \(L\) defined in equation (1.25) to construct polarizations of regular adjoint orbitis, we would have to show essentially things:

1) The Lie algebra \(L\) has dimension \(d/2\) and the points \(y \in \mathcal{O}_X\) for which \(\dim L = d/2\) form a submanifold of \(\mathcal{O}_X\) on which the assignment
\[
y \to \text{span}\{(\xi_{f_{i,j}})_y|i_1 \leq i \leq n - 1, 1 \leq j \leq \text{rk} G_i\}
\]
defines a distribution.

2) Every regular adjoint orbit \(\mathcal{O}_x\) contains elements \(x\) for which the subspace of the tangent space \(\text{span}\{(\xi_{f_{i,j}})_x|i_1 \leq i \leq n - 1, 1 \leq j \leq \text{rk} G_i\} \subset T(\mathcal{O}_x)\) is of maximal dimension \(d/2\).

Since, the Lie algebra \(L\) is commutative, the distribution would be integrable, but we also want to have natural coordinates on the leaves so that we can describe the geometry of the polarizations. We will see that we will actually be able to show the vector fields in \(L\) integrate to a global action of \(\mathbb{C}^{d/2}\) on \(\mathfrak{g}\). We will also observe that the tangent space to a point \(x\) in one of these orbits is given by
\[
\text{span}\{(\xi_{f_{i,j}})_x|i_1 \leq i \leq n - 1, 1 \leq j \leq \text{rk} G_i\}
\]

In the case of \(\mathfrak{g} = M(n)\) 1) and 2) were proved by Kostant and Wallach in [KW]. In the case of \(\mathfrak{g} = \mathfrak{so}(n)\), we will show that the Lie algebra in equation (1.25) can be integrated to a group action on all of \(\mathfrak{so}(n)\), whose orbits of maximal dimension can be used to construct polarizations of certain regular adjoint orbitis. However, it is still open if 2) is valid for \(\mathfrak{g} = \mathfrak{so}(n)\) and if one can use this Gelfand-Zeitlin system to construct polarizations of all regular adjoint orbitis in \(\mathfrak{so}(n)\).

Let us now turn our attention to the work of Kostant and Wallach in the case of \(\mathfrak{g} = M(n)\) which was the foundation of our current research.
2 The group $A$ and the polarization of generic adjoint orbits in $M(n)$

2.1 Integration of commuting vector fields from Gelfand Zeitlin theory and the group $A$

We first need to introduce some terminology that will be used throughout the rest of the thesis. If $X \in M(n)$ we call the $i \times i$ upper left-hand corner of $X$ the $i \times i$ cutoff of $X$ and denote in by $X_i$. We think of $X_i \in M(n)$ via the embedding $M(i) \hookrightarrow M(n)$ which sends $Y \in M(i)$ to

$$
\begin{bmatrix}
Y & 0 \\
0 & 0_{n-i}
\end{bmatrix}
$$

Recall the algebra $J(g)$ defined in equation (1.24) in section 1.7.

$$
J(g) = P(g_1)^{G_1} \otimes \cdots \otimes P(g_n)^{G_n} \subset P(g)
$$

where $g_i = M(i)$ and $G_i = GL(i)$. Recall that we think of $G_i$ as embedded in $GL(n)$ in the top left hand corner. We can define a commutative Lie algebra of vector fields by

$$
V = \{(\xi_f) | f \in J(M(n))\}
$$
By evaluating at \( X \in M(n) \), we get a collection of tangent spaces

\[
X \rightarrow V_X = \{(\xi_f)_X | f \in J(M(n))\}
\]

In Remark 2.8 in [KW, pg 20] the authors observe that if \( J(M(n)) \) is generated by \((q_i)_{i \in I}\), then \( V_X \) is spanned by the tangent vectors \((\xi_{q_i})_X\) for \( i \in I \). Thus, by our discussion in section 1.7, we have that

\[
\dim V_X \leq d/2
\]

In the case of \( M(n) \) we have that \( d/2 = \binom{n}{2} \). To integrate the Lie algebra \( L \) in equation 1.25 to a specific action of \( \mathbb{C}^{(n)} \), we choose a special set of generators for \( J(M(n)) \). For \( 1 \leq i \leq n-1 \) and \( 1 \leq j \leq i \) let \( f_{i,j} = \text{tr}(X^j_i) \). Denote by \( a \) the commutative Lie algebra of vector fields \( L \) on \( M(n) \) for this choice of generators. i.e. we have that

\[
a = \text{span}\{\xi_{f_{i,j}} | 1 \leq i \leq n-1, 1 \leq j \leq i\}
\]

Kostant and Wallach prove the following theorem in [KW, pgs 32-34].

**Theorem 2.1.1.** The Lie algebra \( a \) integrates to an action of a commutative complex analytic group \( A \simeq \mathbb{C}^{(n)} \) on \( M(n) \) with \( \text{Lie}(A) = a \). For \( X \in M(n) \), the tangent space to the \( A \) orbit of \( X \) is the subspace \( V_X \).

**Remark 2.1.1.** One can actually realize the \( A \) orbits more explicitly as the composition of the commuting flows of the complete vector fields \( \xi_{f_{i,j}} \). One can compute that the vector field \( \xi_{f_{i,j}} \) integrates to a global action of \( \mathbb{C} \) on \( M(n) \) given by

\[
\text{Ad} \left( \begin{bmatrix} \exp(tjX^{j-1}_i) & 0 \\ 0 & \text{Id}_{n-i} \end{bmatrix} \right) \cdot X
\]

Since the vector fields \( \xi_{f_{i,j}} \) commute, we can compose these flows in any order we like to get an action of \( \mathbb{C}^{(n)} \) on \( M(n) \) whose orbits are the orbits of \( A \).

**Remark 2.1.2.** We note that since each vector field \( \xi_f \) is tangent to an adjoint orbit \( O_X \), we see that the action of \( A \) stabilizes the adjoint orbits.
Remark 2.1.3. One may ask what would happen if we had chosen a different set of
generators for $J(M(n))$. For example, we could have chosen $f_{i,j}$ to be the coefficient
of $t^{j-1}$ in the characteristic polynomial for $X_i$. Then, one can show that in this
case the Lie algebra $L$ in (1.25) integrates to the action of an isomorphic group
$A'$ on $M(n)$ and for $X \in M(n)$, $T_X(A' \cdot X) = V_X$. The action of $A'$ commutes
with the action of $A$ and has the same orbit structure as that of the action of $A$
on $M(n)$. Since we are concerned with studying the geometry of these orbits, we
lose nothing by choosing to study the action of the particular group $A$. For more
details see Theorem 3.5 in [KW, pgs 34-35].

Since any $A \cdot X$ has tangent space equal to $V_X$ for any $X \in M(n)$, we see
that $\dim(A \cdot X) \leq \binom{n}{2}$. We will now turn our attention to studying those elements
of $M(n)$ whose $A$ orbits are of maximal dimension.

### 2.2 Strongly Regular Elements and Polarizations

We say that an element $X \in M(n)$ is strongly regular if and only if its $A$
orbit is of maximal dimension $\binom{n}{2}$. We denote the set of strongly regular elements
of $M(n)$ by $M^{sreg}(n)$. Note that

$$\dim A \cdot X = \binom{n}{2} \Leftrightarrow \dim V_X = \binom{n}{2}$$

Strongly regular is related to the term regular by the following theorem (see
Theorem 2.7 in [KW, pg 19]). (Recall Definition 1.7.1).

**Theorem 2.2.1.** $X \in M^{sreg}(n)$ if and only if the tangent vectors $(\xi_{f_{i,j}})_x \in T_x(O_x)$
are linearly independent for $1 \leq i \leq n-1$, $1 \leq j \leq i$ and if and only if the
differentials $(df_{i,j})_x$ are linearly independent for all $i$, $1 \leq i \leq n$ and $1 \leq j \leq i$.

Now, we recall a fundamental theorem of Kostant (see [K2, pg 382]).

**Theorem 2.2.2.** Let $\mathfrak{g}$ be a semi-simple Lie algebra. Let $\phi_1, \cdots, \phi_l$ (where $l$ is the
rank of $\mathfrak{g}$) be a basic set of adjoint invariants, then $X \in \mathfrak{g}$ is regular if and only if

$$d\phi_1(X) \wedge \cdots \wedge d\phi_l(X) \neq 0$$
Using Theorem 2.2.2, we see that if $X \in M^{sreg}(n)$ each cutoff $X_i$ is regular. However, this is not sufficient for $X \in M^{sreg}(n)$. We also note that the condition that $df_{1,1}(X) \wedge \cdots \wedge df_{n,n}(X) \neq 0$ is a Zariski open condition on $M(n)$. Thus, $M^{sreg}(n)$ is a Zariski open subset of $M(n)$.

We now describe a canonical set of strongly regular elements.

**Definition 2.2.1.** A matrix $X \in M(n)$ is said to be upper Hessenberg if it is of the form:

$$
X = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\
1 & a_{22} & \cdots & a_{2n-1} & a_{2n} \\
0 & 1 & \cdots & a_{3n-1} & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a_{nn}
\end{bmatrix}
$$

with $a_{ij} \in \mathbb{C}$ for $i \leq j$. We denote the set of upper Hessenberg matrices by $\text{Hess}$.

**Remark 2.2.1.** Note that $\text{Hess}$ is nothing other than the affine variety $f + b$ where $b$ is the standard Borel subalgebra of upper triangular matrices and $f$ is the sum of negative simple root vectors.

The beautiful fact about Hessenberg matrices is that they provide a cross-section to the following regular map.

$$\Phi_n : M(n) \to \mathbb{C}^{n(n+1)/2}$$

given by

$$\Phi_n(X) = (p_{1,1}(X_1), \cdots, p_{i,j}(X_i), \cdots, p_{n,n}(X_n))$$

(Where $p_{i,j}$ is the elementary symmetric polynomial of degree $i - j + 1$ in the eigenvalues of $X_i$). For $c \in \mathbb{C}^{n(n+1)/2}$ we refer to $\Phi_n^{-1}(c)$ as $M_c(n)$. Note that $X, Y \in M_c(n)$ if and only if each of their cutoffs $X_i$ and $Y_i$ have the same eigenvalues counting multiplicity. Thus, for $c \in \mathbb{C}^{n(n+1)/2}$ we write $c = (c_1, \cdots, c_i, \cdots, c_n)$ with
\( c_i \in \mathbb{C} \) and think of \( c_i \) as representing the coefficients of a monic polynomial of degree \( i \) (excluding the leading term).

The theorem concerning the Hessenberg matrices is the following one (see Theorem 2.3 in [KW, pg 16]):

**Theorem 2.2.3.** The map \( \Phi \) is surjective and its restriction to Hess is an algebraic isomorphism. In particular the differentials \( dp_{i,j}(X) \) for \( 1 \leq i \leq n, \ 1 \leq j \leq i \) are linearly independent for \( X \in \text{Hess} \).

**Remark 2.2.2.** Thus, the fibre \( \Phi^{-1}(c) = M_c(n) \) for \( c \in \mathbb{C}^{n+1 \choose 2} \) contains a unique Hessenberg matrix. This says that if one is given any sequence of \( {n+1 \choose 2} \) complex numbers, then there exists a matrix \( X \in M \) with those numbers as eigenvalues of the cutoffs \( X_i \) for \( 1 \leq i \leq n \), and moreover that matrix is unique if we insist that it is upper Hessenberg. The amazing fact is that there is no compatibility condition between the eigenvalues of neighbouring cutoffs. Compare this with interlacing condition that is required in Hermitian case (see [HJ, pg 185]).

Since the differentials \( dp_{i,j}(X) \) for \( 1 \leq i \leq n, \ 1 \leq j \leq i \) are linearly independent, we have :

\[
\text{Hess} \subset M^{\text{reg}}(n)
\]

**Remark 2.2.3.** Since the differentials \( dp_{i,j}(X) \), \( 1 \leq i \leq n, \ 1 \leq j \leq i \) are linearly independent for \( X \in \text{Hess} \), we have that the functions \( p_{i,j} \) are algebraically independent for \( 1 \leq i \leq n, \ 1 \leq j \leq i \). Thus, we see that the Gelfand-Zeitlin Algebra \( GZ(\mathfrak{g}) \) of \( \mathfrak{g} = M(n) \) defined in equation 1.22 and the algebra \( J(M(n)) = P(M(1))^{GL(1)} \otimes \cdots \otimes P(M(n))^{GL(n)} \) are both commutative algebras which are free on \( {n+1 \choose 2} \) generators and hence they are abstractly isomorphic as we promised in section 1.6.

As we will see below, it will be important to understand the interaction of the group \( A \) with the fibres \( M_c(n) \) of the map \( \Phi_c \). From the Poisson commutativity of the functions \( f_{i,j} \), it immediately follows that we have (see Proposition 3.6 in [KW, pg 35]).

**Proposition 2.2.1.** The action of \( A \) stabilizes the fibres \( M_c(n) \) for any \( c \in \mathbb{C}^{n+1 \choose 2} \).
Sketch of Proof:
This proposition follows immediately from the fact that two functions on a symplectic manifold Poisson commute if and only if they are invariant under each others’ Hamiltonian flows (see Remark 1.4.1) in section 1.4. We recall that the $A$ orbits are realized as the composition of the Hamiltonian flows of the vector fields $\xi_{f_{i,j}}$.
Q.E.D.

We want in particular to understand the $A$ orbits of maximal dimension in the fibre $M_c(n)$. We denote

$$M_c(n) \cap M^{sreg}(n) = M^{sreg}_c(n)$$

and we note that $M^{sreg}_c(n)$ is a Zariski open subset of $M_c(n)$. Note that it is clear that the action of $A$ preserves the fibres $M^{sreg}_c(n)$.

Now, by Remark 2.1.2, the adjoint orbit $O_X$ is a union of $A$ orbits. Now, we can use the existence of strongly regular elements to construct polarizations of all generic adjoint orbits in $M(n)$. The crucial theorem is the following one (see Theorem 3.36 in [KW, pg 53]).

**Theorem 2.2.4.** Let $X \in M(n)$ and let $O^{sreg}_X$ denote the set of strongly regular elements in $O_X$ i.e. $O^{sreg}_X = O_X \cap M^{sreg}(n)$. Then $O^{sreg}_X$ is non-empty if and only if $X$ is regular, in which case $O^{sreg}_X$ is a Zariski open subset of $O_X$ and is therefore a symplectic manifold (in the complex sense). Moreover, the $A$ orbits of dimension $\binom{n}{2}$ in $O_X$ are the leaves of a polarization of $O^{sreg}_X$.

Sketch of Proof:
If $O^{sreg}_X$ is non-empty, then by Theorem 2.2.1 it is clear that $X$ is regular. For the converse note that if $X$ is regular then, it is similar to a companion matrix, which is an upper Hessenberg matrix and thus a strongly regular element in $M(n)$.

The tangent space to each $A$ orbit in $O_X$ is an isotropic subspace of the symplectic vector space $T_x(O_X)$ by the Poisson commutativity of the functions $f_{i,j}$ and by Proposition 1.5.2. Thus, for $X \in O^{sreg}_X$, $T_X(A \cdot X)$ is maximal isotropic, since it is isotropic and of dimension exactly half the dimension of the ambient
manifold $\mathcal{O}_X^{\text{reg}}$. Recall that $\binom{n}{2}$ is exactly half the dimension of the regular adjoint orbit $\dim \mathcal{O}_X = n^2 - n$. Thus, the $A$ orbits in $\mathcal{O}_X^{\text{reg}}$ are Lagrangian submanifolds of $\mathcal{O}_X^{\text{reg}}$ and we have our desired polarization. Q.E.D.

Thus, we will turn our attention to studying the structure of the $A$ orbits of maximal dimension in a given regular adjoint orbit $\mathcal{O}_X$. To do that, we need to study the $A$ orbit structure of the fibres $M_c^{\text{reg}}(n)$. The reason for this being that if $X, Y \in M(n)$ are regular, then $Y \in \mathcal{O}_X$ if and only if $X, Y$ have the same characteristic polynomial. Thus, to describe the strongly regular elements in $\mathcal{O}_X$ it is enough to describe $M_c^{\text{reg}}(n)$ for $c = (c_1, \cdots, c_n)$ with $c_n \in \mathbb{C}^n$ determined by $\mathcal{O}_X$. We can then get a description of the leaves of the polarization of $\mathcal{O}_X^{\text{reg}}$ by describing the $A$ orbit structure on the fibres $M_c^{\text{reg}}(n)$ for this particular choice of $c_n \in \mathbb{C}^n$.

In [KW], the authors abstractly describe the fibres $M_c^{\text{reg}}(n)$ as smooth quasi-affine subvarieties of $M(n)$ with finitely many irreducible components which coincide exactly with the $A$ orbits in $M_c^{\text{reg}}(n)$. (see Propositions 3.10, 3.11 and Theorem 3.12 in [KW, pgs 37-38]).

Theorem 2.2.5. Let $M_c^{\text{reg}}(n) = \bigcup_{i=1}^{N(c)} M_{c,i}^{\text{reg}}(n)$ be the irreducible component decomposition of the variety $M_c^{\text{reg}}(n)$. Then each irreducible component $M_{c,i}^{\text{reg}}(n)$ is smooth in $M_c^{\text{reg}}(n)$ and is of dimension $\binom{n}{2}$ so that $M_c^{\text{reg}}(n)$ is of pure dimension $\binom{n}{2}$. Moreover the irreducible components $M_{c,i}^{\text{reg}}(n)$ are precisely the $A$ orbits in $M_c^{\text{reg}}(n)$. Hence for $X \in M^{\text{reg}}(n)$, $A \cdot X$ is an irreducible, non-singular variety of dimension $\binom{n}{2}$.

In [KW] the authors go on to determine the $A$ orbit structure of $M_c^{\text{reg}}(n)$ for $c$ in a certain principal open subset of $\mathbb{C}(\binom{n+1}{2})$. The goal of much of our work was to understand in more detail the $A$ orbit structure of $M_c^{\text{reg}}(n)$ for any $c \in \mathbb{C}$ and to also come up with an algebraic parametrization for the $A$ orbits in $M^{\text{reg}}(n)$ and the larger set of matrices $M(n) \cap S$, where $S$ denotes the set of matrices each of whose cutoffs $X_i$ for $1 \leq i \leq n-1$ are regular. Let us first, consider the generic matrices studied by Kostant and Wallach.
2.3 The action of $A$ on generic matrices

Kostant and Wallach describe the orbit structure of the group $A$ on a special set of matrices $\Omega_n$, consisting of matrices for which each $X_i$ is regular diagonalizable (has distinct eigenvalues) and the cutoffs $X_i$ and $X_{i+1}$ have no eigenvalues in common. Recall that for $c \in \mathbb{C}^{(n+1)/2}$ we write $c = (c_1, \ldots, c_i, \ldots, c_n)$ with $c_i \in \mathbb{C}$ and think of $c_i$ as representing the coefficients of a monic polynomial of degree $i$ (excluding the leading term). We consider the set of $c \in \mathbb{C}^{(n+1)/2}$ such that the polynomial represented by $c_i$ has distinct roots and the monic polynomials represented by $c_i$ and $c_{i+1}$ have no roots in common. We say that such a $c \in \mathbb{C}^{(n+1)/2}$ satisfies the eigenvalue disjointness condition. We note that $\Omega_n = \bigcup_c M_c(n)$, where the union is taken over all $c$ that satisfy the eigenvalue disjointness condition. One can show that $\Omega_n$ is a Zariski principal open subset of $M(n)$ (see Remark 2.16 in [KW, pgs 24-25]). The theorems concerning the action of $A$ on $\Omega_n$ proved by Kostant and Wallach are Theorems 3.23 in [KW, pg 45] and Theorem 3.28 in [KW, pg 49]. We combine the important facts of both results in a single statement below.

**Theorem 2.3.1.** The elements of $\Omega_n$ are strongly regular. Let $c \in \mathbb{C}^{(n+1)/2}$ satisfy the eigenvalue disjointness condition, then the fibre $M_c(n) = M^{\text{sreg}}_c(n)$ is exactly one $A$ orbit. As a variety $M_c(n) \simeq (\mathbb{C}^*)^{(n+1)/2}$. Moreover, the fibre $M_c(n)$ is a single orbit of a free, algebraic action of the torus $(\mathbb{C}^*)^{(n+1)/2}$.

We were able to generalize this result to a larger class of matrices $\Theta_n$, which contain $\Omega_n$ as an open, dense subset. For matrices in $\Theta_n$, we relax the condition that each cutoff $X_i$ is diagonalizable, but still insist that it is regular. We also still insist that the spectra of two adjacent cutoffs have no intersection. Our result concerning $\Theta_n$ is the following theorem.

**Theorem 2.3.2.** The elements of $\Theta_n$ are strongly regular. For $c \in \mathbb{C}^{n(n+1)/2}$ such that $c_i$ and $c_{i+1}$ represent coefficients of relatively prime monic polynomials, we have $M^{\text{sreg}}_c(n) = M_c(n) \cap \Theta_n$ is exactly one $A$ orbit. Thus on $\Theta_n$ the upper Hessenberg matrices form a cross-section to the $A$ action. Moreover, the fibre $M^{\text{sreg}}_c(n) = A \cdot X \ (X \in M^{\text{sreg}}_c(n))$ is a single orbit of a free, algebraic action of the
complex, commutative connected algebraic group $G = G_{X_1} \times G_{X_2} \times \cdots \times G_{X_{n-1}}$. (Here $G_{X_i} \subset \text{Gl}(i)$ denotes the group of the centralizer of $X_i$. We think of $\text{Gl}(i) \hookrightarrow \text{Gl}(n)$ as embedded in the top left hand corner.)

2.4 The action of $A$ on degenerate matrices

We now describe our results for the $A$ orbit structure on an arbitrary fibre $M_{c}^{\text{reg}}(n)$, which give descriptions of polarizations of all regular adjoint orbits in $M(n)$.

Recall from the previous section that we defined the set of generic matrices $\Theta_n$ to be matrices whose cutoffs $X_i$ are regular for $1 \leq i \leq n$ and with the property that the cutoffs $X_i$ and $X_{i+1}$ do not share any eigenvalues. We now consider the $A$ orbit structure on sets of matrices whose cutoffs are regular, but the spectra of two adjacent cutoffs $X_i$ and $X_{i+1}$ intersect. Although such matrices can often be realized as simple deformations of matrices in $\Theta_n$, their $A$ orbit structure is radically different. The main results is:

**Theorem 2.4.1.** Let $c = (c_1, c_2, \cdots, c_i, \cdots, c_n) \in \mathbb{C}^{n(n+1)/2}$ with $c_i \in \mathbb{C}^i$. Suppose there are $0 \leq j_i \leq i$ roots in common between the monic polynomials represented by $c_i$ and $c_{i+1}$, $1 \leq i \leq n-1$. Then the number of $A$ orbits in $M_{c}^{\text{reg}}(n)$ is exactly $2 \sum_{i=1}^{n-1} j_i$. Once again, we have that on $M_{c}^{\text{reg}}(n)$ the action of $A$ has the same orbits of a free, algebraic action of a complex, commutative connected algebraic group $G = G_{X_1} \times G_{X_2} \times \cdots \times G_{X_{n-1}}$. ($X \in M_{c}^{\text{reg}}(n)$).

Now, we have several corollaries to Theorem 2.4.1.

**Corollary 2.4.1.** The action of $A$ is transitive on $M_{c}^{\text{reg}}(n)$ if and only if $c = (c_1, \cdots, c_n)$ with $c_i$ and $c_{i+1}$ representing coefficients of relatively prime monic polynomials. Thus, $\Theta_n$ is the largest set of strongly regular matrices for which the $A$ action is transitive on the fibres $M_{c}^{\text{reg}}(n)$ over $\Theta_n$.

**Corollary 2.4.2.** The nilfibre $M_0^{\text{reg}}(n)$ contains $2^{n-1}$ $A$ orbits.

We will see below in Chapter 6, section 6.2 that the nilfibre $M_0^{\text{reg}}(n)$ has a much richer structure than the corollary indicates.
2.5 Summary of the $\Gamma_n$ Construction

Let us now discuss the construction that was invented to analyze the structure of the $A$ orbits of dimension $\binom{n}{2}$ in the fibres $M_c(n)$. Let $S$ be the Zariski open subset of $M(n)$ for which each $i \times i$ cutoff $X_i$ is regular for $1 \leq i \leq n - 1$. The key idea is to parameterize the $A$ orbits in $M^\text{reg}_c(n)$ and more generally in $M_c(n) \cap S$ using a connected commutative algebraic group which is a subgroup of the product of centralizers $G_{X_1} \times \cdots \times G_{X_{n-1}}$ for $X \in M_c(n) \cap S$.

In order to construct this parametrization, we need to see that we if we are given $X \in M(i+1)$, with $X_i$ in regular Jordan form with an arbitrary characteristic polynomial that we choose the values of the last column and row of $X$ so that $X$ can have any monic polynomial of degree $i + 1$ as its characteristic polynomial. Theorem 2.2.3 indicates that this can be done for a Hessenberg matrix, but we will show that it can be accomplished for a much larger set of matrices.

More specifically, we want to consider the following type of extension problem. Suppose that we are given an $(i + 1) \times (i + 1)$ matrix of the form

\[
\begin{bmatrix}
\lambda_1 & 1 & \cdots & 0 \\
0 & \lambda_1 & \ddots & \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_1 \\
& & & & y_{1,1} \\
& & & & \vdots \\
& & & & y_{1,n_1} \\
\end{bmatrix}
\begin{bmatrix}
0 \\
\vdots \\
0 \\
\lambda_r & 1 & \cdots & 0 \\
0 & \lambda_r & \ddots & \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_r \\
y_{r,1} & & & y_{r,n_r} \\
\end{bmatrix}
\begin{bmatrix}
z_{1,1} & \cdots & \cdots & z_{1,n_1} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
z_{r,1} & \cdots & \cdots & z_{r,n_r} \\
w & & & w
\end{bmatrix}
\]

with $\lambda_j \neq \lambda_k$ for $k \neq j$. We want to choose the $z_{i,j}$ and $y_{i,j}$ so that this matrix has characteristic polynomial given by $p(t) \in \mathbb{C}[t]$ with $p(t)$ monic and $\deg p(t) = i + 1$. We will see that such solutions can always be found regardless of the spectrum of the $i \times i$ cutoff or the choice of polynomial $p(t)$. The set of such solutions will be referred to as the solution variety at level $i$ and denoted by $\Xi^i_{c_i,c_{i+1}}$ (Where we
think of \( c_i \in \mathbb{C}^i \) as denoting the coefficients of a monic polynomial of degree \( i \), similarly for \( c_{i+1} \). We note that the group of the centralizer of the \( i \times i \) cutoff in \( GL(i) \), \( G_i \) acts algebraically on the solution variety \( \Xi_{c_i,c_{i+1}}^i \). We will then take the orbits in \( \mathcal{O}_{a_j}^i \subset \Xi_{c_i,c_{i+1}}^i \) of \( G_i \) that consist of regular elements of \( M(i+1) \) for \( 1 \leq i \leq n-2 \). At level \( n-1 \) we can choose any orbits of \( G_{n-1} \) in \( \Xi_{c_{n-1},c_n}^{n-1} \). We will then use these orbits to construct a regular map:

\[
\Gamma_{a_1,a_2,\ldots,a_{n-1}}^n : \mathcal{O}_{a_1}^1 \times \cdots \times \mathcal{O}_{a_{n-1}}^{n-1} \to M_c(n) \cap S
\]
given by:

\[
\Gamma_{a_1,a_2,\ldots,a_{n-1}}^n(x_1, \ldots, x_{n-1}) = \text{Ad}(g_{1,2}(x_1)^{-1}g_{2,3}(x_2)^{-1} \cdots g_{n-2,n-1}(x_{n-2}))x_{n-1}
\]

where \( g_{i,i+1}(x_i) \) conjugates \( x_i \) into Jordan canonical form. The crucial theorem which we will prove that will allow us to describe and count the number of \( A \) orbits in \( M_c^{\text{reg}}(n) \) for any \( c \in \mathbb{C}^{(n+1)/2} \) is the following one.

**Theorem 2.5.1.** The image of the map \( \Gamma_{a_1,a_2,\ldots,a_{n-1}}^n \) is exactly one \( A \) orbit in \( M_c(n) \cap S \).

With this goal in mind, we now set out to study the extension problem and the solution varieties \( \Xi_{c_i,c_{i+1}}^i \).
3 Solution Varieties

3.1 Extension Problems

We consider the following extension problem which we will use to define the solution variety at level $i$, $\Xi_{c_i,c_{i+1}}^i$ (see section 2.5). Recall that we are thinking of $c_i \in \mathbb{C}^i$ as representing the coefficients of a monic polynomial of degree $i$ excluding the leading leading term (viz. if $p(t) = z_1 + z_2t + \cdots + z_it^{i-1} + t^i$, then $c_i = (z_1, \cdots, z_i)$). Suppose the polynomial represented by $c_i$ factors as $\prod_{j=1}^{r_i}(\lambda_j - t)^{n_j}$ and the one represented by $c_{i+1}$ factors as $\prod_{j=1}^{s}(\mu_j - t)^{m_j}$. Suppose we are given an $(i + 1) \times (i + 1)$ matrix of the form:

\[
\begin{bmatrix}
\lambda_1 & 1 & \cdots & 0 \\
0 & \lambda_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_1 \\
\end{bmatrix}
\begin{bmatrix}
\vdots \\
y_{i,1} \\
0 \\
y_{i,n_i} \\
\vdots \\
0 \\
y_{r,1} \\
\vdots \\
y_{r,n_r} \\
0 \\
y_{i,1} \\
\vdots \\
y_{i,n_i} \\
0 \\
y_{r,1} \\
\vdots \\
y_{r,n_r} \\
0 \\
y_{i,1} \\
\vdots \\
y_{i,n_i} \\
0 \\
y_{r,1} \\
\vdots \\
y_{r,n_r} \\
\end{bmatrix}
\]

(3.1)

(\text{where } \lambda_i \neq \lambda_j). Now, we want to find values for the $z_{i,j}$ and the $y_{i,j}$ such that the characteristic polynomial of the above matrix is equal to

\[
\prod_{j=1}^{s}(\mu_j - t)^{m_j} \text{ where } \sum_{j=1}^{s} m_j = i + 1
\]
We call the variety of all such solutions to this extension problem the solution variety at level $i$ and denote it by $\Xi_{c_i, c_{i+1}}$. Notice that when we define the solution variety, we have inherently fixed an ordering of the Jordan blocks of the $i \times i$ cutoff $X_i$. To have a well-defined way of writing down elements of the solution variety $\Xi_{c_i, c_{i+1}}$, we introduce a lexicographical ordering in $\mathbb{C}$ defined as follows. Let $z_1, z_2 \in \mathbb{C}$, we say that $z_1 > z_2$ if and only if $\Re z_1 > \Re z_2$ or if $\Re z_1 = \Re z_2$ then $\Im z_1 > \Im z_2$. We assume, unless otherwise stated, that the eigenvalues of the $i \times i$ cutoff of an element in $\Xi_{c_i, c_{i+1}}$ are in decreasing lexicographical order.

We now show that $\Xi_{c_i, c_{i+1}}$ is non-empty. The first step is to compute explicitly the characteristic polynomial of the matrix in (3.1).

**Proposition 3.1.1.** The characteristic polynomial of the matrix in (3.1) is given by:

$$
(w - t) \prod_{k=1}^r (\lambda_k - t)^{n_k} + \sum_{j=1}^r (-1)^{n_j} \prod_{k=1, k \neq j}^r (\lambda_k - t)^{n_k} \sum_{l=0}^{n_j-1} \sum_{j'=1}^{n_j-l} y_{j, j'} (t - \lambda_j)^{n_j-l-1} (3.3)
$$

We prove Proposition (3.1.1) by first proving two special cases which we can then use in considering the general case. The first and simplest case is when the $i \times i$ cutoff of the matrix in (3.1) is a principal nilpotent Jordan block.

We consider the following $(i + 1) \times (i + 1)$ matrix:

$$
\begin{bmatrix}
0 & 1 & \cdots & 0 & y_1 \\
0 & 0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & 1 & \vdots & \\
0 & \cdots & \cdots & 0 & y_i \\
z_1 & \cdots & \cdots & z_i & w
\end{bmatrix} (3.4)
$$

The characteristic polynomial of the matrix in (3.4) is given in the following proposition.

**Proposition 3.1.2.** The characteristic polynomial of the matrix in (3.4) is given
by:

\[ = (-1)^{i+1}t^{i+1} + (-1)^iw^t \]

\[ +(-1)^{it} \sum_{j=1}^{i} z_j y_j + (-1)^{it-2} \sum_{j=1}^{i-1} z_j y_{j+1} + (-1)^{it-3}(\sum_{j=1}^{i-2} z_j y_{j+2}) + \cdots \]

\[ +(-1)^i t \sum_{j=1}^{2} z_j y_{j+(i-2)} + (-1)^i z_1 y_i \]

(3.5)

Proof:

The proof proceeds by induction on \( i \), the size of the principle nilpotent cutoff. One can easily work out the cases \( i = 2, 3 \) either by hand or with a computer algebra system. Thus, we may assume that the above formula holds for the case \( j = i \), and we will show that it holds for \( i + 1 \). We evaluate the above determinant by expanding by cofactors along the last column in an elementary fashion. Consider the \( j \)-th cofactor for \( 1 \leq j < i + 1 \).

\[
(-1)^{i+j+1}y_j \begin{vmatrix}
-t & 1 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & -t & 1 & \cdots \\
\cdots & 0 & 0 & -t & 1 \\
\vdots & \vdots & \vdots & 0 & 0 & -t \\
\vdots & \vdots & \ddots & \ddots & -t \\
z_1 & z_2 & \cdots & z_j & \cdots & z_i
\end{vmatrix}
\]

Now, we evaluate the above cofactor by applying a sequence of elementary operations. We move the \( j \)-th column to the position of the \( i \)-th column by interchanging it successively with all columns \( C_l \) for \( j < l \leq i \). This results in the following de-
Now, we finally add the last column to the \( j \)-th column in (3.6) to get the following:

\[
\begin{vmatrix}
-1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & & \\
\vdots & -1 & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & -1 & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 1 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 \\
z_1 & z_2 & \cdots & z_j & z_{j+1} & \cdots & z_i & z_j
\end{vmatrix}
\]

(3.6)

Now, we are finally in a position to use the inductive hypothesis on (3.7). Now, to use the inductive hypothesis, we note carefully that our matrix is of the form in the statement of Proposition 3.1.2 with \( w = z_i + t \), \( y_{j-1} = 1 \), \( y_j = 0 \), for all \( i \) \neq \( j \) and \( \tilde{z}_l = z_l \) for \( 1 \leq l \leq j - 1 \), \( \tilde{z}_j = z_j + z_j + z_j = z_{l+1} \) for \( j + 1 \leq l \leq i - 1 \). We also note that we take \( \tilde{i} = i - 1 \), since the we are now dealing with an \( i \times i \) matrix and its \( (i - 1) \times (i - 1) \) cutoff. Thus, applying the induction hypothesis, the above determinant becomes:

\[
(-1)^{i+j+1}y_j(-1)^{i-j}[(-1)^i t^i + (-1)^{i-1} (z_j + t) t^{i-1} + (-1)^{i-1} t^{i-2} (z_{j-1}) +
+ (-1)^{i-1} t^{i-3} (z_{j-2}) + (-1)^{i-1} t^{i-4} (z_{j-3}) + \ldots + t^{i-2-(j-2)} z_1]
\]
(where we let $z_k = 0$ for $k \leq 0$). Simplifying, we have that the above becomes:

$$(-1)^i y_j [(z_j t^{i-1} + t^{i-2}(z_{j-1}) + t^{i-3}(z_{j-2}) + \cdots + t^{i-j}z_1]$$

$$= (-1)^i [z_j y_j t^{i-1} + t^{i-2}(z_{j-1}y_j) + t^{i-3}(z_{j-2}y_j) + \cdots + t^{i-j}z_1y_j]$$

Now, we have to treat the $(i + 1)$ cofactor separately. However, this is the easiest cofactor to deal with. It is simply given by:

$$(w - t)(-t)^i$$

So putting everything together, we get:

$$(-1)^{i+1}t^{i+1} + (-1)^iw^i + (-1)^i \sum_{j=1}^{i-l} y_j z_{j-l} t^{i-1-l} \quad (3.8)$$

Now, to be able to connect this with the formula in the statement of this proposition, we have to interchange the order of summation in (3.8). Interchanging the order of summation in the last sum (3.8), we get

$$\sum_{l=0}^{i-1} \sum_{j=l+1}^{i} y_j z_{j-l} t^{i-1-l} \quad (3.9)$$

Substituting this into (3.8), we finally arrive at:

$$(-1)^{i+1}t^{i+1} + (-1)^iw^i + (-1)^i \sum_{l=0}^{i-1} \sum_{j=l+1}^{i} y_j z_{j-l} t^{i-1-l} \quad (3.10)$$

Consider the sum

$$\sum_{j=l+1}^{i} y_j z_{j-l} t^{i-1-l}$$

We can make a simple change of variables in the sum by putting $j' = j - l$, then sum we get is:

$$\sum_{j'=1}^{i-l} y_{j'+l} z_{j'} t^{i-1-l}$$

Now, making this change in (3.10), we see that we get:

$$(-1)^{i+1}t^{i+1} + (-1)^iw^i + (-1)^i \sum_{l=0}^{i-1} \sum_{j'=1}^{i-l} y_{j'+l} z_{j'} t^{i-1-l}$$
Comparing this equation with the one in the statement of Proposition 3.1.2, we see that we have reached the desired result.

Q.E.D.

Now, using Proposition 3.1.2, we can easily find the characteristic polynomial of a matrix of the form (3.1), when the $i \times i$ cutoff is a single Jordan block, but not necessarily nilpotent.

**Proposition 3.1.3.** The characteristic polynomial of the matrix

$$
\begin{bmatrix}
\lambda & 1 & \cdots & 0 & y_1 \\
0 & \lambda & \ddots & \vdots & \\
\vdots & \ddots & 1 & \vdots \\
0 & \cdots & \cdots & \lambda & y_i \\
z_1 & \cdots & \cdots & z_i & w
\end{bmatrix}
$$

is given by

$$
(-1)^{i+1}(t-\lambda)^{i+1} + (-1)^i(w-\lambda)(t-\lambda)^i + (-1)^i \sum_{l=0}^{i-1} \sum_{j'=1}^{i-l} y_{j'+l} z_{j'} (t-\lambda)^{i-l} - 1 \quad (3.11)
$$

Proof:

We are interested in computing the following determinantal equation:

$$
\det
\begin{bmatrix}
\lambda - t & 1 & \cdots & 0 & y_1 \\
0 & \lambda - t & \ddots & \vdots & \\
\vdots & \ddots & 1 & \vdots \\
0 & \cdots & \cdots & \lambda - t & y_i \\
z_1 & \cdots & \cdots & z_i & w - t
\end{bmatrix}
$$

We reduce this to the principal nilpotent case by making the change of variables $t' = t - \lambda \Leftrightarrow t = t' + \lambda$ and $w' = w - \lambda$. Then we use the result of Proposition 3.1.2 with $t$ replaced by $t'$ to get that the above determinant is given by

$$
(-1)^{i+1} t'^{i+1} + (-1)^i w't'^i + (-1)^i \sum_{l=0}^{i-1} \sum_{j'=1}^{i-l} y_{j'+l} z_{j'} (t'-\lambda)^{i-l} =
$$

$$
(-1)^{i+1}(t-\lambda)^{i+1} + (-1)^i(w-\lambda)(t-\lambda)^i + (-1)^i \sum_{l=0}^{i-1} \sum_{j'=1}^{i-l} y_{j'+l} z_{j'} (t-\lambda)^{i-l}
$$
This completes the proof.
Q.E.D.

Now, we can use Proposition 3.1.3 to prove Proposition 3.1.1.

Proof of Proposition 3.1.1:
Recall that we are interested in computing \( \det(X - t \text{Id}) \) where \( X \) is the following matrix:

\[
\begin{bmatrix}
\lambda_1 & 1 & \cdots & 0 \\
0 & \lambda_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_1
\end{bmatrix}
\]

with \( \lambda_i \neq \lambda_j \). We reduce it to the case of one Jordan block in the following manner. We consider the \( l \)-th cofactor corresponding to the \( j \)-th Jordan block of \( X_i \), where \( 1 \leq l \leq n_j \) and \( 1 \leq j \leq r \). We can write this cofactor as follows. We let \( k_j = \sum_{i=1}^{j-1} n_i \).

\[
(-1)^{k_j+l+i+1} y_{j,l}
\]

\[
\begin{bmatrix}
J_{\lambda_1} - t \\
& \ddots \\
& & C_{j,l} \\
& & & J_{\lambda_r} - t
\end{bmatrix}
\]

\[
\begin{bmatrix}
J_{\lambda_1} - t \\
& \ddots \\
& & C_{j,l} \\
& & & J_{\lambda_r} - t
\end{bmatrix}
\]

\[
\begin{bmatrix}
z_1 \\
& \ddots \\
& & z_j \\
& & & z_r
\end{bmatrix}
\]

\( C_{j,l} \) denotes the \((n_j - 1) \times n_j\) matrix given by removing the \( l \)-th row from the \( j \)-th Jordan block of \( X_i - t \text{Id}_i \), \( J_{\lambda_i} - t \) denotes the \( n_1 \times n_1 \) Jordan block of eigenvalue
\[ \lambda - t, \text{ and } z_j \in \mathbb{C}^{n_j} \text{ is the row vector given by } z_j = (z_{j,1}, \cdots, z_{j,n_j}) \text{ (similarly for } z_1, \cdots, z_r \text{ and } J_{\lambda_r} - t) \]

Now, we claim that we can reduce the enormous determinant above to the following much more manageable \( n_j \times n_j \) determinant.

\[
( -1)^{l-n_j+1} y_{j,l} \prod_{k=1, k \neq j}^r (\lambda_k - t)^{n_k} \begin{vmatrix}
C_{j,l} & z_{j,1} & \cdots & z_{j,n_j} \\
z_{1,1} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
z_{1,n_1} & \cdots & \cdots & \cdots \\
\end{vmatrix}
\]

To see this, one expands the determinant in equation (3.12) by cofactors starting with the \( i-1 \)st row. This yields the following determinant.

\[
( -1)^{k_j+l+i+1} ( -1)(\lambda_r - t) y_{j,l} \prod_{r=k=1}^{j+1} (\lambda_k - t)^{n_k} \begin{vmatrix}
J_{\lambda_1} - t & \cdots & \\
\cdots & C_{j,l} & \cdots \\
\cdots & \cdots & J_{\lambda_r} - t \\
z_1 & \cdots & z_j & \cdots & z_r' \\
\end{vmatrix}
\]

where the Jordan block for \( \lambda_r - t \) is now of size \((n_r - 1) \times (n_r - 1)\) and \( z_r' = (z_{r,1}, \cdots, z_{r,n_r-1}) \). Now, we expand the above determinant by expanding by cofactors along the \( i-2 \) row, etc. Continuing in this fashion and working our way up to the position of the matrix \( C_{j,l} \), we arrive at the following determinant.

\[
( -1)^{k_j+l+i+1} ( -1)^{\sum_{k=j+1}^{r} n_r} y_{j,l} \prod_{k=j+1}^{r} (\lambda_k - t)^{n_k} \begin{vmatrix}
\lambda_1 - t & 1 & \cdots & 0 \\
0 & \lambda_1 - t & \cdots & : \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \lambda_1 - t \\
\end{vmatrix}
\]

where the Jordan block for \( \lambda_r - t \) is now of size \((n_r - 1) \times (n_r - 1)\) and \( z_r' = (z_{r,1}, \cdots, z_{r,n_r-1}) \). Now, we expand the above determinant by expanding by cofactors along the \( i-2 \) row, etc. Continuing in this fashion and working our way up to the position of the matrix \( C_{j,l} \), we arrive at the following determinant.

\[
( -1)^{k_j+l+i+1} ( -1)^{\sum_{k=j+1}^{r} n_r} y_{j,l} \prod_{k=j+1}^{r} (\lambda_k - t)^{n_k} \begin{vmatrix}
\lambda_1 - t & 1 & \cdots & 0 \\
0 & \lambda_1 - t & \cdots & : \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \lambda_1 - t \\
\end{vmatrix}
\]

Notice that \( ( -1)^{k_j+l+i+1} ( -1)^{\sum_{k=j+1}^{r} n_r} = ( -1)^{l-n_j+l+i+1} = ( -1)^{l-n_j+1} \). Now, the determinant in (3.14) is evaluated the same way by expanding by cofactors first
along the row above the matrix $C_{j,l}$ and then working up to the top left hand corner. A simple induction shows that this yields (3.13). Now, (3.13) we notice is the product of $\prod_{k=1, k\neq j}^{r} (\lambda_k - t)^{n_k}$ with the $(l, n_j + 1)$-th cofactor in the expansion of the following determinant.

$$\det \begin{bmatrix}
\lambda_j - t & 1 & \cdots & 0 & y_{j,1} \\
0 & \lambda_j - t & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots \\
0 & \cdots & \cdots & \lambda_j - t & y_{j,n_j} \\
z_{j,1} & \cdots & \cdots & z_{j,n_j} & w - t
\end{bmatrix}$$

This is because $(-1)^{l-n_j+1} = (-1)^{l+n_j+1}$. Thus, by equation (3.9) in the proof of Proposition 3.1.2 and using the change of variables used to deduce Proposition 3.1.3 the contribution from the $j$-th Jordan block of $X_i - tId_i$ to the desired determinant is given by

$$( -1)^{n_j} \prod_{k=1, k\neq j}^{r} (\lambda_k - t)^{n_k} \sum_{l=0}^{n_j-1} \sum_{j'=1}^{n_j-1} y_{j,j'+l} z_{j',j'} (t - \lambda_j)^{n_j-1-l}$$

(3.15)

Now, we sum (3.15) over all the Jordan blocks, $1 \leq j \leq r$ to get:

$$\sum_{j=1}^{r} \left[ (-1)^{n_j} \prod_{k=1, k\neq j}^{r} (\lambda_k - t)^{n_k} \sum_{l=0}^{n_j-1} \sum_{j'=1}^{n_j-1} y_{j,j'+l} z_{j',j'} (t - \lambda_j)^{n_j-1-l} \right]$$

(3.16)

Now, it remains only to consider the cofactor coming from the $(i+1, i+1)$ entry, but this is easily calculated to be:

$$(w - t) \prod_{k=1}^{r} (\lambda_k - t)^{n_k}$$

Adding this result to (3.16), we get the desired result at last.

$$(w - t) \prod_{k=1}^{r} (\lambda_k - t)^{n_k} + \sum_{j=1}^{r} \left[ (-1)^{n_j} \prod_{k=1, k\neq j}^{r} (\lambda_k - t)^{n_k} \sum_{l=0}^{n_j-1} \sum_{j'=1}^{n_j-1} y_{j,j'+l} z_{j',j'} (t - \lambda_j)^{n_j-1-l} \right]$$

Q.E.D.
Now, we want to find values for the $z_{i,j}$ and the $y_{i,j}$ so that the characteristic polynomial in (3.3) is equal to one of the form

$$\prod_{j=1}^{s}(\mu_j - t)^{m_j}$$

where $\sum_{j=1}^{s} m_j = i + 1$

Let us call the above polynomial $h$ and the one in (3.3) $g$.

We can easily solve for $w$ by insisting that the trace of the matrix (3.1) is equal to $\sum_{j=1}^{s} m_j \mu_j$. However, it is too difficult to solve for $y_{i,j}$ and the $z_{i,j}$ by equating coefficients of the polynomials $g$ and $h$. We will instead equate certain derivatives of these polynomials, making use of the fact that $i \times i$ cutoff of the matrix in (3.1) is a regular Jordan form. In the next section we will study in detail the following system of equations:

$$g^{(n_1-1)}(\lambda_1) = h^{(n_1-1)}(\lambda_1)$$
$$g^{(n_2-1)}(\lambda_1) = h^{(n_2-1)}(\lambda_1)$$
$$\vdots$$
$$g(\lambda_1) = h(\lambda_1)$$
$$\vdots$$
$$g^{(n_r-1)}(\lambda_r) = h^{(n_r-1)}(\lambda_r)$$
$$g^{(n_r-2)}(\lambda_r) = h^{(n_r-2)}(\lambda_r)$$
$$\vdots$$
$$g(\lambda_r) = h(\lambda_r)$$

(3.17)

Recall that $n_j$ is the size of the Jordan block corresponding to the eigenvalue $\lambda_j$ in the $i \times i$ cutoff of the matrix in (3.1).

**Remark 3.1.1.** Let us note that solving the above systems of equations is both necessary and sufficient for $g = h$, so long as we set $w = \sum_{j=1}^{s} m_j \mu_j - \sum_{k=1}^{r} n_k \lambda_k$. Then we note that the polynomial $q = g - h$ is of at most degree $i - 1$, since both $g$ and $h$ have leading coefficient $(-1)^{i+1}$ and we have chosen $w$ so that the trace of the matrix in (3.1) is $\sum_{j=1}^{s} m_j \mu_j$. Now, suppose that we have a polynomial
$q \in \mathbb{C}[t]$ such that
\[
q(\lambda_1) = 0 \\
q'(\lambda_1) = 0 \\
\vdots \\
q^{(n_1-1)}(\lambda_1) = 0
\]
Then it is easy to see that $(t - \lambda_1)^{n_1}$ must divide $q$. Thus, if the equations in (3.17) are satisfied, $q = g - h$ is divisible by the product $\prod_{i=1}^{r}(t - \lambda_i)^{n_i}$, since $\lambda_i \neq \lambda_j$ for $i \neq j$. Now, in our case $q = g - h$ has degree at most $i - 1$. But, $\sum_{j=1}^{r} n_j = i$ so that $q = 0$ and $g = h$.

### 3.2 The generic solution variety

We preserve the notation of that last section. Now, we discuss how we can use the above system of equations (3.17) to describe the solution variety $\Xi^{e_i,e_{i+1}}_{c_i}$ in various cases. We first describe the most generic case where $\lambda_k \neq \mu_j$ for any $j, k$.

Consider first the equations involving only $\lambda_1$, i.e. consider:
\[
\begin{align*}
g^{(n_1-1)}(\lambda_1) &= h^{(n_1-1)}(\lambda_1) \\
g^{(n_1-2)}(\lambda_1) &= h^{(n_1-2)}(\lambda_1) \\
\vdots \\
g(\lambda_1) &= h(\lambda_1)
\end{align*}
\]

Where we recall that $g$ is the characteristic polynomial of the matrix in (3.1), which we restate here for the convenience of the reader.

\[
(w-t) \prod_{k=1}^{r}(\lambda_k-t)^{n_k} + \sum_{j=1}^{r} (-1)^{n_j} \prod_{k=1,k\neq j}^{r}(\lambda_k-t)^{n_k} \sum_{l=0}^{n_j-1} \sum_{j'=1}^{n_j} \sum_{l'=1}^{n_j} y_{j,j'+l}z_{j,j'}(t-\lambda_j)^{n_j-1-l}
\]

A closer inspection of the equations in (3.18) shows that this system only involves coefficients from the first Jordan block. That is to say that the LHS of the system of equations in (3.18) involves only the variables $z_{1,j}$ and $y_{1,j}$ for $1 \leq j \leq n_1$. This is because all of the other terms in the polynomial $g$ are multiplied by a factor of $(t - \lambda_1)^{n_1}$ and therefore evaluate to 0 in $g^{(k)}(\lambda_1)$ for $0 \leq k \leq n_1 - 1$. Thus, in
solving the system in (3.18), we need now only consider the following terms from \( g \):

\[
\prod_{k=2}^{r} (\lambda_k - t)^{n_k} (-1)^{n_1} \sum_{l=0}^{n_1-1} \sum_{j'=1}^{n_1-l} y_{1,j'+l} z_{1,j'} (t - \lambda_1)^{n_1-1-l}
\] (3.19)

We notice that the coefficient in (3.19) of \((-1)^{n_1} (t - \lambda_1)^{n_1-j} \prod_{k=2}^{r} (t - \lambda_k)^{n_k}\) is given by the \( j \)–th row of the matrix product:

\[
\begin{bmatrix}
z_{1,1} & z_{1,2} & \cdots & z_{1,n_1} \\
0 & z_{1,1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & z_{1,2} \\
0 & \cdots & 0 & z_{1,1}
\end{bmatrix}
\begin{bmatrix}
y_{1,1} \\
\vdots \\
y_{1,n_1}
\end{bmatrix}
\] (3.20)

Note that the equation \( g(\lambda_1) = h(\lambda_1) \) yields

\[
(-1)^{n_1} z_{1,1} y_{1,n_1} \prod_{j=2}^{r} (\lambda_j - \lambda_1)^{n_j} = \prod_{k=1}^{s} (\mu_k - \lambda_1)^{m_k}
\]

\[
\Rightarrow z_{1,1} y_{1,n_1} = (-1)^{n_1} \frac{\prod_{k=1}^{s} (\mu_k - \lambda_1)^{m_k}}{\prod_{j=2}^{r} (\lambda_j - \lambda_1)^{n_j}}
\]

\[
\Rightarrow z_{1,1}, y_{1,n_1} \in \mathbb{C}^x
\] (3.21)

since \( \lambda_j \neq \lambda_1, 2 \leq j \leq r \) and \( \mu_k \neq \lambda_1, 1 \leq k \leq s \).

Now, the equation \( g'(\lambda_1) = h'(\lambda_1) \) allows us to solve for \( y_{1,n_1-1} \) uniquely as a regular function of \( z_{1,1} \) and \( z_{1,2} \in \mathbb{C} \), since \( \lambda_1 \neq \lambda_j \). We now proceed by induction to see that we can solve for \( y_{1,n_1-s} \) uniquely as a regular function of \( z_{1,1}, \ldots, z_{1,s+1} \). We first observe that the \( s \)–th derivative of

\[
\begin{bmatrix}
(-1)^{n_1} \prod_{k=1,k \neq 1}^{r} (\lambda_k - t)^{n_k} \sum_{l=0}^{n_1-1} \sum_{j'=1}^{n_1-l} y_{1,j'+l} z_{1,j'} (t - \lambda_1)^{n_1-1-l}
\end{bmatrix}
\]

evaluated at \( t = \lambda_1 \) only involve coefficients of terms of the form \((t - \lambda_1)^q \prod_{k=1,k \neq 1}^{r} (\lambda_k - t)^{n_k}\) for \( q \leq s \). Thus, the \( s \)-th derivative consists of terms of the form \( z_{1,j} \) and \( y_{1,n_1-j+1} \) for \( 1 \leq j \leq s+1 \). Now, by induction we can assume that we have solved for \( y_{1,n_1-j+1} \) for \( 1 \leq j < s+1 \) as uniquely as a regular function of \( z_{1,1} \in \mathbb{C}^x \) and \( z_{1,2}, \ldots, z_{1,j} \in \mathbb{C} \). So that we need only solve for \( y_{1,n_1-s} \). Now, the
triangular nature of the system in (3.20) allows us to see that \( y_{1,n_1-s} \) first occurs in the coefficient of \((-1)^{n_1} \prod_{k=2}^r (\lambda_k - t)^{nk} (t - \lambda_1)^s \) with coefficient \( z_{1,1} \in \mathbb{C}^x \). Thus, it first appears when one takes the \( s \)-th derivative and it has coefficient \((-1)^{n_1} s! \prod_{k=2}^r (\lambda_k - \lambda_1)^{nk} z_{1,1} \). Now, since \( z_{1,1} \in \mathbb{C}^x \) and \( \lambda_1 \neq \lambda_k \) for \( k \neq 1 \) then using the induction hypothesis and the triangular nature of the system in (3.20), we see that we can solve for \( y_{1,n_1-s} \) uniquely as a function of the \( z_{1,1}, \cdots, z_{1,s+1} \) with \( z_{1,s+1} \in \mathbb{C} \).

We can argue analogously for the other \( \lambda_k \) replacing the system in (3.18) by the analogous system for \( \lambda_k \).

**Remark 3.2.1.** Notice that we could have interchanged the roles of the \( z_{1,j} \) and the \( y_{1,j} \) and argued in a similar manner. Now, suppose that we take \( y_{1,n_1} \in \mathbb{C}^x \). Then the coefficient of \((-1)^{n_1} (t - \lambda_1)^{n_1-j} \prod_{k=2}^r (\lambda_k - t)^{nr} \) can be represented as the \( n_1 - j + 1 - th \) column of the matrix product.

\[
\begin{bmatrix}
  z_{1,1} & \cdots & \cdots & z_{1,n_1} \\
  y_{1,n_1} & y_{1,n_1-1} & \cdots & y_{1,1} \\
  0 & y_{1,n_1} & \cdots & \vdots \\
  \vdots & \ddots & \ddots & y_{1,n_1-1} \\
  0 & \cdots & 0 & y_{1,n_1}
\end{bmatrix}
\]

Thus, we can use the same induction argument to get that \( z_{1,s+1} \) can be solved uniquely as a regular function of \( y_{1,n_1}, \cdots, y_{1,n_1-s} \) with

\[
y_{1,n_1} \in \mathbb{C}^x \text{ and } y_{1,n_1-1}, \cdots, y_{1,n_1-s+1} \in \mathbb{C}
\]

We have now proven the following theorem.

**Theorem 3.2.1.** The generic solution variety \( \Xi_{c_i,c_i+1} \) is non-empty. Moreover, we have actually shown that \( \Xi_{c_i,c_i+1} \cong (\mathbb{C}^x)^r \times \mathbb{C}^{i-r} \subset \mathbb{C}^i \). Where the isomorphism is given by

\[
\Phi : \Xi_{c_i,c_i+1} \rightarrow (\mathbb{C}^x)^r \times \mathbb{C}^{i-r}
\]

is given by

\[
\Phi(A) = (z_{1,1}, z_{1,2}, \cdots, z_{r,1}, \cdots, z_{r,n_r}) \text{ for } A \in \Xi_{c_i,c_i+1}
\]
Proof:

An \( X \in \Xi_{c_i,c_i+1} \subset M(i + 1) \) is of the form

\[
\begin{bmatrix}
\lambda_1 & 1 & \cdots & 0 \\
0 & \lambda_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_1 \\
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & \cdots & y_{1,1} \\
0 & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & y_{1,n_1} \\
\end{bmatrix}
\begin{bmatrix}
\lambda_r & 1 & \cdots & 0 \\
0 & \lambda_r & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_r \\
\end{bmatrix}
\begin{bmatrix}
y_{r,1} \\
0 & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & w \\
\end{bmatrix}
\begin{bmatrix}
z_{1,1} & \cdots & \cdots & z_{1,n_1} & \cdots & \cdots & z_{r,1} & \cdots & \cdots & z_{r,n_r} \\
\end{bmatrix}
\]

where the bottom row \( \mathbf{z} = (z_{1,1}, z_{1,2}, \cdots, z_{r,1}, \cdots, z_{r,n_r}) \in (\mathbb{C}^*)^r \times \mathbb{C}^{i-r} \) is arbitrary and the \( y_{i,j} \) are regular functions in \( \mathbf{z} \) determined in our work above.

Thus the map \( \Phi \) is bijective. It is clearly regular and the inverse is easily seen to be regular. This completes the proof.

Q.E.D

Now, we observe that the group of the centralizer of the \( i \times i \) cutoff, which we denote by \( G_i \) acts regularly on \( \Xi_{c_i,c_i+1} \) by conjugation. Here we are thinking of \( Gl(i) \hookrightarrow Gl(i + 1) \) via the embedding

\[
Y \hookrightarrow \begin{bmatrix} Y & 0 \\ 0 & 1 \end{bmatrix}
\]

We now show:

**Proposition 3.2.1.** \( G_i \) acts simply transitively on \( \Xi_{c_i,c_i+1} \) and thus we have \( \Xi_{c_i,c_i+1} \simeq G_i \) as varieties.

Proof:

We first show that \( G_i \) acts freely on \( \Xi_{c_i,c_i+1} \). To show this we have to analyze the \( G_i \) action a little more closely. Now, since \( X_i \) is a regular Jordan block, we have
that $G_i$, as an algebraic group, is a product $G_i = G_{J_1} \times \cdots \times G_{J_r}$ where $G_{J_k}$ is
the centralizer of the Jordan block corresponding to $\lambda_k$. Elementary linear algebra
tells us that $G_{J_k}$ consists of invertible upper triangular Toeplitz matrices.

$$G_{J_k} = \begin{bmatrix}
a_{k,1} & a_{k,2} & \cdots & a_{k,n_k} \\
0 & a_{k,1} & \ddots & \\
\vdots & \ddots & \ddots & a_{k,2} \\
0 & \cdots & \cdots & a_{k,1}
\end{bmatrix}$$

with $a_{k,1} \in \mathbb{C}^\times$ and $a_{k,j} \in \mathbb{C}$ for $2 \leq j \leq n_k$. It is easy to see that when $G_i$ acts
on $\Xi^i_{c_i,c_{i+1}}$ via conjugation it is acting via the standard representation of $G_i$ on the
last column of $X$ and the dual representation on the bottom row. In this case, it is
enough to consider the action of $G_i$ on the last column. Since $G_i = G_{J_1} \times \cdots \times G_{J_r}$
the standard action on the left column is the standard diagonal action. i.e. we have:

$$\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n_1} \\
0 & a_{1,1} & \ddots & \\
\vdots & \ddots & \ddots & a_{1,2} \\
0 & \cdots & \cdots & a_{1,1}
\end{bmatrix} \Rightarrow 0$$

$$\begin{bmatrix}
a_{r,1} & a_{r,2} & \cdots & a_{i,n_r} \\
0 & a_{r,1} & \ddots & \\
\vdots & \ddots & \ddots & a_{r,2} \\
0 & \cdots & \cdots & a_{r,1}
\end{bmatrix} \begin{bmatrix}
y_{1,1} \\
y_{1,n_1} \\
y_{r,1} \\
y_{r,n_r}
\end{bmatrix}$$

which reduces to $r$ matrix products of the form

$$\begin{bmatrix}
a_{j,1} & a_{j,2} & \cdots & a_{j,n_j} \\
0 & a_{j,1} & \ddots & \\
\vdots & \ddots & \ddots & a_{j,2} \\
0 & \cdots & \cdots & a_{j,1}
\end{bmatrix} \begin{bmatrix}
y_{j,1} \\
y_{j,n_j}
\end{bmatrix}$$

for $1 \leq j \leq r$. Now, we suppose that $g \in G_i$ is in the stabilizer of $X \in \Xi^i_{c_i,c_{i+1}}$. 
Then we get $r$ matrix equations of the form
\[
\begin{bmatrix}
    a_{j,1} & a_{j,2} & \cdots & a_{j,n_j} \\
    0 & a_{j,1} & \ddots & \vdots \\
    \vdots & \ddots & \ddots & a_{j,2} \\
    0 & \cdots & \cdots & a_{j,1}
\end{bmatrix}
\begin{bmatrix}
    y_{j,1} \\
    \vdots \\
    \vdots \\
    y_{j,n_j}
\end{bmatrix}
= 
\begin{bmatrix}
    y_{j,1} \\
    \vdots \\
    \vdots \\
    y_{j,n_j}
\end{bmatrix}
\]

Now, we have to make use of the fact that $a_{j,1}, y_{j,n_j} \in \mathbb{C}^\times$. The above matrix equation becomes the following system of linear equations
\[
\sum_{k=1}^{n_j} a_{j,k} y_{j,k} = y_{j,1} \\
\sum_{k=1}^{n_j-1} a_{j,k} y_{j,k+1} = y_{j,2} \\
\vdots \\
\sum_{j=1}^{n_j-l+1} a_{j,k} y_{j,k+l-1} = y_{j,l} \\
\vdots \\
a_{j,1} y_{j,n_j} = y_{j,n_j}
\]

Now, we solve the above system easily using induction. The last equation in the system clearly gives us that $a_{j,1} = 1$, since $y_{j,n_j} \neq 0$. Now, assume that we have been able to show that $a_{j,1} = 1, a_{j,2} = 0, \ldots, a_{j,n_j-l+2} = 0$. We show that this forces $a_{n_j-l+1} = 0$. Consider the equation:
\[
\sum_{k=1}^{n_j-l+1} a_{j,k} y_{j,k+l-1} = y_{j,l}
\]

Using our inductive assumption, this equation becomes:
\[
a_{1,j} y_{j,l} + a_{n_j-l+1,j} y_{j,n_j} = y_{j,l}
\]

Since, $a_{j,1} = 1$, we have $a_{j,n_j-l+1} y_{j,n_j} = 0$. Again since $y_{j,n_j} \neq 0$, we must have $a_{j,n_j-l+1} = 0$.

Repeating this argument for each $1 \leq j \leq r$, we easily see that we have $g \in Stab(X) \Rightarrow g = Id$.

Thus, $G_i$ acts freely on $\Xi_1^i, \Xi_{c_i+1}^i$. So that each $G_i$ orbit in $\Xi_1^i, \Xi_{c_i+1}^i$ is of dimension $i$. Now, we have that $G_i$ being the centralizer of a regular element in $M(i)$ is a connected algebraic group so that each $G_i$ orbit in $\Xi_1^i, \Xi_{c_i+1}^i$ is an irreducible
variety of dimension \( i \). This implies that each orbit must be open since \( \Xi^i_{c_i,c_{i+1}} \) is itself an irreducible variety of dimension \( i \) by Theorem 3.2.1. Hence, \( G_i \) must act transitively on \( \Xi^i_{c_i,c_{i+1}} \), since any two non-empty subsets in an irreducible variety must intersect non-trivially. This completes the proof of the theorem.

Q.E.D.

We now turn our attention to investigating the structure of the solution variety \( \Xi^i_{c_i,c_{i+1}} \) when we allow for overlaps between the roots of the polynomials represented by \( c_i \) and \( c_{i+1} \).

### 3.3 \( \Xi^i_{c_i,c_{i+1}} \) in the case of overlap

Now suppose that \( \lambda = (\lambda_1, \cdots, \lambda_r) \) are eigenvalues of the \( i \times i \) cutoff and we want to extend this to an \( (i + 1) \times (i + 1) \) matrix whose eigenvalues are given by \( (\mu_1, \cdots, \mu_s) \). Now, suppose that for \( 1 \leq j \leq \min(r, s) \), we have that \( \lambda_1 = \mu_1, \cdots, \lambda_j = \mu_j \). Without loss of generality, we can assume that the overlaps occur in this manner. Now, an inspection of the equations in (3.17) shows that we must have the following equations holding:

\[
\begin{align*}
z_{k,1}y_{k,n} &= 0 \text{ for } 1 \leq k \leq j \\
z_{k,1}y_{k,n} &\in \mathbb{C}^x \text{ for } j < k \leq r
\end{align*}
\]

Now, we claim that we can find \( 2^j \) disjoint subsets of the solution variety on which \( G_i \) acts simply transitively. Suppose without loss of generality that in (3.23) we choose \( y_{1,n_1} = 0 \) and \( z_{1,1} \in \mathbb{C}^x \). Then we claim that we can still solve the system in (3.17) for the \( y_{1,j} \) as regular functions of the \( z_{1,j} \). This follows from our discussion in section 3.2. Similarly from Remark 3.2.1 in 3.2 we see that the same holds if we had chosen instead \( z_{1,1} = 0 \) and \( y_{1,n_1} \in \mathbb{C}^x \). The same argument works for all \( 1 \leq k \leq j \). Thus, the solution variety \( \Xi^i_{c_i,c_{i+1}} \) contains at least \( 2^j \) disjoint quasi-affine subvarieties, determined by choosing either \( z_{k,1} = 0 \) or \( y_{k,n_k} = 0 \) for each \( k, 1 \leq k \leq j \) but not both 0. We need a notation for these subvarieties. First, we define an index \( a_i \) that can take two values \( a_i = U, L \) (\( U \) for upper and \( L \) for lower). Then \( Z_{a_1,\cdots,a_j} \) is the subvariety determined as follows. Consider the
conditions in (3.23). If for \( k, 1 \leq k \leq j \), we choose the subvariety with \( z_{k,1} \neq 0 \), we think of this as being the lower subvariety at position \( k \) and assign \( a_k \) the value \( L \). Conversely, if we had chosen \( y_{k,n_k} \neq 0 \), we would think of this as being the upper subvariety at position \( k \) and assign \( a_k = U \). In this way it is clear that we get all of the \( 2^j \) subvarieties. The first observation about the subvarieties \( Z_{a_1,\ldots,a_j} \) is the following

**Proposition 3.3.1.** The subvarieties \( Z_{a_1,\ldots,a_j} \) are isomorphic to \((\mathbb{C}^*)^r \times \mathbb{C}^{i-r}\) as varieties. Thus, each \( Z_{a_1,\ldots,a_j} \) is a smooth, irreducible affine variety of dimension \( i \).

Proof:

The proof is completely analogous to the proof of Theorem 3.2.1, except one has to modify slightly the definition of the mapping \( \Phi : \Xi_{c_i,c_{i+1}}^i \to (\mathbb{C}^*)^r \times \mathbb{C}^{i-r} \) in that proof. Now, for \( Z_{a_1,\ldots,a_j} \) we define an isomorphism \( \Phi_{a_1,\ldots,a_j} : Z_{a_1,\ldots,a_j} \to (\mathbb{C}^*)^r \times \mathbb{C}^{i-r} \) given as follows.

\[
\Phi_{a_1,\ldots,a_j}(Z) = (x_1, \ldots, x_j, \overline{x_{j+1}}, \ldots, \overline{x_r})
\]

for \( Z \in Z_{a_1,\ldots,a_j} \) where \( \overline{x_k} = (z_{k,1}, \ldots, z_{k,n_k}) \) if \( a_k = L \) or \( \overline{x_k} = (y_{k,1}, \ldots, y_{k,n_k}) \) if \( a_k = U \) and with \( \overline{x_k} = (z_{k,1}, \ldots, z_{k,n_k}) \) for \( j < k \leq r \). With this definition of \( \Phi_{a_1,\ldots,a_k} \) and our work above in Remark 3.2.1, the same argument as we used in the proof of Theorem (3.2.1) shows us that this map is the desired isomorphism. This completes the proof.

Q.E.D

We again have in this case that the centralizer of the \( i \times i \) cutoff \( G_i \) acts on \( \Xi_{c_i,c_{i+1}}^i \). Now, we claim that we have the following proposition:

**Proposition 3.3.2.** \( G_i \) preserves each of the above subvarieties and acts simply transitively on them.

Proof:

We note first that it is easy to see that \( G_i \) preserves each one of these subvarieties. We show that \( G_i \) must act freely on the subvariety \( Z_{a_1,\ldots,a_j} \). Then we note that by
Proposition 3.3.1 gives us that each \( Z_{a_1, \ldots, a_j} \) is an irreducible variety of dimension \( i \). Thus, any \( G_i \) orbit in \( Z_{a_1, \ldots, a_j} \) must in fact be open and the action of \( G_i \) must be transitive.

The proof that the action of \( G_i \) is free is very similar to the proof of Proposition 3.2.1 in the generic case. If \( Z_{a_1, \ldots, a_j} \) has a \( l = U \) for some \( 1 \leq l \leq j \), then to see that component of the stabilizer of \( z \in Z_{a_1, \ldots, a_j} \) in \( G_{J_{\lambda_i}} \) must be trivial, we use the exactly same argument as in the proof of Proposition 3.2.1. If we have, on the other hand, that \( a_l = L \), then we argue that \( g \in G_i \) stabilizes \( z \) if and only if \( g^{-1} \) stabilizes \( z \). Then since \( g \) acts on the bottom row of \( z \in Z_{a_1, \ldots, a_j} \) via right translation by the inverse, a totally analogous argument to the one in Proposition 3.2.1, gives that the component of \( g^{-1} \) in \( G_{J_{\lambda_i}} \) must be trivial. More explicitly, we can consider the matrix equation

\[
\begin{bmatrix}
  z_{j,1} & \cdots & z_{j,n_j}
\end{bmatrix} \begin{bmatrix}
  a_{j,1} & a_{j,2} & \cdots & a_{j,n_j} \\
  0 & a_{j,1} & \ddots & \vdots \\
  \vdots & \ddots & \ddots & a_{j,2} \\
  0 & \cdots & \cdots & a_{j,1}
\end{bmatrix} = \begin{bmatrix}
  z_{j,1} & \cdots & z_{j,n_j}
\end{bmatrix}
\]

with \( z_{j,1} \in \mathbb{C}^\times \), \( z_{j,l} \in \mathbb{C} \) for \( l \geq 2 \) and \( a_{j,1} \in \mathbb{C}^\times \), \( a_{j,l} \in \mathbb{C} \) for \( l \geq 2 \).

Equating the \((1,1)\) entries on both the LHS and the RHS, we see that we have the equation \( z_{j,1} a_{j,1} = z_{j,1} \Rightarrow a_{j,1} = 1 \), since \( z_{j,1} \in \mathbb{C}^\times \). Now, we can proceed inductively as we did in the proof of Proposition 3.2.1. The equation for the \((1,k)\) entry is (for \( k \leq n_j \)):

\[
z_{j,1} a_{j,k} + z_{j,2} a_{j,k-1} + \cdots + z_{j,k-1} a_{j,2} + z_{j,k} a_{j,1} = z_{j,k}
\]

By induction, we have that this equation becomes:

\[
z_{j,1} a_{j,k} + z_{j,k} = z_{j,k} \Rightarrow z_{j,1} a_{j,k} = 0
\]

which yields that \( a_{j,k} = 0 \), since \( z_{j,1} \in \mathbb{C}^\times \). Thus, by induction the component of \( g^{-1} \) in \( G_{J_{\lambda_k}} \) is trivial. This completes the proof.

Q.E.D.
4 The $\Gamma_n$ Construction

4.1 Structure of Stabilizer groups $\text{Stab}(X)$ for $X \in \Xi^i_{c_i,c_{i+1}}$

In this chapter, we adopt the following notation. For $c_i \in \mathbb{C}^i$, we think of $c_i$ as representing coefficients of a monic polynomial of degree $i$ (excluding the leading term). Then $\Xi^i_{c_i,c_{i+1}} \subset M(i + 1)$ is the solution variety at level $i$ defined in the previous chapter with the $i \times i$ cutoff of $X \in \Xi^i_{c_i,c_{i+1}}$ having characteristic polynomial prescribed by $c_i$ and $X$ has characteristic polynomial given by $c_{i+1}$. Let us say that the polynomial represented by $c_i$ is of the form $\prod_{j=1}^{r} (t - \lambda_j)^{n_j}$ and the roots of the one represented by $c_{i+1}$ are $\mu_1, \ldots, \mu_s$. Then we recall that $X \in \Xi^i_{c_i,c_{i+1}}$ is a matrix of the form:

$$
\begin{bmatrix}
\lambda_1 & 1 & \cdots & 0 \\
0 & \lambda_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_1 \\
\vdots & & & & \\
0 & & & & 0 \\
\vdots & & & & & \ddots \\
\end{bmatrix}

\begin{bmatrix}
y_{1,1} \\
\vdots \\
y_{1,n_1} \\
\vdots \\
y_{r,1} \\
\vdots \\
y_{r,n_r} \\
\end{bmatrix}

(4.1)

\begin{bmatrix}
\lambda_r & 1 & \cdots & 0 \\
0 & \lambda_r & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_r \\
\vdots & & & & \\
\end{bmatrix}

\begin{bmatrix}
z_{1,1} & \cdots & \cdots & z_{1,n_1} \\
\vdots & \cdots & \cdots & \vdots \\
z_{r,1} & \cdots & \cdots & z_{r,n_r} \\
\end{bmatrix}

w$$
Let $G_i$ denote the centralizer of the $i \times i$ cutoff of the matrix in (4.1). Recall that $G_i$ acts algebraically on $\Xi^i_{c_i,c_i+1}$. To understand the structure of the orbits $O \subset \Xi^i_{c_i,c_i+1}$ of $G_i$ on $\Xi^i_{c_i,c_i+1}$, we need to understand the possible structure of the isotropy groups for this action. We recall that $G_i$ is an abelian connected algebraic group, which is the product of groups $\prod_{j=1}^r G_{J_\lambda}^\text{J}$ where $G_{J_\lambda}$ denotes the centralizer of the Jordan block corresponding to the eigenvalue $\lambda_j$ as above. We also recall that the action of $G_i$ is the diagonal action of the product $\prod_{j=1}^r G_{J_\lambda}$ on the last column of $X \in \Xi^i_{c_i,c_i+1}$ and the dual action on the last row of $X$. Recall also that $G_{J_\lambda} \simeq \mathbb{C}^\times \times \mathbb{C}^{n_j-1}$. To that affect we have the following Lemma.

**Lemma 4.1.1.** Let $X \in \Xi^i_{c_i,c_i+1}$ be as above, and let $\text{Stab}(X) \subset G_i$ be the isotropy group or stabilizer of $X$ under the action of $G_i$ on $\Xi^i_{c_i,c_i+1}$. Then, without loss of generality, we have that

$$Stab(X) = \prod_{j=1}^q G_{J_\lambda} \times \prod_{j=q+1}^r U_j$$

(4.2)

where $U_j \subset G_{J_\lambda}$ is a unipotent Zariski closed subgroup (possibly trivial) for some $q$, $0 \leq q \leq r$.

**Proof:**

For ease of notation, we let $G_{J_k} = G_{J_k}$. Recall from the proofs of Propositions 3.2.1 and 3.3.2 that if for some $k$, $1 \leq k \leq r$ we have $y_{k,n_k} \in \mathbb{C}^\times$ or $z_{1,k} \in \mathbb{C}^\times$, then the component of $\text{Stab}(X)$ in $G_{J_k}$ is trivial. Thus, we may as well assume that we have

$$z_{k,1} = y_{k,n_k} = 0$$

for all $k$.

We consider the component of $\text{Stab}(X)$ in $G_{J_k}$. Suppose that we have $y_{k,i} \neq 0$, but for $i < l \leq n_k$, we have that $y_{k,l} = 0$. Then we consider the matrix equation:

$$A_k \cdot y = y$$
where \( A_k \in G_{J_k} \) is the matrix given by

\[
A_k = \begin{bmatrix}
  a_{k,1} & a_{k,2} & \cdots & a_{k,n_k} \\
  0 & a_{k,1} & \ddots & \vdots \\
  \vdots & \ddots & \ddots & a_{k,2} \\
  0 & \cdots & \cdots & a_{k,1}
\end{bmatrix}
\]

with \( a_{k,1} \in \mathbb{C}^\times, a_{k,l} \in \mathbb{C} \) for \( 2 \leq l \leq n_k \), and \( y \in \mathbb{C}^{n_k} \) is the column vector \( y = (y_{k,1}, \cdots, y_{k,i}, 0, \cdots, 0)^T \). This gives rise to a system of linear equations which one can solve easily by back substitution. The equation for the \( i \)th row is:

\[
a_{k,1} y_{k,i} = y_{k,i}
\]

Since \( y_{k,i} \in \mathbb{C}^\times \), this forces \( a_{k,1} = 1 \). So that the stabilizer, in \( G_{J_k} \) consists of unipotent, upper triangular matrices. We note that the component of \( Stab(X) \) in \( G_{J_k} \) is automatically a Zariski closed subgroup, since it is the intersection of the closed subgroups \( Stab(X) \) and \( G_k \). A similar computation produces the result if we had assumed that we had instead \( y_{j,l} = 0 \) for all \( l \) and \( z_{k,i} \neq 0 \), but \( z_{k,l} = 0 \) for \( 1 \leq l < i \).

The other case to consider is the extreme case in which one has \( y_{k,l} = 0 \) for all \( l \) and \( z_{k,l} = 0 \) for all \( l \). In this case it is easy to check that the component of \( Stab(X) \) in \( G_{J_k} \) is exactly \( G_{J_k} \) itself. Repeating this analysis for each \( k, 1 \leq k \leq r \) and after possibly reordering the Jordan blocks of \( X_i \), we get the desired result.

Q.E.D.

We have an immediate corollary to the Lemma.

**Corollary 4.1.1.** For any \( X \in \Xi_{c_i,c_{i+1}}^i \), the isotropy group of \( X \), denoted by \( Stab(X) \) under the action of \( G_i \) on \( \Xi_{c_i,c_{i+1}}^i \) is connected.

**Proof:**

Upon reordering the eigenvalues, we can always assume that \( Stab(X) \) has the form given in (4.2) in Lemma 4.1.1. This proves the result, since unipotent algebraic groups are always connected and the groups \( G_{J_j} \) are all connected, being centralizers of regular elements in \( M(n_j) \).
Using Lemma 4.1.1, we can prove a basic structural result about the group $G_i$ when it acts on the variety $\Xi^{i}_{c_i,c_{i+1}}$. We now show that as an algebraic group over $\mathbb{C}$, it is a direct product of an isotropy group of an element in $\Xi^{i}_{c_i,c_{i+1}}$ and a connected Zariski closed subgroup.

**Theorem 4.1.1.** Let $X \in \Xi^{i}_{c_i,c_{i+1}}$. Let $\text{Stab}(X) \subset G_i$ denote the isotropy group of $X$ under the action of $G_i$ on $\Xi^{i}_{c_i,c_{i+1}}$. Then as an algebraic group

$$G_i = \text{Stab}(X) K$$

With $K \subset G_i$ a Zariski closed, connected subgroup (possibly trivial) and $K \cap \text{Stab}(X) = \{e\}$ so that as an algebraic group, we have

$$G_i \simeq \text{Stab}(X) \times K$$

**Proof:**

For the purposes of this proof we denote by $G$ the group $G_i$ and by $H$ the group $\text{Stab}(X)$. Without loss of generality, we assume $\text{Stab}(X)$ is as given in (4.2). Let $\mathfrak{g} = \text{Lie}(G)$ and let $\mathfrak{h} = \text{Lie}(H)$. Now, by Lemma 4.1.1, we know that $\mathfrak{h}$ is given by

$$\mathfrak{h} = \bigoplus_{j=1}^{q} \mathfrak{g}_{J_\lambda_j} \oplus \bigoplus_{j=q+1}^{r} \mathfrak{n}_j$$

(4.3)

Where $\mathfrak{g}_{J_\lambda_j}$ is the Lie algebra of the abelian algebraic group $G_{J_\lambda_j}$ and $\mathfrak{n}_j = \text{Lie}(U_j)$ is a Lie subalgebra of $\mathfrak{n}_{n_j}$, the strictly upper triangular matrices in $M(n_j)$. For convenience we denote $\mathfrak{g}_{J_\lambda_j}$ by $\mathfrak{g}_j$ and $G_{J_\lambda_j}$ by $G_j$. Now, we want to make a special choice of a Lie algebra compliment to $\mathfrak{h}$ in $\mathfrak{g}$. That is to say that we want to find a Lie subalgebra $V \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus V$$

Where the direct sum is a direct sum of Lie algebras. Because $\mathfrak{g}$ is abelian, we need only choose $V$ to be a vector space compliment to $\mathfrak{h}$ and it is automatically a Lie algebra compliment. We choose a vector space compliment $V$ as follows. Let
\[ V = \bigoplus_{j=q+1}^{r} m_j \] with \( m_j \) a vector space compliment to \( n_j \) in \( g_j \). i.e. \( g_j = n_j \oplus m_j \) for \( q + 1 \leq j \leq r \). Then \( g = h \oplus V \).

Now, we claim that we can choose the compliments \( m_j \) so that \( V \) is actually an algebraic subalgebra of \( g \). This means that there exists a unique Zariski closed connected subgroup \( K \subset G \) with \( \text{Lie}(K) = V \). Now, since each subalgebra \( g_j \subset g \) is abelian, we have a decomposition into subalgebras \( g_j = g_j^{ss} \oplus g_j^n \) with \( g_j^n \) being the set of nilpotent elements of \( g_j \) and \( g_j^{ss} \) being the set of semi-simple elements of \( g_j \). Note that \( g_j^n \) and \( g_j^{ss} \) are actually subalgebras, since the sum of two commuting nilpotent endomorphisms is nilpotent and the sum of two commuting semi-simple endomorphisms is semi-simple. Now, since \( n_j \subset g_j \) consists of nilpotent endomorphisms, it must be a subalgebra of \( g_j^n \). Now, since \( g_j^n \) is abelian, we can again find a vector space compliment to \( n_j \) in \( g_j^n \) which is a nilpotent Lie subalgebra. Let us call this compliment \( \tilde{n}_j \), so that we have \( g_j^n = n_j \oplus \tilde{n}_j \). So, we now choose our \( m_j \) to be the subalgebra \( m_j = g_j^{ss} \oplus \tilde{n}_j \).

We would now like to see that \( m_j \) so defined are algebraic Lie subalgebras of \( g_j \), as this will give us that \( V \) is an algebraic subalgebra of \( g \). We first note that \( \tilde{n}_j \) is algebraic. This follows from the following general Lemma.

**Lemma 4.1.2.** Let \( n \subset M(n) \) be the subalgebra of nilpotent upper triangular matrices. Let \( v \subset n \) be a subalgebra, then \( v \) is algebraic.

**Proof:**

For simplicity, we prove the result for \( v \) is abelian. The proof can be modified to give the result for any Lie algebra \( v \) consisting of nilpotent linear transformations. We consider the regular map

\[ \exp : v \rightarrow GL(n) \]

Let \( N \subset GL(n) \) be the subgroup of unipotent upper triangular matrices. We note that the map \( \exp : n \rightarrow N \) is an algebraic isomorphism, since it has regular inverse \( \log : N \rightarrow n \). Now, \( v \subset n \) is closed, so the image of \( \exp(v) = T \) is a subgroup that is Zariski closed and connected.

We claim that \( \text{Lie}(T) = v \). To see this, we use Theorem 1.3.2. in [GW, pg 36] which states that if \( G \subset Gl(n) \) is a Zariski closed subgroup and \( A \in M(n) \)
is nilpotent, then \( A \in \text{Lie}(G) \Leftrightarrow \exp(A) \in G \). Now, let \( X \in \mathfrak{v} \), then \( X \) is nilpotent, and \( \exp(X) \in T \), so \( X \in \text{Lie}(T) \). Thus \( \mathfrak{v} \subset \text{Lie}(T) \). For the reverse inclusion, let \( X \in \text{Lie}(T) \), then \( X \) is nilpotent since \( T \subset N \). Thus, we have that \( \exp(X) \in T \), so that \( \exp(X) = \exp(v) \) with \( v \in \mathfrak{v} \). Taking logarithms, we have \( \log(\exp(X)) = \log(\exp(v)) \Rightarrow X = v \). Thus, \( \text{Lie}(T) \subset \mathfrak{v} \). This completes the proof of the lemma.

Q.E.D.

**Remark 4.1.1.** Note that the above Lemma actually holds for any Lie subalgebra \( \mathfrak{v} \) of \( M(n) \) consisting of nilpotent linear transformations, since by conjugation we may always assume that \( \mathfrak{v} \subset \mathfrak{n} \).

Now, in our case we have that \( \widetilde{n}_j \) is a Lie algebra consisting of nilpotent upper triangular matrices, since \( \mathfrak{g}_j \) consists of upper triangular matrices. Thus, by the above lemma it is an algebraic subalgebra. Let \( \widetilde{N}_j \) be the unique Zariski closed connected subgroup of \( G_j \) with \( \text{Lie}(\widetilde{N}_j) = \widetilde{n}_j \) (i.e. \( \widetilde{N}_j = \exp(\widetilde{n}_j) \)). Now, the subalgebra \( \widetilde{n}_j \oplus \mathfrak{g}_j^{ss} \) is the Lie algebra of the closed, connected subgroup \( \mathbb{C}^\times \times \widetilde{N}_j \), where \( \mathbb{C}^\times \) is the semi-simple part of the abelian algebraic group \( G_j \). Indeed, note that \( \text{Lie} (\mathbb{C}^\times \times \widetilde{N}_j) = \text{Lie} (\mathbb{C}^\times) \oplus \text{Lie} (\widetilde{N}_j) = \mathfrak{g}_j^{ss} \oplus \widetilde{n}_j = \mathfrak{m}_j \). Thus, \( V = \bigoplus_{j=q+1}^r \mathfrak{m}_j \) is an algebraic subalgebra of \( \mathfrak{g} \). Namely, it is the Lie algebra of the Zariski closed, connected subgroup of \( G \) given by \( K = K_{q+1} \times \cdots \times K_r \) with \( K_j = \mathbb{C}^\times \times \widetilde{N}_j \subset G_j \).

Now, we have our candidate for the connected subgroup \( K \) in the statement of the theorem. First, we show that \( K \cap H = \{e\} \). The argument proceeds in two steps.

(A) \( K \cap H \) is discrete (i.e. finite).

(B) \( K \cap H \) is unipotent.

Since any unipotent group is connected, a finite unipotent group must be trivial.

We get (A) easily from the fact that we have constructed \( K \) with \( \text{Lie}(K) = V \). Now, we know for algebraic subgroups of \( H, K \subset GL(n) \) that we have \( \text{Lie}(K \cap H) = \text{Lie}(K) \cap \text{Lie}(H) \), see Corollary 1.2.12 in [GW, pg 31]. However in our case
\textit{Lie}(K) \cap \textit{Lie}(H) = V \cap \mathfrak{h} = 0 \text{ since } V \text{ is a Lie algebra compliment to } \mathfrak{h}. \text{ Thus } \dim(\textit{Lie}(K \cap H)) = \dim K \cap H = 0 \Rightarrow K \cap H \text{ is finite. This gives (A).}

Now, (B) follows directly from the fact that $K \subset \prod_{j=q+1}^r G_j$. Thus $K \cap H \subset \prod_{j=q+1}^r G_j \cap H = \prod_{j=q+1}^r U_j$ by equation (4.2). Thus, $K \cap H$ must be unipotent.

So now, we have (B). Thus, we have shown that $K \cap H = \{e\}$ is trivial.

Thus, as an abstract group, it is easy to see that $HK \simeq H \times K$. We want to see that $HK \simeq H \times K$ as an algebraic group. First, we note that $HK$ is a Zariski closed subgroup of $G$. $HK$ is the image of $H \times K$ under the morphism $\phi : H \times K \rightarrow G$ given by $\phi((h,k)) = hk$. Hence, $HK$ is a constructible subset of $G$. The fact that $HK$ is closed follows from the following Proposition (see [Hum 1,p. 54]) for a proof.

**Proposition 4.1.1.** Let $T$ be a constructible subgroup of an algebraic group $G'$, then $T = \overline{T}$.

Hence, $HK$ is a closed subgroup of $G$. Now, it is easy to see that $HK \simeq H \times K$ as algebraic groups. Note that by Corollary 4.1.1 the group $H = \text{Stab}(X)$ is connected. In our work above, we have seen that $K$ is connected. The map that achieves the isomorphism $\phi : (h,k) \rightarrow hk$ is clearly a bijective homomorphism of connected algebraic groups, so it must be an isomorphism of algebraic groups (see Corollary 11.1.3 in [GW, pg 467]). So we have that $\dim HK = \dim(H \times K) = \dim H + \dim K$.

We can now show easily that $G = HK$. From that last paragraph we have that $HK$ is a connected, algebraic subgroup of $G$ of dimension $\dim(H) + \dim(K) = \dim(\mathfrak{h}) + \dim(V)$. But since we have chosen $V$ so that $\mathfrak{g} = \mathfrak{h} \oplus V$, we have that $\dim(H) + \dim(K) = \dim(G)$. Thus $HK$ is a closed, irreducible subvariety of dimension $\dim(G)$ of the irreducible variety $G$. Hence, we must have that $G = HK$.

Thus, $G = HK \simeq H \times K$ as algebraic groups as desired. This completes the proof of the theorem. Q.E.D.
4.2 The $\Gamma_n$ maps

Now, using Theorem 4.1.1, we can define a parameterization of certain $A$ orbits in the fibres $M_c(n)$. Let $S \subset M(n)$ be the set of matrices such that each cutoff $X_i$ for $1 \leq i \leq n - 1$ is regular. We note that $M_c(n) \cap S$ naturally has the structure of a quasi-affine variety, since $S$ is Zariski open in $M(n)$. Indeed, $S$ is given by the non-vanishing of the $i$-forms $d\phi_{i,1} \wedge \cdots \wedge d\phi_{i,i}$ for $1 \leq i \leq n - 1$ (here the $\phi_{i,j}$ are fundamental adjoint invariants for $M(i)$). We note that by Theorem 2.14 in [KW, pg 23] that $M^{sreg}_c(n) \subset M_c(n) \cap S$ so that $M_c(n) \cap S$ is non-empty.

Our goal is to parameterize the $A$ orbits in $M_c(n) \cap S$ and through that to describe the $A$ orbits in $M^{sreg}_c(n)$. Recall that we write $c \in \mathbb{C}^{(n+1)/2}$ as $c = (c_1, \cdots, c_i, \cdots, c_n)$ with $c_i \in \mathbb{C}^i$ and think of $c_i$ as representing the coefficients of a monic polynomial of degree $i$ (excluding the leading term). Let $G_i$ act on $\Xi^i_{c_i,c_{i+1}}$, the solution variety at level $i$, as in the previous sections. Let $G = G_1 \times \cdots \times G_{n-1}$. We will show that the $A$ orbits are given by orbits of certain subgroups of $G$ acting on certain analytic submanifolds of $M(n)$. On $M^{sreg}_c(n)$ the action of $A$ will be realized by an algebraic action of the full group $G$.

In order to define the mappings that parameterize $A$ orbits in $M_c(n) \cap S$, we need to be able to conjugate elements of a $G_i$ orbit $O \subset \Xi^i_{c_i,c_{i+1}}$ into Jordan form for $1 \leq i \leq n - 2$ in an algebraic manner. In other words, we need to find a function from the smooth variety $O \rightarrow Gl(i+1)$ given by $X \rightarrow g(X)$ with the property that $\text{Ad}(g(X)) \cdot X$ is in Jordan form.

We can do this precisely because of Theorem 4.1.1. Fix an $X \in O$ as above. Then we know that $O \simeq G_i/\text{Stab}(X)$. But by Theorem 4.1.1, we have that $G_i = \text{Stab}(X) K$ with $K$ a connected, closed subgroup of $G_i$. This means that $O \simeq K$ and is an orbit under $K$ with $K$ acting freely. More specifically, given $Y \in O$, $Y = k \cdot X$ for a unique $k \in K$ and the function $O \rightarrow K$ given by $Y \rightarrow k$ is an algebraic isomorphism (See Theorem 25.1.2 in [TY, pg 387]). (We note that the connected group $K$ acts simply transitively on $O$, so that we can apply part (iv) of the above theorem to the orbit map $k \rightarrow k \cdot X$, which is a bijective $K$-equivariant morphism of affine varieties which are both homogenous $K$-spaces).
Now, if we think of $\mathcal{O}$ as being the $K$ orbit of $X$, then we can first fix a $g \in GL(i+1)$ such that $\text{Ad}(g) \cdot X$ is in Jordan canonical form. Then for any $Y \in \mathcal{O}$ with $Y = k \cdot X$ for a unique $k \in K$, the matrix $g \cdot k^{-1} \in GL(i+1)$ conjugates $Y$ into Jordan form and the function $Y \to g \cdot k^{-1}$ is regular on $\mathcal{O}$ by our observations above.

Now, suppose we are given $G_i$ orbits in $\Xi_{c_i,c_{i+1}}^i$, $\mathcal{O}_{a_i}^i = K_{a_i} \cdot x_{a_i} \cong K_{a_i}$ with $x_{a_i} \in \Xi_{c_i,c_{i+1}}^i$ for $1 \leq i \leq n - 1$, with $\mathcal{O}_{a_i}^i$ consisting of regular elements of $M(i+1)$ for $1 \leq i \leq n - 2$. We can then define the following regular map:

$$\Gamma_{a_1,\ldots,a_{n-1}}^n : K_{a_1} \times \cdots \times K_{a_{n-1}} \to M_c(n) \cap S \hookrightarrow M(n)$$

given by:

$$\Gamma_{a_1,\ldots,a_{n-1}}^n(k_1, \ldots, k_{n-1}) = \text{Ad}(k_1g_{1,2}(x_{a_1})^{-1}k_2g_{2,3}(x_{a_2})^{-1} \cdots k_{n-2}g_{n-2,n-1}(x_{a_{n-2}})^{-1}k_{n-1})x_{a_{n-1}}$$

(4.4)

where $g_{i,i+1}(x_{a_i}) \in Gl(i+1)$ conjugates the base point of the orbit $\mathcal{O}_{a_i}^i$, $x_{a_i}$ into Jordan canonical form.

We now make a few observations about the mapping defined in (4.4). First, we need the following definition.

**Definition 4.2.1.** Let $X \in M_c(n) \cap S$. We say that $z \in \Xi_{c_i,c_{i+1}}^i \subset M(i+1)$ is a projection of $X$ into the solution variety $\Xi_{c_i,c_{i+1}}^i$ at level $i$ if $z$ is the $(i+1) \times (i+1)$ cutoff of a matrix of the form $\text{Ad}(g) \cdot X$ with $g \in Gl(i) \hookrightarrow Gl(n)$ is such that $(\text{Ad}(g) \cdot X)_i = \text{Ad}(g) \cdot X_i$ is in Jordan canonical form with eigenvalues in decreasing lexicographical order.

**Remark 4.2.1.** We claim that $X \in Im\Gamma_{a_1,\ldots,a_{n-1}}^n$ if and only if there exists a projection into the solution variety at level $j$, $z_j \in \Xi_{c_i,c_{i+1}}^i$ such that $z_j \in \mathcal{O}_{a_j}^j$.

Note that the sufficiency is trivial from the definition of the mapping $\Gamma_{a_1,\ldots,a_{n-1}}^n$. To see the necessity, we first make the following observation. If we have $\text{Ad}(g)Y = B = \text{Ad}(h)Y \Leftrightarrow \text{Ad}(hg^{-1})B = B \Leftrightarrow hg^{-1} \in GL(n)^B \Leftrightarrow h = kg, k \in GL(n)^B$ with $GL(n)^B$ the centralizer of $B$ in $GL(n)$. So that if two matrices put a given matrix into the same Jordan canonical form then these matrices
are in the same right coset of the centralizer of the Jordan form. Thus, if there exists a projection \( z_j \) that is in the orbit \( O_{a_j}^j \) in \( \Xi_{c_j, c_{j+1}} \), then any projection into the solution variety at level \( j \) must be in the orbit \( O_{a_j}^j \). Now, if \( Y \in M_c(n) \cap S \) satisfies the property that any projection \( z_j \) into the solution variety at level \( j \) is in the orbit \( O_{a_j}^j \), then \( Y_2 \in O_{a_1}^1 \subset \Xi_{c_1, c_2}^1 \). Thus, \( [\text{Ad}(g_{1,2}(x_{a_1}))k_1^{-1} \cdot Y]_3 \in O_{a_2}^2 \) for a unique \( k_1 \in K_{a_1} \). (We show the uniqueness for a general \( k_j \) below. Exactly the same argument shows the uniqueness of \( k_1 \).) Now, by induction we can assume that we have that the \((j + 1) \times (j + 1)\) cutoff of

\[
Z = \text{Ad}(g_{j-1,j}(x_{a_j-1})) \text{Ad}(k_1^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})) \text{Ad}(k_1^{-1}) \cdot Y
\]

is in the orbit \( O_{a_j}^j \) for unique choice of \( k_1, \ldots, k_{j-1} \in K_{a_1}, \ldots, K_{a_{j-1}} \) respectively. Since the \((j + 1) \times (j + 1)\) cutoff of \( Z \) is in \( O_{a_j}^j \), then there exists a \( k_j \in K_{a_j} \) such that the \((j + 2) \times (j + 2)\) cutoff of \( \text{Ad}(g_{j+1,j}(x_{a_j})) \text{Ad}(k_1^{-1}) \cdot Z \) is in \( O_{a_{j+1}}^{j+1} \). This is because \( [\text{Ad}(g_{j+1,j}(x_{a_j})) \text{Ad}(k_1^{-1}) \cdot Z]_{j+2} \) is a projection of \( Y \) into the solution variety at level \( j + 1 \).

Now, we claim that \( k_1, \ldots, k_j \) are the unique elements of \( K_{a_1}, \ldots, K_{a_j} \) respectively such that the \((j + 2) \times (j + 2)\) cutoff of

\[
Z' = \text{Ad}(g_{j+1,j}(x_{a_j})) \text{Ad}(k_1^{-1}) \cdots \text{Ad}(g_{j-1,j}(x_{a_j-1})) \text{Ad}(k_1^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})) \text{Ad}(k_1^{-1}) \cdot Y
\]

is in the orbit \( O_{a_{j+1}}^{j+1} \).

Indeed, suppose that we have another set of elements \( t_1, \ldots, t_j \) in \( K_{a_1}, \ldots, K_{a_j} \) respectively such that the \((j + 2) \times (j + 2)\) cutoff of

\[
T = \text{Ad}(g_{j+1,j}(x_{a_j})) \text{Ad}(t_1^{-1}) \cdots \text{Ad}(g_{j-1,j}(x_{a_j-1})) \text{Ad}(t_1^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})) \text{Ad}(t_1^{-1}) \cdot Y
\]

is in the orbit \( O_{a_{j+1}}^{j+1} \). Then, we have that there exists an element \( k_{j+1} \in K_{a_{j+1}} \) such that

\[
Z_{j+2}' = \text{Ad}(k_{j+1})T_{j+2}
\]

But since \( k_{j+1} \) centralizes the \((j + 1) \times (j + 1)\) cutoff of an element in \( O_{a_{j+1}}^{j+1} \), the \((j + 1) \times (j + 1)\) cutoffs of both sides of the equation (4.5) are equal. This yields

\[
\text{Ad}(g_{j+1,j}(x_{a_j})) \text{Ad}(k_1^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})) \text{Ad}(k_1^{-1}) \cdot Y_{j+1} = \text{Ad}(g_{j+1,j}(x_{a_j})) \text{Ad}(t_1^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})) \text{Ad}(t_1^{-1}) \cdot Y_{j+1}
\]

Indeed, suppose that we have another set of elements \( t_1, \ldots, t_j \) in \( K_{a_1}, \ldots, K_{a_j} \) respectively such that the \((j + 2) \times (j + 2)\) cutoff of

\[
T = \text{Ad}(g_{j+1,j}(x_{a_j})) \text{Ad}(t_1^{-1}) \cdots \text{Ad}(g_{j-1,j}(x_{a_j-1})) \text{Ad}(t_1^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})) \text{Ad}(t_1^{-1}) \cdot Y
\]

is in the orbit \( O_{a_{j+1}}^{j+1} \). Then, we have that there exists an element \( k_{j+1} \in K_{a_{j+1}} \) such that

\[
Z_{j+2}' = \text{Ad}(k_{j+1})T_{j+2}
\]
(since $g_{j+1,j}(x_{aj}) \in GL(j + 1)$). Simplifying, the above becomes:

$$\text{Ad}(t_j)\text{Ad}(k_j^{-1})\text{Ad}(g_{j-1,j}(x_{aj-1}))\cdot \text{Ad}(g_{1,2}(x_{a1}))\text{Ad}(k_1^{-1}) \cdot Y_{j+1} =$$

$$= \text{Ad}(g_{j-1,j}(x_{aj-1}))\text{Ad}(t_{j-1})\cdot \text{Ad}(g_{1,2}(x_{a1}))\text{Ad}(t_1^{-1}) \cdot Y_{j+1}$$

(4.6)

Notice that the LHS of equation (4.6) is in the orbit $O_{aj}^j$. Thus, by the induction assumption applied to the RHS of equation (4.6), we must have $t_1 = k_1$, $t_2 = k_2$, $\ldots$, $t_{j-1} = k_{j-1}$. Thus, equation (4.6) becomes:

$$\text{Ad}(t_j k_j^{-1}) \cdot Z_{j+1} = Z_{j+1} \Rightarrow$$

$$t_j (k_j)^{-1} \in \text{Stab}(Z_{j+1}) \cap K_{aj} = \text{Stab}(x_{aj}) \cap K_{aj}$$

But since $K_{aj} \cap \text{Stab}(x_{aj})$ is trivial, we get $k_j = t_j$. This completes the inductive step.

Thus, by induction, we can conclude that there exists unique $k_1, \ldots, k_{n-2} \in K_{a1}, \ldots, K_{an-2}$ respectively so that the element

$$y_{an-1} = \text{Ad}(g_{n-2,n-1}(x_{an-1}))\text{Ad}(k_{n-2}^{-1})\cdot \text{Ad}(g_{1,2}(x_{a1}))\text{Ad}(k_1^{-1}) \cdot Y$$

is in the orbit $O_{an-1}^{n-1}$. Thus, there exists an unique $k_{n-1} \in K_{an-1}$ such that $\text{Ad}(k_{n-1}^{-1}) \cdot y_{an-1} = x_{an-1}$ which is the base point for the map $\Gamma_{an-1}^{a1,\ldots,a_{n-1}}$. We conclude that there exist unique $k_1, \ldots, k_{n-1}$ in $K_{a1}, \ldots, K_{an-1}$ respectively such that $Y = \Gamma_{an}^{a1,\ldots,a_{n-1}}(k_1, \ldots, k_{n-1})$.

Note that the argument for the necessity in Remark 4.2.1 lets us see that $\Gamma_{an}^{a1,\ldots,a_{n-1}}$ is injective, since the $k_j \in K_{aj}$ that we produce at each step of the procedure are unique.

For the remainder of the section, we work mainly in the analytic category. The reason being that the group $A$ is an analytic group, so that to analyze the relationship between $\text{Im} \Gamma_{n}^{a1,\ldots,a_{n-1}}$ and the action of $A$ we have to use analytic techniques. We now consider the topology on subsets of $\mathbb{C}^n$ induced by the Euclidean distance, and we refer to it as the analytic topology. Let us observe at this point that everything we have proven so far, goes over to the analytic category. This is because smooth varieties naturally have complex analytic structure...
and morphisms between them are holomorphic maps (see [Shaf, Ch 7,8]). Also if a group \( G \) is a connected algebraic group over \( \mathbb{C} \), then it is naturally a complex Lie group and connected in the Lie group topology (see Theorem 11.1.2 [GW, pg 480]). Thus the groups \( G_i, \text{Stab}(x_{a_i}) \), and \( K_{a_i} \) can all be considered as connected complex analytic groups that act holomorphically on the analytic submanifold \( O_{a_i}^i \) of \( M(n) \). (In what follows below the word “continuous” means continuous in the analytic topology).

The argument in Remark 4.2.1 also tells us how to construct the inverse map as a map \( \text{Im} \Gamma_{a_1,\ldots,a_{n-1}}^n \rightarrow K_{a_1} \times \cdots \times K_{a_{n-1}} \). We want to see that the inverse map is continuous in the analytic topology when \( \text{Im} \Gamma_{a_1,\ldots,a_{n-1}}^n \) is given the subspace topology. We can see that in the following way. We work inductively as in Remark 4.2.1. Given \( Y \in \text{Im} \Gamma_{a_1,\ldots,a_{n-1}}^n \), we have that \( Y_2 = k_1 \cdot x_{a_1} \) for unique \( k_1 \in K_{a_1} \).

1) We assume inductively that the map:

\[
\Phi_0 : \text{Im} \Gamma_{a_1,\ldots,a_{n-1}}^n \rightarrow M(n)
\]

given by

\[
Y \rightarrow \text{Ad}(g_{j,j-1}(x_{a_{j-1}})k_{j-1}^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})\text{Ad}(k_1^{-1})) \cdot Y
\]

is continuous and that \( k_l \in K_{a_l} \) for \( 1 \leq l \leq j - 1 \) have already been computed as continuous functions of \( Y \). Where \( k_1, \ldots, k_{j-1} \) are the unique elements of \( K_{a_1}, \ldots, K_{a_{j-1}} \) respectively such that \( \text{Ad}(g_{j,j-1}(x_{a_{j-1}})k_{j-1}^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})\text{Ad}(k_1^{-1})) \cdot Y \) is in the orbit \( O_{a_j}^j \) as in Remark 4.2.1.

(2) It then follows that the map \( \text{Im} \Gamma_{a_1,\ldots,a_{n-1}}^n \rightarrow O_{a_j}^j \) given by

\[
\Phi_1(Y) = [\text{Ad}(g_{j,j-1}(x_{a_{j-1}})k_j^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})k_1^{-1}) \cdot Y]_{j+1}
\]
is continuous. Now, $\Phi_1(Y) \in \mathcal{O}_{a_j}^j = k_j \cdot x_{a_j}$ for unique $k_j$ as in Remark 4.2.1. Now, we use again that the canonical map $\mathcal{O}_{a_j}^j \rightarrow K_{a_j}$ is a biholomorphism and hence continuous to see that we can obtain $k_j$ as a continuous function of $Y$.

(3) Now, we note that we have a natural holomorphic map on the product.

$\Phi_3 : GL(j + 1) \times M(n) \rightarrow M(n)$

given by

$(g, Z) \rightarrow \text{Ad}(g) \cdot Z$

(4) We get a continuous map

$\Phi_4 : \text{Im}\Gamma_{n}^{a_1, \cdots, a_{n-1}} \rightarrow Gl(j + 1) \times M(n)$

given by

$Y \rightarrow (g_{j+1,j}(x_{a_j})k_j^{-1}(Y), \Phi_0(Y))$

(5) Lastly we note that the compositions of the continuous maps $\Phi_3$ and $\Phi_4$ is the map

$\Psi(Y) = \Phi_3 \circ \Phi_4 = \text{Ad}(g_{j+1,j}(x_{a_j})k_j^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1})k_1^{-1}) \cdot Y$

This completes the inductive step. And we thus get that all $k_i$ are continuous functions of $Y \in \text{Im}\Gamma_{n}^{a_1, \cdots, a_{n-1}}$ and by our discussion in Remark 4.2.1 $Y = \Gamma_{n}^{a_1, \cdots, a_{n-1}}(k_1, \cdots, k_{n-1})$. Thus, the inverse map is continuous in the analytic topology.

### 4.3 The analytic structure of $Im\Gamma_{n}$

We retain the notation of the last section. Our goal is to show that the subsets $\text{Im}\Gamma_{n}^{a_1, \cdots, a_{n-1}}$ of $M_c(n) \cap S$ are $A$ orbits in $M_c(n) \cap S$. To that end, we need to see that the sets $\text{Im}\Gamma_{n}^{a_1, \cdots, a_{n-1}}$ are embedded complex submanifolds of $M(n)$. From our work in the last section we have seen that the map

$\Gamma_{n}^{a_1, \cdots, a_{n-1}} : K_{a_1} \times \cdots \times K_{a_{n-1}} \rightarrow M_c(n) \cap S \hookrightarrow M(n)$
defined in equation (4.4) is a topological embedding in the analytic category. To see that $\text{Im} \Gamma_{a_1, \ldots, a_n}^{a_{n-1}}$ is an embedded complex submanifold of $M(n)$, we need to see that the map $\Gamma_{a_1, \ldots, a_n}^{a_{n-1}}$ is an immersion. To that effect, we have the following proposition.

**Proposition 4.3.1.** The map

$$\Gamma_{a_1, \ldots, a_n}^{a_{n-1}} : K_{a_1} \times \cdots \times K_{a_{n-1}} \to M_e(n) \cap S \hookrightarrow M(n)$$

defined by

$$\text{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} k_2 g_{2,3}(x_{a_2})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}$$

is an immersion (see equation (4.4)). Thus $\text{Im} \Gamma_{a_1, \ldots, a_n}^{a_{n-1}}$ is a complex embedded submanifold of $M(n)$.

**Proof:**

The proof proceeds by explicitly computing the differential of the map $\Gamma_{a_1, a_2, \ldots, a_n}^{a_{n-1}}$. We will also be able to use this computation in the next section to see that tangent space to the image of $\Gamma_{a_1, a_2, \ldots, a_n}^{a_{n-1}}$ is a subspace of $V_X = \{ \xi f_{i,j} | 1 \leq i \leq n - 1, 1 \leq j \leq i \}$ with $f_{i,j}(X) = \text{tr}(X^j)$, which we recall is precisely the tangent space to the $A$ orbit $A \cdot X$.

For ease of notation, we denote $\Gamma_{a_1, a_2, \ldots, a_n}^{a_{n-1}}$ simply by $\Gamma_n$. Let $X \in M(n) = \Gamma_n(id_1, \ldots, id_{n-1})$ with $id_j \in K_{a_j}$ denoting the identity element. We observe that

$$\Gamma_n(k_1, \ldots, k_{n-1}) = \text{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} k_2 g_{2,3}(x_{a_2})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}$$

with $x_{a_{n-1}}$ given by

$$x_{a_{n-1}} = \text{Ad}(g_{n-2,n-1}(x_{a_{n-2}})) \cdots \text{Ad}(g_{1,2}(x_{a_1})) \cdot X$$

Making this substitution the above becomes:

$$\text{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} k_2 g_{2,3}(x_{a_2})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) \text{Ad}(g_{n-2,n-1}(x_{a_{n-2}})) \cdots$$

$$\text{Ad}(g_{1,2}(x_{a_1})) \cdot X$$

(4.7)
Now, let $Y \in \text{Im}\Gamma_n$, so that $Y = \Gamma_n(k_1, \ldots, k_{n-1})$. Now, we can define a map $\Psi_Y : K_{a_1} \times \cdots \times K_{a_{n-1}} \to \text{Im}\Gamma_n$ to be a $\Gamma_n$ map but “based” at $Y$ instead of at $X$. i.e. $\Psi_Y$ would be defined as

$$\Psi_Y(\tilde{k}_1, \ldots, \tilde{k}_{n-1}) = \text{Ad}(\tilde{k}_1)\text{Ad}(k_1)\text{Ad}(g_{1,2}(x_{a_1})^{-1})\text{Ad}(\tilde{k}_2)\text{Ad}(k_2)\text{Ad}(g_{2,3}(x_{a_2})^{-1}) \cdots$$

$$\text{Ad}(g_{n-2,n-1}(x_{a_{n-2}})^{-1})\text{Ad}(\tilde{k}_{n-1})y_{a_{n-1}}$$

(4.8)

Where $y_{a_{n-1}}$ is a projection of $Y$ into the orbit $O_{a_{n-1}}^{n-1}$ and is given by

$$y_{a_{n-1}} = \text{Ad}(g_{n-2,n-1}(x_{a_{n-2}}))\text{Ad}(k_{n-2}^{-1}) \cdots \text{Ad}(g_{1,2}(x_{a_1}))\text{Ad}(k_1^{-1}) \cdot Y$$

(4.9)

as computed in Remark 4.2.1. Now, using that $y_{a_{n-1}} = \text{Ad}(k_{n-1})x_{a_{n-1}}$, we can write that above as

$$\text{Ad}(\tilde{k}_1 k_1 g_{1,2}(x_{a_1})^{-1}\tilde{k}_2 k_2 g_{2,3}(x_{a_2})^{-1} \cdots$$

$$\cdots \tilde{k}_{n-2} k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1}\tilde{k}_{n-1} k_{n-1} g_{n-2,n-1}(x_{a_{n-2}}) \cdots g_{1,2}(x_{a_1}) \cdot X$$

(4.10)

Comparing equations (4.10) and (4.7), we see that we have

$$\Psi_Y(\tilde{k}_1, \ldots, \tilde{k}_{n-1}) = \Gamma_n(\tilde{k}_1 k_1, \ldots, \tilde{k}_{n-1} k_{n-1})$$

So that we have the relation:

$$\Psi_y = \Gamma_n \circ L_{\tilde{k}}$$

(4.11)

where $\tilde{k} \in K_{a_1} \times \cdots \times K_{a_{n-1}}$ is given by $\tilde{k} = (k_1, \ldots, k_{n-1})$ and $L_{\tilde{k}}$ denotes left translation by $\tilde{k}$. Thus, taking differentials at the identity in $K_{a_1} \times \cdots \times K_{a_{n-1}}$ we obtain

$$(d\Psi_Y)_{id} = (d\Gamma_n)_{\tilde{k}} \circ (dL_{\tilde{k}})_{id}$$

(4.12)

By equation (4.12), we see that, since $L_{\tilde{k}}$ is a differomorphism, $(d\Gamma_n)_{\tilde{k}}$ is of full rank if and only if $(d\Psi_Y)_{id}$ has full rank. Thus, to compute the rank of $(d\Gamma_n)_{\tilde{k}}$ we just compute the rank of $(d\Psi_Y)_{id}$, which is much easier, since we only have to worry about differentiating at the identity in $K_{a_1} \times \cdots \times K_{a_{n-1}}$. For ease of notation, in the remainder of the proof we write $g_{i, i+1}$ for $g_{i, i+1}(x_{a_i})$. 
Now, we focus on differentiating the map $\Psi_Y$ at the identity. Using equation (4.9), we can write $\Psi_Y$ as

$$\text{Ad}(\tilde{k}_1 k_1 g_{1,2}^{-1} k_2 g_{2,3}^{-1} \cdots g_{n-2,n-1}^{-1} k_{n-2}^{-1} g_{n-2,n-1}^{-1} k_{n-2}^{-1} \cdots g_{1,2}^{-1} k_1^{-1}) \cdot Y$$

Let $\text{Lie}(K_a)$ have basis $\{\alpha_{i,1}, \cdots, \alpha_{i,s(i)}\}$ with $s(i) = \dim \text{Lie}(K_a) \leq i$. Because the group $K_a$ is abelian and connected as a complex analytic group, we note that we can represent $k_i \in K_a$ as $\exp(\sum_{j=1}^{s(i)} t_{i,j} \alpha_{i,j})$ for $t_{i,j} \in \mathbb{C}$. Thus, to compute the differential of $\Psi_Y$ at the identity, we need to consider the differentials

$$\frac{\partial}{\partial t_{i,j}}|_{t=0} \text{Ad}(\exp(t_{1,1} \alpha_{1,1})k_1 g_{1,2}^{-1} \cdots \exp(\sum_{j=1}^{s(i)} t_{i,j} \alpha_{i,j})k_i g_{i,i+1}^{-1} \cdots \cdots g_{n-2,n-1}^{-1} \exp(\sum_{j=1}^{s(n-1)} t_{n-1,j} \alpha_{n-1,j})g_{n-2,n-1}^{-1} k_{n-2}^{-1} \cdots g_{1,2}^{-1} k_1^{-1}) \cdot Y$$

(4.13)

(where $\tilde{t}$ is the tuple $\tilde{t} = (t_{i,j})_{1 \leq i \leq n-1, 1 \leq j \leq s(i)}$) Now, using the fact that we are differentiating at the identity, the above becomes (in coordinates):

$$ad(k_1 g_{1,2}^{-1} k_2 \cdots g_{i-1,i}^{-1} \alpha_{i,j} k_i g_{i,i+1}^{-1} k_{i+1}^{-1} \cdots k_{n-2}^{-1} g_{n-2,n-1}^{-1} k_{n-2}^{-1} \cdots \cdots k_{i+1}^{-1} g_{i,i+1}^{-1} k_{i-1}^{-1} g_{i-1,i}^{-1} \cdots g_{1,2}^{-1} k_1^{-1}) \cdot Y$$

(4.14)

The above simplifies to

$$v_{i,j} = ad(k_1 g_{1,2}^{-1} k_2 g_{2,3}^{-1} \cdots k_{i-1,i}^{-1} \alpha_{i,j} k_{i-1,i} k_{i-1}^{-1} \cdots g_{1,2}^{-1} k_1^{-1}) \cdot Y$$

(with $v_{i,j} \in T(M(n))_Y$). Now, let $h_i = g_{i-1,i}^{-1} \cdots g_{1,2}^{-1} k_1^{-1} \in GL(i) \hookrightarrow GL(n)$ and note that the $i \times i$ cutoff of $\text{Ad}(h_i) \cdot Y$ is in Jordan canonical form by construction. Now $\alpha_{i,j} \in \text{Lie}(K_a) \subset g_i$ with $g_i = \text{Lie}(G_i)$. Hence $h_i^{-1} \alpha_{i,j} h_i \in GL(i)$ centralizes the $i \times i$ cutoff of $Y$. With this notation the above equation becomes:

$$v_{i,j} = ad(h_i^{-1} \alpha_{i,j} h_i) \cdot Y$$

(4.15)

Now, we suppose by way of contradiction that the $v_{i,j}$ are dependent. We may assume that we have a strictly increasing sequence of indices $i_1 < i_2 < \cdots < i_t$ so that the following linear relation holds.

$$ad(h_{i_1}^{-1} \alpha_{i_1} h_{i_1}) \cdot Y + \cdots + ad(h_{i_t}^{-1} \alpha_{i_t} h_{i_t}) \cdot Y = 0$$

(4.16)
where \( \widetilde{\alpha}_{ij} = \sum_{l=j_j}^{j_l} c_l \alpha_{i,l} \in \text{Lie}(K_{a_i}) \) with \( c_l \in \mathbb{C}^\times \) and \( \{j_1, \ldots, j_m\} \subset \{1, 2, \ldots, s(i)\} \) a strictly increasing sequence. We observe that equation (4.16) implies that

\[
\sum_{j=1}^{t} h_{ij}^{-1} \alpha_{ij}^{-} h_{ij} \in M(n)^{Y} \tag{4.17}
\]

with \( M(n)^{Y} \) the centralizer of \( Y \in M(n) \). We also claim that since \( i_j > i_1 \) for all \( j > 1 \), we have

\[
\sum_{j=2}^{t} [h_{ij}^{-1} \alpha_{ij}^{-}, h_{ij}, Y]_{i_1+1} = 0 \tag{4.18}
\]

We use an argument very similar to the one in the proof of Theorem 2.14 in [KW, pg 24]. To see (4.18) we have to show that for \( j \geq 2 \), we have

\[
[h_{ij}^{-1} \alpha_{ij}^{-}, h_{ij}, Y]_{i_1+1} = 0 \tag{4.19}
\]

This follows from the fact that \( M(n) = M(i_j) \oplus M(i_j)^\perp \) (where \( M(i_j)^\perp \) denotes the orthogonal compliment of \( M(i_j) \) with respect to the trace form). The component of \( Y \) in \( M(i_j) \) is clearly \( Y_{i_j} \), but we observed above that \( [h_{ij}^{-1} \alpha_{ij}^{-}, h_{ij}, Y_{i_j}] = 0 \). Thus, since \( \text{ad} M(i_j) \) stabilizes the components of the above decomposition, we have that \( [h_{ij}^{-1} \alpha_{ij}^{-}, h_{ij}, Y] \in M(i_j)^\perp \). Now, since \( i_j \geq i_1 + 1 \), we have that \( M(i_j)^\perp \subset M(i_1 + 1)^\perp \). But then we get easily equation (4.19) and thus equation (4.18) by linearity. Equations (4.17) and (4.18), then imply that

\[
[h_{i_1}^{-1} \alpha_{i_1}, h_{i_1}, Y]_{i_1+1} = 0
\]

But we have that \( h_{i_1}^{-1} \alpha_{i_1} h_{i_1} \in M(i_1) \) so that the above becomes:

\[
[h_{i_1}^{-1} \alpha_{i_1}, h_{i_1}, Y_{i_1+1}] = 0
\]

But this becomes

\[
[\alpha_{i_1}, \text{Ad}(h_{i_1}) \cdot Y_{i_1+1}] = 0 \tag{4.20}
\]

But \( y_{i_1+1} = \text{Ad}(h_{i_1}) \cdot Y_{i_1+1} \) is a projection of \( Y \in \text{Im} \Gamma_n \) into the solution variety \( \Xi^{i_1, c_{i_1+1}} \) and it must therefore lie in the orbit \( O_{a_{i_1}}^{i_1} \) by Remark 4.2.1. Thus, \( \alpha_{i_1} \in \text{Lie}(K_{a_{i_1}}) \cap \text{Lie}(\text{Stab}(y_{i_1+1})) = \text{Lie}(K_{a_{i_1}}) \cap \text{Lie}(\text{Stab}(x_{a_{i_1}})) \). Where the last equality
follows because the group $G_i$ is abelian so that $\text{Stab}(x_{a_i}) = \text{Stab}(y_{a_i}) \subset G_i$ for all $i$.

But, we have chosen $K_{a_i}$ such that $\text{Lie}(K_{a_i}) \cap \text{Lie}(\text{Stab}(x_{a_i})) = 0$ (see Theorem 4.1.1 above). Thus, $\tilde{\alpha}_{a_i} = 0 \Rightarrow \sum_{l=j_1}^{j_m} c_l \alpha_{i,l} = 0 \Rightarrow c_l = 0$ for all $l$, a contradiction. Thus, the tangent vectors $v_{i,j}$ in equation (4.15) are linearly independent. Since $Y \in \text{Im} \Gamma_n$ was chosen arbitrarily this gives us that we have $(d\Gamma_n)_{\tilde{k}}$ is of full rank by equation (4.12). Hence $\Gamma_n$ is an immersion. We have already observed that it is a topological embedding in the analytic topology, so that $\text{Im} \Gamma_n$ is a complex embedded submanifold of $M(n)$, as desired. This completes the proof. Q.E.D.

**Remark 4.3.1.** In the above proof, we have used that $\text{Lie}(\text{Stab}(x_{a_i})) = \{Z \in g_i | [Z, x_{a_i}] = 0\}$. We can see this in the following way. We again work in the analytic category. Since $\text{Stab}(x_{a_i})$ is a Lie group over $\mathbb{C}$, we have that $Z \in g_i, Z \in \text{Lie}(\text{Stab}(x_{a_i}))$ if and only if $\exp(tZ) \in \text{Stab}(x_{a_i})$ for all $t \in \mathbb{C}$. We first suppose that $\text{Ad}(\exp(tZ)) \cdot x_{a_i} = x_{a_i}$, then differentiating at $t = 0$ yields $\text{ad}(Z) \cdot x_{a_i} = 0$. Conversely, suppose that we have $\text{ad}(Z) \cdot x_{a_i} = 0$, then we note that for any $t \in \mathbb{C}$ we have that $\text{Ad}(\exp(tZ)) \cdot x_{a_i} = \exp(\text{ad}(tZ)) \cdot x_{a_i}$.

**Remark 4.3.2.** With this analytic structure on $\text{Im} \Gamma_n^{a_1, \cdots, a_{n-1}}$, the inverse map $(\Gamma_n^{a_1, \cdots, a_{n-1}})^{-1}$ defined in the previous section is easily seen to be holomorphic, as $\Gamma_n^{a_1, \cdots, a_{n-1}}$ has non-singular differential and is bijective onto its image, so it is thus a biholomorphism.

Now, we note that on the submanifold $\text{Im} \Gamma_n^{a_1, \cdots, a_{n-1}}$, we have a holomorphic action of the analytic group $K = K_{a_1} \times \cdots \times K_{a_{n-1}}$. Given as follows:

If $(\Gamma_n^{a_1, a_2, \cdots, a_{n-1}})^{-1}(Y) = (k_1, \cdots, k_{n-1})$ then

$$(k'_1, \cdots, k'_{n-1}) \cdot Y = \Gamma_n^{a_1, a_2, \cdots, a_{n-1}}(k'_1 k_1, \cdots, k'_{n-1} k_{n-1}) \quad (4.21)$$

This action is easily seen to be a holomorphic action as follows. We write for $\tilde{k} = (k_1, \cdots, k_{n-1}) \in K = K_{a_1} \times \cdots \times K_{a_{n-1}}$. We consider the following holomorphic maps.
\[\Phi_1 : K \times \text{Im} \Gamma_n^{a_1, a_2, \cdots, a_{n-1}} \to K \times K\]

given by

\[((k_1, \cdots, k_{n-1}), Y) \to ((k_1, \cdots, k_{n-1}), (\Gamma_n^{a_1, a_2, \cdots, a_{n-1}})^{-1}(Y))\]

\[\Phi_2 : K \times K \to K\]

given by

\[((k_1', \cdots, k'_{n-1}), (k_1, \cdots, k_{n-1})) \to (k_1' k_2, \cdots, k'_{n-1} k_{n-1})\]

\[\Phi_3 = \Gamma_n^{a_1, a_2, \cdots, a_{n-1}}\]

Then the action map is given by the composition \(\Phi_3 \circ \Phi_2 \circ \Phi_1\).

Since the action map is a composition of holomorphic maps, we have that the action is holomorphic.

Remark 4.3.3. We note that the action of \(K\) on \(\text{Im} \Gamma_n^{a_1, a_2, \cdots, a_{n-1}}\) is a free and transitive action.

Our goal is to see that for \(X \in \text{Im} \Gamma_n^{a_1, a_2, \cdots, a_{n-1}}\) the \(A\) orbit of \(X\) is the same as its \(K\) orbit.

4.4 The tangent space to \(\text{Im} \Gamma_n\)

Our goal in this section is to show that \(A\) acts on the manifold \(\text{Im} \Gamma_n^{a_1, a_2, \cdots, a_{n-1}}\). The following result from differential geometry will be of use later in this section.

**Proposition 4.4.1.** Let \(M\) be a complex (or real differentiable) manifold. Let \(N \subset M\) be a submanifold. Let \(V\) be a complete holomorphic (resp smooth) vector field on \(M\) with the property that for each \(p \in N\) \(V_p\) is tangent to \(N\) i.e. \(V_p \in \)
\[ di_p T_p (N) \text{ (with } i : N \hookrightarrow M \text{ denoting the inclusion). Then the integral curve of } V \text{ starting at } p \in N \text{ lies in } N. \]

This is a standard consequence of the uniqueness theorem for ordinary differential equations.

Thus, if we can show that for \( X \in Im \Gamma_n^{a_1, \ldots, a_{n-1}} \) the tangent space \( T_X (Im \Gamma_n^{a_1, \ldots, a_{n-1}}) = V_X \), then Proposition (4.4.1) will give us that \( A \) acts on \( Im \Gamma_n^{a_1, \ldots, a_{n-1}} \).

During the proof of Proposition 4.3.1, we have computed the tangent space to the embedded submanifold \( Im \Gamma_n^{a_1, a_2, \ldots, a_{n-1}} \subset M(n) \). We can see this as follows. Recall the map \( \Psi_Y \) defined in equations (4.8) and (4.10) for \( Y \in Im \Gamma_n^{a_1, a_2, \ldots, a_{n-1}} \), \( Y = \Gamma_n^{a_1, a_2, \ldots, a_{n-1}} (k_1, \ldots, k_{n-1}) \). In the course of that proof we computed the differential of the map \( \Psi_Y \) at the identity element in \( K = K_{a_1} \times \cdots \times K_{a_{n-1}} \). But we note that this map is simply the orbit map \( \theta_Y : K \rightarrow Im \Gamma_n^{a_1, a_2, \ldots, a_{n-1}} \) at \( Y \) (see equation (4.11)). Thus, differentiating \( \theta_Y = \Psi_Y \) at the identity, should give us \( T_Y (K \cdot Y) = T_Y (Im \Gamma_n^{a_1, a_2, \ldots, a_{n-1}}) \). Now, we computed the image of the differential of \( \theta_Y \) in equation (4.15) to be

\[ v_{i,j} = ad(h_i^{-1} \alpha_i h_i) \cdot Y \]

for \( 1 \leq i \leq n - 1, 1 \leq j \leq \dim(K_{a_i}) \) (with \( \{ \alpha_{i,1}, \ldots, \alpha_{i,\dim(K_{a_i})} \} \) a basis for \( Lie(K_{a_i}) \)). Recall that \( h_i Y_{i+1} h_i^{-1} \) is a projection of \( Y \) into the solution variety \( \Xi_{c_i,c_{i+1}}^i \) at level \( i \) and thus is in the orbit \( \mathcal{O}_{a_i}^i \). We observe that \( Lie(K_{a_i}) \subset \mathfrak{g}_i = Lie(G_i) \). Recalling that \( Lie(G_i) \) is the centralizer of the \( i \times i \) cutoff of \( x_{a_i} \in \Xi_{c_i,c_{i+1}}^i \), we see that we have that since \( h_i \in Gl(i) \hookrightarrow Gl(n) \), \( h_i^{-1} \alpha_{i,j} h_i \) centralizes the \( i \times i \) cutoff of \( Y \). Now, recall that \( Y \in M_c(n) \cap S \), so that the \( i \times i \) cutoff of \( Y \) is regular. Thus, the centralizer of \( Y_i \) in \( M(i) \) is spanned by the powers \( \{ Id_i, Y_i, \ldots, Y_i^{r_i-1} \} \). We recall Theorem 2.13 in [KW, pg 23] which states that the subspace \( V_X = \{ (\xi f)_X | f \in J(M(n)) \} \) given by the Hamiltonian fields from the analogue Gelfand-Zeitlin algebra is equal to (in coordinates):

\[ V_X = span \{ [z, X] | z \in Z_X \} \]

for \( X \in M(n) \). Here \( Z_X \) is defined to be \( \bigoplus_{i=1}^{n-1} Z_{X_i} \), where \( Z_{X_i} \) is the associative subalgebra of \( M(i) \hookrightarrow M(n) \) generated by \( Id_i \) and \( X_i \). Thus, we have that \( v_{i,j} \in V_Y \).
and thus
\[ T_Y(Im\Gamma_n^{a_1,\ldots,a_n-1}) \subset V_Y \] (4.22)

We want to show that the inclusion in (4.22) is actually equality. To see that the subspace \( V_X \subset T_X(M(n)) \) is tangent to the manifold \( Im\Gamma_n^{a_1,\ldots,a_n-1} \), we define the following map

\[ \Phi_Y : G_1 \times G_2 \cdots \times G_{n-1} \to Im\Gamma_n^{a_1,\ldots,a_n-1} \]

given by

\[ (h_1, \ldots, h_{n-1}) \mapsto \]
\[ \to Ad(h_1 k_1 g_{12}(x_{a_1})^{-1} h_2 k_2 g_{23}(x_{a_2})^{-1} \cdots h_{n-2} k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} h_{n-1}) \cdot y_{a_{n-1}} \] (4.23)

with \( h_i \in G_i \), where \((k_1, \ldots, k_{n-1}) \in K_{a_1} \times \cdots \times K_{a_{n-1}}\) is fixed and \( y_{a_{n-1}} \) is a projection of \( Y \) into the orbit \( O_{a_{n-1}}^{n-1} \) and is given by

\[ y_{a_{n-1}} = Ad(g_{n-2,n-1}(x_{a_{n-2}})) Ad(k_{n-2}^{-1}) \cdots Ad(g_{12}(x_{a_1})) Ad(k_1^{-1}) \cdot Y \]

with \( Y = \Gamma_n^{a_1,\ldots,a_{n-1}}(k_1, \ldots, k_{n-1}) \). First, we observe that from Remark 4.2.1 \( \Phi_Y \) actually maps into \( Im\Gamma_n^{a_1,\ldots,a_{n-1}} \). Since for \( Z \in Im\Phi_Y \) a projection into the solution variety at level \( j \) is given by

\[ [Ad(g_{j,j-1}(x_{a_{j-1}}) k_{j-1}^{-1} h_{j-1}^{-1}) \cdots Ad(g_{1,2}(x_{a_1}) k_1^{-1} h_1^{-1}) \cdot Z]_j \]
\[ = [Ad(h_j) Ad(k_j) Ad(g_{j,j+1}(x_{a_j})^{-1}) \cdots Ad(h_{n-1} y_{a_{n-1}})]_j \]

is in the orbit given by \( O_{a_j}^j \) for all \( 1 \leq j \leq n-1 \). Now, note that \( \Phi_Y : G_1 \times \cdots \times G_{n-1} \to \Gamma_n^{a_1,\ldots,a_{n-1}} \) is a holomorphic map into the embedded submanifold \( \Gamma_n^{a_1,\ldots,a_{n-1}} \). This follows easily from the fact that \( \Phi_Y \) is a holomorphic map into \( M(n) \) and since \( Im\Gamma_n^{a_1,\ldots,a_{n-1}} \) is an embedded submanifold, it follows automatically that the map is holomorphic into \( Im\Gamma_n^{a_1,\ldots,a_{n-1}} \) (see Corollary 8.25 in [Lee, pg 191]). We also note that \( \Phi_Y \) is a submersion, since its restriction to the subgroup \( K_{a_1} \times \cdots \times K_{a_{n-1}} \) of \( G_1 \times \cdots \times G_{n-1} \) is the map \( \Psi_Y = \Gamma_n^{a_1,\ldots,a_{n-1}} \circ L_k \) defined in the proof of Proposition
4.3.1 in equation (4.8) which by construction is a diffeomorphism onto its image. (Recall $L_k$ is left translation by $k = (k_1, \ldots, k_{n-1})$). The upshot is then that we can compute the tangent space to $Y$ by differentiating $\Phi_Y$ at the identity.

This computation is totally analogous to the one done in the proof of Proposition 4.3.1 in equations (4.13) and (4.14). The same computations as in those equations, now produces the following result.

$$w_{i,j} = ad(h_i^{-1}\gamma_{i,j}h_i) \cdot Y \in T_Y(Im\Gamma_{n_1,\ldots,n_{n-1}})$$ (4.24)

Where $h_i \in Gl(i)$ is as in equation (4.15) in the proof of Proposition 4.3.1. But now $\gamma_{i,j} \in g_i = Lie(G_i)$ and is part of a basis $\{\gamma_{i,1}, \ldots, \gamma_{i,i}\}$ for $g_i$. Thus the elements of the form

$$\{h_i^{-1}\gamma_{i,j}h_i | 1 \leq j \leq i\}$$

form a basis for the centralizer in $M(i)$ of the $i \times i$ cutoff of $Y$ for $1 \leq i \leq n - 1$.

Since $Y \in Im\Gamma_{n_1,\ldots,n_{n-1}} \subset M_c(n) \cap S$, we have that

$$span\{w_{i,j} | 1 \leq i \leq n-1, 1 \leq j \leq i\} = span\{[z,Y] | z \in Z_Y\} = V_Y$$

as in the beginning of the section. Since $Y \in Im\Gamma_{n_1,\ldots,n_{n-1}}$ is arbitrary, we have proven the following result.

**Proposition 4.4.2.** The commuting Lie algebra of vector fields $\{\xi_{f_{i,j}} | 1 \leq i \leq n-1, 1 \leq j \leq i\}$ (where $f_{i,j} = tr(X_i^j)$) is tangent to the submanifold $Im\Gamma_{n_1,\ldots,n_{n-1}}$. Thus $A$ acts on $Im\Gamma_{n_1,\ldots,n_{n-1}}$.

Proof:

The only statement that remains to be proven is the last statement, but this follows from the fundamental result in differential geometry in Proposition 4.4.1 and the fact that the $A$ action is obtained by composing that action of the commuting flows of the commuting vector fields $\xi_{f_{i,j}}$.

Q.E.D.

4.5 The submanifolds $Im\Gamma_n$ and $A$ orbits

In the last section, we showed in Proposition 4.4.2 that each $Im\Gamma_{n_1,\ldots,n_{n-1}}$ is a union of $A$ orbits. Now let $Y \in Im\Gamma_{n_1,\ldots,n_{n-1}}$ and consider the $A$ orbit of $Y$, $A \cdot Y$. 

We know that the dim $A \cdot Y = \dim V_Y = \dim Im \Gamma_n^{a_1, \ldots, a_{n-1}}$. (Recall from [KW] that for $X \in M(n)$, $T_X(A \cdot X) = V_X$.) Thus for $Y \in Im \Gamma_n^{a_1, \ldots, a_{n-1}}$, $A \cdot Y \subset Im \Gamma_n^{a_1, \ldots, a_{n-1}}$ is a submanifold of the same dimension as $Im \Gamma_n^{a_1, \ldots, a_{n-1}}$, hence it must be open. Thus, $Im \Gamma_n^{a_1, \ldots, a_{n-1}}$ is a disjoint union of open $A$ orbits, so we write

$$Im \Gamma_n^{a_1, \ldots, a_{n-1}} = \coprod_{i \in I} A \cdot X(i)$$

with $X(i) \in Im \Gamma_n^{a_1, \ldots, a_{n-1}}$. We note that $A \cdot X(j), j \in I$ is given by $Im \Gamma_n^{a_1, \ldots, a_{n-1}} \setminus (\coprod_{i \neq j, i \in I} A \cdot X(i))$, since the union is disjoint. But $\coprod_{i \neq j, i \in I} A \cdot X(i)$ is open, so that $A \cdot X(j)$ is closed. Thus $A \cdot X(j)$ is both open and closed in the connected manifold $Im \Gamma_n^{a_1, \ldots, a_{n-1}}$. Hence since $A \cdot X(j) \neq \emptyset$, we must have $Im \Gamma_n^{a_1, \ldots, a_{n-1}} = A \cdot X(j)$ is exactly one $A$ orbit.

We have now reached our goal and have:

**Theorem 4.5.1.** The embedded complex submanifold $Im \Gamma_n^{a_1, \ldots, a_{n-1}}$ is exactly one $A$ orbit in $M_c(n) \cap S$. Moreover, every $A$ orbit in $M_c(n) \cap S$ is given as the image of a mapping of the form $\Gamma_n^{a_1, \ldots, a_{n-1}}$ where $a_i$ denotes a choice of $G_i$ orbit in the solution variety $\Xi_i^{c_i, c_{i+1}}$ at level $i$.

**Proof:**

Only the second statement needs to be addressed. It follows easily from the fact that $M_c(n) \cap S$ is covered by sets of the form $Im \Gamma_n^{a_1, \ldots, a_{n-1}}$ for some choice of orbits $a_i$ (see Remark 4.2.1).

Q.E.D.

We have an immediate corollary.

**Corollary 4.5.1.** The $A$ orbits in $M_c(n) \cap S$ are irreducible Zariski constructible subsets of $M(n)$ that are isomorphic as analytic manifolds to a product of abelian connected algebraic groups over $\mathbb{C}$, which is a subgroup of the group $G_1 \times \cdots \times G_{n-1}$.

**Remark 4.5.1.** The first statement of the corollary appears already in Theorem 3.7 in [KW, pg 36]. However, in that reference the authors only establish a bijection in the case of strongly regular elements (see Theorem 3.14 in [KW, pg 40]). This result is thus a strengthening of Theorem 3.7 and gives a partial generalization of Theorem 3.14 to elements in $M_c(n) \cap S$ which are not strongly regular.
4.6 The $\Gamma_n$ construction and $M_{c}^{sreg}(n)$

Recall that if $X \in M(n)$ is such that $\dim A \cdot X = \binom{n}{2}$, then $X$ is said to be strongly regular and the strongly regular elements in the fibre $M_c(n)$ are denoted by $M_{c}^{sreg}(n)$. Theorem 2.4 in [KW, pg 23] states that $M_{c}^{sreg}(n) \subset M_c(n) \cap S$. Thus, by Theorem 4.5.1, an $A$ orbit in $M_{c}^{sreg}(n)$, must be an image of a map $Im\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ for some choice of orbits $a_i \in \Xi_i^{c_i,c_{i+1}}$.

**Theorem 4.6.1.** The submanifold $Im\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ is an $A$ orbit of maximal dimension $\binom{n}{2}$ if and only if the orbit $O_{a_i}^i \subset \Xi_{c_i,c_{i+1}}^i$ is of maximal dimension $i$, in which case $G_i$ acts freely on $O_{a_i}^i$.

Proof:

By Lemma 4.1.1 $O_{a_i}^i$ is of maximal dimension $i$ if and only if $Stab(x_{a_i})$ is trivial and hence $G_i$ acts freely. In this case $K_{a_i} = G_i$ and is of dimension $i$ so that the manifold $Im\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ has dimension $\binom{n}{2}$. Thus $Im\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ is an $A$ orbit of dimension $\binom{n}{2}$.

Conversely, if $Im\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ is of dimension $\binom{n}{2}$, then since $Im\Gamma_{n}^{a_1,\cdots,a_{n-1}} = \sum_{i=1}^{n-1} \dim K_{a_i} \Rightarrow \dim K_{a_i} = i$ for all $i$. This follows from the fact that $\dim K_{a_i} \leq i$ for all $i$. Thus, $O_{a_i}^i$ is of dimension $i$.

Q.E.D.

In the remainder of this section $Im\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ denotes a strongly regular $A$ orbit. Now, in the case where $X \in M_{c}^{sreg}(n)$ is strongly regular $A \cdot X$ is a smooth and irreducible variety (see Theorem 3.12 in [KW, pg 48]). We want to see that in this case $\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ is actually an isomorphism of varieties. This follow from the following corollary to Zariski’s Main Theorem (see [TY, pg. 229]).

**Proposition 4.6.1.** Let $u : X \rightarrow Y$ be a bijective morphism of irreducible varieties. If $Y$ is normal, then $u$ is an isomorphism.

Now, we have observed that we have a regular map $\Gamma_{n}^{a_1,\cdots,a_{n-1}} : G_1 \times \cdots \times G_{n-1} \rightarrow A \cdot X$. Where $X$ is a fixed point in the image of $\Gamma_{n}^{a_1,\cdots,a_{n-1}}$. We have observed that $\Gamma_{n}^{a_1,\cdots,a_{n-1}}$ is bijective. Since non-singular varieties are normal, and
the groups $G_i$ for $1 \leq i \leq n - 1$ are all connected, the above corollary to Zariski’s main theorem gives us that $\Gamma_n^{a_1, \ldots, a_{n-1}}$ is actually an isomorphism of varieties. To summarize we have:

$$G_1 \times \cdots \times G_{n-1} \xrightarrow{\Gamma_n^{a_1, \ldots, a_{n-1}}} \text{Im} \Gamma_n^{a_1, \ldots, a_{n-1}}$$

(4.25)

where the isomorphism is actually biregular. Now, recall the action of $G$ that we defined in equation (4.21). Now, the maps defining the action map can be all be seen to be morphisms, since $(\Gamma_n^{a_1, \ldots, a_{n-1}})^{-1}$ is a morphism. Thus we have an algebraic action of $G$ on $\text{Im} \Gamma_n^{a_1, \ldots, a_{n-1}}$ which has the same orbit structure as the $A$ action. Now, by Theorem 3.12 in [KW, pg 38] the $A$ orbits in $M_c^{\text{reg}}(n)$ are the irreducible components on $M_c^{\text{reg}}(n)$ and since they are disjoint they are both open and closed in $M_c^{\text{reg}}(n)$ (in the Zariski topology on $M_c^{\text{reg}}(n)$). Following the notation of [KW], we index these components by $M_{c,i}^{\text{reg}}(n) = A \cdot X(i)$ with $X(i) \in M_c^{\text{reg}}(n)$. Now, we have regular maps $\phi_i : G \times M_{c,i}^{\text{reg}}(n) \to M_c^{\text{reg}}(n)$ given by the action of $G$ on $\text{Im} \Gamma_n^{a_1, \ldots, a_{n-1}}$. We note that the sets $G \times M_{c,i}^{\text{reg}}(n)$ are (Zariski) open in the product $G \times M_c^{\text{reg}}(n)$ and are disjoint. Thus the morphisms $\phi_i$ glue to a unique morphism

$$\Phi : G \times M_c^{\text{reg}}(n) \to M_c^{\text{reg}}(n)$$

such that

$$\Phi|_{G \times M_{c,i}^{\text{reg}}(n)} = \phi_i$$

The morphism $\Phi$ gives us an algebraic action of the group $G$ on $M_c^{\text{reg}}(n)$ whose orbits are the orbits of $A$ in $M_c^{\text{reg}}(n)$. We have thus proved the following theorem.

**Theorem 4.6.2.** On $M_c^{\text{reg}}(n)$ the orbits of the group $A$ are given by orbits of an algebraic action of the connected abelian algebraic group $G = G_1 \times \cdots \times G_{n-1}$ with $G_i = G_{X_i}$ for $X \in M_c^{\text{reg}}(n)$. (Recall that $G_{X_i}$ denotes the centralizer of $X_i$ in $GL(i)$.)

Now that we have developed a way to construct $A$ orbits in $M_c^{\text{reg}}(n)$, we can count the number of $A$ orbits in $M_c^{\text{reg}}(n)$ for any $c \in \mathbb{C}^{\binom{n+1}{2}}$ and explicitly
describe the orbits. However, first we have to see that for any \( c \in \mathbb{C}^{\binom{n+1}{2}} \) that the solution variety \( \Xi_{c_i,c_{i+1}} \) has \( G_i \) orbits \( O^i \) which consist of regular elements of \( M(i + 1) \) on which \( G_i \) acts freely. We will actually be able to show that if \( O^i \) is an orbit in \( \Xi_{c_i,c_{i+1}} \) on which \( G_i \) acts freely, then it must consist of regular elements of \( M(i + 1) \). This will be the content of the next chapter.
5 Regular elements and maximal orbits of the solution variety

\[ \Xi^i_{c_i, c_i+1} \]

We first consider the case where \( c_i \in \mathbb{C}^i \) represents a monic polynomial of the form \((t - \lambda)^i\). So that the \( i \times i \) cutoff of \( X \in \Xi^i_{c_i, c_i+1} \) is a single Jordan block given by

\[
\begin{bmatrix}
\lambda & 1 & \cdots & 0 \\
0 & \lambda & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda
\end{bmatrix}
\]

and we let \( \lambda_{i+1} = (\lambda_{i+1,1}, \cdots, \lambda_{i+1,r}) \in \mathbb{C}^r \) be chosen arbitrarily. Then we saw from our work in section 3.2 that the solution variety \( \Xi^i_{c_i, c_i+1} \) contains a \( G_i \) orbit on which the \( G_i \) action is free. This orbit contains an element of the form

\[
\begin{bmatrix}
\lambda & 1 & \cdots & 0 & f_1(1, \cdots, 0) \\
0 & \lambda & \ddots & \vdots & \vdots \\
\vdots & \ddots & 1 & \vdots & \vdots \\
0 & \cdots & \cdots & \lambda & f_i(1) \\
1 & \cdots & \cdots & 0 & w
\end{bmatrix}
\]  (5.1)

This solution is clearly a regular element in \( M(i + 1) \). We have an obvious choice of a cyclic vector, \( e_i \), where \( e_i \) denotes the \( i-th \) standard basis vector in \( \mathbb{C}^{i+1} \). Now, the difficulty comes in showing that more general solution varieties consist of regular elements.
Suppose now that we have a solution variety like the one considered in equation (3.1). For example consider an element of $M(i + 1)$ of the form

\[
\begin{bmatrix}
\lambda_1 & 1 & \cdots & 0 \\
0 & \lambda_1 & \ddots & \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda_1
\end{bmatrix}
\begin{bmatrix}
y_{1,1} \\
0 \\
0 \\
y_{1,n_1}
\end{bmatrix}
\]

with $\lambda_i \neq \lambda_j$, but otherwise they are allowed to be arbitrary. We suppose that the Jordan block corresponding to the eigenvalues $\lambda_j$ is of size $n_j$. In sections 3.2 and 3.3 we showed that if we are given any monic polynomial of degree $i + 1$, say $p(t)$, then we can choose the $z_{i,j}$ and $y_{i,j}$ in the above matrix so that it has characteristic polynomial the given polynomial. We also observed that we can find sets of solutions to the extension problem on which $G_i$ (the centralizer of the $i \times i$ Jordan block) acts simply transitively. We claim now that if $O \subset \Xi_{c_i,c_{i+1}}$ is such an orbit of $G_i$ in $\Xi^i_{c_i,c_{i+1}}$, then $O$ consists of regular elements of $M(i + 1)$.

We show this in a slightly indirect fashion using the description of strongly regular orbits in $M(i + 1)$ developed in Chapter 4, section 4.6. (Recall that we denote by $S$ the set of elements in $M(i + 1)$ such that each cutoff $X_j$ for $1 \leq j \leq i$ is regular.) We consider the fibre $M_c(i+1) \cap S$ where $c \in \mathbb{C}^{(i+1)^2}$ is such that $c_j, c_{j+1}$ for $1 \leq j \leq i - 2$ represent coefficients of relatively prime monic polynomials and $c_j$ for $1 \leq j \leq i - 1$ represents coefficients of polynomial $(t - \mu_j)^j$. That is to say that $X \in M_c(i+1)$ has that the characteristic polynomial of $X_j$ is equal to $(t - \mu_j)^j$ for $1 \leq j \leq i - 1$, and $\mu_j \neq \mu_{j+1}$ for $1 \leq j \leq i - 2$. Now, we let $c_i$ correspond to the characteristic polynomial of the $i \times i$ cutoff of the matrix in (3.1). (i.e. the polynomial represented by $c_i$ factors as $\prod_{j=1}^i (t - \lambda_j)^{n_j}$). We also let $c_{i+1}$ be the
coefficients (excluding the leading term) of the polynomial $p(t)$.

By our work in Chapter 4, section 4.2, we can use the orbit of $\mathcal{O} \subset \Xi^j_{c_i,c_{i+1}}$ to construct a $\Gamma_{i+1}$ mapping onto a subset of $M_c(i+1) \cap S$. Since for the construction of such a $\Gamma_{i+1}$ mapping, we recall that we only need the elements of the solution varieties at levels below $i$ to be regular. Only elements in $\Xi^j_{c_j,c_{j+1}}$ for $1 \leq j \leq i-1$ that are conjugated into Jordan form.

So now, we have a map:

$$\Gamma_{i+1}^{a_1,a_2,\ldots,a_i} : \mathcal{O}_{a_1}^1 \times \cdots \times \mathcal{O}_{a_{i-1}}^{i-1} \times \mathcal{O} \to M_c(i+1) \cap S$$

where each orbit $\mathcal{O}_{a_j}^j \subset \Xi^j_{c_j,c_{j+1}}$ is an orbit of $G_j$ on which the $G_j$ action is free. We know that for $1 \leq j \leq i-2$ that there is only one choice of orbit given in section 3.2. However, for $\mathcal{O}_{a_{i-1}}^{i-1}$ we can choose a $G_i$ orbit of the form (5.1). Now, by Theorem 4.6.1, the image of the above map is an $A$-orbit of dimension $\binom{i+1}{2}$, so that it consists of strongly regular elements. Now, we recall Thm 2.14 in [KW, pg 23] that $X$ is strongly regular implies that $X_m$ is regular for all $1 \leq m \leq i + 1$.

We recall the definition of $\Gamma_{i+1}^{a_1,a_2,\ldots,a_i}$:

$$\Gamma_{i+1}^{a_1,a_2,\ldots,a_i}(h_1, \cdots, h_i) = \text{Ad}(h_1g_1(x_{a_1})^{-1}h_2g_2(x_{a_2})^{-1} \cdots h_{i-1}g_{i-1}(x_{a_{i-1}})h_i)x_{a_i}$$

where $g_{j,j+1}(x_{a_j})$ conjugates the “base point” of the orbit $\mathcal{O}_{a_j}^j$, $x_{a_j}$ into Jordan canonical form for $1 \leq j \leq i - 1$ and $(h_1, \cdots, h_i) \in G_1 \times \cdots \times G_i$.

We see that any $y \in \mathcal{O} \subset M(i+1)$ ($y = \text{Ad}(h_i) \cdot x_{a_i}$ for unique $h_i \in G_i$) is conjugate to an $X \in \text{Im}\Gamma_{i+1}^{a_1,a_2,\ldots,a_i}$. Since $X \in M_{sreg}(n)$ we have that $X$ is regular so that $x_{a_i}$ must be regular. To summarize we have now proven the following proposition.

**Proposition 5.0.2.** Let $\mathcal{O} \in \Xi^i_{c_i,c_{i+1}}$ be a $G_i$ orbit of maximal dimension $i$, on which $G_i$ necessarily acts freely by Lemma 4.1.1, then the elements of $\mathcal{O}$ are regular elements of $M(i+1)$. 
6 Counting $A$ orbits in $M^\text{sreg}_c(n)$

We preserve the notation of section 3.3. We can use our results in the previous Chapters to count $A$ orbits in arbitrary fibres in $M^\text{sreg}_c(n)$. The main result is the following.

**Theorem 6.0.3.** Let $c = (c_1, c_2, \cdots, c_i, c_{i+1}, \cdots, c_n) \in \mathbb{C}^{n(n+1)/2}$. Suppose there are $0 \leq j_i \leq i$ roots in common between the monic polynomials represented by $c_i$ and $c_{i+1}$. Then the number of $A$ orbits in $M^\text{sreg}_c(n)$ is exactly $2^{\sum_{i=1}^{n-1} j_i}$. We have that on $M^\text{sreg}_c(n)$ the orbits of $A$ are the orbits of an algebraic action of the complex, commutative connected algebraic group $G = G_1 \times G_2 \times \cdots \times G_{n-1}$.

Proof: We suppose that $c_i \in \mathbb{C}^i$ are the coefficients of the polynomial $\prod_{i=1}^{r}(t-\lambda_j)^{n_j}$ so that elements in the solution variety $\Xi^i_{c_i,c_{i+1}}$ are of the form (3.1). Then we know by equations from section 3.3

\begin{align*}
 z_{k,1}y_{k,n_k} &= 0 \text{ for } 1 \leq k \leq j_i \\
 z_{k,1}y_{k,n_k} &\in \mathbb{C}^\times \text{ for } j < k \leq r
\end{align*}

that there are at least $2^{j_i}$ $G_i$ orbits of maximal dimension in the solution variety $\Xi^i_{c_i,c_{i+1}}$ at level $i$. (We can suppose without loss of generality that the overlaps occur in this order). Where each orbit is given by a choice of either $y_{k,n_k} \neq 0$ or $z_{k,1} \neq 0$ for $1 \leq k \leq j_i$. Now, I claim that these are all the orbits of dimension $i$ in $\Xi^i_{c_i,c_{i+1}}$. Without loss of generality, suppose that we take $z_{1,1} = y_{1,n_1} = 0$. If there are solutions in this case, it is easy to see that $G_i$ stabilizes them. However, it is easy to see that they would have a non-trivial isotropy group in $G_i$ that would contain
the subgroup:

\[
\begin{bmatrix}
1 & 0 & \cdots & c \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & 0 \\
0 & \cdots & \cdots & 1
\end{bmatrix}
\]

where \( c \in \mathbb{C} \). Thus, if such solutions exist they could not belong to orbits of maximal dimension. Hence there are exactly \( 2^j \) orbits of dimension \( i \) in \( \Xi^i_{c_i, c_{i+1}} \). We recall that by Proposition 5.0.2 these orbits all consist of regular elements of \( M(i+1) \). Thus, we can construct \( 2^{\sum_{i=1}^{n-1} j_i} \) mappings of the form

\[
\Gamma^{a_1, \ldots, a_{n-1}}_n : G_1 \times \cdots \times G_{n-1} \to M^{sreg}_{c}(n)
\]

with \( a_i \) a choice of orbit \( Z_{b_1, \ldots, b_{j_i}} \) with \( b_l = U, L \) as in section 3.3. By Theorem 4.6.1 the images of such mappings are precisely the \( A \) orbits of dimension \( \begin{pmatrix} n \\ 2 \end{pmatrix} \) in \( M_{c}^{sreg}(n) \). Now, recall from Remark 4.2.1, that \( X \in Im^{a_1, \ldots, a_{n-1}}_n \) if and only if every projection \( z_j \) into the solution variety at level \( j \) is in the orbit \( O_{a_j}^j \in \Xi^i_{c_i, c_{i+1}} \) for each \( j, 1 \leq j \leq n - 1 \). It follows that the images of the different \( \Gamma^{a_1, \ldots, a_{n-1}}_n \) mappings are disjoint. Thus, we have precisely \( 2^{\sum j_i} \) orbits in \( M_{c}^{sreg}(n) \). The last statement of the theorem then follows directly from Theorem 4.6.2. This gives us the desired result.

Q.E.D.

Now, we have several important corollaries.

**Corollary 6.0.1.** The action of \( A \) is transitive on \( M^{sreg}_{c}(n) \) if and only if

\[ c = (c_1, \cdots, c_n) \] with \( c_i \) and \( c_{i+1} \) representing coefficients of relatively prime polynomials.
Proof:
The proof is immediate from Theorem 6.0.3.

**Corollary 6.0.2.** The nilfibre $M_0^{sreg}(n)$ contains $2^{n-1}$ $A$-orbits.

Proof:
This follows again from Theorem 6.0.3, since there is now exactly one overlap at each level.

### 6.1 The new set of generic matrices $\Theta_n$

Corollary 6.0.1 allows us to identify the set of matrices in $M(n)$ on which the action of $A$ is transitive. This expands the results of Kostant and Wallach in [KW]. Let $R_i = \text{set of regular elements of } M(i)$. We define a set of matrices

$$\Theta_n = \{X \in M(n) | X_i \in R_i, \sigma(X_i) \cap \sigma(X_{i+1}) = \emptyset, 1 \leq i \leq n - 1\}$$

Here $\sigma(X_i)$ denotes the spectrum of $X_i$. It follows from Remark 2.16 in [KW, pgs 24-25], that $\Theta_n \subset M(n)$ is Zariski open. The result concerning $\Theta_n$ is the following. First, we need some extra terminology. We say that $c \in \mathbb{C}^{n(n+1)/2}$ satisfies the eigenvalue disjointness condition if $c = (c_1, \cdots, c_n)$ with $c_i \in \mathbb{C}^i$ such that $c_i$ and $c_{i+1}$ represent coefficients of relatively prime monic polynomials.

**Theorem 6.1.1.** The elements of $\Theta_n$ are strongly regular. Thus, for $X \in \Theta_n$, $X_i$ is regular for all $1 \leq i \leq n$. For $c \in \mathbb{C}^{n(n+1)/2}$ satisfying the eigenvalues disjointness condition, we have $M_c^{sreg}(n) = M_c(n) \cap \Theta_n$ is exactly one $A$ orbit. Thus, on $\Theta_n$ the upper Hessenberg matrices form a cross-section to the $A$ action.

Proof:
Only the first statement needs to be proven. The second automatically follows from the first. To see that elements of $\Theta_n$ are strongly regular, note that if $X \in \Theta_n$, then $X \in M_c(n) \cap S$ where $c \in \mathbb{C}^{(n+1)/2}$ satisfies the eigenvalues disjointness condition. Thus, $X \in \text{Im} \Gamma_n^{a_1, \cdots, a_{n-1}}$ where $a_i = \mathcal{O}^i = \Xi^i_{c_i, c_{i+1}}$ is the unique orbit of $G_i$ in $\Xi^i_{c_i, c_{i+1}}$ which is necessarily of dimension $i$ by Proposition 3.2.1. But then $X \in \text{Im} \Gamma_n^{a_1, \cdots, a_{n-1}}$ with $\dim \text{Im} \Gamma_n^{a_1, \cdots, a_{n-1}} = \binom{n}{2}$. Thus by Theorem 4.5.1, $X$ is
Corollary 6.1.1. For $c \in \mathbb{C}^{(n+1)/2}$ with $c$ satisfying the eigenvalues disjointness condition, we have that

$$M_{c}^{sreg}(n) \simeq G_1 \times \cdots \times G_{n-1}$$

as algebraic varieties.

Proof:
This follows immediately from the fact that for such $c \in \mathbb{C}^{(n+1)/2}$, $M_{c}^{sreg}(n)$ is exactly one $A$ orbit and then the isomorphism follows by equation (4.25).

Q.E.D.

Remark 6.1.1. This theorem and its corollary generalize Theorems 3.23 and 3.28 in [KW, pgs 45, 49]. In these theorems the authors prove that analogous result for a Zariski open subset $\Omega_n \subset \Theta_n$ where each cutoff $X_i$ is regular semi-simple.

Remark 6.1.2. We note that for a matrix $X \in M_c(n)$ where $c \in \mathbb{C}^{(n+1)/2}$ satisfies the eigenvalue disjointness condition its strictly upper triangular part is determined by its strictly lower triangular part. This follows directly from the form the solution varieties $\Xi_{\epsilon_1, \cdots, \epsilon_{n-1}}$ in section 3.2 and the definition of the mappings $\Gamma_{\alpha_1, \cdots, \alpha_{n-1}}$.

6.2 The nilfibre $M_{0}^{sreg}(n)$

The nilfibre carries with it some very interesting structure beyond the result in Corollary 6.0.2. Consider the solution variety $\Xi_{0,0}^{i}$, i.e. Consider extending an
$(i+1) \times (i+1)$ matrix of the form:

\[
\begin{bmatrix}
0 & 1 & \cdots & 0 & y_1 \\
0 & 0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & 1 & \vdots & \\
0 & \cdots & \cdots & 0 & y_i \\
z_1 & \cdots & \cdots & z_i & w
\end{bmatrix}
\]

to an $(i+1) \times (i+1)$ principal nilpotent. $\Xi_{0,0}^i$ has exactly two $G_i$ orbits of dimension $i$.

\[
O_L^i = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & 1 & \vdots & \\
0 & \cdots & \cdots & 0 & 0 \\
z_1 & \cdots & \cdots & z_i & 0
\end{bmatrix}
\]

with $z_1 \in \mathbb{C}^\times$ and $z_j \in \mathbb{C}, 2 \leq j \leq i$. And

\[
O_U^i = \begin{bmatrix}
0 & 1 & \cdots & 0 & y_1 \\
0 & 0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & 1 & \vdots & \\
0 & \cdots & \cdots & 0 & y_i \\
0 & \cdots & \cdots & 0 & 0
\end{bmatrix}
\]

with $y_i \in \mathbb{C}^\times$ and $y_j \in \mathbb{C}, 1 \leq j \leq i - 1$.

This allow us to construct $2^{n-1}$ mappings of the form $\Gamma_{n,0}^{a_1,a_2,\ldots,a_{n-1}}$ where $a_i = O_U^i, O_L^i$ is a choice of $G_i$ orbit of dimension $i$ in $\Xi_{0,0}^i$. By Theorem 4.5.1, the image of each such mapping gives us an A orbit in $M_0^{sreg}(n)$.

We want to see that the closure of an A orbit in $M_0^{sreg}(n)$ is a nilradical of a Borel subalgebra of $M(n)$. The main tool we use here is Gerstenhaber’s theorem, which states the following (see [Ger]):

**Theorem 6.2.1.** (Gerstenhaber)
A linear space \( L \) of nilpotent matrices in \( M(n) \) that has dimension \(^n_2\) is a nilradical of a Borel subalgebra \( b \subset M(n) \) which is conjugate to the standard Borel by some member \( w \in S_n \), the Weyl group of \( M(n) \).

We use this theorem to prove the following theorem.

**Theorem 6.2.2.** Let \( A \cdot X \) be an orbit in \( M_0^{v_{reg}}(n) \). Then \( A \cdot X \simeq \mathbb{C}^{(n)} \) both as a variety and as a vector space over \( C \). More explicitly, if the \( A \) orbit is given by \( \Gamma_{n}^{a_1,a_2,\cdots,a_{n-1}} \) where \( a_i = O_{U}^i \) or \( O_{L}^i \), then \( A \cdot X \) is the set of matrices of the following form

\[
\left\{ X : X_i = \begin{bmatrix} b_1 \\ X_{i-1} \\ \vdots \\ b_{i-1} \\ 0 \end{bmatrix} \right\}
\]

with \( b_j \in \mathbb{C} \) if \( a_{i-1} = O_{U}^{i-1} \) or if \( a_i = O_{L}^{i-1} \) then we would have

\[
\left\{ X : X_i = \begin{bmatrix} X_{i-1} \\ 0 \\ b_1 \cdots b_{i-1} \end{bmatrix} \right\}
\]

with \( b_j \in \mathbb{C} \).

Before proving this theorem, let us consider an example in the case of \( M(3) \).

**Example 6.2.1.** Suppose we have chosen \( a_2 = U \) and \( a_1 = L \). The description in the above theorem would then give us that

\[
A \cdot X = \begin{bmatrix} 0 & 0 & a \\ b & 0 & c \\ 0 & 0 & 0 \end{bmatrix}
\]

Note that \( A \cdot X \simeq \mathbb{C}^{3} \) both as a vector space and as a variety.

Proof (of Theorem 6.2.2):
Without loss of generality, let us assume that we have made the choice \( a_{n-1} = U \). The case of \( a_{n-1} = L \) is very similar. The proof proceeds by induction on \( n \), the case \( n = 1 \) being trivial. It follows from the definition of the \( \Gamma_{n}^{a_1,\cdots,a_{n-1}} \) maps in Chapter.
4 that \((A \cdot X)_{n-1} = (A_{n-1} \cdot X)_{n-1}\) where \(A_{n-1} \simeq \mathbb{C}^{(n-2)/2}\) is the group generated by integrating the vector fields \(\{\xi_{f,ij}, 1 \leq i \leq n-2, 1 \leq j \leq i\}\) on \(M(n-1)\). More specifically, one can see directly from the definition of the map \(\Gamma^{a_1, \cdots, a_{n-1}}_n\) that 
\[
\text{Im}(\Gamma^{a_1, \cdots, a_{n-1}}_n)_{n-1} = \text{Im}(\Gamma^{a_1, \cdots, a_{n-2}}_n)_{n-1} = \text{Im}(\Gamma^{a_1, \cdots, a_{n-2}}_{n-1}) \subset (A \cdot X)_{n-1} = \overline{A_{n-1} \cdot X_{n-1}}.
\]

Let \(X \in M_0^{sreg}(n)\). Now, if we consider the Zariski continuous map \(X \to X_{n-1}\), we get \((A \cdot X)_{n-1} \subseteq (A \cdot X)_{n-1} = \overline{A_{n-1} \cdot X_{n-1}}\). Now, we will use a dimension argument. We first define a set of matrices \(\mathcal{C}\) as all matrices of the form

\[
\begin{bmatrix}
A_{n-1} \cdot X_{n-1} & b_1 \\
\vdots \\
b_{n-1} \\
0 & 0
\end{bmatrix}
\]

with \(b_i \in \mathbb{C}\). Now \(\mathcal{C}\) is an irreducible variety of dimension \(\binom{n}{2}\), since 
\[
\dim \overline{A_{n-1} \cdot X_{n-1}} = \binom{n-1}{2} \quad \text{and} \quad \overline{A_{n-1} \cdot X_{n-1}} \quad \text{is irreducible by Theorem 3.7 in [KW, pg 36].}
\]
(Note that \(\dim \overline{A_{n-1} \cdot X_{n-1}} = \binom{n-1}{2}\), since \(X_{n-1}\) is clearly strongly regular if \(X\) is.) Now, we note from our work above that \(\overline{A \cdot X} \subset \mathcal{C}\), since \((A \cdot X)_{n-1} \subseteq \overline{A_{n-1} \cdot X_{n-1}}\) and the obvious fact from the \(\Gamma_n\) construction above that in this case of \(a_{n-1} = U\) we have:

\[
\overline{A \cdot X} \subset \begin{bmatrix}
M(n-1) & b_1 \\
\vdots \\
b_{n-1} \\
0 & 0
\end{bmatrix}
\]

But then, by Theorem 3.7 in [KW, pg 36], we again have that \(\overline{A \cdot X}\) is a closed, irreducible subvariety of \(\mathcal{C}\) of dimension \(\binom{n}{2}\). Thus, by basic properties of dimension, we must have \(\overline{A \cdot X} = \mathcal{C}\). But by induction \(\overline{A_{n-1} \cdot X_{n-1}}\) is a linear space with the form described in the theorem. And hence \(\mathcal{C}\), being the product of linear spaces is linear and has exactly the form described in the theorem.

Q.E.D.

Now, we know that the closure of each orbit in \(M_0^{sreg}(n)\) is exactly the nilradical of a Borel subalgebra. We adopt the following notation. If \(X \in M_0^{sreg}(n)\) and \(A \cdot X\) is given by the image of \(\Gamma^{a_1, a_2, \cdots, a_{n-1}}_n\) with \(a_i = O_{U_i}, O_{L_i}\), then we set

\[
\overline{A \cdot X} = n_{a_1, \cdots, a_{n-1}}.
\]

We would now like to find the element of the Weyl group that
conjugates the nilradical $n_{a_1, \cdots, a_{n-1}}$ into the standard nilradical, $n^-$ (i.e. the strictly lower triangular matrices).

First, let us see how to do it in the Example 6.2.1 that we considered above.

**Example 6.2.2.** Recall that we chosen $a_2 = U$ and $a_1 = L$ and the closure we obtained was

$$
\begin{bmatrix}
0 & 0 & a \\
b & 0 & c \\
0 & 0 & 0
\end{bmatrix}
$$

One can check that the matrix that conjugates $n^-$ into $A \cdot X$ is the permutation matrix representing the permutation $\sigma = (132)$. We note that $\sigma = (12)(13)$ is the product of the long elements for Type $A_2$ and for Type $A_3$, respectively.

Now, we give an explicit recipe for finding the permutation $\sigma$ that conjugates $n^-$ into the closure of any $A$ orbit in $M_0^{sreg}(n)$. First, we develop a technique to conjugate one of the two standard nilradicals ($n^-$ or $n^+$) into the closure of an $A$ orbit in $M_0^{sreg}(n)$. ($n^+$ denotes the strictly upper triangular matrices). The theorem is the following:

**Theorem 6.2.3.** Let the $A$ orbit be given by the image of the map $\Gamma^{a_1, a_2, \cdots, a_{n-1}}$. where $a_i = O_{U}^i$ or $O_{L}^i$. (To abbreviate the notation, we say $a_i = L$ or $U$). If $a_{n-1} = U$, the following procedure produces a permutation $\sigma$ that conjugates $n^+$ into $n_{a_1, \cdots, a_{n-1}}$ and if $a_{n-1} = L$ it produces a permutation which conjugates $n^-$ into $n_{a_1, \cdots, a_{n-1}}$.

**Step 1**) Start with the identity permutation $\sigma_0 \in S_n$.

**Step 2**) Counting down from $n - 1$, find the first $i_1$ such that $a_{i_1} \neq a_{n-1}$. (Note that if no such $i$ exists then $A \cdot X = n^-$ if $a_{n-1} = L$ or $A \cdot X = n^+$ if $a_{n-1} = U$. In either case, take $\sigma$ to be the indentity permutation $\sigma_0$.) Now take $\sigma_{i_1} = w_{i_1, 0} \sigma_0$, where $w_{i_1, 0}$ is the long element in $S_{i_1+1}$.

**Step 3**) Starting with $\sigma_{i_1}$, $a_{i_1}$ repeat steps 1 and 2.

**Step 4**) This procedure must eventually stop at some index $i_k < n$, i.e. $a_1 = a_2 = \cdots = a_{i_k}$. Then take $\sigma = \sigma_{i_k}$.
Furthermore, the permutation constructed in this manner is the unique permutation that maps either $n^-$ or $n^+$ into $n_{a_1, \ldots, a_{n-1}}$.

Proof:
The proof is by induction on $n$, noting that the case $n = 2$ is trivial. Without loss of generality let us suppose that $a_{n-1} = L$ (so that we start with $n^-$). Now suppose that $a_i = U$, but $a_j = L$, for $i < j \leq n - 1$. Then we conjugate $n^-$ by the long element $w_{i,0} \in S_{i+1}$. We note that since a permutation $S_\sigma$ acts on $E_{i,j}$ via $S_\sigma E_{i,j} S_\sigma^{-1} = E_{\sigma(i), \sigma(j)}$, $w_{i,0}$ permutes the elements $E_{k,l}$ for $i + 2 \leq k \leq n$, $1 \leq l \leq k - 1$ among themselves, since $k - 1 \geq i + 1$. Thus $\text{Ad}(w_{i,0}) \cdot n^-$ is a matrix of the form

$$
\begin{bmatrix}
A_{i+1} & 0 \\
* & \\
* & \ddots \\
* & 0
\end{bmatrix}
$$

where $A_{i+1} \in M(i + 1)$ is strictly upper triangular. We think of $*$ as representing the remaining rows which are still in strictly lower triangular form. Now, by Theorem 6.2.2, $X \in n_{a_1, \ldots, a_{n-1}}$ has the form

$$
\begin{bmatrix}
& b_1 \\
& \vdots \\
& b_i \\
& 0 & 0 \\
& * & \\
& * & \ddots \\
& * & * & 0
\end{bmatrix}
$$

Now, by Theorem 6.2.2, it is clear that the $(i+1) \times (i+1)$ cutoff of the above matrix is just a nilradical of the form $n_{a_1, \ldots, a_i} \subset M(i + 1)$, i.e. is just $A_{i+1} \cdot x_{i+1}$ where $A_{i+1} \simeq C^{(i)}$ is the group generated by the vector fields $\{\xi_{f_l,j}, 1 \leq l \leq i, 1 \leq j \leq l\}$ on $M(i + 1)$. Thus, we can apply the induction hypothesis to find a permutation $\tau_i \in S_{i+1}$ with $\tau_i = w_{j_1,0} \cdots w_{j_k,0}$ as given in the statement of the theorem such that $\text{Ad}(\tau_i) n^+_{i+1} = n_{a_1, \ldots, a_i}$ with $n^+_{i+1}$ the strictly upper triangular matrices in $M(i + 1)$. 
Now, we notice again that \( \tau_i \in S_{i+1} \) permutes the elements \( E_{k,l} \) for \( i+2 \leq k \leq n, \ 1 \leq l \leq k-1 \) amongst themselves. Thus, \( \text{Ad}(\tau_i w_{i+1,0}) \cdot n^- = n_{a_1,\ldots,a_{n-1}} \). This completes the induction and the existence part of the proof of the theorem.

The uniqueness follows easily from that fact that if we have for example \( \sigma n^- \sigma^{-1} = \tau n^- \tau^{-1} \), then \( \sigma^{-1} \tau n^- \tau^{-1} \sigma = n^- \). By properties of the Weyl group (see Lemma 2.5.5 in [GW, pg 97]) we must have \( \sigma = \tau \) (Similarly for \( n^+ \)).

Q.E.D.

**Corollary 6.2.1.** To conjugate \( n^- \) into \( n_{a_1,\ldots,a_{n-1}} \) we take \( \sigma \) as in the theorem if \( a_{n-1} = L \) and if \( a_{n-1} = U \), we take \( \tilde{\sigma} = \sigma w_{n,0} \) where \( \sigma \) is given by the theorem and \( w_{n,0} \in S_n \) is the long element.

**Remark 6.2.1.** If we start with \( n^- \), then we note that what we are doing to get to \( n_{a_1,\ldots,a_{n-1}} \) is either choosing or not choosing to conjugate by the long element of the Weyl group at the various levels \( M(i) \) for \( 1 \leq i \leq n-1 \). Thus in this way we are selecting \( 2^{n-1} \) special elements of the Weyl group for Type \( A_n \) that parameterize the \( 2^{n-1} \) \( A \) orbits in \( M_{0}^{\text{sreg}}(n) \).
7 Gelfand-Zeitlin Theory in Types $B_l$ and $D_l$

7.1 Preliminaries

We briefly recall our discussion in sections 1.6 and 1.7. We realize the complex orthogonal Lie algebra $\mathfrak{so}(n)$ as $n \times n$ skew symmetric matrices over $\mathbb{C}$. We can think of $\mathfrak{so}(i)$ as embedded in $\mathfrak{so}(n)$ in the top left-hand corner for $X \in \mathfrak{so}(i)$. For $X \in \mathfrak{so}(n)$, we refer to the $i \times i$ skew-symmetric matrix in the upper left-hand corner of $X$ as the the $i \times i$ cutoff of $X$ and denote it by $X_i \in \mathfrak{so}(i)$. We also think of $SO(i) \hookrightarrow SO(n)$ via the embedding

$$g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & Id_{n-i} \end{bmatrix}$$

for $g \in SO(i)$. (Recall in this case that $SO(i)$ consists of $i \times i$ orthogonal matrices with complex entries). Let $P(\mathfrak{so}(n))$ denote the algebra of polynomials on $\mathfrak{so}(n)$. Recall from section 1.6 that $P(\mathfrak{so}(i)) \subset P(\mathfrak{so}(n))$ is a Poisson subalgebra. This allowed us to define a classical analogue to the Gelfand-Zeitlin algebra of $U(\mathfrak{so}(n))$ in $P(\mathfrak{so}(n))$,

$$J(\mathfrak{so}(n)) = P(\mathfrak{so}(2))^{{SO}(2)} \otimes \cdots \otimes P(\mathfrak{so}(n))^{{SO}(n)}$$

which is Poisson commutative. We saw further in section 1.7 that using $J(\mathfrak{so}(n))$, 

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we could define a commutative Lie algebra of Hamiltonian vector fields:

\[ V = \{ \xi_f | f \in J(\mathfrak{so}(n)) \} \]

Choosing a set of generators for the algebra \( J(\mathfrak{so}(n)) \), \( \{ f_{i,j} \in P(\mathfrak{so}(i))^{SO(i)} | 2 \leq i \leq n-1, 1 \leq j \leq \text{rk } i \} \) (where \( \text{rk } i \) denotes the rank of \( \mathfrak{so}(i) \) and \( f_{i,j} \in P(\mathfrak{so}(i))^{SO(i)} \) is a fundamental adjoint invariant), we obtained a commutative Lie algebra of vector fields

\[ L = \{ \xi_{f_{i,j}} | 2 \leq i \leq n-1, 1 \leq j \leq \text{rk } i \} \]  \hspace{1cm} (7.1)

of dimension at most \( d/2 \), where \( d \) is the dimension of a regular adjoint orbit in \( \mathfrak{so}(n) \).

Using the Lie algebra \( V \), we can then define a collection of subspaces:

\[ X \to V_X = \{ (\xi_f)_X | f \in J(\mathfrak{so}(n)) \} \subset T_X(\mathfrak{so}(n)) \]

We now show that the Lie algebra \( L \) integrates to a global action of \( \mathbb{C}^{d/2} \) on \( \mathfrak{so}(n) \). If we let \( B = \mathbb{C}^{d/2} \) and \( X \in \mathfrak{so}(n) \), then we will show \( T_X(B \cdot X) = V_X \).

Note that

\[ V_X = \text{span}\{ (\xi_{f_{i,j}})_X | 2 \leq i \leq n-1, 1 \leq j \leq \text{rk } i \} \]  \hspace{1cm} (7.2)

c.f. Remark 2.8 in [KW, pg 20].

### 7.2 The integration of vector fields derived from the classical analogue of the Gelfand-Zeitlin algebra for Types \( B_l \) and \( D_l \)

To integrate the Lie algebra \( L \) in equation (7.1), we need to see that each vector field \( \xi_{f_{i,j}} \) integrates to a global action of \( \mathbb{C} \) on \( \mathfrak{so}(n) \). This will follow from the following proposition.

**Proposition 7.2.1.** Let \( p \in P(\mathfrak{so}(i))^{SO(i)} \) where \( P(\mathfrak{so}(i)) \) is viewed as a Poisson subalgebra of \( P(\mathfrak{so}(n)) \) as in section 1.6. The the vector field \( \xi_p \) integrates to global action of \( \mathbb{C} \) on \( \mathfrak{so}(n) \) given by the formula:

\[ t \cdot X = Ad\exp(-tdp(X_i)) \cdot X \]  \hspace{1cm} (7.3)
for $t \in \mathbb{C}$, $X \in \mathfrak{so}(n)$.

Proof:
Recall that for any reductive Lie algebra $\mathfrak{g}$, and any polynomial $f$ on $\mathfrak{g}$ we think of the differential of $df(X)$ at $X \in \mathfrak{g}$ as an element of $\mathfrak{g}$ via the formula

$$\frac{d}{dt}|_{t=0} f(X + tZ) = \beta(df(X), Z)$$

where $Z \in \mathfrak{g}$ is arbitrary and $\beta(\cdot, \cdot)$ is the $\mathfrak{g}$-equivariant form on $\mathfrak{g}$. We now claim that $p$ viewed as a polynomial on $\mathfrak{so}(n)$ has differential $dp(X) = dp(X_i) \in \mathfrak{so}(i) \hookrightarrow \mathfrak{so}(n)$ given by

$$\begin{bmatrix} dp(X_i) & 0 \\ 0 & 0 \end{bmatrix}$$

for $X \in \mathfrak{so}(n)$. To see this consider $\frac{d}{dt}|_{t=0} p(X + tY)$ for $Y \in \mathfrak{so}(n)$. Now, we recall from section 1.6 that $\mathfrak{so}(n) = \mathfrak{so}(i) \oplus \mathfrak{so}(i)^\perp$ where $\mathfrak{so}(i)^\perp$ is the orthogonal compliment of $\mathfrak{so}(i)$ with respect to the trace form on $\mathfrak{so}(n)$. If we write $Y \in \mathfrak{so}(n)$ with respect to this decomposition as $Y = Y_i + Y_i^\perp$, we see that $p(X + tY) = p(X_i + tY_i)$ for any $t \in \mathbb{C}$. Thus,

$$\frac{d}{dt}|_{t=0} p(X + tY) =$$

$$\frac{d}{dt}|_{t=0} p(X_i + tY_i) =$$

$$tr(dp(X_i)Y_i) =$$

$$tr(dp(X_i)Y_i + dp(X_i)Y_i^\perp) =$$

$$tr(dp(X_i)Y)$$

as desired. By equation (1.9), we have

$$(\xi_p)_X = (\partial^{-1}[dp(X_i), X])_X$$

for $X \in \mathfrak{so}(n)$. Now, it is well known that for $p \in P(\mathfrak{so}(i))^{SO(i)}$, $dp(Z) \in \mathfrak{so}(i)^Z$ for $Z \in \mathfrak{so}(i)$ (where $\mathfrak{so}(i)^Z$ denotes the centralizer in $\mathfrak{so}(i)$ of $Z$) (see [K1, pg 1]).
Using this fact, we can show that

$$\theta(t, X) = \text{Ad}(\exp(-t \, dp(X_i))) \cdot X$$

is the integral curve for the vector field $\xi_p$ starting at $X \in \mathfrak{so}(n)$. To see this, we compute the differential to the curve $\theta(t, X)$ at an arbitrary $t_0 \in \mathbb{C}$.

$$\left. \frac{d}{dt} \right|_{t=t_0} \text{Ad}(\exp(-t \, dp(X_i))) \cdot X = \left. \frac{d}{dt} \right|_{t=t_0} \text{exp}(t \, \text{ad}(-dp(X_i))) \cdot X = \text{ad}(-dp(X_i))(\text{exp}(t_0 \, \text{ad}(-dp(X_i))) \cdot X)$$

Now, if we let

$$Y = \exp(t_0 \, \text{ad}(-dp(X_i))) \cdot X = \text{Ad}(\exp(-t_0 \, dp(X_i))) \cdot X = \theta(t_0, X),$$

then we note that since $-dp(X_i)$ centralizes $X_i$, we have that $(\theta(t, X))_i = \theta(t, X)_i$ for all $t \in \mathbb{C}$. In particular, we have $Y_i = X_i$. We thus obtain that

$$\text{ad}(-dp(X_i))(\exp(t_0 \, \text{ad}(dp(X_i))) \cdot X) = \text{ad}(-dp(X_i)) \cdot Y = (\xi_p)_Y$$

(in coordinates) by equation (7.5), which verifies the claim. This completes the proof of the theorem.

Q.E.D

Now, we can use Proposition 7.2.1 to integrate the Lie algebra $L$ in equation (7.1).

The fact that any two vector fields $\xi_f, \xi_g$ with $f, g \in J(\mathfrak{so}(n))$ commute easily allows us to integrate $L$ to an action of $C^{d/2}$. Recall that we denote by $d$ the dimension of a regular adjoint orbit in $\mathfrak{so}(n)$. Note that we have

$$d/2 = \begin{cases} l^2 - l & \text{if } n = 2l \\ l^2 & \text{if } n = 2l + 1 \end{cases}$$

We now have the following analogue of Theorem 3.4 in [KW, pg 34].

**Theorem 7.2.1.** Let $L$ be the commutative Lie algebra of vector fields given by:

$$L = \{\xi_{f,i} | 2 \leq i \leq n - 1, 1 \leq j \leq rk i\}$$
Then $L$ integrates to an action of a group $B = \mathbb{C}^{d/2}$ on $\mathfrak{so}(n)$. If $X \in \mathfrak{so}(n)$ and $B \cdot X$ denotes its $B$ orbit, then $T_X(B \cdot X) = V_X$ where $V_X \subset T_X(\mathfrak{so}(n))$ is the subspace given in equation (7.2).

Proof:
By Proposition 7.2.1 we have that each field $\xi_{f_{i,j}}$ integrates to a global action of $\mathbb{C}$ on $\mathfrak{so}(n)$. Since the fields $\{\xi_{f_{i,j}}|2 \leq i \leq n-1, 1 \leq j \leq \text{rk } i\}$ pairwise commute the composition of the corresponding flows defines a group action of $\mathbb{C}^{d/2}$ on $\mathfrak{so}(n)$. All that remains to be seen is that the tangent space to these orbits is the subspace $V_X$. This follows from the standard fact that orbits of a group action are submanifolds with tangent space spanned by the vector fields given by the infinitesimal action of the Lie algebra.

Q.E.D.

Remark 7.2.1. We note that the action of $B$ stabilizes adjoint orbits by Proposition (4.4.1), since the vector fields $\xi_{f_{i,j}}$ for $2 \leq i \leq n-1, 1 \leq j \leq \text{rk } i$ are tangent to adjoint orbits.

Remark 7.2.2. One notes that group $B$ that we obtain above in Theorem 7.2.1 is obtained by fixing a set of generators $\{f_{i,j} \in P(\mathfrak{so}(i))^{SO(i)}|2 \leq i \leq n-1, 1 \leq j \leq \text{rk } i\}$ for the ring $P(\mathfrak{so}(2))^{SO(2)} \times \cdots \times P(\mathfrak{so}(n-1))^{SO(n-1)}$ with $f_{i,j} \in P(\mathfrak{so}(i))^{SO(i)}$ a fundamental adjoint invariant. Choosing a different set of generators for $J(\mathfrak{so}(n))$, $\{f'_{i,j}|1 \leq i \leq n-1, 1 \leq j \leq \text{rk } i\}$ with $f'_{i,j} \in P(\mathfrak{so}(i))^{SO(i)}$ a fundamental adjoint invariant, defines a different Lie algebra $L' = \{\xi'_{f'_{i,j}}|2 \leq i \leq n-1, 1 \leq j \leq \text{rk } i\}$. By Theorem 7.2.1, the Lie algebra $L'$ integrates to an action of $\mathbb{C}^{d/2}$ on $\mathfrak{so}(n)$. One can show that Theorem 3.5 in [KW, pg 34] remains valid in the orthogonal case, so that the two actions of $\mathbb{C}^{d/2}$ commute with one another and have the same orbit structure on $\mathfrak{so}(n)$. Since we are concerned with studying the geometry of these orbits, we lose nothing by choosing a specific set of invariants $f_{i,j}$. We will discuss a convenient choice of invariants below.

We say that $X \in \mathfrak{so}(n)$ is strongly regular if its $B$ orbit is of maximal dimension $d/2$ (just as in the case of $M(n)$). The set of strongly regular elements
is denoted by $\mathfrak{so}^{\text{reg}}(n)$. Note that we have:

$$X \in \mathfrak{so}^{\text{reg}}(n) \iff \dim B \cdot X = d/2 \iff \dim V_X = d/2$$

Now, one can easily show that Proposition 2.6 and Theorem 2.7 in [KW, pg 19] carry over to the orthogonal case.

**Theorem 7.2.2.** $X \in \mathfrak{so}^{\text{reg}}(n)$ if and only if the differentials $(df_{i,j})_x$ are linearly independent for $2 \leq i \leq n$ and $1 \leq j \leq rki$. Thus, the set of strongly regular elements in $\mathfrak{so}(n)$ is Zariski open.

**Remark 7.2.3.** $\mathfrak{so}^{\text{reg}}(n)$ is Zariski open, as in the case of $M(n)$, but we do not know apriori that it is not empty, as we do not have an analogue of Theorem 2.3 in [KW, pg 16] (i.e. Hessenberg matrices) as in the case of $M(n)$. We will show in later sections that $\mathfrak{so}^{\text{reg}}(n)$ is non-empty.

Now, to fix an action of $\mathbb{C}^{d/2}$, we make a choice of generators of the Poisson commutative algebra $J(\mathfrak{so}(n))$. We take as fundamental invariants for $\mathfrak{so}(2l)$ the following polynomials:

$$p_j(X) = \text{tr}(X^{2j}) \text{ for } 1 \leq j \leq l - 1$$

$$p_l(X) = \text{Pfaff}(X)$$

(7.6)

For $\mathfrak{so}(2l + 1, \mathbb{C})$, we take

$$p_j(X) = \text{tr}(X^{2j}) \text{ for } 1 \leq j \leq l$$

(7.7)

Now, to be able to write down this action of $\mathbb{C}^{d/2}$ by composing the commuting actions of $\mathbb{C}$ in equation (7.3), we need to know the differentials of the functions $\text{tr}(X_i^{2j})$ and $\text{Pfaff}(X_i)$. The differential of the former is easily computed to be $2jX_i^{2j-1}$ so that equation (7.3) becomes in this case:

$$t \cdot X = \text{Ad}(\exp(-t 2j X_i^{2j-1})) \cdot X$$

Obtaining a formula for the differential of the polynomial $\text{Pfaff}(X_i)$ is more difficult and is addressed in the next section. The actual formula will not be used in the following sections, but it is derived for completeness.


7.2.1 The differential of the Pfaffian

We now compute the differential of $\text{Pfaff}(X_i)$.

Remark 7.2.4. We note that by our discussion in the proof of Proposition 7.2.1 (see equation (7.4)) that it suffices to consider the case $n = 2l$ and compute $d(\text{Pfaff}(X))$.

To compute $d\text{Pfaff}(X)$, we use the fact that $\det(X) = \text{Pfaff}(X)^2$ and compute $d(\det)(X)$ on the Zariski open set of non-singular elements in $\mathfrak{so}(2l)$. Supposing that $X \in \mathfrak{so}(2l)$ is invertible, we consider $\frac{d}{dt}(\det(X + tY)|_{t=0})$. Since $X$ is invertible this becomes

$$\det(X) \frac{d}{dt}(\det(I + t^{-1}Y)|_{t=0}) = \det(X)\text{tr}(X^{-1}Y)$$

(7.8)

Where the last equality follows from the fact that the differential of the determinant at the identity it is the identity. But now we claim that (7.8) is nothing more that $\text{tr}(\mathcal{A}(X)Y)$ where $\mathcal{A}(X)$ denotes the classical adjoint matrix. This follows from the definition of $\mathcal{A}(X)$ which gives $X^{-1} = \det(X)^{-1}\mathcal{A}(X)$. Thus, for $X \in \mathfrak{so}(2l)$ non-singular, we have that $d(\det)(X) = \mathcal{A}(X)$.

Now, we want to observe that $\mathcal{A}(X)$ is skew-symmetric for $X \in \mathfrak{so}(2l)$. It will suffice to prove the claim for $X \in \mathfrak{so}2l$ invertible, because being skew-symmetric is a polynomial condition on the entries of $\mathcal{A}(X)$, which are themselves globally defined regular functions on $\mathfrak{so}(2l)$. Suppose that $X \in \mathfrak{so}(2l)$ invertible. We have that $\det(X)Id = X\mathcal{A}(X)$. We transpose this equation and use the fact that $X$ is skew-symmetric to get $\det(X)Id = -\mathcal{A}(X)^tX$. Setting these two equations equal and canceling $X$ yields $\mathcal{A}(X) = -(\mathcal{A}(X))^t$, as desired.

Now, for $X \in \mathfrak{so}(2l)$ is non-singular, we have that $d(\text{Pfaff})(X)$ is given by:

$$d(\det(X))(Y) = 2\text{Pfaff}(X)d(\text{Pfaff})(X)(Y)$$

hence $d\text{Pfaff}(X) = \frac{1}{2}\mathcal{A}(X)/\text{Pfaff}(X)$

Thus, on a Zariski dense set in $\mathfrak{so}(2l)$, we have

$$d\text{Pfaff}(X)\text{Pfaff}(X) = \frac{1}{2}\mathcal{A}(X)$$

(7.9)
Now, the entries of $Q = d\text{Pfaff}(X)$ are all polynomial functions on $\mathfrak{so}(2l)$. Specifically $Q_{ij} = -2 \frac{\partial g}{\partial v_{ij}} |_X$ (where $v_{ij}$ is the basis dual to the basis $E_{ij} - E_{ji}$ of $\mathfrak{so}(2l)$ and $g = \text{Pfaff}(X)$). Since the polynomials in (7.9) are globally defined and agree on a Zariski dense set, they must agree on all of $\mathfrak{so}(n)$. We claim that this implies that $\text{Pfaff}(X)$ divide the entries of the matrix $A(X)$ for any $X \in \mathfrak{so}(2l)$.

This follows from the following fact see Scholium B.2.12 in [GW, page 630].

**Proposition 7.2.2.** The polynomial $g = \text{Pfaff}(X)$ on $\mathfrak{so}(2l)$ is irreducible as a polynomial in $\{x_{ij} | 1 \leq i < j \leq 2l\}$.

Now, using (7.9) one has that $\text{Pfaff}(X) = 0$ implies $\Delta_{ji}(X) = 0$ for all $i, j$ (where $\Delta_{ji}(X)$ denotes the $2l - 1 \times 2l - 1$ minor of $X$ obtained by deleting the $j$th row and the $i$th column.). One can also obtain this fact directly without using (7.9). Simply observe that the rank of a $2l \times 2l$ skew-symmetric matrix is always even.

By Proposition 7.2.2 $g = \text{Pfaff}$ is irreducible. So that if we have $g(X) = 0$ implies $\Delta_{ji}(X) = 0$, then $g | \Delta_{ji}(X)$ by unique factorization. Thus, $g(X) = \text{Pfaff}(X)$ is, up to associates, an irreducible factor of $\Delta_{ij}$ for all $i, j$. Now, to obtain a more explicit description of $d(\text{Pfaff}(X))$, we want to figure out what the product of the remaining irreducible factors of $\Delta_{ij}$ are. We claim that we have

$$\Delta_{ij} = k_{ij} \text{Pfaff}(X) \text{Pfaff}(X(i, j)) \quad (7.10)$$

where $X(i, j)$ denotes the $2l - 2 \times 2l - 2$ skew symmetric matrix obtained from $X$ by deleting both the $i$-th row and column and the $j$-th row and column of $X$ and $k_{ij} \in \mathbb{C}^\times$. Note that $\text{Pfaff}(X(i, j))$ is irreducible as an element of the polynomial ring $\mathbb{C}[x_{ij}]_{1 \leq i < j \leq l}$. Without loss of generality assume that $i < j$. We note that by Proposition 7.2.2 that it is clearly irreducible in the subring $\mathbb{C}[x_{km}]_{1 \leq k < m \leq l, k \neq i, j, l \neq i, j}$ of $\mathbb{C}[x_{ij}]_{1 \leq i < j \leq l}$. If the polynomial was reducible in the ring $\mathbb{C}[x_{ij}]_{1 \leq i < j \leq l}$, i.e. $\text{Pfaff}(X(i, j)) = fg$ with $f, g \in \mathbb{C}[x_{ij}]_{1 \leq i < j \leq l}$, it would have to have monomials involving positive powers of $x_{am}$ with $a = i, j$ for some $m$ or $x_{kb}$ with $b = i, j$ for some $k$ in the factorization, which is impossible. Thus, we will have (7.10) if we can show

$$\text{Pfaff}(X(i, j)) = 0 \implies \Delta_{ij}(X) = 0 \quad (7.11)$$
Since then we have $\text{Pfaff}(X(i, j))\text{Pfaff}(X)|\Delta_{ij}(X)$. Equation (7.10) then follows from the fact that the degree of $\Delta_{ij}(X)$ is $2l - 1$, the degree of $\text{Pfaff}(X)$ is $l$, and the degree of $\text{Pfaff}(X(i, j))$ is $l - 1$.

Now, we turn our attention to showing equation (7.11). We note that by the skew-symmetry of the matrix $A(X)$, it is enough to show (7.11) for $i < j$ and hence (7.10) for $i < j$. Simply take $k_{ji} = -k_{ij}$ in (7.10). We make use of a rank argument. The essential fact to keep in mind is that the rank of a $2k \times 2k$ skew-symmetric matrix is even. Let $Z_{ij}$ be the $2l - 1 \times 2l - 1$ matrix obtained from $X \in \mathfrak{so}(2l)$ by deleting the $i$ the row and $j$ - th column so that $\Delta_{ij} = \det(Z_{ij})$. Then one notes that $X(i, j)$ is obtained from $Z_{ij}$ by deleting the $i$-th column and the $j - 1$-th row. Now, suppose the rank of $X(i, j)$ is $k \leq 2l - 2$. Then consider the matrix $C_{ij}$ of size $2l - 1 \times 2l - 2$ which is obtained by reinserting the $j - 1$ the row of $Z_{ij}$ without the entry in the $i$-th position in the $j - 1$-th position. It can have at most rank $k + 1$. This follows clearly from the fact that if $\{v_1, \cdots, v_k\}$ is a basis for the row space of $X(i, j)$ and $w \in \mathbb{C}^{2l - 2}$ is the reinserted row then the dimension of the span of $\{v_1, \cdots, v_k\} \cup \{w\}$ can be at most $k + 1$. Now, we obtain $Z_{ij}$ from $C_{ij}$ by reinserting the $i$-th column of $Z_{ij}$ in $C_{ij}$ (in the $i$-th position). The same argument again shows that the rank of $Z_{ij}$ can be at most $k + 2$. Now, suppose that $\text{Pfaff}(X(i, j)) = 0$. This implies that $X(i, j)$ is singular. Since $X(i, j) \in \mathfrak{so}(2l - 2)$, this gives us that $\text{rk} X(i, j) \leq 2l - 4$, thus $\text{rk} Z_{i,j} \leq 2l - 2$. But $Z_{ij}$ is of size $2l - 1 \times 2l - 1$. Thus $Z_{ij}$ is singular, and we have $\Delta_{ij}(X) = \det Z_{ij} = 0$ as desired. Thus, we have equation (7.11) for $i < j$.

We have now proven the following result.

**Proposition 7.2.3.**

$$
(\xi_{\text{Pfaff}(X)})_X = (\partial[\text{Pfaff}(X)_X]) = (\partial[\text{Pfaff}(X)/\text{Pfaff}(X)]_X)
$$

for $X \in \mathfrak{so}(n)$ where $A(X)$ is the classical adjoint of $X$. Moreover

$$
(A(X)/\text{Pfaff}(X))_{ij} = \Delta_{ji}(X)/\text{Pfaff}(X) = k_{ji}\text{Pfaff}(X(i, j))
$$

where $\text{Pfaff}(X(i, j))$ is the Pfaffian of the $2l - 2 \times 2l - 2$ skew-symmetric matrix obtained from $X$ by deleting both the $i$-th column and row and the $j$-th column and row and $k_{ij} \in \mathbb{C}^x$. 
We now have the following immediately corollary which follows from Remark 7.2.4.

**Corollary 7.2.1.**

$$\left(\xi_{\text{Pfaff}(X_j)}\right)_X = (\partial [^{-d\text{Pfaff}(X_j)}])_X = (\partial [^{-1/2A(X_j)/\text{Pfaff}(X_j)}])_X$$

with

$$(A(X_j)/\text{Pfaff}(X_j))_{kl} = \Delta_{lk}(X_j)/\text{Pfaff}(X_j) = c_{lk}\text{Pfaff}(X_j(k,l))$$

and $c_{lk} \in \mathbb{C}^\times$ (for $X \in \mathfrak{so}(n)$ $j < n$ even).

### 7.3 The generic matrices for the orthogonal Lie algebras

We want to generalize the result that Kostant and Wallach obtained for certain regular semi-simple matrices in $M(n)$ to the orthogonal case (see Theorems 3.23, 3.28 in [KW, pg 45, 49]). First, we define the following regular map:

$$\Phi : \mathfrak{so}(n) \to \mathbb{C}^{d/2+\text{rk}n}$$

$$\Phi(X) = (f_{2,1}(X_2), f_{3,1}(X_3), \ldots, f_{n,\text{rk}n}(X))$$

with $f_{i,j} \in P(\mathfrak{so}(i))^{SO(i)}$ a fundamental adjoint invariant. For $c \in \mathbb{C}^{d/2+\text{rk}n}$, let $\mathfrak{so}(n)_c = \Phi^{-1}(c)$.

**Proposition 7.3.1.** The action of $B$ preserves the fibres $\mathfrak{so}(n)_c$.

Proof:

The proof here is exactly the same as the proof of Proposition 3.6 in [KW, pg 35]. Since the functions $\{f_{i,j} | 2 \leq i \leq n-1, 1 \leq j \leq \text{rk}i\}$ all Poisson commute, $f_{i,j}$ is constant along the Hamiltonian flow of $f_{k,l}$ (see [Lee, pg 489]). Thus, $f_{i,j}$ is invariant under the action of the group $B$. Hence, the fibres $\mathfrak{so}_c(n)$ are invariant under the action of $B$.

Q.E.D.
We define $\Omega_n \subset \mathfrak{so}(n)$ to be the set of matrices where each cutoff $X_i$ is regular semi-simple, and the spectrum of $X_i$ and $X_{i+1}$ have no intersection. We say that $c \in \mathbb{C}^{d/2+rk_n}$ satisfies the eigenvalue disjointness condition if $c = (c_1, \ldots, c_i, \ldots, c_n)$ with each $c_i \in \mathbb{C}^{rk_i}$ representing a regular semi-simple element of $\mathfrak{so}(i)$ (up to the action of the Weyl group, $W_i$) and the elements represented by $c_i$ and $c_{i+1}$ have no eigenvalues in common.

(Here we are making use of the canonical isomorphism

$$\Phi_i : \mathfrak{h}_i/W_i \rightarrow \mathbb{C}^{rk_i}$$

given by

$$\Phi_i(X) = (f_{i,1}(X), \ldots, f_{i,rk_i}(X))$$

where $\mathfrak{h}_i \subset \mathfrak{so}(i)$ is a Cartan subalgebra, $W_i$ is the Weyl group and $f_{i,j}$ are fundamental adjoint invariants for $\mathfrak{so}(i)$).

For $c \in \mathbb{C}^{d/2+rk_n}$ satisfying the eigenvalue disjointness condition we claim that we have that $\mathfrak{so}_c(n) \subset \Omega_n$. This follows from the following general result.

**Proposition 7.3.2.** Let $g$ be a semi-simple Lie algebra of rank $l$ and let $\phi_1, \ldots, \phi_l$ be a basic set of adjoint invariants. Let $\mathfrak{h} \subset g$ be a Cartan subalgebra and let $\mathfrak{h}^{reg}$ denote the set of regular elements in $\mathfrak{h}$. Let $X \in g$ and suppose that there exists a $Y \in \mathfrak{h}^{reg}$ such that $\phi_1(X) = \phi_1(Y), \ldots, \phi_l(X) = \phi_l(Y)$, then $X \in G \cdot Y$ (where $G$ denotes the adjoint group of $g$). In particular $X$ is regular semi-simple.

**Proof:**

We write $X \in g$ in its Jordan decomposition, $X = X_s + X_n$ where $X_s$ and $X_n$ are the semi-simple and nilpotent parts of $X$ respectively. Now, one knows that for any invariant polynomial $\phi \in P(g)^G$, we have that $\phi(X) = \phi(X_s)$ (see [K2, pg 360]). Then we have that $\phi_i(Y) = \phi_i(X_s)$ for all fundamental invariants $\phi_i$. Thus, since both $Y$ and $X_s$ are semi-simple, we have that $X_s = g \cdot Y$ for some $g \in G$. Hence $X_s$ must be regular semi-simple. But then from the Jordan decomposition of $X$, we know $[X_s, X_n] = 0$ so that $X_n \in C_g(X_s)$. But since $X_s$ is regular semi-simple $C_g(X_s) = \mathfrak{h}$ where $\mathfrak{h}$ is the unique Cartan subalgebra containing $X_s$. This
gives us that $X_n = 0$ and that $X = X_s = g \cdot Y$, which is the desired result.

Q.E.D

With this proposition, we also clearly have $\Omega_n = \bigcup_{c \in \mathbb{C}^{d/2 + rk_n}} \mathfrak{so}_c(n)$ with $c$ satisfying the eigenvalues disjointness condition. We note also that the action of $B$ preserves the set $\Omega_n$. This is clear by the argument in the proof of Proposition (7.3.1), which states that each function $f_{i,j}$ is invariant under the action of $B$. So that if $X \in \mathfrak{so}_c(n)$ with $c \in \mathbb{C}^{d/2 + rk_n}$ satisfying the eigenvalue disjointness condition, then $B \cdot X \in \mathfrak{so}_c(n) \subset \Omega_n$.

Our goal in the next several sections will be to describe the action of $B$ on the set $\Omega_n$. We will accomplish this by considering an analogous construction to the $\Gamma_n$ construction that we had for $M(n)$. As in the case of $M(n)$, the first step will be to describe the structure of the solution varieties for this generic situation.

### 7.4 Generic solution varieties for orthogonal Lie algebras

First, we recall the form of regular semi-simple elements in Types $B_l$ and $D_l$. Consider first the case of type $B_l$ i.e. $\mathfrak{so}(2l + 1)$. In this case, we can choose a Cartan subalgebra consisting of matrices of the form:

\[
\begin{bmatrix}
0 & a_1 & & & \\
-a_1 & 0 & & & \\
& & 0 & a_2 & \\
& & -a_2 & 0 & \\
& & & & \ddots \\
& & & & 0 & a_l \\
& & & & -a_l & 0 \\
& & & & & 0 \\
\end{bmatrix}
\tag{7.13}
\]

with $a_i \in \mathbb{C}$. We will refer to the Cartan subalgebra in (7.13) as the standard Cartan subalgebra in $\mathfrak{so}(2l + 1)$.

Now, regular semi-simple elements in this particular Cartan $\mathfrak{h}$ are exactly the elements which do not have repeated eigenvalues. i.e. the elements of the
form (7.13) with \( a_i \neq 0 \) and \( a_i \neq (+/−) a_j \) for \( i \neq j \). To see this, we must recall the following. Let \( \mathfrak{g} \) be a semi-simple Lie algebra over \( \mathbb{C} \) and \( \mathfrak{h} \subset \mathfrak{g} \) be a Cartan subalgebra. We recall that \( X \in \mathfrak{g} \) is regular if and only if \( dp_1(X) \wedge dp_2(X) \wedge \cdots \wedge dp_l(X) \neq 0 \), with \( p_j \in P(\mathfrak{g})^G \) fundamental adjoint invariants (see Theorem 9 [K2, pg 382]). Now, let \( q_i = p_i|_{\mathfrak{h}} \) for \( 1 \leq i \leq l \) and \( X \in \mathfrak{h} \). We claim that we have

\[
dq_1(X) \wedge dq_2(X) \wedge \cdots \wedge dq_l(X) \neq 0
\]

if and only if

\[
dp_1(X) \wedge dp_2(X) \wedge \cdots \wedge dp_l(X) \neq 0
\]

(7.14)

(Here we are thinking of the \( dp_i(X) \) and \( dq_i(X) \) as covectors.) The sufficiency is obvious. To see the necessity, note that if \( X \in \mathfrak{h} \) is regular, and \( \mathfrak{g}^X \) denotes the centralizer of \( X \in \mathfrak{g} \), then \( \mathfrak{g}^X = \mathfrak{h} \), see ([CM, pgs 20-21]). Now let \( \Phi(\mathfrak{g}, \mathfrak{h}) \) be a system of roots for \( \mathfrak{g} \) relative to \( \mathfrak{h} \). Now, since \( \mathfrak{g}^X = \mathfrak{h} \), we must have that \( \alpha(X) \neq 0 \) for all \( \alpha \in \Phi(\mathfrak{g}, \mathfrak{h}) \). But now, we make use of the fact that [see Hum 2, pg 69]:

\[
dq_1(X) \wedge dq_2(X) \wedge \cdots \wedge dq_l(X) = k \prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \alpha(X)
\]

(7.15)

where \( k \in \mathbb{C}^\times \) and \( \Phi^+(\mathfrak{g}, \mathfrak{h}) \) denotes the set of positive roots with respect to the chosen Cartan \( \mathfrak{h} \). Thus, \( dq_1(X) \wedge dq_2(X) \wedge \cdots \wedge dq_l(X) \neq 0 \), completing the proof of the claim.

Now, we let \( \mathfrak{g} = \mathfrak{so}(2l + 1) \) and suppose that \( X \in \mathfrak{h} \) is regular where \( \mathfrak{h} \subset \mathfrak{so}(2l + 1) \) is the Cartan subalgebra consisting of elements of the form (7.13). We want to understand what the RHS of equation (7.15) means in this case. To see the root system for a Lie algebra of type \( B_l \), we have to consider another realization of \( \mathfrak{so}(2l + 1) \). We let \( \mathfrak{g}' = \mathfrak{so}(2l + 1) \) thought of as matrices which are skew-symmetric about the skew diagonal. In other words, \( \mathfrak{g}' \) consists of matrices in \( M(2l + 1) \) which are skew-adjoint with respect to the symmetric bilinear form defined by the matrix \( S \in M(2l + 1) \) where \( S \) is given by:

\[
S = \begin{bmatrix}
0 & 0 & s_t \\
0 & 1 & 0 \\
s_t & 0 & 0
\end{bmatrix}
\]

(7.16)
with $s_i$ being the $l \times l$ matrix with ones down the skew-diagonal and zeros elsewhere. Let $\mathfrak{h}' \subset \mathfrak{g}'$ be the standard Cartan. This means that $\mathfrak{h}'$ consists of diagonal matrices which are skew-symmetric about the skew-diagonal i.e. matrices of the form

$$\text{diag}[x_1, \cdots, x_{l-1}, x_l, 0, -x_l, -x_{l-1}, \cdots, -x_1]$$

with $x_i \in \mathbb{C}$. Then one computes that the right side of equation (7.15) is the polynomial

$$f(X) = 2^l l! x_1 \cdots x_l \prod_{1 \leq i < j \leq l} (x_j^2 - x_i^2)$$

(\text{cf \cite{Hum 2, pg 68}}). Now, if we let $\mathfrak{g}$ be the $2l+1 \times 2l+1$ skew-symmetric matrices that we are considering then $\mathfrak{g} \simeq \mathfrak{g}'$. In fact, there exists a matrix $g \in Gl(2l+1)$ such that $g \mathfrak{g} g^{-1} = \mathfrak{g}'$. We can describe $g$ as a matrix with the property that $g^t S g = Id$ where $S \in M(2l+1)$ is as given in equation (7.16) (see \cite{GW, pg 3}).

We also note that the associated adjoint groups are conjugate by $g$ i.e. $G' = g G g^{-1}$ [see GW, pg 3] (where $G$ denotes the adjoint group of $\mathfrak{g}$ and $G'$ the adjoint group of $\mathfrak{g}'$). Using this fact, it is easy to see that $X \in \mathfrak{g}$ is regular if and only if $gXg^{-1} \in \mathfrak{g}'$ is regular. Thus, if $X \in \mathfrak{g}$ is regular semi-simple, then $Y = gXg^{-1} \in \mathfrak{g}'$ is regular semi-simple. By the discussion above $Y \in \mathfrak{g}'$ is regular semi-simple if and only if when $Y$ is conjugated into $Z \in \mathfrak{h}'$, $f(Z) \neq 0$. To summarize, we have the following chain of implications.

Given $X \in \mathfrak{g}$ there exists $Z \in \mathfrak{h}' \subset \mathfrak{g}'$ such that $hXh^{-1} = Z$, $h \in GL(n)$, and we have that

$$X \text{ is regular semi-simple } \Leftrightarrow Z \in \mathfrak{g}' \text{ is regular semi-simple } \Leftrightarrow$$

$$f(Z) \neq 0 \Leftrightarrow$$

$$Z \text{ has distinct eigenvalues } \Leftrightarrow$$

$$\Leftrightarrow X \text{ has distinct eigenvalues}$$

A simple calculation shows that the eigenvalues of the matrix in (7.13) are $(+/-)a_j i$, $0$ for $1 \leq j \leq l$. Thus, the condition that $a_i \neq 0$ and $a_i \neq (+/-) a_j$ for $i \neq j$ is exactly the condition that the matrix in (7.13) have distinct eigenvalues.
The case of type $D_l$ (i.e. $\mathfrak{so}(2l)$) is very similar. In this case, we choose our Cartan subalgebra $\mathfrak{h}$ to be matrices of the form:

$$
\begin{bmatrix}
0 & a_1 & & \\
-a_1 & 0 & & \\
 & & 0 & a_2 \\
 & & -a_2 & 0 \\
& & & & \ddots \\
& & & & & 0 & a_l \\
& & & & & -a_l & 0
\end{bmatrix}
$$

with $a_i \in \mathbb{C}$ for $1 \leq i \leq l$. We will refer to (7.18) as the standard Cartan subalgebra in $\mathfrak{so}(2l)$.

Again the regular semi-simple elements are the elements with distinct eigenvalues i.e. the elements of the form (7.18) for which $a_i \neq (+/-)a_j$ for $i \neq j$. We can argue this exactly as we did in type $B_l$. We note that if we realize $\mathfrak{so}(2l)$ as $2l \times 2l$ matrices which are skew-symmetric about the skew-diagonal, then the standard Cartan is given by diagonal matrices of the form

$$
\text{diag}[x_1, x_2, \cdots, x_l, -x_l, \cdots, -x_2, -x_1]
$$

with $x_i \in \mathbb{C}$, and in this case, one computes that the determinant (7.15) is given by (cf. [Hum 2, pg. 68]):

$$
2^{l-1} (l-1)! \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2).
$$

In the case of orthogonal Lie algebras $\mathfrak{so}(n)$ there are two different types of extension problems to consider. We have to consider extending from Type $D_l$ to Type $B_l$ and from Type $B_l$ to Type $D_{l+1}$. 
7.4.1 Extending from Type $B_l$ to Type $D_{l+1}$

We consider the following $2l + 2 \times 2l + 2$ skew-symmetric matrix.

\[
\begin{bmatrix}
0 & a_1 & 0 & \cdots & 0 & z_{l+1} \\
a_1 & 0 & a_2 & \cdots & 0 & z_{l+1} \\
0 & a_2 & 0 & \cdots & 0 & z_{l+1} \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_l & 0 & z_{l+1} \\
-z_{11} & -z_{12} & -z_{21} & -z_{22} & \cdots & -z_{l1} -z_{l2} -z_{l+1} & 0
\end{bmatrix}
\]

with $a_i \in \mathbb{C}$ for $1 \leq i \leq l$ and $a_i \neq 0$ for all $i$ and $a_i \neq (+/-)a_j$ for $i \neq j$. We want to choose the $z_{ij}$ and $z_{l+1}$ in the matrix in (7.19) so that it has characteristic polynomial given by

\[
l + 1 \prod_{i=1}^{l+1} (t^2 + b_i^2)
\]

with $b_i \neq 0, b_i \neq (+/-)b_j$ and $b_i \neq (+/-)a_j$ for $i \neq j$.

To do this we first have to find the characteristic polynomial of the matrix in (7.19). To that affect we have the following proposition.

**Proposition 7.4.1.** The characteristic polynomial of the matrix in (7.19) is given by the following polynomial.

\[
\sum_{i=1}^{l} \left( (z_{11}^2 + z_{22}^2) t^2 \left[ \prod_{j=1, j \neq i} (t^2 + a_j^2) \right] \right) + z_{l+1}^2 \prod_{i=1}^{l} (t^2 + a_i^2) + t^2 \prod_{i=1}^{l} (t^2 + a_i^2) \quad (7.20)
\]

Proof:
We are interested in computing the following determinant.

\[
\begin{vmatrix}
  t & -a_1 & -z_{11} \\
  a_1 & t & -z_{12} \\
  & t & -a_2 & -z_{21} \\
  & a_2 & t & -z_{22} \\
  & & & \ddots \\
  & & & t & -a_l & -z_{l1} \\
  & & & a_l & t & -z_{l2} \\
  & & & & t & -z_{l+1}
\end{vmatrix}
\]

(7.21)

We expand by cofactors down the last column, keeping careful track of the signs involved. We consider the cofactor corresponding to \(-z_{j1}\). After eliminating the row containing \(-z_{j1}\) (i.e. the \(2(j - 1) + 1\)-th row), we get the following:

\[
(-1)^{2j-1+2l+2}(-z_{j1}) \times
\begin{vmatrix}
  t & -a_1 \\
  a_1 & t \\
  & t & -a_2 \\
  & a_2 & t \\
  & & & \ddots \\
  & & & t & -a_{j-1} \\
  & & & a_{j-1} & t & 0 & 0 \\
  & & & 0 & a_j & t \\
  & & & & & & \ddots \\
  & & & & & & t \\
  z_{11} & z_{12} & z_{21} & z_{22} & \cdots & \cdots & \cdots & z_{j1} & z_{j2} & z_{l+1}
\end{vmatrix}
\]

(7.22)

Now, we want to compare this cofactor to the one from \(-z_{j2}\), since some of the
Now, we expand the $2l + 1 \times 2l + 1$ determinant in (7.22) by cofactors along the $2j - 1$-th column and the one in (7.23) by cofactors along the $2j$-th column. The claim is that the terms involving $a_j$ from (7.22) and from (7.23) cancel and the remaining terms are the desired ones. We concentrate first on the terms involving the $a_j$. Consider (7.22). We get the following determinant

$$(-1)^{2j-1+2l+2}(-z_{j1}) \times$$

$$
\begin{align*}
\begin{bmatrix}
 t & -a_1 \\
 a_1 & t \\
 & t & -a_2 \\
 & & a_2 & t \\
 & & & \ddots \\
 & & & & t & -a_{j-1} \\
 & & & & a_{j-1} & t & 0 & 0 \\
 & & & & & 0 & t & -a_j \\
 & & & & & & \ddots & \ddots \\
 z_{11} & z_{12} & z_{21} & z_{22} & \cdots & \cdots & \cdots & z_{j1} & z_{j2} & z_{l+1} \\
\end{bmatrix}
\end{align*}
$$

(7.24)
Now, we look at the cofactor from $-a_j$ in (7.23). We get:

$$(-1)^{2j-1+2j}(-z_{j2})(-a_j) \times$$

$$
\begin{vmatrix}
\begin{array}{cccccc}
t & -a_1 \\
a_1 & t \\
t & -a_2 \\
a_2 & t \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
t & -a_{j-1} \\
a_{j-1} & t & 0 & 0 \\
0 & 0 & t & -a_{j+1} \\
a_{j+1} & t & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
z_{11} & z_{12} & z_{21} & z_{22} & \cdots & \cdots & \cdots & z_{j1} & z_{j+1} & \cdots & \cdots & \cdots & \cdots & z_{l+1}
\end{array}
\end{vmatrix}
$$

(7.25)

Finally, we evaluate the determinant in (7.24) and (7.25) by expanding by cofactors along the $2j - 1$-th column. For (7.24), we get the following:

$$(-1)^{2j-1}(-z_{j1})a_j(-1)^{2l+2j-1}z_{j2} \det(X)$$

where $X \in M(2l - 1)$ is the $2l - 1 \times 2l - 1$ matrix given by taking the matrix the $2l + 1 \times 2l + 1$ cutoff of the matrix in (7.21) and removing the $2j - 1$ and $2j$ rows and columns.

For (7.25), we get the following:

$$(-z_{j2})(a_j)(-1)^{2j-1+2l}z_{j1} \det(X)$$

One observes that the above two equations cancel one another.

Now, it only remains to compute the cofactors in (7.22) comming from $z_{j1}$ and the one in (7.23) comming from $z_{j2}$. First, consider (7.22): We get
\((-1)^{2j-1}(-z_{j1})(-1)^{2j-1+2l+1}(z_{j1})\times\)

\[
\begin{vmatrix}
t & -a_1 \\
a_1 & t \\
. & . & . \\
t & -a_{j-1} \\
a_{j-1} & t \\
. & . & . & . \\
t & -a_{j+1} \\
a_{j+1} & t \\
\end{vmatrix}
\]

And from (7.23), we get:

\((-1)^{2j+2l+2}(-z_{j2})z_{j2}(-1)^{2l+1+2j}\times\)

\[
\begin{vmatrix}
t & -a_1 \\
a_1 & t \\
. & . & . \\
t & -a_{j-1} \\
a_{j-1} & t \\
. & . & . & . \\
t & -a_{j+1} \\
a_{j+1} & t \\
\end{vmatrix}
\]

These two determinants evaluate to:

\((z_{j1}^2 + z_{j2}^2) \left[ t^2 \prod_{i=1, i \neq j}^l (t^2 + a_i^2) \right] \) (7.26)
In evaluating the determinant in (7.21), we now only have two more cofactors to consider, namely the one from \( z_{l+1} \) and the one in the \( 2l + 2 \times 2l + 2 \) position. Let us first consider the one from \( z_{l+1} \). It is easily seen to be:

\[
z_{l+1}^2 \prod_{i=1}^{l} (t^2 + a_i^2) \tag{7.27}
\]

The final cofactor is also easily seen to be:

\[
t^2 \prod_{i=1}^{l} (t^2 + a_i^2) \tag{7.28}
\]

Adding the cofactors from (7.26) for \( 1 \leq j \leq l \) to (7.27) and (7.28). We arrive at

\[
\sum_{j=1}^{l} \left( z_{j1}^2 + z_{j2}^2 \right) \left[ t^2 \prod_{i=1, i\neq j}^{l} (t^2 + a_i^2) \right] + z_{l+1}^2 \prod_{i=1}^{l} (t^2 + a_i^2) + t^2 \prod_{i=1}^{l} (t^2 + a_i^2) \tag{7.29}
\]

which is the desired result at last.

Q.E.D.

Now, we actually want to see that we can use this result to extend from the given semi-simple orbit in \( \mathfrak{so}(2l+1) \) to an arbitrary semi-simple orbit in \( \mathfrak{so}(2l+2) \). Recall that we want to choose the \( z_{ij} \) and \( z_{l+1} \) so that the matrix in (7.19) has characteristic polynomial given by

\[
\prod_{i=1}^{l+1} (t^2 + b_i^2) \tag{7.30}
\]

with \( b_i \neq 0 \), \( b_i \neq (+/-)b_j \) and \( b_i \neq (+/-)a_j \) for \( i \neq j \).

If we can solve this extension problem, then the matrix in (7.19) will be similar in \( O(2l + 2) \) to a matrix of the form

\[
\begin{bmatrix}
0 & b_1 & & \\
-b_1 & 0 & & \\
& & 0 & b_2 \\
& & -b_2 & 0 \\
& & & \ddots & \\
& & & & 0 & b_{l+2} \\
& & & & -b_{l+2} & 0
\end{bmatrix}
\]
with \( b_i \in \mathbb{C} \) for \( 1 \leq i \leq l + 2 \) and satisfying \( b_i \neq 0, b_i \neq (+/-) b_j \) and \( b_i \neq (+/-) a_j \) for \( i \neq j \).

Let us call the polynomial in (7.29) \( f(t) \) and the one in (7.30) \( g(t) \). Now \( f(t), g(t) \) are both polynomials of degree \( 2l + 2 \). However, since both polynomials are monic and since they are both characteristic polynomials of matrices with zero trace, their difference \( h(t) = f(t) - g(t) \) is a polynomial of degree at most \( 2l \). Now, we want to find conditions on the \( z_{ij} \) and \( z_{l+1} \) such that \( h(t) \equiv 0 \). If we can choose the \( z_{ij} \) and \( z_{l+1} \) such that there exist \( 2l + 1 \) distinct complex numbers \( x_1, \ldots, x_{2l+1} \) with the property that \( f(x_i) = g(x_i) \) for \( 1 \leq i \leq 2l + 1 \), then, since \( f - g \) is of degree at most \( 2l \), we would be forced to have \( f - g \equiv 0 \Rightarrow f = g \).

We choose for our \( x_i \) the distinct \( 2l + 1 \) eigenvalues of the \( 2l + 1 \times 2l + 1 \) cutoff of the matrix in (7.19). This means that we take \( x_1 = a_1, x_2 = -a_1, \ldots, x_{2l-1} = a_l, x_{2l} = -a_l, x_{2l+1} = 0 \). Now, if we substitute \( y = (+/-) a_i \) into the equation \( f(t) = g(t) \), for \( 1 \leq i \leq l \), we obtain:

\[
f(y) = g(y) \Leftrightarrow z_{11}^2 + z_{12}^2 = \frac{\prod_{j=1}^{l+1} (b_j^2 - a_j^2)}{-a_i^2 \prod_{j=1, j \neq i}^l (a_j^2 - a_i^2)} \tag{7.31}
\]

for \( 1 \leq i \leq l \). We note that this is defined precisely because our assumption of regularity on the \( 2l + 1 \times 2l + 1 \) cutoff gives us that \( a_i \neq (+/-) a_j \) and \( a_i \neq 0 \). Now, by our hypothesis on the disjointness between the eigenvalues of the adjacent cutoffs, we actually have the RHS of (7.31) is non-zero, so we can rewrite (7.31) as:

\[
f(y) = g(y) \Leftrightarrow (z_{11}^2 + z_{12}^2) = d_i^2 \tag{7.32}
\]

with \( d_i \neq 0, d_i = \sqrt{\frac{\prod_{j=1}^{l+1} (b_j^2 - a_j^2)}{-a_i^2 \prod_{j=1, j \neq i}^l (a_j^2 - a_i^2)}} \). Observe that \( d_i \) only depends upon the values of the \( a_j \) and the \( b_k \).

Now, to find the condition on \( z_{l+1} \), we substitute \( t = 0 \) into the equation \( f(t) = g(t) \) to get:

\[
f(0) = g(0) \Leftrightarrow z_{l+1}^2 = \frac{\prod_{j=1}^{l+1} b_j^2}{\prod_{i=1}^l a_i^2}
\]

hence \( z_{l+1} = (+/-) \frac{\prod_{j=1}^{l+1} b_j}{\prod_{i=1}^l a_i} \). Once again the RHS of (7.33) is defined since \( a_i \neq 0 \) for \( 1 \leq i \leq l \). We note that we have some ambiguity in the choice of \( z_{l+1} \). We will see that the choice
of sign for \( z_{l+1} \) corresponds to the choice of sign of the Pfaffian of the matrix in (7.19) Now, we note that if (7.32) and (7.33) are satisfied then we have that
\[
h(x_i) = f(x_i) - g(x_i) = 0 \text{ for } 1 \leq i \leq 2l + 1 \Rightarrow h(t) \equiv 0 \Rightarrow f(t) = g(t) \text{ for all } t.
\]
And we solved the extension problem.

Let us make some observations about the structure of the solutions. For now, let us fix a choice of sign for \( z_{l+1} \). We want to consider the equation (7.32) for \( i = 1, \cdots, l \). Consider the closed subvariety of \( \mathbb{C}^2 \) defined by:
\[
X_i = \{ (z_{i1}, z_{i2}) \in \mathbb{C}^2 | z_{i1}^2 + z_{i2}^2 = d_i^2 \neq 0 \}
\]
We claim that \( X_i \) is isomorphic to \( SO(2) \) (thought of as \( 2 \times 2 \) complex orthogonal matrices). Note also that we clearly have an action of \( G = SO(2) \) on \( X_i \) given as follows. Let \( g \in SO(2) \) so that
\[
g = \begin{bmatrix} u & -w \\ w & u \end{bmatrix}
\]
with \( w^2 + u^2 = 1 \). Then \( g \) acts on \( X_i \) by left translation (thinking of the elements of \( X_i \) as column vectors).
\[
g \cdot \begin{bmatrix} z_{i1} \\ z_{i2} \end{bmatrix} = \begin{bmatrix} z_{i1}u - wz_{i2} \\ z_{i1}w + uz_{i2} \end{bmatrix}
\]
Now, we observe \( (z_{i1}u - wz_{i2})^2 + (z_{i1}w + uz_{i2})^2 = z_{i1}^2 (u^2 + w^2) + z_{i2}^2 (u^2 + w^2) = z_{i1}^2 + z_{i2}^2 = d_i^2 \), since \( u^2 + w^2 = 1 \). Thus, we have a regular action of \( SO(2) \) on \( X_i \).

With this action on \( X_i \), we have a \( G \)-equivariant isomorphism, \( \Phi_i \) from \( X_i \) to \( G \), where the action on \( G \) is given by left translation. The map is given by
\[
\Phi_i \left( \begin{bmatrix} z_{i1} \\ z_{i2} \end{bmatrix} \right) = \frac{1}{d_i} \begin{bmatrix} z_{i1} & -z_{i2} \\ z_{i2} & z_{i1} \end{bmatrix} \tag{7.34}
\]
An easy computation shows us that this map is \( G \)-equivariant. Consider
\[
\Phi_i \left( g \cdot \begin{bmatrix} z_{i1} \\ z_{i2} \end{bmatrix} \right). \text{ By our above work this is just}
\]
\[
\Phi_i \left( \begin{bmatrix} z_{i1}u - wz_{i2} \\ z_{i1}w + uz_{i2} \end{bmatrix} \right) = \frac{1}{d_i} \begin{bmatrix} z_{i1}u - wz_{i2} & -(z_{i1}w + uz_{i2}) \\ z_{i1}w + uz_{i2} & z_{i1}u - wz_{i2} \end{bmatrix}
\]
But we have that
\[
g \cdot \Phi_i \left( \begin{bmatrix} z_{i1} \\ z_{i2} \end{bmatrix} \right) = \begin{bmatrix} u & -w \\ w & u \end{bmatrix} \frac{1}{d_i} \begin{bmatrix} z_{i1} & -z_{i2} \\ z_{i2} & z_{i1} \end{bmatrix}
\]
which becomes
\[
\frac{1}{d_i} \begin{bmatrix} z_{i1}u - wz_{i2} & -(z_{i1}w + uz_{i2}) \\ wz_{i1} + uz_{i2} & uz_{i1} - wz_{i2} \end{bmatrix}
\]
Thus, we have \( \Phi_i \) is a \( G \)-equivariant isomorphism.

The upshot of this discussion is that the set of solutions to the extension problem outlined at the beginning of the section are isomorphic to \( SO(2)^l \) (once we have fixed a choice of sign for \( z_{l+1} \)) and the centralizer of the \( 2l + 1 \times 2l + 1 \) cutoff acts diagonally by \( l \) copies of the standard left regular representation of \( SO(2) \) on itself with this identification. This follows immediately from the observation that the centralizer of the \( 2l + 1 \times 2l + 1 \) cutoff of the matrix in (7.19) is simply given by
\[
G_{2l+1} = \begin{bmatrix}
SO(2) & & \\
& SO(2) & \\
& & \ddots \\
& & \\
SO(2) & & \\
& & \\
1 & & 
\end{bmatrix}
\]
(7.35)

Now, we want to observe that once we have fixed a choice of sign for \( z_{l+1} \) that \( G_{2l+1} \) acts freely on the set of solutions given by (7.32) and (7.33) and hence such solutions form one \( G_{2l+1} \) orbit. Denote such a set of solutions by \( Z \subset \mathfrak{so}(2l+2) \).

First of all, note that from the description of \( G_{2l+1} \) in (7.35), it is easy to see that the action preserves the exact value of \( z_{l+1} \), so \( G_{2l+1} \) acts on \( Z \). The fact that the action is free now follows easily from our above description of the action and the fact that the action of \( SO(2) \) on itself by left translation is free. Now, to see that the group acts transitively, we observe that \( Z \) forms an irreducible variety of dimension \( 2l \) that is isomorphic to \( SO(2l) \) by our above work. Hence, since \( G_{2l+1} \simeq SO(2)^l \) acts freely any \( G_{2l+1} \) orbit is open in the set of solutions. Since two non-empty open subsets of an irreducible variety must intersect, we have that the action of \( G_{2l+1} \) on \( Z \) is transitive.
Now, let us discuss the significance of the choice of sign of $z_{l+1}$ in equation (7.33). Let $X$ be the matrix in (7.19). Then by Proposition (7.4.1) the determinant of this matrix is simply

$$\det(X) = z_{l+1}^2 \prod_{i=1}^{l} a_i^2$$

(Since the degree of the characteristic polynomial in this case is even, we have that the constant term is simply $\det(X)$.) Thus, $\text{Pfaff}(X) = (+/-)z_{l+1} \prod_{i=1}^{l} a_i$. Suppose first that $\text{Pfaff}(X) = z_{l+1} \prod_{i=1}^{l} a_i$. Then choosing the positive sign in equation (7.33), gives $\text{Pfaff}(X) = \prod_{i=1}^{l+1} b_i$. We claim that the matrix in equation (7.19) is now similar in $SO(2l + 2)$ to a regular semi-simple element of the form:

$$\begin{bmatrix}
0 & b_1 \\
-b_1 & 0 \\
& & 0 & b_2 \\
& & -b_2 & 0 \\
& & & & \ddots \\
& & & & & 0 & b_{l+1} \\
& & & & & -b_{l+1} & 0
\end{bmatrix}$$

with $b_i \neq (+/-)b_j$. Note that the Pfaffian of such a matrix is exactly $\prod_{i=1}^{l+1} b_i$ and we have chosen the $z_{ij}$ and the $z_{l+1}$ so that the matrix $X$ now takes the same values on the fundamental adjoint invariants

$$\psi_1(X) = tr(X^2), \psi_2(X^4), \cdots, \psi_l(X) = tr(X^{2l}), \psi_{l+1}(X) = \text{Pfaff}(X)$$

as this regular semi-simple element. Now, the claim follows from Proposition 7.3.2.

Now, suppose that we choose the negative sign for $z_{l+1}$ in (7.33). Then $\text{Pfaff}(X) = -\prod_{i=1}^{l+1} b_i$. This means that we have chosen the extension so that the matrix $X$ is conjugate via an element of $SO(2l + 2)$ to the element in $\mathfrak{so}(2l + 2)$.
given by

\[
\begin{bmatrix}
0 & -b_1 \\
-b_1 & 0 \\
0 & -b_2 \\
-b_2 & 0 \\
\ddots & \ddots \\
0 & -b_{l+1} \\
-b_{l+1} & 0 \\
\end{bmatrix}
\]

If we suppose instead that \( \text{Pfaff}(X) = -z_{l+1} \prod_{i=1}^{l} a_i \), then choosing the positive sign in equation (7.33) gives \( \text{Pfaff}(X) = -\prod_{i=1}^{l+1} b_i \) and choosing the negative sign in (7.33) gives \( \text{Pfaff}(X) = \prod_{i=1}^{l+1} b_i \), so that the situation is reversed. The point is that we can choose a sign in equation (7.33) so that we can extend from any regular semi-simple orbit \( \mathfrak{so}(2l+1) \) to any regular semi-simple orbit in \( \mathfrak{so}(2l+2) \) satisfying the eigenvalue disjointness condition.

We now summarize all that we have shown in the following theorem.

**Theorem 7.4.1.** Given a matrix of the form (7.19) i.e.

\[
\begin{bmatrix}
0 & a_1 & \cdots & z_{11} \\
-a_1 & 0 & \cdots & z_{12} \\
0 & a_2 & \cdots & z_{21} \\
-a_2 & 0 & \cdots & z_{22} \\
\ddots & \ddots & \ddots & \ddots \\
0 & a_l & \cdots & z_{l1} \\
-a_l & 0 & \cdots & z_{l2} \\
-z_{11} & -z_{12} & -z_{21} & -z_{22} & \cdots & -z_{l1} & -z_{l2} & -z_{l+1} & 0 \\
\end{bmatrix}
\]

with the \( 2l+1 \times 2l+1 \) cutoff regular semi-simple, we can choose the \( z_{ij} \) and \( z_{l+1} \) such that this matrix has characteristic polynomial given by

\[
\prod_{i=1}^{l+1}(t^2 + b_i^2)
\]
with the $b_i$ satisfying the disjointness condition that $b_i \neq 0$, $b_i \neq (+/-)b_j$, $b_i \neq (+/-)a_j$. Moreover, we can choose the sign of $z_{l+1}$ such that the Pfaffian of the matrix in (7.19) is either $\prod_{i=1}^{l+1} b_i$ or $-\prod_{i=1}^{l+1} b_i$. Thus, we can extend from any regular semi-simple orbit in $\mathfrak{so}(2l + 1)$ to any regular semi-simple orbit in $\mathfrak{so}(2l + 2)$ satisfying the disjointness condition. Moreover, each of these extensions is isomorphic as a variety to the torus $SO(2)^l$ and the centralizer $G_{2l+1} = SO(2)^l$ acts simply transitively on the extensions.

### 7.4.2 Extending from Type $D_l$ to Type $B_l$

We consider the following $2l + 1 \times 2l + 1$ skew-symmetric matrix.

\[
\begin{bmatrix}
0 & a_1 & & & z_{11} \\
-a_1 & 0 & & & z_{12} \\
 & 0 & a_2 & & z_{21} \\
 & & -a_2 & 0 & z_{22} \\
 & & & \vdots & \vdots \\
 & & & 0 & a_l & z_{l1} \\
 & & & -a_l & 0 & z_{l2} \\
-z_{11} & -z_{12} & -z_{21} & -z_{22} & \cdots & -z_{l1} & -z_{l2} & 0 \\
\end{bmatrix} \tag{7.36}
\]

with $a_i \in \mathbb{C}$ for $1 \leq i \leq l$ and $a_i \neq 0$ for all $i$ and $a_i \neq (+/-)a_j$ for $i \neq j$.

We want to choose the $z_{ij}$ so that this matrix has characteristic polynomial given by:

\[
t \prod_{i=1}^{l}(t^2 + b_i^2) \tag{7.37}
\]

with $b_i \in \mathbb{C}$ satisfying the conditions that $b_j \neq (+/-)a_k$, $b_i \neq (+/-)b_j$, and $b_i \neq 0$.

To do this we first have to find the characteristic polynomial of the matrix in (7.36). To that affect we have the following proposition.

**Proposition 7.4.2.** The characteristic polynomial of the matrix in (7.36) is given by:

\[
t \left[ \prod_{i=1}^{l}(t^2 + a_i^2) + \sum_{i=1}^{l}(z_{1i}^2 + z_{2i}^2) \prod_{j=1,j \neq i}^{l}(t^2 + a_j^2) \right] \tag{7.38}
\]
Proof:

We can actually derive the result quickly using Proposition 7.4.1. We note that Proposition 7.4.1 implies that the determinant

$$\begin{vmatrix}
  t & -a_1 & \ldots & -z_{11} \\
  a_1 & t & \ldots & -z_{12} \\
  \vdots & \ddots & \ddots & \vdots \\
  t & -a_l & \ldots & -z_{l1} \\
  a_l & t & \ldots & -z_{l2} \\
  z_{11} & z_{12} & \cdots & 0 \\
  z_{21} & z_{22} & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots \\
  t & z_{l1} & z_{l2} & 0 \\
  \end{vmatrix}$$

is given by:

$$\sum_{j=1}^{l} \left( (z_{j1}^2 + z_{j2}^2) \left[ t^2 \prod_{i=1, i \neq j}^l (t^2 + a_i^2) \right] \right) + z_{l+1}^2 \prod_{i=1}^l (t^2 + a_i^2) + t^2 \prod_{i=1}^l (t^2 + a_i^2)$$

If we set \( z_{l+1} = 0 \) above then

$$\begin{vmatrix}
  t & -a_1 & \ldots & -z_{11} \\
  a_1 & t & \ldots & -z_{12} \\
  \vdots & \ddots & \ddots & \vdots \\
  t & -a_l & \ldots & -z_{l1} \\
  a_l & t & \ldots & -z_{l2} \\
  z_{11} & z_{12} & \cdots & 0 \\
  z_{21} & z_{22} & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots \\
  t & z_{l1} & z_{l2} & 0 \\
  \end{vmatrix}$$

is equal to

$$\sum_{j=1}^{l} \left( (z_{j1}^2 + z_{j2}^2) \left[ t^2 \prod_{i=1, i \neq j}^l (t^2 + a_i^2) \right] \right) + t^2 \prod_{i=1}^l (t^2 + a_i^2)$$

Now, we evaluate the determinant in (7.39) by expanding by cofactors along the \( 2l + 1 \) column to get
where $A$ is the matrix in (7.36). We know that by our above work this is equal to (7.40). Canceling $t$ then gives us the desired result.

Q.E.D.

Remark 7.4.1. We note that the polynomial in (7.38) is invariant under sign changes $a_j \rightarrow -a_j$. So that the above computations and the ones that follow are independent of the Pfaffian of the $2l \times 2l$ cutoff of the matrix in (7.36).

Now, we want to see that we can choose the $z_{ij}$ in the matrix in (7.36) so that it is conjugate in $\mathfrak{so}(2l+1)$ to an element of the form:

\[
\begin{bmatrix}
0 & b_1 \\
-b_1 & 0 \\
& & 0 & b_2 \\
& & -b_2 & 0 \\
& & & & \ddots \\
& & & & 0 & b_l \\
& & & & -b_l & 0 \\
& & & & & 0
\end{bmatrix}
\]

(7.41)

with $b_i \in \mathbb{C}$, $b_i \neq 0$, for all $i$ and $b_i \neq (+/-)b_j$ and satisfying the disjointness condition that $a_i \neq (+/-)b_j$.

We argue in a similar manner as we did in the case of extending Type $B_l$ to Type $D_{l+1}$. We denote the polynomial in equation (7.38) by $f(t)$ and the one in equation (7.37) by $g(t)$. Then we note that since both polynomials are monic and both the above matrix and the matrix in (7.36) have trace 0, the polynomial $h(t) = f(t) - g(t)$ is of degree at most $2l - 1$. Now, we want to find conditions on the $z_{ij}$ such that $h(t) \equiv 0$. As in the previous case, if we can choose the $z_{ij}$ such that there exist $2l$ distinct complex numbers $x_1, \cdots, x_{2l}$ such that $f(x_i) = g(x_i)$ for $1 \leq i \leq 2l$, then, since $f - g$ is of degree at most $2l - 1$, we would be forced to have $f - g \equiv 0 \Rightarrow f = g$. 

\[
t \det(A)
\]
We choose for our $x_i$ the $2l$ distinct eigenvalues of the $2l \times 2l$ cutoff of the matrix in (7.36). i.e. We take $x_{1,1} = a_1$, $x_{1,2} = -a_1, \ldots, x_{l,1} = a_l, x_{l,2} = -a_l$. Note that these are distinct precisely because of the regularity condition on the $2l \times 2l$ cutoff of (7.36) and because no $a_i = 0$, by the eigenvalue disjointness condition.

We have that

$$f(x_{j,i}) = g(x_{j,i}) \Leftrightarrow (z_{j1}^2 + z_{j2}^2) = \frac{\prod_{i=1}^{l}(b_i^2 - a_j^2)}{\prod_{i=1, i \neq j}^{l}(a_i^2 - a_j^2)}$$

(7.42)

for $i = 1, 2$.

The RHS of the (7.42) is defined precisely because the $2l \times 2l$ cutoff of the matrix in (7.36) is regular and it is non-zero because of the disjointness condition on the $a_i$ and $b_j$. If we let

$$d_j = \sqrt{\frac{\prod_{i=1}^{l}(b_i^2 - a_j^2)}{\prod_{i=1, i \neq j}^{l}(a_i^2 - a_j^2)}}$$

(7.43)

then we have that $f(t) = g(t)$ if and only if

$$z_{j1}^2 + z_{j2}^2 = d_j^2$$

for $1 \leq j \leq l$ and with $d_j \neq 0$. Again, we note that $d_j$ depends only on the values of the $a_j$ and $b_i$. Thus, $X \in \mathfrak{so}(2l+1)$ satisfying (7.42) is conjugate in $SO(2l+1)$ to an element of the form (7.41) by Proposition 7.3.2.

As in the case of extending from Type $B_l$ to Type $D_{l+1}$, we see that the set of extensions form an irreducible, smooth affine variety of dimension $l$ that is algebraically isomorphic to $SO(2)^l$. We refer to these solutions as $Z$. Now, in this case the centralizer in $SO(2l)$ of the $2l \times 2l$ cutoff of the matrix in (7.36) consists of matrices of the form

$$\begin{pmatrix}
SO(2) \\
SO(2) \\
\ddots \\
SO(2)
\end{pmatrix}$$

(7.44)
Thus, $G_{2l} \simeq SO(2)^l$ as algebraic groups. $G_{2l}$ acts on the solution variety $Z$ by the adjoint action. The same reasoning as used in the previous case shows that this action is nothing more than $l$ copies of the left regular representation of $SO(2)$ acting along the last column of the matrix in (7.36). Thus, the same analysis in the case of extending from Type $B_l$ to Type $D_{l+1}$ allows us to arrive at the following theorem.

**Theorem 7.4.2.** Given a matrix of the form (7.36) i.e.

$$
\begin{bmatrix}
0 & a_1 & & & z_{11} \\
-a_1 & 0 & & & z_{12} \\
& 0 & a_2 & & z_{21} \\
& -a_2 & 0 & & z_{22} \\
& & & \ddots & \vdots \\
& & & 0 & a_l & z_{l1} \\
& & & -a_l & 0 & z_{l2} \\
& -z_{11} & -z_{12} & -z_{21} & -z_{22} & \cdots & -z_{l1} & -z_{l2} & 0
\end{bmatrix}
$$

with the $2l \times 2l$ cutoff regular semi-simple, we can choose the $z_{ij}$ such that this matrix has characteristic polynomial given by

$$
t \left[ \prod_{i=1}^{l} (t^2 + b_i^2) \right]
$$

with $b_i \neq 0$ for all $i$, $b_i \neq (+/-)b_j$, and with the $a_j$ and the $b_i$ satisfying the disjointness condition that $b_i \neq (+/-)a_j$ and $a_j \neq 0$ for all $j$. By Remark (7.4.1), this computation is independent of the sign of the Pfaffian of the $2l \times 2l$ cutoff. Thus, we can extend from any regular semi-simple orbit in $\mathfrak{so}(2l)$ to any regular semi-simple orbit in $\mathfrak{so}(2l + 1)$ satisfying the disjointness condition. Moreover, each of these extensions is isomorphic as a variety to the torus $SO(2)^l$ and the centralizer $G_{2l} = SO(2)^l$ acts simply transitively on the extensions.
The construction proceeds in a similar manner to the case of $M(n)$ discussed in detail in Chapter 4. In this case, however, we only define an algebraic action of $SO(2)^{d/2}$ on each fibre $\mathfrak{so}(n)$, where $c \in \mathbb{C}^{d/2+rkn}$ satisfies the eigenvalues disjointness condition. (Recall $d$ is half the dimension of a generic adjoint orbit of $SO(n)$ in $\mathfrak{so}(n)$.) Now for each such $c \in \mathbb{C}^{d/2+rkn}$ we write $c = (c_1, c_2, \cdots, c_i, \cdots, c_n)$ with $c \in \mathbb{C}^{rk_i}$ where $rk_i$ is the rank of $\mathfrak{so}(i)$. Then each $c_i$ represents a regular semi-simple element in the standard Cartan $\mathfrak{h}_i \subset \mathfrak{so}(i)$ up to the action of the Weyl group $W_i$ and the elements represented by $c_i$ and $c_i+1$ have no eigenvalues in common. Recall that we identify $\mathbb{C}^{rk_i}$ with $\mathfrak{h}_i/W_i$ via the map $\Phi_i(X) = (f_{i,1}(X), \cdots, f_{i,rk_i}(X))$ which takes $X \in \mathfrak{h}_i$ to its values on the fundamental Weyl group invariants. (Recall also that for $i = 2l + 1$ the standard Cartan $\mathfrak{h}_i \subset \mathfrak{so}(i)$ is given by matrices of the form (7.13) and for $i = 2l$ $\mathfrak{h}_i$ is given by matrices of the form (7.18).) We consider the principal subset of $\mathfrak{h}_i/W_i$ given by the quotient by the action of $\mathfrak{W}_i$ on the principal open subset of $\mathfrak{h}_i$ defined by the non-vanishing of the polynomial $\pi^2$ with $\pi = \prod_{1 \leq j < k \leq rk_i} (x_k^2 - x_j^2) \prod_{l=1}^{rk_i} x_l$. (Here we take as coordinates on $\mathfrak{h}_i$ the entries on the first super diagonal.) By our previous work, this principal open subset corresponds to the set of regular semi-simple $\mathfrak{W}_i$ orbits in $\mathfrak{h}_i$ in the case of $i = 2l + 1$ and in the case where $i = 2l$ is even, we have added the further condition that Pfaff$(X) \neq 0$ for $X \in \mathfrak{h}_i$. We will denote this principal open subset of $\mathfrak{h}_i/W_i$ by $\mathfrak{h}'_i/W_i$.

As in the Type A case, it is convenient, although not essential, to have a well-defined way of choosing a representative in the quotient, $(\mathfrak{h}_i)/\mathfrak{W}_i$. In fact, there is a canonical fundamental domain for the action of $\mathfrak{W}_i$ on the regular elements in $\mathfrak{h}_i$. We can construct this fundamental domain for any reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a choice of Cartan subalgebra. Let $\Phi(\mathfrak{g}, \mathfrak{h}) = \Phi$ be a system of roots relative to the Cartan subalgebra $\mathfrak{h}$, let $\Phi^+$ denote a choice of positive roots, and let $\mathfrak{W}$ be the Weyl group relative to $\mathfrak{h}$. Given $z \in \mathfrak{h}$, we can write $z = Rez + i Imz$, with $Rez \in \mathfrak{h}_R$ and $Imz \in \mathfrak{h}_R$. Then it is a standard result
that a fundamental domain for the action of $\mathcal{W}$ on the set of regular elements $\mathfrak{h}^{\text{reg}}$ in $\mathfrak{h}$ is given by:

$$D = \{ z \in \mathfrak{h}^{\text{reg}} | \alpha(Re z) \geq 0, \text{ if } \alpha(Re z) = 0, \alpha(Im z) > 0 \text{ for all } \alpha \in \Phi^+ \} \quad (7.45)$$

Consider the case $g = M(i)$, $\mathfrak{h}$ the $i \times i$ diagonal matrices, and $\mathcal{W} = S_i$, the symmetric group on $i$ letters. If we take the standard positive roots for $g$, then it is easy to see that fundamental domain we have just described is nothing other than the set

$$\{ (z_1, \ldots, z_i) \in \mathfrak{h}^{\text{reg}} | z_1 > z_2 > \cdots > z_i \}$$

where “$>$” denotes the lexicographical ordering on $\mathbb{C}$ that we introduced in the beginning of Chapter 3 and used to write down elements of the solution varieties $\Xi^i_{c_i, c_{i+1}} \subset M(i+1)$.

Now, given $c \in \mathbb{C}^{d/2 + rk_n}$ satisfying the eigenvalue disjointness condition, we write $c = (c_1, c_2, \cdots, c_i, \cdots, c_n)$ with $c \in \mathbb{C}^{rk_i}$ as above. We want to define a regular isomorphism

$$\Gamma^n_c : SO(2)^{d/2} \to \mathfrak{so}(n)_c$$

We define $\Xi^i_{c_i, c_{i+1}} \subset \mathfrak{so}(i+1)$ the orthogonal solution variety at level $i$ as follows. Recall that each $c_i$ represents a regular semi-simple element in $(\mathfrak{h}_i)'/\mathcal{W}_i$. We choose a representative of $(\mathfrak{h}_i)'/\mathcal{W}_i$ and $(\mathfrak{h}_{i+1})'/\mathcal{W}_{i+1}$ using the fundamental domain given equation (7.45). This gives us a canonical way of writing down elements of the standard Cartan subalgebras $\mathfrak{h}_i \subset \mathfrak{so}(i)$ and $\mathfrak{h}_{i+1} \subset \mathfrak{so}(i+1)$ that satisfy the eigenvalue disjointness condition whose values on the fundamental Weyl group invariants are as prescribed by $c_i \in \mathbb{C}^{rk_i}$ and $c_{i+1} \in \mathbb{C}^{rk_{i+1}}$. Once the representatives for $c_i$ and $c_{i+1}$ have been chosen $\Xi^i_{c_i, c_{i+1}}$ is defined to be the set of solutions to the extension problem given in Theorem 7.4.1 if $i$ is odd and if $i$ is even then $\Xi^i_{c_i, c_{i+1}}$ is given by Theorem 7.4.2.

Let $l = \text{rk} (\mathfrak{so}(i))$. First, we need to see that we can find an element $g(z) \in SO(i+1)$ with the property that $g(z)$ conjugates the element $z \in \Xi^i_{c_i, c_{i+1}}$ to the element in the standard Cartan in $\mathfrak{so}(i+1)$ determined by $c_{i+1}$ and lying in the fundamental domain $D_{i+1} \subset \mathfrak{h}_{i+1}$ in equation (7.45) and with the property that the function $z \to g(z)$ is regular on $\Xi^i_{c_i, c_{i+1}}$. If $i$ is odd, then $g(z)$ would conjugate
the element

\[
\begin{bmatrix}
0 & a_1 & z_{11} \\
-a_1 & 0 & \vdots \\
0 & a_2 & \vdots \\
-a_2 & 0 & \vdots \\
0 & a_l & z_{l1} \\
-a_l & 0 & z_{l2} \\
0 & z_{l+1} \\
\end{bmatrix}
\]

into the element

\[
\begin{bmatrix}
0 & b_1 & 0 \\
-b_1 & 0 & \vdots \\
0 & b_2 & \vdots \\
-b_2 & 0 & \vdots \\
0 & b_{l+1} & \vdots \\
-b_{l+1} & 0 & \end{bmatrix}
\in \mathcal{D}_{i+1}
\]

If \(i\) is even, \(g(z)\) would conjugate the element in \(z \in \Xi_{c_i, c_{i+1}}\) given by

\[
\begin{bmatrix}
0 & a_1 & z_{11} \\
-a_1 & 0 & \vdots \\
0 & a_2 & \vdots \\
-a_2 & 0 & \vdots \\
0 & a_l & z_{l1} \\
-a_l & 0 & z_{l2} \\
-z_{l1} - z_{l2} - z_{l2} - \cdots - z_{l1} - z_{l2} & 0 \\
\end{bmatrix}
\]

into the element
This can be done non-explicitly as follows. By Theorems 7.4.1 and 7.4.2, we know that \( \Xi_{i_1, c_{i+1}} \simeq SO(2)^l \) and that \( G_i = SO(2)^l \) acts simply transitively on \( \Xi_{i_1, c_{i+1}} \). Thus, we have a natural notion of an identity element in \( \Xi_{i_1, c_{i+1}} \) under this identification. More explicitly, in the case where \( i \) is even, we can take the identity element to be the following matrix

\[
\begin{bmatrix}
0 & b_1 & & & & \\
-b_1 & 0 & & & & \\
& 0 & b_2 & & & \\
& -b_2 & 0 & & & \\
& & \ddots & & & \\
& & & 0 & b_l & \\
& & & -b_l & 0 & \\
& & & & & 0
\end{bmatrix} \in D_{i+1}
\]

where \( d_j \) is given by equation (7.43). We can do a similar construction in the odd case.
In the case where \( i \) is even, we can act by the element

\[
\begin{bmatrix}
\frac{1}{d_1}z_{11} & -\frac{1}{d_1}z_{12} \\
\frac{1}{d_1}z_{12} & \frac{1}{d_1}z_{11} \\
\frac{1}{d_2}z_{21} & -\frac{1}{d_2}z_{22} \\
\frac{1}{d_2}z_{22} & \frac{1}{d_2}z_{21} \\
\vdots & \\
\frac{1}{d_l}z_{l1} & -\frac{1}{d_l}z_{l2} \\
\frac{1}{d_l}z_{l2} & \frac{1}{d_l}z_{l1}
\end{bmatrix}
\]

in \( G_i \) to obtain the arbitrary element of the solution variety

\[
\begin{bmatrix}
0 & a_1 & z_{11} \\
-a_1 & 0 & z_{12} \\
0 & a_2 & z_{21} \\
-a_2 & 0 & z_{22} \\
\vdots & \vdots & \vdots \\
0 & a_l & z_{l1} \\
-a_l & 0 & z_{l2} \\
-z_{11} & -z_{12} & -z_{21} & -z_{22} & \cdots & -z_{l1} & -z_{l2} & 0
\end{bmatrix}
\]

The strategy is as follows. We concentrate on the case where \( i = 2l \) is even. The case of \( i \) odd is similar. Given an element \( z \in \Xi_{c_i, c_{i+1}}^i \) we write \( z = \text{Ad}(k(z)) \cdot X \) for a unique \( k(z) \in G_i \) where \( X \in \Xi_{c_i, c_{i+1}}^i \) is the matrix given in (7.46). We then fix a choice of matrix \( h \in SO(i + 1) \) such that \( \text{Ad}(h) \cdot X \) is in the fundamental domain \( D_{i+1} \) in \( \mathfrak{h}_{i+1} \subset \mathfrak{so}(i+1) \). Notice that such a choice is independent of the coordinates \( z_{ij} \) on \( \Xi_{c_i, c_{i+1}}^i \). We then observe that the matrix \( g(z) = h k(z)^{-1} \) is such that \( \text{Ad}(g(z)) \cdot z \) is in the fundamental domain \( D_{i+1} \subset \mathfrak{h}_{i+1} \). The matrix \( k(z)^{-1} \) is the inverse of the matrix in equation (7.47), which is clearly a regular function of \( z \in \Xi_{c_i, c_{i+1}}^i \). Thus the function \( z \rightarrow g(z) \) is regular on \( \Xi_{c_i, c_{i+1}}^i \).

Now, we are ready to define the mapping \( \Gamma_c^i \) for \( c \in \mathbb{C}^{d/2+rk} \) satisfying the eigenvalues disjointness condition. Again, we write \( c = (c_1, \cdots, c_i, \cdots, c_n) \) with \( c_i \in \mathbb{C}^{rk_i} \). We identify the solution variety at level \( i \), \( \Xi_{c_i, c_{i+1}}^i \) with \( SO(2)^{rk_i} \) via
Theorems (7.4.1) and (7.4.2). For \( z \in \Xi_{c_i,c_{i+1}} \), we let \( g_{i,i+1}(z) \in SO(i+1) \) be as defined in the last paragraph. Then we can define the following regular map

\[
\Gamma_c^n : SO(2) \times \cdots \times SO(2)^{rk_1} \cdots \times SO(2)^{rk_{n-1}} \to \mathfrak{so}(n)_c
\]

given by:

\[
\Gamma_c^n(z_2, \cdots, z_{n-1}) = Ad(g_{2,3}(z_2)^{-1} g_{2,3}(z_2)^{-1} \cdots g_{n-2,n-1}(z_{n-2}))z_{n-1}
\]

where \( z_i \in \Xi_{c_i,c_{i+1}} \simeq SO(2)^{rk_i} \) for \( 2 \leq i \leq n-1 \). Recall also that in this definition we are thinking of \( SO(j) \hookrightarrow SO(n) \) via the embedding

\[
g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & Id_{n-j} \end{bmatrix}
\]

A few observations are in order about the map defined in (7.48). The existence of the map \( \Gamma_c^n \), which is based on the existence of the solution varieties \( \Xi_{c_i,c_{i+1}} \) for \( 2 \leq i \leq n \), gives that the fibre \( \mathfrak{so}(n)_c \) is non-empty for \( c \in \mathbb{C}^{d/2+rk_n} \) satisfying the eigenvalue disjointness condition. It is easy to see that \( \Gamma_c^n \) is surjective onto the fibre \( \mathfrak{so}(n)_c \). For given an \( X \in \mathfrak{so}(n)_c \), we have that \( X_3 \in \Xi_{c_2,c_3}^2 \). This defines an element \( z_2 \in SO(2) \). Then we conjugate by \( g_{2,3}(z_2) \in SO(3) \) to get a new element \( Y \). The 4 \times 4 cutoff of \( Y \) is then in the solution variety \( \Xi_{c_3,c_4}^3 \). This defines for us a \( z_4 \). We then conjugate \( Y \) by \( g_{3,4}(z_4) \in SO(4) \). Proceeding in this fashion we obtain that \( X = \Gamma_c^n(z_2, \cdots, z_{n-1}) \). This argument is totally analogous to the one we used to characterize the map \( \Gamma_n \) defined for \( M(n) \) in Chapter 4 (see Remark 4.2.1 in that chapter). We can also use an analogous argument to the one used in the case of \( M(n) \) to see that the map \( \Gamma_c^n \) is injective using the fact that \( SO(i)^{rk_i} \) is always acting freely on the solution variety \( \Xi_{c_i,c_{i+1}}^i \).

In this way, we have also defined an inverse to the map \( \Gamma_c^n \). We want to also observe that this map is regular as a map from \( \mathfrak{so}(n)_c \to SO(2)^{d/2} \). The argument is analogous to the one we used in Chapter 4 to see that the inverse to the map \( \Gamma_n^{a_1, \cdots, a_{n-1}} \) is continuous. We proceed by induction. The point is that on the fibre the eigenvalues of each of the cutoffs are fixed. Given \( X \in \mathfrak{so}(n)_c, X_3 \in \Xi_{c_2,c_3}^2 \).
So that $z_2 = (x_{13}, x_{23}) \in \Xi_{c_2, c_3}^2 \simeq SO(2)$. From our above discussion, we know that the element $g_{2,3}(z_2) \in SO(3)$ which conjugates $X_3$ into the standard Cartan in $\mathfrak{so}(3)$ and $g_{2,3}$ depends regularly on $z_2$ and thus on $X$. Thus, the map from $\mathfrak{so}(n)_c \to \mathfrak{so}(n)$ given by
\[
X \to \text{Ad}(g_{2,3}(z_2)) \cdot X
\]
is regular. This starts off the induction. We now follow the following steps.

1. We assume inductively that the map:
\[
\Phi_0 : \mathfrak{so}(n)_c \to \mathfrak{so}(n)
\]
given by
\[
X \to \text{Ad}(g_{j,j-1}(z_{j-1})) \cdots \text{Ad}(g_{2,3}(z_2)) \cdot X
\]
is regular and $z_l \in \Xi_{c_l, c_{l+1}}^l$ for $2 \leq l \leq j - 1$ have already been computed as regular functions of $X$.

2. It then follows that the map $\mathfrak{so}(n)_c \to \Xi_{c_j, c_{j+1}}^j$ given by
\[
\Phi_1(X) = [\text{Ad}(g_{j,j-1}(z_{j-1})) \cdots \text{Ad}(g_{2,3}(z_2)) \cdot X]_{j+1}
\]
is regular. Thus, we have obtained $z_{j+1}$ as a regular function of $X$.

3. Now, we have a regular map $\Xi_{c_j, c_{j+1}}^j \to SO(j+1, \mathbb{C})$ given by taking $z_j \to g_{j,j+1}(z_j)$ (where $g_{j,j+1}(z_j)$ is the element of $SO(j+1)$ that conjugates $z_j$ into the standard Cartan in $\mathfrak{so}(j+1)$).

4. Composing the map from (2) with the map from (3), we have a regular map:
\[
\Phi_2 : \mathfrak{so}(n)_c \to SO(j+1)
\]
given by
\[
X \to g_{j,j+1}(z_j)
\]

5. Now, we note that we have a natural regular map on the product defined as
follows:

$$\Phi_3 : SO(j + 1) \times so(n) \to so(n)$$

given by

$$(g, Z) \to Ad(g) \cdot Z$$

This is nothing more than the adjoint action of the subgroup $SO(j + 1) \hookrightarrow SO(n)$ on $so(n)$. (Here we have used the embedding of $SO(j + 1)$ mentioned above).

(6) Using the universal property of the product in the category of varieties we get a regular map

$$\Phi_4 : so(n)_c \to SO(j + 1) \times so(n)$$

given by

$$X \to (\Phi_2(X), \Phi_0(X))$$

(7) Lastly, we note that the compositions of the regular maps $\Phi_3$ and $\Phi_4$ is the map

$$\Psi(X) = \Phi_3 \circ \Phi_4 = Ad(g_{j,j+1}(z_j)) \cdots Ad(g_{2,3}(z_2)) \cdot X$$

is regular. This completes the inductive step. Thus by, induction we can find all of the $z_j \in \Xi_{c_j, c_{j+1}}$ as regular functions of $X$ and hence we can define a regular inverse to $\Gamma_n^c$.

We have now proven the following theorem.

**Theorem 7.5.1.** Let $c \in \mathbb{C}^{d/2 + rk n}$ satisfy the orthogonal eigenvalue disjointness condition. Then the fibre $so(n)_c$ is non-empty. Moreover, as an affine variety we have that

$$SO(2)^{d/2} \cong so(n)_c$$

(Where $\Gamma_n^c$ is the map defined in equation (7.48).) Thus, $so(n)_c$ is a smooth, affine variety of dimension $d/2$. (Where $d$ is the dimension of a generic adjoint orbit of $SO(n)$ in $so(n)$).
Remark 7.5.1. Note that as an algebraic group, $SO(2) \cong (\mathbb{C}^\times)^{d/2}$. Thus, Theorem 7.5.1 is the orthogonal analogue of Theorem 3.23 in [KW, pg 45].

Remark 7.5.2. In the case of orthogonal Lie algebras, it is not automatic that the fibre $\mathfrak{so}_c(n)$ is non-empty for $c \in \mathbb{C}^{d/2+rk \ n}$ satisfying the eigenvalue disjointness condition, unlike in the case of $M(n)$. In the case of $M(n)$, we know that all fibres $M_c(n)$ are non-empty for $c \in \mathbb{C}^{(n+1)/2}$ because the map

$$\Phi_n : M(n) \to \mathbb{C}^{n(n+1)/2}$$

given by

$$\Phi_n(X) = (p_{1,1}(X_1), \cdots, p_{i,j}(X_i), \cdots, p_{n,n}(X_n))$$

(Where $p_{i,j}$ is the elementary symmetric polynomial of degree $i - j + 1$ in the eigenvalues of $X_i$) is surjective by Theorem 2.3 in [KW, pg 16].

7.6 The action of $B$ on generic fibres $\mathfrak{so}(n)_c$

We want to use the isomorphism $\Gamma^n_c$ to define an algebraic action of the group $SO(2)^{d/2}$ on the fibre $\mathfrak{so}(n)_c$ for $c \in \mathbb{C}^{d/2+rk \ n}$ satisfying the eigenvalue disjointness condition. We define the action as follows. Let $X \in \mathfrak{so}(n)_c$. If $(\Gamma^n_c)^{-1}(X) = (z_2, \cdots, z_{n-1})$ then

$$g \cdot X = \Gamma^n_c(z'_2z_2, \cdots, z'_{n-1}z_{n-1})$$

(7.49)

for $g = (z_2, \cdots, z_{n-1}) \in SO(2)^{d/2}$ with $z_i \in SO(2)^{rk \ i}$. (Here are identifying the action of $G_i = SO(2)^{rk \ i}$ on $\Xi_i^c \cong SO(2)^{rk \ i}$ via conjugation action of $SO(2)^{rk \ i}$ on itself by left translation, as in the previous sections).

Remark 7.6.1. We note that the above action of $SO(2)^{d/2}$ on $\mathfrak{so}(n)_c$ is a simply, transitive group action on $\mathfrak{so}(n)_c$. We also note that this is an algebraic action of $SO(2)^{d/2}$ on $\mathfrak{so}(n)_c$. We can see this as follows. We want to see that the map $SO(2)^{d/2} \times \mathfrak{so}(n)_c \to \mathfrak{so}(n)_c$ defining the action $(\tilde{z}, X) \to \tilde{z} \cdot X$ with $\tilde{z} \in SO(2)^{d/2}$ is regular. We can realize this map as the composition of the following regular
maps: (We write for $\mathcal{Z} = (z_2, \cdots, z_{n-1})$ with $z_i \in SO(2)^k$.)

\[ \Phi_1 : SO(2)^{d/2} \times \mathfrak{so}(n)_c \to SO(2)^{d/2} \times SO(2)^{d/2} \]

given by

\[ (((z_2, \cdots, z_{n-1}), X) \to ((z_2, \cdots, z_{n-1}), (\Gamma^c_n)^{-1}(X)) \]

\[ \Phi_2 : SO(2)^{d/2} \times SO(2)^{d/2} \to SO(2)^{d/2} \]

given by

\[ (((z'_2, \cdots, z'_{n-1}), (z_2, \cdots, z_{n-1})) \to (z'_2 z_2, \cdots, z'_{n-1} z_{n-1}) \]

\[ \Phi_3 = \Gamma^c_n \]

then the action map is given by the composition

\[ \Phi_3 \circ \Phi_2 \circ \Phi_1 \]

Now, we can easily describe this action by noticing that

\[ g(z'_j z_j) = g(z_j)(z'_j)^{-1} \]  \hspace{1cm} (7.50)

This follows from our definition of $g(z_j)$ as $g(z_j) = h k(z_j)^{-1}$ where $h \in SO(j + 1)$ conjugates the “identity element” given in equation (7.46) into the standard Cartan and $k(z_j)$ is given in equation (7.47). Equation (7.50) follows from the identity $k(z'_j z_j) = z'_j k(z_j)$. This identity follows from the fact that the map in (7.34) was
an $SO(2)$ equivariant isomorphism. Using (7.50), we can write the action as

$$(z'_2, \cdots, z'_{n-1}) \cdot X$$

is given by

$$\text{Ad}(z'_2g_{2,3}^{-1}(z_2)z'_3g_{3,4}(z_3)^{-1}\cdots g_{n-2,n-1}(z_{n-2})z'_n-1 \cdots g_{n-2,n-1}(z_{n-2}) \cdots g_{2,3}(z_2)) \cdot X$$

(7.51)

For $X \in \mathfrak{so}_c(n) \cdot X = \Gamma^n_c(z_2, \cdots, z_{n-1})$. Note the similarities with the Type $A$ case.

The $B$ action on the generic fibres $\mathfrak{so}(n)_c$ is now given by the following theorem.

**Theorem 7.6.1.** For $c \in \mathbb{C}^{d/2+rk_n}$ satisfying the eigenvalue disjointness condition, we have $\mathfrak{so}_c(n)$ is exactly one $B$ orbit of maximal dimension $d/2$. Thus, elements of $\Omega_n$ are strongly regular. On the fibre $\mathfrak{so}_c(n)$ the action of $B$ has the same orbits as a simply transitive algebraic action of the torus $SO(2)^{d/2} \simeq (\mathbb{C}^\times)^{d/2}$.

**Proof:**

We claim that the action of the torus $SO(2)^{d/2}$ defined in (7.49) on the fibre $\mathfrak{so}(n)_c$ has the same orbit structure as the $B$ action on the fibre. To see this, we differentiate the action defined in (7.49) and compute the tangent space to $\mathfrak{so}(n)_c$. We note that since the action of $SO(2)$ is algebraic it is in particular holomorphic on $\mathfrak{so}(n)_c$, which, as a smooth affine subvariety of $\mathfrak{so}(n)$ naturally has the structure of an embedded complex submanifold of $\mathfrak{so}(n)$. To differentiate the action in (7.49), it is easier to use the description of the action given in equation (7.51). We note that differentiating the orbit map given in equation (7.51) at the identity gives us a description of $T_X(\mathfrak{so}(n)_c)$, for $X \in \mathfrak{so}_c(n)$, since the orbit map of a holomorphic group action is a submersion. In this case, it is in fact a diffeomorphism since the action is free (see [Lee, pg 230]).

To differentiate this action of $SO(2)$, we want to have a simple coordinate system on $SO(2)$ in a neighbourhood of the identity. The coordinate system we
use is the following one. Let \( g \in SO(2) \), then \( g \) has the form

\[
\begin{bmatrix}
u & -v \\
v & u
\end{bmatrix}
\]

for \( u, v \in \mathbb{C} \) with \( u^2 + v^2 = 1 \). We claim that for \( g \) in a neighbourhood of the identity that \( g \) can be written as

\[
\begin{bmatrix}
cos(z) & -\sin(z) \\
\sin(z) & \cos(z)
\end{bmatrix}
\]

for a unique \( z \in \mathbb{C} \). We can see this as follows. Consider the map \( \Psi : \mathbb{C} \rightarrow SO(2) \) given by

\[
\Psi(z) = \begin{bmatrix}
cos(z) & -\sin(z) \\
\sin(z) & \cos(z)
\end{bmatrix}
\]

Now, \( \Psi(0) = Id \) and if we differentiate this map at \( z = 0 \) we see that we get

\[
\Psi'(0) = \begin{bmatrix}0 & -1 \\1 & 0\end{bmatrix}
\]

So that the map is non-singular. Now, applying the inverse function theorem, we see that \( \Psi \) is a diffeomorphism from a neighbourhood of \( 0 \in \mathbb{C} \) onto a neighbourhood of the identity in \( SO(2) \).

Since we are only interested in differentiating the orbit map in (7.51) at the identity, we can use this coordinate representation. Thus, we can use as a basis for \( \text{Lie}(SO(2)) \)

\[
\Psi'(0) = \left. \frac{\partial}{\partial z} \right|_{z=0}
\]

For the convenience of the reader, we restate the expression of the orbit map in (7.51) below.

\[
(z_2', \cdots, z_{n-1}') \cdot X
\]

is given by

\[
\text{Ad}(z_2'g_{2,3}(z_2)z_3'g_{3,4}(z_3)^{-1} \cdots g_{n-2,n-1}^{-1}(z_{n-2})z_{n-1}' \cdots g_{n-2,n-1}(z_{n-2}) \cdots g_{2,3}(z_2)) \cdot X
\]
Using the above coordinates, we represent $z_i' \in SO(2)^{rk_i}$ as follows. Let $l = rk_i$ (recall $rk_i$ represents the rank of $\mathfrak{so}(i)$). For $i$ odd we have

$$
\begin{bmatrix}
\cos(z_{i1}) & -\sin(z_{i1}) \\
\sin(z_{i1}) & \cos(z_{i1}) \\
\cos(z_{i2}) & -\sin(z_{i2}) \\
\sin(z_{i2}) & \cos(z_{i2}) \\
\vdots \\
\cos(z_{il}) & -\sin(z_{il}) \\
\sin(z_{il}) & \cos(z_{il})
\end{bmatrix}
$$

For $i$ even we have:

$$
\begin{bmatrix}
\cos(z_{i1}) & -\sin(z_{i1}) \\
\sin(z_{i1}) & \cos(z_{i1}) \\
\cos(z_{i2}) & -\sin(z_{i2}) \\
\sin(z_{i2}) & \cos(z_{i2}) \\
\vdots \\
\cos(z_{il}) & -\sin(z_{il}) \\
\sin(z_{il}) & \cos(z_{il})
\end{bmatrix}
$$

(Recall that we are thinking of $SO(i, \mathbb{C}) \hookrightarrow SO(n, \mathbb{C})$ as embedded in the top left hand corner). To get the differential of the orbit map in coordinates we compute:

$$
\frac{\partial}{\partial z_{ij}} \bigg|_{z_{ij}=0} \text{Ad}(z_i'g_{2,3}^{-1}(z_2) \cdots g_{n-2,n-1}^{-1}(z_{n-2})z_{n-1}' \cdots g_{n-2,n-1}^{-1}(z_{n-2}) \cdots g_{2,3}(z_2)) \cdot X
$$

For $2 \leq i \leq n-1, 1 \leq j \leq l$. For the case of $i = 2l$ even, we get

$$
ad(g_{2,3}^{-1}(z_2) \cdots g_{i-1,i}^{-1}(z_{i-1})A_{ij}g_{i-1,i}(z_{i-1}) \cdots g_{2,3}(z_2)) \cdot X \quad (7.52)
$$
where $A_{ij} \in \mathfrak{so}(i) \hookrightarrow \mathfrak{so}(n)$ is the matrix given by

$$
A_{ij} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
& \
\ddots \\
0 & -1 \\
1 & 0 \\
& \ddots \\
0 & 0 \\
0 & 0
\end{bmatrix}
$$

Similarly, for the case where $i = 2l + 1$ is odd. Now, in the case where $i = 2l$ is even the element $h_i = g_{i-1,i}(z_{i-1}) \cdots g_{3,4}(z_3)g_{2,3}(z_2) \in SO(i)$ conjugates the $i \times i$ cutoff of $X, X_i$ into the standard Cartan in $\mathfrak{so}(i)$. Namely, it conjugates it into a matrix of the form

$$
\begin{bmatrix}
0 & b_1 \\
b_1 & 0 \\
& \\
0 & b_2 \\
-b_2 & 0 \\
& \ddots \\
0 & b_l \\
-b_l & 0
\end{bmatrix}
$$

Recall that we are assuming that each cutoff $X_i$ is regular, so that the centralizer of such a matrix in $\mathfrak{so}(2l)$ is given by:

$$
\begin{bmatrix}
\mathfrak{so}(2) \\
\mathfrak{so}(2) \\
& \ddots \\
\mathfrak{so}(2)
\end{bmatrix}
$$

which has basis given by matrices of the form $A_{ij}$ for $1 \leq j \leq l$. Thus, we have that equation (7.52) gives us tangent vectors of the form

$$
\partial_X^{[h_i^{-1}A_{ij}h_i,X]} \quad \text{(7.53)}
$$
Now, the $A_{ij}, 1 \leq j \leq l$ form a basis for the centralizer of the matrix $h_i X h_i^{-1}$, so that the elements $h_i^{-1} A_{ij} h_i, 1 \leq j \leq l$ form a basis of the centralizer in $\mathfrak{so}(i)$ of $X_i$.

Performing this argument for all $2 \leq i \leq n - 1$, we see that the tangent space at $X \in \mathfrak{so}(n)_c$ is given by

$$T_X(\mathfrak{so}(n)_c) = \text{span}\{\partial_X^{\mathfrak{so}(i)_X,X}\}$$

(where $\mathfrak{so}(i)_X$ denotes the centralizer of $X_i$ in $\mathfrak{so}(i)$).

Now, for $X \in \Omega_n \subset \mathfrak{so}(n)$ this is none other than the subspace $V_X$. Recall that we had in equation (7.2)

$$V_X = \text{span}\{(\partial X^{(d_{i,j})(X_i,X)})_X|2 \leq i \leq n - 1, 1 \leq j \leq \text{rk } i\}$$

where $f_{i,j}$ are fundamental adjoint invariants for $\mathfrak{so}(i)$.

Now if $X_i$ is regular, then we have that $(d f_{i,j})(X_i), 2 \leq j \leq \text{rk } i$ form a basis for the centralizer of $X_i$ (see [K2, pg 382]). Thus, we have

$$T_X(\mathfrak{so}(n)_c) = \text{span}\{\partial_X^{\mathfrak{so}(i)_X,X}\} = V_X$$

(7.54)

Now, we also clearly have that $\dim T_X(\mathfrak{so}(n)_c) = d/2 = \dim SO(2)^{d/2}$. Thus, for $X \in \mathfrak{so}(n)_c$, $\dim V_X = d/2$. So that $\mathfrak{so}(n)_c \subset \mathfrak{so}(n)^\text{sreg}$ and $\mathfrak{so}(n)^\text{sreg}$ is non-empty.

We know from Proposition 7.3.1 that for any $X \in \mathfrak{so}(n)_c$, we have $B \cdot X \subset \mathfrak{so}(n)_c$. Thus, we can write $\mathfrak{so}(n)_c$ as a disjoint union of $B$ orbits,

$$\mathfrak{so}(n)_c = \bigsqcup_{i \in I} B \cdot X(i)$$

with $X(i) \in \mathfrak{so}(n)_c$. Now, the dimension of each $B$ orbit is exactly $d/2$, since

$$\dim B \cdot X(i) = \dim V_X = \dim T_X(\mathfrak{so}(n)_c) = d/2$$

Hence, each $B$ orbit $B \cdot X(i)$ is an open submanifold of $\mathfrak{so}(n)_c$. Now, we argue as we did in the case of $M(n)$. We note that $B \cdot X(i), i \in I$ is given by $\mathfrak{so}(n)_c \setminus \left(\bigsqcup_{j \neq i, j \in I} B \cdot X(j)\right)$, since the union is disjoint. But $\bigsqcup_{j \neq i, j \in I} B \cdot X(j)$ is open, so that $B \cdot X(i)$ is closed. Thus $B \cdot X(i)$ is both open and closed in the connected manifold $\mathfrak{so}(n)_c$. Hence, since $B \cdot X(i) \neq \emptyset$, we must have

$$\mathfrak{so}(n)_c = B \cdot X(i)$$
is exactly one \( B \) orbit. This completes the proof of the theorem.

Q.E.D.

**Remark 7.6.2.** Theorem 7.6.1 generalizes Theorems 3.23 and 3.28 in [KW, pgs 45, 48] for the analogous set of regular semi-simple matrices in \( M(n) \) to the orthogonal Lie algebras \( \mathfrak{so}(n) \).

We have a corollary to Theorem 7.6.1.

**Corollary 7.6.1.** The functions \( \{ f_{i,j} | 2 \leq i \leq n-1, 1 \leq j \leq \text{rk} i \} \) are algebraically independent.

**Proof:**

We now know that \( \mathfrak{so}^{s\text{reg}}(n) \) is non-empty from Theorem 7.6.1, since \( \Omega_n \subset \mathfrak{so}(n)^{s\text{reg}} \). Then, we can apply Theorem 2.7 in [KW, pg 19], which carries over to the orthogonal case by Theorem 7.2.2, to see that there exist a non-empty Zariski open set of elements for which the differentials \( (df_{i,j})_x \) for \( 2 \leq i \leq n, 1 \leq j \leq \text{rk} i \) are linearly independent. The algebraic independence follows immediately.

Q.E.D.

**Remark 7.6.3.** Using this corollary, we see that the Gelfand-Zeitlin Algebra \( GZ(\mathfrak{g}) \) of \( \mathfrak{g} = \mathfrak{so}(n) \) defined in equation (1.22) and the algebra \( J(\mathfrak{g}) = P(\mathfrak{so}(2))^{SO(2)} \otimes \cdots \otimes P(\mathfrak{so}(n))^{SO(n)} \) are both commutative algebras which are free on \( d/2 + \text{rk} n \) generators and hence they are abstractly isomorphic as we promised in section 1.6.

As another corollary to Theorem 7.6.1, we see that we can polarize open submanifolds of adjoint orbits of elements in \( \Omega_n \subset \mathfrak{so}(n) \).

**Corollary 7.6.2.** Let \( X \) in \( \Omega_n \subset \mathfrak{so}(n) \) and let \( \mathcal{O}_X \) be its adjoint orbit. Then we have that \( \mathcal{O}^{s\text{reg}}_X = \mathcal{O}_X \cap \mathfrak{so}(n)^{s\text{reg}} \) is non-empty Zariski open subset of \( \mathcal{O}_X \). The \( B \) orbits of dimension \( d/2 \) in \( \mathcal{O}_X \) form the leaves of a polarization of \( \mathcal{O}^{s\text{reg}}_X \).

**Proof:**

From Theorem 7.6.1, we know that \( \mathcal{O}^{s\text{reg}}_X \) is a non-empty Zariski open subset of \( \mathcal{O}_X \), since \( X \in \mathfrak{so}(n)^{s\text{reg}} \). One argues in exactly the same manner as Theorem 3.36.
in [KW, pg 86] using Proposition 1.5.2 to see that the $B$ orbits of dimension $d/2$ in $\mathcal{O}_X$ are Langrangian submanifolds of $\mathcal{O}_X^{\text{reg}}$. 
Bibliography


