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CONFINEMENT AT LARGE-N

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CONFINEMENT AT LARGE-N

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ABSTRACT
Recent numerical results indicate that QCD in the limit of an infinite number (N) of colors also has confinement and moreover that it looks rather similar to normal QCD with N = 3 colors. This imposes severe restrictions on what the mechanism of confinement can be.

1. INTRODUCTION
Quantum Chromodynamics (QCD) is presently believed to be the fundamental theory that underlies the observed strong interactions. It is surprisingly simple to say what this theory is: a SU(3) gauge theory with (anti-)quarks in the (3)3 representation. Apart from the current masses of the quarks, which are quite negligible anyway for the up and down flavors, there are no free parameters in the theory. The gauge coupling constant g is dimensionally transmuted to a mass scale and all the mass ratios of hadrons and glueballs can, in principle, be calculated. On the one hand this makes QCD a powerful theory, but on the other hand this makes it, short of a complete solution, difficult to understand what goes on, since we cannot study the response of the theory under variation of its parameters.

Already in 1974 't Hooft suggested that we could consider the number (N) of colors as such a free parameter and that in particular it might be worthwhile to study the limit N → ∞, where the theory simplifies dramatically, only "planar" Feynman graphs contributing. The assumption was, of course, that variation with N is not too drastic a change and that the N → ∞ limit exists. It turned out to
be relatively easy to establish\textsuperscript{3)} that planar QCD has many of the
desired features of real QCD, such as the existence of infinitely many
light and narrow meson resonances and an explanation of Zweig's rule.
Baryons are thought to arise\textsuperscript{4)} as solitons of the effective meson
theory and their mass diverges as $N = 1/(1/N)$ just as the 't Hooft-
Polyakov magnetic monopole has a mass $\propto 1/\alpha$, where $\alpha \equiv g^2/4\pi$ is
the expansion parameter. In fact, this provides the field theoretic
underpinning of Skyrme's successful idea that the baryons are solitons
of the effective Lagrangian of the pions.\textsuperscript{5)} It seems that we can
write symbolically

\begin{equation}
\text{"QCD"} = [1 + 1/N + 1/N^2 + \ldots]_{N=3}
\end{equation}

and that taking the leading term is quite good an approximation
already. Perhaps the \textit{raison d'être} of (1) is that the starting point
planar QCD is a better defined theory, i.e. having a less divergent
perturbation series,\textsuperscript{6)} compared with the QCD "theory". Since confine-
ment is thought to be largely a matter of the gauge fields, appropri-
ately called gluons, it is reasonable to consider just the pure gauge
theory with quarks acting as sources only and not as dynamical fields
(quenched approximation\textsuperscript{*}). In this case the expansion parameter is
$1/N^2$, so that the leading term alone in (1) is an even better
approximation. Henceforth we consider only the pure gauge theory.

The successful phenomenology of mesons and baryons from the
large-$N$ point of view depends on one crucial assumption:
confinement. This is less trivial than one might think at first. It
is possible to show\textsuperscript{7)} (see Section 4 for details) that the two string
tensions for heavy quarks in the adjoint and fundamental
representations of SU($N$) are related at $N = \infty$ by

\begin{equation}
[\sigma_A = 2\ \sigma_F]_{N=\infty}
\end{equation}

\textsuperscript{*}For $N_F$ flavors each quark loop in a Feynman graph carries a factor
$N_f/N$, so that the quenched approximation seems reasonable for
$N >> N_f$. But in the real world $N = 3$ and $N_f \sim 2$, so that this
argument is rather weak. Still, we do observe valence quarks, so
that the quenched approximation must not be too bad.
The popular confinement scheme with $Z(N)$ vortices, \(^{8}\) where $Z(N)$ is the discrete center of the $SU(N)$ gauge group, implies $\sigma_\lambda = 0$, since the adjoint representation is blind to $Z(N)$. Were this the mechanism of confinement we would conclude from (2) that the large-$N$ limit of QCD loses confinement, $\sigma_F(N = \infty) = 0$. We see that a study of large-$N$ QCD may tell us something about the nature of the mechanism that operates confinement in the real world ($N = 3$), assuming the mechanism to be basically the same for all $N \geq 3$. But the numerical simulations would be much more time consuming than the present simulations of the pure $SU(3)$ lattice gauge theory by a factor of order $(N^2 - 1)/8$, which is the ratio of the number of degrees of freedom. Surprisingly Eguchi and Kawai\(^{9}\) discovered a few years ago that when $N$ is large the volume to be used can be much reduced, even to a single-point lattice. Let us compare then for a fixed large number $N$ of colors the number of degrees of freedom ($D$ and $V$ are the dimensionality and volume of the space-time lattice)

\[
\text{lattice: } \quad \text{d.o.f. } \sim D N^2 V \\
\text{single point model: } \quad \text{d.o.f. } \sim \frac{D N^2 V_{\text{eff}}}{V_{\text{eff}}}
\]

where $V_{\text{eff}} \sim N^2$ is the effective volume mimicked by the reduced model (see Section 2 for details). Compared to a lattice with volume $V = V_{\text{eff}}$ the reduced model gains a factor $V_{\text{eff}} \sim N^2$, which for calculations at $N \sim 100$ is substantial. This improvement makes simulations of large-$N$ QCD feasible on present day CYBERS and CRAYS. In Section 2 we briefly review the reduced model with a twist to it and in Section 3 we discuss the numerical results obtained from simulations of this model. The results indicate that there is indeed confinement in large-$N$ QCD and that it looks rather similar to what happens at $N = 3$. As mentioned above this has important implications for the nature of the mechanism of confinement and in Section 4 we discuss them.
2. REDUCTION AND TWIST

We start with a pure SU(N) gauge theory on a 4-dimensional hypercubic lattice with spacing \(a\), which we set equal to 1 (sometimes we will gain in clarity by keeping \(a\) explicit). The group variables are associated with the links between neighbouring lattice sites and we write them as \(U_\mu(x)\) for the link from site \(x\) to site \(x + \hat{\mu}\), where \(\hat{\mu}\) is a directional vector of unit length. The expectation value of an observable \(O(U_\mu(x))\) is calculated as follows:

\[
\langle O \rangle_L = Z_L^{-1} \int dU_\mu(x) \times \exp^{-N S_W} . \tag{3a}
\]

\[
S_W = \sum_x \sum_{\mu \neq \nu} \text{Tr}(1 - U_\mu(x)U_\nu(x + \hat{\mu})U_\nu(x + \hat{\nu})^{-1}) . \tag{3b}
\]

In (3a) \(dU\) is the Haar measure over the group manifold (in our case SU(N)) and the normalization factor \(Z_L\) is just the same integral with \(O = 1\). In (3b) we use the simple plaquette action introduced in Wilson's seminal paper. The inverse coupling \(\beta = 1/\lambda = 1/N g^2\) is held fixed in the large-N limit.

To calculate the same expectation value in the reduced model we simply collapse the variables in (3) as follows

\[
\forall x : U_\mu(x) \rightarrow U_\mu \tag{4}
\]

Eguchi and Kawai proved that in leading order \(\langle O \rangle_R\) and \(\langle O \rangle_L\) obey the same Schwinger-Dyson equations, which are closed at \(N = \infty\), so that

\[
\langle O(U_\mu(x)) \rangle_L = \langle O(U_\mu) \rangle_R + \text{terms down by } 1/N^2 . \tag{5}
\]

Instead of calculating the expectation value with the lattice variables we may as well use the reduced model without explicit space-time dependence, as long as \(N\) is sufficiently large. But there is a snag: at weak coupling a \(Z(N)^4\) phase symmetry \((U_\mu \rightarrow Z(\mu)U_\mu)\) of the reduced model is broken spontaneously and spurious terms in the Schwinger-Dyson equations get a non-zero expectation value, so that (5) breaks down. One of these terms is, for example, \(N^{-1} \text{Tr} U_\mu U_\nu \).
which need not vanish if the phase symmetry is broken, whereas on the lattice $<N^{-1} \text{Tr} U_0(x)U_1(x + \hat{\sigma})>$, vanishes for all couplings due to gauge invariance. We need "something" to prevent the breaking of the phase symmetry. An elegant choice for the "something" is $Z(N)$ twist as introduced by 't Hooft in a different context. The twisted reduced model is given by\(^{12}\)

\[
\begin{align*}
<0(U_{\mu})>_{\text{TR}} &= Z_{\text{TR}}^{-1} \int \prod_{\mu} d U_{\mu} O(U_{\mu}) e^{-NBS_{\text{TR}}}, \\
S_{\text{TR}} &= \sum_{\mu \neq \nu} \text{Tr}(1 - Z_{\mu \nu} U_{\mu} U_{\nu}^{-1} U_{\mu}^{-1}),
\end{align*}
\]

(6a) (6b)

where the 6 elements $Z_{\mu \nu} = Z^*_{\nu \mu}$ take values in the center $Z(N)$ of the gauge group $SU(N)$. The art is to choose an appropriate twist $Z_{\mu \nu}$, which kills the spurious terms. For such twist the spurious terms vanish because at weak coupling the $U_{\mu}$ fluctuate around the non-trivial zero action configuration $\Omega$, whose products have vanishing trace. In short, $<N^{-1} \text{Tr} U_0 U_1>_{\text{TR}} = N^{-1} \text{Tr} \Omega_0 \Omega_1 = 0$. For two types of twist the reduced model is equivalent to the lattice gauge theory on a box with periodic boundary conditions and dimensions $N_{\mu}$. The following sketch (for $N_0 = 2$, $N_1 = 6$ and periodically continued) shows which Wilson lines starting at the lower left hand corner and ending elsewhere vanish (0) or not (1) at weak coupling:

\[
\begin{array}{cccccccc}
\bullet & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\
\bullet & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\
\end{array}
\]

For example:

\[
\begin{align*}
<N^{-1} \text{Tr} U_0 U_0^6 U_0>_{\text{TR}} & \neq 0, \\
<N^{-1} \text{Tr} U_0^3 U_1>_{\text{TR}} &= <N^{-1} \text{Tr} U_0 U_1>_{\text{TR}} = 0, \\
<N^{-1} \text{Tr} U_0 U_1 U_0^6 U_1 U_0^{-1} U_0^{-1} U_0>_{\text{TR}} &= 0
\end{align*}
\]

In order to simulate the gluon degrees of freedom correctly one must have $V_{\text{eff}} = \prod_{\mu} N_{\mu} = N^2$. Also it is clear that there is only an effective volume as long as the $\Omega_{\mu}$ are relevant, i.e. in the weak
The two types of twist announced above have the following characteristics:

1. **Symmetric twist**\(^{12}\): for \(N = L^2\), with \(L\) an integer, the box size is \(N_u = L(1,1,1,1)\);

2. **Hot twist**\(^{14}\): for \(N = N_0 \lambda_1 \lambda_2 \lambda_3\), where the integers \(\lambda_i\) are pairwise prime (for example 1,2,3 or 1,3,8), the box size is \(N_u = N_0(1,\lambda_2 \lambda_3, \lambda_3 \lambda_1, \lambda_1 \lambda_2)\).

This last twist was constructed for finite temperature studies where one keeps one dimension \((N_0)\) much less than the others, hence its name.

It is possible to keep certain directions unreduced, as long as one takes care that in the reduced subspace the \(N^2 - 1\) gluon d.o.f. are reproduced.\(^{15}\) In particular models with one unreduced direction \((\hat{0})\) have been used to calculate the finite temperature \((aT = N_0^{-1})\) behaviour and the mass gap of large-\(N\) QCD.

For further details of reduction at large-\(N\) we refer the reader to the original articles and to two recent reviews.\(^{16}\)

3. **MONTE CARLO RESULTS**

Fig. 1 shows the results\(^{17}\) for the Creutz\(^{18}\) ratios \(\chi \sim g_F^2\) for large loops, but it appears that the loops are not large enough to show an "envelope." As it stands we only have an upper bound on the string tension \(g_F\). Remark that if one is interested in certain loop sizes \(N\) should be sufficiently large, otherwise \(1/N^2\) errors accumulate and give the wrong behaviour (see \(\chi(4,3)\) for \(N = 64\) in Fig. 1). From these results alone it is not clear whether or not large-\(N\) confines.

We turn now to the behaviour at finite temperature, where the lattice size in one direction \((\hat{0})\) is much less than those of the "spatial" directions, \(N_s \gg N_0 = (aT)^{-1}\). For given \(N_0\) deconfinement

---

\*The fields fluctuate around this saddle-point (and around other saddle-points with non-vanishing action) for values of the coupling constant up to the cross-over to the strong coupling regime. Numerical results for the 2d chiral SU(\(N\)) model show this explicitly.\(^{13}\)
Fig. 1

Monte Carlo results (Fabricius and Haan, Ref. 17) for the Creutz ratio $\chi(1,1)$ in the reduced model with symmetric twist. For $1, J \rightarrow \chi \rightarrow \sigma_F a^2$, where $\sigma_F$ is the string tension between heavy quarks (in the usual fundamental representation of SU(N)) and $a$ is the lattice spacing.
High temperature deconfinement transition of large-$N$ QCD (dots) from simulations of reduced models with starts at weak coupling. Details of the models used are presented in Table 1. The bulk transition is drawn at $\beta = 0.345$, a compromise value between Ref. 19 and 21, but for symmetric lattice spacings ($\xi \equiv a/a_0 = 1$) the transition is really at $\beta = 0.350$ (see Appendix). For comparison the $N = 3$ results$^{21}$ are shown also (open squares).
occurs at a critical coupling $\beta_c(N_0)$. Fig. 2 summarizes the best results to date and the details of the reduced models used are given in Table 1. The $\beta_c(N_0)$ were determined from the change* in the thermal line expectation value $<N^{-1} Tr U_0^N>$, which is (non-) zero in the (de)confined phase. For $N_0 = 4$ the energy density $\varepsilon$ has been calculated\textsuperscript{19} also: at $\beta_c(4)$ it has a discontinuity and for $\beta \to \infty$ ($T \to \infty$) it approaches the Stefan-Boltzman value. It looks as if for $\beta_c > 0.35$ asymptotic scaling** sets in, so that there is indeed a physical deconfinement temperature. We are tempted to rephrase the saying about smoke and fire: if there is deconfinement, there must be confinement somewhere. Furthermore, we compare in Fig. 2 the large-$N$ results with those\textsuperscript{23} for $N = 3$, which may start to scale for $N_0 > 10$ (see Kuti in these Proceedings). To this author it looks as if the rapid change for the $N = 3$ theory is the "shadow" of the genuine phase transition of large-$N$. In fact the value of $\beta$ where the sudden deviation from the asymptotic scaling behaviour sets in is practically the same for $N = 3$ and $\infty$ (see Section 4 for further discussion).

Finally we report some preliminary results\textsuperscript{24} of the massgap as calculated in a partially reduced model with $N_0 = 12$ time slices. From the correlation between two spatial loops (of 3 different types) at different time slices the massgap $\Delta M$ can be extracted. Table 2 presents the values for $\Delta M / \Lambda$, which give an average of $\Delta M / \Lambda \sim 1248 \pm 135$. Note that the $\beta$ values 0.36 and 0.37 are on the weak coupling side of the bulk transition, but still in the confinement phase (from Fig. 2 one gets $\beta_c(10) \sim 0.375$).

To summarize, the preliminary Monte Carlo results are

\*The $\beta_c(N_0)$ shown in Fig. 2 are from runs started at weak coupling, which is where the twists impart knowledge of $N_0$ to the model (see Section 2). $\beta_c$ from runs started in strong coupling are irrelevant, since there the model does not know about $N_0$. We hope that there is not too much hysteresis so that $\beta_c(N_0)$ from cold starts is only a little below the correct value. Furthermore it would be interesting to check if the latent heat $\Delta \varepsilon$ calculated\textsuperscript{19} in the single-point model is the same as in a partially reduced model with $N_0 = 4$ time slices.

\**Das and Kogut\textsuperscript{21} claim to have seen some evidence for asymptotic scaling with respect to the asymmetry parameter $\xi \equiv a_s / a_0$.\*
Table 1: Deconfinement transition of large-N QCD

<table>
<thead>
<tr>
<th>$1/aT_c (= N_0)$</th>
<th>$a^3 V_{s, \text{effective}}$</th>
<th>$N$</th>
<th>$\beta_c(N_0)$</th>
<th>model</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$6 \times 10 \times 15$</td>
<td>30</td>
<td>0.19</td>
<td>hot twist</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>$4 \times 6 \times 12$</td>
<td>24</td>
<td>0.335</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>$6 \times 9 \times 18$</td>
<td>54</td>
<td>0.34</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>$8 \times 12 \times 24$</td>
<td>96</td>
<td>0.35</td>
<td></td>
<td>19</td>
</tr>
<tr>
<td>6*</td>
<td>$9 \times 9 \times 9$</td>
<td>81</td>
<td>0.353</td>
<td>symm.twist, $a/a_0 = 1.5$</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>$15 \times 20 \times 24$</td>
<td>60</td>
<td>0.365</td>
<td>partially reduced</td>
<td>22</td>
</tr>
</tbody>
</table>

*here $N_u = 9$, so that $1/aT_c = (a_0/a)N_0 = 6$.

Table 2: Massgap/A of large-N QCD*

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$1 \times 1$</th>
<th>$1 \times 2$</th>
<th>$2 \times 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.36</td>
<td>1282</td>
<td>1245</td>
<td>1231</td>
</tr>
<tr>
<td>0.37</td>
<td>1294</td>
<td>1294</td>
<td>1141</td>
</tr>
</tbody>
</table>

*Monte Carlo results from a partially reduced model with $N = 30$, $N_0 = 12$ and the effective spatial volume $V_{s, \text{eff}} = 10 \times 12 \times 15$. The typical error in $\Delta M/A$ values is 300.
and the deconfinement transition at high temperature appears to be first order. For $N = 3$ the approximate values on the right hand side of (7) would be $90, 50$ and $250.*$ Of course, one should only compare the two ratios $\Delta M/T_c$ and $\sqrt{\sigma_F}/T_c$ and the $N = 3$ and $\infty$ values appear to be in the same ballpark.

4. CONFINEMENT MECHANISM

The tentative conclusion from the results presented in the previous section is: yes, there is confinement at large-$N$, moreover it looks rather similar to that at $N = 3$. Excluding some pathological behaviour we are led to think that the same mechanism is responsible for confinement at all $N^{**}$. This rather obvious statement excludes in fact many "schemes" and "pictures" for confinement. We take the liberty to distinguish between a "scheme," which identifies the crucial set of configurations, and a mere "picture," which does not so explicitly, but which may still have some value. The property that makes life at large-$N$ tough for many a scheme is factorization

$$<O_i O_j>_N = \frac{<O_i> <O_j>}{N},$$

(8)

where $O_i$ are gauge invariant observables, which for the pure gauge theory are simply the Wilson loops (unless we say otherwise, the fundamental representation is understood)

$$w(C) \equiv N^{-1} \text{Tr} U(C) \equiv N^{-1} \prod_C U(x)$$

(9)

Eq. (8) holds to all orders in planar perturbation theory and also in

*Present glueball masses are doubtful even if they show asymptotic scaling, since they are calculated at too large couplings ($\beta \sim 0.31$) where universality is lost, viz. $\sigma$ and $T_c$ do not scale asymptotically.

**The case $N = 2$ may be somewhat different, since, for example, the high-$T$ deconfinement transition is second order.
the strong coupling expansion (except perhaps for some higher order clusters). Hence, we assume factorization to be present at all couplings. In fact, there is a close analogy with statistical mechanics, where the fluctuations in a macroscopic variable vanish in the thermodynamic limit \( V \rightarrow \infty \). That the variable should be macroscopic means in our case that the \( O_i \) should "feel" the whole gauge group, while the thermodynamic limit corresponds to our limit \( N^2 \rightarrow \infty \). Note that factorization is crucial for the phenomenon of reduction: (i) only then form the Schwinger-Dyson equations a closed set, and (ii) it makes the spurious contributions to the S-D equations of the reduced model vanish (also with some help from the twist).

For the adjoint (A) and fundamental (F) representations of SU(N) the following relation holds

\[
N^{-2} \text{Tr}_A U(C) = N^{-1} \text{Tr}_F U(C) \cdot N^{-1} \text{Tr}_F U(C)^\dagger - N^{-2},
\]

while for U(N) the last term would be absent. Taking the expectation value of (10) and using factorization (B) one obtains if the area law holds the relation (2) mentioned in the Introduction.

We are now ready to discuss the restrictions on the mechanism of confinement that follows from large-N. The widely accepted confinement scheme of Z(N) vortices is in trouble. The idea of the scheme is that \( \langle w(C) \rangle \) decreases rapidly by summation over many different configurations of Z(N) vortices, which pierce the minimal area \( A(C) \) and thereby multiply the trace of (9) by a Z(N) factor, so that in the sum substantial cancellations occur. In short,

\[
\langle \begin{array}{c}
\end{array} \rangle \sim \sum_v \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} - \sum_v z(v)w(C) \sim e^{-\sigma_A(C)},
\]

provided the vortices are abundant enough. All this is very nice, but does not work for an adjoint trace, which does not pick up the Z(N) factors, so that only a perimeter law results, \( \sigma_A = 0 \). But the numerical results of Section 3 using the relation (2) contradict this: \( \sigma_A(N = \infty) \neq 0 \). (Below we will see that also \( \sigma_A(N = 2) \neq 0 \) for not too large separation of the quarks). Any scheme that uses
instantons is in trouble also, since their weight factor vanishes: \[ \exp(-N \frac{8\pi^2}{\lambda}) \to 0 \] for \( N \to \infty \). Even surviving extrema (weight \( \neq 0 \)) are irrelevant, since all give the same contribution to the path integral by factorization.\(^{27}\) The crucial physics probably occurs at \( \beta \approx 0.35 \) where the connection with the strong coupling regime is made (Figs. 1, 2). This might tie in with Greensite's claim\(^{28}\) that high order planar graphs form a string at a certain critical value of the coupling. On the other hand Neuberger\(^{29}\) has shown that for the lattice instantons, although absent in weak coupling, could suddenly turn on at a value \( \beta^* \) (see the Appendix for its value). For completeness we also mention Ref. 30, but we think that their discussion has little bearing on the first order transition we are concerned with here.\(^*\)

What scheme does work then? The present author does not know, but at least there is a "picture" that appears to have some relevance. This is the so called stochastic confinement,\(^{31}\) which in pictures runs as follows

\[
\langle \square \rangle^{4d} (\beta^{4d}) \sim \langle \square \rangle^{4d}_{\text{stochastic}} (\beta^{4d}) \sim \langle \square \rangle^{2d} \beta^{2d}.
\]  

(12)

The question "why confinement?" has been replaced in (12) by "how does the stochastic background arise?", so we have not gained in understanding. But the second relation in (12) of dimensional reduction \( D(=4) \to 2 \) has some heuristic arguments in favor of it.\(^{32}\) Note that in this reduction the lattice spacing \( a \) and all physical quantities, e.g. \( \sigma \), remain unchanged, only \( \beta^{4d} \) is renormalised to \( \beta^{2d} \). Monte Carlo results have confirmed this dimensional reduction.

---

\(^*\)Green and Samuel\(^{30}\) use the order parameter \( D \equiv \langle \text{det} U(C) \rangle^{1/N} \), which only makes sense for \( U(N) \), and from the strong coupling expansion find a second (or higher) order phase transition at \( \beta = 0.396 \). We know now that before \( \beta \) is reached a first order transition from the \( SU(N) \) dynamics occurs at \( \beta_c = 0.35 \), for which transition their alleged mechanism, where the \( U(1) \) part of \( U(N) \) is crucial, must be irrelevant. Perhaps the \( U(N) \) theory is different from the \( SU(N) \) theory in having a second transition at \( \beta \), where \( D(\beta) \) is non-analytic, but it appears more likely that these transition points will coincide at \( \beta = 0.35 \).
1. Consider the distribution \( \langle \rho_C(\alpha) \rangle \) of the eigenphases \( \alpha_m \) of the Wilson loop (9) given by

\[
U(C) = \Omega \text{ diagonal}(e^{i\alpha_1}, \ldots, e^{i\alpha_N}) \Omega^{-1}
\]

(13)

\[
\langle N^{-1} \text{Tr} \ U(C) \rangle = \int_\pi^{-\pi} d\alpha \ \langle \rho_C(\alpha) \rangle \ e^{i\alpha}
\]

(14)

where in the last equation we used a continuum notation \( (N \rightarrow \infty) \). It turns out that \( \langle \rho_C(\alpha) \rangle^4_{4d}(\beta^4_{4d}) \) as calculated numerically\(^{33}\) is the same as the 2d distribution at a shifted coupling constant \( \beta^2_{2d}(\beta^4_{4d}) \). Since the 2d theory is trivial \( \langle \rho^2_{2d} \rangle \) can be calculated analytically\(^{34}\). These results are for \( N = \infty \), but at \( N = 2 \) they have been observed also\(^{32}\).

2. For finite \( N \) the adjoint string will break at a certain distance, which grows with \( N \), but before that there may or may not be a linear potential between heavy adjoint quarks. In fact, for \( SU(2) \) it has been observed for representations \( j \leq 2 \) that

\[
\left[ x_j(I,J)/x_1(I,J) - j(j+1)/\frac{1}{2}(\frac{1}{2}+1) \right]_{N=2}
\]

(15)

which is the ratio of the quadratic Casimirs. Note that the factor 2 in (2) is also the ratio of the Casimirs! Since confinement in 2d is perturbative, the observed ratios in (2) and (15) indicate that dimensional reduction has occurred (but the Casimirs could also arise from a bag picture).

Let us return to the picture (12). For our purpose, the most important is Olesen's identification\(^{31}\) of the "random fluxes" as the distribution of the eigenphases. The meaning of having a background of random and additive fluxes is that

\[
\langle \rho_C(\alpha) \rangle \approx \int \ldots \ d\beta_1 d\beta_2 \ldots \quad \langle \rho_C(\beta_1) \rangle \langle \rho_C(\beta_2) \rangle \ldots \ x
\]

\[
A_1 + A_2 + \ldots = A(C)
\]

\[
x \delta_{2\pi} (\alpha - \beta_1 - \beta_2 \ldots)
\]

(16)
where \( A(C) \) is the minimal area of curve \( C \). From (16) the area law for (14) follows immediately. But only at \( N = \infty \) are the fluxes truly random, since (16) implies that for all \( n < w(C^n) \sim \exp(-\alpha n |A(C)|) \), which would be incorrect at finite \( N \) (for \( n > N \) a singlet can be formed on the perimeter). The appeal of this picture is that it involves only the group, not a sheet of magnetic monopoles, say, and thus appears consistent with the reduction of all space-time structure (4) at \( N = \infty \). As is well known planar QCD on a 2d lattice has a third order phase transition\(^{34}\) at \( \beta_c = 1/2 \), but the transition vanishes for finite \( N \). For \( N = \infty \) QCD on a tetrahedron\(^{35}\) \( \beta_c = \pi/8 \), but this is not really 3 dimensional, but rather 2d with a special topology. The 2d string tension is known analytically\(^{34}\)

\[
\chi^{2d} = \begin{cases} 
-\ln \beta^{2d} & , \quad \beta^{2d} \leq 1/2 \\
-\ln (1 - (4\beta^{2d})^{-1}) & , \quad \beta^{2d} \geq 1/2 
\end{cases} (17)
\]

and with the Monte Carlo results for \( \chi(\beta^{4d}) \) (Fig. 1) we get the relation \( \beta^{2d}(\beta^{4d}) \) as shown in Fig. 3. This relation is not particularly enlightening in itself, except that it indicates that the \( N = \infty \) phase transition at 2d is "contained" in the 4d one. Note that, in principle, the curve in Fig. 3 could have been higher or lower, so that \( \beta_c^{2d} \) would be on the strong or weak coupling branch, instead it is in the "middle" of the jump (probably consistency requires this). Knowledge of the specific configurations that lead to the first equality in (12) is required to see why the transition in 4d is first order (for example instantons, see the Appendix). With regard to high temperature deconfinement and glueballs dimensional reduction is not very helpful, since 2d does not have them, and again the relevant configurations are needed to investigate these facts. Still the picture of (12,16), although certainly not complete, does not run into insurmountable problems at large-\( N \), which is a rare virtue as we have seen above.
In summary, what do we have from the impressive numerical results (7) for QCD at large-N? In our opinion: even less reason to think we know what confinement is all about. Socrates could have lived with this, but can we really?

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APPENDIX: INSTANTONS IN TRANSITION

We have redone Neuberger's calculation\(^{29}\) for the lattice coupling \(\beta^*\), where the instantons suddenly turn on for \(N = \infty\), and extended it to the case of an asymmetric lattice. In weak coupling the packing fraction of instantons up to size \(\rho\) contains a factor \((\exp - [..])^{2N}\), where the expression [..] involves the running coupling constant \(g(\rho)\). For \(\beta > \beta^*\) [..] is positive and the density vanishes at \(N \to \infty\), but at \(\beta = \beta^*\), where [..] = 0, the instantons contribute. Neuberger found \(\beta^* = 0.16\), but had used a wrong value for \(\Lambda_{PV}/\Lambda_{L}\); with the correct value\(^{36}\) we get \(\beta^* = 0.25\). We can try to get a higher value for \(\beta^*\) by changing the cutoff of the instanton size integral in the magnetic susceptibility, from which the \(\beta\)-function is calculated, to \(\rho_{\text{cutoff}} = f a\), with \(f\) now a number \#1. The equation for \(\beta^*\) is

\[
[6.95 - \frac{1}{2}(y^* + \delta) + \ln(y^* + \delta)] = 0
\]

where \(y(\rho) \equiv 8\pi^2/g(\rho)^2 N\) and \(\delta \equiv -(11/3)\ln f\). From (A1) we get the desired \(\beta^* = 0.35\) for \(f = f \equiv 8.27\). This value \(f\) is much larger than the expected\(^{37}\) value of \(\approx 0.7\). On the other hand the precise value of our \(f\) is rather sensitive to the method used (for example \(f = 2.6\) if we keep the coupling constant bare in the factor of the zero modes of the instanton). Anyway, our main interest in this exercise is the qualitative behavior with respect to the lattice asymmetry \(\xi\), to which we turn now.

Consider an asymmetric lattice with a smaller lattice spacing in the \(\hat{0}\) direction, \(\xi \equiv a/a_0 > 1\). We can make two different approximations for the \(\rho\) integral:

1. if \(a_0\) is the relevant scale we set \(\rho_{\text{cutoff}} = f a_0\);
2. remembering that the instanton is a 4d object we set \(\rho_{\text{cutoff}} = f (a_0^3)a^{-1/4} = f a \xi^{-1/4}\).

The needed ratio \(\Lambda_L(\xi)/\Lambda_L(1)\) has already been calculated.\(^{38}\) Fig. 4 shows \(\beta^*\) as a function of \(\xi\) for approximation 1 (dashed) and 2 (full) and we expect the correct value for \(\beta^*\) to be close to (and perhaps somewhat to the left of) the full line.
Numerical simulations\textsuperscript{21) }show that for $\xi$ up to a certain value $\Xi$ the first order bulk transition stays at 0.35 or a little below (0.3425 for $\xi = 1.5$ and $N = 64$), but becomes of higher order or disappears for $\xi > \Xi$. We see that the location of the transition for $\xi < \Xi$ is in agreement with Fig. 4, but the instanton scheme as outlined here cannot explain the change of order. Still, it could be that the change in the phase transition is a finite-$N$ effect, since the values $\Xi \approx 1.75$, $2.25$, $2.5$, $3$ and $>3$ at $N = 16$, $25$, $35$, $49$ and $64$, respectively, could mean that $\Xi \rightarrow \infty$ for $N \rightarrow \infty$ (perhaps $\Xi^2 \sim N$).

So we see that instantons may (partly) explain the rapid change in the $\xi$-function at coupling 0.35 (Figs. 1 and 2), but it is somewhat worrying that their success depends on the regularization scheme used, i.e. it works for the lattice but not for Pauli-Villars, say. This instanton argument would also account for the fact that in Figs. 1 and 2 the jump is roughly equal ($\Delta a \sim \sqrt{7}$), which means that although asymptotic scaling no longer holds there is a vestige of universality. Of course, other configurations than instantons may contribute also. Needless to say, all this may have some relevance for $N = 3$, which as we have argued in Section 3 appears to show similar behaviour.
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