Supersymmetric sigma models, partition functions and the Chern-Gauss-Bonnet Theorem

by

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Abstract

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In the search for a geometric model for elliptic cohomology and the Witten genus, Stolz and Teichner have defined a class of supersymmetric Euclidean field theories in various superdimensions $d|\delta$. With collaborators, they have shown that 1|1-dimensional theories encode the $\hat{A}$-genus and $KO$-theory, and in dimension 2|1 they expect similar constructions to lead to the Witten genus and TMF, the universal elliptic cohomology theory. In this thesis, we investigate super Euclidean field theories in a variety of dimensions with the goal of understanding their role in algebraic topology. We focus on two aspects: (1) the appearance of invariants like the $\hat{A}$-genus and (2) the relationship between field theories and cohomology theories.

Beginning in the early 80s, physicists observed that partition functions of supersymmetric sigma models could frequently be identified with manifold invariants like the Euler characteristic, signature and $\hat{A}$-genus. Making these arguments precise culminated in the heat kernel proof of the index theorem. The first result in this thesis is a structural one: partition functions of supersymmetric sigma models always furnish manifold invariants. We prove this in Chapter 2. This leaves open the difficult question of how one might construct these supersymmetric sigma models.

Chapter 3 is a proof of the Chern-Gauss-Bonnet Theorem, which comes in two parts. We construct the 0|2-sigma model, which we know a priori leads to a manifold invariant. Then, via computations in supergeometry, we identify this invariant with the Euler characteristic. By coupling the sigma model to a constant times a Morse function we obtain a variety of expressions that compute the partition function, and by equating two of these we derive the Chern-Gauss-Bonnet formula.

In Chapter 4, we investigate the relationship between field theories and cohomology theories. Our results are various no-go theorems for when field theories give cocycles for cohomology theories for $\delta > 1$. In dimensions 0|\delta we are able to prove fairly strong statements to this effect, but for $d > 0$ we need to make assumptions motivated by our desired connection between invariants coming from sigma models and the putative cohomology theory coming from the space of field theories.
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Chapter 1

Introduction and background

Supersymmetric field theories provide a language that encodes many familiar objects from the geometry and topology of manifolds. Within this language, there are obvious generalizations that lead one to consider things like Dirac operators and de Rham complexes on loop spaces. Although rigorous constructions of these have yet to be completed, there are many indications from physics that such objects exist and possess many interesting mathematical properties.

Unfortunately, attempting to make the physical reasoning precise requires one to confront and solve problems in higher-category theory: in order for the desired geometric objects to be locally defined on a smooth manifold, one requires that the associated field theories be fully extended or fully local. In low dimensions, e.g., dimension 2, the definitions are quite subtle but seem within reach. Unfortunately robust computations using these tentative definitions are beyond the scope of the current technology. The philosophy of this thesis is to extract 0-categorical data from these hoped-for local field theories in an effort to understand aspects of the general structure that allow for explicit computation.

Our main technical tool is a thorough understanding of 0|δ-dimensional field theories, and a large part of the thesis focusses on these simple examples. We relate this information to general field theories by constructing dimensional reduction functors which have as their source an arbitrary category of field theories and as target 0|δ-dimensional ones. Although information is lost through this map, the target can be computed directly using tools from supergeometry. We leverage these computations to deduce various facts about general supersymmetric field theories.

Chapter 2 seeks to understand the role of supersymmetric partition functions in producing manifold invariants. The following result helps us understanding the general structure behind these examples.

Theorem 1.0.1. The partition function of a twisted supersymmetric field theory is a concordance invariant.

In Chapter 3 we use the above and an explicit construction of the 0|2-dimensional sigma model to prove the Chern-Gauss-Bonnet theorem. The following result is the first important step in this direction.
Theorem 1.0.2. The partition function of the 0|2-dimensional sigma model with target $X$ computes the Euler characteristic of the $X$.

In Chapter 4, we turn to the question of when field theories are related to multiplicative cohomology theories in the manner suggested by Stolz and Teichner [33]. The minimal necessary data to make sense of this question is a sequence of twisted field theories, $d|\delta$-EFT, and desuspension maps, $\int$. Subject to an (essential) assumption motivated by the expected existence of perturbative quantization, we show that such situations are special.

Theorem 1.0.3. Let $(d|\delta$-EFT, $\int, Q)$ be a sequence of twisted field theories equipped with desuspension maps and a compatible perturbative quantization map. If $(d|\delta$-EFT, $\int)$ gives multiplicative cocycles for a cohomology theory, then $\delta = 0$ or 1.

It will take a little while to explain all the terms above results; we will start with the big picture and some motivating questions.

1.1 Motivating examples and questions

Examples of supersymmetric field theories in smooth geometry

In this subsection we outline the point of view that supersymmetric field theories generalize well-known objects in the geometry and topology of manifolds.

Let $M$ be a Riemannian manifold and $E \to M$ a metrized $\mathbb{Z}/2$-graded vector bundle with compatible connection. Quantum mechanics on $M$ twisted by $E$ is the data of the $\mathbb{Z}/2$-graded Hilbert space of sections $L^2(M, E)$ and the 1-parameter family of operators $e^{-t\Delta}$ where $\Delta$ is the Laplacian gotten from the metric on $M$ and connection on $E$. Quantum mechanics is a 1-dimensional QFT in the sense that there is an action of the 1-parameter semigroup $t \in \mathbb{R}_{>0}$ that we think of as “time translation.” Supersymmetric quantum mechanics requires that the infinitesimal generator $\Delta$ of this action have one (or several) odd “square roots.” For example, if $M$ is assumed to be spin and $E$ is the spinor bundle, then the Laplacian has an odd square root, namely the Dirac operator. This is an example of 1|1-dimensional SUSY quantum mechanics, because we have one square root of the Laplacian. An example of 1|2-dimensional SUSY quantum mechanics is furnished by letting $E$ be the bundle of differential forms on $M$, wherein the Hodge Laplacian has two odd square roots, namely $d + d^*$ and $i(d - d^*)$ for $d$ the de Rham operator and $d^*$ its adjoint. Stolz and Teichner repackage this data using the language due to Atiyah and Segal: quantum mechanics on $M$ is a symmetric monoidal functor from a certain bordism category to the category of vector spaces. The relevant bordism category has as objects superpoints and as morphisms super Euclidean paths; we denote this category by $1|\delta$-EBord and the associated field theories by $1|\delta$-EFT.

One might hope for a 2|1-EFT that encodes the spinor bundle on loop space and its Dirac operator, and a 2|2-EFT encoding the differential forms on loop space together with analogs of $d + d^*$ and $i(d - d^*)$. Indeed, it is this sort of reasoning that led to Witten’s construction of
elliptic genera \[37\] which are bordism invariants of string manifolds taking values in modular forms. Although we have yet to understand the full mathematical structure of these theories, physical arguments strongly suggest they will deepen our understanding of manifolds and their free loop spaces.

Families of Dirac operators and de Rham complexes parametrized by a manifold \(X\) probe interesting topology of \(X\) related to \(K\)- and \(L\)-theory, respectively. The connection between \(K\)-theory and \(1|1\)-field theories was made in \[17\], and in this thesis we give some preliminary evidence relating \(1|2\)-field theories to Poincaré duality complexes. Emboldened by this, one might hope families of higher-dimensional SUSY field theories would be similarly interesting; after dealing with several subtle choices, one could define a \(d\)-category of \(d|\delta\)-Euclidean field theories over \(X\) by

\[
d|\delta\text{-EFT}(X) := \text{Fun}^{\otimes}(d|\delta\text{-EBord}(X), d\text{Vect}),
\]

where the right hand side is functors from a bordism \(d\)-category comprised of \(d|\delta\)-dimensional Euclidean supermanifolds to some algebraic \(d\)-category that deloops the category of topological vector spaces.

The \(d\)-categorical definition above is often referred to as a local field theory to distinguish it from the 1-categorical definition due to Atiyah and Segal. There are two reasons why we require this modification. The first is that we wish the invariants of manifolds obtained from field theories to be locally defined, much as the index of the Dirac operator is computed by the integral of a differential form. For \(d > 1\), one cannot expect this from the 1-categorical definition. The second reason stems from the cobordism hypothesis as proved by J. Lurie \[27\]: local topological field theories are more rigid and hence simpler than non-local ones, and we expect this philosophy to persist for super Euclidean field theories.

**Partition functions and manifold invariants**

Dirac operators are studied by topologists because their indices carry topological information about a space, e.g., the \(\hat{A}\)-genus. In the language of supersymmetric field theories, the index of \(D\) is the partition function of the \(1|1\)-Euclidean field theory associated to spinors on a manifold. Similarly, the Euler characteristic \(\chi\) and signature \(\sigma\) are encoded by the partition function of the \(1|2\)-Euclidean field theory associated to the de Rham complex of a manifold. Physical generalizations of these invariants led to the definition of the Witten genus and a construction of the elliptic genus \[37\] as partition functions of certain supersymmetric field theories. This leads to the following question.

**Question 1.1.1.** When is the partition function of a super Euclidean field theory a manifold invariant?

In Chapter 2 we give a flexible definition for the class of super Euclidean field theories that encode geometric data about manifolds (these field theories are called sigma models), and prove that the partition functions of these field theories always give manifold invariants. In this sense, one can view partition functions of supersymmetric sigma models as
generalizations of the familiar invariants $\hat{A}$, $\chi$ and $\sigma$. The remaining obstacle is the difficult problem of actually constructing these sigma models. In Chapter 3 we unwind our definitions and construct the $0|2$-dimensional sigma model; the upshot of this is a proof of the Chern-Gauss-Bonnet Theorem.

Field theories and cohomology theories

One reason Dirac operators are especially powerful in topology is their relation to $K$-theory: the tools of Mayer-Vietoris sequences and other spectral sequences aid many computations. One might hope for a similar situation with field theories, i.e., computational techniques similar to Mayer-Vietoris sequences that allow one to understand partition functions. In this subsection we will briefly explain a relation between field theories and cohomology theories known to exist in certain super dimensions. We will take up the general question in more detail in Chapter 4.

There is an equivalence relation $\sim$ on field theories called concordance such that the assignment $X \mapsto d|\delta\text{-EFT}(X)/\sim$ is a functor from manifolds to pointed sets that preserves homotopy equivalences. Furthermore, after fixing a dimension there exists a Brauer group for field theories whose elements Stolz and Teichner call twists. In good cases they construct a sequence of twists for $\ell \in \mathbb{Z}$ and produce (or conjecture) a bijection $h^\ell(X) \cong d|1\text{-EFT}^\ell(X)/\sim$ for $h^\bullet$ a cohomology theory; in dimensions $0|1$, $1|1$ and $2|1$ they obtain de Rham cohomology, $K$-theory, and (conjecturally) the universal elliptic cohomology, TMF. This leads us to the following question.

**Question 1.1.2.** When is there a natural isomorphism of rings

$$h^\bullet(X) \cong d|\delta\text{-EFT}^\bullet(X)/\sim, \tag{1.1}$$

for $h^\bullet$ a multiplicative cohomology theory, and $d|\delta\text{-EFT}^\bullet$ a sequence of twisted super Euclidean field theories?

In Chapter 4, with some additional assumptions we answer this question in the negative when $\delta > 1$. We emphasize that these assumptions do not shut the door completely on a positive answer to the question, and we have set up the discussion to provide useful lemmas for checking the isomorphism (1.1) in particular examples.

1.2 A brief review of super geometry

Before proceeding we need to pin down some notation and conventions. All tensor products of super vector spaces and super algebras will be $\mathbb{Z}/2$-graded, and the braiding comes with the usual minus sign of super linear algebra. We write $\text{SMfld}$ for the category of (real) supermanifolds, and refer the reader to [9, 19, 25] for preliminaries. In brief, objects in this category are locally ringed spaces, $M^{n|m} = ([M]^n, C^\infty)$, where $C^\infty$ is a sheaf of real
superalgebras locally isomorphic to $C^\infty(\mathbb{R}^n) \otimes \Lambda^*(\mathbb{R}^m)^*$. We write $|M|$ for the $n$-manifold $([M], C^\infty/\text{nilpotents})$, called the reduced manifold of $M$. Since smooth manifolds admit partitions of unity, morphisms of supermanifolds are determined by the induced map on global sections of the sheaf $C^\infty$, whence the slogan “supermanifolds are affine.” We will use this fact without comment throughout. We will sometimes consider a variant of the above, wherein $C^\infty$ is a sheaf of $\mathbb{C}$-algebras; these objects form a category of cs-manifolds, and when we need to draw a distinction we will denote this category by $\text{csMfld}$. Many discussions work equally well in either category, so when we wish to be flexible $\text{SM}$ will denote either of these two categories.

We will frequently use the parity reversal functor $\pi$. It has a few incarnations:

1. for $A$ a superalgebra, $\pi: \text{Mod}_A \to \text{Mod}_A$ takes a (left or right) $A$-module to the parity reversed (left or right) module;

2. $\pi: \text{SVBund} \to \text{SVBund}$ takes a super vector bundle over a supermanifold to the parity reversed bundle; and

3. $\pi: \text{SVBund} \to \text{SM}$ takes a super vector bundle to the total space of the parity reversed bundle.

When these distinctions matter we will be explicit.

**Generalized supermanifolds and the functor of points**

Let $\text{SM}(M, N)$ denote the set of maps between supermanifolds $M$ and $N$, and $\text{SM}(M, N)$ the inner hom, i.e., the functor

$$\text{SM}(M, N): \text{SM}^{\text{op}} \to \text{SET}, \quad S \mapsto \text{SM}(S \times M, N).$$

Similarly, we define $\text{Diff}(M)$ as the functor

$$\text{Diff}(M)(S) = \left\{ \begin{array}{c} S \times M \\ \cong \\ \circ \\ S \end{array} \right\}. \quad (1.2)$$

The above are examples of functors of points, or for emphasis, functors of $S$-points. These functors may not be representable as supermanifolds, meaning there may not exist a natural isomorphism with a functor

$$\tilde{M}: \text{SM}^{\text{op}} \to \text{SET}, \quad S \mapsto \text{SM}(S, M) =: M(S),$$

where $M$ is some supermanifold. Still, much of supermanifold theory utilizes the functor of points rather than the supermanifold itself, and a surprising amount can be done with
nonrepresentable supermanifolds. We denote the category of set-valued presheaves and morphisms of presheaves by $gSM$, and refer to its objects as generalized supermanifolds.

In particular, one can identify the smooth functions on a generalized supermanifold: the functor $C^\infty$ that takes a supermanifold $M$ to its superalgebra of functions with the functor $M \mapsto SM(M, \mathbb{R}^{1|1})$ where addition and multiplication on the image are defined using structures on $\mathbb{R}^{1|1}$. The grading on this algebra comes the the involution $\alpha$ of $\mathbb{R}^{1|1}$ determined by $\alpha^* : C^\infty(\mathbb{R})[\theta] \to C^\infty(\mathbb{R})[\theta], \theta \mapsto -\theta$

where we have identified $C^\infty(\mathbb{R}^{1|1}) \cong C^\infty(\mathbb{R})[\theta]$. Using the Yoneda Lemma, the morphisms $SM(M, \mathbb{R}^{1|1})$ are determined by natural transformations between the functors $M$ and $\mathbb{R}^{1|1}$. Such a natural transformation is a map of sets

$SM(S, M) \to SM(S, \mathbb{R}^{1|1}) \cong C^\infty(S)$.

Hence, maps of sets $M(S) \to C^\infty(S)$ natural in $S$ are in bijection with functions on $M$. This makes sense for $M$ a generalized supermanifold. Being a functor valued in algebras, generalized supermanifolds have an algebra of functions. In fact, since objects in the category of generalized super manifolds can be written as a coequalizer of supermanifolds and equalizers of nuclear vector spaces are nuclear, one can talk about the nuclear super vector space of functions on a generalized supermanifold. This fact was explained to me by Dmitri Pavlov.

Even if a generalized supermanifold is representable, whenever we refer to a point $\Phi$ of $M$, we implicitly mean a map $\Phi : S \to M$, so $\Phi \in M(S)$. The reason for this is that although the ordinary points of a supermanifold tell us very little (namely, $|M|$) the $S$-points of $M$ tell us everything by the Yoneda lemma.

**Integration on supermanifolds, by example**

Sections of $\Lambda^{\top}X$ on an ordinary manifold can be integrated; similarly, sections of the Berezinian line on a supermanifold can be integrated. In this thesis we will need rudimentary aspects of this integration theory. For a throughout treatment see the article by Deligne and Morgan [9]. Here we just give a few examples.

**Example 1.2.1** (Relative integration). The important case to consider is relative integration for the trivial bundle $\mathbb{R}^{n|m} \times S \to S$. For any other family $M \to S$, the relative integration is locally of this form. On this bundle, the relative Berezinian is a $C^\infty_{\mathbb{R}^{n|m}}$-module of rank 1 if $m$ is even, and rank 0 if $m$ is odd. A choice of coordinates $\theta_1, \ldots, \theta_m$ of $\mathbb{R}^{0|m}$ induces a trivialization of this module, and we denote trivializing section by $[d\theta_1 \cdots d\theta_m]$. If we tensor the relative Berezinian with the relative orientation bundle, we get a map

$$\int_{\mathbb{R}^{n|m} \times S/S} : C^\infty(\mathbb{R}^{n|m} \times S) \to C^\infty(S).$$

Consider first the case where $n = 0$. To Evaluate on a function, we Taylor expand in $\theta_i$ and project to the component of $\theta_1 \cdots \theta_m$, obtaining a function on $S$. When $n \neq 0$, first we

\begin{align*}
\int_{\mathbb{R}^{n|m} \times S/S} : C^\infty(\mathbb{R}^{n|m} \times S) & \to C^\infty(S) \\
\int_{\mathbb{R}^{n|m} \times S/S} C^\infty(\mathbb{R}^{n|m} \times S) & \to C^\infty(S) \\
\int_{\mathbb{R}^{n|m} \times S/S} [d\theta_1 \cdots d\theta_m] & \to [d\theta_1 \cdots d\theta_m]
\end{align*}
project, obtaining a function on $C^\infty(\mathbb{R}^n \times S)$. Next we use an orientation form on $\mathbb{R}^n$ to integrate down to $C^\infty S$.

**Example 1.2.2** (Fermionic Gaussians). The following is standard and can be found, for example, in [15]. Let $q$ be a quadratic form on a purely odd supervector space $V$ of even dimension. Note that for vectors $\omega, \eta \in V$, super quadratic means

$$q(\omega, \eta) = -q(\eta, \omega).$$

Thinking of functions on $V$ as being an exterior algebra, we see $q$ is in the 2nd antisymmetric power, so in particular, $q \in C^\infty(V)$. We claim

$$\int_V \exp \left( -\frac{1}{2} \tilde{q} \right) = \det(q)^{1/2}.$$

To see this, choose coordinates $C^\infty(V) \cong \mathbb{R}[\theta_1, \ldots, \theta_{2n}]$ such that $q$ is a skew matrix of the form

$$q = \begin{pmatrix}
\lambda_1 J & 0 & \ldots & 0 \\
0 & \lambda_2 J & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n J
\end{pmatrix}, \quad J = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}. \quad (1.3)$$

The Berezinian integral projects to the top component of $\exp(-q/2)$, which is

$$(-2)^n \frac{1}{n!} q^n = \lambda_1 \cdots \lambda_n \theta_1 \cdots \theta_n.$$

Thus, the value of the integral is the product of the $\lambda_i$, whereas the determinant of $q$ is

$$\det(q) = \lambda_1^2 \cdots \lambda_n^2,$$

which verifies the claim.

**Remark 1.2.3.** Compare the above with the usual Gaussian integral,

$$(2\pi)^{m/2} \int_W \exp \left( -\frac{1}{2} \tilde{q} \right) = \frac{1}{\det(q)^{1/2}}.$$ 

for $W$ an even (i.e., bosonic) $m$-dimensional vector space.

### 1.3 Smooth stacks

Throughout, a **prestack** will be a presheaf of essentially small groupoids on the category of generalized supermanifolds. We will often implicitly identify a supermanifold $S$ with a
prestack via the Yoneda embedding: \( S \) defines a presheaf of sets, which we view as a presheaf of discrete groupoids. We can make the category of generalized supermanifolds into a site by declaring a covering to be a morphism \( Y \to M \) of generalized supermanifolds such that for any representable \( S \) and morphism \( \phi: S \to M \), the pullback covering \( \phi^*Y \to S \) is a surjective submersion of ordinary supermanifolds. We remark that this site is equivalent to the site of (representable) supermanifolds and surjective submersions. A stack is a prestack satisfying descent; see \([5, 10, 20]\) for nice introductions.

A (symmetric) monoidal stack \( \mathcal{G} \) is a presheaf of essentially small (symmetric) monoidal groupoids that satisfies descent in the symmetric monoidal sense: the natural functor from \( \mathcal{G}(S) \) to the descent category associated to a cover of \( S \) is an equivalence of (symmetric) monoidal categories. See Definition 41 of \([18]\).

A stack \( \mathcal{G} \) is called smooth or differentiable if there is a surjective morphism \( S \to \mathcal{G} \) for \( S \) a generalized super manifold regarded as a stack; this morphism is called an atlas for \( \mathcal{G} \). We will typically view smooth stacks as objects in the bicategory associated to the double category of super groupoids and left principle bibundles in generalized supermanifolds. This allows us to identify a groupoid \( \mathcal{G} \) with a stack we denote by \([\mathcal{G}]\). An introduction to the relation between smooth stacks and groupoids can be found in Section 2 of C. Blohmann’s article \([5]\). In brief, a smooth stack (up to isomorphism) is a Morita equivalence class of a Lie groupoid.

Let \( \mathcal{G} \) denote a groupoid with objects \( \mathcal{G}_0 \) and morphisms \( \mathcal{G}_1 \). Source, target and unit maps will be denoted \( s, t, u \), respectively. Most of our examples of stacks will arise as quotient groupoids, denoted \( \mathcal{G} = M//G \) coming from the action of a group \( G \) on a (generalized super) manifold \( M \); here \( \mathcal{G}_0 \cong M, \mathcal{G}_1 \cong M \times G \), the source map is the projection, \( s = p \), the target map is the action of \( G \) on \( M \), \( t = \text{act} \) and the unit map is the inclusion of \( M \) into \( M \times G \) along the identity element of \( G \).

### Vector bundles on quotient stacks

We wish to review the notion of vector bundle over a smooth stack of the form \([M//G]\). One can define an \( n \)-dimensional vector bundle on a stack as a left principle bibundle morphism to the stack \([\text{pt}//\text{Gl}_n]\); however, a more useful (though equivalent) definition for our purposes is that a vector bundle is a natural assignment to any \( S \)-point of the stack a vector bundle over \( S \), and natural isomorphisms of vector bundles for any morphism in the groupoid associated to the \( S \)-points of the stack. For example, a vector bundle over a quotient stack is precisely a \( G \)-equivariant vector bundle over \( M \).

A useful way of constructing vector bundles over \( M//G \) takes as input data a smooth \( M \)-family of \( G \)-representations. The construction is most easily explained in the functor of points. An \( S \)-point of \( M//G \) is a principal \( G \)-bundle \( P \) over \( S \) equipped with a \( G \)-equivariant map \( P \to M \). Hence, from a family of representations \( \rho: G \times M \to \text{End}(V) \) we can form an associated bundle \( P \times_{\rho} V \) to the \( S \)-point \( P \), and recover a vector bundle over \( S \) that is natural in \( S \). We observe that any isomorphism of principle \( G \)-bundles over \( S \) leads to an isomorphic associated bundle. This defines a vector bundle in our functor of points definition. We have
the following pair of useful formulae for determining sections of line bundles $L_\rho$ arising from this construction for $\rho: G \to \mathbb{C}^\times$ or $\rho: G \to \mathbb{R}^\times$:

\[
\Gamma(M//G, L_\rho) \cong \{ x \in C^\infty(M) | \text{act}^*(x) = p_1^*(x) \cdot p_2^*(\rho) \in C^\infty(M \times G) \},
\]

\[
\Gamma(M//G, \pi L_\rho) \cong \{ x \in C^\infty(M)^{\text{odd}} | \text{act}^*(x) = p_1^*(x) \cdot p_2^*(\rho) \in C^\infty(M \times G) \},
\] (1.4)

where $p_1: M \times G \to M$, $p_2: M \times G \to G$ are the projections and act: $M \times G \to M$ is the action; see Corollary 39 of [18] for a proof.

**Concordance for prestacks**

Let $\mathcal{F}: \text{Man}^{\text{op}} \to \text{Grpd}$ be a prestack.

**Definition 1.3.1.** Two sections $\phi_+, \phi_- \in \mathcal{F}(X)$ are **concordant** if there exists a section $\Phi \in \mathcal{F}(X \times \mathbb{R})$ such that $i^*_\pm \Phi \cong \pi^*_\pm \phi_\pm$ where

\[
i_+: X \times (1, \infty) \hookrightarrow X \times \mathbb{R}, \quad \pi_+: X \times (1, \infty) \to X
\]

\[
i_-: X \times (-\infty, -1) \hookrightarrow X \times \mathbb{R}, \quad \pi_-: X \times (-\infty, -1) \to X
\]

are the usual inclusion and projection maps. Concordance defines an equivalence relation; we denote the set of sections up to concordance by $\mathcal{F}[X]$ and the concordance class of a section $\phi$ by $[\phi]$.

Taking concordance classes gives a contravariant functor from manifolds to sets, and we observe that for a smooth map $f: X \to Y$, the map $\mathcal{F}[f]$ only depends on the smooth homotopy class of the map $f$.

### 1.4 Super Euclidean and super conformal geometries

We begin by briefly explaining rigid geometries, see [17], Section 6.3 for details. As input data we take a pair $(\mathbb{M}, G)$ for $\mathbb{M}$ a supermanifold with an action by a (generalized) super Lie group $G$. Then an $(\mathbb{M}, G)$-supermanifold is a supermanifold $Y$ together with a maximal atlas whose charts are diffeomorphic to open submanifolds of $\mathbb{M}$ and whose transition functions arise from restricting the action of $G$ to these open submanifolds. When the action of $G$ is not effective, one must impose a cocycle condition. For details, see Section 2.5 of [33]. Families of $(\mathbb{M}, G)$-manifolds parametrized by a supermanifold $S$ are defined similarly: locally there are diffeomorphisms to $S \times V_i \subset S \times \mathbb{M}$, and gluing data comes from $S$-families of group elements, i.e., smooth maps $S \to G$.

Isometries $Y \to Y'$ of families of $(\mathbb{M}, G)$-manifolds are commutative squares

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & Y' \\
\downarrow & & \downarrow \\
S & \xrightarrow{f} & S'
\end{array}
\]
that respect the fiberwise model geometry as follows: for charts \((U, \phi)\) of \(Y\) and \((U', \phi')\) of \(Y'\) such that \(\hat{f}(U) \subset U'\), the map \(\phi' \circ \hat{f} \circ \phi^{-1}\) is of the form \((s, m) \mapsto (f(s), g(s)m)\) for some \(g \in G\) acting on a subset of \(M\). For commutative squares above with \(S = S'\) and \(Y = M \times S\) for a \((M, G)\)-manifold \(M\), the above defines an \(S\)-point of a group we denote by \(\text{Iso}(M)\).

The above sketches the definition of the category of families of \((M, G)\)-manifolds and their isometries. We remark that this defines a stack: to \(S\) we assign the groupoid of \((M, G)\)-manifolds and isometries over \(S\).

Super model spaces

In this subsection we describe super Euclidean and super conformal model spaces, following Freed [12] and Stolz-Teichner [33]. We give definitions for both real supermanifolds and cs-manifolds.

**Definition 1.4.1.** Given data:

1. a real \(d\)-dimensional inner product space \(V\);
2. a real (resp. complex) spinor representation \(\Delta\) of \(\text{Spin}(V)\) having real (resp. complex) dimension \(\delta\);
3. a \(\text{Spin}(V)\)-equivariant, nondegenerate symmetric pairing \(\Gamma: \Delta^* \otimes \Delta^* \to V\) (resp. \(\Gamma: \Delta^* \otimes \Delta^* \to V \otimes \mathbb{C}\)) where \(\text{Spin}(V)\) acts on \(V\) via the double covering \(\text{Spin}(V) \to \text{SO}(V)\)

the super Euclidean space is the supermanifold \(E := V \times \pi \Delta\) with an action of the super group \(\text{Euc}(E) := E \rtimes \text{Spin}(V)\) where \(\text{Spin}(V)\) acts on \(\Delta\) via the given spinor representation and on \(V\) through the double cover \(\text{Spin}(V) \to \text{SO}(V)\) and \(E\) is given a multiplication

\[(v, \sigma) \cdot (w, \tau) = (v + w + \Gamma(\sigma, \tau), \sigma + \tau), \quad (v, \sigma), (w, \tau) \in E(S)\]

where we interpret the above equation in terms of the functor of points for supermanifolds (resp. cs-manifolds). We call \(\text{Euc}(E)\) the super Euclidean group. The pair \((E, \text{Euc}(E))\) defines a model super Euclidean geometry. There is an analogously defined unoriented super Euclidean geometry where the \(\text{Spin}(V)\)-representation is replaced by a \(\text{Pin}(V)\)-representation.

The vector subspace \(\pi \Delta^* \subset V \oplus \pi \Delta^* \cong E\) defines a subspace of the tangent space of \(E\) at the identity element. Left-translating this subspace generates a distribution on \(E\) we denote (in an abuse of notation) by \(\Delta\). By construction, this distribution is preserved by the left action of \(E\) on itself.

**Definition 1.4.2.** The super conformal group, denoted \(\text{Conf}(E)\) is the group of orientation preserving diffeomorphisms of \(E\) that preserve the odd distribution determined by \(\Delta\) at the origin in \(E\); the pair \((E, \text{Conf}(E))\) determine a model super conformal geometry. The unoriented super conformal group consists of diffeomorphisms preserving \(\Delta\) that are not required to preserve the orientation.
As usual, the super Lie algebra of a super Lie group consists of the left-invariant vector fields. Since $\mathbb{E}$ is a super group structure on a super vector space, the Lie algebra of $\mathbb{E}$ is isomorphic (as a super vector space) to $V \oplus \pi\Delta^*$. Since $V < \mathbb{E}$ is central, the Lie bracket is trivial except for

$$[\sigma, \tau] = -2\Gamma(\sigma, \tau), \quad \sigma, \tau \in \pi\Delta^*.$$ 

In particular an odd vector $\sigma$ has $[\sigma, \sigma] = -2\sigma^2 \in V$ gives an “odd square root” of an even translation in $V$, as per the informal discussion in the introduction to Chapter 1. Below we can give explicit descriptions of this Lie algebra structure by choosing coordinates on $\mathbb{R}^{d|\delta}$ and computing the left-invariant vector fields.

We will frequently use the notation $\mathbb{E}^{d|\delta}$ for the superspace $\mathbb{E}$ to emphasize the dimensions involved. In some cases real supermanifolds or cs-manifolds will be preferred in the definition of the super Euclidean model geometries, as we will see in examples below. For emphasis, in these examples we will use the notation $\mathbb{R}^{d|\delta}$ to denote the supermanifold $(\mathbb{R}^d, \Lambda^*(\mathbb{R}^\delta)^*)$ and $\mathbb{R}^{d|\delta}_{cs}$ to denote the cs-manifold, $(\mathbb{R}^d, \Lambda^*_c(\mathbb{R}^\delta)^*)$.

Example 1.4.3 (1|1-dimensional geometries). The group $\text{Spin}(1) \cong \mathbb{Z}/2$ acts on a 1-dimensional vector space by reflection. We choose a pairing $\Gamma$ that gives rise to the supergroup,

$$(t, \theta) \cdot (t', \theta') = (t + t' + \theta\theta', \theta + \theta'), \quad (t, \theta), (t', \theta') \in \mathbb{R}^{1|1}(S),$$

yielding the super Euclidean group,

$$\text{Euc}(\mathbb{E}^{1|1}) \cong \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2,$$

and the model geometry $(\mathbb{E}^{1|1}, \text{Euc}(\mathbb{E}^{1|1}))$. This model geometry makes sense for both real and cs-supermanifolds.

An unoriented Euclidean geometry requires an action of $\text{Pin}_-(1) \cong \mathbb{Z}/4$ on $\mathbb{R}$; for this to be interesting we need to complexify, i.e., use cs-manifolds where the generator acts on $\mathbb{R}^{1|1}_{cs}$ by $(t, \theta) \mapsto (-t, i\theta)$.

The Lie algebra of the translation group, $\mathbb{E}^{1|1}$, consists of the left-invariant vector fields $D := \partial_0 - \theta \partial_t$ and $\partial_t$. We observe that $[D, D] = -2\partial_t$, so that the Lie algebra of $\mathbb{E}^{1|1}$ is free on the single generator $D$.

The distribution $\Delta$ defining the superconformal group is generated by the vector field $D$. The group preserving this distribution is infinite dimensional. As in the Euclidean case, an unoriented version requires that one use cs-manifolds.

Example 1.4.4 (1|2-dimensional geometries). Again, we have $\text{Spin}(1) \cong \mathbb{Z}/2$, and we choose a pairing that leads to the following group structure on $S$-points of $\mathbb{R}^{1|2}$:

$$(t, \theta_1, \theta_2) \cdot (t', \theta'_1, \theta'_2) = (t + t' + \theta_1\theta'_1 + \theta_2\theta'_2, \theta_1 + \theta'_1, \theta_2 + \theta'_2), \quad (t, \theta_1, \theta_2), (t', \theta'_1, \theta'_2) \in \mathbb{R}^{1|2}(S).$$

Hence, the Euclidean group is $\text{Euc}(\mathbb{E}^{1|2}) \cong \mathbb{R}^{1|2} \rtimes \mathbb{Z}/2$. An unoriented version of this takes the action of $\text{Pin}_-(1)$ on $\mathbb{R}^{1|2}_{cs}$ to be generated by

$$(t, \theta_1, \theta_2) \mapsto (-t, i\theta_1, i\theta_2).$$
There are numerous variations on these model geometries in dimension 1|2: any subgroup $R < O(2)$ acts on $\mathbb{R}^2$ through the usual 2-dimensional representation. We call this group $R$ the $R$-symmetry group. This leads to a model geometry via the supergroup $\text{Euc}_R(\mathbb{R}^{1|2}) \cong \mathbb{R}^{1|2} \rtimes R$

where the standard Euclidean geometry is a special case with $R = \{ \pm \text{Id} \}$. The classical 1|2-dimensional sigma model has a group of symmetries of this form, where $R = O(2)$.

There is an unoriented version of these geometries with isometry group coming from $\tilde{R} < \text{Pin}(2)$ sitting in the pushout diagram

$$
\begin{array}{ccc}
\mathbb{Z}/2 & \rightarrow & \mathbb{Z}/4 \\
\downarrow & & \downarrow \\
R & \rightarrow & \tilde{R}
\end{array}
\rightarrow 
\begin{array}{c}
\mathbb{Z}/2
\end{array}
$$

(1.5)

where we assume $\{ \pm \text{Id} \} < R$. We will explain the diagram for the universal case that $R = O(2)$; then $\mathbb{Z}/2 \leftarrow O(2)$ as $1 \mapsto -\text{id}$, and $\mathbb{Z}/2$ includes into $\mathbb{Z}/4$ by sending 1 to twice the generator of $\mathbb{Z}/4$. The representation of $\tilde{R}$ on $\mathbb{R}^2 \otimes \mathbb{C}$ is determined by the standard representation of $O(2)$ on $\mathbb{R}^2$, and the generator of $\mathbb{Z}/4$ acting as in the standard unoriented 1|2-Euclidean geometry. This gives an unoriented model geometry with $R$-symmetry, $\text{Euc}_{\tilde{R}}(\mathbb{E}^{1|2}) \cong \mathbb{R}^{1|2} \rtimes \tilde{R}$.

**Example 1.4.5** (2|1-dimensional geometries). Let $V = \mathbb{R}^2$ and $\Delta$ be the 1-dimensional complex representation of $\text{Spin}(2) \cong U(1)$ where $U(1)$ acts by clockwise rotation, and the pairing $\Gamma$ is the composition

$$
\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \hookrightarrow \mathbb{C} \oplus \mathbb{C} \cong \mathbb{R}^2 \otimes \mathbb{C}
$$

where the middle arrow includes $\mathbb{C}$ into the copy of $\mathbb{C}$ where $U(1)$ acts by clockwise rotation. This gives a super Euclidean space $\mathbb{E}^{2|1}$ with Euclidean super group $\mathbb{R}^{2|1}_c \rtimes \text{Spin}(2)$; the group structure on $\mathbb{R}^{2|1}_c$ is

$$(z, \bar{z}, \theta) \cdot (z', \bar{z}', \theta') = (z + z', \bar{z} + \bar{z}' + \theta \theta', \theta + \theta'), \quad (z, \bar{z}, \theta), (z', \bar{z}', \theta') \in \mathbb{R}^{2|1}_c$$

where we are using the standard notation where $z, \bar{z} \in C^\infty(\mathbb{R}^{2|1}_c)$ are complex conjugates in $C^\infty(\mathbb{R}; \mathbb{C})$ after taking the quotient by the ideal generated by nilpotent functions. We observe that cs-manifolds are required for this example, as there are no spinor representations of Spin(2) of real dimension 1.

**Example 1.4.6** (2|2-dimensional geometries). Let $V = \mathbb{R}^2$ and $\Delta$ be the 2-dimensional complex representation of $\text{Spin}(2) \cong U(1)$ where $U(1)$ acts by clockwise rotation on one copy of $\mathbb{C}$, and counterclockwise rotation on the other. We take as pairing $\Gamma$ the composition

$$(\mathbb{C} \oplus \mathbb{C}) \otimes (\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{C}^4 \rightarrow \mathbb{C} \oplus \mathbb{C} \cong \mathbb{R}^2 \otimes \mathbb{C}$$
where the middle projects to the copy of $\mathbb{C} \oplus \mathbb{C}$ where $U(1)$ acts by clockwise rotation on one copy and counterclockwise rotation on the other. This gives a super Euclidean space $\mathbb{E}_{2|2}$ with Euclidean super group $\mathbb{R}_{cs}^{2|2} \rtimes \text{Spin}(2)$; the group structure on $\mathbb{R}_{cs}^{2|2}$ is

$$(z, \bar{z}, \theta_1, \theta_2) \cdot (z', \bar{z}', \theta_1', \theta_2') = (z + z' + \theta_1 \theta_1', \bar{z} + \bar{z}' + \theta_2 \theta_2', \theta_1 + \theta_1', \theta_2 + \theta_2').$$

In this example different choices of spinor representation and pairing $\Gamma$ give different model geometries; these are either isomorphic to the above under “conjugation,” i.e., the exchange $z \leftrightarrow \bar{z}$ (though this is not conjugation) or the chiral model geometries that have the flavor of a product geometry between the ordinary Euclidean $\mathbb{R}$ and the above $1|2$-Euclidean geometry, e.g., we have the group structure on $\mathbb{E}_{2|2}$,

$$(z, \bar{z}, \theta_1, \theta_2), \ (z', \bar{z}', \theta_1', \theta_2') \in \mathbb{R}_{cs}^{2|2}.$$
Felix Klein, determine a (rigid) geometry. For simplicity, in this subsection we will discuss
the situation for supermanifolds; the story for the category of cs-manifolds is identical.

With a choice of rigid geometry \((\mathcal{M}, G)\), we can sketch the definition of the bordism
category. First we fix some family parameter \(S\). The objects and morphisms (respectively)
parametrized by \(S\) in \(d|\delta\)-\(\text{Bord}\) are fiber bundles with base \(S\) whose fibers are \(d|\delta\)-
and \((d−1)|\delta\)-supermanifolds (respectively) equipped with a rigid geometry. Both the objects
and the morphisms form a groupoid, where we take bundle automorphisms that are isometries
in the fibers with respect to the geometry. The symmetric monoidal structure—as usual—
comes from disjoint union, but now this happens fiberwise over \(S\). The above structures can
be pulled back along any map of supermanifolds \(S' \rightarrow S\), and in fact \(d|\delta\)-\(\text{Bord}\) is a category
internal to symmetric monoidal stacks on the site of supermanifolds; see [33] Definition 2.21
and Definition 2.46 for details.

To define field theories we need a target category. For the purposes of this thesis, we have
quite a lot of flexibility in our choice; nearly all of our arguments apply to any category of
vector spaces whose symmetric monoidal unit is \(\mathbb{R}\) or \(\mathbb{C}\). However, for the sake of concreteness
one can consider the fibered category \(TV\) ([33] Definition 2.47) whose object symmetric
monoidal stack is comprised of bundles of locally convex, Hausdorff topological super vector
spaces over supermanifolds and vector bundle automorphisms; the morphism stack consists
of super vector bundle morphisms with only identity automorphisms. The graded projective
tensor product turns these stacks into symmetric monoidal ones. We will have occasion to
consider both real and complex vector spaces, and will refer to the associated field theories
as field theories over \(\mathbb{R}\) or \(\mathbb{C}\), respectively. When the bordism category has as objects and
morphisms cs-manifolds, we always work over \(\mathbb{C}\).

The category \(TV\) has a \(\mathbb{Z}/2\)-action or \text{flip} coming from the grading involution on super
vector spaces (see [33], Section 2.6). Often the bordism category will also have a flip arising
from some kind of natural geometric involution, in which case one may wish to define field
theories as flip preserving functors. For example, categories of bordisms with spin structures
can be equipped with the spin flip, and identifying this with the grading involution on super
vector spaces is a form of the spin statistics theorem from physics. Super Euclidean bordism
categories have a similar flip, coming from the element of Spin(\(V\)) that acts trivially on \(V\).
Although crucial in certain applications, in this thesis the flips will play a very minor role.

A final key idea due to Segal is to equip each \(S\)-family of bordisms with a map to a
fixed smooth manifold \(X\), resulting in a bordism category over \(X\) denoted \(d|\delta\)-\(\text{Bord}(X)\).
We can also consider bordisms over any generalized manifold, or in fact any stack on the
site of manifolds. For example, classical gauge theories are related to field theories over the
stack \(\text{pt}///G\); see Example 2.3.19 below.

In summary, we define field theories over \(X\) as flip preserving, fibered, symmetric monoidal
functors between internal categories ([33] Definitions 2.48 and 4.12)

\[
d|\delta\text{-GFT}(X) := \text{Fun}^{\mathcal{G}}_{\text{SM}}(d|\delta\text{-Bord}(X), TV)^{\mathbb{Z}/2},
\]

(1.6)

where \(\text{SM} = \text{SMfld}\) or \(\text{csMfld}\).
Examples of $1|\delta$-dimensional super Euclidean field theories

In this subsection, we give a variety of motivational examples of $1|\delta$-dimensional field theories for $\delta = 1, 2, 4$. The super Euclidean geometries are always taken to be unoriented. Many of our descriptions of field theories rely on results of Hohnhold, Stolz and Teichner in [17].

Example 1.5.1 ($1|1$-EFTs over the point). In [17], the authors show that examples of flip preserving functors

$$1|1\text{-EFT}(\text{pt}) \cong \text{Fun}_{\text{SM}}^\otimes(1|1\text{-EBord}(\text{pt}), \text{sVect})^{\mathbb{Z}/2}$$

are determined by

1. a vector space $V$ with inner product $\langle -, - \rangle$ and $\mathbb{Z}/2$-grading, and
2. an odd, self-adjoint Fredholm operator $D$ acting on $V$.

The flip-preserving condition means that the spin flip on $1|1$-Euclidean manifolds acts by the grading involution on the Hilbert space. The odd operator $D$ determines a $1|1$-dimensional semigroup; geometrically this comes from gluing together super Euclidean intervals in the bordism category.

The prototypical example of this structure is the Dirac operator $D$ acting on the spinors of an even dimensional manifold. The associated semigroup is $\exp(-tD^2 + \theta D)$ for $(t, \theta) \in \mathbb{R}_{>0}^{1|1}$ acting on the sections of the spinor bundle.

Example 1.5.2 ($1|1$-EFTs over $X$). Examples of field theories $E \in 1|1\text{-EFT}(X)$ arise from $\mathbb{Z}/2$-graded vector bundles with grading-preserving connection over $X$. By constructing super parallel transport, Florin Dumitrescu showed that these are in fact examples of $1|1$-conformal field theories over $X$ [11], and there are evident maps,

$$\text{sVect}(X) \hookrightarrow 1|1\text{-CFT}(X) \hookrightarrow 1|1\text{-EFT}(X).$$

Explicitly, to an $S$-family of (connected) objects $\phi: S \times \mathbb{R}^{0|1} \to X$ we assign the pullback of the bundle $E$ along $\phi$ and to a family of morphisms connecting these objects, we assign the morphism of vector bundles induced by super parallel transport along the $S$-family of paths.

Example 1.5.3 ($1|2$-EFTs). As discussed above in Example 1.4.4, there are a number of possible $1|2$-dimensional geometries, and part of our goal in this and the next few examples is to explain the way in which these choices are reflected in algebraic structures on chain complexes. We begin with the simplest example of the $1|2$-Euclidean geometry where $\text{Euc}(1|2) \cong \mathbb{R}_{cs}^{1|2} \rtimes \mathbb{Z}/2$. We observe from the group structure on $\mathbb{R}_{cs}^{1|2}$ defined in Example 1.4.4 the Lie algebra of $\mathbb{R}_{cs}^{1|2}$ is presented by a pair of commuting odd elements, denoted $D_1$ and $D_2$ such that $[D_1, D_1] = [D_2, D_2]$. Any (smooth) representation of $\mathbb{R}_{cs}^{1|2}$ can therefore be identified with an action of $D_1, D_2$ satisfying these relations.
Following flip-preserving functors

\[ 1\text{-}2\text{-EFT}(X) := \text{Fun}_{\text{SM}}(1\text{-}2\text{-EBord}(X), \text{sVect})^{\mathbb{Z}/2} \]

can be built from

1. a vector space \( V \) with a pairing \( \langle -, - \rangle \) and a \( \mathbb{Z}/2 \)-grading, and

2. a pair of commuting, self-adjoint Fredholm odd operators \( D_1 \) and \( D_2 \) acting on \( V \) that satisfy \( D_1^2 = D_2^2 \).

As in the 1|1-dimensional case, flip preserving means the grading involution on \( V \) arises from the \( \mathbb{Z}/2 \)-subgroup of the model geometry and the \( \mathbb{R}_{>0} \) semigroup generated by \( D_1 \) and \( D_2 \) comes from gluing superintervals in the bordism category. An example of this structure is the de Rham complex of an oriented Riemannian manifold where

\[ V := \Omega^\bullet(X), \quad D_1 = d + d^*, \quad D_2 = i(d - d^*) \]

We observe that the operators \( D_i \) are self adjoint and \( V \) is \( \mathbb{Z}/2 \)-graded by the mod 2 reduction of the usual \( \mathbb{Z} \)-grading. The pairing on \( V \) is the usual Hodge pairing,

\[ \langle \omega, \eta \rangle = \int_X \omega \wedge * \bar{\eta} \]

where * is the Hodge star.

Remark 1.5.4. We observe that the linear combinations

\[ d := \frac{1}{2} (D_1 - iD_2), \quad d^* := \frac{1}{2} (D_1 + iD_2) \]

give a pair of square zero operators on \( V \) and (as suggested by the notation) \( d^* \) is the adjoint of \( d \). Hence, from a 1|2-EFT as above, we can extract a \( \mathbb{Z}/2 \)-graded vector space and odd square zero operator.

Example 1.5.5 (1|2-Euclidean field theories with \( SO(2) \)-symmetry). Suppose that we take \( R = SO(2) \) in the definition of the \( R \)-extended Euclidean geometry in Example 1.4.4. Translating this into the data of a field theory, we obtain an action of \( SO(2) \) on \( V \) which determines a \( \mathbb{Z} \)-grading. The operators \( d \) and \( d^* \) defined in Remark 1.5.4 have \( \mathbb{Z} \)-grading +1 and −1 respectively.

The de Rham complex of an oriented Riemannian manifold again gives an example of this structure, where now \( \Omega^\bullet(X) \) has its \( \mathbb{Z} \)-grading. More generally, any cochain complex in topological vector spaces, \((V^\bullet, d)\), where \( V \) has an inner product and \( d + d^* \) is Fredholm gives an example of a 1|2-EFT with \( SO(2) \)-symmetry.
Example 1.5.6 (1|2-Euclidean field theories with $O(2)$-symmetry). If we take $R = O(2)$, we still have a $\mathbb{Z}$-graded vector space $V$ given by the action of $SO(2) \subset O(2)$. There is an additional $\mathbb{Z}/2$-action coming from the extension to $O(2)$; we denote by $\star$ the action of the element $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in O(2)$. A representation of $O(2)$ that intertwines compatibly with the $\mathbb{R}_{>0}$ semi-group representation satisfies

$$
\star D_1 \star = -D_1 \quad \star D_2 \star = D_2 \\
[r, D_1] = -iD_2 \quad [r, D_2] = iD_1
$$

(1.7)

where $r$ is the generator of the Lie algebra of $SO(2)$.

We claim that an example of this structure again comes from the de Rham complex of an oriented Riemannian manifold, where now the dimension is required to be $2n$. We normalize the Hodge-$\star$ operator to one we denote by $\star$ with the property that $\star^2 = 1$. Explicitly, when acting on $k$-forms

$$
\star := \begin{cases} 
\star & k \text{ even} \\
 i\star & k \text{ odd}
\end{cases}
$$

We observe that $\star$ is self-adjoint. Now we define the operators $D_1$ and $D_2$ that when acting on $k$-forms as

$$
D_1 = \begin{cases} 
d + \star d\star & k \text{ even} \\
 i(d + \star d\star) & k \text{ odd}
\end{cases} \quad D_2 = \begin{cases} 
i(d - \star d\star) & k \text{ even} \\
 d - \star d\star & k \text{ odd}
\end{cases}
$$

The action of $S^1 < O(2)$ is determined by a shifted grading on forms

$$
R(\theta) \cdot \omega = z^{k-n} \omega \quad \omega \in \Omega^k(X).
$$

Notice that middle degree forms are in degree zero. It is easy to check that the required relations among these operators are satisfied, so that this defines a representation of the semigroup $\mathbb{R}_{>0}^{1|2}$ that is compatible with the $O(2)$-action on forms.

More generally, a cochain complex in topological vector spaces $(V^\bullet, d)$ with a hermitian inner product $\langle -, - \rangle$ and involution $\star$ with the property that $d^* = \star d\star$ and $d+d^*$ is Fredholm gives a 1|2-EFT. We observe that the kernel of $(d+d^*)^2$ is always finite dimensional, and since $\star$ commutes with $(d+d^*)^2$, the restriction of the pairing $\langle -, \star - \rangle$ to the kernel is always a perfect pairing.

Other examples of 1|2-EFTs arising from geometry include twisted de Rham complexes of oriented Riemannian manifolds, $(\Omega^\bullet(X), d+\alpha)$ for $\alpha$ a closed 1-form. In particular, we can consider the complex arising from the differential $d + \lambda dh$ for $\lambda \in \mathbb{R}$ and $h$ a Morse function. In the limit $\lambda \to \infty$, one obtains the Morse complex as described by Witten in [36]. This is a 1|2-EFT that has $SO(2)$-symmetry, but not $O(2)$-symmetry: the Hodge-$\star$ “acts” by sending the complex arising from Morse function $h$ to the one from the Morse function $-h$, which is not an involution of the complex.
Example 1.5.7 (1|2-EFTs over X). An example of a field theory in 1|2-EFT(\(X\)) is a \(\mathbb{Z}/2\)-graded vector bundle with flat grading preserving connection. These bundles are in fact objects of 1|2-CFT(\(X\)), and we have maps

\[
\text{SVect}_{\text{flat}}(X) \hookrightarrow 1|2\text{-CFT}(X) \hookrightarrow 1|2\text{-EFT}(X).
\]

To explain the flatness condition, consider a vector bundle with connection \((E, \nabla)\) over \(X\) and a map \(\gamma: \mathbb{S} \times I^{1|2} \to X\). We need to produce maps \(\Gamma(\mathbb{S} \times \mathbb{R}^{0|2}, \gamma_0^*E) \to \Gamma(\mathbb{S} \times \mathbb{R}^{0|2}, \gamma_1^*E)\) where \(\gamma_0\) and \(\gamma_1\) denote the restrictions of the map \(\gamma\) to the start and endpoints of the path, respectively. To do this, we consider the bundle \(\gamma^*E\) over \(\mathbb{S} \times I^{1|2}\) and the vector fields \(D_1 = \partial_{\theta_1} + \theta_1 \partial_t\) and \(D_2 = \partial_{\theta_2} + \theta_2 \partial_t\) along \(I^{1|2}\). We use the connection to lift these vector fields to the total space of \(\gamma^*E\); the flows generated by these vector fields on the total space give maps \(\Gamma(\mathbb{S} \times \mathbb{R}^{0|2}, \gamma_0^*E) \to \Gamma(\mathbb{S} \times \mathbb{R}^{0|2}, \gamma_1^*E)\). However, we need the action by these flows to commute for this to behave well under gluing of super intervals \(I^{1|2}\), i.e., to define the analog of a representation of the super semigroup \(\mathbb{R}^{1|2}_{>0}\). These vector fields commute on the total space if and only if the connection to be flat.

Example 1.5.8 (1|4-EFTs). Analogously to the 1|2-dimensional case, examples of \(E \in 1|4\text{-EFT}(\text{pt})\) arise from the de Rham complex of a Kähler manifold: the 4 commuting square roots of the Laplacian are infinitesimal generators for the action of a 1|4-dimensional semigroup. Here we are considering the simplest Euclidean group, \(\mathbb{R}^{1|2} \rtimes \mathbb{Z}/2\), but one can enlarge this to \(\mathbb{R}^{1|4} \rtimes R\) for various groups \(R\) to encode more of the algebra of the de Rham complex of a Kähler manifold.

Twisted field theories

Below we will also consider twisted field theories. To explain these, first we need to sketch the definition of the bicategory of super algebras, bimodules and intertwiners internal to symmetric monoidal stacks, denoted \(\mathbf{TA}\). The object stack is comprised of bundles of topological super algebras and bundle maps; the 1-morphism stack is bundles of topological bimodules and bundle automorphisms; the 2-morphism stack is that of bimodule maps and identity automorphisms (all \(\mathbb{Z}/2\)-graded). The projective tensor product endows this internal category with a symmetric monoidal structure. For a more careful definition of \(\mathbf{TA}\), see Definition 5.1 of [33].

Definition 1.5.9. Twisted field theories are natural transformations \(E \in d|\delta\text{-GFT}^T(X)\)
where $\mathcal{T}$ is some fixed functor called the \textit{twist} and $\mathbf{1}$ is the constant functor to the symmetric monodial unit of $\mathbf{T} \mathcal{A}$, i.e., $\mathbb{R}$ or $\mathbb{C}$ as a bimodule over itself. In an abuse of notation, above we consider $\mathcal{d}|\delta-\text{Bord}(X)$ as a bicategory internal to symmetric monodial stacks whose objects and 1-morphisms are the same as the usual $\mathcal{d}|\delta-\text{Bord}(X)$, and whose 2-morphisms are isometries of $\mathcal{d}|\delta$-bordisms.

Twists pull back along smooth maps $f: Y \to X$; we simply precompose $\mathcal{T}$ with the induced functor $f_*: \mathcal{d}|\delta-\text{Bord}(Y) \to \mathcal{d}|\delta-\text{Bord}(X)$. In particular, a twist over the point leads to twisted field theories naturally defined over any manifold via the canonical map $Y \to \text{pt}$,

$$f^*: \mathcal{d}|\delta-\text{GFT}^T(\text{pt}) \to \mathcal{d}|\delta-\text{GFT}^{f^*T}(Y) =: \mathcal{d}|\delta-\text{GFT}^T(Y). \quad (1.8)$$

Twisted field theories arise naturally in a variety of contexts: anomalies in Chern-Simons theory [13], central charges and modular functors in conformal field theory [32], or a degree when comparing field theories to cohomology theories [33].

\section*{Examples of 1-dimensional twisted field theories}

**Example 1.5.10.** Ignoring the family parameter for the moment, we can define $\mathcal{T}_A$ as the functor that assigns the algebra $A$ to any connected object of $\mathcal{d}|\delta-\text{Bord}(X)$ and $A$ as an $A$-$A$ bimodule to any morphism in $\mathcal{d}|\delta-\text{Bord}(X)$. Then a $\mathcal{T}_A$-twisted field theory is more-or-less an ordinary field theory valued in $A$-modules. Putting the families back in, $\mathcal{T}_A$ assigns to any $S$-family of connected $0|\delta$-manifolds an $S$-family of the algebra $A$. Using the fact that stackification is the left adjoint to the forgetful functor from stacks to prestacks, it suffices to define $\mathcal{T}_A$ for trivial $S$-families of connected $0|\delta$-manifolds; in this case the value of $\mathcal{T}_A$ is the bundle of algebras $S \times A$. Similarly, we get $S \times A$ as a bundle of $A$-$A$-bimodules for any $S$-family of superintervals. Using the monoidal structure, this defines the twist functor $\mathcal{T}_A$.

**Example 1.5.11** (Clifford linear $1|1$-EFTs). We can define a twist, denoted $n$, for $1|1$-EFTs via the $n$-dimensional Clifford algebra, $\text{Cl}_n$, and the resulting field theories $E \in 1|1-\text{EFT}^n(\text{pt})$ will have as data \textit{Clifford linear} odd operators $D$ on a $\mathbb{Z}/2$-graded Hilbert space. The isomorphism $\text{Cl}_n \otimes \text{Cl}_m \cong \text{Cl}_{n+m}$ gives maps

$$\otimes: 1|1-\text{EFT}^n \times 1|1-\text{EFT}^m \to 1|1-\text{EFT}^{n+m},$$

gotten from using the tensor product on the category $\mathcal{T} \mathcal{V}$. This is closely related to the cup product in K-theory. Since $\text{Cl}_8$ is Morita equivalent to $\mathbb{R}$, we observe that this twist is $8$-periodic.

We can produce examples of this structure via the $\text{Cl}_n$-linear Dirac operator on a manifold. For a closed Riemannian spin manifold $M$ of dimension $n$ with principle spin bundle $P$, we define the associated bundle,

$$\$(M) := \text{Spin}_M \times_{\text{Spin}(n)} \text{Cl}_n.$$
This is a bundle of $\mathbb{Z}$/2-graded vector spaces with fibers as a point $x$ irreducible $Cl(T_x M)$-$Cl_n$-bimodules. The Dirac operator for this bundle is $Cl_n$-linear, so taking sections we obtain a $\mathbb{Z}$/2-graded vector space $\Gamma(S(M))$ with an action of $Cl_n$ and a $Cl_n$-linear odd operator $D$ acting on this vector space.

By adding a positivity restriction on the functors defining 1|1-EFTs, a classifying space of $Cl_n$-linear 1|1-EFTs can be identified with the $KO$-theory spectrum (see [17]).

**Example 1.5.12** (Clifford linear 1|2-EFTs). In this example we wish to mimic the story for $Cl_n$-linear 1|1-EFTs, but we will motivate our choices from the 1|2-dimensional sigma model. We claim that geometric quantization of this sigma model requires an orientation $Cl$ structure for a section $v$ whose fiber at a point is a ($Cl$ $\Omega$ where $\iota$ dual to skew-adjoint (compare Roe [30] Lemma 3.21) and that

$$i$$ The Dirac operators associated to this pair of Clifford actions are $D_R$ $R$ basis of $\Omega$ and we obtain maps $\Lambda$ $\delta$ $d$ $\iota$ $Cl$: 1|2-EFTs via the $R$-symmetry in the above discussion: the group $O(2)$ acts as in Example [4.5.6] on $\Omega^*(X)$, and in a similar fashion on the exterior algebra $\Lambda^*(\mathbb{R}^n)$, using an orientation of $\mathbb{R}^n$ to define a Hodge star.

With this geometric motivation in mind, we define a degree $n$ twist for 1|2-EFTs via the algebra $Cl_n \otimes Cl_n$. We observe that this twist is 4-periodic since

$$(Cl_n \otimes Cl_n) \otimes (Cl_m \otimes Cl_m) \cong Cl_{n+m} \otimes Cl_{n+m},$$

and we obtain maps

$$\otimes: 1|2\text{-EFT}^n \times 1|2\text{-EFT}^m \to 1|2\text{-EFT}^{n+m}.$$
CHAPTER 1. INTRODUCTION AND BACKGROUND

We observe that super Euclidean, conformal, or super Poincaré geometries have tori. Given a model geometry with tori, we can consider the moduli stack of tori in the given geometry, denoted $\mathcal{M}_T$, whose $S$-points are families of geometric manifolds, where the supermanifold underlying the total space is isomorphic to a bundle with fibers super tori, $T^{d|\delta} \cong \mathbb{R}^{d|\delta}/\mathbb{Z}^d$. This defines a subcategory of the bordism category, and the inclusion $\mathcal{M}_T \to d|\delta\text{-GBord}(pt)$ gives a functor

$$d|\delta\text{-GFT}(pt) \to C^\infty(\mathcal{M}_T)^{ev}.$$ 

**Definition 1.5.14.** The partition function $Z_E \in C^\infty(\mathcal{M}_T)$ of a field theory $E \in d|\delta\text{-GFT}(pt)$ coming from a model geometry with tori is the value of $E$ on tori $\mathbb{R}^{d|\delta}/\mathbb{Z}^d$ for $\mathbb{Z}^d \subset \mathbb{R}^d \subset \mathbb{R}^{d|\delta}$ a lattice. Explicitly, for each $S$-family of such tori in the bordism category, a field theory returns an even function on $S$, so that we have a natural transformation

$$Z_E(S): \text{SMfld}(S, \mathcal{M}_T) \to C^\infty(S)^{ev},$$

i.e., an even function on $\mathcal{M}_T$. For a fixed torus $\mathbb{T}^{d|\delta}$ viewed as a family over $S = pt$, the number $E(\mathbb{T}^{d|\delta})$ is the partition function evaluated at $\mathbb{T}^{d|\delta}$. For twisted field theories, the partition function is a section of the line bundle over $\mathcal{M}_T$ determined by the twist.

**Remark 1.5.15.** The above definition is a bit restrictive: we are only considering the value of the field theory on tori (as opposed to all closed $d|\delta$-manifolds) and we are only considering the tori that come from particular actions of $\mathbb{Z}^d$ on $\mathbb{R}^{d|\delta}$. The former restriction isn’t so terrible for $2|\delta$-Euclidean field theories, since the only closed, flat 2-manifolds are tori. However, the latter is a real restriction that we use as a technical simplification. This is in keeping with the conventions in [33] for partition functions of $2|1$-dimensional Euclidean field theories. However, one may wish to generalize this definition in certain future applications.

**Example 1.5.16.** Let $1|1\text{-EFT}(X)$ denote the groupoid of field theories over $X$ determined by the unoriented super Euclidean geometry of Example 1.4.3. For field theories $E \in 1|1\text{-EFT}(pt)$ constructed as in Example 1.5.1, we claim that the partition function is the index of the operator $D$. The crucial pieces of the argument come from the proof of Proposition 3.1.1 in [34]. First we cut the circle $S_t^{1|1} \cong \mathbb{R}^{1|1}/t \cdot \mathbb{Z}$ “in half,” meaning we factorize the original morphism $S_t^{1|1}$ from the empty set to the empty set as a morphism from the empty set to a pair of super points and then back to the empty set:

$$\emptyset \xrightarrow{L_{t/2}} \text{spt} \coprod \text{spt} \xrightarrow{\alpha \coprod \text{id}} \text{spt} \coprod \text{spt} \xrightarrow{R_{t/2}} \emptyset,$$

where $L_{t/2}$ is the left half of the circle, $R_{t/2}$ is the right half of the circle, and $\alpha \coprod \text{id}$ is the coproduct of the spin flip and the identity automorphism of the point; we require this last morphism to obtain a super circle $\mathbb{R}^{1|1}/t \cdot \mathbb{Z}$ rather than the circle where the generator of $\mathbb{Z}$ acts by reflection in the odd direction. This amounts to the underlying ordinary Euclidean circle $[\mathbb{R}^{1|1}/t \cdot \mathbb{Z}]$ having the periodic spin structure rather than the antiperiodic one.
Now we wish to apply our field theory $E$ to the cobordism in the above displayed equation. Let the value of the field theory on the super point be $V$, a topological vector space. Then the factorization gives a morphism $\mathbb{C} \to V \otimes V$ determined by $1 \mapsto e^{-tD^2/2} \in V \otimes V$, where we identify $V \otimes V$ with trace class operators on the Hilbert space $V$, and a morphism $V \otimes V \to \mathbb{C}$ given by the formula $(v \otimes w) \mapsto \langle e^{-tD^2/2}v, e^{-tD^2/2}w \rangle$. These two morphisms are “glued” along $\alpha \Box \text{id}$ where $\alpha$ is the grading involution on $V$ (where we are using the field theory are flip-preserving functors). The composition returns the trace of $\alpha \circ e^{-tD^2}$, which we identify with the super trace of $e^{-tD^2}$. By the McKean-Singer formula this is the index of $D$. In particular, we observe that this number is independent of $t$, and so the partition function is a constant function on the moduli space of 1|1-Euclidean circles of the form $\mathbb{R}^{1|1}/t \cdot \mathbb{Z}$.

We observe that in the case that $D$ is a Dirac operator acting on the spinor bundle of a manifold, the partition function is the $\hat{A}$-genus.

**Example 1.5.17.** In complete parallel to the above, partition functions of field theories $E \in 1|2$-EFT$(\text{pt})$ arising from the construction in Example 1.5.3 are also indices of the operator $D_1$ (or equivalently, $D_2$). In the example of the de Rham complex of a manifold described above, this index is the Euler characteristic as a constant function on the moduli space of 1|2-Euclidean circles of the form $\mathbb{R}^{1|2}/\mathbb{Z}$. In more detail, we utilize the same factorization of the cobordism as in the previous example (though now super points have two odd dimensions rather than one) and observe that $D^2$ is the Laplacian, $\Delta = (d + d^*)^2 = (i(d - d^*))^2$, and so the value of the field theory on super circles $\mathbb{R}^{1|2}/t \cdot \mathbb{Z}$ is the super trace of the Laplacian, which is the index of the de Rham operator $d + d^*$. Again by the McKean-Singer formula, this is the Euler characteristic.
Chapter 2

Partition functions and manifold invariants

A motivating object of study in supersymmetric field theories is the partition function of a sigma model: one can identify invariants such as the $\hat{A}$-genus or the Euler characteristic with certain supersymmetric partition functions. The physically-motived constructions of these familiar invariants can be made mathematically precise in a number of ways, and often physical reasoning allows one to observe interesting interplays between the geometry and topology of manifolds. For example, the heat kernel proofs of the index theorem arose from making arguments about the path integral precise. We will give another (perhaps easier) example of this philosophy in a proof of the Chern-Gauss-Bonnet theorem via the 0|2-dimensional sigma model in the next chapter.

Natural generalizations of these physical constructions lead to more complicated invariants, such as the Witten genus and the elliptic genus; see Table 2.1 where conjectures are marked by a “?”.

<table>
<thead>
<tr>
<th>dim</th>
<th>partition function</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
</tr>
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<td>2</td>
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<td>1</td>
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Table 2.1: Supersymmetric partition functions in various dimensions.
Throughout this chapter, $d\delta\text{-GFT}^T : \text{Man}^{\text{op}} \to \text{Grpd}$ will be a prestack arising from a $\mathcal{T}$-twisted field theory associated to a model geometry that has tori in the sense of Definition 1.5.13.

A main result of this chapter is Theorem 1.0.1 that we restate here.

**Theorem 2.0.18.** The partition function of a twisted supersymmetric field theory $E \in d\delta\text{-GFT}^T(\text{pt})$ is a concordance invariant.

To tie this result to manifold invariants, we will give a definition of a sigma model below; the input data is a smooth manifold $X$ together with some geometric data $g$ on $X$ such as a Riemannian metric, an orientation, an almost complex structure, or a spin structure. We call the pair $(X, g)$ the *target* of the sigma model.

**Corollary 2.0.19.** The partition function of a quantized supersymmetric sigma model with target $(X, g)$ is an invariant of $(X, [g])$ where $[g]$ denotes the concordance class of $g$.

The proofs of these results are not merely formal consequences of our definitions; instead they rely on a construction of *dimensional reduction functors*

$$\text{red}_T : d\delta\text{-GFT}^* (X) \to 0\delta\text{-GFT}^* (X)$$

for each super torus $T$, together with a sort of Stokes' theorem for $0\delta$-dimensional field theories. Dimensional reduction functors generalize the construction of F. Han [16] that realizes a geometric lift of the Chern character as a functor from $1\delta$-Euclidean field theories to $0\delta$-dimensional ones.

**The idea of the proof of Theorem 2.0.18**

For simplicity, we will restrict to the case of untwisted field theories for this discussion; the twisted case is completely analogous.

A conceptual ingredient in our proof of Theorem 2.0.18 is to view $d\delta$-dimensional geometric field theories ($d\delta$-GFTs, for short) as a generalized space whose functor of points is

$$X \mapsto d\delta\text{-GFT}(X) := \text{Fun}_{\text{SM}}^\otimes (d\delta\text{-GBord}(X), TV).$$

The groupoid $d\delta\text{-GFT}(X)$ has as objects symmetric monoidal functors fibered over the category of supermanifolds and as morphisms natural isomorphisms.

The partition function extracts from a field theory its value on tori, so produces a function on the moduli stack $\mathcal{M}_T$ of tori equipped with a $G$-geometry. We can evaluate this function at a particular torus, $\mathbb{T}$, yielding a commutative diagram

$$d\delta\text{-GFT}(\text{pt}) \xrightarrow{Z} C^\infty(\mathcal{M}_T) \xrightarrow{\circ} C^\infty(\mathcal{M}_T) \xrightarrow{\text{ev}_T} \mathbb{K}$$

(2.3)
where the map $\text{ev}_T$ denotes evaluation, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ depending on the target category of vector spaces. Since the moduli stack $\mathcal{M}_T$ is comprised of ordinary lattices in $\mathbb{R}^d$ modulo an action determined by the geometry, the values $\text{ev}_T(Z_E)$ for varying $T$ encode the partition function of $E$. Theorem 2.0.18 states that if we are given a field theory $\tilde{E} \in d|\delta\text{-GFT}(\mathbb{R})$ and denote the restrictions to $\{0,1\} \in \mathbb{R}$ by $E_0$ and $E_1$, we have $Z_{E_0} = Z_{E_1}$ as functions. We view $\tilde{E}$ as a path in the prestack $d|\delta\text{-GFT}$ connecting the points $E_0$ and $E_1$. We would like to associate to $\tilde{E}$ a path in some space of partition functions; however, a priori, the map $Z$ is not packaged as a morphism of prestacks.

The main construction that proves Theorem 2.0.18 is a lift of diagram 2.3 to one of prestacks

$$
\begin{array}{ccc}
    d|\delta\text{-GFT} & \xrightarrow{Z} & d|\delta\text{-GFT}_0 \\
    \text{red}_T \downarrow & \circlearrowleft & \downarrow \text{ev}_T \\
    0|\delta\text{-GFT} & \rightarrow &
\end{array}
$$

where $\text{red}_T$ is the dimensional reduction map (2.1) and $d|\delta\text{-GFT}_0, 0|\delta\text{-GFT}$ are presheaves of sets with $d|\delta\text{-GFT}_0(\text{pt}) \cong C^\infty(\mathcal{M}_T)$ and $0|\delta\text{-GFT}(\text{pt}) \cong \mathbb{K}$. In a certain sense, $0|\delta$-dimensional field theories form a discrete space for $\delta > 0$, and so the above factorization implies partition functions are invariant under concordances.

More precisely, Proposition 2.2.18 implies the above diagram gives an induced map on concordance classes,

$$
\begin{array}{ccc}
    d|\delta\text{-GFT}[\text{pt}] & \xrightarrow{[Z]} & d|\delta\text{-GFT}_0[\text{pt}] \\
    \downarrow \text{[red}_T] & \circlearrowleft & \downarrow \text{[ev}_T] \\
    0|\delta\text{-GFT}[\text{pt}] & \rightarrow &
\end{array}
$$

and we show that the quotient map $0|\delta\text{-GFT}(\text{pt}) \rightarrow 0|\delta\text{-GFT}[\text{pt}]$ is an isomorphism. Hence, the values of the partition function at various tori $T$ can be read off from the concordance class of the field theory, and so the partition function itself is a concordance invariant.

**Remark 2.0.20.** In order for the assignment (2.2) to be a stack, field theories need to be local, i.e., need to be $d$-functors between suitable defined $d$-categories $d|\delta\text{-GBord}(X)$ and $\mathbb{T}V$. For these (super) geometric bordism categories, such $d$-categorical definitions are still in development. Our constructions will only make use of the “top level” of the categories in question (i.e., the closed bordisms) and so our results apply to any bordism $d$-category that extends the current 1-categorical definition.

### 2.1 $0|\delta$-Dimensional field theories

We begin this chapter with a study of very low-dimensional theories and leverage this in order to say something about higher dimensional ones. The rationale for this approach is
that in dimensions $0|\delta$ all higher categorical complexities disappear—$0|\delta$-field theories are just certain functions on a finite dimensional supermanifold.

**Notation 2.1.1.** The following applies to both real supermanifolds and cs-manifolds, so (following the notation in Chapter 1) $\text{SM}$ denotes either of these two categories. We will use $\mathbb{K}$ to denote $\mathbb{R}$ in the setting of real super manifolds and $\mathbb{C}$ in the setting of cs-manifolds; the super space $\mathbb{R}^{0|\delta}$ will have as functions $\mathbb{K}[\theta_1, \ldots, \theta_\delta]$.

In [18] the authors show that the internal 0-category of $0|\delta$-geometric bordisms over $X$ is the free symmetric monoidal category on the connected bordisms. This comes from the geometric fact that every $0|\delta$-dimensional supermanifold is a coproduct of connected ones. We denote this category by $0|\delta\text{-}\text{GBord}_{\text{conn}}(X)$ (with “conn” standing for “connected”), and emphasize that it has no symmetric monoidal structure. Furthermore, this 0-category internal to stacks has a presentation by the quotient groupoid in supermanifolds

$$0|\delta\text{-}\text{GBord}_{\text{conn}}(X) \cong \text{SM}(\mathbb{R}^{0|\delta}, X)/\text{ISO}(\mathbb{R}^{0|\delta}),$$

where $\text{SM}(\mathbb{R}^{0|\delta}, X)$ denotes the inner hom in generalized supermanifolds and $G := \text{ISO}(\mathbb{R}^{0|\delta}) < \text{Diff}(\mathbb{R}^{0|\delta})$ is a chosen group of isometries of $\mathbb{R}^{0|\delta}$ defining a model geometry. We observe that $\text{ISO}(\mathbb{R}^{0|\delta})$ is the internal automorphism group of the unique object of $0|\delta\text{-}\text{GBord}_{\text{conn}}(\text{pt})$.

With a little work, one can understand fibered functors to $\mathbb{K}$ as functions,

$$0|\delta\text{-}\text{GFT}(X) = \text{Fun}_\text{SM}^\otimes(0|\delta\text{-}\text{GBord}(X), \mathbb{K})$$

$$\cong \text{Fun}_\text{SM}(0|\delta\text{-}\text{GBord}_{\text{conn}}(X), \mathbb{K}) \cong C^\infty(0|\delta\text{-}\text{GBord}_{\text{conn}}(X))^{\text{ev}},$$

where $\mathbb{K}$ is the representable stack given by the supermanifold $\mathbb{K}$ and has the structure of a commutative monoid $(\mathbb{K}, \times)$ when considering symmetric monoidal functors. The first isomorphism uses the fact that the free functor is left adjoint to the forgetful functor. Functions on a groupoid are the invariant functions, $C^\infty(0|\delta\text{-}\text{GBord}_{\text{conn}}(X)) \cong C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))^{\text{ISO}(\mathbb{R}^{0|\delta})}$, and for this thesis we take this as a definition of $0|\delta$-dimensional field theories:

$$0|\delta\text{-}\text{GFT}(X) := \left( C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))^{\text{ISO}(\mathbb{R}^{0|\delta})} \right)^{\text{ev}}.$$
0|0-\text{EFT}(X) \cong C^\infty(\text{SM}(\mathbb{R}^{0|0}, X)/\{\text{id}\}) \cong C^\infty(X)^\text{ev}.

\textbf{Example 2.1.3 (0|1-\text{EFT}(X)).} Following \cite{[18]}, we choose the Euclidean group 
\text{Euc}(\mathbb{R}^{0|1}) := \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2 < \text{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times,

and use the fact that $C^\infty(\text{SM}(\mathbb{R}^{0|1}, X)) \cong C^\infty(\pi TX) \cong \Omega^\bullet(X)$, where differential forms are regarded as being $\mathbb{Z}/2$-graded via mod 2 reduction of the usual de Rham grading. We claim

$0|1-\text{EFT}(X) \cong C^\infty(\pi TX/((\mathbb{R}^{0|1} \rtimes \mathbb{Z}/2))^\text{ev} \cong \Omega^\text{ev}_\text{cl}(X),

where $\Omega^\text{ev}_\text{cl}$ denotes the sheaf of closed, even differential forms. Indeed, functions on the quotient groupoid are functions on $\pi TX$ invariant under the group action; an exercise (sketched in Example 2.1.20) shows that the infinitesimal action of $\mathbb{R}^{0|1}$ on $\Omega^\bullet(X)$ is precisely the de Rham $d$, and the $\mathbb{Z}/2$ action is by the grading involution. Hence, functions fixed under these two actions are $d$-closed and of even degree. These computations are carried out in detail in \cite{[18]}.

To obtain the odd forms we require \textit{twisted} field theories. Specializing definition 1.5.9 a \textit{twist} is a line bundle on $0|\delta-\text{GBord}_{\text{conn}}(X)$, and a \textit{twisted field theory} is a section of this line bundle. Note that a section of the trivial line bundle is just a function, which was our notion of an (untwisted) field theory above. In summary, we define

$0|\delta-\text{GFT}^\mathcal{L}(X) := \Gamma(\text{SM}(\mathbb{R}^{0|\delta}, X)//\text{Iso}(\mathbb{R}^{0|\delta}), \mathcal{L})^\text{ev}.$

We observe that for any smooth map $f: X \to Y$ we have a functor $0|\delta-\text{GBord}_{\text{conn}}(X) \to 0|\delta-\text{GBord}_{\text{conn}}(Y)$ which in turn induces $f^* : 0|\delta-\text{GFT}^\mathcal{L}(Y) \to 0|\delta-\text{GFT}^\mathcal{L}(X)$. Hence, if $\mathcal{L}$ is a line bundle on $0|\delta-\text{GBord}_{\text{conn}}(pt)$, the canonical map $p: X \to pt$ produces a line bundle $p^* \mathcal{L}$ on $0|\delta-\text{GBord}_{\text{conn}}(X)$ for each $X$. Such line bundles $\mathcal{L}$ come from a homomorphism of super Lie groups, $\rho: \text{Iso}(\mathbb{R}^{0|\delta}) \to K^\times$; when such a 1-dimensional representation $\rho$ is fixed, we will use the notation $\mathcal{L}_\rho$. Sections of $p^* \mathcal{L}_\rho$ are functions on $\text{SM}(\mathbb{R}^{0|\delta}, X)$ that are \textit{equivariant} with respect to the induced action of $\text{Iso}(\mathbb{R}^{0|\delta})$ on $K$ via $\rho$.

The above description shows that the assignment $X \mapsto 0|\delta-\text{GFT}^{p^* \mathcal{L}}(X)$ is natural in $X$ and defines a sheaf on manifolds. In fact, we can obtain a sheaf of graded rings on manifolds whose degree $k$ part on $X$ is $0|\delta-\text{GFT}^{p^* \mathcal{L}_\rho^k}(X)$; multiplication in this ring comes from tensor products of line bundles. We observe that the degree 0 part is the same as twisted field theories gotten from choosing $\rho$ to be the trivial homomorphism; this gives ordinary (or untwisted) field theories since equivariant functions with respect to the trivial action on $\mathbb{R}$ are exactly the invariant functions. When the line bundle $\mathcal{L}_\rho$ is understood, we use the notation

$0|\delta-\text{GFT}^\bullet(X) := 0|\delta-\text{GFT}^{p^* \mathcal{L}_\rho^0}(X).$
Example 2.1.4. When $\delta = 1$, we choose the projection $
abla: \mathbb{R}^{0|1} \times \mathbb{Z}/2 \to \mathbb{Z}/2 \subset \mathbb{K}^\times$, to build a line bundle $\pi L_\nabla$. The functions on $\mathcal{S}M(\mathbb{R}^{0|1}, X)$ equivariant with respect to the action of $\mathbb{R}^{0|1} \rtimes \mathbb{Z}/2$ are precisely the closed forms in the $-1$ eigenspace of the grading involution, i.e., the odd forms. Hence

$$0|1\text{-EFT}^\rho \pi L_\nabla(X) \cong \Omega^\text{odd}_{\text{cl}}(X).$$

We’ve sketched the proof of the following result.

**Theorem 2.1.5** (Hohnhold-Kreck-Stolz-Teichner [18]). There are isomorphisms of abelian groups

$$0|1\text{-EFT}^k(X) \cong \begin{cases} \Omega^\text{even}_{\text{cl}}(X) & k = \text{even}, \\ \Omega^\text{odd}_{\text{cl}}(X) & k = \text{odd}. \end{cases}$$

These isomorphisms are compatible with the graded ring structure on both sides, namely tensor products of field theories on the left (i.e., multiplication of functions on $\mathcal{S}M(\mathbb{R}^{0|1}, X)$) and wedge products of forms on the right.

Example 2.1.6 (0|δ-Euclidean geometries). We can generalize the above example to 0|δ-Euclidean field theories by declaring $\mathcal{Euc}(\mathbb{R}^{0|\delta}) := \mathbb{R}^{0|\delta} \rtimes O(\delta)$ to be the isometry group, and take $\rho: \mathbb{R}^{0|\delta} \rtimes O(\delta) \to \mathbb{Z}/2 \subset \mathbb{K}^\times$ to define a twist where $\rho$ first projects to $O(\delta)$, then applies the determinant homomorphism. When $\delta$ is even, we take $L_\rho$ as the twist, and when $\delta$ is odd we take $\pi L_\rho$; this is because $-\text{id} \in O(\delta)$ acts by the grading involution on functions on $\mathcal{S}M(\mathbb{R}^{0|\delta}, X)$. These theories are in some sense a generalization of closed differential forms; aspects of this perspective are explained by Kochan and Ševera in [22], though they consider the action of the entire diffeomorphism group of $\mathbb{R}^{0|\delta}$.

For certain applications, we will need to impose a certain regularity assumption for the behavior of field theories under the renormalization group flow.

**Definition 2.1.7.** The renormalization group (RG) action on 0|δ-GFT$^\star(X)$ is the action induced from dilating $\mathbb{R}^{0|\delta}$ by $(\mathbb{R}_{>0}, \times)$, which in turn gives an action on 0|δ-GBord$_{\text{conn}}(X)$, and hence 0|δ-GFTs.

Since 0|δ-dimensional field theories over $X$ form a vector space, we can consider the field theories that are in the $\lambda$-eigenspace of the RG-action, i.e., where $r \in \mathbb{R}_{>0}$ acts by $r^\lambda$. The action of $\mathbb{R}_{>0}$ extends to one by the monoid $\mathbb{R}$, so it turns out that $\lambda$ is necessarily a natural number.

**Lemma 2.1.8.** Suppose that the monoid $(\mathbb{R}, \times)$ acts smoothly on a super manifold $M$. Then the eigenvalues of the infinitesimal induced action on $C^\infty M$ are positive integers.
Proof. Given an \( \mathbb{R}_{>0} \)-action on \( C^\infty(M) \), suppose that we are given a subspace of where the action is by \( r^\lambda \) for \( r \in \mathbb{R}^\times \) and \( \lambda \in \mathbb{R} \) fixed. If this action extends smoothly to zero, this requires that \( \lambda \) be nonnegative. Now let \( k \in \mathbb{N} \) be large enough so that \( \lambda - k \) is negative. Then the \( k \)th derivative of the action is differentiable if and only if \( \lambda \) is an integer. This proves the lemma. \( \square \)

Definition 2.1.9. Define polynomial functions on \( \text{SM}(\mathbb{R}^0|\delta, X) \) as

\[
C^\infty_{\text{pol}}(\text{SM}(\mathbb{R}^0|\delta, X)) := \bigoplus_{k \in \mathbb{N}} \{ f \in C^\infty(\text{SM}(\mathbb{R}^0|\delta, X)) \mid r \cdot f = r^k f, \ r \in \mathbb{R}_{>0} \},
\]

where \( r \cdot f \) denotes the action of \( \mathbb{R}_{>0} \) on functions on \( \text{SM}(\mathbb{R}^0|\delta, X) \) induced by the dilation action of \( \mathbb{R}_{>0} \) on \( \mathbb{R}^0|\delta \).

Definition 2.1.10. Renormalizable (or polynomial) \( 0|\delta \)-dimensional field theories over \( X \) are

\[
0|\delta\text{-GFT}_{\text{pol}}(X) := 0|\delta\text{-GFT}(X) \bigcap C^\infty_{\text{pol}}(\text{SM}(\mathbb{R}^0|\delta, X)) \cong \bigoplus_{k \in \mathbb{N}} \{ E \in 0|\delta\text{-GFT}(X) \mid r \cdot E = r^k E, \ r \in \mathbb{R}_{>0} \},
\]

where \( r \cdot E \) denotes the action of the renormalization group on field theories. We observe that the above direct sum consists of those field theories that diverge polynomially with respect to the RG action, whence the notation pol. There is an evident inclusion of the renormalizable field theories into all field theories, \( 0|\delta\text{-GFT}_{\text{pol}}(X) \hookrightarrow 0|\delta\text{-GFT}(X) \).

Remark 2.1.11. One can rephrase the polynomial growth condition by putting \( \omega \) in Equation 3.7 into the exponent: then renormalizability translates into at most logarithmic growth with the RG action which (ignoring formal details) agrees with renormalizability in Kevin Costello’s sense; see Definition 7.2.1 in [8].

\( \text{SM}(\mathbb{R}^0|\delta, X) \) and its functions

Using the functor of points, we can identify an element of \( C^\infty(\text{SM}(\mathbb{R}^0|\delta, X)) \) with maps of sets \( \text{SM}(\mathbb{R}^0|\delta, X)(S) \to C^\infty(S) \) natural in \( S \). Throughout, unless stated otherwise \( X \) will be an ordinary smooth manifold, viewed as a purely even supermanifold. We start with the familiar example of differential forms.

Functions on \( \text{SM}(\mathbb{R}^0|1, X) \)

The computations below can be found in many places, for example [9, 18, 23]. We compute the \( S \)-points,

\[
\text{SM}(\mathbb{R}^0|1, X)(S) \cong \{ \Phi : S \times \mathbb{R}^0|1 \to X \} \cong \{ \Phi^* : C^\infty X \to C^\infty(S) \otimes C^\infty(\mathbb{R}^0|1) \}.
\]
CHAPTER 2. PARTITION FUNCTIONS AND MANIFOLD INVARIANTS

Choosing a coordinate $\theta$ on $\mathbb{R}^{0|1}$ allows for a decomposition,

$$C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|1}) \cong C^\infty S \oplus C^\infty S \cdot \theta$$

so we may express $\Phi$ in terms of the Taylor components

$$\Phi^* = f + \phi \theta.$$ 

Enforcing the condition that $\Phi^*$ be an algebra homomorphism we find $f: C^\infty X \to C^\infty S$ is a grade-preserving algebra homomorphism and $\phi: C^\infty X \to C^\infty S$ is a grading-reversing map that is an odd derivation with respect to $f$,

$$\phi(ab) = \phi(a)f(b) + (-1)^{p(a)}f(a)\phi(b), \quad a,b \in C^\infty(X).$$

But this is the standard description [9] of $\pi TX$ in terms of its $S$-points, which recovers the isomorphism $SM(\mathbb{R}^{0|1}, X) \cong \pi TX$. We can define functions on this space for any $x \in C^\infty(X)$ by assigning their values on $S$-points as

$$x(\Phi) := f(x), \quad dx(\Phi) := \phi(x).$$

These are the zero- and one-forms in $\Omega^*(X) \subset C^\infty(\pi TX)$, respectively. For $X$ an ordinary manifold, these generate $C^\infty(SM(\mathbb{R}^{0|1}, X))$ as an algebra. The dilation action of $\mathbb{R}_{>0}$ on $\mathbb{R}^{0|1}$ gives an action on $C^\infty(SM(\mathbb{R}^{0|1}, X))$ whose eigenspaces are indexed by $\mathbb{N}$; the $k$th eigenspace for $k \in \mathbb{N}$ is exactly the degree $k$ differential forms, $\Omega^k(X)$.

**Remark 2.1.12.** Following the remark on page 74 of [9], we can describe differential forms on supermanifolds in terms of functions on the odd tangent bundle.

**Proposition 2.1.13.** Let $M$ be a supermanifold. Then there is an isomorphism of sheaves,

$$\Omega^*(M) \cong C^\infty_{pol}(SM(\mathbb{R}^{0|1}, M)),$$

between polynomial functions on $\pi TM$ and differential forms on $M$.

We emphasize that the polynomial condition is essential; for example, the algebra of smooth functions on $\pi T(\mathbb{R}^{0|1}) \cong \mathbb{R}^{1|1}$ is $C^\infty(\mathbb{R})[\theta]$, which is much larger than the algebra of differential forms, $\Omega^*(\mathbb{R}^{0|1}) \cong \mathbb{R}[\theta, d\theta]$.

**Functions on $SM(\mathbb{R}^{0|\delta}, X)$**

We choose coordinates $\{\theta_1, \ldots, \theta_\delta\}$ on $\mathbb{R}^{0|\delta}$, which we think of as an isomorphism $C^\infty(S \times \mathbb{R}^{0|\delta}) \cong C^\infty(S)[\theta_1, \ldots, \theta_\delta]$. A map $\Phi$ of supermanifolds is determined by a map $\Phi^*: C^\infty X \to C^\infty(S \times \mathbb{R}^{0|\delta})$ of superalgebras. We can express $\Phi^*$ in terms of its Taylor components,

$$\Phi^* = f + \sum \phi_I \theta_I,$$
where \( I = \{i_1 < \cdots < i_k\} \) is a nonempty increasing subset of \( \{1, \ldots, \delta\} \), \( \theta_I = \theta_{i_1} \cdots \theta_{i_k} \), and \( f, \phi_I: C^\infty(X) \to C^\infty(S) \) are linear maps with restrictions that make \( \Phi^* \) an algebra homomorphism. Notice that \( f \) induces a map of supermanifolds \( S \times \text{pt} \to X \).

Given any \( x \in C^\infty X \), we define a function \( x \in C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X)) \) whose value at an S-point \( \Phi \) is \( x(\Phi) = f(x) \). For a map \( s: S' \to S \), the value of the function \( x \) at the \( S' \)-point is \( x(s \circ \Phi) = (s \circ f)(x) \in C^\infty(S') \), so that \( x \) is indeed natural in \( S \) and therefore defines an honest function on \( \text{SM}(\mathbb{R}^{0|\delta}, X) \). This gives an inclusion of algebras

\[
C^\infty X \hookrightarrow C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X)).
\]  

(2.6)

Other examples of functions are denoted by \( d_I x \) for \( x \in C^\infty X \), whose value at an S-point is defined as \( (d_I x)(\Phi) := \phi_I(x) \). We note that the dilation action of \( \mathbb{R}_{>0} \) on \( \mathbb{R}^{0|\delta} \) induces an action on \( d_I x \) by \( r^{|I|} \) for \( r \in \mathbb{R}_{>0} \).

We can form arbitrary smooth functions in the variables \( d_I x \), in the sense that if \( \{x^j\} \) are local coordinates on \( X \), \( \{d_I x^j\} \) are local coordinates on \( \text{SM}(\mathbb{R}^{0|\delta}, X) \). Working locally on \( X \) and invoking the sheaf property, one can show that all functions arise this way; the \( d_I x \) turn out to give coordinates on \( \text{SM}(\mathbb{R}^{0|\delta}, \mathbb{R}^n) \cong (\pi T)^{\delta} \mathbb{R}^n \), which proves the following.

**Proposition 2.1.14.** The algebra \( C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X)) \) is generated by smooth functions in \( d_I x \) for \( x \in C^\infty X \) and \( I \) varying over all multi-indices \( \{i_1, \ldots, i_k\} \). The algebra \( C^\infty_{\text{pol}}(\text{SM}(\mathbb{R}^{0|\delta}, X)) \) is freely generated by the variables \( d_I x \) for \( x \in C^\infty X \).

**Remark 2.1.15.** The polynomials in the \( \mathbb{N} \)-graded functions are called differential gorms in \([22]\) for \( \delta = 2 \) and differential worms for higher \( \delta \).

We now consider the example when \( \delta = 2 \) in more detail.

**Example 2.1.16.** Consider the S-points with a choice of coordinate,

\[
\text{SM}(\mathbb{R}^{0|2}, X)(S) \cong \{\Phi^*: C^\infty(X) \to C^\infty(S)[\theta_1, \theta_2]\},
\]

which allows us to write Taylor components

\[
\Phi^* = f + \phi_1 \theta_1 + \phi_2 \theta_2 + E \theta_1 \theta_2,
\]

where \( \phi_i: C^\infty X \to (C^\infty S)^{\text{odd}} \) and \( f, E: C^\infty X \to (C^\infty S)^{\text{even}} \). A computation shows that \( \Phi^* \) being an algebra homomorphism requires

\[
\begin{align*}
    f(ab) &= f(a)f(b) \\
    \phi_i(ab) &= \phi_i(a)f(b) - f(a)\phi_i(b) & i = 1, 2 \\
    E(ab) &= E(a)f(b) + f(a)E(b) + \phi_1(a)\phi_2(b) + \phi_1(b)\phi_2(a).
\end{align*}
\]  

(2.7)

so \( f \) is an algebra homomorphism, \( \phi_1 \) and \( \phi_2 \) are odd derivations with respect to \( f \), and \( E \) the above quadratic identity.
CHAPTER 2. PARTITION FUNCTIONS AND MANIFOLD INVARIANTS

The polynomial function on $\text{SM}(\mathbb{R}^{0|2}, X)$ are generated as an algebra by

$$x(\Phi) = f(x), \quad (d_1x)(\Phi) = \phi_1(x), \quad (d_2x)(\Phi) = \phi_2(x), \quad (d_2d_1x)(\Phi) = E(x). \quad (2.8)$$

The functions $d_1x$ and $d_2x$ have $\mathbb{N}$-grading +1, whereas $x$ and $d_2d_1x$ have $\mathbb{N}$-gradings 0 and +2 respectively. We remark that unlike $d_1x$ and $d_2x$, the element $d_2d_1x$ is not nilpotent, so polynomials in the above variables are a strict subset of $C^\infty(\text{SM}(\mathbb{R}^{0|2}, X))$.

**Group actions on $\text{SM}(\mathbb{R}^{0|\delta}, X)$**

Let $A \in \text{Diff}(\mathbb{R}^{0|\delta})(S)$ and $\Phi \in \text{SM}(\mathbb{R}^{0|\delta}, X)(S)$, i.e., $A_\sharp : S \times \mathbb{R}^{0|\delta} \rightarrow S \times \mathbb{R}^{0|\delta}$ and $\Phi : S \times \mathbb{R}^{0|\delta} \rightarrow X$, where $A$ is a map of bundles over $S$. By restricting $A \in \text{Euc}(\mathbb{R}^{0|\delta})(S) \subset \text{Diff}(\mathbb{R}^{0|\delta})(S)$, we define an action on $S$-points as

$$\text{SM}(\mathbb{R}^{0|\delta}, X)(S) \times \text{Euc}(\mathbb{R}^{0|\delta})(S) \xrightarrow{\Phi, A} \text{SM}(\mathbb{R}^{0|\delta}, X)(S) \xleftarrow{\Phi \cdot A = \Phi \circ A.}$$

Let $\text{euc}(\mathbb{R}^{0|\delta})$ denote the Lie algebra of $\text{Euc}(\mathbb{R}^{0|\delta})$. The infinitesimal action of odd translations leads to odd vector fields on $\text{SM}(\mathbb{R}^{0|\delta}, X)$ that raise the $\mathbb{N}$-degree of functions by 1. For a chosen basis of $\mathbb{R}^{0|\delta}$, let $D_i$ denote odd vector field associated to the action by the $i$th basis vector. We have a homomorphism of Lie algebras from $\mathbb{R}^{0|\delta}$ into vector fields on $\text{SM}(\mathbb{R}^{0|\delta}, X)$, and so we also get an induced homomorphism of universal enveloping algebras from $\text{Sym}(\mathbb{R}^{0|\delta})$ into differential operators on $\text{SM}(\mathbb{R}^{0|\delta}, X)$; here we are using that $\mathbb{R}^{0|\delta}$ is a superabelian Lie algebra, so its universal enveloping algebra is the (graded) symmetric algebra on its Lie algebra. We observe that $D_iD_j = -D_jD_i$. Let $D_I$ denote the differential operator obtained from the composition $D_{i_1} \cdots D_{i_k}$ for a given ordered set $I = \{i_1, \ldots, i_k\}$. The following characterizes the action of $D_I$ on $C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$.

**Lemma 2.1.17.** Let $x \in C^\infty(X) \subset C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$. Then $D_Ix = d_Ix$, where the left hand side is the action of the differential operator $D_I$ on the function $x$, and the right hand side is the function $d_Ix$ defined in the previous subsection.

**Proof.** Consider the action of $D_i$ on $d_Ix \in C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$ in terms of the functor of points. At an $S$-point $\Phi$, $D_i$ acts as

$$D_I : C^\infty(X) \xrightarrow{\Phi^*} C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}) \xrightarrow{id \otimes \partial_{\theta_i}} C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}). \quad (2.9)$$

where $\partial_{\theta_i}$ is the vector field on $\mathbb{R}^{0|\delta}$ associated with infinitesimal translations in the $\theta_i$-direction. If we express an $S$-point in terms of its Taylor expansion and consider the action of $D_i$ on the function $d_Ix$, we find

$$\sum_{i} \phi_i \theta_i \xrightarrow{D_i} \sum_{i} \phi_i \theta_i \xrightarrow{d_I} \phi_i \cup I(x)$$
so that $D_i(d_Ix) = d_{i,I}x$. Given $I = \{i_1, \ldots, i_k\}$, we can iterate the above action on the function $x \in C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$, finding

$$D_I(x)(\Phi) = (d_Ix)(\Phi) = \phi_I(x),$$

as claimed.

**Notation 2.1.18.** Following the previous lemma, we use $d_Ix$ to denote both the function $d_Ix \in C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$ and an operator $d_I := D_I$ acting on the function $x$. In particular, the action of the $i$th basis vector of $\mathbb{R}^{0|\delta}$ is denoted by $d_i$, and these operators have $\mathbb{N}$-grading $+1$. Let $\Delta$ denote the composition $d_\delta \cdots d_1$; it has $\mathbb{N}$-grading $+\delta$.

The previous lemma together with our characterization of $C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$ makes it possible to describe the action by the Euclidean group. The action of odd translations $\partial_{\theta_i} \in \mathbb{R}^{0|\delta}$ is the expected one given by composition, $\partial_{\theta_i} \cdot (d_Ix) = (d_{i,I}d_I)x$; we also have the relations $d_id_j = -d_jd_i$. To compute the action of $O(\delta)$, observe that its action on the Lie superalgebra $\mathbb{R}^{0|\delta}$ naturally extends to one on the universal enveloping algebra of $\mathbb{R}^{0|\delta}$, $\text{Sym}(\mathbb{R}^{0|\delta})$. Hence $A \in O(\delta)$ acts on a function $d_Ix$ by $A(d_Ix) = (Ad_I)x$. Concretely, this is the standard action of $O(\delta)$ on $\text{Sym}(\mathbb{R}^{0|\delta})$, which (ignoring gradings) is an exterior algebra on $\mathbb{R}^\delta$.

The other action we need to understand is defined at an $S$-point by

$$\text{SM}(\mathbb{R}^{0|\delta}, X)(S) \times \text{SM}(\mathbb{R}^{0|\delta}, \text{Diff}(X))(S) \xrightarrow{\Phi, \mathcal{G}} \text{SM}(\mathbb{R}^{0|\delta}, X)(S) \xleftarrow{\Phi \cdot \mathcal{G}},$$

where we view $\mathcal{G}$ as an automorphism of the trivial bundle over $S \times \mathbb{R}^{0|\delta}$ with fiber $X$, $S \times \mathbb{R}^{0|\delta} \times X \xrightarrow{\mathcal{G}} S \times \mathbb{R}^{0|\delta} \times X$, and can turn $\Phi$ into a section of this bundle via

$$\text{id} \times \Phi \in \Gamma(S \times \mathbb{R}^{0|\delta}, S \times \mathbb{R}^{0|\delta} \times X),$$

and finally we define $\Phi \cdot \mathcal{G}$ as the composition

$$S \times \mathbb{R}^{0|\delta} \xrightarrow{\text{id} \times \Phi} S \times \mathbb{R}^{0|\delta} \times X \xrightarrow{\mathcal{G}} S \times \mathbb{R}^{0|\delta} \times X \xrightarrow{p} X$$

where $p$ is projection. We can consider the corresponding infinitesimal action at the level of the Lie algebra, $\text{SM}(\mathbb{R}^{0|\delta}, \Gamma(TX))$. For our purposes we need only consider the action of elements $\mathcal{L}_v, I_w \in \text{SM}(\mathbb{R}^{0|\delta}, \Gamma(TX))$ defined for $v, w \in \Gamma(TX)$; these operators have Taylor components at an $S$-point take of the form

$$\mathcal{L}_v^* := v, \quad I_w^* := w\theta_1 \ldots \theta_\delta.$$

It is straightforward to check that $\mathcal{L}_v$ has $\mathbb{N}$-grading $0$ and $I_w$ has $\mathbb{N}$-grading $-\delta$. As suggested by the notation, $\mathcal{L}_v$ acts by the Lie derivative, and the relevancy of the operator $I_w$ comes from the formula

$$[d_\delta, \ldots, [d_2, [d_1, I_w]]\ldots] = \mathcal{L}_w,$$

(2.10)
generalizing the Cartan formula. To explain the above equality, we consider the action of the left side on the function $d_j x$. Expanding the expression $[d_\delta, \ldots, [d_2, [d_1, I_w]]]$, we get a sum of terms of the form $d_K I_w d_L$ for $K \cup L \cong \{1, \ldots, n\}$ (though not necessarily as ordered sets). If $i \in J$ and $i \in L$, then $d_K I_w d_L (d_j x) = d_K I_w(0) = 0$, using the fact that $d_i^2 = 0$ for all $i$. As usual, the value of $d_j x$ at an $S$-point $\sum \phi_i \theta_i$ is $\phi_j(x)$, and we can understand the action of the operator $I_w$ on $d_L d_j x$ by precomposing with the action on the $S$-point. If $i \notin J$ and $i \notin L$, then $I_w (d_L(d_j x)) = 0$, using the definition of $I_w$ and the fact that $\theta_i^2 = 0$ for all $i$. From this it follows that any nontrivial action of $[d_\delta, \ldots, [d_2, [d_1, I_w]]]$ on $d_j x$ arises from terms where $L \cup J \cong \{1, \ldots, n\}$ and hence $K \cong J$, where again these isomorphisms may not preserve the ordering of these sets. In this case we compute

$$d_K I_w d_L (d_j x) = d_K I_w(\Delta x) = d_K (wx) = d_j (wx) = L_w d_j x,$$

where there are possible signs we have suppressed, owing to the non-ordered isomorphisms $K \cong J$ and $L \cup J \cong \{1, \ldots, n\}$. However, these sign ambiguities exactly cancel, so that

$$[d_\delta, \ldots, [d_2, [d_1, I_w]]] \ldots d_j x = L_w d_j x$$

Since $[d_\delta, \ldots, [d_2, [d_1, I_w]]] \ldots$ is a derivation (being an iterated Lie bracket of derivations) the above characterizes its action on functions via the Leibniz rule and we have proved formula (2.10).

**Remark 2.1.19.** The action by the Euclidean group is natural in $X$ since a map $X \to Y$ induces a $\text{Euc}(\mathbb{R}^0,\delta)$-equivariant map $\text{SM}(\mathbb{R}^0, X) \to \text{SM}(\mathbb{R}^0, Y)$. This generalizes the usual naturality of de Rham $d$. In particular, $\Delta$ acts naturally, which will be important in the next subsection. Another way to package this information is that a map $X \to Y$ induces a functor between the groupoids $0|\delta\text{-EBord}_{\text{conn}}(X) \to 0|\delta\text{-EBord}_{\text{conn}}(Y)$.

We now explain how the above actions give rise to familiar algebraic structures on differential forms when $\delta = 1$, and then we explain in detail the situation for $\delta = 2$.

**Example 2.1.20.** We have that $C^\infty(\text{SM}(\mathbb{R}^0, X)) \cong \Omega^*(X)$; the Euclidean group in this example acts through an $\mathbb{R}^0$-action and a $O(1) \cong \mathbb{Z}/2$-action. The infinitesimal generator of $\mathbb{R}^0$ acts by the de Rham $d$, and the $\mathbb{Z}/2$-action is by $+1$ on even forms and $-1$ on odd forms. We get an infinitesimal action from the (infinite dimensional) Lie algebra

$$\text{Lie}(\text{SM}(\mathbb{R}^0, \text{Diff}(X))) \cong \text{SM}(\mathbb{R}^0, \Gamma(TX)) \cong \Gamma(TX) \oplus \pi \Gamma(TX)$$

where in the above we view $\Gamma(TX)$ as a generalized manifold whose functor of points is $C^\infty(S, \Gamma(TX))$, i.e., smooth functions with values in $\Gamma(TX)$. Then the $S$-points of the inner hom $\text{SM}(\mathbb{R}^0, \Gamma(TX))$ can be identified with $C^\infty(S, \Gamma(TX))[\theta]$; Taylor expanding in $\theta$ we get a term in $\Gamma(TX)$ and a term in $\pi \Gamma(TX) \cong \Gamma(TX) \otimes \mathbb{R}^0$, which gives the second isomorphism in the above displayed equation. We claim that $v \in \Gamma(TX)$ acts by the Lie derivative, $L_v$, and $\psi \in \pi \Gamma(TX)$ acting by interior multiplication, $\iota_\psi$, and these change N-degrees by 0 and $-1$, respectively. We see this by computing the composition that defines the action,

$$C^\infty(S)[\theta] \otimes C^\infty X \xrightarrow{\iota_\psi} C^\infty S[\theta] \otimes C^\infty X \xrightarrow{\Phi^*} C^\infty(S)[\theta]$$
where $G^* = v + \psi \theta$, $(v, \psi) \in \Gamma(TX) \oplus \pi \Gamma(TX)$, and $\Phi = f + \phi \theta$. Then we find on functions

$$(G^* x)(\Phi) = f(v x) = (L_v x)(\Phi), \quad (G^* dx)(\Phi) = \phi(v x) + f(\psi x) = (L_v dx)(\Phi) + (\iota_\psi dx)(\Phi),$$

which follows from the action of $G$ on the $S$-point, $f + \phi \theta \xrightarrow{G^*} f \circ v + (f \circ \psi) \theta + (\phi \circ v) \theta$. Since $G^* \in \text{Lie}(\mathcal{SM}(\mathbb{R}^{0|1}, \text{Diff}(X)))$ acts by derivations, the above formulas determine the action uniquely on $C^\infty(\mathcal{SM}(\mathbb{R}^{0|1}, X)) \cong \Omega^*(X)$. An identical (though simpler) argument as in the proof of Equation 2.10 proves the usual Cartan identity, $[d, \iota_v] = L_v$.

**Example 2.1.21.** Functions on $\mathcal{SM}(\mathbb{R}^{0|2}, X)$ are generated by the monomials in Equation 2.8. The odd translations, $\mathbb{R}^{0|2}$, act in the predictable way that was described before, using that $d_1 d_2 = -d_2 d_1$ and both $d_i$ square to zero. For $x \in C^\infty X$, the rotations $O(2)$ act via the usual 2-dimensional representation on the span of $d_1 x, d_2 x$; act trivially on $x$; and act through the determinant homomorphism on $d_2 d_1 x$.

Next we wish to understand the action by

$$\mathcal{SM}(\mathbb{R}^{0|2}, \Gamma(TX)) \cong \Gamma(TX) \oplus \pi \Gamma(TX) \oplus \pi \Gamma(TX) \oplus \Gamma(TX).$$

As in the previous example, we consider the composition

$$C^\infty(S)[\theta_1, \theta_2] \otimes C^\infty X \xrightarrow{\mathcal{G}^*} C^\infty(S)[\theta_1, \theta_2] \otimes C^\infty X \xrightarrow{\Phi^*} C^\infty(S)[\theta_1, \theta_2],$$

where $G^* = v + \psi_1 \theta_1 + \psi_2 \theta_2 + w \theta_1 \theta_2$, $\Phi^* = f + \phi_1 \theta_1 + \phi_2 \theta_2 + E \theta_1 \theta_2$, and $v, w \in \Gamma(TX)$, $\psi_1, \psi_2 \in \pi \Gamma(TX)$. The action of $v$ is by the Lie derivative, $L_v$,

$$(L_v x)(\Phi) = f(v x), \quad (L_v d_1 x)(\Phi) = \phi_1(v x), \quad (L_v d_2 d_1 x)(\Phi) = E(v x).$$

Most of the action of $\psi_1$ and $\psi_2$ can be computed by considering inclusions $\mathbb{R}^{0|1} \hookrightarrow \mathbb{R}^{0|2}$: when restricting to the subspaces generated by $\{x, d_1 x\}$ or $\{x, d_2 x\}$, we get copies of the Cartan algebra. Explicitly, we denote the action of $\psi_i$ by $\iota_{\psi_i}$ and compute

$$\iota_{\psi_1} x(\Phi) = 0, \quad \iota_{\psi_1} d_1 x(\Phi) = (L_{\psi_1} x)(\Phi), \quad \iota_{\psi_1} d_2 x(\Phi) = 0, \quad \iota_{\psi_1} d_2 d_1 x(\Phi) = d_2 x(\Phi).$$

Similar formulas hold for the action of $\psi_2$. We denote the action of $w$ by $I_w$ and compute

$$I_w x = 0, \quad I_w d_1 x = 0, \quad I_w d_2 x = 0, \quad (I_w d_2 d_1 x)(\Phi) = f(w x) = (L_w x)(\Phi),$$

most of which can be deduced by the fact that $I_w$ lowers a function’s $N$-degree by 2. Finally, following the argument proving (2.10) we note the identity $[d_2, [d_1, I_w]] = L_w$.

### 2.2 Dimensional reduction

In this section we prove Theorem 2.0.18 by constructing functors

$$d|\delta\text{-GFT}^T(X) \to 0|\delta\text{-GFT}^T(X),$$
natural in $X$. The $0|\delta$-geometry and the twist will be produced explicitly from the corresponding data in the $d|\delta$-dimensional field theory, and when $X = \text{pt}$ the functor will evaluate the partition function of a given field theory on a specified closed manifold $\Sigma^{d|\delta}$.

Naively, reduction should take as input a $d$-manifold $\Sigma^d$ and be induced by a morphism between bordism categories gotten from “crossing with $\Sigma^d$”

$$0|\delta-\text{Bord}(X) \times_{\Sigma^d} d|\delta-\text{Bord}(X) \quad \text{(Naive!)} \quad (2.12)$$

so that a bordism of $0|\delta$-supermanifolds is sent to a bordism of $d|\delta$-supermanifolds. However, this doesn’t make sense for most super geometric structures. For example, when $\delta > 0$ a $d|\delta$-dimensional Euclidean manifold never splits as a product of a $0|\delta$-dimensional super geometric manifold and a $d$-dimensional Euclidean manifold. The relevant example to bear in mind is that of super tori $R^{d|\delta}/Z^d$: if this supermanifold were to geometrically decompose as a product, $(|E|^d/Z^d) \times R^{0|\delta}$, its isometry group would also be a product of isometries of $|E|^d/Z^d$ with isometries of $R^{0|\delta}$. However, in the case of super Euclidean model geometries, the torus $E^{d|\delta}/Z^d$ has isometries that mix the odd and even parts, coming from the square roots of even translations. Another way to see the nonexistence of this geometric splitting is to observe that there are no nontrivial homomorphisms $R^{0|\delta} \to E^{d|\delta}/Z^d$ when $\delta > 0$, and so there can be no way to define the functor (2.12). Therefore, super Euclidean tori cannot be geometric products when $\delta > 0$ and in our examples of interest there is no hope of constructing a functor between bordism categories using (2.12). Below we will define and construct dimensional reduction for geometric tori $R^{d|\delta}/Z^d$, and at the end of the section comment on why extending the construction to arbitrary closed $d|\delta$-manifolds is not possible in general.

**Defining dimensional reduction**

In order to set up the problem we begin with a definition.

**Definition 2.2.1.** Let $\Sigma^{d|\delta}$ be a closed super manifold equipped with a rigid geometry of type $G$. Dimensional reduction along $\Sigma^{d|\delta}$ or, more simply, dimensional reduction is a morphism of prestacks

$$\text{red}_{\Sigma^{d|\delta}} : d|\delta-\text{GFT}(X) \to 0|\delta-\text{GFT}(X)$$

for some sheaf $0|\delta-\text{GFT}$ with the condition that when $X = \text{pt}$, $\text{red}_{\Sigma^{d|\delta}}$ is the evaluation on $\Sigma^{d|\delta}$,

$$\text{red}_{\Sigma^{d|\delta}} : d|\delta-\text{GFT}(\text{pt}) \to 0|\delta-\text{GFT}(\text{pt}) \subset K, \quad E \mapsto E(\Sigma^{d|\delta}) \in K.$$

**Proposition 2.2.2.** Let $(M, G)$ be a model geometry with tori in the sense of Definition 1.5.13. For any torus $T := R^{d|\delta}/Z^d$ there is a sheaf $0|\delta-\text{GFT}$ and a dimensional reduction functor

$$\text{red}_T : d|\delta-\text{GFT}(X) \to 0|\delta-\text{GFT}(X).$$

Furthermore, this dimensional reduction admits a factorization as in Equation 2.4.
As we will describe below, the definition of 0|δ-GFT comes directly from the model geometry G defining the prestack d|δ-GFT, and in some (imprecise) sense is the most rigid geometry that admits a dimensional reduction functor from d|δ-GFT. There is also a twisted analog of Proposition [2.2.2] that we describe in Section [2.2].

We prove the above proposition in two steps. The first is basically a categorical construction (a looping of the category d|δ-GBord(X) to an internal 0-category) whereas the second uses the specific geometric assumptions on the model geometry and super torus.

Looping via closed connected objects

Given the data of a bordism category d|δ-GBord(X) and some closed d|δ-dimensional supermanifold Σ, we can form a subcategory d|δ-GBordΣ(X) ⊂ d|δ-GBord(X). This subcategory has as object stack the empty stack: objects are S-families of the empty manifold with only identity automorphisms. The morphism stack of d|δ-GBordΣ(X) is comprised of families of (ℳ, G)-manifolds with a map to X, whose fiber is diffeomorphic Σ; isomorphisms are bundle automorphisms that are isometries of the fibers. The subcategory d|δ-GBordΣ(X) does not have a symmetric monoidal structure.

Remark 2.2.3. Supposing we had a definition of a d|δ-bordism d-category in which the k-morphisms formed a symmetric monoidal stack, the above construction would proceed identically: the 0- through (d − 1)-morphism stacks would be the empty stack over SMfd, and the d-morphism stack would be the same as above.

The inclusion of internal categories d|δ-GBordΣ(X) → d|δ-GBord(X) induces a functor

\[ d|δ-GFT(X) \cong \text{Fun}_SM(d|δ-GBord(X), TV) \to \text{Fun}_SM(d|δ-GBordΣ(X)), \mathcal{K}, \]  

(2.13)

where \( \mathcal{K} \) denotes the representable stack determined by the manifold \( \mathcal{K} \). The arrow is well-defined because the object stack in d|δ-GBordΣ(X) pulls back from the empty family over the point (which is the symmetric monoidal unit of d|δ-GBord(X)) whose image under a fibered functor must be a family that pulls back from the trivial line \( \mathcal{K} \) on the point (which is the symmetric monoidal unit of TV).

We can restrict the above to Σ with a fixed (ℳ, G)-geometry, obtaining an evaluation map

\[ ev_Σ: \text{Fun}_SM(d|δ-GBord_Σ(X)), \mathcal{K} \to C^∞(SM(Σ, X)//Iso(Σ))^{ev}. \]  

(2.14)

To see this, consider the prestack whose objects over \( S \) are product families, \( Σ \times S \to S \) equipped with a map to \( X \) and whose morphisms over \( S \) are gauge transformations of this bundle that restrict to isometries in each fiber. This prestack is represented by the quotient super Lie groupoid \( SM(Σ, X)//Iso_G(Σ) \), which basically follows from the adjunction

\[ gSM(S, SM(Σ, X)) \cong gSM(S × Σ, X). \]

Using the fact that stackification is left adjoint to forget, functors out of the stackification of this prestack are the same as even functions on this groupoid. The stackification of this
prestack includes into \(d|\delta\text{-GBord}_{\Sigma}(X)\), and this induces the above evaluation map. This argument is a mild elaboration on the proof of Lemma 55 in [18].

The remaining data of dimensional reduction is an algebra map

\[
C^\infty(\text{SM}(\Sigma, X) / \text{Iso}_G(\Sigma))^{ev} \to C^\infty(\text{SM}(\mathbb{R}^0|\delta, X) / \text{Iso}_G(\mathbb{R}^0|\delta))^{ev} =: 0|\delta\text{-GFT}(X) \tag{2.15}
\]

for some \(\text{Iso}_G(\mathbb{R}^0|\delta)\) acting on \(\mathbb{R}^0|\delta\), to be defined below.

**Remark 2.2.4.** If we take the target category \(\text{TV}\) to have a flip in the sense discussed before Equation 1.6 and in [33], the symmetric monoidal unit is necessarily fixed under this involution. Hence, the restriction of flip preserving functors to a closed connected bordism gives a function on the same groupoid as above.

### The target of dimensional reduction and the triangle 2.4

With the data in Proposition 2.2.2, we want to produce a map as in Equation 2.15. To that end, we define a \(0|\delta\text{-geometry as}

\[
\text{Iso}_G(\mathbb{R}^0|\delta) := N(|T|)/|T| < \text{Diff}(\mathbb{R}^0|\delta) \tag{2.16}
\]

where \(N\) denotes the normalizer in \(\text{Iso}_G(T)\). To spell out this definition, let \(|T| := |\mathbb{R}^d|/\mathbb{Z}^d|\) be the reduced group of \(T\). We have the canonical inclusion \(|T| \hookrightarrow T\) as a normal subgroup by assumption (see Definition 1.5.13) and so \(|T|\) acts on \(T\), and (again, by Definition 1.5.13) this action is by isometries. This allows us to form \(N(|T|)\). Furthermore, since the subgroup \(|T| < T\) is normal, the group \(N(|T|)/|T|\) acts on the quotient \(T/|T| \cong \mathbb{R}^0|\delta\). Hence, the above defines a model geometry, \((\mathbb{R}^0|\delta, \text{Iso}_G(\mathbb{R}^0|\delta))\).

**Remark 2.2.5.** The above definition of \(\text{Iso}_G(\mathbb{R}^0|\delta)\) was inspired by a remark in [12] in the last two paragraphs of the second section of Lecture 3. By the construction below, dimensional reduction can also be defined for any subgroup of \(N(|T|)/|T|\).

To define the presheaf \(d|\delta\text{-GFT}_0\), we use the projection \(q: S \times T \to S \times \mathbb{R}^0|\delta\) induced by the quotient \(T \to T/|T| \cong \mathbb{R}^0|\delta\). Let \(d|\delta\text{-GBord}_0(X)\) denote the subcategory of the bordism category consisting of families of tori equipped with a map to \(X\) that factors locally on \(S\) as

\[
S \times T \xrightarrow{q} S \times \mathbb{R}^0|\delta \to X.
\]

Then we define

\[
d|\delta\text{-GFT}_0: \text{Man}^op \to \text{Set}, \quad X \mapsto \text{Fun}_{\text{SM}}(d|\delta\text{-GBord}_0(X), \mathbb{R})
\]

and we observe there is a natural transformation of presheaves

\[
Z: d|\delta\text{-GFT} \to d|\delta\text{-GFT}_0.
\]

Proposition 2.2.2 will now follow from a pair of lemmas.
Lemma 2.2.6. There exists a map of groupoids
\[ \phi: \mathrm{SM}(\mathbb{R}^0, X) // N(|T|) \to \mathrm{SM}(T, X) // \mathrm{Iso}_G(T), \]  
(2.17)
natural in \( X \).

Lemma 2.2.7. There exists an isomorphism of algebras
\[ C^\infty(\mathrm{SM}(\mathbb{R}^0, X) // N(|T|)) \cong C^\infty(\mathrm{SM}(\mathbb{R}^0, X) // \mathrm{Iso}_G(\mathbb{R}^0)). \]  
(2.18)
natural in \( X \).

Proof of Lemma 2.2.6. Considering \( \mathbb{R}^0 \) as the quotient \( T/|T| \cong \mathbb{R}^0 \), we obtain an action of \( N(|T|) < \mathrm{Iso}_G(T) \) on \( T \) and since \( |T| < N(|T|) \) acts trivially on \( T/|T| \) (using that \( |T| \) is normal in \( T \)) this produces an action by \( N(|T|)/|T| \) on \( \mathbb{R}^0 \). This leads to a morphism of action groupoids,
\[ \mathrm{SM}(\mathbb{R}^0, X) // N(|T|) \to \mathrm{SM}(T, X) // N(|T|) \to \mathrm{SM}(T, X) // \mathrm{Iso}_G(T), \]
where the first map is induced by the projection, \( T \to T/|T| \) and the second by the inclusion of groups \( N(|T|) \hookrightarrow \mathrm{Iso}_G(T) \).

Proof of 2.2.7. By definition the action of \( |T| \triangleleft N(|T|) \) on \( \mathrm{SM}(\mathbb{R}^0, X) \) is trivial, and so when passing to functions we have
\[ C^\infty(\mathrm{SM}(\mathbb{R}^0, X)^{ev} // N(|T|)) \cong C^\infty(\mathrm{SM}(\mathbb{R}^0, X)/(N(|T|)/|T|)^{ev}). \]
Using the definition (2.16), the Lemma is proved.

Proof of Proposition 2.2.2. The map (2.15) can be gotten by converting (2.17) to a map on functions, and post-composing with (2.18). Together with (2.13), this produces the desired dimensional reduction functor. Dimensional reduction picks out the subcategory of \( d|\delta\)-\( \mathcal{G}_{\text{Bord}} \Sigma \) given by \( S \)-families of tori all isomorphic to a fixed torus, and hence the triangle 2.4 automatically commutes by construction.

Dimensional reduction of twisted field theories

Now we wish to explain the situation for reduction of twisted theories. Suppose that we have a twist as in Equation 1.8. Restricting the data of \( \mathcal{T} \) to its value on \( d|\delta\)-\( \mathcal{G}_{\text{Bord}} \Sigma \langle \text{pt} \rangle \), we obtain a line bundle \( \tilde{L} \) over the groupoid \( \mathrm{SM}(\Sigma^{d|\delta}, \text{pt}) // \mathrm{Iso}_G(\Sigma) \cong \text{pt} // \mathrm{Iso}_G(\Sigma) \); see Remark 38 in [18]. Such line bundles arise as pullbacks from line bundles over \( \text{pt} // \mathrm{Iso}_G(\Sigma) \), and hence there is an equivalence of categories between line bundles and line bundle automorphisms, and homomorphisms \( \rho: \mathrm{Iso}_G(\Sigma) \to \mathbb{K}^x \) and intertwiners of these representations.
Proposition 2.2.8. Let $(\mathcal{M}, G)$ be a model geometry with tori and $\mathcal{T}$ a twist satisfying \ref{1.8}. Suppose that $|T| := |\mathcal{T}| < \text{Isog}(\mathcal{T})$ is in the kernel of the homomorphism $\rho: \text{Isog}(\mathcal{T}) \to \mathbb{K}^\times$ that defines $\mathcal{T}$ over $\text{SM}(\mathcal{T}, X)/\text{Isog}(\mathcal{T})$. Then there exists a dimensional reduction functor $d|\delta\cdot\text{GFT}^T \to 0|\delta\cdot\text{GFT}^L$ where $L$ is uniquely determined by $\mathcal{T}$.

Proof. First consider $d|\delta\cdot\text{GFT}^T(X) \to \Gamma \left( \text{SM}(\mathcal{T}, X)/\text{Isog}(\mathcal{T}); \tilde{\mathcal{L}} \right)^{ev} \to \Gamma \left( \text{SM}(\mathbb{R}^0|\delta, X)/N(|\mathcal{T}|); \phi^*\tilde{\mathcal{L}} \right)^{ev}$ where the first map uses the construction of Section \ref{2.2} the second map is induced by the map of groupoids $\phi$ in Equation \ref{2.17} we observe that $\phi^*\tilde{\mathcal{L}}$ is defined by the composition of homomorphisms $N(|\mathcal{T}|) \hookrightarrow \text{Isog}(\mathcal{T}) \xrightarrow{\rho} \mathbb{K}^\times$. Assuming that $|T|$ is in the kernel of the composition, we get an isomorphism,\[ \Gamma \left( \text{SM}(\mathbb{R}^0|\delta, X)/N(|\mathcal{T}|); \tilde{\mathcal{L}} \right)^{ev} \cong \Gamma \left( \text{SM}(\mathbb{R}^0|\delta, X)/\text{Isog}(\mathbb{R}^0|\delta); \mathcal{L} \right) \cong 0|\delta\cdot\text{GFT}^L(X)^{ev} \] where $\mathcal{L}$ is the line bundle associated to $\rho_0: N(|\mathcal{T}|)/T^d \to \mathbb{K}^\times$. Composing, we get $d|\delta\cdot\text{GFT}^T(X) \to 0|\delta\cdot\text{GFT}^L(X)$, as claimed. \hfill \Box

Corollary 2.2.9. Let $(\mathcal{M}, G)$ be a model geometry that has tori in the sense of Definition \ref{1.5.13} and $\mathcal{T}$ be a twist of $d|\delta\cdot\text{GFT}$ over $\mathbb{R}$ satisfying \ref{1.8} Then there exists a dimensional reduction functor $d|\delta\cdot\text{GFT}^T \to 0|\delta\cdot\text{GFT}^L$ for a canonically defined twist $L$.

Proof. Since $|T|$ is compact and connected, it must be in the kernel of $\rho$ since the only compact connected subgroup of $\mathbb{R}^\times$ is the trivial group. \hfill \Box

Observation 2.2.1. For twists $k$ defined via the $k$th tensor power of a particular twist $\mathcal{T}$ we get a graded multiplication, $d|\delta\cdot\text{EFT}^k(X) \times d|\delta\cdot\text{EFT}^l(X) \to d|\delta\cdot\text{EFT}^{k+l}(X)$.

Dimensional reduction first restricts to the closed connected groupoid of the bordism category, wherein the twists are line bundles. Sections of tensor powers of these line bundles have a graded multiplication via tensor products of line bundles. Hence, by construction, dimensional reduction respects products in that the diagram commutes
\[ \begin{array}{ccc}
 d|\delta\cdot\text{EFT}^k(X) \times d|\delta\cdot\text{EFT}^l(X) & \xrightarrow{\text{red}} & 0|\delta\cdot\text{EFT}^k(X) \times 0|\delta\cdot\text{EFT}^l(X) \\
 \downarrow & & \downarrow \\
 d|\delta\cdot\text{EFT}^{k+l}(X) & \xrightarrow{\text{red}} & 0|\delta\cdot\text{EFT}^{k+l}(X). \\
\end{array} \] (2.19)
The commutativity is strict since $0|\delta\cdot\text{EFT}$s form a zero category, i.e., a set.
Supersymmetric field theories and examples of reduction

Below we consider dimensional reduction for a few examples of model geometries. First we make a definition critical to the statement of Theorem 2.0.18.

**Definition 2.2.10.** A model geometry with tori is called *supersymmetric* if \( \mathbb{R}^{0|\delta}/|T| \cong R^0|\delta \times G_0 \) where \( \mathbb{R}^{0|\delta} \) is equipped with its standard super commutative group structure. A geometric field theory is *supersymmetric* when it is defined via a supersymmetric model geometry.

**Remark 2.2.11.** We know of no geometrically-motivated field theories with isometry groups that are not supersymmetric, nor of interesting examples of higher-dimensional field theories that dimensionally reduce to ones with more general model geometries. The more general geometries may be intrinsically interesting, but they do not appear to be relevant in our attempt to understand \( d|\delta \)-EFTs and \( d|\delta \)-CFTs for \( d > 0 \). Many of our results apply to arbitrary model geometries on \( \mathbb{R}^{0|\delta} \), but some of the arguments become much more delicate in the general situation.

**Example 2.2.12.** The super Euclidean and super conformal geometries define bordism categories over \( X \), and we wish to do dimensional reduction along the tori \( T := M/\mathbb{Z}^d \).

First we need to compute \( \text{Iso}(T) \), which can be divided into two pieces: the translations and the discrete isometries. As explained previously, \( \mathbb{R}^{d|\delta}/\mathbb{Z}^d < \text{Iso}(T) \), and furthermore we have the short exact sequence of groups

\[
0 \to \mathbb{R}^d \hookrightarrow \mathbb{R}^{d|\delta} \to \mathbb{R}^{0|\delta} \to 0
\]

where the first arrow is the canonical inclusion of the reduced group into the super Euclidean group and the second is the projection onto the odd part. This shows that \( \mathbb{R}^d \) is normal, and hence after taking the quotient by a lattice, the normalizer of \( |T| \) in \( T \) is \( T \) itself.

Let \( \text{Spin} = \text{Spin}(d); \) we now compute which elements of \( \text{Spin} \) act by isometries. Generically, the only nontrivial element that commutes with the the \( \mathbb{Z}^d \)-action is the spin flip, i.e., the element of \( \text{Spin} \) that acts by \(-1\) on \( \mathbb{R}^{0|\delta} \) and by the identity on \( \mathbb{R}^d \). However, for lattices that contain extra symmetry, we get a larger group we denote by \( H \) that contains \( \mathbb{Z}/2 \) as a subgroup and we have

\[
\text{Iso}(T) \cong (\mathbb{R}^{d|\delta}/\mathbb{Z}^d) \rtimes H.
\]

We observe that \( H \) is a finite group, being the intersection of a compact group (\( \text{Spin} \)) with a discrete group (automorphisms of the lattice).

**Remark 2.2.13.** The above argument goes through verbatim for \( \text{Spin} = \text{Spin}(d-1,1) \) for super Poincaré geometries, though \( H \) need not be finite since in this case \( \text{Spin} \) is not compact.

We have that \( N(|T|) \cong \text{Iso}(T) \), so following (2.16) we define

\[
\text{Iso}(\mathbb{R}^{0|\delta}) := N(|T|)/|T| \cong \text{Iso}(T)/|T| \cong \mathbb{R}^{0|\delta} \rtimes H,
\]
where $\mathbb{R}^{0|\delta}$ is super abelian and $H < \text{Spin}$ acts according to the spinor representation on $\mathbb{R}^{0|\delta}$. When $\delta = 1$, the generic case of $H = \mathbb{Z}/2$ is the Euclidean geometry chosen in [18]. This computation shows that the super Euclidean and super conformal model geometries are indeed supersymmetric.

The following is a corollary to Theorem 2.0.18.

**Corollary 2.2.14.** Partition functions of super Euclidean and super conformal field theories are concordance invariants.

### Concordance classes of $0|\delta$-GFTs

Given a closed differential form, one can extract a topological invariant by considering the de Rham cohomology class that form represents. Analogously, $0|\delta$-GFTs give us new supergeometric objects generalizing closed forms, and one can extract topological information from these. The correct notion that allows for this passage from the geometric to the topological is that of *concordance*. We specialize Definition 1.3.1 to the current setting.

**Definition 2.2.15.** Two field theories $E_+, E_- \in 0|\delta\text{-GFT}^\bullet(X)$ are *concordant* if there exists a field theory $\bar{E} \in 0|\delta\text{-GFT}^\bullet(X \times \mathbb{R})$ such that $i_\pm^* \bar{E} = \pi_\pm^* E_\pm$, where

$$i_\pm: X \times (\pm 1, \pm \infty) \hookrightarrow X \times \mathbb{R}, \quad \pi_\pm: X \times (\pm 1, \pm \infty) \to X$$

are the usual inclusion and projection maps, respectively. We denote the set of field theories up to concordance by $0|\delta\text{-GFT}^\bullet[X]$ and the concordance class of a field theory $E$ by $[E]$.

It is easy to check that concordance defines an equivalence relation. In fact, for field theories twisted by a line bundle $\mathcal{L}_\rho$ the graded ring structure on $0|\delta\text{-GFT}^\bullet(X)$ descends to concordance classes. Field theories up to concordance furnish an additive contravariant functor from manifolds to graded rings,

$$0|\delta\text{-GFT}^\bullet[-] : \text{Man}^{op} \to \text{GRing}.$$

It is straightforward to check that smoothly homotopic maps between manifolds are assigned the same homomorphism between graded rings. In particular, the graded ring $0|\delta\text{-GFT}^\bullet[X]$ is a homotopy invariant of $X$.

An application of Stokes’ Theorem shows the following (see also Proposition 2.2.17).

**Theorem 2.2.16** (Hohnhold-Kreck-Stolz-Teichner). There is an isomorphism of abelian groups

$$0|1\text{-EFT}^k[X] \cong \begin{cases} H^\text{ev}_{dR}(X) & k = \text{even}, \\ H^\text{odd}_{dR}(X) & k = \text{odd}, \end{cases}$$

compatible with the ring structures on both sides, where $H^\bullet_{dR}$ is de Rham cohomology with its usual cup product.
Although 0|δ-dimensional theories share many similar structures with de Rham cohomology, we will show in Chapter 4 that the functor $X \mapsto 0|\delta$-EFT$^*[X]$ is a cohomology theory only when $\delta = 0$ or 1.

Following the discussion in Section 2.1 below we will identify 0|δ-dimensional twisted field theories with equivariant functions on $\text{SM}(\mathbb{R}^{0|\delta}, X)$. The following proposition is the key to computing concordance classes.

**Proposition 2.2.17.** Let $\delta > 0$ and let $G$ be a supersymmetric 0|δ-dimensional model geometry. Then two twisted 0|δ-dimensional field theories over $X$ are concordant if and only if they are $\Delta$-cohomologous:

$$[E_-] = [E_+] \iff E_+ - E_- = \Delta e,$$

for $e \in C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$ where $e$ satisfies equivariance properties such that $\Delta e$ is a twisted field theory. An identical statement also holds for $E_-$ and $E_+$ being twisted renormalizable field theories.

**Proof.** All the functions and operators employed below respect the N-grading on the algebra $C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$, so the argument automatically applies to renormalizable field theories.

Let $\lambda$ be a coordinate on $\mathbb{R}$. If $E_+ - E_- = \Delta e$, then define

$$\tilde{E}(\lambda) := E_- + \Delta(b(\lambda) \cdot e) \in C^\infty(\text{SMfld}(\mathbb{R}^{0|\delta}, X \times \mathbb{R}))$$

for $b$ a smooth bump function that is equal to 0 on $(-\infty, -1]$ and to +1 on $[1, \infty)$. The action of $G$ on $E_\pm$ is through the 1-dimensional representation $\rho$ determined by the twist, and since $E_+ - E_- = \Delta e$, we have that $G$ acts on $\Delta e$ through this 1-dimensional representation.

We claim that $\tilde{E}$ is a field theory of the appropriate twist: following Definition 2.2.10 we observe that the 1-dimensional representation defining the twist factors through a representation of a subgroup of $GL(\delta)$. Since $GL(\delta)$ acts trivially on the subspace $C^\infty(M) \subset C^\infty(\text{SMfld}(\mathbb{R}^{0|\delta}, M))$ for any $M$, it acts trivially on $b(\lambda)$. The action of $G$ is through algebra automorphisms, and using the fact that the operator $\Delta$ comes from the action of $\mathbb{R}^{0|\delta} \times GL(\delta)$ we find that the action of $G$ on $\Delta(b(\lambda)e)$ is through the same 1-dimensional representation as the action of $G$ on $\Delta e$. Hence, $\tilde{E}$ is a twisted field theory of the appropriate degree. Examining the various pullbacks, $\tilde{E}$ gives a concordance.

Now suppose that $\tilde{E}$ is a concordance from $E_+$ to $E_-$, and let $\partial_\lambda$ be a nonvanishing vector field on $\mathbb{R}$ associated to a choice of coordinate $\lambda$. We run the usual Stokes-type argument; namely we shall define a linear map

$$Q : 0|\delta\text{-GFT}^*(X \times \mathbb{R}) \to C^\infty(\text{SM}(\mathbb{R}^{0|\delta}, X))$$

with the property $\Delta Q = i^*_+ - i^*_-$, allowing $Q(\tilde{E}) =: e$. Let

$$Q(\tilde{E}) := \int_{-1}^{1} i^*_\lambda I_{\partial_\lambda} \tilde{E} d\lambda,$$
where we view the integral as a $0|\delta\text{-GFT}(X)$-valued function on $\mathbb{R}$. We compute

$$\mathcal{L}_{\partial_\lambda} \tilde{E} = \left[ d_\delta, \ldots, [d_2, [d_1, I_{\partial_\lambda}]] \ldots \right] \tilde{E} = \Delta I_{\partial_\lambda} \tilde{E}$$

where the first equality uses Equation 2.10 and the second expands the Lie brackets into a sum of operators acting on $\tilde{E}$; uses the fact that $d_k \tilde{E} = 0$ for all $k$, and observes that the only remaining nonzero terms is $\Delta I_{\partial_\lambda} \tilde{E}$.

We calculate

$$\Delta Q \tilde{E} = \int_{-1}^1 i_\lambda^* \Delta I_{\partial_\lambda} \tilde{E} d\lambda = \int_{-1}^1 i_\lambda^* \mathcal{L}_{\partial_\lambda} \tilde{E} d\lambda = i_\lambda^* \tilde{E} - i_\lambda^* \tilde{E},$$

where the first equality is differentiation under the integral together with naturality of $\Delta$, and the last is the fundamental theorem of calculus. Thus, we have shown that $E_+$ and $E_-$ are $\Delta$-cohomologous.

The following lemma allows us to use morphisms of presheaves to study concordance classes.

**Proposition 2.2.18.** Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves taking values in some category. Any morphism $R: \mathcal{F} \to \mathcal{G}$ respects concordance classes.

**Proof.** By naturality, we have commutative diagrams,

$$
\begin{array}{ccc}
\mathcal{F}(X \times \mathbb{R}) & \xrightarrow{R} & \mathcal{G}(X \times \mathbb{R}) \\
\downarrow i_\pm^* & & \downarrow i_\pm^* \\
\mathcal{F}(X \times (\pm 1, \pm \infty)) & \xrightarrow{R} & \mathcal{G}(X \times (\pm 1, \pm \infty)),
\end{array}
\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{R} & \mathcal{G}(X) \\
\downarrow \pi_\pm^* & & \downarrow \pi_\pm^* \\
\mathcal{F}(X \times (\pm 1, \pm \infty)) & \xrightarrow{R} & \mathcal{G}(X \times (\pm 1, \pm \infty))
\end{array}
$$

so that a concordance $\tilde{E}$ between sections $E_+, E_- \in \mathcal{F}(X)$ maps to a concordance $R(\tilde{E})$ between sections $R(E_+)$ and $R(E_-)$ in $\mathcal{G}(X)$. \qed

**Corollary 2.2.19.** Suppose that $\delta > 0$ and a chosen geometry $\mathcal{G}$ satisfies the hypothesis of Proposition 2.2.17. Then the quotient map that sends a field theory to its concordance class $0|\delta\text{-GFT}(\text{pt}) \sim 0|\delta\text{-GFT}[\text{pt}]$ is an isomorphism.

**Proof.** We begin with the untwisted case and compute:

$$0|\delta\text{-GFT}(\text{pt}) \cong C^\infty(\text{SM}(\mathbb{R}^0|\delta, \text{pt})/G)^{ev} \cong C^\infty(\text{pt}/G)^{ev} \cong (C^\infty(\text{pt})^{ev})^G.$$

Since the $G$-action is trivial, $C^\infty(\text{pt})^G \cong \mathbb{K}$. By Proposition 2.2.17

$$\mathbb{K} \cong 0|\delta\text{-GFT}(\text{pt}) \sim 0|\delta\text{-GFT}[\text{pt}] \cong \mathbb{K}$$
where we use that the action of $\Delta$ on $K$ is by zero, so two field theories over the point are concordant if and only if they are equal, i.e., $0|\delta$-$\text{GFT}(pt) \cong 0|\delta$-$\text{GFT}[pt]$.

For twisted field theories the story is very similar. By assumption $\mathbb{R}^{0|\delta} < G$ is in the kernel of the homomorphism $\rho: G \to K \times$ that defines the twist, so Proposition 2.2.17 still applies. We compute,

$$0|\delta$-$\text{GFT}^{\rho}(pt) \cong \Gamma(\text{SM}(\mathbb{R}^{0|\delta}, pt)//G; \mathcal{L}_\rho)^{ev} \cong \Gamma(pt//G; \mathcal{L}_\rho)^{ev},$$

and the sections of this line bundle are either 0 or $K$. In the former case, there is only one connected component of $0|\delta$-$\text{GFT}^{\rho}[pt]$; in the latter case the argument proceeds identically to the untwisted version. In summary,

$$0|\delta$-$\text{GFT}^{\rho}[pt] \cong \left\{ \begin{array}{ll} K & \\
0 & \end{array} \right.$$ with the options depend on the existence of nonzero sections of $\mathcal{L}_\rho$ over $pt//G$.

\[\text{Remark 2.2.20.}\] We wish to explain heuristically the way in which the above statements can be interpreted in terms of the topology of the generalized spaces $d|\delta$-$\text{GFT}$, and how this can be used to deduce Theorem 2.0.18. We observe that

$$\text{PreSt}(X, d|\delta$-$\text{GFT}) \cong d|\delta$-$\text{GFT}(X), \quad [X, d|\delta$-$\text{GFT}] \cong d|\delta$-$\text{GFT}[X]$$

where the first equation computes the groupoid of morphisms in the bicategory of prestacks, and the second equation computes smooth homotopy classes of maps. Lemma 2.2.18 shows that morphisms of prestacks induce morphisms of concordance classes, so in particular dimensional reduction as in Definition 2.2.1 induces a map on homotopy groups

$$(\text{red}_T)_* : \pi_0(d|\delta$-$\text{GFT}) \to \pi_0(0|\delta$-$\text{GFT}), \quad \pi_0(d|\delta$-$\text{GFT}) := [S^0, d|\delta$-$\text{GFT}]_{ptd} \cong d|\delta$-$\text{GFT}[pt],$$

where $[-, -]_{ptd}$ denotes smooth homotopy classes of pointed maps and we use the fact that $d|\delta$-$\text{GFT}$ is pointed by the 0 field theory. By Corollary 2.2.19 the composition

$$\text{PreSt}(pt, 0|\delta$-$\text{GFT}) \cong 0|\delta$-$\text{GFT}(pt) \to 0|\delta$-$\text{GFT}[pt] \cong [pt, 0|\delta$-$\text{GFT}] \cong \pi_0(0|\delta$-$\text{GFT})$$

is an isomorphism; in this sense, $0|\delta$-$\text{GFT}$ as a discrete space. By Proposition 2.2.18 the map $Z$ in (2.4) takes field theories $E_0, E_1 \in d|\delta$-$\text{GFT}(pt)$ in the same path component (i.e., concordant field theories) to concordant sections of $d|\delta$-$\text{GFT}_0(pt) \cong C^\infty(\mathcal{M}_T)$; but by Corollary 2.2.19 these concordant sections must be equal when evaluated at every torus $T$, and hence must be equal as functions on $\mathcal{M}_T$, so $Z(E_0) = Z(E_1)$.
The proof of Theorem 2.0.18 and its twisted version

Proof of Theorem 2.0.18. By Proposition 2.2.18, dimensional reduction gives a commutative square

\[
\begin{array}{ccc}
d|\delta\text{-GFT}(pt) & \xrightarrow{\text{red}_T} & 0|\delta\text{-GFT}(pt) \\
\downarrow & & \downarrow \\
d|\delta\text{-GFT}[pt] & \xrightarrow{\text{red}_T} & 0|\delta\text{-GFT}[pt].
\end{array}
\] (2.20)

The assumption that the $d|\delta$-geometry is supersymmetric means that the dimensionally reduced geometry satisfies the hypothesis of Proposition 2.2.17 and so the vertical arrow on the right side is the identity map by Corollary 2.2.19. It suffices to show that for any $E \in d|\delta\text{-GFT}(pt)$ the number $\text{red}_T(E) \in \mathbb{R}$ is a concordance invariant, but by the diagram above, we have

\[\text{red}_T(E) = [\text{red}_T([E])]\]

The right hand side is obviously a concordance invariant, so the left side must be as well. By varying $T$, we see that the partition function is a concordance invariant for each torus, and hence the function itself is a concordance invariant. \qed

Using Proposition 2.2.8, an identical argument works for twisted field theories satisfying the required hypothesis.

Theorem 2.2.21. Let $(M, G)$ be a supersymmetric model geometry, and $T$ be a twist satisfying the hypothesis of Proposition 2.2.8 when restricted to all $(M, G)$-tori of the form $\mathbb{R}^{d|\delta}/\mathbb{Z}^d$. Then the partition function of a twisted field theory $E \in d|\delta\text{-GFT}^T(pt)$ is a concordance invariant.

Remark 2.2.22. The twists of $1|1$- and $2|1$-Euclidean field theories considered by Stolz and Teichner satisfy the hypothesis of Proposition 2.2.8, so their partition functions are necessarily concordance invariants.

For super Euclidean, Poincaré and conformal geometries, there is a natural dilation action on the model geometry, given by the action on $\mathbb{R}^{d|\delta}$,

\[(v, \theta) \mapsto (q^2 v, q\theta), \quad (v, \theta) \in \mathbb{R}^{d|\delta}(S), \quad q \in \mathbb{R}_{>0}(S).\]

We observe that dimensional reduction as defined above is $\mathbb{R}_{>0}$-equivariant for these model geometries. This allows for the following definition.

Definition 2.2.23. Let $G$ define a Euclidean, Poincaré, conformal Euclidean, or conformal model geometry. A field theory $E \in d|\delta\text{-GFT}^\bullet(X)$ is renormalizable if its image under dimensional reduction to $\text{red}_T(E) \in 0|\delta\text{-GFT}^\bullet(X)$ is renormalizable in the sense of Definition 2.1.10. The groupoid of renormalizable field theories, denoted $d|\delta\text{-GFT}^\bullet_{\text{pol}}(X)$ is the
homotopy pullback
\[
d|\delta\text{-GFT}_{\text{pol}}(X) \leftrightarrow d|\delta\text{-GFT}^*(X) \\
\downarrow \quad \downarrow \text{red}_T \\
0|\delta\text{-GFT}_{\text{pol}}(X) \leftrightarrow 0|\delta\text{-GFT}^*(X),
\]
taken in (pre)stacks.

**Remark 2.2.24.** Because 0|\delta\text{-GFT}^* is discrete in the sense that it is a sheaf of a sets, the homotopy pullback above is equivalent to an ordinary pullback.

Since Proposition 2.2.17 applies to renormalizable field theories, the proof of Theorem 2.2.21 proceeds without alteration for renormalizable field theories.

**Remarks on extending dimensional reduction**

In this section we discuss generalizations of the construction of dimensional reduction in the sense of Definition 2.2.1. First we claim that one cannot hope to construct such maps for arbitrary \(\mathbb{T}\). To set up our counter-example, the commutative square (2.20) implies that if reduction along a d|\delta\text{-manifold} \(\Sigma\) is possible, then \(E(\Sigma)\) is a concordance invariant of \(E \in d|\delta\text{-GFT}(pt)\) when \(\delta > 0\).

**Example 2.2.25.** As described in Example 1.5.1 the prototypical example of a 1|1-Euclidean field theory is a Dirac operator \(D\) acting on sections of a spinor bundle, \(\mathcal{S}(M)\). The value of this field theory on closed 1|1-manifolds is determined (using the symmetric monoidal structure) by the value of the field theory on circles. For a fixed superlength \((t, \theta)\) there are two classes of circles in the 1|1-Euclidean bordism category, denoted \(S_p^{1|1}\) and \(S_{ap}^{1|1}\), corresponding to the periodic and anti-periodic spin structures on \(S^1\); in the super Euclidean language, the former arises as \(\mathbb{R}^{1|1}/\mathbb{Z}\) for \(\mathbb{Z}\) being generated by a preferred translation and the latter arises as \(\mathbb{R}^{1|1}/\mathbb{Z}\) which is generated by a preferred translation together with the generator of \(\mathbb{Z}/2 < \text{Euc}(\mathbb{R}^{1|1})\). The value of the field theory on these circles is
\[
E(S_p^{1|1}) = \text{Str}(e^{-tD^2 + \theta D}), \quad E(S_{ap}^{1|1}) = \text{Tr}(e^{-tD^2 + \theta D}),
\]
where \(\text{Str}\) denotes the supertrace and \(\text{Tr}\) the trace. The former is a concordance invariant quantity whereas the latter is not: the supertrace is the index of \(D\) so independent of, e.g., the choice of metric, whereas the trace will depend on the metric. Hence, we should only attempt dimensional reduction from 1|1 to 0|1 field theories along circles with *periodic* spin structure. These are precisely the circles of the form \(\mathbb{R}^{1|1}/t\cdot\mathbb{Z}\).

More generally, we anticipate the existence of dimensional functor reduction along tori,
\[
\text{red}_T: d|\delta\text{-EFT} \to 0|(\delta - k)\text{-EFT},
\]
where \((\delta - k)\) of the spinor bundles over \(\mathbb{T}\) are endowed with periodic boundary conditions. When \((\delta - k) > 0\), the corresponding partition functions will be concordance invariants, following the same arguments as above.
2.3 Quantization and partition functions

In this section we use Theorem 2.0.18 to prove a few things about sigma model partition functions. We begin with a brief description of sigma models before introducing a preliminary definition of a (quantized) supersymmetric sigma model, motivated by ideas of Stolz and Teichner [33] in the introduction to Section 1. We wrap up with the proof of Corollary 2.0.19.

Cartoon of sigma models

Classical sigma models are a natural generalization of classical mechanics; rather than studying paths in a Riemannian manifold $X$, we study maps of $d|\delta$-dimensional manifolds into $X$. The sigma model action functional generalizes the energy of a path: we use certain geometric structures on the source manifold together with the metric on $X$ to make sense out of \( \int_{\Sigma} |d\Phi|^2 \) for $\Phi: \Sigma \to X$.

We can quantize classical mechanics via the path integral, or more precisely, using the Wiener measure on the paths in $X$. As shown in [2, 3], one can view this measure as a limit of finite dimensional measures which is how one makes sense out of formulas like

\[
\langle O \rangle = \int_{LX} O e^{-|\gamma|^2} D\gamma,
\]

where $\gamma: S^1 \to X$ is a path, and $O \in C^\infty(LX)$ is a function (or classical observable). Sadly, analogous measures for $d > 1$ have yet to be constructed rigorously. However, in many nice examples physicists are able to describe an object that behaves very much like such a measure. Below we formalize some desired properties and give mathematical examples.

There are two important features of the Wiener measure we wish to emphasize and incorporate in the structure below. First, the classical energy is an essential ingredient: $D\gamma$ itself does not exist. Hence, if we hope to construct a quantization map we will require some geometric data on $X$ that allows us to write a kinetic energy term in the classical action, which we think of (at least philosophically) as defining a Gaussian measure on the corresponding mapping space. In some sense, the classical sigma model is this measure. Second, continuously varying the Riemannian metric on $X$ results in a continuous family of Wiener measures, which allows us to consider quantization in families. For example, a 1-parameter family of metrics on $X$ results in a 1-parameter family of quantum mechanical theories, and more generally a $Y$-family of metrics results in a $Y$-family of quantum theories. Our definition of quantization is in some sense the most general definition with these two ingredients.

Geometric orientations, quantizations and pushforwards

Consider the category whose objects are submersions of manifolds $p: X \to Y$ and whose morphisms are fiberwise isomorphisms, i.e., smooth maps such that the canonical map to the pullback is an isomorphism. We give this category the structure of a site by declaring a
family \( \{ f_i : p_i \to p \} \) to be a covering family if the map from the coproduct \( \coprod_i p_i \) is surjective. Denote this site by \( \text{Subm} \). There is a similar site defined by restricting attention to \textit{proper} submersions; we denote this site by \( p\text{Subm} \).

**Definition 2.3.1.** Let a prestack \( \mathcal{G} \) on the site \( \text{Subm} \) be given. A \( \mathcal{G} \)-oriented submersion is a submersion \( p : X \to Y \) together with an object \( g \in \mathcal{G}(p) \). For a manifold \( X \), a \( \mathcal{G} \)-structure on \( X \) is object of \( \mathcal{G} \) applied to the submersion \( X \to \text{pt} \). We define \( \mathcal{G} \)-oriented proper submersions similarly via the site \( p\text{Subm} \).

Many examples arise as discrete stacks, i.e., sheaves of sets.

**Example 2.3.2.** The sheaf \( \text{Or} \) on the site \( \text{Subm} \) assigns to a submersion \( p : X \to Y \) the set of smoothly varying orientations on the fibers. One can also view orientations as a sheaf of categories where the objects are oriented bases of the vertical tangent bundle, denoted \( Vp \), of the submersion \( p \), and morphisms are changes of bases with positive determinant. The first notion of \( \text{Or} \) is exactly (the categorical) \( \pi_0 \) of the second. The correct definition of a sheaf of orientations is largely determined by the example one has in mind; for the examples below, the sheaf of sets definition will suffice.

**Example 2.3.3.** The sheaf \( \text{Vol} \) on the site \( \text{Subm} \) assigns to \( p : X \to Y \) a smoothly varying family of volume forms, i.e., a nonvanishing section of \( \Lambda^{\text{top}}(Vp) \).

**Example 2.3.4.** The sheaf \( \text{Riem} \) on the site \( \text{Subm} \) assigns to a submersion \( p : X \to Y \) the set of fiberwise Riemannian metrics, i.e., an inner product on the vertical tangent bundle. A \( \text{Riem} \)-structure on \( X \) is exactly a Riemannian metric on \( X \).

**Example 2.3.5.** The sheaf \( \mathcal{C}^\infty \) on the site \( \text{Subm} \) assigns to a submersion \( p : X \to Y \) smooth function on the total space, viewed as the set of smoothly varying smooth functions on the fibers. Alternatively, we can take the sheaf \( \mathcal{C}^\infty_{\text{Morse}} \) that assigns the set of smoothly varying Morse functions.

**Example 2.3.6.** The stack \( \text{Spin} \) on the site \( \text{Subm} \) that assigns to a submersions \( p : X \to Y \) the groupoid of smoothly varying \( Cl(n)-Cl(Vp) \)-bimodules where \( Vp \) is the vertical tangent bundle and the fiber of \( X \to Y \) has dimension \( n \). The automorphisms in this groupoid are bimodule maps. We note that implicit in the data \( Cl(Vp) \) is a section of the sheaf \( \text{Riem} \) defined above.

**Definition 2.3.7.** For \( X \) a compact manifold and \( \mathcal{G} \) a presheaf on the site \( \text{Subm} \), define a presheaf

\[
\mathcal{G}_X(Y) := \mathcal{G}(X \times Y \to Y).
\]

We think of \( \mathcal{G}_X \) as the generalized smooth space of \( \mathcal{G} \)-structures on \( X \).
Quantization of sigma models

Below we’ll give our preliminary definition of quantization, and then explain how the various pieces work in a couple of examples.

**Definition 2.3.8** (Preliminary). Given a source prestack of field theories $d|\delta\widetilde{\text{GFT}}$ and target prestack of field theories $d|\delta\text{-GFT}$, a $\mathcal{G}$-oriented quantization is an assignment to each $\mathcal{G}$-oriented proper submersion $(p,g)$ a morphism of groupoids

$$\rho_g : d|\delta\widetilde{\text{GFT}}(X) \to d|\delta\text{-GFT}(Y).$$

The map $\rho_g$ is called quantization along $p$ with respect to geometric orientation data $g$. A variant of this definition is to consider quantization of prestacks of renormalizable field theories in the sense of Definition 2.2.23.

**Remark 2.3.9.** The main reason the above definition is preliminary is that we expect modifications when dealing with extended $d|\delta$-dimensional field theories; then $\mathcal{G}_X$ should be a sheaf of $d$-groupoids. We also expect quantization to possess a (somewhat complicated) form of naturality which is not captured in the preliminary definition.

**Remark 2.3.10.** The above definition mimics wrong-way maps in cohomology theories: given a cohomology theory $h$ and a proper submersion $p : X \to Y$ whose fibers are equipped with an $h$-orientation, we get a map $p : h^*(X) \to h^{*-n}(Y)$ where the fiber dimension of $p$ is $n$. From this perspective we think of the presheaf $\mathcal{G}$ as being a geometric refinement of $h$-orientation data. The shift in degree also appears in some of the field theory examples below.

**Definition 2.3.11.** The sigma model (with target $X$), denoted $\sigma_X(g)$, is the image of $1 \in d|\delta\widetilde{\text{GFT}}(X)$ under quantization along $X \to \text{pt}$.

**Lemma 2.3.12.** Concordant geometries produce concordant sigma models, i.e., $[g] = [g'] \in \mathcal{G}_X[\text{pt}]$ implies that $[\sigma_X(g)] = [\sigma_X(g')] \in d|\delta\text{-GFT}[\text{pt}]$.

**Proof.** If we restrict Definition 2.3.8 to the submersions $p : X \times Y \to Y$, we can consider quantization as a morphism of presheaves in the variable $Y$,

$$p : \mathcal{G}_X(Y) \times d|\delta\widetilde{\text{GFT}}(X \times Y) \to d|\delta\text{-GFT}(Y).$$

Together with Proposition 2.2.18 the lemma follows.

To build an invariant of $(X, [g])$ we need only extract a concordance invariant of $\sigma_X(g)$. With dimensional reduction functors in hand, one such invariant is the partition function.
Proof of Corollary 2.0.19. Using Lemma 2.3.12 and the notation therein, from Theorem 2.0.18 we see that for any fixed \( g \in G \) the image of \( 1 \) under the composition

\[
G_X(pt) \times d|\delta\text{-GFT}(X) \overset{\sigma_X(g) \mapsto \text{red}_T(\sigma_X(g))}{\longrightarrow} 0|\delta\text{-GFT}(pt) \cong \mathbb{R},
\]

induces a morphism on concordance classes,

\[
G_X[pt] \times d|\delta\text{-GFF}[X] \overset{\sigma_X(g) \mapsto \text{red}_T(\sigma_X(g))}{\longrightarrow} 0|\delta\text{-GFT}[pt] \cong \mathbb{R},
\]

since \( [\text{red}_T(\sigma_X(g))] = \text{red}_T(\sigma_X(g)) \). Hence the partition function is a concordance invariant of the pair \((X, [g])\). The identical proof applies to twisted and renormalizable field theories.

Now we will survey some examples of quantization in the sense of Definition 2.3.8.

Example 2.3.13. Ordinary integration on manifolds is an example of Definition 2.3.8: as shown in [18] Lemma 47, there is an isomorphism of sheaves \( 0|0\text{-TFT}(X) \cong C_\infty(X) \). Let \( G := \text{Vol} \) from Example 2.3.3 be the sheaf on \( \text{Subm} \) whose value on \( p: X \to Y \) is the set of (smoothly varying) top forms on the fibers of \( p \). Then there is a natural map

\[
p_t(g): 0|0\text{-TFT}(X) \to 0|0\text{-TFT}(Y) \iff p_t(g): C_\infty(X) \to C_\infty(Y)
\]

coming from integration of functions along the fibers \( p: X \to Y \) with respect to the chosen fiberwise volume form. In this case the sigma model is just a number \( \sigma_X(g) \in 0|0\text{-TFT}(pt) \cong \mathbb{R} \), and equal to its own partition function. We observe that the integral of a top form on \( X \) is not a concordance invariant, since all field theories in \( 0|0\text{-TFT}(pt) \cong \mathbb{R} \) are concordant.

Example 2.3.14. A simple and familiar example of quantization that includes supersymmetry comes from integration of differential forms. We take the source category of field theories to be \( 0|1\text{-TFT}^k \) and the target to be \( 0|1\text{-TFT}^{k-n} \), as defined in [18] via the geometry with \( \text{Iso}(\mathbb{R}_{01}) \cong \text{Diff}(\mathbb{R}_{01}) \cong \mathbb{R}_{01} \times \mathbb{R}^\times \). As geometric orientation data, we take the sheaf \( \text{Or} \) on \( \text{Subm} \) defined in Example 2.3.2. Using the isomorphism of sheaves \( 0|1\text{-TFT}^\ell \cong \Omega^\ell_{cl} \) with closed forms ([18], Theorem 2), integration over the fibers yields a map

\[
p_t(g): 0|1\text{-TFT}^\bullet(X) \to 0|1\text{-TFT}^{\bullet-n}(Y), \quad \omega \mapsto \int_{X/Y} \omega
\]

where \( \omega \) is a differential form on \( X \), and \( \int \) denotes the integral over the fibers of \( X \to Y \). An identical construction works for \( 0|1\text{-Euclidean field theories}; these have } \text{Euc}(\mathbb{R}_{01}) \cong \mathbb{R}_{01} \times \mathbb{Z}/2, \text{ and as a sheaf are isomorphic to even and odd forms depending on the twist degree modulo } 2. \text{ In these cases the partition function of the sigma model is } 0 \text{ unless } X \text{ is a } 0\text{-manifold, in which case it is a signed cardinality of } X. \text{ These numbers are indeed invariants of oriented manifolds.
Example 2.3.15. In the next chapter we will construct a less familiar but still finite dimensional example arising in $0|2$-Euclidean field theories. We take a geometry with $\text{Euc}(\mathbb{R}^{0|2}) \cong \mathbb{R}^{0|2} \rtimes O(2)$ and geometric orientation data the sheaf $\text{Riem} \times C^\infty$ defined in examples 2.3.4 and 2.3.5. We show that there exists a map $p_t(g): 0|2\text{-EFT}_\text{pol}(X) \to 0|2\text{-EFT}_\text{pol}(Y)$.

In this example there is an honest Gaussian measure $e^{-S}D\Phi$ for a certain classical action, and $D\Phi$ a canonical section of the Berezinian line for the finite dimensional super space of maps, $\text{SM}(\mathbb{R}^{0|2}, X)$. Since the space of metrics and smooth functions are contractible, the sigma model—again, just a number equal to the partition function—is an invariant of the manifold $X$. Indeed, we find that it is the Euler characteristic.

Moving away from finite dimensional examples, we can consider $1|\delta$-dimensional field theories, i.e., mechanics. In these examples, defining quantization requires that one identify the source category of field theories with familiar geometric objects such as vector bundles. Some of the details in such an identification have yet to be written down, but with this caveat in place we will proceed to describe the expected quantization functors.

Example 2.3.16. For ordinary mechanics, the relevant geometric orientation data is the sheaf $\text{Riem}$ of Example 2.3.4. Then we have a functor $p_t(g): 1\text{-TFT}(X) \to 1\text{-EFT}(Y)$, where the source category of field theories is that of 1-dimensional topological field theories, and the target is 1-dimensional Euclidean field theories. Dumitrescu and Stolz have shown that the stack $1\text{-TFT}$ on manifolds is equivalent to the stack of (finite dimensional) vector bundles with connection. We describe $p_t(g)$ for $Y = \text{pt}$: a vector bundle with connection $(E, \nabla)$ is sent to the quantum mechanical theory defined by the Hilbert space $L^2(X; E)$ with Hamiltonian the Hodge Laplacian twisted by $\nabla$. The sigma model for $p_t(g)(1)$ is $L^2(X)$ with the Laplacian $\Delta$, and the partition function is $\text{Tr}(e^{-t\Delta})$. This number is not a concordance invariant, as it can change as we very the input metric data.

Example 2.3.17. Adding some supersymmetry to the previous example, we claim there is a Spin-oriented (in the sense of Example 2.3.6) quantization $p_t(g): 1|1\text{-CFT}^k(X) \to 1|1\text{-EFT}^{*-n}(Y)$, where the source category is that of $1|1$-conformal field theories over $X$ and the target is the category of $1|1$-Euclidean field theories. We will sketch the expected structures of such a quantization.

A field theory in $1|1\text{-CFT}^k(X)$ is equivalent to a super vector bundle $E$ of $\text{Cl}_k$-representations with compatible super connection $\nabla$ ([11], Theorem 1.1). We define $p_t$ by mapping such a bundle to the $E$-twisted spinor bundle along the fibers of the projection with its twisted
The Cl-linear Dirac operator, $(E \otimes \mathcal{S}(X), \nabla \otimes D)$. When $Y = \text{pt}$ it is shown in \cite{17} that this data gives a degree $n$-twisted $1|1$-Euclidean field theory.

The $1|1$-sigma model is supersymmetric quantum mechanics with Hilbert space the spinors on $X$, and the Cl-linear Dirac operator furnishes a square root of the Hamiltonian. The partition function of this field theory is the index of $D$, i.e., $\hat{A}(X)$, which is a concordance invariant. In fact, as shown in \cite{18}, there is a bijection

$$\pi_0(1|1\text{-EFT}^0) \cong \mathbb{Z}$$

that identifies the connected component of a given field theory with the value of its partition function.

**Example 2.3.18.** A similar example comes from considering $1|2$-Euclidean field theories. We claim there is a Riem-oriented quantization using the Euclidean geometry described in Example \cite{1.4.4} for $G = SO(2)$ and the associated field theories in Example \cite{1.5.5}

$$p_!(g): 1|2\text{-CFT}(X) \to 1|2\text{-EFT}(\text{pt}),$$

where we claim the source field theories are equivalent to flat $\mathbb{Z}$-graded super vector bundles with super connection $(E, \nabla)$. Quantization of this returns the twisted de Rham complex, $\Omega^\bullet(X; E)$, with its twisted de Rham operator $\nabla \otimes d$, together with its adjoint, $\nabla^* \otimes d^*$. The sigma model of this quantization is the quantum mechanical system considered by Witten in \cite{36}. The partition function of this field theory is the Euler characteristic of the complex $\Omega^\bullet(X; E)$, which is the number we get under reduction along the circle $\mathbb{R}|2/\mathbb{Z}$ to $0|2\text{-EFT}^0(\text{pt}) \cong \mathbb{R}$.

We can think of closed 1-forms $\alpha$ as yielding objects in $1|2\text{-CFT}(X)$, being the connections of flat line bundles. The image of these field theories under quantization is the de Rham complex with differential $d + \alpha$. If we consider the 1-parameter family of $1|2\text{-CFTs}$ coming from $\alpha = \lambda dh$ for $h$ a Morse function and $\lambda \in \mathbb{R}$ a parameter, we obtain the family of $1|2$-Euclidean field theories considered by Witten in \cite{36}, wherein it is argued that the $\lambda \to \infty$ limit of the sigma model is the Morse complex of $X$.

Alternatively, we could define a $C^\infty_{\text{Morse}}$-oriented quantization (see Example \cite{2.3.5})

$$p_!(g): 1|2\text{-CFT}(X) \to 1|2\text{-EFT}(\text{pt}).$$

This quantization takes a flat vector bundle with connection $(E, \nabla)$ to the Morse-Smale complex of $X$ twisted by this bundle. In more detail, let $\{p_i\}$ be the critical locus of the chosen Morse function. Then let take the vector space $V := \oplus_i E_{p_i}$ that is the sum over the fibers of $E$ at the critical points. We get a differential on $V$ by using parallel translation along gradient flow lines, together with the usual signs in the definition of the Morse differential.

One can also interpret previous constructions in gauge theories with finite gauge groups in terms of these quantization functors.
Example 2.3.19. The appropriate way to view gauge theories within this framework is to consider bordism categories over smooth stacks: elements of $d|\delta\text{-}\mathrm{GBord}(\text{pt}//G^\nabla)$ are precisely $G$ bundles with connection on $d|\delta$-dimensional geometric bordisms, and so are the classical fields for gauge theory with gauge group $G$.

We can construct a quantization for such categories in the case of 2-dimensional Yang-Mills theory. For simplicity, let $G$ be a finite group. One can show that $2\text{-}\mathrm{TFT}(\text{pt}//G)\cong \hat{H}^3(\text{pt}//G)\cong \hat{H}^2(G)$, where $\hat{H}^\bullet$ denotes differential cohomology and the field theories are invertible 2-dimensional topological field theories. These are field theories taking values in the Picard stack, $\text{Pic}$, rather than $\text{Vect}$. In [13], D. Freed, M. Hopkins, J. Lurie and C. Teleman construct a quantization

$$p_i^{\text{YM}}: 2\text{-}\mathrm{TFT}(\text{pt}//G)\to 2\text{-}\mathrm{TFT}(\text{pt}).$$

We can define $Q_{\text{YM}}$ as taking an element of $\hat{H}^2(G)$, forming the associated central extension of $G$, and then constructing the group algebra of this centrally extended group. By [31] this algebra (which happens to be a Frobenius algebra) determines a 2-dimensional topological field theory, in fact a fully extended one. One can think of $p_i^{\text{YM}}$ as the path integral for 2-dimensional Yang-Mills theory.
Chapter 3

The Chern-Gauss-Bonnet Theorem

In this Chapter we use ideas from supersymmetric sigma models and ingredients from Chapter 2 to prove the Chern-Gauss-Bonnet Theorem. There are two main pieces to the proof: (1) construction of quantization of the 0|2-Euclidean sigma model and (2) computing the partition function of this quantization. The latter gives the desired pair of expressions for the Euler characteristic, while the former proves they are equal. Below we will begin by outlining (2).

For any choice of Riemannian metric \( g \) on \( X \), we identify a function \( S_0 \) on a super space (of fields) \( \text{SMfld}(\mathbb{R}^{0|2}, X) \cong \pi T \pi TX \) defined in complete analogy to the energy of a path: to a map \( \phi \in \text{SMfld}(\mathbb{R}^{0|2}, X) \) we have

\[
S_0(\phi) = \int_{\mathbb{R}^{0|2}} \|T\phi\|^2.
\]

This comes from the fact that \( T\phi \), the differential of the map \( \phi \), is a section of Berezinian of \( \mathbb{R}^{0|2} \) so the above integral is defined.\(^1\) We also show that the Berezinian line of \( \text{SMfld}(\mathbb{R}^{0|2}, X) \) is canonically trivialized, allowing us to integrate functions. The first result is a restatement of Theorem 1.0.1.

**Theorem 3.0.20.** The integral

\[
Z_X^{0|2}(g) := \int_{\text{SMfld}(\mathbb{R}^{0|2}, X)} \exp (-S_0) = (2\pi)^{-n/2} \int_X \text{Pf}(R),
\]

equals the integral of the Pfaffian of the curvature on \( X \), so computes the Euler characteristic.

Next we consider a modification of \( S_0 \),

\[
S_h(\phi) := \int_{\mathbb{R}^{0|2}} (\|T\phi\|^2 + \phi^* h \cdot \text{Ber}),
\]

\(^1\)We are being sloppy: as usual in supergeometry, for the expression to be interesting we need \( S \)-families of maps, \( \phi: \mathbb{R}^{0|2} \times S \to X \), and so the integral is over the fibers of a projection \( \mathbb{R}^{0|2} \times S \to S \).
for $h \in C^\infty(X)$ and Ber a section of the Berezinian on $\mathbb{R}^{0|2}$. We define

$$Z_X^{0|2}(g, \lambda h) := \int_{\text{SMfld}(\mathbb{R}^{0|2}, X)} \exp (-\mathcal{S}_{\lambda h}),$$

and use this to obtain the other side of the Chern-Gauss-Bonnet formula.

**Theorem 3.0.21.** Let $h \in C^\infty(X)$ be a Morse function and $\lambda \in \mathbb{R}_{>0}$ be a parameter. Then

$$\lim_{\lambda \to \infty} Z_X^{0|2}(g, \lambda h) = \text{Index}(\nabla h)$$

where the right hand side is the Hopf index of the gradient vector field $\nabla h$.

In order to prove that the two sides of the Chern-Gauss-Bonnet formula are equal, we will show that $Z_X^{0|2}(g, \lambda h)$ comes as a piece of larger structure, namely quantization of families of 0|2-Euclidean field theories,

$$0|2\text{-EFT}_{\text{pol}}^\bullet(X) \xrightarrow{Q(g, h)} 0|2\text{-EFT}_{\text{pol}}^\bullet(\text{pt}),$$

$$1 \quad \mapsto \quad Z_X^{0|2}(g, h)$$

as defined in the previous chapter. The existence of this quantization proves the following.

**Corollary 3.0.22.** As a function on metrics $g$ and smooth functions $h$, $Z_X^{0|2}(g, h)$ is locally constant.

From the above three results we deduce the following.

**Corollary 3.0.23** (Chern-Gauss-Bonnet).

$$(2\pi)^{-n/2} \int_X \text{Pf}(R) = \text{Index}(\nabla h).$$

Ours is not the first quantum field theory proof of the Chern-Gauss-Bonnet Theorem. The original physical arguments are due to Alvarez-Gaume [1] and Witten [36]. This inspired mathematical arguments using heat kernels by Berline, Getzler, and Vergne [4], Lott [26], and Roe [30]. There were also more algebraic approaches such as the Mathai-Quillen formalism [29]. Morally, all of these proofs compute an integral over free loop space—as an infinite-dimensional manifold—and use various combinations of difficult analysis and appeals to physical reasoning to confirm the Chern-Gauss-Bonnet formula. This is motivated by the path integral in 1|2-dimensional (alias, $N = 2$ supersymmetric) quantum mechanics; from our perspective this is a quantization of a certain 1|2-Euclidean field theory. Below, by working with 0|2-Euclidean field theories we manage to keep all spaces of fields finite-dimensional, so the functional integral that defines quantization is just an ordinary integral. In some sense, our argument bridges the gap between Chern’s original one [7]—which manifestly takes place in finite-dimensional geometry—and supersymmetric field theory arguments.
3.1 The space of fields

First we need a geometric characterization of the relevant mapping space.

**Lemma 3.1.1.** Given a connection on $X$, there exists an isomorphism of supermanifolds

$$\text{SM}(\mathbb{R}^{0|2}, X) \cong p^*(\pi(TX \oplus TX))$$

where $p: TX \to X$ is the usual projection. Hence, after a choice of connection, a point in $\text{SM}(\mathbb{R}^{0|2}, X)$ is a point of $X$, two odd tangent vectors, and one even tangent vector; we denote this quadruple as $(x, \phi_1, \phi_2, F)$.

We prove Lemma 3.1.1 by giving a bijection on $S$-points,

$$p^*\pi(TX \oplus TX)(S) \cong \text{SMfld}(\mathbb{R}^{0|2}, X)(S).$$

To define the map, there is some preliminary work to be done. Recall that the ordinary covariant Hessian is a map of vector bundles over $X$

$$\text{Hess}: TX \otimes TX \to \text{Diff}^{\leq 2}(X)$$

where $\text{Diff}^{\leq 2}(X)$ denotes the bundle of differential operators of order at most 2. We can also define the Hessian on pairs of odd tangent vectors via the isomorphism

$$\pi TX \otimes \pi TX = (\mathbb{R}^{0|1} \otimes TX) \otimes (\mathbb{R}^{0|1} \otimes TX) \cong (\mathbb{R}^{0|1} \otimes \mathbb{R}^{0|1}) \otimes (TX \otimes TX) \cong TX \otimes TX$$

where we use that $\pi TX := \mathbb{R}^{0|1} \otimes TX$, $\mathbb{R}^{0|1} \otimes \mathbb{R}^{0|1} \cong \mathbb{R}$, and $\sigma$ denotes the braiding isomorphism. Precomposing Hess with the above gives a map of vector bundles over $X$,

$$\text{Hess}: \pi TX \otimes \pi TX \to \text{Diff}^{\leq 2}(X). \quad (3.1)$$

Given an $S$-point $f: S \to X$, we can pull back to obtain a map over $S$

$$(f^*\pi TX) \otimes (f^*\pi TX) \to f^*\text{Diff}^{\leq 2}(X).$$

Recall that an $S$-point of $\text{SM}(\mathbb{R}^{0|2}, X)$ is a quadruple $(f, \phi_1, \phi_2, E)$, where

$$\Phi^* = f + \phi_1 \theta_1 + \phi_2 \theta_2 + E\theta_1 \theta_2, \quad \Phi^* \in \text{ALG}(C^\infty X, C^\infty S[\theta_1, \theta_2]). \quad (3.2)$$

We can plug $\phi_1$ and $\phi_2$ into the above map and get

$$(f^*\text{Hess})(\phi_1, \phi_2) \in \Gamma(f^*\text{Diff}^{\leq 2}(X)).$$

We note $S$-points of $\text{Diff}^{\leq 2}(X)$ are maps of vector spaces $C^\infty X \to C^\infty S$ satisfying some additional conditions. Explicitly, on $X$ there is the evaluation map

$$\Gamma(\text{Diff}^{\leq 2}(X)) \otimes R C^\infty X \to C^\infty X,$$
which is a map of sheaves of $C^\infty X$-modules via the left action of $C^\infty X$ on differential operators. Using the map $f^* : C^\infty X \to C^\infty S$, we obtain a map of sheaves of $C^\infty S$-modules

$$C^\infty(S) \otimes_{f^*} \Gamma(f^*\Diff^2(X)) \otimes_{\mathbb{R}} C^\infty(X) \to C^\infty(S) \otimes_{f^*} C^\infty(X) \cong C^\infty(S).$$

So in particular, given $f^*\Hess(\phi_1, \phi_2) \in \Gamma(f^*\Diff^2(X))$ and $x \in C^\infty X$, we get a function in $C^\infty(S)$.

**Lemma 3.1.2.** Let $(f, \phi_1, \phi_2, E)$ be an $S$-point of $\text{SM}(\mathbb{R}^{0|2}, X)$. Then

$$(f, \phi_1, \phi_2, E - (f^*\Hess(\phi_1, \phi_2))$$

is an $S$-point of $p^*\pi(TX \oplus TX)$. Equivalently, $F := E - (f^*\Hess(\phi_1, \phi_2))$ is an even derivation with respect to $f$.

**Proof of Lemma 3.1.1 using Lemma 3.1.2.** The map in Lemma 3.1.2 is natural in $S$ so gives the required map on the functor of points. This map is invertible, implying Lemma 3.1.1. \qed

**Proof of Lemma 3.1.2.** The proof follows from direct computation. We assume that $X$ is an ordinary manifold, though a similar result holds with some extra signs for a general supermanifold target.

First we note that the Hessian is $C^\infty X$-linear in both vectors, and so is a map of sheaves of $C^\infty X$-modules. We have the formula\(^2\)

$$\Hess(\phi_1, \phi_2)(ab) = (\Hess(\phi_1, \phi_2)a) \cdot b + a \cdot \Hess(\phi_1, \phi_2)b + \phi_1(a) \cdot \phi_2(b) + \phi_1(b) \cdot \phi_2(a).$$

on $X$, and so when we pull back the Hessian to $S$, for $\phi_1, \phi_2 \in \Gamma(f^*\pi TX)$, and $a, b \in C^\infty X$ we find

$$f^*\Hess(\phi_1, \phi_2)(ab) = (f^*\Hess(\phi_1, \phi_2)(a)) \cdot f(b) + f(a) \cdot f^*\Hess(\phi_1, \phi_2)(b) + f_1(a) \cdot \phi_2(b) + f_1(b) \cdot \phi_2(a),$$

where both sides are elements of $C^\infty S$. The above argument is the functor of points version that Hess—being a tensor—is determined by its value (and well-defined) at points.

We recall the conditions for $E$ to be a component of an $S$-point of $\text{SMfd}(\mathbb{R}^{0|2}, X)$:

$$E(ab) = E(a)f(b) + f(a)E(b) + \phi_1(a)\phi_2(b) + \phi_1(b)\phi_2(a).$$

Upon subtracting, $F := E - f^*\Hess(\phi_1, \phi_2)$ is thus an even derivation,

$$(E + f^*\Hess(\phi_1, \phi_2))(ab) = E(a)f(b) + (f^*\Hess(\phi_1, \phi_2)(a))f(b) + f(a)E(b) + f(a)f^*\Hess(\phi_1, \phi_2)(b) + f(a)F(b).$$

This completes the proof. \qed

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\(^2\)This can be verified directly with the classical formula for the Hessian, $v \otimes w \mapsto vw - \nabla_v w$, with some care not to introduce extra signs from the braiding isomorphism, $\sigma$. 

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*This can be verified directly with the classical formula for the Hessian, $v \otimes w \mapsto vw - \nabla_v w$, with some care not to introduce extra signs from the braiding isomorphism, $\sigma$. 

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Remark 3.1.3. Lemma 3.1.1 also allows us to say something about spaces of fields for other \(d|2\) field theories. Notice

\[
\text{SM}(\Sigma^{d|2}, X) \cong \text{SM}(\Sigma^{d|0}, \text{SM}(\mathbb{R}^{0|2}, X)),
\]

where we’ve assumed the odd plane bundle on \(\Sigma\) is the topologically trivial one. Thus for a fixed connection on \(X\), the \(S\)-points of \(\text{SMfld}(\Sigma^{d|2}, X)\) can be identified with quadrupoles

\[
f: \Sigma^d \to X, \phi_1 \in \Gamma(f^*\pi TX), \phi_2 \in \Gamma(f^*\pi TX), F \in \Gamma(f^*TX).
\]

When \(d = 2\), these are the usual component fields for the 2-dimensional sigma model with two supersymmetries. We can restate the data of (3.3) as

\[
f \in \text{SMfld}(\Sigma, X), \phi_1 \in \pi T_f(\text{SMfld}(\Sigma, X)), \phi_2 \in \pi T_f(\text{SMfld}(\Sigma, X)), F \in T_f(\text{SMfld}(\Sigma, X)).
\]

This shows that Lemma 3.1.1 holds for generalized manifolds that arise as mapping spaces.

### 3.2 The classical action

As is usual in Lagrangian mechanics the action functional is defined in terms of a Lagrangian density, which we will state in terms of \(S\)-points \(\Phi\),

\[
S(\Phi) = \int_{S \times \mathbb{R}^{0|2}/S} \mathcal{L}, \quad \Phi \in \text{SMfld}(S \times \mathbb{R}^{0|2}, X),
\]

where \(\mathcal{L} \in \text{Ber}(S \times \mathbb{R}^{0|2}/S)\) is a relative density. The Lagrangian will contain two parts,

\[
\mathcal{L} := \frac{1}{2} \|T\Phi\|^2 - \lambda \Phi^* h \cdot \text{Ber}
\]

for \(h \in C^\infty X\) and \(\text{Ber} \in \Gamma(\text{Ber}(S \times \mathbb{R}^{0|2}/S))\). We think of \(\|T\Phi\|^2\) as the kinetic energy of the map \(\Phi\), and \(\Phi^* h\) as its potential energy.

First we focus on defining and understanding the kinetic part. We will give a coordinate-independent definition of \(\mathcal{L}\) before fixing coordinates to set up for computations that follow. Let \(\Phi \in \text{SMfld}(\mathbb{R}^{0|2}, X)(S)\). Then \(T\Phi \in \Gamma(S \times \mathbb{R}^{0|2}, \text{Hom}_{S \times \mathbb{R}^{0|2}}(T\mathbb{R}^{0|2}, \Phi^* TX))\), where (in an abuse of notation) \(T\mathbb{R}^{0|2}\) denotes the vertical tangent bundle to \(S \times \mathbb{R}^{0|2} \to S\) and \(\text{Hom}_{S \times \mathbb{R}^{0|2}}(T\mathbb{R}^{0|2}, \Phi^* TX)\) denotes the vector bundle of fiberwise linear maps from \(T\mathbb{R}^{0|2}\) to \(\Phi^* TX\) over \(S \times \mathbb{R}^{0|2}\). The metric on \(X\) gives a pairing

\[
\langle -, \cdot \rangle: \Phi^* TX \otimes \Phi^* TX \to C^\infty(S \times \mathbb{R}^{0|2}),
\]

which we apply to \(T\Phi \otimes T\Phi \in \Gamma(S \times \mathbb{R}^{0|2}, \text{Hom}_{S \times \mathbb{R}^{0|2}}(T\mathbb{R}^{0|2}, \Phi^* TX)^{\otimes 2})\) to obtain

\[
\|T\Phi\|^2 := \langle T\Phi \otimes T\Phi \rangle \in \Gamma(S \times \mathbb{R}^{0|2}, \text{Hom}_{S \times \mathbb{R}^{0|2}}(T\mathbb{R}^{0|2} \otimes T\mathbb{R}^{0|2}, \mathbb{R})).
\]
where \( \mathbb{R} \) is the trivial bundle on \( S \times \mathbb{R}^{0|2} \). By the symmetry of the pairing \( \langle -,- \rangle \), we find that

\[
\|T\Phi\|^2 \in \Gamma(\text{Sym}^2((T\mathbb{R}^{0|2})^*) \subset \Gamma((T\mathbb{R}^{0|2} \otimes T\mathbb{R}^{0|2})^*) \cong \Gamma(\text{Hom}_{S\times \mathbb{R}^{0|2}}(T\mathbb{R}^{0|2} \otimes T\mathbb{R}^{0|2}, \mathbb{R}))
\]

where \( \text{Sym}^2((T\mathbb{R}^{0|2})^*) \) is the second symmetric power of the super vector bundle \( (T\mathbb{R}^{0|2})^* \). This bundle is precisely \( \text{Ber}(S \times \mathbb{R}^{0|2}/S) \), verifying that \( \|T\Phi\|^2 \) is indeed a section of the relative Berezinian.

There are some important symmetries of this Lagrangian density. The action of \( \text{Euc}(\mathbb{R}^{0|2}) \cong \mathbb{R}^{0|2} \rtimes O(2) \) on \( \text{SMfld}(\mathbb{R}^{0|2}, X) \) by precomposition induces an action on on functions on this space. The action on \( \|T\Phi\|^2 \) is entirely through the action on \( \text{Sym}^2((T\mathbb{R}^{0|2})^*) \). The action by translations of \( \mathbb{R}^{0|2} \) is trivial, and given an \( S \)-point \( A \) of \( O(2) \), it acts by \( 1/\det(A) \) on the bundle \( \text{Sym}^2((T\mathbb{R}^{0|2})^*) \). This is the identical action to that of \( \text{Euc}(\mathbb{R}^{0|2}) \) on \( \text{Ber}(S \times \mathbb{R}^{0|2}/S) \), so the map

\[
\|T\Phi\|^2 : \text{SMfld}(\mathbb{R}^{0|2}, X) \to \text{Ber}(S \times \mathbb{R}^{0|2}/S)
\]

is \( \text{Euc}(\mathbb{R}^{0|2}) \)-equivariant, and so is an invariant \( \text{Ber}(S \times \mathbb{R}^{0|2}/S) \)-valued function.

Above we understood \( \|T\Phi\|^2 \) in terms of the supergeometry of \( \text{SMfld}(\mathbb{R}^{0|2}, X) \); presently we wish to describe \( \|T\Phi\|^2 \) via local geometry on \( X \) (e.g., curvature and the Riemannian metric). First we trivialize the relative Berezinian of \( S \times \mathbb{R}^{0|2} \to S \) by choosing coordinates \( \theta_1, \theta_2 \), which gives us trivializing sections \( \partial_{\theta_1}, \partial_{\theta_2} \) of \( T\mathbb{R}^{0|2} \), so

\[
\Phi \mapsto \langle T\Phi(\partial_{\theta_1}), T\Phi(\partial_{\theta_2}) \rangle [d\theta_1 d\theta_2].
\]

Below we will focus on computing \( \langle T\Phi(\partial_{\theta_1}), T\Phi(\partial_{\theta_2}) \rangle \). We require the following.

**Lemma 3.2.1.** Let \( f : N \to M \) be a map of supermanifolds. Then

\[
\text{Der}(C^\infty M, C^\infty M) \otimes_f C^\infty N \cong \text{Der}_f(C^\infty M, C^\infty N).
\]

**Proof.** For \( W \otimes n \in \text{Der}(C^\infty M, C^\infty M) \otimes_f C^\infty N \) we define a map

\[
W \otimes n \mapsto n \cdot V, \quad V(m) := (f^*W)m.
\]

One can show that map is bijective abstractly, but we will need an explicit inverse map for computations below. As usual with maps into a tensor product, this inverse is somewhat less natural and we need coordinates \( \{x^i\} \) on \( M \) to define it. We will show the above isomorphism holds in each coordinate patch, and the sheaf property will prove the result.

With a choice of coordinates in effect, given \( V \in \text{Der}_f(C^\infty M, C^\infty N) \), we get a map

\[
V \mapsto \sum (\partial_{x^i}) \otimes_f V(x^i).
\]

One can check explicitly that this defines an inverse in the given chart \( \{x^i\} \).
To calculate $\langle T\Phi(\partial_{t_1}), T\Phi(\partial_{t_2}) \rangle$, we apply Lemma 3.2.1 to $N = S \times \mathbb{R}^{0|2}$ and $M = X$. For an $S$-point $\Phi \in \text{SMfld}(\mathbb{R}^{0|2}, X)(S)$, we will examine the composition

$$
\left( \text{Der}(C^\infty X, C^\infty(S \times \mathbb{R}^{0|2})) \otimes_{C^\infty(S \times \mathbb{R}^{0|2})} \right) \to \text{Der}(C^\infty X, C^\infty(S \times \mathbb{R}^{0|2}))
$$

where in the last line we use the action of $C^\infty X$ on $C^\infty(S \times \mathbb{R}^{0|2})$ by $\Phi^*$, and in the second to last line the metric on $X$ is thought of as

$$
g : \text{Der}(C^\infty X) \otimes_{C^\infty X} \text{Der}(C^\infty X) \to C^\infty X.
$$

Now we compute for an $S$-point $\Phi : S \times \mathbb{R}^{0|2} \to X$,

$$
T\Phi(\partial_{t_1}) = \psi_1 + \theta_2 F, \quad T\Phi(\partial_{t_2}) = \psi_2 - \theta_1 F,
$$

so that $T\Phi(\partial_{t_i}) \in \text{Der}_\Phi(C^\infty X, C^\infty(S \times \mathbb{R}^{0|2}))$. Lemma 3.2.1 gives us an isomorphism

$$
T\Phi(\partial_{t_i}) \in \text{Der}(C^\infty X, C^\infty(S \times \mathbb{R}^{0|2})) \cong \text{Der}(C^\infty X, C^\infty X) \otimes_{C^\infty} C^\infty(S \times \mathbb{R}^{0|2})
$$

and using the proof of the lemma we find

$$
T\Phi(\partial_{t_1}) \mapsto \sum_i \frac{\partial}{\partial x^i} \otimes_{C^\infty} (d_1 x^i + \theta_2 d_2 d_1 x^i)(\Phi),
$$

$$
T\Phi(\partial_{t_2}) \mapsto \sum_j \frac{\partial}{\partial x^j} \otimes_{C^\infty} (d_2 x^j - \theta_1 d_2 d_1 x^j)(\Phi),
$$

where as usual we are identifying functions with their natural transformations in the notation of Equation 2.8. Then we can apply the pairing $g$,

$$
\langle T\Phi(\partial_{t_1}), T\Phi(\partial_{t_2}) \rangle = \sum_{ij} g_{ij} \otimes_{C^\infty} (d_1 x^i + \theta_2 d_2 d_1 x^i)(d_2 x^j - \theta_1 d_2 d_1 x^j)(\Phi)
$$

where $g_{ij}$ are the components of $g$ in the given coordinates.

So now we need to understand how the pulled back metric, $\Phi^* g_{ij}$, acts on functions on $\text{SM}(\mathbb{R}^{0|2}, M)$. So we compute $\Phi^* (g_{ij})$, getting

$$
\Phi^* (g_{ij}) = g_{ij}(\Phi) + \theta_1 d_1 g_{ij}(\Phi) + \theta_2 d_2 g_{ij}(\Phi) + \theta_1 \theta_2 d_2 d_1 g_{ij}(\Phi).
$$
Putting this together we obtain an element of $C^\infty(\text{SM}(\mathbb{R}^{0|2}, M))$, whose value at an $S$-point $\Phi$ is

$$\langle T\Phi(\partial_{\theta_1}), T\Phi(\partial_{\theta_2}) \rangle = \sum_{i,j} \left((g_{ij} + \theta_1 d_1 g_{ij} + \theta_2 d_2 g_{ij} + \theta_1 \theta_2 d_2 d_1 g_{ij})(\Phi) \cdot (d_1 x^i + \theta_2 d_2 d_1 x^i)(d_2 x^j - \theta_1 d_2 d_1 x^j)(\Phi)\right).$$ (3.4)

This formula will have more obvious geometric meaning when we pass from the Lagrangian density to the action functional.

**Lemma 3.2.2.** The action functional for the $0|2$-sigma model with potential $h$ evaluated at a point $\Phi = (x, \phi_1, \phi_2, F) \in \text{SM}(\mathbb{R}^{0|2}, X)$ is

$$S(\Phi) = \frac{1}{2} \|F\|^2 + R(\phi_1, \phi_2, \phi_1, \phi_2) - \langle F, \nabla h \rangle - \text{Hess}(h)(\phi_1, \phi_2).$$

**Proof.** We accomplish this by computing the integral

$$S(\Phi) := \int_{S \times \mathbb{R}^{0|2}/S} \frac{1}{2} \|T\Phi\|^2 = \int_{S \times \mathbb{R}^{0|2}/S} \frac{1}{2} \langle T\Phi(\partial_{\theta_1}), T\Phi(\partial_{\theta_2}) \rangle [d\theta_1 d\theta_2]$$

using Equation (3.4). Recall that

$$\int_{S \times \mathbb{R}^{0|2}/S} \theta_1 \theta_2 [d\theta_1 d\theta_2] = 1 \in C^\infty(S),$$

so if we expand Equation (3.4) and project to the $\theta_1 \theta_2$ component, we get

$$S(\Phi) = \frac{1}{2} \sum_{i,j} \left(g_{ij} d_2 d_1 x^i d_2 d_1 x^j + d_1 g_{ij} d_2 d_1 x^i d_2 d_1 x^j + d_2 g_{ij} d_1 x^i d_2 d_1 x^j + d_2 d_1 g_{ij} d_1 x^i d_2 x^j\right)(\Phi).$$

$$= \frac{1}{2} \sum_{i,j} \left(g_{ij} d_2 d_1 x^i d_2 d_1 x^j + \frac{\partial g_{ij}}{\partial x^k} d_1 x^k d_2 d_1 x^i d_2 x^j + \frac{\partial g_{ij}}{\partial x^k} d_2 x^k d_1 x^i d_2 d_1 x^j + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} d_2 x^k d_1 x^i d_2 x^j\right)(\Phi)$$

Using Lemma 3.1.1, we can interpret the above in terms of more familiar Riemmanian geometry of $X$. For the rest of the proof, formulas will employ the index summation convention. First we collect the terms in (3.5) that have a first derivative of $g_{ij}$ and we observe

$$\left(\frac{\partial g_{ij}}{\partial x^k} d_1 x^k d_2 d_1 x^i d_2 x^j + \frac{\partial g_{ij}}{\partial x^k} d_2 x^k d_1 x^i d_2 d_1 x^j + \frac{\partial g_{ij}}{\partial x^k} d_2 d_1 x^k d_1 x^i d_2 x^j\right)(\Phi)$$

$$= \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j}\right) d_2 d_1 x^k d_1 x^i d_2 x^j(\Phi) = 2\Gamma_{ijk} d_2 d_1 x^k d_1 x^i d_2 x^j(\Phi)$$
where $\Gamma_{ijk}$ denotes the Christoffel symbol. Next we notice that for $x^k$ a coordinate,

$$\text{Hess}(\phi_1, \phi_2)(x^k)(\Phi) = -\Gamma^k_{ij} \phi_1(x^i)\phi_2(x^j) = -\Gamma^k_{ij} dx^i dx^j(\Phi) \quad (3.6)$$

using the fact that the second derivative of a coordinate function vanishes. Making the identifications

$$\phi_1(x^i) = dx^i(\Phi), \quad \phi_2(x^j) = dx^j(\Phi), \quad (F + \text{Hess}(\phi_1, \phi_2))(x^k) = d_2 d_1 x^k(\Phi),$$

we compute

$$2S(\Phi) = g_{ij}(F + \text{Hess}(\phi_1, \phi_2))(x^i)(F + \text{Hess}(\phi_1, \phi_2))(x^j)$$

$$+ 2\Gamma_{ijk} (F + \text{Hess}(\phi_1, \phi_2))(x^k)\phi_1(x^i)\phi_2(x^j) + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} \phi_1(x^i)\phi_2(x^j)\phi_1(x^k)\phi_2(x^l)$$

$$= g_{ij} F(x^i) F(x^j) + g_{ij} F(x^i) \text{Hess}(\phi_1, \phi_2)(x^j) + g_{ij} \text{Hess}(\phi_1, \phi_2)(x^i) F(x^j)$$

$$+ g_{ij} \text{Hess}(\phi_1, \phi_2)(x^i) \text{Hess}(\phi_1, \phi_2)(x^j) + 2\Gamma_{ijk} F(x^i) \phi_1(x^j)\phi_2(x^k)$$

$$+ 2\Gamma_{ijk} \text{Hess}(\phi_1, \phi_2)(x^i)\phi_1(x^j)\phi_2(x^k) + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} \phi_1(x^i)\phi_2(x^j)\phi_1(x^k)\phi_2(x^l)$$

$$= g_{ij} F(x^i) F(x^j) + 2\Gamma_{ijk} \text{Hess}(\phi_1, \phi_2)(x^i)\phi_1(x^j)\phi_2(x^k)$$

$$+ g_{ij} \text{Hess}(\phi_1, \phi_2)(x^i) \text{Hess}(\phi_1, \phi_2)(x^j) + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} \phi_1(x^i)\phi_2(x^j)\phi_1(x^k)\phi_2(x^l)$$

$$= \langle F, F \rangle + R(\phi_1, \phi_2, \phi_1, \phi_2).$$

where in the last two equalities we use Equation 3.6 and the standard expression for the curvature tensor in local coordinates. Thus we have

$$S(\Phi) = \frac{1}{2} \left( \langle F, F \rangle + R(\phi_1, \phi_2, \phi_1, \phi_2) \right),$$

proving Lemma 3.2.2 when $h = 0$.

For $h \neq 0$, we compute

$$\Phi^* h = f(h) + \theta_1 \phi_1(h) + \theta_2 \phi_2(h) + \theta_1 \theta_2 E(h),$$

and when we integrate

$$\int_{S \times \mathbb{R}^{0|2}} \Phi^* h[d\theta_1 d\theta_2] = E(h) = F(h) + \text{Hess}(\phi_1, \phi_2) h = \langle F, \nabla h \rangle + \text{Hess}(\phi_1, \phi_2) h.$$

This computation together with the above shows

$$S_\lambda(\Phi) = \int_{S \times \mathbb{R}^{0|2}} \left( \frac{1}{2} \|T\Phi\|^2 - \lambda(\Phi^* h) \text{Ber}_{\mathbb{R}^{0|2}} \right)$$

$$= \frac{1}{2} \langle F, F \rangle + \frac{1}{2} R(\phi_1, \phi_2, \phi_1, \phi_2) - \lambda \langle F, \nabla h \rangle - \lambda \text{Hess}(\phi_1, \phi_2) h,$$

which concludes the proof of Lemma 3.2.2. \qed
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Remark 3.2.3. We can also complete the above computation for the case \( h = 0 \) in Riemann normal coordinates, so that \( E(x^i) = F(x^i) \) and \( d_2d_1g_{ij}dx^kdx^l = R_{ijkld_1x^md_1x^nd_1x^o} \), owing to the vanishing of the Christoffel symbols. However, this requires one make sense out of the vanishing of \( \Gamma_{ijk} \) at a point of \( X \) in terms of \( S \)-points of \( \mathbf{SMfld}(\mathbb{R}^{0|2}, X) \), which is a bit subtle.

Remark 3.2.4. The previous lemma is in agreement with the “finesse” utilized in \([9, 12, 14]\) to obtain the \( 1|2 \)-supersymmetric quantum mechanics action functional from the Lagrangian density. The kinetic term in this action has the form

\[
S(\gamma, \phi_1, \phi_2, F) = \frac{1}{2} \int_\gamma \left( \|\dot{\gamma}\|^2 + \langle \nabla_\gamma \phi_1, \phi_1 \rangle + \langle \nabla_\gamma \phi_2, \phi_2 \rangle + R(\phi_1, \phi_2, \phi_1, \phi_2) + \|F\|^2 \right) d\gamma
\]

where \( \gamma \) is a path in \( X \), \( \phi_i \in \Gamma(\gamma^*\pi TX) \) and \( F \in \Gamma(\gamma^*TX) \). The dimensional reduction from \( 1|2 \) to \( 0|2 \) has the effect of only considering the constant paths in \( X \) for which \( \dot{\gamma} = 0 \). This recovers the action function we derived above.

### 3.3 Constructing the quantization

**Stacky integrals and quantization of \( 0|\delta \)-EFTs**

As demonstrated in the previous Chapter, quantization is a way to produce manifold invariants from supersymmetric field theories. In dimensions \( 0|\delta \) the space of fields, \( \mathbf{SM}(\mathbb{R}^{0|\delta}, X) \), is a finite-dimensional supermanifold, so with a bit of work we can make the integration defining quantization completely rigorous. However, when \( \delta > 0 \) fields are not a space, but a stack. Weinstein has explained integrals on smooth stacks \([35]\), and our construction can be interpreted in terms of his definition. The stacky aspects end up being comparably easy, so we’ll be brief in our description.

When computing an integral on a stack, one wants to include isotropy in the computation; for example, for a stack \( pt//G \) arising from the action of a finite group \( G \) on the point \( pt \), there is a counting measure such that the volume of stack is \( 1/|G| \). In the case where a (nondiscrete) Lie group \( G \) acts on a manifold \( M \), Weinstein shows \((35), Theorem 3.2)\) that a measure on the stack \( M//G \) arises from a section of the line \( \Lambda^{\text{top}} g \otimes^{\text{top}} T^*M \) on \( M \) invariant under the action of \( G \), where \( g \) is the Lie algebra of \( G \) and we use the inclusion of Lie algebras \( g \to \Gamma(TM) \) given by the action.

In the case of \( 0|\delta \)-dimensional field theories, we have \( M = \mathbf{SMfld}(\mathbb{R}^{0|\delta}, X) \) and \( G = \mathbf{Euc}(\mathbb{R}^{0|\delta}) \). Proposition 3.3.1 will show that the supergeometric analog of \( \Lambda^{\text{top}} T^*M \) (namely, the Berezinian line on \( \mathbf{SMfld}(\mathbb{R}^{0|\delta}, X) \)) is trivialized by a choice of metric on \( X \); we denote this trivialization by \( D\Phi \). The bundle on \( \mathbf{SMfld}(\mathbb{R}^{0|\delta}, X) \) coming from the Lie algebra of \( \mathbf{Euc}(\mathbb{R}^{0|\delta}) \) is trivial, so is trivialized by a function. We will take this function to be \( \exp(-S) \). If \( S \) is invariant under the action of \( \mathbf{Euc}(\mathbb{R}^{0|\delta}) \), then so is \( \exp(-S)D\Phi \) and this data gives us a Weinstein volume form on the stack \( \mathbf{SMfld}(\mathbb{R}^{0|\delta}, X)// \mathbf{Euc}(\mathbb{R}^{0|\delta}) \).
In our case, Weinstein’s formula tells us that to integrate a function, we compute

$$\pi_t(\omega, g) = \int_{SMfld(\mathbb{R}^0|\delta)} \omega(\Phi) \exp(-S(\Phi)) \frac{D\Phi}{N},$$

(3.7)

where $N$ is some finite normalization constant into which we’ve absorbed the volume of $Euc(\mathbb{R}^0|\delta)$. When the above is defined, we are guaranteed that the result is an invariant of the stack $SMfld(\mathbb{R}^0|\delta) // Euc(\mathbb{R}^0|\delta)$ together with our choice of measure.

However, $SMfld(\mathbb{R}^0|\delta) // Euc(\mathbb{R}^0|\delta)$ is not a proper stack, so a priori there might be very few integrable functions: $SMfld(\mathbb{R}^0|\delta, X)$ is noncompact for $\delta > 1$. Indeed, this forces $S \neq 0$ for $\delta > 1$ if we want, e.g., 1 to be integrable—we require an interesting action functional to obtain a partition function. As we will describe, all functions that are polynomial “at infinity” will be integrable. This polynomial behavior can be described in terms of the renormalization group action on field theories.

### Integration on $SMfld(\mathbb{R}^0|\delta, X)$

Integration of functions on $SMfld(\mathbb{R}^0|\delta, X)$ is particularly easy, owing to a canonically trivialized Berezinian line. Explicitly, integration is the composition

$$C^\infty(SMfld(\mathbb{R}^0|1, X)) \approx \Omega^\bullet(X) \xrightarrow{\text{project}} \Omega^{\text{top}}(X) \rightarrow \mathbb{R},$$

where the last arrow requires an orientation on $X$. We claim that a similar situation holds for $SMfld(\mathbb{R}^0|\delta, X), \delta > 1$. The key result is the following.

**Proposition 3.3.1.** Let $\delta > 0$. Given a choice of connection on $TX$, there is an isomorphism

$$SMfld(\mathbb{R}^0|\delta, X) \cong \pi T(T^{\delta-1}X)$$

as supermanifolds. For $\delta > 2$ this isomorphism requires a framing of $\mathbb{R}^0|\delta$.

With this proposition, we can define integration (for compactly supported or Schwartz functions, denoted cs) as

$$C^\infty_{cs}(SMfld(\mathbb{R}^0|\delta, X)) \cong \Omega^\bullet_{cs}(T^{\delta-1}X) \xrightarrow{\text{project}} \Omega^{\text{top}}_{cs}(T^{\delta-1}X) \rightarrow \mathbb{R}.$$

Furthermore, since $TM$ is canonically oriented for any manifold $M$, this integration map has no topological obstruction on $X$ when $\delta > 1$.

**Proof of Proposition 3.3.1.** We’ve already proved this for $\delta = 1$ (without assuming the existence of a connection). Next we prove the proposition for $\delta = 2$. A connection on $X$ splits $T(TX)$ into horizontal and vertical subspaces,

$$T(TX) \cong H(TX) \oplus V(TX) \cong p^*(TX \oplus TX)$$
where we get the second isomorphism from maps $Tp : H(TX) \to TX$ and the canonical map $V(TX) \to TX$. Sprinkling in the parity reversal functor we get

$$\pi T(TX) \cong p^*(\pi (TX \oplus TX)) \cong SMfd(\mathbb{R}^{0|2}, X)$$

where the second isomorphism uses Lemma [3.1.1] concluding the proof for $\delta = 2$. Now we iterate the above isomorphism,

$$SMfd(\mathbb{R}^{0|\delta}, X) \cong (\pi T)^{\delta} X \cong \pi T(T^{\delta-1} X),$$

where the first isomorphism requires a framing on $\mathbb{R}^{0|\delta}$, and we use the fact that $\pi T\pi TX \cong SMfd(\mathbb{R}^{0|2}, X) \cong \pi T(TX)$.

Remark 3.3.2. Following [22], we can show that the following is a section

$$d_1 x^1 d_1 \xi^1 \cdots d_1 x^n d_1 \xi^n d_2 x^1 d_2 \xi^1 \cdots d_2 x^n d_2 \xi^n \in \Gamma(Ber(TSMfd(\mathbb{R}^{0|2}, U)))$$

which is independent of the choice of coordinates, verifying that the Berezinian of $SMfd(\mathbb{R}^{0|2}, U)$ is canonically trivialized independent of the choice of connection used above. However, since the integration map from the sigma model uses the metric and connection to define a Gaussian measure, we prefer the coordinate-free argument above. One can check that the two trivializations of the Berezinian are in fact the same.

The quantization

In this subsection we will prove the following.

**Theorem 3.3.3.** Let $G = Riem \times C^\infty$ be the sheaf parametrizing metrics and smooth functions on the fibers of submersions. Using the classical action of the $0|2$-sigma model, Equation 3.7 defines a quantization

$$\pi_! : 0|2\cdot\text{EFT}_{pol}(X) \to 0|2\cdot\text{EFT}_{pol}^{*\n}(Y).$$

To prove the above, we will first define a map

$$C^\infty_{pol}(SMfd(\mathbb{R}^{0|2}, X \times Y)) \xrightarrow{(-)\circ} C^\infty_{pol}(SMfd(\mathbb{R}^{0|2}, Y)),$$

and show that it restricts to field theories. Consider

$$C^\infty_{pol}(SMfd(\mathbb{R}^{0|2}, X \times Y)) \xrightarrow{\exp(-S) \otimes id} C^\infty(\text{SMfd}(\mathbb{R}^{0|2}, X)) \otimes C^\infty_{pol}(\text{SMfd}(\mathbb{R}^{0|2}, Y)) \cong \Omega^{*}(TX) \otimes C^\infty_{pol}(\text{SMfd}(\mathbb{R}^{0|2}, Y))$$

$$\xrightarrow{\text{project} \times \text{id}} \Omega^{*\text{top}}(TX) \otimes C^\infty_{pol}(\text{SMfd}(\mathbb{R}^{0|2}, Y))$$

$$\xrightarrow{\frac{1}{\pi} \int (-) \otimes \text{id}} C^\infty_{pol}(\text{SMfd}(\mathbb{R}^{0|2}, Y)).$$
where the second line uses Proposition 3.3.1 and \( \mathcal{S} \) is the action of the 0|2 sigma model on \( X \).
What remains is to check convergence of this integral. However, by how we’ve set things up, the image in \( \Omega^\bullet(TX) \) consists of functions with polynomial growth in the noncompact direction, as discussed in 2.1. We claim that the Gaussian measure \( \exp(-\mathcal{S}) \) allows us to integrate all functions in the image.

It suffices to work locally on \( X \) to verify this claim, and furthermore we can set \( Y = \text{pt} \) for this part. So let \( U \subset (\mathbb{R}^n, g) \) be a bounded open submanifold with coordinates \( \{x^i\} \). Then \( \text{SMfld}(\mathbb{R}^{0|2}, U) \cong U \times \mathbb{R}^n \times R^{0|2n} \). Polynomial functions at an \( S \)-point \( \Phi = (x, \phi_1, \phi_2, F) \) have the form
\[
G(\Phi) = g(x)P(F)\omega(\phi_1)\eta(\phi_2) \in C_{\text{pol}}(\text{SMfld}(\mathbb{R}^{0|2}, U))
\]
for \( P, \omega \) and \( \eta \) polynomials in \( n \)-variables. First we multiply by \( \exp(-\mathcal{S}) \), so we have
\[
G(\Phi) \exp(-\mathcal{S}(\Phi)) = e^{-F^2}e^{-R(\phi_1,\phi_2,\phi_1,\phi_2)}g(x)P(F)\omega(\phi_1)\eta(\phi_2) \tag{3.9}
\]
Expanding in coordinates as in the previous subsection, we project to the coefficient of \( \Pi_{i=1}^n d_1 x^i d_2 x^i \), which we identify with the a section of the line bundle \( \Omega^{\text{top}}(TU) \). The only problem we might encounter in convergence of the integral is in the \( F \)-variable. But \( P(F)e^{-|F|^2} \) is integrable on \( \mathbb{R}^n \) for any metric \( g \), which completes the local argument.

Now we need to show that this map respects the action by \( \text{Euc}(\mathbb{R}^{0|2}) \cong \mathbb{R}^{0|2} \times O(2) \), and for this we can no longer set \( Y = \text{pt} \). There are various way to do this; we prefer a relatively direct one.

**Proposition 3.3.4.** \( \langle - \rangle \) restricts to a map on \( SO(2) \times \mathbb{R}^{0|2} \)-invariant functions.

**Proof.** Since \( \exp(-\mathcal{S}) \) is \( \mathbb{R}^{0|2} \times O(2) \)-invariant, we may assume that we are given an element in \( \omega \in C^\infty(\text{SMfld}(\mathbb{R}^{0|2}, X \times Y)) \) that is invariant and integrable. It suffices to check the claim locally, so together with Fubini’s theorem we can restrict to the case that \( X = Y = \mathbb{R} \).
Choosing a coordinate \( x \) on \( X = \mathbb{R} \), we Taylor expand \( \omega \) to obtain
\[
\omega = \omega_0 + \omega_1 d_1 x + \omega_2 d_2 x + \omega_{12} d_1 x d_2 x, \\
\omega_i = \eta_i \cdot P_i(d_2 d_1 x) f_i(x)
\]
where \( \eta_i \in C^\infty(\text{SMfld}(\mathbb{R}^{0|2}, Y)) \), \( P_i \) is a polynomial, and \( f_i \in C^\infty(X) = C^\infty(\mathbb{R}) \). The map \( \langle - \rangle \) projects to the last term and integrates over the \( x \) and \( d_2 d_1 x \) variables. So proving that \( \langle \omega \rangle \) is invariant amounts to showing that either \( \eta_{12} \) is invariant or that the integral is zero.

Since \( d_1 x d_2 x \) and \( \omega \) are \( SO(2) \)-invariant, \( \omega_{12} \) must be as well. Any function of \( x \) and \( d_2 d_1 x \) is also \( SO(2) \)-invariant, so \( \eta_{12} \) must also be \( SO(2) \)-invariant.

Now suppose that \( \eta_{12} \) is not invariant under one of the \( d_i \), say \( d_1 \). Then for \( \omega \) to be invariant (i.e., \( d_1 \omega = 0 \)) we require that
\[
d_1(\omega_2) d_2 x = d_1(\omega_{12} d_1 x d_2 x).
\]
Computing we find that this implies \( f_{12}(x) = f_2'(x) \) is a total derivative, and so the integral vanishes, and so \( \langle \omega \rangle \) is \( \mathbb{R}^{0|2} \)-invariant. \( \square \)
Proof of Theorem 3.3.3. We have the isomorphism
\[ 0|2\text{-EFT}_{\text{pol}}(X \times Y) \cong C^\infty_{\text{pol}}(\text{SMfld}(\mathbb{R}^{0|2}, X \times Y))^{SO(2) \times \mathbb{R}^{0|2}}, \]
where the residual \( \mathbb{Z}/2 \)-action remaining from the original \( O(2) \) action on each side above recovers the grading via the \( \pm 1 \)-eigenspaces. We have shown there is a map \( 0|2\text{-EFT}_{\text{pol}}(X \times Y) \to 0|2\text{-EFT}_{\text{pol}}(Y) \), and it remains to analyze the grading shift. Since \( \exp(-S) \) is \( O(2) \)-invariant, we consider the situation for integrable functions invariant under \( \mathbb{R}^{0|2} \times SO(2) \) and in the \( \pm 1 \) eigenspace of the \( \mathbb{Z}/2 \)-action. Again, this is a local computation and by Fubini we can restrict to the case that \( X = Y = \mathbb{R} \).

Using the notation from the proof of the previous proposition, we have \( \langle \omega \rangle = c \cdot \eta_{12} \) for \( c \in \mathbb{R} \), so the change in grading is precisely the grade of \( f_{12}(x)P_{12}(d_2d_1x)d_2xd_1x \). Since \( f_{12} \) always has even degree and \( d_1xd_2x \) always has odd degree, we may assume that \( P \) is homogeneous even or odd. If \( P(d_2d_1x) \) is odd, the coefficient \( c \) will be zero (e.g., by Wick’s Lemma and Equation 3.9). If \( P \) is even, then the degree is lowered by 1, which is indeed the dimension of \( \mathbb{R} \).

Lastly, we fix the normalization constant \( N = (2\pi)^n \), which completes the construction of the quantization.

### 3.4 Computing the partition function

Now we insert a parameter \( \lambda \in \mathbb{R} \) in front of the potential term \( h \) in the classical action, denoting the resulting 1-parameter family of action functions by \( S_\lambda \), and the quantization with respect to this action by \( \pi(g_\lambda) \). We’d like to calculate the partition function,
\[ Z_{X}^{0|2}(g, \lambda h) = (2\pi)^{-n/2} \int_{\text{SMfld}(\mathbb{R}^{0|2}, X)} \exp(-S_\lambda(\Phi)) d\Phi. \]

We begin with the case that \( \lambda = 0 \).

Proof of Theorem 3.0.20. First we integrate over the fibers in the horizontal direction in the diagram
\[ \text{SMfld}(\mathbb{R}^{0|2}, X) \xrightarrow{\pi} TX \oplus TX \]
which amounts to a Gaussian integral in the \( F \)-variable. The result is
\[ Z_{X}^{0|2}(g, 0) = (2\pi)^{-n/2} \int_{\pi(TX \oplus TX)} \exp(-R(\phi_1, \phi_2, \phi_1, \phi_2)). \]
The odd integral will project $\exp(-R)$ onto the “top component.” We claim this picks out the Pfaffian. Explicitly, we compute in coordinates so

$$R(\phi_1, \phi_2, \phi_1, \phi_2) = R_{klij}d_1 x^k d_2 x^l d_1 x^i d_2 x^j(\Phi)$$

and if $\int (-) D\Phi$ denotes the projection to the component of $d_1 x^1 d_2 x^2 \cdots d_1 x^n d_2 x^n$,

$$\int e^{-R(\phi_1, \phi_2, \phi_1, \phi_2)/2} D\Phi = \frac{(-1)^{n/2}}{(n/2)!2^{n/2}} \sum \epsilon_{i_1 \cdots i_n} R_{i_1 i_2 i_1 i_2} \cdots R_{i_{n-1} i_n i_{n-1} i_n} = \text{Pf}(R),$$

as claimed.

Next we consider the limit $\lambda \to \infty$; heuristically we see that the integral for $Z^0_X(g, \lambda h)$ will be supported on small neighborhoods of the set where $\nabla h = 0$. Assuming $h$ is Morse, there is a now-standard argument inspired by [36] and due to [29] showing

$$\lim_{\lambda \to \infty} Z^0_X(g) = \sum_{\nabla h = 0} \text{index}(\nabla h) = \chi(X),$$

i.e., the integral computes the Hopf index of $\nabla h$, leading to Theorem 3.0.21. We now proceed to explain the details in this calculation.

**Proof of Theorem 3.0.21.** As in the previous proof, we first integrate out the $F$-variable

$$\int_{\text{SMfld}(\mathbb{R}^{0|2}, X)} \exp(-S) \frac{D\Phi}{N} = \frac{1}{N} \int_{\pi TX \oplus \pi TX} \left( \exp \left( -R(\phi_1, \phi_2, \phi_1, \phi_2)/2 + \lambda \text{Hess}(h)(\phi_1, \phi_2) \right) \right.$$

$$\cdot \int_{\text{SMfld}(\mathbb{R}^{0|2}, X)/\pi TX \oplus \pi TX} \exp \left( -(F, F)/2 - \lambda \langle \nabla h, F \rangle \right) \right)$$

so we compute the Gaussian integral

$$\int_{\mathbb{R}^n} e^{-\langle F, F \rangle/2 - \lambda \langle \nabla h, F \rangle} dF = (2\pi)^{n/2} e^{-\lambda^2 \|\nabla h\|^2}.$$
it remains to evaluate the integral near the critical points of \( h \). Focusing our attention on one such point \( p \) (and possibly shrinking \( U_p \)), we choose coordinates on \( U_p \) and via a concordance deform the metric to the standard one on \( \mathbb{R}^n \). Since this metric is flat, \( R = 0 \). By Lemma 2.3.12 and Theorem 3.3.3, this concordance does not affect the value of the integral. We get the simplified form

\[
\int_{\text{SMfld}(\mathbb{R}^0, U_p)} \exp(-S) \frac{D\Phi}{N} = \int_{\text{SMfld}(\mathbb{R}^0, U_p)} \exp \left( -\frac{\lambda^2}{2} \| \nabla h \|^2 + \lambda \text{Hess}(h)(\phi_1, \phi_2) \right)
\]

This integration amounts to a pair of Gaussian integrals. The odd integral is a fermionic Gaussian integral with respect to the pairing \( \text{Hess}(h) \). So we find

\[
\int_{\pi TU_p \oplus \pi TU_p} \exp(\lambda \text{Hess}(h)(\phi_1, \phi_2)) D\Phi = \lambda^n \det(\text{Hess}(h)),
\]

where the right hand side is understood to be a top-form on \( U_p \). Hence,

\[
\int_{\text{SMfld}(\mathbb{R}^0, U_p)} \exp(-S) D\Phi = \lambda^n \int_{U_p} \exp \left( -\frac{\lambda^2}{2} \| \nabla h \|^2 \right) \det(\text{Hess}(h)).
\]

There are coordinates \( \{ x^i \} \) where the vector field \( \nabla h \) on \( U_p \) can be represented by a matrix \( H_p \), where we get a vector field on \( \mathbb{R}^n \) by \( x \mapsto H_p x \). Note that in these coordinates, \( \text{Hess}(h) = H_p \) is symmetric and nondegenerate. We can repackage the above as

\[
\int_{\text{SMfld}(\mathbb{R}^0, U_p)} \exp(-S) \frac{D\Phi}{N} = \lambda^n \text{sgn}(\det(H_p)) \int_{U_p} \exp \left( -\frac{\lambda^2}{2} \| H_p x \|^2 \right) |\det(H_p)| dx^1 \wedge \cdots \wedge x^n.
\]

This is another Gaussian integral, and in the limit \( \lambda \to \infty \) the value of the integral on \( U_p \) approaches the value of the integral on \( \mathbb{R}^n \), which is

\[
\int_{\text{SMfld}(\mathbb{R}^0, U_p)} \exp(-S) \frac{D\Phi}{N} = (2\pi)^{n/2} \text{sgn}(\det \text{Hess}(h)).
\]

Summing over critical points

\[
Z_X^{0/2} = \sum_{p \in \text{zero}(dh)} \text{sgn}(\det \text{Hess}(h)) = \text{Index}(\nabla h) = \chi(X),
\]

we obtain the desired result.

\[ \square \]

### 3.5 A proof of the Chern-Gauss-Bonnet Theorem

**Proposition 3.5.1.** If a quantization of renormalizable 0|\( \delta \)-EFTs uses the sheaf \( \mathcal{G} = \text{Riem} \times C^\infty \) defined in Examples 2.3.4 and 2.3.5 then \( Z_X^{0/\delta}(g) \) is independent of the metric and smooth function on \( X \).
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Proof. By Lemma 2.3.12 we need only prove that any pair of metric and smooth function is concordant to any other pair. This basically amounts to the contractibility of the space of metrics and smooth functions. We will construct concordances explicitly. First, let \( g_\lambda \) be a smooth 1-parameter family of metrics connecting \( g_0 \) and \( g_1 \) that is constant on \( g_1 \) for \( \lambda \geq 1 \) and constant on \( g_0 \) for \( \lambda \leq -1 \). Put the metric \( \tilde{g} := g_\lambda \otimes d\lambda^2 \) on \( X \times \mathbb{R} \), where \( \lambda \) is identified with a global coordinate on \( \mathbb{R} \). Similarly, let \( h_\lambda \) be a 1-parameter family of functions on \( X \) connecting \( h_0 \) and \( h_1 \), which we can then promote to a function \( \tilde{h} \) on \( X \times \mathbb{R} \), \( \tilde{h}(x, \lambda) = h_\lambda(x) \). Together, this shows that \( \mathcal{G}_X[\text{pt}] \) is the 1-point set, and so the Proposition follows. \( \square \)

Proof of Corollary 3.0.23. By Theorems 3.3.3 and Proposition 3.5.1 we have that the partition function \( Z_X^{0/2}(g, \lambda h) \) is constant as a function of a metric \( g \) and a parameter \( \lambda \). Then by Theorems 3.0.20 and 3.0.21 we have the equality,

\[
(2\pi)^{-n/2} \int_X \text{Pf}(R) = Z_X^{0/2}(g, 0) = \lim_{\lambda \to \infty} Z_X^{0/2}(g, \lambda h) = \sum_{\nabla h = 0} \text{index}(\nabla h),
\]

which is the Chern-Gauss-Bonnet formula. \( \square \)

Remark 3.5.2. There is a physical interpretation of the above argument as a toy-model of a path integral localization: the integral of the Pfaffian comes from an integral over all fields, whereas the sum over critical points is an integral over a formal neighborhood of the lowest action classical solutions of the sigma model action, which are called the classical vacua. The former computation can be thought of as a 0-dimensional Feynman integral, whereas the latter is a stationary phase approximation. The fact that this approximation is exact seems to persist in other facets of Euclidean field theories with two supersymmetries. For example the 1|2-Euclidean field theory described by Witten in [36] shows how the de Rham complex localizes onto the Morse complex.
Chapter 4

No-go results for field theories as cocycles for cohomology theories

In this chapter we ask when there can be an isomorphism

\[ h^k(X) \cong \{d|\delta\text{-dimensional field theories of degree } k \text{ over } X\}/\text{concordance} \quad (4.1) \]

where \( h \) is a cohomology theory. For appropriately defined degrees, the right-hand-side will have a product structure with unit and for most of the chapter we will restrict our attention to multiplicative isomorphisms \((4.1)\), i.e., cases where \( h \) is multiplicative cohomology theory. Hohnhold, Kreck, Stolz and Teichner have produced such an isomorphism for 2-periodic de Rham cohomology and 0|1-Euclidean field theories \([18]\), and Hohnhold, Stolz and Teichner have produced highly suggestive evidence for such an isomorphism for \( K \)-theory and 1|1-Euclidean field theories \([17]\). Stolz and Teichner conjecture that 2|1-Euclidean field theories correspond to TMF, the universal elliptic cohomology theory of topological modular forms.

We will begin by making precise what we mean by an isomorphism between concordance classes of field theories and a cohomology theory. Then we will characterize multiplicative isomorphisms \((4.1)\) for 0|\(\delta\)-dimensional field theories, and arbitrary isomorphisms for renormalizable (or polynomial) 0|\(\delta\)-dimensional field theories; in these cases we find that the isomorphism exists if and only if \( \delta = 0, 1 \). Turning our attention to higher dimensions, we formulate a condition that can be checked in finite dimensions for field theories to give cocycles for a cohomology theory. The condition is motivated by the existence of perturbative quantization (in the language of field theories) or the desire for a local index theorem (in the language of cohomology theories). For field theories satisfying this condition we again find that multiplicative isomorphisms \((4.1)\) can occur only when \( \delta = 0, 1 \).

Terminology

Throughout, a cohomology theory will be a generalized cohomology theory on manifolds; the problem of extending these to all topological spaces is discussed by Kreck and Singhof in \([24]\).
Below we will assume that field theories are supersymmetric in the sense of Definition 2.2.10. In particular, super conformal or super Euclidean geometries qualify.

### 4.1 Cocycles for cohomology theories

In this section we will describe the theory of cocycles due to Stolz and Teichner. This will give conditions under which \( X \mapsto d|\delta\text{-GFT}^*[X] \) is a cohomology theory. Let \( \mathcal{F} \) be a stack taking values in essentially small pointed groupoids, pointed by \( * \).

**Definition 4.1.1.** The support of a section \( \tau \in \mathcal{F}(X) \) is the complement of the union over open sets \( U \) where \( \tau|_U \cong * \). Given a stack, we can consider its compactly supported sections on \( X \), denoted \( \mathcal{F}_{cs}(X) \). Given a stack on \( X \times Y \), we can consider the sections with compact vertical support, denoted \( \mathcal{F}_{cvs}(X \times Y) \); these are sections \( \tau \in \mathcal{F}(X \times Y) \) that have compact support when restricted to any \( K \times Y \subset X \times Y \) for \( K \subset X \) compact.

**Definition 4.1.2.** A cocycle theory \( \mathcal{F} \) is a sequence of stacks \( \mathcal{F}_k \), for \( k \in \mathbb{Z} \) and natural desuspension maps

\[
\int: \mathcal{F}_{cvs}^{k+1}(X \times \mathbb{R}) \to \mathcal{F}_k(X)
\]

where cvs denotes compact vertical support in the \( \mathbb{R} \)-direction. A multiplicative cocycle theory has in addition natural maps

\[
\times: \mathcal{F}_k(X) \wedge \mathcal{F}_\ell(Y) \to \mathcal{F}_{k+\ell}(X \times Y),
\]

that are associative up to concordance and are compatible with desuspension,

\[
\int(\alpha \times \beta) = \int(\alpha) \times \beta, \quad \int(\beta \times \alpha) = \beta \times \int(\alpha)
\]

for \( \alpha \in \mathcal{F}_{cs}^{k+1}(X \times \mathbb{R}) \) and \( \beta \in \mathcal{F}_\ell(Y) \). A cocycle theory is called linear when \( \int \) is a bijection on concordance classes. Morphisms of cocycle theories are sequences of morphisms of stacks that commute with the structure maps. The cross-product above defines a cup product via the composition

\[
\mathcal{F}_k(X) \wedge \mathcal{F}_\ell(x) \xrightarrow{\times} \mathcal{F}_{k+\ell}(X \times X) \xrightarrow{\Delta^*} \mathcal{F}_{k+\ell}(X)
\]

for \( \Delta: X \to X \times X \) the diagonal map. A multiplicative cocycle theory has a unit if there is an element \( 1 \in \mathcal{F}_0(pt) \) so that when pulled back along \( X \to pt \) has the obvious properties for a unit with respect to \( \cup \).

We observe that a cup product \( \cup: \mathcal{F}_k(X) \times \mathcal{F}_\ell(Y) \to \mathcal{F}_{k+\ell}(X) \) gives rise to a cross product \( \times: \mathcal{F}_k(X) \times \mathcal{F}_\ell(Y) \to \mathcal{F}_{k+\ell}(X \times Y) \) in the usual manner: we define

\[
\omega \times \eta := p_1^* \omega \cup p_2^* \eta, \quad \omega \in \mathcal{F}_k(X), \quad \eta \in \mathcal{F}_\ell(Y), \quad p_1: X \times Y \to X, \quad p_2: X \times Y \to Y.
\]
Definition 4.1.3. Given a multiplicative cocycle theory \((F, \int, \times)\) with unit, denote by \(\sigma \in F^1_{\text{cs}}(\mathbb{R})\) a preimage of a representative of \([1] \in F^0[\text{pt}]\) under \(\int\). We call \(\sigma\) a suspension class.

For a linear, multiplicative cocycle theory, the map
\[
F^k(X) \xrightarrow{\times \sigma} F^{k+1}_{\text{cvs}}(X \times \mathbb{R})
\]
induces an isomorphism on concordance classes, with inverse induced by \(\int\).

Theorem 4.1.4 (Stolz-Teichner). If \((F^\bullet, \int)\) is a linear cocycle theory, then the assignment
\[
X \mapsto F^\bullet[X]
\]
is a cohomology theory. If the cocycle theory is multiplicative, so is the cohomology theory, and an inverse to \(\int\) is given by multiplication with a suspension class, \(\sigma\).

A sketch of a proof for sheaves of sets without multiplicative properties. Our sketch makes use of a result of I. Madsen and M. Weiss [28] (see Appendix A.1): any sheaf \(F\) has a naturally defined classifying space \(|F|\) with the property that
\[
F[X] \cong [X, |F|],
\]
where \([-,-]\) denotes homotopy classes of maps. The morphisms of sheaves \(\sigma : F(-) \to F_{\text{cvs}}(- \times \mathbb{R})\) give homotopy equivalences \(|F^k| \to \Sigma |F^{k+1}|\), so that the sequence of spaces has the structure of a spectrum, and homotopy classes of maps into it form a cohomology theory.

Remark 4.1.5. We emphasize that with the current definition, twisted field theories \(d|\delta\text{-GFT}_k\)
are prestacks of pointed groupoids, pointed by the zero field theory, and are only stacks for \(d = 0, 1\).

Definition 4.1.6 (Preliminary!). A sequence of twisted \(d|\delta\text{-EFTs}\) give cocycles for a cohomology theory if there exists maps \(\int : d|\delta\text{-GFT}_{\text{cvs}}^{k+1}(X \times \mathbb{R}) \to d|\delta\text{-GFT}^k(X)\) such that if we stackify the data \((d|\delta\text{-GFT}^*(X), \int)\), we obtain a linear cocycle theory in the sense of Definition 4.1.2. Field theories giving multiplicative cocycles are defined analogously.

Remark 4.1.7. This is clearly the wrong definition when \(d > 1\); stackification will spoil many geometrical properties of a prestack of field theories. We remark, however, that all the results of this Chapter will apply to any \(d\)-categorical definition of field theory satisfying properties:

1. The \(d\)-categorical definition deloops the top level of the current definition, meaning the stack of closed \(d\)-morphisms of the bordism category is isomorphic to the currently defined stack of closed morphisms made up of families of \(d|\delta\text{-Euclidean manifolds, and similarly twists at the top level arise from functors to the Picard stack of \(\mathbb{Z}/2\text{-graded line bundles on the site of supermanifolds.}}\)
2. The assignment \( X \mapsto d|\delta\)-GFT\(^{\bullet}\)(X) \) is a \( d \)-stack on manifolds.

With this caveat in mind, we proceed.

**Notation 4.1.8.** In an abuse of notation, we will use the same notation, \( d|\delta\)-GFT\(^{\bullet}\), to denote the stack associated to the prestack \( X \mapsto d|\delta\)-GFT\(^{\bullet}\)(X). To be explicit, the above discussion shows that isomorphism 4.1 requires data:

1. a super model geometry that defines a super geometric bordism category;
2. a twist \( T \), the \( k \)th power of which gives degree \( k \) field theories \( d|\delta\)-GFT\(^{\bullet}\)\(^{k}\); and
3. natural desuspension maps, \( \int: d|\delta\)-GFT\(^{k+1}\)(X × \( \mathbb{R} \)) → d|\delta\)-GFT\(^{k}\)(X) \) that induce isomorphisms on concordance classes.

The first two pieces of data result in a sequence of functors \( d|\delta\)-GFT\(^{k}\) that have a graded multiplication:

\[
\cup: d|\delta\)-GFT\(^{k}\)(X) × d|\delta\)-GFT\(^{\ell}\)(X) → d|\delta\)-GFT\(^{k+\ell}\)(X).
\]

These data need to satisfy the compatibility between \( \int \) and \( \cup \) described above. We can ask for isomorphisms 4.1 that respect this product, or we can ignore this extra structure.

We wish to emphasize an important constraint on (1) we explained in Remark 1.4.7: for \( \delta > 0 \) there are finitely many possible super Euclidean geometries for a fixed fermionic dimension \( \delta \). For example, super dimensions \( d|1 \) require that \( d = 0, 1 \) or 2; see Freed [12] for details. Furthermore, it seems that \( d|0\)-Euclidean field theories frequently give rise to the 0-cohomology theory, as shown for \( d = 0 \) and 1 in [18] and [17] respectively.

**Definition 4.1.9.** We define an object \( 1 \in d|\delta\)-GFT\(^{0}\)(X) \) called the **unit field theory**: to any trivial \( S \)-family of \((d-1)|\delta\)-manifolds \( 1 \) assigns the trivial bundle \( S × \mathbb{K} \), and to any \( S \)-family of bordisms, \( 1 \) assigns the identity map. We observe that this is a unit for the product \( \cup \) of field theories.

**Example 4.1.10 (Dimension 0|1).** As shown in [18], there is a twist for 0|1-Euclidean field theories so that even degree theories over a manifold are isomorphic to even dimensional closed forms, and odd degree theories to odd dimensional closed forms,

\[
0|1\text{-EFT}^{\text{ev/odd}}(X) \cong \Omega^{\text{ev/odd}}_{\text{cl}}(X).
\]

Furthermore, the product structures on each side agree. Up to concordance, these give de Rham cohomology classes so that there is an isomorphism of \( \mathbb{Z}/2 \)-graded rings,

\[
0|1\text{-EFT}^{\bullet}(X) \cong H^{\bullet}_{\text{dR}}(X).
\]

As a cocycle theory, the extra data is the map that integrates forms along the fiber \( X × \mathbb{R} → X \). The suspension class sigma is any closed, compactly supported 1-form on \( \mathbb{R} \) that integrates to 1.
CHAPTER 4. NO-GO RESULTS FOR FIELD THEORIES AS COCYCLES FOR COHOMOLOGY THEORIES

Example 4.1.11 (Dimension 1|1). One-dimensional bordism categories give rise to local field theories, so we needn’t worry about higher categorical issues. Twists for such field theories arise from (topological) algebras; in the case of 1|1-EFTs, we take the Clifford algebras \(Cl_n\). As proved by Stolz and Teichner, there is an isomorphism

\[
1|1-\text{EFT}^*[X] \cong K^*(X)
\]

between concordance classes of 1|1-EFTs and \(K\)-theory. Since \(Cl_n \cong \bigotimes_n Cl_1\), there is a product structure on the left-hand-side, and it agrees with the cup product in the \(K\)-theory. Furthermore, by Bott periodicity, this defines a periodic twist. Morally, the map \(\int\) should be defined in terms of a fiberwise superpath integral for the fibers of the map \(X \times \mathbb{R} \to X\); this is currently being pursued by A. Prat-Waldron.

Example 4.1.12 (Dimension 2|1). Two-dimensional Euclidean bordism categories remain a good deal more illusive than the above examples, largely because of the higher-categorical complexities alluded to in Chapter 1. Stolz and Teichner \cite{33} have described twists for these theories and conjecture an isomorphism of graded rings,

\[
2|1-\text{EFT}^*[X] \cong \text{TMF}^*(X).
\]

They have shown that for the non-local definition of 2|1-EFTs, there is a twist of periodicity 48. In order for the above conjecture to hold, the hypothesized fully-local definition of 2|1-EFTs needs to support a twist of periodicity 24^2.

4.2 0|\(\delta\)-dimensional field theories and cohomology

In this section, let \((M,G)\) be a model geometry for \(M = \mathbb{R}^{0|\delta}\) and suppose that \(\mathbb{R}^{0|\delta} < G\) acts on \(M\) by translations. We observe that if only a subgroup of translations \(\mathbb{R}^{0|\delta'} < \mathbb{R}^{0|\delta}\) acts by isometries, we have an equivalence of groupoids

\[
\text{SMfld}(\mathbb{R}^{0|\delta}, X)/\mathbb{R}^{0|\delta'} \cong \text{SMfld}(\mathbb{R}^{0|\delta'}, (\pi T)^{\delta-\delta'} X)/\mathbb{R}^{0|\delta'}
\]

where \(\pi T\) is the functor that takes a manifold \(X\) to the total space of its odd tangent bundle. There is a homotopy equivalence\footnote{Homotopy equivalences of supermanifolds are defined in complete analogy to the case of smooth manifolds: we require a pair of maps whose compositions are homotopic to the identity.} \(\pi TX \simeq X\) (in fact, a deformation retraction) and therefore concordance classes of functions on the above groupoids will define the same functor from manifolds to sets as a certain \(0|\delta\)-dimensional field theory. This justifies our assumption that \(\mathbb{R}^{0|\delta} < G\).

In this section we will prove the following pair of results.

Theorem 4.2.1. Let \(\delta > 0\) be even and \(G\) be a supersymmetric model geometry. The functor \(X \mapsto 0|\delta-\text{GFT}^*_{\text{pol}}(X)/\text{concordance}\) is not a cohomology theory.
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Theorem 4.2.2. The functor $X \mapsto 0|\delta\text{-GFT}^*(X)$ gives multiplicative cocycles for a cohomology theory if and only if $\delta = 0, 1$.

Both theorems are proved by contradiction. The first is direct, avoiding the language of cocycles completely. The second proceeds by showing that there can be no suspension class $\sigma$ satisfying the condition \([4.1]\).

Remark 4.2.3. The ideas in the proof of Theorem \([4.2.1]\) can be used more widely, but require some further assumptions on the geometry and twist. For example, one can show that the geometries with $G = R^{0|\delta} \rtimes O(\delta)$ and with twist given by the determinant homomorphism only give rise to cohomology theories for $\delta = 0, 1$.

Proof of Theorems \([4.2.1]\) and \([4.2.2]\)

Notation 4.2.4. As in Chapters 2 and 3, we choose a basis for $R^{0|\delta}$ and denote by $d_i$ the action of infinitesimal translation in the $i$th direction on $C^\infty(\text{SMfld}(R^{0|\delta}, X))$; these act by odd derivations on functions. For some $I = \{i_1, \ldots, i_k\}$, we denote by $d_I$ the composition $d_{i_1} \cdots d_{i_k}$ and $\Delta$ the composition $d_{j}d_{\delta-1} \cdots d_{2}d_{1}$. The fact that $R^{0|\delta}$ is a superabelian group means that $d_{j}d_{j} = -d_{j}d_{i}$, which gives relations among various operators $d_I$.

Example 4.2.5. In this example we establish some notation and carry out computations on $T^2 = S^1 \times S^1$ that will be used in the proof of the main theorem. Let $\theta_1$ and $\theta_2$ be standard coordinates on $R^2$ and let $f : R^2 \to T^2$ be the universal covering of the torus, $\theta_k \mapsto e^{i\theta_k}$, $k = 1, 2$. The functions on $\text{SMfld}(R^{0|\delta}, T^2)$ are then furnished by the functions on $\text{SMfld}(R^{0|\delta}, R^2)$, invariant under the action of the lattice $2\pi \cdot Z \times 2\pi \cdot Z \subset R^2$. We compute

$$\text{SMfld}(R^{0|\delta}, R^2) \cong (\pi T)^{\delta}R^2 \cong R^{2|2^\delta}.$$ 

Counting dimensions, the functions $d_I\theta_1$ and $d_I\theta_2$ for $I, J \subset \{1, \ldots, \delta\}$ give coordinates on $R^{2|2^\delta}$. Under the identification of $\{\theta_1, \theta_2\}$ as coordinates on $T^2$, the $d_I\theta_1, d_I\theta_2$ give coordinates on $\text{SMfld}(R^{0|\delta}, T^2)$. Field theories are functions in these that are polynomial in generators $d_I\theta_k$ for $I$ nonempty and invariant under a group $G$ containing the translation group, $R^{0|\delta}$, as a subgroup.

Lemma 4.2.6. Let $\delta > 0$. If there is an isomorphism $0|\delta\text{-GFT}^*[X] \cong h^*[X]$ for some cohomology theory $h$, then $h^k(X) = \bigoplus_{i+j=k} H^i_{\text{dR}}(X; V^j)$ is de Rham cohomology with coefficients in a graded vector space $V^\bullet$.

Proof. We can compute the expected coefficient ring of $h$ directly. Using Corollary \([2.0.19]\) above and its proof,

$$0|\delta\text{-GFT}^k[pt] \cong \begin{cases} K & \text{if } L^{\otimes k} \cong K \\ 0 & \text{else,} \end{cases}$$

where $L$ is the chosen twist, and $K$ denotes the trivial bundle on $pt//G$. Hence, if $0|\delta\text{-GFT}$s are cocycles for a cohomology theory the coefficient ring is a sum of copies of $K$. This is
a rational coefficient ring, so the proposed cohomology theory must come from a wedge of real or complex Eilenberg-MacLane spectra, i.e., a generalized Eilenberg-MacLane spectrum determined by a graded vector space $V^\bullet$, e.g., see Boardman [6]. On manifolds, such a cohomology theory is computed by de Rham forms with coefficients in $V^\bullet$.

**Proof of Theorem 4.2.2.** Arguing by contradiction, the previous lemma allows us to assume $X \mapsto 0|\delta\text{-GFT}^\bullet_{\text{pol}}[X]$ is de Rham cohomology with coefficients in $V^\bullet$. The computation $0|\delta\text{-GFT}^0(\text{pt}) \cong C^\infty(\text{pt}) \cong \mathbb{R}$ shows that $V^\bullet$ contains a 1-dimensional vector space of degree 0.

Let $T^2$ denote the 2-torus. Since $V^0 \cong \mathbb{R}$, there is a nonzero class $[\omega] \in 0|\delta\text{-GFT}^\bullet_{\text{pol}}[T^2]$ that for any map $\phi: T^2 \to T^2$ has the property that $[\omega] \mapsto \deg(\phi)[\omega]$ where $\deg(\phi) \in \mathbb{Z}$ is the degree of the map. We will produce a contradiction when $\delta > 1$ by showing that no representative $\omega \in 0|\delta\text{-GFT}^\bullet_{\text{pol}}(T^2)$ can exist.

Given a candidate $\omega \in C^\infty_{\text{pol}}(\text{SMfld}(\mathbb{R}^0, T^2))$, by assumption it takes the form

$$\omega = P(d_1 \theta_1, d_2 \theta_2, d_2 d_1 \theta_1, d_3 d_1 \theta_1, \ldots d_\delta d_1 \theta_1, d_\delta \cdots d_1 \theta_2),$$

for some polynomial $P$ with coefficients in $C^\infty(T^2)$. If $\omega$ is invariant under the action of $T^2$ on itself, then these coefficients are constant functions. We claim that any candidate $\omega$ is concordant to one invariant under the action of $T^2$ on itself by integrating over the action. For note that for $g \in T^2$, $\omega - g^* \omega$ is null-concordant where the concordance is a choice of path from $g$ to $e \in T^2$. By Proposition 2.2.17, $\omega - g^* \omega = \Delta \omega$. Now let $\alpha$ denote the map that averages over the action of $T^2$. By differentiation under the integral sign, the difference $\omega - \alpha(\omega)$ is also null-concordant. This allows us to assume that $\omega$ is invariant under this action, and so is a polynomial (with constant coefficients) in the variables $d_I \theta_1, d_I \theta_2$ for $I$ nonempty.

Now, let $\phi_{n,m}$ denote the $n \cdot m$-fold covering map whose lift to the universal cover is $\theta_1 \mapsto m \cdot \theta_1, \theta_2 \mapsto n \cdot \theta_2$ with $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. We observe that if it exists, $\omega$ satisfies $[\omega] \mapsto mn[\omega]$. Hence, $[\omega]$ must be quadratic in the sense that it has a representative of the form

$$\omega = \sum_{I,J} c_{IJ} d_I \theta_1 d_J \theta_2,$$

where $c_{IJ}$ are constants.

Since the universal cover is contractible, the pullback $f^* \omega$ is concordant to zero so by Proposition 2.2.17 there exists some $\Theta \in C^\infty_{\text{pol}}(\text{SMfld}(\mathbb{R}^d, \mathbb{R}^2))$ with $f^* \omega = \Delta(\Theta)$. Considering the lifted action of $\phi_{n,m}$ to $\mathbb{R}^2$, we find $\Theta = \sum c_{IJ} d_I \theta_1 d_J \theta_2$. One of $I'$ or $J'$ in each summand must be empty since otherwise $\Theta$ would be $\mathbb{Z} \times \mathbb{Z}$ invariant, and hence globally defined on $T^2$ proving $[\omega] = [\Delta \Theta] = 0$ by Proposition 2.2.17, a contradiction. Therefore, we may assume that $\Theta = \sum c_I d_I \theta_1 d_J \theta_2 + \sum b_I (d_I \theta_1 \theta_2)$. We observe that for $k > 1$,

$$(d_k \cdots d_1 \theta_1)(d_k \cdots d_1 \theta_2) = (-1)^k (d_k \cdots d_1)(d_1 \theta_1 d_k \cdots d_2 \theta_2),$$

so that

$$\Delta(c_I d_I \theta_1 d_J \theta_2) = c_I d_K (d_I \theta_1 d_I \theta_2) = c_I d_K (d_I (d_1 \theta_1 d_k \cdots d_2 \theta_2)) = c_I \Delta(d_1 \theta_1 d_k \cdots d_2 \theta_2)$$
for $I = \{i_1, \ldots, i_k\}$ and $K$ being the complement of $I$ in $\{1, \ldots, \delta\}$. Hence $[\Delta(c_I \theta_1 d_I \theta_2)] = 0$ and $[\Delta(b_I (d_I \theta_1) \theta_2)] = 0$ if $|I| > 1$, and the remaining candidates for $\omega$ with $[\omega] \neq 0$ are of the form

$$
\omega = \Delta \left( \sum_{k=1}^{\delta} c_k \theta_1 d_k \theta_2 + \sum_{k=1}^{\delta} b_k (d_k \theta_1) \theta_2 \right). \tag{4.2}
$$

We observe that when $\delta$ is even, the above function $\omega$ is odd as a function on the supermanifold $\text{SMfld}(\mathbb{R}^{0|\delta}, X)$. Since, by assumption, $\omega$ is a section of an even tensor power of a line bundle over $\text{SMfld}(\mathbb{R}^{0|\delta}, X)/G$ coming from a homomorphism $G \rightarrow \mathbb{K}^\times$, $\omega$ is necessarily an even function on $\text{SMfld}(\mathbb{R}^{0|\delta}, X)$. Therefore, $[\omega] = 0$, producing the desired contradiction. \qed

Now we proceed to collect the ingredients for the proof of Theorem 4.2.2.

**Lemma 4.2.7.** Let $\delta > 1$ and take nonconstant $\omega \in 0|\delta\text{-GFT}^\bullet(X)$ and nonconstant $\eta \in 0|\delta\text{-GFT}^\bullet(Y)$. Then $\omega \otimes \sigma \in 0|\delta\text{-GFT}^\bullet(X \times Y)$ is concordant to zero.

**Proof.** The key technical step is finding functions $d_\delta \omega_\delta \in C^\infty(\text{SMfld}(\mathbb{R}^{0|\delta}, X))$ and $d_1 \eta_1 \in C^\infty(\text{SMfld}(\mathbb{R}^{0|\delta}, Y))$ such that $\omega = d_1 \cdots d_\delta \omega_\delta$ and $\eta = d_\delta \cdots d_1 \eta_1$. The existence of such functions requires that $\delta > 1$. Then we write

$$
\omega \otimes \eta = -\Delta(d_\delta \omega_\delta \otimes d_{\delta-1} \cdots d_1 \eta_1),
$$

so that the product is concordant to zero by Proposition 2.2.17.

To obtain $d_\delta \omega_\delta$, fix a good cover for $X$. Since $d_1 \omega = 0$, we can view $\omega$ as a de Rham class on the supermanifold $(\pi T)^{\delta-1}(X)$, and applying the functor $(\pi T)^{\delta-1}$ to the original good cover, we obtain a good cover of this supermanifold. When restricted to open sets in the cover, we have $\omega = d_1 \omega_1$ by the Poincare Lemma. On overlaps $\omega_1 \mapsto \omega_1 + d_1 \tilde{\omega}_1$. Since $\omega$ is $d_2$-closed and the $d_1$ surcommute, $\omega_1$ and $\tilde{\omega}_1$ are also $d_2$-closed and so can also be viewed as a de Rham cohomology class on $(\pi T)^{\delta-1}X$. Therefore we can write $\omega = d_1 d_2 \omega_2$, and on overlaps $\omega_2 \mapsto \omega_2 + d_2 \tilde{\omega}_2$. Iterating this construction, we obtain $\omega = d_1 \cdots d_\delta \omega_\delta$, where on overlaps $\omega_\delta \mapsto \omega_\delta + d_\delta \tilde{\omega}_\delta$. Hence, $d_\delta \omega_\delta$ is globally defined, and so we have produced the claimed expression for $\omega$. To construct $d_1 \eta_1$ the process is identical, except that we start the iteration with the operator $d_\delta$ rather than $d_1$. \qed

**Proof of Theorem 4.2.2.** Assume that $X \mapsto 0|\delta\text{-GFT}^\bullet(X)$ defines multiplicative cocycles for a cohomology theory. Let $\sigma \in 0|\delta\text{-GFT}^1_{cs}(\mathbb{R})$ be a suspension class. Then for $\beta \in 0|\delta\text{-GFT}^\bullet(X)$ we have

$$
\int (\sigma \times \beta) = \int (\sigma) \times \beta = \beta, \quad \int (\beta \times \sigma) = \beta \times \int (\sigma) = \beta
$$

for all $\beta$. But since $\sigma$ is compactly supported, it is either zero or nonconstant. If $\sigma = 0$, then $[\beta] = 0$ cannot be an isomorphism, so assume $\sigma$ is nonconstant. Then by Lemma 4.2.7 $[\sigma \times \beta] = 0$ for all nonconstant $\beta$. By the above formula, this implies $[\beta] = 0$ for all nonconstant $\beta$. Therefore, the functor $X \mapsto 0|\delta\text{-GFT}^\bullet[X]$ sends $X$ to the constant functions on $\text{SMfld}(\mathbb{R}^{0|\delta}, X)$; we can identify this functor with the one that sends $X$ to the constant functions on $X$ as a ring. But this is not a cohomology theory. \qed
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4.3 Perturbative quantization and local index theorems

let $X$ be a spin manifold with Dirac operator $D$, $E$ a vector bundle with connection on $X$, and $\alpha$ a differential form representative of the $\hat{A}$-class. The motivation for results in this section is the interpretation of the formula

$$\text{index}(D \otimes E) = \int_X \alpha \cdot \text{ch}(E)$$

(4.3)

in terms of field theories. The starting point is the result of Hohnhold, Stolz and Teichner that concordance classes of 1|1-EFTs over $X$ are the KO-theory of $X$ [17]. Hohnhold, Kreck, Stolz and Teichner have shown that 0|1-EFT$^\bullet(X)$ is isomorphic as a graded algebra to closed differential forms on $X$ [18]. Han showed in his thesis that dimensional reduction of 1|1-EFTs gives a lift of the Chern character map, denoted ch, to field theories [16]. This suggests we compare the following commuting squares:

$$
\begin{array}{ccc}
1|1\text{-EFT}_{cs}^{k+n}(X) & \xrightarrow{\text{red}_\tau} & 0|1\text{-EFT}_{cs}^{k+n}(X) \\
\downarrow_{p!} & & \downarrow \int \alpha \cdot (-) \\
1|1\text{-EFT}^k(pt) & \xrightarrow{\text{red}_\tau} & 0|1\text{-EFT}^k(pt). \\
\end{array}
\quad
\begin{array}{ccc}
\text{KO}_{cs}^{k+n}(X) & \xrightarrow{\text{ch}} & \text{H}_{dR,cs}^{k+n}(X) \\
\downarrow [p!] & & \downarrow \lfloor \int \alpha \cdot (-) \rfloor \\
\text{KO}^k(pt) & \xrightarrow{\text{ch}} & \text{H}_{dR}^k(pt). \\
\end{array}
$$

(4.4)

On the left we have a diagram consisting of compactly supported field theories over $X$ and over the point, and at the right we have the square that encodes (4.3). Taking concordance classes of the left hand square gives the four abelian groups at the corners of the right hand square. Furthermore, abusing notation and using the result of Hohnhold-Kreck-Stolz-Teichner [18], we can consider field theories as closed differential forms and use integration of forms to define the map $\omega \mapsto \int \alpha \cdot \omega$. Taking concordance gives the relevant map on de Rham cohomology. We can give this map an intrinsic field theoretic interpretation via perturbative quantization as discussed by Witten [9] pages 477-485. The difficult (and as-yet unfinished) construction is the map $p!$; it is expected that this will be defined using quantization of field theories over $X$.

More generally, we expect sigma models and their quantization to give squares

$$
\begin{array}{ccc}
d|\delta\text{-EFT}_{cs}^{k+n}(X) & \xrightarrow{\text{red}_\tau} & 0|\delta\text{-EFT}_{cs}^{k+n}(X) \\
\downarrow_{p!} & & \downarrow Q \\
d|\delta\text{-EFT}^k(pt) & \xrightarrow{\text{red}_\tau} & 0|\delta\text{-EFT}^k(pt), \\
\end{array}
$$

(4.5)

for some map $Q$. In practice, $Q$ comes from Berezinian integration against some function $\alpha \in C^\infty(\text{SMfld}(\mathbb{R}^{0|\delta}, X))$. This is relatively easy to define: both $\alpha$ and the Berezinian integral belong to finite-dimensional supergeometry and can usually be constructed directly from the action functional of a $d|\delta$-dimensional sigma model. From the above we would obtain a formula

$$\text{red}_\tau(p!E) = Q(\text{red}_\tau(E))$$
which we could view as a field theory generalization of Equation 4.3 and the index theorem. Given a map $Q$, we wish to determine whether the numbers $Q(red_T(E))$ and concordance classes of $d|\delta$-EFTs are related to an index theorem for a cohomology theory in the sense of the commutative square 4.5.

We observe that when $X = \mathbb{R}^n$, the map $p_!$ gives part of the data of the desuspension map $\int$ in the definition of a cocycle theory. It is exactly this identification between $p_!$ and $\int$ that interweaves the data of the cohomology theory and the index theory from the wrong way map $p_!$. We will now formalize the pieces of this story we need to prove Theorem 1.0.3.

**Definition 4.3.1.** Perturbative quantization is a map $Q: 0|\delta$-GFT$^*_{cs}(\mathbb{R}^2) \to 0|\delta$-GFT$^{*-2}_{cs}(pt)$. Desuspension maps $\int$ are compatible with the perturbative quantization $Q$ if the diagram commutes

$$
\begin{array}{ccc}
d|\delta$-GFT$^2_{cs}[\mathbb{R}^2] & \overset{red_\delta}{\longrightarrow} & 0|\delta$-GFT$^2_{cs}[\mathbb{R}^2] \\
\int & \downarrow & Q \\
d|\delta$-GFT$^0[pt] & \overset{red_\delta}{\longrightarrow} & 0|\delta$-GFT$^0[pt].
\end{array}
$$

(4.6)

**Remark 4.3.2.** One can ask for a stronger versions of compatibility; e.g., one can ask for a commutative diagram for any $\mathbb{R}^n$ or for field theories of all degrees. However, the above definition is enough to prove our no-go result.

**Proof of Theorem 1.0.3** Supposing that $(d|\delta$-GFT$^*$, $\int$) gives multiplicative cocycles, consider a representative for the suspension class, $\sigma$. By hypothesis, we have a commutative diagram

$$
\begin{array}{ccc}
d|\delta$-GFT$^2_{cs}[\mathbb{R}^2] & \overset{red_\delta}{\longrightarrow} & 0|\delta$-GFT$^2_{cs}[\mathbb{R}^2] \\
\int & \downarrow & Q \\
d|\delta$-GFT$^0[pt] & \overset{red_\delta}{\longrightarrow} & 0|\delta$-GFT$^0[pt].
\end{array}
$$

(4.7)

By tracing the field theory $\sigma^2 \in d|\delta$-GFT$^2_{cs}(\mathbb{R}^2)$ around the above diagram we will produce a contradiction when $\delta > 1$. First, by assumption we have that $\int(\sigma^2) \cong 1$, the unit field theory, and the image of the unit field theory in $0|\delta$-GFT$^0(pt) \cong \mathbb{R}$ is 1. If instead we first dimensionally reduce $\sigma^2$, we obtain an element $red(\sigma)^2 \in 0|\delta$-GFT$^2_{cs}(\mathbb{R}^2)$. Below we will show that such an element is concordant to zero when $\delta > 1$. Therefore, it must map to an element concordant to zero under $Q$; but by Proposition 2.2.17 when $\delta > 0$ the field theory $1 \in 0|\delta$-GFT$^0(pt)$ is not concordant to zero, so the diagram fails to commute when $\delta > 1$.

To prove $red(\sigma)^2$ is concordant to zero, there are two cases to consider: the first is where $red(\sigma)$ is a constant function on $\mathbb{R}$. However, the only constant compactly supported function on $\mathbb{R}$ is the 0 function. So suppose that $red(\sigma)$ is not a constant function on $\mathbb{R}$. Then Lemma 4.2.7 implies that $red(\sigma)^2$ is concordant to zero when $\delta > 1$.

**Example 4.3.3.** An example of diagram 4.5 comes from the results of Chapter 3: the action functional for the 0|2-sigma model can be extracted from the 1|2, 2|2 or 3|2-dimensional sigma
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model, and one might ask if the square

\[
\begin{array}{ccc}
d|2\text{-EFT}^{k+n}_{cs}(X) & \overset{\text{red}_S}{\longrightarrow} & 0|2\text{-EFT}^{k+n}_{cs}(X) \\
p_i \downarrow & \quad & \downarrow \int \exp(-S) \cdot (-) \\
d|2\text{-EFT}^k(pt) & \overset{\text{red}_S}{\longrightarrow} & 0|2\text{-EFT}^k(pt),
\end{array}
\]

(4.8)

has an interpretation in terms of an index theorem for a cohomology theory, for any notion of twist of \(d|2\text{-EFTs}\) and any possible desuspension map. In this example, \(\exp(-S) \in C^\infty(\text{SMfld}(\mathbb{R}^{0|2}, X))\) is the function defined by Theorem 3.3.3. By Theorem 1.0.3, there is no desuspension map lifting the map defined by \(\exp(-S)\) such that concordance classes of \(d|2\text{-EFTs}\) form a multiplicative cocycle theory. We remark that the expected sigma model quantization when \(d = 1, 2\) does indeed lead to a theory whose partition function is the Euler characteristic, so the map defined by Theorem 3.3.3 is a reasonable one to attempt to lift if \(p_i\) and \(\int\) are to be defined via quantization.

**Example 4.3.4.** Another example comes from \(2|1\text{-EFTs}\) and the Witten genus. We can consider the square

\[
\begin{array}{ccc}
2|1\text{-EFT}^{k+n}_{cs}(X) & \overset{\text{red}_S}{\longrightarrow} & 0|1\text{-EFT}^{k+n}_{cs}(X) \\
p_i \downarrow & \quad & \downarrow \int \text{Wit}(T) \cdot (-) \\
2|1\text{-EFT}^k(pt) & \overset{\text{red}_S}{\longrightarrow} & 0|1\text{-EFT}^k(pt),
\end{array}
\]

(4.9)

where \(\text{Wit}(T)\) is the Witten class of \(X\) evaluated at the lattice associated to the torus \(T \cong \mathbb{R}^{2|1}/\mathbb{Z}\). We view \(\text{Wit}\) as a de Rham cohomology class valued in modular forms, and since the lattice \(\mathbb{Z}^2 \subset \mathbb{R}^2 \subset \mathbb{R}^{2|1}\) defines an ordinary lattice, we can evaluate the modular form, so the integration map makes sense. In this example we expect concordance classes of \(2|1\text{-EFTs}\) to give TMF, and the above square to be related to a local index theorem for the Witten genus. Theorem 1.0.3 does not put any restrictions on this possibility.
Bibliography


