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Authors
Etzion, T
Vardy, A

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Automorphisms of Codes in the Grassmann Scheme

TUVI ETZION* ALEXANDER VARDY†

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Abstract

Two mappings in a finite field, the Frobenius mapping and the cyclic shift mapping, are applied on lines in PG(n,p) or codes in the Grassmannian, to form automorphisms groups in the Grassmanian and in its codes. These automorphisms are examined on two classical coding problems in the Grassmannian. The first is the existence of a parallelism with lines in the related projective geometry and the second is the existence of a Steiner structure. A computer search was applied to find parallelisms and codes. A new parallelism of lines in PG(5,3) was formed. A parallelism with these parameters was not known before. A large code which is only slightly short of a Steiner structure was formed.

Keywords: Cyclic shifts, cyclotomic cosets, Frobenius mapping, Grassmannian scheme, parallelism, q-analog, spreads, Steiner structures.

*Department of Computer Science, Technion, Haifa 32000, Israel, e-mail: etzion@cs.technion.ac.il.
†Department of Electrical Engineering, University of California San Diego, La Jolla, CA 92093, USA, e-mail: avardy@ucsd.edu.
1 Introduction

The Grassmannian $G_q(n, k)$ is the set of all $k$-dimensional subspaces of an $n$-dimensional subspace over the finite field $F_q$. A code in $G_q(n, k)$ is a subset of $G_q(n, k)$. There has been a lot of interest in these codes in the last five year due to their application in network coding [6]. Our motivation for this work also came from this application in network coding.

Some of the coding problems in the Grassmannian were formulated in the past in terms of projective geometry or $q$-analog of block design. In this paper we will consider two of these problems. The first is the existence of a parallelism with lines in PG$(n, p)$ and the second is the existence of a Steiner structure.

Steiner structures are known also as $q$-analog of Steiner systems. A Steiner structure $S_q[t, k, n]$ is a set $S$ of $k$-dimensional subspaces of $F_q^n$ such that each $t$-dimensional subspace of $F_q^n$ is contained in exactly one subspace of $S$. Steiner structures were considered in many papers [1, 4, 10, 11, 12], where they have other names as well. An $S_q[t, k, n]$ can be readily constructed for $t = k$ and for $k = n$. If $t = 1$ these structures are called $k$-spreads and they are known to exist if and only if $k$ divides $n$. These structures are also considered to be trivial. The first nontrivial case is a Steiner structure $S_2[2, 3, 7]$. The possible existence of this structure was considered by several authors, and some conjectured [8] that it doesn’t exist and that generally nontrivial Steiner structures do not exist.

$k$-spreads were considered in numerous papers. These can be viewed as partition of the points set of PG$(n - 1, q)$ into disjoint $(k - 1)$-dimensional subspaces (in the geometry). It is called a $(k - 1)$-spread in the geometry. Two disjoint spreads are called parallel spreads and a partition of the $\frac{q^n - 1}{q - 1}$ points of PG$(n - 1, q)$ into disjoint $(k - 1)$-spreads is called a parallelism. The only known parallelism with $(k - 1)$-spreads, $k > 2$, is for $k = 3$, $q = 2$, and $n = 6$ [9]. It is known that parallelisms with 1-spreads exist for $q = 2$ and all even $n$ [2, 13], and for each prime power $q$, where $n = 2^m$ [3]. No other parallelisms are known.

In this paper we consider solutions for these two problems, the existence of parallelisms with 1-spreads in PG$(n - 1, p)$ (which are 2-spreads in $G_p(n, 2)$) and the existence of nontrivial Steiner structures. We will use two types of mappings, the Frobenius mappings and cyclic shift mappings to form an automorphisms group in codes and as a mechanism to obtain disjoint spreads and Steiner structures. We examine 1-spreads in which the Frobenius mappings form an automorphism group and a parallelism is obtained by using cyclic shifts on two disjoint cycles of PG$(n - 1, p)$. The method was found to be successful for $p = 3$ and $n = 6$. We conjecture that the method would be successful whenever some necessary conditions are satisfied. To form a Steiner structure $S_2[2, 3, 13]$ fifteen representatives are needed. It turns out that fourteen such representatives are easy to obtain and it is an open problem whether fifteen representatives exist.

The rest of this paper is organized as follows. In Section 2 we define the two types of mappings and state some of their properties. In Section 3 we use the two types of mappings for the construction of a parallelism of lines in PG$(n, p)$. In Section 4 we use the two types of mappings for an attempt to construct nontrivial Steiner structures. Conclusion and problems for further research are given in Section 5.
2 Mappings in the Grassmannian

Let $\mathbb{F}_{p^n}$ be a finite field with $p^n$ elements, where $p$ and $n$ are primes, and let $\alpha$ be a primitive element in $\mathbb{F}_{p^n}$.

The Frobenius mapping $\Upsilon_\ell$, $0 \leq \ell \leq n - 1$, $\Upsilon_\ell : \mathbb{F}_{p^n} \setminus \{0\} \rightarrow \mathbb{F}_{p^n} \setminus \{0\}$ is defined by $\Upsilon_\ell(x) \stackrel{\text{def}}{=} x^{p^\ell}$ for each $x \in \mathbb{F}_{p^n} \setminus \{0\}$.

The cyclic shift mapping $\Phi_j$, $0 \leq j \leq p^n - 2$, $\Phi_j : \mathbb{F}_{p^n} \setminus \{0\} \rightarrow \mathbb{F}_{p^n} \setminus \{0\}$ is defined by $\Phi_j(\alpha^i) \stackrel{\text{def}}{=} \alpha^{i+j}$, for each $0 \leq i \leq p^n - 2$.

The two types of mappings $\Upsilon_\ell$ and $\Phi_j$ can be applied on a subset or a subspace, by applying the mapping on each element of the subset or subspace, respectively. Formally, given two integers $0 \leq \ell \leq n - 1$ and $0 \leq j \leq p^n - 2$,

$$\Upsilon_\ell\{x_1, x_2, \ldots, x_r\} \stackrel{\text{def}}{=} \{\Upsilon_\ell(x_1), \Upsilon_\ell(x_2), \ldots, \Upsilon_\ell(x_r)\},$$

$$\Phi_j\{x_1, x_2, \ldots, x_r\} \stackrel{\text{def}}{=} \{\Phi_j(x_1), \Phi_j(x_2), \ldots, \Phi_j(x_r)\}.$$

Lemma 1. The mappings $\Upsilon_\ell$ and $\Phi_j$ are invertible.

Proof. Clearly, $\Upsilon_\ell^{-1} = \Upsilon_{n-\ell}$ and $\Phi_j^{-1} = \Phi_{2^n-1-j}$.

For a given integer $s \in \mathbb{Z}_{p^n-1}$, the cyclotomic coset $C_s$ is defined by

$$C_s \stackrel{\text{def}}{=} \{s \cdot p^i : 0 \leq i \leq n-1\}.$$

The smallest element in a cyclotomic coset is called the coset representative. Let $C(s)$ denote the coset representative of $C_s$, i.e. if $r$ is the coset representative for the coset of $s$, $C_s$, then $r = C(s)$. The first lemma is well known and easy to verify.

Lemma 2. The size of a cyclotomic coset is either $n$ or one. There are exactly $\frac{p^n-1}{n}$ different cyclotomic cosets of size $n$.

The definitions of the Frobenius mappings and the cyclotomic cosets imply the following lemma.

Lemma 3. When applied on $\mathbb{F}_{p^n} \setminus \{0\}$ the Frobenius mappings form an equivalence relation on $\mathbb{F}_{p^n} \setminus \{0\}$, where an equivalence class contains the powers of $\alpha$ which are exactly the elements of one cyclotomic cosets of $\mathbb{Z}_{p^n-1}$.

Another well known result is the following lemma.

Lemma 4. The finite field $\mathbb{F}_{p^n}$ and the vector space $\mathbb{F}_p^n$ are isomorphic.

In view of Lemma 4, we can apply the Frobenius mappings and the cyclic shifts mapping on $\mathbb{F}_{p^n}$ exactly as they are applies on $\mathbb{F}_p^n$. If $h : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_p^n$ is the isomorphism from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p^n$ such that $y = h(x)$ for $y \in \mathbb{F}_p^n$ and $x \in \mathbb{F}_{p^n}$ then

$$\Upsilon_\ell(y) \stackrel{\text{def}}{=} h(\Upsilon_\ell(x)) \text{ and } \Phi_j(y) \stackrel{\text{def}}{=} h(\Phi_j(x)),$$

for every $0 \leq \ell \leq n - 1$ and $0 \leq j \leq p^n - 2$. 

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**Lemma 5.** Let $X$ and $Y$ be two $k$-dimensional subspaces of $\mathbb{F}_p^n$ such that there exist two integers, $\ell_1$, $0 \leq \ell_1 \leq n - 1$, and $j_1$, $0 \leq j_1 \leq p^n - 1$, such that $Y = \Phi_{j_1}(T_{\ell_1}(X))$. Then there exist two integers, $\ell_2$, $0 \leq \ell_2 \leq n - 1$, and $j_2$, $0 \leq j_2 \leq p^n - 1$, such that $Y = \Upsilon_{\ell_2}(\Phi_{j_2}(X))$.

**Corollary 1.** The combination of the Frobenius mappings and the cyclic shifts mappings induce an equivalence relation of the set of all $k$-dimensional subspaces of $\mathbb{F}_p^n$.

### 3 Parallelism of lines in $\text{PG}(n, p)$

Let $\alpha$ be a primitive element in $\mathbb{F}_p^n$, where $p$ and $n$ are primes. Let $\mathcal{P}_0 = \{(\alpha^i, 0) : 0 \leq i \leq \frac{p^n - 1}{p - 1} - 1\}$ and $\mathcal{P}_1 = \{(x, 1) : x \in \mathbb{F}_p^n\}$. The points of the projective geometry $\text{PG}(n, p)$ are represented by $\mathcal{P}_0 \cup \mathcal{P}_1$.

The Frobenius mapping $\Upsilon_{\ell}$ is applied on a subset $X \in \mathcal{P}_0 \cup \mathcal{P}_1$ as follows. For a given point $(\alpha^i, 0) \in \mathcal{P}_0$, and an integer $\ell$, we define $\Upsilon_{\ell}((\alpha^i, 0)) \overset{\text{def}}{=} (\alpha^{\ell p^i}, 0)$, where the exponent is taken modulo $\frac{p^n - 1}{p - 1}$. For a given point $(y, 1) \in \mathcal{P}_1$, we have $y = \alpha^i$, for some $0 \leq i \leq p^n - 2$. We define $\Upsilon_\ell((y, 1)) \overset{\text{def}}{=} (\alpha^{\ell p^i}, 1)$, where the exponent is taken modulo $p^n - 1$; if $y = 0$ then we define $\Upsilon_\ell((0, 1)) \overset{\text{def}}{=} (0, 1)$. If $X = \{x_1, x_2, \ldots, x_m\}$ then $\Upsilon_\ell(X) \overset{\text{def}}{=} \{\Upsilon_\ell(x_1), \Upsilon_\ell(x_2), \ldots, \Upsilon_\ell(x_m)\}$.

Similarly, the cyclic shift mapping is applied on a subset $X \in \mathcal{P}_0 \cup \mathcal{P}_1$. For a given point $(\alpha^i, 0) \in \mathcal{P}_0$, and an integer $j$, we defined $\Phi_j((\alpha^i, 0)) \overset{\text{def}}{=} (\alpha^{i + j}, 0)$, where the exponent is taken modulo $\frac{p^n - 1}{p - 1}$. For a given point $(y, 1) \in \mathcal{P}_1$, we have $y = \alpha^i$, for some $0 \leq i \leq p^n - 2$. We define $\Phi_j((y, 1)) \overset{\text{def}}{=} (\alpha^{i + j}, 1)$, where the exponent is taken modulo $p^n - 1$; if $y = 0$ then we define $\Phi_j((0, 1)) \overset{\text{def}}{=} (0, 1)$. If $X = \{x_1, x_2, \ldots, x_m\}$ then we define $\Phi_j(X) \overset{\text{def}}{=} \{\Phi_j(x_1), \Phi_j(x_2), \ldots, \Phi_j(x_m)\}$.

The results on the two types of mappings given in Section 2 hold also for this representation of $\text{PG}(n, p)$ and these modified mappings. In particular each one of the two types of mappings and the combination of the two induce an equivalence relation on the set of all $k$-dimensional subspaces of $\mathbb{F}_p^n$, also for this representation of $\text{PG}(n, p)$.

A line in $\text{PG}(n, p)$ consists of $p + 1$ points. Either all the $p + 1$ points are contained in $\mathcal{P}_0$ or exactly one point is contained in $\mathcal{P}_0$ and $p$ points are contained in $\mathcal{P}_1$. A $1$-spread (or spread in short) contains $p^{n-1} + p^{n-3} + \cdots + p^2 + 1$ lines, from which exactly $p^{n-1}$ contain points from $\mathcal{P}_1$.

We construct the first spread as follows. The spread contains three types of lines.

1. The first type contains only one line. This line contains the points $(\alpha^0, 0)$, $(0, 1)$, and $(\alpha^0, 1)$, for all $0 \leq i \leq p - 2$.

2. The second type contains $p^{n-1} - 1$ lines, each line has exactly one point from $\mathcal{P}_0$ and $p$ points from $\mathcal{P}_1$. $X$ is a line of this type if and only if $\Upsilon_\ell(X)$ is a line of this type, for every $\ell$, $0 \leq \ell \leq n - 1$. The lines of this type are partitioned into $p - 1$ sets. Each set has $\frac{p^{n-1} - 1}{p - 1}$ lines. For each line $X$ in the $i$-th set there is a line $Y$ in the $(i + 1)$-th set such that $Y = \Phi_{(p - 1)i + 1}(X)$ for some integer $j$. For every two distinct given lines of this type, $X$ and $Y$, there is no $j$ such that $Y = \Phi_{(p - 1)j}(X)$. Therefore, we can consider for this type $\frac{p^{n-1} - 1}{(p - 1)m}$ lines as base lines. Each such line is shifted $p - 1$ shifts.
to form base lines for each set, one for each residue modulo $p − 1$. On these shifts we apply the Frobenius mappings. Note, that since $t \cdot p \equiv t \pmod{p − 1}$ the Frobenius mappings preserve the set in which the subspace of this type is contained.

3. The third type contains $\frac{p^n - 1}{p^2 - 1} = p^{n-3} + p^{n-5} + \cdots + p^2 + 1$ lines, where all the points of each line are contained in $P_0$. $X$ is a line of this type if and only if $Y_1(X)$ is a line of this type. For each two given distinct lines of this type, $X$ and $Y$, there is no $j$ such that $Y = \Phi_j(X)$. Therefore, we can view $p^{n-3} + p^{n-5} + \cdots + p^2 + 1$ of these lines as generator lines. The Frobenius mappings are applied on these generator lines.

The base line and the generator lines should have some difference properties. We skip the description of these difference properties and just mention that they are similar to the ones described in Section 4.

The $(i + 1)$-th spread, $S_{i+1}$, is generated from the $i$-th spread, $S_i$, as follows. If $X$ is a line of $S_i$ then $\Phi_{p-1}(X)$ is a line in $S_{i+1}$. In this way we generate a total of $\frac{p^n - 1}{p-1}$ pairwise disjoint spreads. It can be verified that if we want to obtain distinct lines in the $\frac{p^n - 1}{p-1}$ spreads then $p - 1$ must be relatively prime to $\frac{p^n - 1}{p-1}$.

Clearly, our description of the construction implies some divisibility conditions. Some of these conditions always hold and only the necessary proof should be given. The conditions which are not always satisfied are summarized in the following lemma.

**Lemma 6.** For the construction of $\frac{p^n - 1}{p-1}$ pairwise disjoint spreads, given in this section, the following conditions must hold.

1. $n$ is odd.
2. $n$ divides $\frac{p^n - 1}{p^2 - 1}$.
3. $\gcd(p - 1, \frac{p^n - 1}{p-1}) = 1$.

We conjecture that if the conditions of Lemma 6 are satisfied then there is a parallelism of lines in PG$(n, p)$ obtained by the method given in this section.

We demonstrate the construction to one of a numerous number of solutions for $p = 3$ and $n = 5$. Let $\alpha$ be a root of the primitive polynomial $x^5 + 2x^4 + 1$ over $\mathbb{F}_3$. There are 8 base lines of the second type. They are shifted by 1, 5, 29, 111, 61, 187, 129, and 125, respectively for the odd shifts, and by 218, 8, 12, 230, 150, 132, 202, 40, respectively for the even shifts. The outcome are the sixteen lines given in the following table.

The generator lines of the third type are

$$\{(\alpha^8, 0), (\alpha^9, 0), (\alpha^{13}, 0), (\alpha^{77}, 0)\}$$

and

$$\{(\alpha^{73}, 0), (\alpha^{75}, 0), (\alpha^{119}, 0), (\alpha^{26}, 0)\}.$$  

Thus, a parallelism of lines with 121 disjoint spreads in PG$(5,3)$ is obtained.
4 Steiner structures

We suggest to construct a set $S$ of 3-dimensional subspaces of $\mathbb{F}_2^n$, $n$ prime, in which the cyclic shifts mappings and the Frobenius mappings form its automorphism group. The nonzero elements of the field will be represented as one cycle for this construction. Given a 3-dimensional subspace

$$\{0, \alpha^i, \alpha^i \alpha^j, \alpha^i \alpha^j \alpha^k, \alpha^i \alpha^j \alpha^k \alpha^l, \alpha^i \alpha^j \alpha^k \alpha^l \alpha^m, \alpha^i \alpha^j \alpha^k \alpha^l \alpha^m \alpha^n \}$$

in $S$, we require that for each $0 \leq \ell \leq n - 1$ and $0 \leq j \leq 2^n - 2$,

$$\{0, \alpha^{i2^\ell+j}, \alpha^{i2^\ell+j}, \alpha^{i2^\ell+j}, \alpha^{i2^\ell+j}, \alpha^{i2^\ell+j}, \alpha^{i2^\ell+j}, \alpha^{i2^\ell+j} \}$$

will be also a 3-dimensional subspace of $S$. In other words, $X \in S$ if and only if $\Phi_j(\Upsilon_\ell(X)) \in S$, for every $0 \leq \ell \leq n - 1$ and $0 \leq j \leq 2^n - 2$.

For a given 3-dimensional subspace

$$X = \{0, \alpha^i, \alpha^i \alpha^j, \alpha^i \alpha^j \alpha^k, \alpha^i \alpha^j \alpha^k \alpha^l, \alpha^i \alpha^j \alpha^k \alpha^l \alpha^m, \alpha^i \alpha^j \alpha^k \alpha^l \alpha^m \alpha^n \}$$

of $\mathbb{F}_2^n$, let the difference set of $X$, $\Delta(X)$, be the set of integers defined by

$$\Delta(X) \text{ def } = \{i_r - i_s : 1 \leq r, s \leq 7, r \neq s\}.$$

**Lemma 7.** If $X$ is a 3-dimensional subspace of $\mathbb{F}_2^n$, where $2^n - 1 \equiv 0 \pmod 7$, then the cyclic shifts of $X$ form $2^n - 1$ distinct 3-dimensional subspaces.

A 3-dimensional subspace $X$ of $\mathbb{F}_2^n$ will be called complete if $|\Delta(X)| = 42$. Two complete 3-dimensional subspaces $X$, $Y$ of $\mathbb{F}_2^n$ will be called disjoint complete if $\Delta(X) \cap \Delta(Y) = \emptyset$. Each 3-dimensional subspace $X$ of $\mathbb{F}_2^n$ contains seven two-dimensional subspaces of $\mathbb{F}_2^n$. If $|\Delta(X)| = 42$ then no two of these seven two-dimensional subspaces are cyclic shifts of each other.

**Lemma 8.** If $X$ is a 3-dimensional subspace of $\mathbb{F}_2^n$ and $|\Delta(X)| = 42$ then the cyclic shifts of $X$ form $2^n - 1$ distinct 3-dimensional subspaces. The $7 \cdot (2^n - 1)$ two-dimensional subspaces of these $2^n - 1$ distinct 3-dimensional subspaces are all distinct.
Theorem 1. If \( n \cong 1 \pmod{6} \) and there exist \( \frac{2^{n-2}}{42} \) pairwise disjoint complete 3-dimensional subspaces then there exists a Steiner structure \( S_2[2, 3, n] \).

For \( n = 7 \) there exist only two pairwise disjoint complete 3-dimensional subspaces of \( \mathbb{F}_2^7 \) instead of three needed for a Steiner structure \( S_2[2, 3, 7] \). If \( n = 13 \) then \( \frac{2^{13}-2}{42} = 195 \) and the search for 195 pairwise disjoint complete 3-dimensional subspaces of \( \mathbb{F}_2^{13} \) looks to be out of reach. Therefore, we try to increase the size of the automorphism group by adding the Frobenius mappings into the equation.

For a given 3-dimensional subspace

\[
X = \{0, \alpha^i, \alpha^j, \alpha^k, \alpha^l, \alpha^m, \alpha^n\}
\]

of \( \mathbb{F}_2^n \) let the coset difference set of \( X \), \( C(\Delta(X)) \), be the set of integers defined by

\[
C(\Delta(X)) \overset{\text{def}}{=} \{C(i_r - i_s) : 1 \leq r, s \leq 7, r \neq s\}.
\]

A 3-dimensional subspace \( X \) of \( \mathbb{F}_2^n \) will be called coset complete if \( |C(\Delta(X))| = 42 \). Two coset complete 3-dimensional subspaces \( X, Y \) of \( \mathbb{F}_2^n \) will be called disjoint coset complete if \( C(\Delta(X)) \cap C(\Delta(Y)) = \emptyset \).

Theorem 2. If \( n \cong 1 \pmod{6} \) is a prime and there exist \( \frac{2^{n-2}}{42n} \) pairwise disjoint coset complete 3-dimensional subspaces then there exists a Steiner structure \( S_2[2, 3, n] \).

A search for pairwise disjoint coset complete 3-dimensional subspaces of \( \mathbb{F}_2^n \) is done as follows. \( \{0, \alpha^i, \alpha^j, \alpha^k\} \) is a two-dimensional subspace if and only if \( \alpha^i + \alpha^j + \alpha^k = 0 \). Also, \( \{0, \alpha^i, \alpha^j, \alpha^k\} \) is a two-dimensional subspace if and only if \( \{0, \alpha^{i+j}, \alpha^{i+j+k}, \alpha^{i+j+k}\} \) is a two-dimensional subspace for every integer \( j \). Therefore, \( C(i_2 - i_1), C(i_1 - i_2), C(i_3 - i_1), C(i_1 - i_3), C(i_3 - i_2), C(i_2 - i_3) \), always appear together in a coset difference set. It follows that we can partition the \( \frac{2^{n-2}}{6n} \) cyclotomic cosets of size \( n \), into \( \frac{2^{n-2}}{6n} \) groups of cosets and instead of 42 integers in a coset difference set we should consider only 7 such integers. We form a graph \( G(V, E) \) as follows. The set \( V \) of vertices for \( G \) are represented by 7-subsets of the set of \( \frac{2^{n-2}}{6n} \) elements which represents the \( \frac{2^{n-2}}{6n} \) groups of cosets. Such a 7-subset \( v \) represents a vertex if and only if there exists a 3-dimensional subspace \( X \) of \( \mathbb{F}_2^n \) whose coset difference set \( C(\Delta(X)) \) is represented by \( v \). For two vertices \( v_1, v_2 \in V \) represented by 7-subsets, there is an edge \( \{v_1, v_2\} \in E \) if and only if \( v_1 \cap v_2 = \emptyset \), i.e. the related 3-dimensional subspaces are disjoint coset complete. A clique with \( m \) vertices in \( G \) represents \( m \) pairwise disjoint coset complete 3-dimensional subspaces. A clique with \( \frac{2^{n-2}}{42n} \) vertices represents a Steiner structure \( S_2[2, 3, n] \).

A program which generates this graph for \( n = 13 \) was written. For \( n = 13 \) we have \( \frac{2^{13}-2}{13} = 630 \) cyclotomic cosets of size 13, and 105 groups of cosets. A clique of size \( \frac{2^{n-2}}{42n} = 15 \) in this graph represents a Steiner structure \( S_2[2, 3, 13] \). Unfortunately, it is not feasible to check even a small fraction of the subsets with 15 vertices of this graph. Cliques of size 14 in the graph were found in a rate of more than 10 such cliques per minute. Such a clique represents a code with 1490762 3-dimensional subspaces of \( \mathbb{F}_2^7 \) and minimum distance 4 (compared to the largest code of size 1221296 known before \([5]\), and the largest code of size 1154931 in which the cyclic shifts form the automorphism group \([7]\)).
5 Conclusion and future work

This work is a preliminary attempt to solve two of the most intriguing and difficult problems connecting design theory and coding theory in the Grassmannian (and the related projective geometry), namely the existence of nontrivial Steiner structure and construction of parallelism of lines in the projective space. A parallelism of lines in PG(5,3) was given and an idea for a general construction of a parallelism in PG(n,p) was suggested. A Steiner structure $S_2[2,3,13]$ was not constructed, but a related code which is the largest known one for the related parameters, was found and the foundations to find such structure were suggested. We would like to see the first nontrivial Steiner structure and a new infinite family of parallelisms with new parameters based on the foundation given in this work.

References


