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Curvature in orbital dynamics

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Abstract

The physical basis and the geometrical significance of the equation for the orbit of a particle moving under the action of external forces is exhibited by deriving this equation in a coordinate-independent representation in terms of the radius of curvature of the orbit. Although this formulation already appeared in Newton's *Principia*, it has been ignored in virtually all modern textbooks on classical mechanics up to the present time. For orbits of small eccentricities, this orbit equation is used to obtain approximate solutions that illustrate the role of curvature. It is shown that this approach leads to a simple graphical method for computing orbits for central forces, which is similar to a method attributed to Newton. He applied it to the case of a constant central force, and sent a diagram of the orbit to Robert Hooke in 1679. The result is compared to the corresponding path of a ball revolving inside an inverted cone that Hooke described in his response to Newton.

Introduction

A basic problem in classical mechanics is the derivation of the equation for the orbit of a particle moving under the action of a central force. In modern textbooks this equation is generally obtained by starting with the equations of motion written in polar coordinates, and then applying the conservation of angular momentum to replace the time by the polar angle as the independent variable. This derivation, however, does not exhibit the physical and geometrical significance of the resulting orbit equation, and it hides the important role of curvature in dynamics. The calculus of curvature, which was developed by Newton and somewhat earlier by Huygens [2], guided Newton in his earliest efforts to develop orbital dynamics [3]- [5]. Newton first introduced the concept of curvature in the first edition (1687) of his *Principia*, and he applied it without explanation to a calculation of motion in dense media and to some lunar inequalities [5] [7]. Later, in the second edition (1713), he used curvature to derive relations for central forces, but only as an alternate method. The role of curvature in dynamics was re-discovered in 1705 by Abraham De Moivre, and was discussed by Jakob Hermann in 1717 and by other contemporary mathematicians [8], but it has been generally ignored by historians of science, and, although it appeared in some older texbooks in classical mechanics [9][10] [11], it has been left out of virtually all modern textbooks.

The main purpose of this paper is to discuss the role of curvature in orbital dynamics from a physics perspective, and to describe its historical background. In section I we give the standard derivation of the orbit equation in polar coordinates as it is presented in modern textbooks of classical mechanics. In addition, we obtain a generalization of this equation for the case of non-central forces. In section II we give a coordinate-independent derivation of the orbit equation in terms of the radius of curvature of the orbit, which exhibits its physical basis and geometrical significance. In section III we discuss Newton's own geometric derivation of this equation for the special case of central forces, as it first appeared in the second edition (1713) of the *Principia* in Prop. 6, Corollary 3 [12]. In section IV we illustrate the role of curvature in understanding the properties of central force orbits by discussing a perturbation solution for orbits of small eccentricity. In section V we describe a simple graphical method based on the curvature equation developed in sections II and III, which was originally developed in a similar form by Newton and applied by him to obtain approximate orbits for various central forces [4]. In the case of a constant central force, Newton sent a diagram of the orbit in a letter to Robert Hooke, and Hooke responded that this diagram corresponded to the orbit of a ball rolling in an inverted cone. We have repeated Hooke's experiment, and in Fig. 6 we show a stroboscopic picture of the orbit which confirms Hooke's observation. This section is followed by a brief summary. In appendix A we present a simple geometrical derivation of the radius of curvature of an orbit in polar coordinates. In Appendix B we discuss an important historical application by Clairaut and d'Alembert of the orbit equation to the lunar inequalities, which is considered to be a major landmark in establishing the universality of Newton's theory of gravitation. In appendix C we calculate by perturbation theory the motion of a point mass moving on the surface of an inverted cone.

I The standard derivation of the orbit equation

For central force motion it was first shown by Newton in his great masterpiece, the *Principia*, that the orbit is confined to a plane and that the angular momentum, which is normal to this plane, is a constant [1]. In modern textbooks of classical mechanics, this conservation law is applied in the derivation of the orbit of a particle of mass m moving under the action of a central force f by writing the equation of motion in polar coordinates r, θ in the plane of the orbit. In this case this equation takes the form

$$\frac{d^2r}{dt^2} - \frac{l^2}{m^2r^3} = \frac{f}{m}, \tag{1}$$

where

$$l = mr^2\frac{d\theta}{dt} \tag{2}$$

is the constant angular momentum. The time variable t in Eq. (1) can be replaced by the polar angle θ as an independent variable by the transformation

$$\frac{dr}{dt} = -\frac{l}{m}\frac{du}{d\theta}, \tag{3}$$

where $u = 1/r$. Taking the time derivative of Eq. (3) and using Eq. (2), we obtain

$$\frac{d^2r}{dt^2} = -\frac{l^2u^2}{m^2}\frac{d^2u}{d\theta^2}. \tag{4}$$

Substituting this relation in Eq. (1) leads to the standard textbook form for the orbit equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{mf}{l^2u^2}. \tag{5}$$

This equation can also be readily generalized for the case that there is a force component g normal to the radial direction. In this case the angular momentum l is not conserved, and its rate of change is

$$\frac{dl}{dt} = rg. \tag{6}$$

Instead of Eq. (4), we now have

$$\frac{d^2r}{dt^2} = -\frac{l^2u^2}{m^2}\frac{d^2u}{d\theta^2} - \frac{lu^2}{m^2}\frac{dl}{d\theta}\frac{du}{d\theta}. \tag{7}$$

Substituting this expression in the equation of motion, Eq. (1), we obtain [4], [13]

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}(f + \frac{g}{u}\frac{du}{d\theta}). \tag{8}$$

Applying the relation $d/dt = (l/mr^2)(d/d\theta)$, Eq. 2, the angular momentum equation, Eq. 6, can also be written in the form

$$\frac{dl^2}{d\theta} = \frac{2mg}{u^3}. \tag{9}$$

with θ as the independent variable.

This orbit equation has a physical and geometrical significance in terms of the radius of curvature of the orbit which is not manifest in this derivation. Consequently, this significance has remained unnoticed, although Newton had considered the role of curvature in the determination of central forces already in 1664, and he later discussed it in his *Principia* [3] [4] [5]. In the next section we apply the Huygens-Newton relation for the radial acceleration of circular orbits to exhibit the geometrical significance of the orbit equation [14].

II Geometrical derivation of the orbit equation

To obtain a geometrical and coordinate-independent form of the orbit equation, Eq. (8), we consider the component a_n of the acceleration which is normal to the tangent of the orbit at a given point of the orbit (see Fig. 1). This component of the acceleration is related to the velocity v and the radius of curvature ρ of the orbit at this point by the well known Huygens-Newton relation for circular motion [14]

$$a_n = \frac{v^2}{\rho}. \tag{10}$$

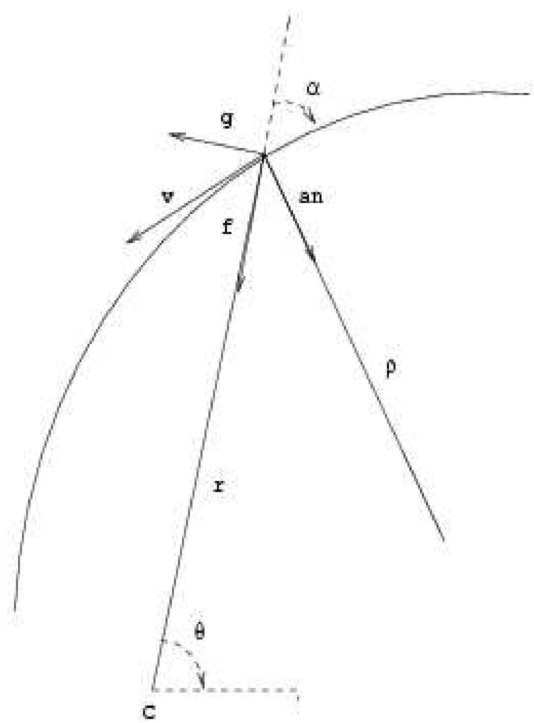

Figure 1: Segment of an orbit with the center of force at C, and radius of curvature ρ, showing the direction of the velocity v, the normal acceleration a_n, the radial component of the force, f (for attractive forces), and the component of the force normal to the radial direction, g.

In terms of the radial and angular components f and g of the force, the normal component of the acceleration is determined by the relation

$$a_n = -\frac{1}{m}[f sin(\alpha) + g cos(\alpha)], \tag{11}$$

where α is the angle between the radial and tangential directions of motion (see Fig. 1). Writing the angular momentum in the form

$$l = mvr sin(\alpha), \tag{12}$$

and applying this expression for l to eliminate the velocity v on the right hand side of Eq. (10), leads to a coordinate independent relation between the radius of curvature ρ of the orbit and the components f and g of the force [3],[4], [5],[7]

$$\frac{1}{\rho sin^3(\alpha)} = -\frac{mr^2}{l^2}(f + g cot(\alpha)). \tag{13}$$

This relation corresponds to the orbit equation, Eq. (8), that we derived in the previous section. Indeed, it can be readily shown (see Appendix A) that the radius of curvature ρ is given in polar coordinates by [15]

$$\frac{1}{\rho sin^3(\alpha)} = (\frac{d^2u}{d\theta^2} + u), \tag{14}$$

where

$$cot(\alpha) = \frac{1}{u}\frac{du}{d\theta}. \tag{15}$$

and $u = 1/r$ [16].

This coordinate-independent derivation exhibits the underlying physical and geometrical significance of the orbit equation, Eq. (8), in terms of the radius of curvature ρ of the orbit, which is not manifest in the conventional derivation given in section I. Remarkably, this relation was applied by Newton to calculate a hypothetical lunar orbit in the first (1687) edition of the *Principia* (Book III, Prop. 28), and appeared later in the second edition (1713) (Book 1, Prop. 6 Cor. 3), as an *alternate* expression for the case of central forces, $g = 0$, [3]. But in modern classical mechanics textbooks, the role of curvature in orbital dynamics has been neglected.

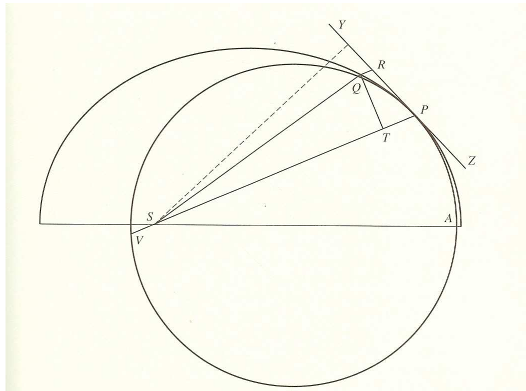

Figure 2: Newton's diagram for the proof of Proposition 6, including the circle of curvature at P.

III. Newton's derivation of the curvature equation

In this section I discuss Newton's original derivation of the curvature equation for the case of central forces. Readers who are not interested in the historical background of Eq. (13) can skip this section without loss of understanding of the underlying physics. From a cryptic remark in his 1664 notebook, called the "Waste book", it is evident that at that time Newton had already considered the relation Eq. (10) for a central force f, by generalizing the expression for the acceleration that he and, independently, Christiaan Huygens had derived for circular motion [14]. In Prop. 6, Book 1, of the *Principia*, Newton derived several geometrical expressions for the central force acting on a body which travels in an orbit satisfying angular momentum conservation (Kepler's area law in Prop. 1,2) [17],[18]. Referring to Fig. 2, which describes the motion of "a body P revolving about a center S along the curve APQ", Newton concluded at the end of Corollary 3 that

> ...the centripetal force will be inversely as the solid $SY^2 \times PV$ [12].

The line SY is perpendicular to the tangent at P, but the point V in Fig. 2 was not defined in Prop. 6. From the text [19] of Corollary 3, however, one infers that PV is the chord of the circle of curvature at P which passes through the center S, and the intersection of this chord with the circle of curvature gives the location of V. Therefore $PV = 2\rho sin(\alpha)$, where ρ is the radius of this circle or the curvature radius of the orbit at P, and α is the angle between the radial line SP and the tangential line PY at P. Since $SY = rsin(\alpha)$, Newton's result can be written in the form

$$force \propto \frac{1}{SY^2 \times PV} = \frac{1}{2\rho r^2 sin^3(\alpha)}, \tag{16}$$

which corresponds to the curvature form of the orbit equation, Eq. (13). The factor l^2, however, is missing in this derivation because Newton defined the small time interval δt for the body to move from P to a nearby point Q on the orbit, see Fig. 2, to be proportional to the area of the triangle PQS,

$$\delta t \propto SP \times QT, \tag{17}$$

where QT is perpendicular to SP. This relation is Kepler's area law, that the area swept by the radial vector is proportional to the time, but Newton left out the constant of proportionality, $1/l$, which depends on the angular momentum l of the orbit.

Newton's expression for the central force also follows from a geometrical relation for curvature, discussed more fully in Lemma 11, which is stated succinctly at the end of corollary 3:

> For PV is equal to QP^2/QR [12].

Here QR is the displacement of the orbit at Q from the tangential line at P in a direction parallel to the radial line SP. Hence, during the small interval of time δt for the body to move from P to Q, the acceleration due to the central force is given by $2QR/(\delta t)^2$, and setting the velocity $v = QP/\delta t$, we have

$$\frac{force}{m} = 2\frac{QR}{(\delta t)^2} = 2(\frac{QP}{\delta t})^2(\frac{1}{PV}) = \frac{v^2}{\rho sin(\alpha)}, \tag{18}$$

which corresponds to Eq. (10). This derivation closely follows Newton's derivation for the central force in the case of circular motion, but before 1679 he did not understand the physical origin (central forces) of angular momentum conservation (Kepler's area law). This conservation law, which Newton understood only after his correspondence that year with Robert Hooke [20], [21], is essential to relate the velocity v to the conserved angular momentum by the relation Eq. (12).

IV. Application of the curvature equation to perturbation theory

To illustrate the usefulness of curvature in understanding orbital motion, we consider the general solution of the orbit equation, Eq. (5), in the neighborhood of a circular orbit of radius r_0 centered at the origin of an attractive force. To first order in the deviation from this orbit, which is measured by the eccentricity parameter ϵ, we assume that [22]

$$\frac{1}{r} = \frac{1}{r_0}(1 + \epsilon cos(\nu\theta)), \tag{19}$$

where ν is a constant. For $\epsilon < 1$ this relation corresponds to the equation of an elliptical orbit whose major axis is rotating continuously and turns by an amount $2\pi(1/\nu - 1)$ during a period of the orbit. Substituting this expression for r in Eq. (5), one finds that for a given value of the angular momentum l, the mean radius r_0 is determined by the balance between centrifugal and central forces for a circular orbit of radius r_0

$$f(r_0) = m\frac{v_0^2}{r_0} = \frac{l^2}{mr_0^3}, \tag{20}$$

where $l = mv_0r_0$, while to first order in ϵ

$$\nu = \sqrt{3 + \frac{r_0 f'(r_0)}{f(r_0)}}, \tag{21}$$

where $f'(r) = df(r)/dr$. According to Eq. (14), the radius of curvature ρ, to first order in ϵ, is given by

$$\rho = r_0[1 - \epsilon(1 - \nu^2)cos(\nu\theta)]. \tag{22}$$

As an example, for an attractive force of the power law form $f(r) = \kappa/r^n$, where κ and the exponent n are constants, we obtain from Eq. (21)

$$\nu = \sqrt{3 - n}. \tag{23}$$

This result implies that for motion confined near a circular orbit we must have $n < 3$. For $n = 2$, corresponding to an inverse square central force, $\nu = 1$, and Eq. (19) is the exact solution, valid to all orders in ϵ, representing a conic section with one of the foci at the origin of the central force. To first order in ϵ, the radius of curvature, Eq. (22), is a constant, and therefore the *shape* of the orbit remains circular in this approximation. This property explains the spectacular success of Ptolemaic astronomy, which for centuries accounted for the best observations of the motion of planets that, fortunately, have rather small eccentricities, by assuming that their orbits where composed of circular motions [23]. For $n = -1$, corresponding to a force which depends linearly on r (Hooke's law), $\nu = 2$ and the orbit is also an ellipse, but now the origin of force is at the center of this ellipse, and the curvature depends linearly on ϵ.

The approximation of orbits with small eccentricity by a precessing ellipse, Eq. (19), was first described graphically by Newton in a letter to Robert Hooke written on Dec. 13, 1679. In this letter Newton included a diagram for an orbit $AFOGHIKL$ shown in Fig. 3, for the case that the central radial force is a constant with center at C. Although this orbit is qualitatively correct, the angle AOC, where O is the point in the orbit closest to C, should be $\pi/\sqrt{3} \approx 104^0$ for small eccentricities, but in the diagram $AOC \approx 130^0$. Evidently Newton made an error which is discussed in detail in references [4] [21]. The correct form of the orbit is shown in Fig. 4, which is well approximated by the solution of a rotating ellipse given in Eq. (19).

By the time, however, that Newton began writing the *Principia*, he was able to demonstrate (see Book I, Proposition 44) that a precessing ellipse, Eq. (19) is the *exact* solution for a central force consisting of a linear combination of an inverse square force and inverse cube force in the form

$$f(r) \propto \frac{\nu}{r^2} + \frac{(1-\nu^2)r_0}{r^3}, \tag{24}$$

where r_0 is the radius of the circular orbit, as can be verified directly by substituting Eq. (19) in Eq. (5). In Proposition 45, example 2, Newton then obtained Eq. (23) for ν for an $1/r^n$ force by approximating this force near $r = r_0$ by Eq. (24), and in Corollary 2, he calculated the rate of precession of the axis for the approximate elliptical orbit of the moon revolving around the earth, due to the gravitational attraction of the Sun [24].

The solution Eq. (19) of the orbit equation, which is valid only to first order in the eccentricity parameter ϵ for a *general* attractive central force in the neigborhood of $r = r_0$, indicates that the corresponding orbit oscillates

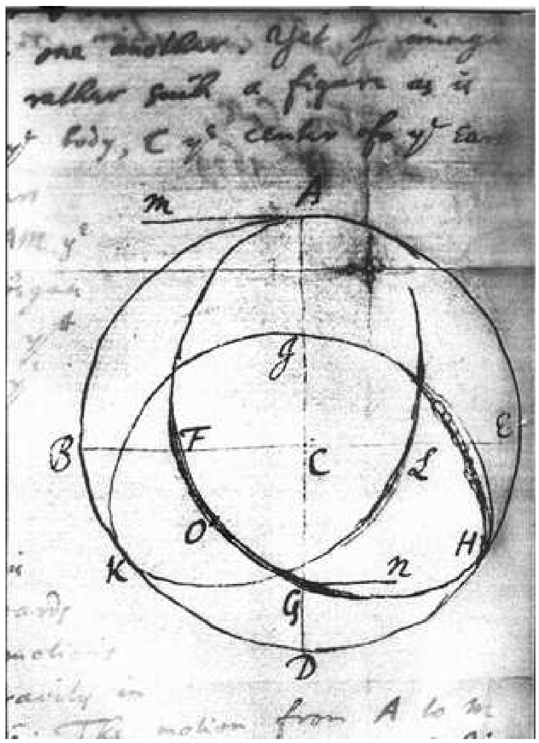

Figure 3: Newton's diagram in his Dec. 13, 1679 letter to Hooke

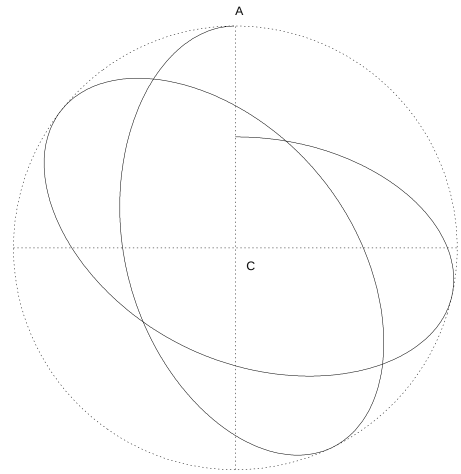

Figure 4: Precessing elliptical orbit for a constant central force.

around the circular orbit at a radius $r = r_0$ as a function of the polar angle θ. Such an orbit, however, is incorrectly illustrated in one of the best known modern textbooks of classical mechanics - see Fig. 5, [25].

In Fig. 5 the radius of curvature ρ of the orbit diverges near $r = r_0$ where it also changes sign. But to first order in ϵ the radius of curvature ρ, Eq. (22), does not diverge. Moreoever, the geometrical form of the orbit equation, Eq. (13), shows that if ρ diverges and changes sign near $r = r_0$, then correspondingly the central force vanishes and changes sign near $r = r_0$. But if such a domain existed, it would lie outside the range of validity of first order perturbation theory, which is the basis for the approximate orbit solution, Eq. (19), and the expression for the radius of curvature, Eq. (22). An understanding of the relation between central forces and orbital curvature, Eq. (14), would have avoided these long standing errors, which have remained during the past half century in all editions of this popular textbook on classical mechanics [25]. After the completion of this paper it was called to my attention by the editor of AJP that recently these errors have also been pointed out by Martin Tiersen [29].

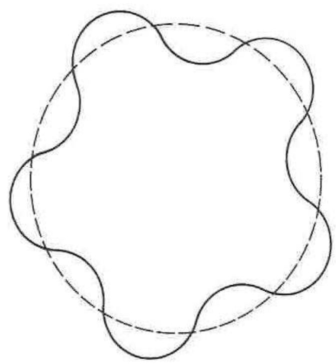

FIGURE 3–13
Orbit for motion in a central force deviating slightly from a circular orbit.

Figure 5: Reproduced from H. Goldstein, "Classical Mechanics"

V. Newton's graphical curvature method

Another application of the curvature equation, Eq. (13), is a simple graphical method for obtaining an approximate orbit which recently has been attributed to Newton [4]. Referring to Fig. 6, suppose that the particle is initially located at A, and that it is moving with a velocity v along the direction Aa perpendicular to the radial direction AO measured from the center of force at O. Then the radius of curvature ρ from Eq. (13) is given by

$$\rho = \frac{l^2}{mr^2 f sin^3(\alpha)}, \tag{25}$$

where l is the angular momentum, $l = mvr$, $r = AO$, and $\alpha = \pi/2$.

The curvature graphical method proceeds as follows: Take a line segment $AQ = \rho$ along the initial radial direction AO, and rotate it counterclockwise about Q by a small angle $\delta\phi$ to obtain a small arc AB of the curve. Next, draw the line OB which determines the radial distance $r_1 = OB$, and then obtain the angle $\alpha_1 = bBO$ between the tangential line bB and the radial line OB. Substituting $r = r_1$ and $\alpha = \alpha_1$ in Eq. (25) gives the radius of curvature ρ_1 of the orbit at B, and drawing the line segment $BP = \rho_1$ along BQ determines the location P of the corresponding center of curvature. Likewise, the next arc BC is obtained by rotating the line BP about P

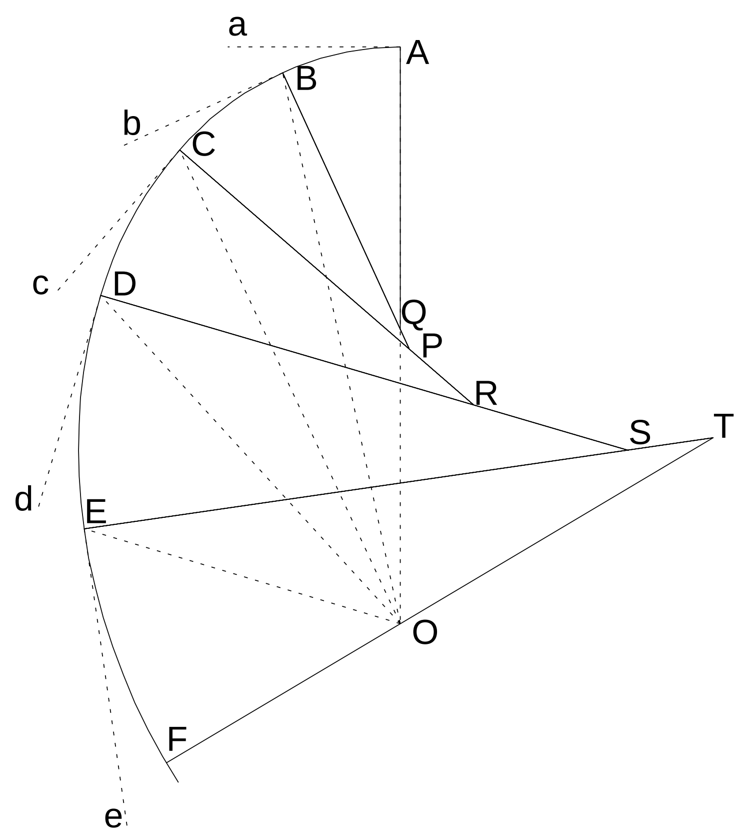

Figure 6: Graphical construction of an orbit by the curvature method

through the angle $\delta\phi$. This graphical operation is repeated as shown in Fig. 6 to give an approximate orbit $ABCDEF$ made up of small arc segments of varying radius of curvature. When the radius of curvature vector intersects the origin O, as is the case here with FT, the angle α becomes again equal to $\pi/2$, and the orbit will have reached its minimum distance to the origin O. Subsequent branches can be obtained by mirror reflections of this orbit about an axis along the direction of this radius of curvature.

In a similar way, Newton was able to approximately calculate orbits for general central forces [26]. This is shown in a diagram, Fig. 3, which is contained in a letter he wrote to Robert Hooke [4], [27] on Dec. 13, 1679, for the special case of a constant radial central force. Hooke responded to Newton in a letter dated Jan 6, 1680, that

> Your Calculation of the Curve by a body attracted by an equall power at all Distances from the center Such as that of a ball Roulling in an inverted Concave Cone is right and the two auges [apsides] will not unite for about a third of a Revolution...[28]

We have verified Hooke's remarkable observation by taking a stroboscopic picture of a ball revolving inside a cone shown in Fig. 7. A theoretical calculation for this motion is discussed in Appendix C.

Summary

The main focus of this paper has been to discuss the role of curvature in orbit dynamics and the underlying physical basis of the orbit equation. For those readers who are interested in the history of physics we have include section III and also several endnotes on the origin of these concepts in the work of Newton and Huygens, but these can be skipped without any loss in understanding the physical ideas. We have shown that the orbit equation, expressed in polar coordinates, Eqs. (5) and (8), has a physical and geometrical significance which is exhibited in terms of the radial acceleration for motion along a circle as a function of the radius of curvature of the orbit. A representation of the orbit equation is given in a coordinate-independent form, as was first formulated by Newton in the second edition (1713) of the *Principia*, and was re-discovered by several mathematicians who were applying the differential calculus to Newton's mechanics later in the eighteenth

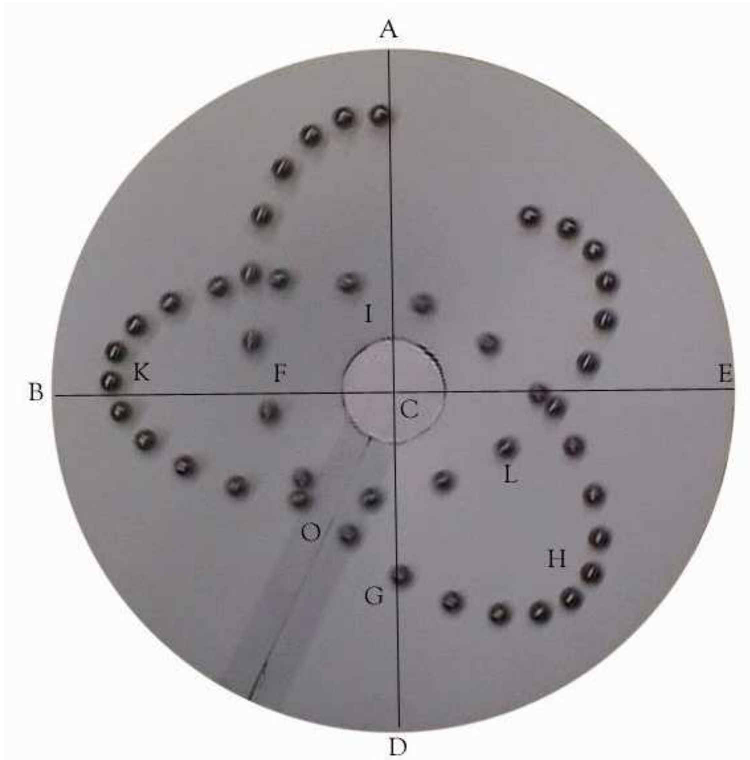

Figure 7: Stroboscopic picture of a ball rolling inside an inverted cone with an aperture $\alpha = 60^0$ degrees

century [8]. Although these results were included in older texbooks on classical mechanics, currently they are ignored in the presentation of orbital motion in virtually all modern texbooks. We have presented here, in modern notation, a derivation of the orbit equation which is based on curvature and closely follows Newton's original derivation, and we have used it to determine the motion of near-circular orbits. We also have shown that the curvature equation suggests a simple and effective graphical method for obtaining the solution of orbits for the case of central forces which originally was developed in similar form by Newton.

Appendix A, Geometric derivation of the radius of curvature

Referring to Fig. 8, consider a small arc of the curve of length $\delta\sigma$, and draw the two normals to the curve from the end points of the arc to their intersection at a point O. The radius of curvature ρ is defined by the relation

$$\delta\sigma \approx \rho\delta\phi \tag{26}$$

where

$$\delta\sigma \approx \sqrt{\delta r^2 + r^2\delta\theta^2} \tag{27}$$

and $\delta\phi$ is the angle at O between the two normals. The angle α between the radial and tangential directions satisfies the relations[1]

$$cot(\alpha) = -\frac{dr}{rd\theta} \tag{28}$$

and

$$sin(\alpha) = \frac{1}{\sqrt{1 + (1/r^2)(dr/d\theta)^2}} \tag{29}$$

The negative sign in Eq. 28 appears because the angle θ is assumed to increase in the counter clockwise direction in Fig. 1. Hence in the limit that $\delta\sigma$ becomes vanishingly small, ρ can be written in the form

$$\rho = \frac{d\sigma}{d\phi} = \frac{r}{sin(\alpha)}\frac{d\theta}{d\phi} \tag{30}$$

From Fig. 1 we see that

$$\delta\phi \approx \delta\theta - \delta\alpha, \tag{31}$$

and from Eq. (28) we obtain

$$\frac{d\alpha}{d\theta} = -sin^2(\alpha)[\frac{1}{r^2}(\frac{dr}{d\theta})^2 - \frac{1}{r}\frac{d^2r}{d\theta^2}]. \tag{32}$$

Hence, substitutinng Eq. (29) for sin(α) we have

$$\frac{d\alpha}{d\theta} = (1 + \frac{2}{r^2}(\frac{dr}{d\theta})^2 - \frac{1}{r}\frac{d^2r}{d\theta^2})/(1 + \frac{1}{r^2}(\frac{dr}{d\theta})^2), \tag{33}$$

[1]the negative sign appears below because the angle θ is assumed to increase in the counter clockwise direction, and then $dr/d\theta$ is negative in Fig. 1

and applying the relation

$$\frac{d^2}{d\theta^2}\frac{1}{r} = \frac{2}{r^3}(\frac{dr}{d\theta})^2 - \frac{1}{r^2}d^2rd\theta^2 \tag{34}$$

we obtain

$$\frac{d\phi}{d\theta} = rsin^2(\alpha)(\frac{d^2}{d\theta^2}+1)\frac{1}{r}. \tag{35}$$

Finally, by substituting this expression into Eq. (30), we obtain

$$\frac{1}{\rho sin^3(\alpha)} = (\frac{d^2}{d\theta^2}+1)\frac{1}{r} \tag{36}$$

This is the expression for the radius of curvature ρ in polar coordinates, Eq. (14), which Newton already obtained in 1671 [15]. Our derivation of the orbit equation, Eq. (13), exhibits the underlying geometrical significance of this differential equation in terms of the curvature of the orbit.

Appendix B: Non-central force: The orbital dynamics of Clairaut and d'Alembert

As an illustration of the application of the curvature equation for the case of non-central forces, Eq. (8), I consider a very important historical solution of this equation: the first correct approximation of the perturbed orbit of the moon due to the effect of the gravitational attraction of the Sun. Such a solution was first obtained in 1751 by Alexis Clairaut, and a few years later by Jean Rond d'Alembert [13], for small values of the eccentricity parameter ϵ. In Propositions 25 and 26, Book 3 of the *Principia*, Newton had shown that the solar perturbation can be approximated by an effective force acting on the moon with a radial component

$$f_p = -\frac{GM_oM_mr}{2R^3}(1+cos(2\psi)) \tag{37}$$

and an angular component

$$g_p = -\frac{3GM_oM_mr}{2R^3}sin(2\psi), \tag{38}$$

where M_o is the mass of the sun, M_m is the mass of the moon, R is the earth-sun distance, and $\psi = (1-m)\theta$ is the angle between the moon and the

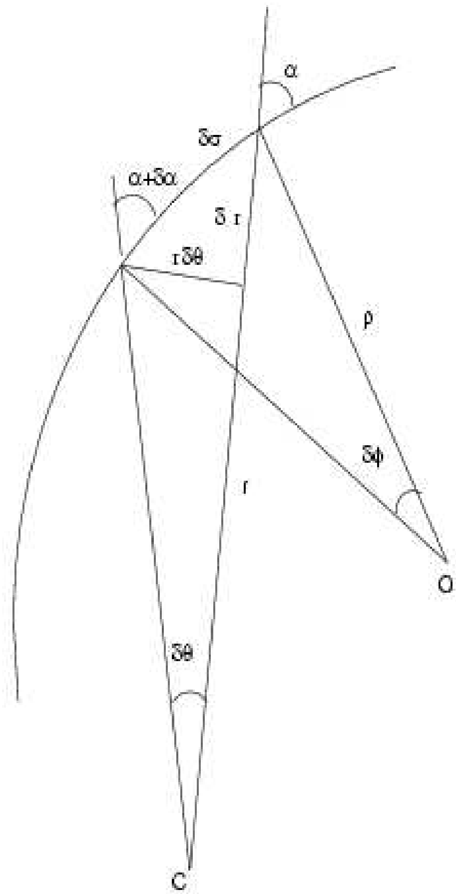

Figure 8: A segment of a curve of length $\delta\sigma$, showing two normals to this curve separated by a small distance $\delta\sigma$, intersecting at O making a small angle $\delta\phi$. The radial directions associated with these two normals intersect at the center C at a small angle $\delta\theta$, and the corresponding change in angle with respect to the tangential direction is $\delta\alpha$

sun, as seen from the earth (Newton's constant G was introduced only later on). Hence the orbit equation, Eq. (8), takes the form

$$(\frac{d^2}{d\theta^2}+1)\frac{1}{r}=\frac{GM)}{h^2}+\frac{r^2}{h^2}[V-\frac{W}{r}\frac{dr}{d\theta}] \tag{39}$$

and

$$h^2=h_0^2+2\int d\theta r^3 W, \tag{40}$$

where $V=f_p/M_m$, $W=g_p/M_m$ and $h=l/M_m$. Using the approximation of a revolving ellipse, Eq. (19), for r as a function of θ in Eqs. (37)- (40), gives

$$r^2V=\frac{m^2GM}{2}[1-3\epsilon cos(\nu\theta)+3cos(2\psi)-\frac{9\epsilon}{2}[cos(2\psi+\nu\theta)+cos(2\psi-\nu\theta)], \tag{41}$$

$$Wr\frac{dr}{d\theta}=\frac{3m^2GM\epsilon}{4}[cos(2\psi+\nu\theta)-cos(2\psi-\nu\theta)], \tag{42}$$

and

$$h^2=h_0^2[1+\frac{3m^2}{2}[cos(2\psi)-\frac{4}{3}\epsilon cos(2\psi+\nu\theta)-4\epsilon cos(2\psi-\nu\theta)]. \tag{43}$$

These expressions indicate that to order em^2, there also must appear terms proportional to $cos(2\psi+\nu\theta)$ and $cos(2\psi-\nu\theta)$ on the left hand side of the orbit equation, Eq. (39). To obtain such terms, Clairaut introduced an improved approximation for r in the form

$$\frac{1}{r}=\frac{1}{r_0}[1+ecos(\nu\theta)+xcos(2\psi)+\delta cos(2\psi+\nu\theta)+\gamma cos(2\psi-\nu\theta)] \tag{44}$$

where δ and γ are new coefficients determined by matching the corresponding terms proportional to $cos(2\psi+\nu\theta)$ and $cos(2\psi-\nu\theta)$ that appear in Eqs. (41)-(43). This matching implies that

$$\gamma=\frac{15m\epsilon}{8}, \tag{45}$$

and

$$\delta=-\frac{5m^2\epsilon}{8}. \tag{46}$$

An unexpected result is that the coefficient γ is only of order $m\epsilon$ instead of order $m^2\epsilon$ as in the case of δ. Therefore the contribution to r of the corresponding cosine term $cos(2\psi - \nu\theta)$ should not be neglected in the evaluation of the right hand side of Eq. (39), and one finds that the contribution of this term gives rise to additional terms proportional to $cos(\nu\theta)$ on the right hand side Eq. (39) which modifies the previous evaluation of ν. In particular the term r^2V gives the added term $-(9m^2GM\gamma/4)cos(\nu\theta)$, he term $Wrdr/d\theta$ contributes $-(3m^2GM\gamma/4)cos(\nu\theta)$, and the term h^2 contributes $-(6m^2\gamma/\nu)cos(\nu\theta)$. Collecting all of these additional terms proportional to $cos(\nu\theta)$ on the right hand side of Eq. (39) leads to the new relation

$$(\nu^2 - 1) = -\frac{3}{2}m^2(1 + \frac{5\gamma}{\epsilon}). \tag{47}$$

Finally, substituting Eq. (45) for γ one obtains Clairaut's and d'Alembert's result

$$\nu = 1 - \frac{3}{4}m^2(1 + \frac{75m}{8}) \tag{48}$$

Although the parameter $m = 1/13.36$ in this expression is small, in the correction term this is compensated by the large factor 75/8 and accounted for Newton's missing factor of two in the rate of rotation of the lunar axis. Today we would regard with suspicion such a large correction due to a higher order perturbation calculation, but at the time Euler proclaimed, in a letter to Clairaut on June 29, 1751, that

> ...the more I consider this happy discovery, the more important it seems to me...for it is very certain that it is only since this discovery that one can regard the law of attraction reciprocal proportional to the squares of the distances as solidly established and on this depends the entire theory of astronomy [13].

Appendix C, Orbital motion in an inverted cone

Approximating the motion of a ball revolving inside an inverted cone by the motion of a point mass, the equations of motion in cylindrical coordinates r, θ, z (see Fig. 9) are

$$\frac{d^2r}{dt^2} - \frac{l^2}{m^2r^3} = -\frac{Fcos(\alpha)}{m} \tag{49}$$

and

$$\frac{d^2z}{d^2t} = \frac{Fsin(\alpha)}{m} - g, \tag{50}$$

where m is the mass of the ball, $l = mr^2d\theta/dt$ is the conserved angular momentum about the axis of symmetry, α is the angle of aperture of the cone, g is the acceleration due to gravity, and F is the effective force acting on the ball in the direction normal to the surface of the cone. The constraint that the ball moves on the inside surface of the inverted cone is given by

$$z = rcotan(\alpha). \tag{51}$$

Applying this equation to eliminate the unknown force F from Eqs. (49) and (50), we obtain the effective radial equation of motion

$$\frac{d^2r}{dt^2} - \frac{l^2sin^2(\alpha)}{m^2r^3} = -\frac{gsin(2\alpha)}{2}. \tag{52}$$

Finally, substituting $d^2r/dt^2 = -l^2u^2m^2d^2u/d\theta$ we have

$$\frac{d^2u}{d\theta^2} + usin^2(\alpha) = \frac{mf}{l^2u^2}, \tag{53}$$

where $u = 1/r$, and $f = mgsin(2\alpha)/2$ is the effective radial force. An approximate solution is obtained by setting

$$u = u_0(1 - \epsilon cos(\nu\theta) \tag{54}$$

for an orbit deviating by a small amount from a circular orbit of radius $r_0 = 1/u_0$, where the eccentricity ϵ is a small parameter. Substituting this expression in Eq. (53) yields

$$\nu = \sqrt{3}sin(\alpha) \tag{55}$$

to first order in ϵ. This expression should be compared with the result obtained for a constant central force, $\nu = \sqrt{3}$. The stroboscopic picture shown in Fig. 7 was obtained with a cone with $\alpha = 60^0$ degrees, which accounts for the difference between the observed orbit and the calculated orbit shown in Fig. 4

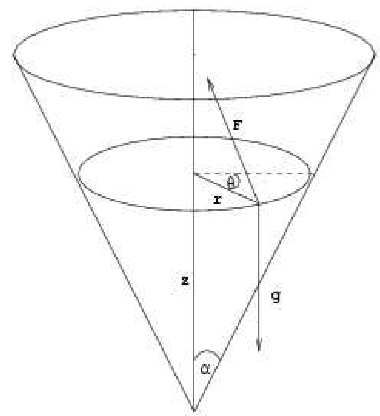

Figure 9: This figure shows the polar coordinates r, θ, z in an inverted cone, and the direction of the force of gravity g and the total force F acting on a body revolving inside this cone.

References

[1] M. Nauenberg, *Kepler's area law in the Principia: filling in some details in Newton's proof of Proposition 1* Historia Mathematica**30** (2003) 441-456.

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[4] M. Nauenberg, *Newton's early computational method for dynamics*, Archives for History of Exact Sciences **46** (1994) 221-252.

[5] J.B. Brackenridge and M. Nauenberg, *Curvature in Newton's dynamics*, in "The Cambridge Companion to Newton" edited by I.B. Cohen and G.E. Smith (Cambridge Univ. Press 2002) pp. 85-137.

[6] An English translation from the Latin by Mary Ann Rossi of Sections 1,2 and 3 of Book I from the first (1687) edition of Newton's *Principia* is given in J.B. Brackenridge, *The Key to Newton's Dynamics: The Kepler Problem and the Principia* (Univ. of California Press 1995) pp. 229-267.

[7] M. Nauenberg, *Newton's Perturbation Methods for the Three-body Problem and their Application to Lunar Motion*, in "Isaac Newton's Natural Philosophy", edited by J.Z. Buchwald and I.B. Cohen, (MIT Press, Cambridge Mass. 2001) p. 215.

[8] N. Guicciardini, *Reading the Principia: The Debate on Newton's Mathematical Methods for Natural Philosophy from 1687 to 1736* (Cambridge Univ. Press 1999) pp. 205-216.

[9] P. Frost, *Newton's Principia, First Book, Sections I,II,III, with notes and illustrations and a collection of problems principally intended as examples of Newton's method* (Cambridge, Macmillan and Co. 1854).

[10] E. T. Whitattker, *A Treatise on the Analytic Dynamics of Particles and Rigid Bodies* (Fourth Edition, 1937 reprinted by Dover Publications, New York) p. 75. Here the radial acceleration $a = f/m$ for a central force f is given in the form

$$a = \frac{h^2 r}{p^3 \rho} \tag{56}$$

where ρ is the radius of curvature, $p = rsin(\alpha)$ and $h = l/m$. This expression, which corresponds to Eq. (13), is attributed by Whittaker to Siacci, *Atti della R. Acc di Torino,* XIV. p.715.

[11] E.J. Routh, *A Treatise on Dynamics of a Particle* (Cambridge Univ. Press, 1898) p. 199. Reprinted by Dover Publications, 1960.

[12] Isaac Newton, *The Principia, Mathematical Principles of Natural Philosophy.* A new translation by I.B Cohen and Anne Whitman preceed by a Guide to Newton's Principia by I.B. Cohen (Univ. of California Press 1999) p. 455.

[13] A derivation along these lines of this general orbit equation was first given in 1752 by Alexis Clairaut, and two years later by Jean le Rond d'Alembert in connection with their first successful theory of lunar motion, F. Tisserand, *Traité de Méchanique Céleste* (Gauthier-Villars et fils, Paris,1894) vol.3, p.40 and 60. See also C. Wilson, *The problem of perturbation analytically treated: Euler, Clairaut, d'Alembert* in R. Taton and C. Wilson " The general History of Astronomy: Planetary astronomy from the Renaissance to the rise of astrophysics" 2B (Cambridge Univ. Press 1995) pp. 89-107.

[14] This relation for the acceleration was first considered by Newton as a generalization of the case of circular motion which had been obtained somewhat earlier by Christian Huygens, see Ref. [2]. In a cryptic remark contained in a 1664 manuscript on circular motion, *Waste Book*, Newton remarked that

> "If the body b moves in an Ellipsis, then its force in each point (if its motion [velocity] in that point be given)may be found by a tangent circle of equal crookedness with that point of the Ellipsis".

The word *crookedness* was Newton's early term for *curvature*, and by *tangent circle* he meant the *osculating circle* first introduced by Huygens [2], and re-discovered and named by Leibniz some 30 years later.

[15] This expression for the radius of curvature ρ in polar coordinates was first obtained by Newton in 1671, see *The Mathematical Papers of Isaac Newton* vol. III, 1670-1673, edited by D.T. Whiteside, (Cambridge Univ. Press, 19..) pp. 169-173.

[16] The general lack of understanding of the geometrical significance of the differential form, Eq. (14), is revealed by the reference in a recent text-book on classical mechanics to the relation $u = 1/r$ as a "Magic transformation", see L.Hand and J. D. Finch, *Analytic Mechanics*, (Cambridge Univ. Press, 1998) p. 142.

[17] B. Pourciau, *Newton's Argument for Proposition 1 of the Principia*, Archive for History of Exact Sciences **57** (2003) 267-311.

[18] M. Nauenberg, *Kepler's area law in the Principia: Filling in some details in Newton's proof of Proposition 1*, Historia Mathematica **30** (2003) 441-456

[19] In Newton's own words, Corollary 3 of Prop. 6,

> If the orbit APQ either is a circle or touches a circle concentrically or cuts it concentrically - that is, if it makes with the circle an angle of contact or of section which is the least possible - and has the same curvature and the same radius of curvature at point P, and if the circle has a chord drawn from the body through the center of forces, then the centripetal force will be inversely as the solid $SY^2 \times PV$. For PV is equal to QP^2/QR [12] p.455.

[20] M. Nauenberg, *Hooke, orbital motion and Newton's Principia*, Am. J. Phys. **62**, 331-350 (1994).

[21] M. Nauenberg, *Robert Hooke's seminal contribution to orbital Dynamics*, Physics in Perspective (to be published 2004).

[22] This general form for an orbit was first given in a geometrical form by Newton in his *Principia*, Book 1, Proposition 43. In this proposition, Newton proved that for this orbit the central force had the form $f(r) \propto \nu^2/r^2 + (1-\nu^2)r_0/r^3$ [12]p. 538.

[23] Some of the difficulties that Kepler experienced in establishing from Tycho Brahe's accurate observations of the orbit of Mars, that this orbit deviated from a circle and instead fitted an ellipse, was due to the fact that these deviations were *second order* effects in the eccentricity of Mars.

[24] Averaging over the period of the moon, this perturbation gives an additional repulsive force $f_p = cr$ between the earth and the moon that depends linearly on the earth-moon distance r, where $c = GM_0M_L/2R^3$, G is Newton's constant,M_0 is the mass of the sun, M_L is the mass of the moon, and R is the earth-sun distance [7]. Substituting for f in Eq. (21), the net attractive force on the moon $f = GM/r^2 - f_p$, one obtains $\nu \approx 1-(3/4)m^2$, where $m = \sqrt{M_0r^3/MR^3}$ is the ratio of the period of the moon rotating around the earth divided by the period of the earth rotating around the sun. "Thus", Newton found, "...in each revolution the upper apsis will move forward through $1^o31'45$", and reluctantly he admitted that the advance of " the apsis of the moon is twice as swift". This discrepancy remained as one of the major unsolved problems in Newton's theory of gravitation, until it was finally resolved in 1752 by a more detail calculation of Alexis Clairaut (see Appendix B). Newton realized early on that averaging the solar perturbation was inadequate, and he made considerable progress in a more detailed calculation along the same direction as Clairaut. But his work remained unpublished until it was found among his papers in the Portsmouth collection [4].

[25] H. Goldstein, *Classical Mechanics* (Addison Wesley 1980) p.92, Fig. 3-13. On the cover of this edition there is also a 'schematic illustration of the nature of the orbits for bounded motion", which appears in the text as Fig. 3-7, in which there is a change of sign in the curvature of the orbit. But the text associated with this figure implies that the central force is purely attractive, which is incorrect. Although it is not mentioned in this textbook, I have found that this figure corresponds to the "perturbative" solution given by Eq. (19), for $\epsilon \approx .49$, and $\nu \approx 3.1$. Clearly this is not

a perturbation solution, but it corresponds to a solution for the central force given by Eq. (24), which changes sign at $r = r_0(\nu^2-1)/\nu^2 \approx .896 r_0$. These errors are also repeated in the latest edition of this book (Addison Wesley 2002), co-authored by H. Goldstein, J.L. Safko and C.P. Poole, as well as in the first edition of this book (Addison Wesley 1950) which also shows the sketch of an untenable orbit for attractive central forces in Fig. 3-37.

[26] At the time, Newton had not yet discovered the conservation law of angular momentum, and to obtain the radius of curvature ρ he had to apply Eq. (10). Then he could have calculated the change of velocity at each step only approximately from the tangential component of the force $f cos(\alpha)$ which would have led to an error in his graphical calculation. Newton completed his diagram by drawing secondary braches from mirror reflection of the initial branch, but he made a further error by not properly aligning the center [4] [21].

[27] According to R.S. Westfall and H.E. Erlichson, Newton's diagram in his Dec. 13, 1679 letter to Hooke was obtained by an application of the geometrical construction in Newton's proof of Kepler's area law, as shown in Prop. 1 , Book 1 of the *Principia*; see Westfall, *Hooke and the law of universal gravitation*, Br. J. Hist. Sci. **3**, 245-261 (1967), and Erlichson, *Newton's 1679/1680 solution of the constant gravity problem*, Am. J. Phys. **59**, 728-733 (1991). But their interpretation cannot be correct, because by Newton's own account, see Ref. [4], he discovered the origin of Kepler's area law only *after* his 1679 correspondence with Hooke.

[28] H. W. Turnbull, *The Correspondence of Isaac Newton* vol II (Cambridge Univ. Press 1960) p. 69.

[29] M. Tiersen, *Errors in Goldstein's Classical Mechanics*, Am. J. Phys **71** (2003) 103.