Title
SINGLE PION PRODUCTION IN THE K+ p CHANNEL FROM 860 TO 1360 MeV/c

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Publication Date
1968-03-11
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Roger Woodward Bland
(Ph. D. Thesis)
March 11, 1968

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SINGLE PION PRODUCTION IN THE K⁺p CHANNEL
FROM 860 TO 1360 MeV/c

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ACKNOWLEDGMENTS

Many people at the Lawrence Radiation Laboratory have made essential contributions to this experiment: Dr. W. Hartsough and the Bevatron staff, Robert Watt and the bubble chamber crews, the computer center staff, and others. The aid and cooperation of Howard S. White, Jr. and the Data Handling Group is gratefully acknowledged.

I wish to thank the many scanners, measurers, programmers, and data handlers of the Trilling-Goldhaber Group, past and present, who have contributed to this experiment.

Primary credit for the completion of this experiment must be given to the physicists involved: Gerson and Sulamith Goldhaber, George Trilling, Michael Bowler, John Brown, John Kadyk, Victor Seeger, and Charles Wohl (see frontispiece).

Most important of all, I wish to thank Professor George Trilling for his teaching, advice and example during the years that I have studied under him.

This work was supported by the U. S. Atomic Energy Commission.

Roger Bland
SINGLE PION PRODUCTION IN THE K⁺p CHANNEL
FROM 860 TO 1360 MeV/c

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ABSTRACT

We have studied single pion production by K⁺ mesons on protons from 860 to 1360 MeV/c incident beam momentum. We find strong N*(1236) production in this region, with its maximum cross section occurring slightly below the 1250 MeV/c peak in the total cross section reported by Cool et al.² A partial wave analysis of the KN* final state indicates that the N* production is dominated by the P₁/₂ and P₃/₂ states in the region of the Cool peak, with no evident phase variation from 860 to 1360 MeV/c. The N* production is qualitatively consistent with the predictions of the magnetic dipole (M₁) ρ-exchange model; near threshold, however, the experimental cross section is too high by a factor of 5 and the detailed partial wave structure deviates considerably from the M₁ predictions. The smooth variation with beam momentum of all properties of the K⁺p system suggests that the K⁺p channel has no strong resonant activity, and should be interpreted in terms of t-channel phenomena.
INTRODUCTION

In 1963 the Bevatron Scheduling Committee approved a high-statistics, low-energy K⁺p experiment, to be done in the 25-inch hydrogen bubble chamber. At the time there was little detailed data on the K⁺p channel above 810 MeV/c. The total cross section was known to rise rapidly from about 13 mb at 810 MeV/c, leveling off at higher momenta at about 18 mb.¹ The only known resonances produced in this region were the K⁺(891) and the N⁺(1235) and possibly the controversial K⁺ meson at 725 MeV. This experiment was intended to study the K⁺p interaction in detail in the region of the single pion threshold and to search for K⁺ production at momenta just below threshold for production of other resonances.

This experiment has been long in analysis. Since the film was taken the K meson has been largely discredited. However, recent very accurate total cross-section measurements (Cool et al.² at Brookhaven and Bugg et al.³ at Rutherford Laboratory) have confirmed the presence of structure just above the single pion production threshold, and suggest the possibility of resonant behavior in the s channel. The high inelasticity of such a resonance, if it exists, and the lack of polarization data in K⁺p elastic scattering make it mandatory to study the inelastic channels in detail. The experiment may be equally interesting in the absence of such a resonance, however. With the lack of strong resonant activity in the s channel and with relatively few resonating particle pairs in the inelastic final states, this channel is ideal for studying the t-channel structure of the KN⁺ and K⁺N production amplitudes.

The plan of this paper is as follows: Section I includes a description of the experimental setup, exposure of the film, and the subsequent.
scanning, measuring, and fitting of events. In Section II we discuss bias corrections and errors and finally calculate cross sections for the various reactions detected in the bubble chamber. Sections III and IV contain a relatively model-independent exposition of the main features of the single pion production data. In Section III the mass distributions are used to determine the cross sections for resonance production and nonresonant background, and in Section IV the angular distributions of the 3-body system are discussed, emphasizing the $N\pi$ and $K\pi$ diparticle pairings. In Section V the $KN^*(1236)$ and $K^*(891)N$ final states are discussed in terms of their exchange model interpretation. A contrasting analysis of the $KN^*$ final state, in terms of partial waves, is presented in Section VI. Section VII discusses the interference between the $KN^*$ and $K^*N$ amplitudes in terms of a previously published model for this process. A summary of the paper and our conclusions are presented in Section VIII.
I. DESCRIPTION OF THE EXPERIMENT

The data for this thesis are part of a 600,000-picture exposure of the Lawrence Radiation Laboratory 25-inch hydrogen bubble chamber at the Bevatron in the spring and summer of 1964. A two-stage variable momentum mass separated beam was built especially for this experiment. The layout of the beam is shown in Fig. 1. The target was in an extracted proton beam, external to the Bevatron field, enabling the beam to operate over a wide range of momenta and for positive or negative beam particles. There were two mass separation stages, each consisting of an electrostatic separator and a vertical mass slit. The central beam momentum and momentum bite (about ±1% in this experiment) were defined at the first horizontal focus by the horizontal aperture of slit 1. The beam was limited to operating momenta at or below about 1,600 MeV/c, due to saturation of the magnets, and the optical separation of $K^+$ mesons from $\pi^+$ mesons and protons was good at all operating momenta. However, there was a substantial beam contamination at higher momenta due to pions and protons passing through the jaws of the slits.

In Table I we give, for the four momenta discussed in this thesis, the numbers of pictures analyzed and numbers of events of each topology included in this analysis. The average fitted beam momentum at the point of the $K^+$ interaction or decay is given in this table; hereafter we will use for reference the nominal momenta of 860, 960, 1200, and 1360 MeV/c. The numbers in Table I refer to the entire analysis. However, due to scanning inconsistencies and other bookkeeping problems, a subsample was chosen at each momentum from which to determine cross sections. At 860 and 1360 MeV/c almost all events were used, and at 960 and 1200 MeV/c,
Fig. 1. Diagram of the beam.
Table I. Number of pictures taken and numbers of events analyzed at the four momenta included in this thesis.

<table>
<thead>
<tr>
<th>Beam momentum (BeV/c)</th>
<th>Number of pictures analyzed</th>
<th>3-prong</th>
<th>2-prong, no $\pi^0$</th>
<th>2-prong, with $\pi^0$</th>
<th>4-prong</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>864</td>
<td>46 000</td>
<td>2 300</td>
<td>13 000</td>
<td>350</td>
<td>20</td>
<td>15 670</td>
</tr>
<tr>
<td>969</td>
<td>{ 25 000</td>
<td>1 100</td>
<td>8 100</td>
<td>500</td>
<td>30</td>
<td>9 730</td>
</tr>
<tr>
<td></td>
<td>17 000&lt;sup&gt;a&lt;/sup&gt;</td>
<td>--</td>
<td>--</td>
<td>200</td>
<td>--</td>
<td>200</td>
</tr>
<tr>
<td>1207</td>
<td>44 000</td>
<td>1 100</td>
<td>12 600</td>
<td>1 100</td>
<td>50</td>
<td>14 850</td>
</tr>
<tr>
<td>1367</td>
<td>66 000</td>
<td>1 500</td>
<td>--</td>
<td>1 800</td>
<td>400</td>
<td>3 700</td>
</tr>
<tr>
<td>all</td>
<td>198 000</td>
<td>6 000</td>
<td>33 700</td>
<td>3 950</td>
<td>500</td>
<td>44 150</td>
</tr>
</tbody>
</table>

<sup>a</sup>Only $\pi^0$ events were measured in this section of film.
slightly over half; the numbers in Tables III, IV, and V refer to this selected cross-section sample. The detailed analysis in subsequent sections is based on events from the entire data sample, however.

The primary measuring device for this experiment was the Berkeley Flying Spot Digitizer (FSD), a rapid automatic film-plane digitizer measuring at a rate of about 100 events per hour. The FSD requires that a rough-digitized three-point "road" be made in advance for each track of an event to be measured. Scanning and rough digitizing proceeds at a rate of about 20 events per hour. A small fraction of the data was measured by two "Franckenstein" film-plane measuring projectors, at a rate of about four events per hour.

All events, with a few exceptions, were first measured by the FSD. After the first measurement certain configuration-unbiased failures were rejected and not considered further in the analysis, in order to reduce the remeasurement task. Gross non-beam events, accidental duplicate measurements and events outside the fiducial volume were also rejected at this stage. All other failures were again predigitized and measured on the FSD. Those still failing were measured on the Franckensteins, some several times, until only a small fraction of the events remained unresolved. The exceptions to this procedure were (1) a small fraction of the events, about 2%, with tracks too short for the FSD to measure, which were recorded by hand and later measured on the Franckensteins; and (2) events from about 70% of the film at 1360 MeV/c, which were measured only on the Franckensteins.

In Table II we give the numbers of passing and failing events for the first two FSD measurements, at 1200 MeV/c, for events with three
Table II. Numbers of passing and failing events at 1200 MeV/c, for first and second measurements on the FSD.

<table>
<thead>
<tr>
<th>Topology</th>
<th>2-prong, no $\nu^0$</th>
<th>3-prong</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number measured</td>
<td>11,960</td>
</tr>
<tr>
<td></td>
<td>Accepted events</td>
<td>7,421</td>
</tr>
<tr>
<td>First</td>
<td>Unbiased FSD rejects</td>
<td>1,148</td>
</tr>
<tr>
<td>FSD measurement</td>
<td>Fiducial rejects</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>Other failures (to be remeasured)</td>
<td>3,091</td>
</tr>
<tr>
<td></td>
<td>% remeasured</td>
<td>26%</td>
</tr>
<tr>
<td>Second</td>
<td>Number measured</td>
<td>3,091</td>
</tr>
<tr>
<td>FSD measurement</td>
<td>Accepted events</td>
<td>1,325</td>
</tr>
<tr>
<td></td>
<td>Fiducial rejects</td>
<td>136</td>
</tr>
<tr>
<td></td>
<td>Other failures (to be remeasured)</td>
<td>1,630</td>
</tr>
<tr>
<td></td>
<td>% remeasured</td>
<td>53%</td>
</tr>
</tbody>
</table>
prongs or two prongs and no $V^0$. The 27\% of events remeasured after the first measurement can be broken down roughly into 17\% bad measurements and 10\% of events apparently well-measured but failing to fit. Most of those in the latter category fit either after the second FSD measurement or after measurement on the Franckenstein, and are due to undetected measurement errors. We have not included events with $V^0$'s in Table II because we did not try to optimize their passing rate. The $V^0$ presented a special problem when measured on the FSD because the direction of the neutral track depends on the reconstruction of the decay vertex, and for short neutral flight paths large errors were often made. This problem was not so great for events measured on the Franckensteins, since difficult decay vertices can be located more reliably by a human measurer. For this reason and to minimize special programming for small numbers of events, four-prong events and events with $V^0$'s were measured primarily on the Franckensteins.

The measurements were analyzed with two sequences of programs. The FOC-CLOUDY-FAIR system was used for FSD measurements, and PACKAGE for the Franckenstein measurements. The only unconventional feature of the fitting was "total beam track editing." This procedure was adopted because the heavy flux of beam tracks, 15-25 per frame, caused a large failure rate for beam tracks measured by the FSD. This measurement of the beam track was therefore discarded, and replaced by an artificial track. The artificial track had for angles and momentum the average beam angles and momentum, and for measurement errors the half-widths of the beam distributions. The beam distributions were determined from a sample of four-constraint events measured on the Franckensteins and analyzed by PACKAGE. Total beam editing
was not adopted in PACKAGE, since for events measured by hand the correct beam track could usually be located satisfactorily. However, except for the sample to determine the beam profile, PACKAGE events were momentum-averaged; for each event the measured momentum and the central beam momentum were averaged with weights inversely proportional to their respective errors, and this "beam-averaged" momentum was used in the fitting.

There are two drawbacks to total beam editing. First, the loss in accuracy of the beam direction increases the number of fitting ambiguities. This did not handicap us severely, since four-constraint hypotheses were still identified uniquely by fitting, and one-constraint fits would have been checked on the scan table for ionization in any case. With inspection of one-constraint events on the scan table the ability to resolve ambiguities was adequate at the momenta considered here. The second problem was that events induced by non-beam tracks could not be eliminated by testing on the measured beam direction or momentum, since these were discarded. Such events usually failed to fit, however, and were detected when measured on the Frankensteins.

**Classification of Events After Fitting**

After measuring and fitting and inspection of ionization where necessary, all events were either accepted, rejected as one of a number of distinct reject types, or remeasured. For an event to be accepted it was first required that all secondaries be well-measured, as indicated by the spread of measured points from the fitted curve. A $X^2$ cutoff was made at $X^2 = 20$ for the elastics (confidence level = 0.06%), and at the
1% level for other hypotheses. Fits to four-constraint hypotheses were accepted regardless of whether the one-constraint hypotheses fit. A check of this procedure on about one-third of all events by inspection of ionization revealed no instance where the inelastic fit should have been chosen. After inspection of ionization there were no ambiguities among the four-constraint hypotheses, and the ambiguities among the one-constraint hypotheses were very few, always less than 2% of the inelastic events. The numbers of unambiguous accepted events are given in Tables IIIa-d, for our four momenta.

Badly measured events were always returned for remeasurement. In addition, well-measured FSD events giving no incident-K⁺ fits were sent to the Franckensteins for remeasurement, since incident-π⁺ fits were not tried in CLOUDY. After measurement on the Franckensteins, well-measured events were either accepted or rejected. The reject types, given in Tables IIIa-d, are defined as follows:

**Unbiased FSD Reject** - an event failing on the first FSD measurement for reasons independent of the final-state configuration, such as a frame number error or an unreadable data box.

**Non-Beam Reject** - an event induced by a track either grossly non-beam or rejected by the PACKAGE beam track criteria. In PACKAGE we required beam tracks to have a measured momentum within three standard errors of the central beam momentum and to have dip and azimuthal angles within ±1.5° of the average beam angles. The latter requirement occasionally lead to rejection of good events with poorly-measured beam tracks. This effect will be discussed further in the next section.
Table IIIa. 860 MeV/c, numbers of events found and their classification after fitting.

<table>
<thead>
<tr>
<th>Topology</th>
<th>3-prong</th>
<th>2-prong no VO</th>
<th>2-prong with VO</th>
<th>4-prong</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number found</td>
<td>2 313</td>
<td>13 009</td>
<td>345</td>
<td>23</td>
<td>15 690</td>
</tr>
<tr>
<td>Accepted, unambiguous</td>
<td>1 865</td>
<td>11 385</td>
<td>239</td>
<td>15</td>
<td>13 504</td>
</tr>
<tr>
<td>Accepted, ambiguous</td>
<td>--</td>
<td>0</td>
<td>--</td>
<td>--</td>
<td>0</td>
</tr>
<tr>
<td>All accepted</td>
<td>1 865</td>
<td>11 385</td>
<td>239</td>
<td>15</td>
<td>13 504</td>
</tr>
<tr>
<td>Unresolved</td>
<td>40</td>
<td>107</td>
<td>32</td>
<td>2</td>
<td>181</td>
</tr>
<tr>
<td>Unbiased FSD rejects</td>
<td>213</td>
<td>866</td>
<td>34</td>
<td>2</td>
<td>1 115</td>
</tr>
<tr>
<td>Non-beam rejects</td>
<td>83</td>
<td>411</td>
<td>29</td>
<td>2</td>
<td>525</td>
</tr>
<tr>
<td>% non-beam rejects</td>
<td>3.6%</td>
<td>3.16%</td>
<td>8.4%</td>
<td>8.7%</td>
<td>3.35%</td>
</tr>
<tr>
<td>False events</td>
<td>68</td>
<td>97</td>
<td>8</td>
<td>1</td>
<td>174</td>
</tr>
<tr>
<td>No fit events</td>
<td>18</td>
<td>60</td>
<td>1</td>
<td>0</td>
<td>79</td>
</tr>
<tr>
<td>Immeasurable</td>
<td>26</td>
<td>55</td>
<td>2</td>
<td>0</td>
<td>83</td>
</tr>
<tr>
<td>Zero-constraint</td>
<td>--</td>
<td>20</td>
<td>--</td>
<td>--</td>
<td>20</td>
</tr>
<tr>
<td>All rejects</td>
<td>408</td>
<td>1 509</td>
<td>74</td>
<td>5</td>
<td>1 996</td>
</tr>
<tr>
<td>Adjusted non-beam rejects</td>
<td>73</td>
<td>411</td>
<td>11</td>
<td>1</td>
<td>496</td>
</tr>
<tr>
<td>Adjusted %</td>
<td>3.16%</td>
<td>3.16%</td>
<td>3.16%</td>
<td>3.16%</td>
<td>3.16%</td>
</tr>
</tbody>
</table>
Table IIIb. 960 MeV/c, numbers of events found and their classification after fitting.

<table>
<thead>
<tr>
<th>Topology</th>
<th>3-prong</th>
<th>2-prong no V°</th>
<th>2-prong with V°</th>
<th>4-prong</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number found</td>
<td>765</td>
<td>5 165</td>
<td>315</td>
<td>16</td>
<td>6 261</td>
</tr>
<tr>
<td>Accepted, unambiguous</td>
<td>626</td>
<td>4 553</td>
<td>253</td>
<td>13</td>
<td>5 445</td>
</tr>
<tr>
<td>Accepted, ambiguous</td>
<td>-</td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>12</td>
</tr>
<tr>
<td>All accepted</td>
<td>626</td>
<td>4 565</td>
<td>253</td>
<td>13</td>
<td>5 457</td>
</tr>
<tr>
<td>Unresolved</td>
<td>35</td>
<td>76</td>
<td>33</td>
<td>2</td>
<td>146</td>
</tr>
<tr>
<td>Unbiased FSD rejects</td>
<td>33</td>
<td>154</td>
<td>4</td>
<td>0</td>
<td>191</td>
</tr>
<tr>
<td>Non-beam rejects</td>
<td>34</td>
<td>195</td>
<td>21</td>
<td>1</td>
<td>251</td>
</tr>
<tr>
<td>% non-beam rejects</td>
<td>4.4%</td>
<td>3.77%</td>
<td>6.7%</td>
<td>7%</td>
<td>4.01%</td>
</tr>
<tr>
<td>False events</td>
<td>32</td>
<td>145</td>
<td>4</td>
<td>0</td>
<td>181</td>
</tr>
<tr>
<td>No fit events</td>
<td>1</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>Immeasurable</td>
<td>4</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>Zero-constraint</td>
<td>-</td>
<td>15</td>
<td>0</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>All rejects</td>
<td>104</td>
<td>524</td>
<td>29</td>
<td>1</td>
<td>658</td>
</tr>
<tr>
<td>Adjusted non-beam rejects</td>
<td>29</td>
<td>195</td>
<td>12</td>
<td>1</td>
<td>237</td>
</tr>
<tr>
<td>Adjusted %</td>
<td>3.77%</td>
<td>3.77%</td>
<td>3.77%</td>
<td>3.77%</td>
<td>3.77%</td>
</tr>
</tbody>
</table>
Table IIIc. 1200 MeV/c, numbers of events found and their classification after fitting.

<table>
<thead>
<tr>
<th>Topology</th>
<th>3-prong</th>
<th>2-prong no V₀</th>
<th>2-prong with V₀</th>
<th>4-prong</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number found</td>
<td>624</td>
<td>6 724</td>
<td>614</td>
<td>65</td>
<td>8 027</td>
</tr>
<tr>
<td>Accepted, unambiguous</td>
<td>488</td>
<td>5 550</td>
<td>475</td>
<td>43</td>
<td>6 556</td>
</tr>
<tr>
<td>Accepted, ambiguous</td>
<td>--</td>
<td>26</td>
<td>--</td>
<td>--</td>
<td>26</td>
</tr>
<tr>
<td>All accepted</td>
<td>488</td>
<td>5 576</td>
<td>475</td>
<td>43</td>
<td>6 582</td>
</tr>
<tr>
<td>Unresolved</td>
<td>11</td>
<td>30</td>
<td>25</td>
<td>8</td>
<td>74</td>
</tr>
<tr>
<td>Unbiased FSD rejects</td>
<td>77</td>
<td>740</td>
<td>85</td>
<td>6</td>
<td>908</td>
</tr>
<tr>
<td>Non-beam rejects</td>
<td>17</td>
<td>258</td>
<td>17</td>
<td>6</td>
<td>298</td>
</tr>
<tr>
<td>% non-beam rejects</td>
<td>2.7%</td>
<td>3.84%</td>
<td>2.8%</td>
<td>9.3%</td>
<td>3.71%</td>
</tr>
<tr>
<td>False events</td>
<td>25</td>
<td>98</td>
<td>8</td>
<td>0</td>
<td>91</td>
</tr>
<tr>
<td>No fit events</td>
<td>5</td>
<td>20</td>
<td>4</td>
<td>1</td>
<td>29</td>
</tr>
<tr>
<td>Immeasurable</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Zero-constraint</td>
<td>--</td>
<td>36</td>
<td>0</td>
<td>0</td>
<td>36</td>
</tr>
<tr>
<td>All rejects</td>
<td>125</td>
<td>1 118</td>
<td>114</td>
<td>14</td>
<td>1 371</td>
</tr>
<tr>
<td>Adjusted non-beam rejects</td>
<td>24</td>
<td>259</td>
<td>24</td>
<td>2</td>
<td>309</td>
</tr>
<tr>
<td>Adjusted %</td>
<td>3.84%</td>
<td>3.84%</td>
<td>3.84%</td>
<td>3.84%</td>
<td>3.84%</td>
</tr>
</tbody>
</table>
Table IIIId. 1360 MeV/c, numbers of events found and their classification after fitting.

<table>
<thead>
<tr>
<th>Topology</th>
<th>3-prong</th>
<th>2-prong with ( V_0 )</th>
<th>4-prong</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number found</td>
<td>1 534</td>
<td>1 756</td>
<td>367</td>
<td>3 657</td>
</tr>
<tr>
<td>Accepted, unambiguous</td>
<td>1 251</td>
<td>1 447</td>
<td>251</td>
<td>2 949</td>
</tr>
<tr>
<td>Accepted, ambiguous</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>All accepted</td>
<td>1 251</td>
<td>1 447</td>
<td>251</td>
<td>2 949</td>
</tr>
<tr>
<td>Unresolved</td>
<td>85</td>
<td>152</td>
<td>23</td>
<td>260</td>
</tr>
<tr>
<td>Unbiased FSD rejects</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Non-beam rejects</td>
<td>87</td>
<td>99</td>
<td>71</td>
<td>257</td>
</tr>
<tr>
<td>% non-beam rejects</td>
<td>5.7%</td>
<td>5.5%</td>
<td>19%</td>
<td>7.0%</td>
</tr>
<tr>
<td>False events</td>
<td>88</td>
<td>43</td>
<td>9</td>
<td>142</td>
</tr>
<tr>
<td>No fit events</td>
<td>13</td>
<td>8</td>
<td>10</td>
<td>31</td>
</tr>
<tr>
<td>Immeasurable</td>
<td>10</td>
<td>6</td>
<td>3</td>
<td>19</td>
</tr>
<tr>
<td>Zero-constraint</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>All rejects</td>
<td>198</td>
<td>157</td>
<td>93</td>
<td>448</td>
</tr>
</tbody>
</table>
**False Event** - this category includes duplicate events, phony events due to accidental coincidence of background tracks in the chamber, and three-prong events coming from a decay of a beam track giving a Dalitz pair and one other charged secondary.

**No Fit** - this classification is reserved for well-measured events giving no acceptable fits, with an incident $K^+$ meson, $\pi^+$ meson, or proton. Most of these events are slightly low in momentum, and a small fraction probably have undetected measurement errors.

**Immeasurable** - obscured by passing tracks, chamber distortion, or film damage.

**Zero-Constraint** - well measured but with a secondary scatter or decay so near the primary vertex that the momentum of the secondary track cannot be measured. This category includes only such events which did not give three-constraint fits to hypotheses with no missing neutrals, and thus presumably have one or more missing neutrals.

For completeness we also include in Tables IIIa-d the number of events never successfully measured, or unresolved.
II. CROSS SECTIONS FOR STABLE FINAL STATES

In the previous section we discussed the detection of events and their classification as fitted events or failures and rejects of various types. In this section we will calculate cross sections for all the reactions observed in this experiment. First, however, we must discuss the biases in our data, correct for them, and estimate the errors in the resulting numbers of events. Because of the difficulty of directly estimating systematic errors from specific sources, we will assign a single systematic error to the number of events in each channel. The errors given for the various corrections made in Tables IV and V will thus be statistical only.

We will use two different methods of normalization in calculating our cross sections. At all four momenta we have measured $K^+$ beam particles decaying by the $\tau$ mode, $K^+ \to \pi^+ \pi^+ \pi^-$, in the same fiducial volume used for interactions. From the number of such decays we can infer the total $K^+$ meson path length, and use this to relate numbers of interactions to cross sections. In addition, at 960 and 1200 MeV/c we can normalize our data to recent very accurate counter measurements of the total $K^+p$ cross section, by Cool et al.\textsuperscript{2} at Brookhaven. Since our data include many more interactions than $\tau$ decays, and since the uncertainty in the $\tau$ decay branching ratio is not negligible, the second method will be more accurate, and will provide a check on the decay normalization procedure. The data of Cool et al. do not extend as low as 860 MeV/c, and at 1360 MeV/c we do not measure the cross sections for all channels, having measured only selected event topologies; at these two momenta we can normalize only to $\tau$ decays.
Before discussing corrections to the data we observe that except in the 3-prong topology they have very little effect. The only corrections with a substantial uncertainty are those dependent only on topology. However, most of our information comes from a single topology, that with two prongs and no $V^0$. The contribution of the 4-prongs is small, with the accuracy of the cross sections for the four-body final state always statistics-limited. The 2-prong with $V^0$ topology contributes only one-third to the $K^0\pi^+$ cross section and at most, at 1200 MeV/c, 10% to the total cross section, and so corrections to this topology introduce much smaller fractional changes in the cross sections. Thus the accuracy of the elastic scattering and single pion production cross sections depends mainly on our distinguishing reliably among the various final states in the 2-prong, no $V^0$ topology.

We can divide the corrections to be made into two categories, internal corrections and bias corrections. The first type of correction involves distributing among the various final states the unresolved events and certain categories of rejects. The biases to be corrected for include beam contamination, scanning loss, and $K^0_S$ escape loss.

**Internal Corrections**

The internal corrections to our data are given in Tables IVa-d. They involve distributing ambiguous and zero-constraint events within the 2-prong, no $V^0$ topology, making a topology-dependent non-beam reject correction, and distributing among all final states the unresolved and various categories of unbiased rejects. We will discuss these corrections in the order in which they occur in Table IV.
<table>
<thead>
<tr>
<th>Topology</th>
<th>Reaction</th>
<th>Accepted events</th>
<th>Corrections for rejected and unresolved events</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3-prong</td>
<td>$K^+ \rightarrow \pi^+\pi^+\pi^-$</td>
<td>1865±43</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^+p \rightarrow K^+p\pi^0$</td>
<td>10278±101</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^-\pi^+\pi^-$</td>
<td>305±17</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^0\pi^+\pi^-$</td>
<td>111±11</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^0$ unseen</td>
<td>657±26</td>
<td>0</td>
</tr>
<tr>
<td>2-prong,</td>
<td>$K^0 \rightarrow \pi^+\pi^-\pi^-$</td>
<td>239±15</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^0 \rightarrow \pi^+\pi^-\pi^-$</td>
<td>14±4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p \rightarrow p\pi^0$</td>
<td>17±4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p \rightarrow p\pi^0$</td>
<td>2±1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$pp \rightarrow pp$</td>
<td>6±3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table IVa. 860 MeV/c, numbers of accepted events for each final state and all internal corrections.
Table IVb. 960 MeV/c, numbers of accepted events for each final state and all internal corrections.

<table>
<thead>
<tr>
<th>Topology</th>
<th>Reaction</th>
<th>Accepted events</th>
<th>Corrections for rejected and unresolved events</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Unambiguous</td>
<td>Ambiguous</td>
</tr>
<tr>
<td>3-prong</td>
<td>$K^+ \rightarrow \pi^+\pi^+\pi^-$</td>
<td>626±25</td>
<td>0</td>
</tr>
<tr>
<td>2-prong, no $V^0$</td>
<td>$K^+p \rightarrow K^+p$</td>
<td>3672±61</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^+p \rightarrow K^+p$</td>
<td>247±16</td>
<td>2±1</td>
</tr>
<tr>
<td></td>
<td>$K^+p \rightarrow K^+p$</td>
<td>88±9</td>
<td>4±2</td>
</tr>
<tr>
<td></td>
<td>$K^0 \rightarrow K^0\pi^+\pi^-$</td>
<td>504±22</td>
<td>4±2</td>
</tr>
<tr>
<td>2-prong</td>
<td>$K^0 \rightarrow K^0\pi^+\pi^-$</td>
<td>253±16</td>
<td>0</td>
</tr>
<tr>
<td>4-prong</td>
<td>$K^+p \rightarrow \pi^+p(\pi^+\pi^-)$</td>
<td>13±4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\pi^+p \rightarrow \pi^+p$</td>
<td>14±3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\pi^+p \rightarrow \pi^+p$</td>
<td>16±4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$pp \rightarrow pp$</td>
<td>13±4</td>
<td>0</td>
</tr>
</tbody>
</table>
Table IVc. 1200 MeV/c, numbers of accepted events for each final state and all internal corrections.

<table>
<thead>
<tr>
<th>Topology</th>
<th>Reaction</th>
<th>Accepted events</th>
<th>Corrections for rejected and unresolved events</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Unambiguous</td>
<td>Ambiguous</td>
</tr>
<tr>
<td>3-prong</td>
<td>$K^+ \rightarrow \pi^+ \pi^+ \pi^-$</td>
<td>488±22</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^+ p \rightarrow K^+ p$</td>
<td>347±58</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K^0 p$</td>
<td>58±24</td>
<td>11±3</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K^0 \pi^- p$,</td>
<td>184±14</td>
<td>4±2</td>
</tr>
<tr>
<td></td>
<td>$K^0$ unseen</td>
<td>1106±33</td>
<td>5±2</td>
</tr>
<tr>
<td>2-prong, no $\phi^0$</td>
<td>$K^+ p \rightarrow K^0 \pi^+ p$, $K_L^0 \rightarrow \pi^+ \pi^-$</td>
<td>475±22</td>
<td>0</td>
</tr>
<tr>
<td>2-prong</td>
<td>$K^+ p \rightarrow \pi^+ p (\pi^+ \pi^-)$</td>
<td>23±5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K^0 \pi^- p$,</td>
<td>6±3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$K^0$ unseen</td>
<td>1111±33</td>
<td>21±5</td>
</tr>
<tr>
<td>4-prong</td>
<td>$\pi^+ p \rightarrow \pi^+ \pi^- p$</td>
<td>14±4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow \pi^+ p$</td>
<td>76±9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow \pi^+ p \pi^0$</td>
<td>88±9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow \pi^+ \pi^- n$</td>
<td>23±5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$pp \rightarrow pp$</td>
<td>5±2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$pp \rightarrow p \pi^+ n$</td>
<td>4±2</td>
<td>0</td>
</tr>
</tbody>
</table>

|          | Zero- constraint rejects             | Non-beam rejects  | Unbiased FSD rejects, unresolved, and immeasurable | Total, all internal corrections made |
|          |                                     |                  |                                                |                                          |
| 3-prong  | 0                                     | -7±2             | 87±9                                           | 568±24                                    |
|          |                                       |                  |                                                |                                          |
| 2-prong, no $\phi^0$ | 10±3                                 | 5±2              | $\times 1.139$                                | 395±70                                    |
|          |                                       |                  |                                                | $\pm 0.005$                                |
|          |                                       |                  |                                                | 220±16                                    |
| 2-prong  | 21±5                                  | -                 | $\times 1.32$                                 | 1290±38                                   |
|          |                                       |                  |                                                |                                          |
| 4-prong  | 0                                     | 2±1              | $\times 1.139$                                | 33±7                                      |
|          |                                       |                  |                                                |                                          |
| 2-prong, no $\phi^0$ | 1±1                                  | 0                 |                                                | 9±3                                       |
|          |                                       |                  |                                                |                                          |


Table IVd. 1360 MeV/c, numbers of accepted events for each final state and all internal corrections.

<table>
<thead>
<tr>
<th>Topology</th>
<th>Reaction</th>
<th>Accepted events</th>
<th>Corrections for rejected and unresolved events</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Unambiguous</td>
<td>Ambiguous</td>
</tr>
<tr>
<td>3-prong</td>
<td>( K^+ \rightarrow \pi^+ \pi^- )</td>
<td>1251±35</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( K^+ p \rightarrow K^0 n^+ p ), ( K^0 \rightarrow \pi^+ \pi^- )</td>
<td>1430±38</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( K^+ p \rightarrow K^0 n^+ p^0 ), ( K^+ p \rightarrow K^0 n^+ n^0 ), ( K^0 \rightarrow \pi^+ \pi^- )</td>
<td>14±4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( K_1^0 \rightarrow \pi^+ \pi^- ), ( K_1 \rightarrow \pi^+ \pi^- )</td>
<td>3±2</td>
<td>0</td>
</tr>
<tr>
<td>2-prong,</td>
<td>( K^+ p \rightarrow K^0 n^+ p(\pi^+ \pi^-)^0 ), ( K^+ p \rightarrow K^0 n^+ n(\pi^+ \pi^-)^0 ), ( K_1 \rightarrow \pi^+ \pi^- )</td>
<td>39±6</td>
<td>0</td>
</tr>
<tr>
<td>with ( V^0 )</td>
<td>( K_1^0 \rightarrow \pi^+ p(\pi^- p)^0 ), ( K_1 \rightarrow \pi^+ p(\pi^- p)^0 )</td>
<td>1±1</td>
<td>0</td>
</tr>
<tr>
<td>4-prong</td>
<td>( K^+ p \rightarrow K^+ n^- p )</td>
<td>58±8</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( \pi^+ p \rightarrow \pi^+ n^- p )</td>
<td>140±12</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( \pi^+ p \rightarrow \pi^+ n^- p^0 )</td>
<td>13±4</td>
<td>0</td>
</tr>
</tbody>
</table>
The number of accepted events for each fitting hypothesis is given in Tables IVa-d. (The notation \( \pi^+\pi^- \)) \( K^0 \) indicates a \( \pi^+\pi^- \) pair coming from a \( K^0 \) decay very near the production vertex. For the fit to be accepted the \( \pi^+\pi^- \) effective mass was required to be consistent with the mass of the \( K^0 \) meson.) The ambiguous events are very few, and are ambiguous only among the three single-pion-production final states. We assign them by choosing the fit with the lowest \( \chi^2 \). A few events are ambiguous between hypotheses with the same final state but interchanged identities for the two charged secondaries; in such cases we also choose the fit with the lowest \( \chi^2 \), but do not consider the event as an ambiguity in Table IV.

The zero-constraint events are rather few, equal to about 2% of the accepted single-pion production events. Since events with two missing neutrals are negligible,\(^6\) we assign all of these to the single-pion-production final states. They are divided among the three charge states in proportion to the probability per cm of a secondary interaction or decay, where for each charge state the interaction and decay probabilities are averaged over our experimentally observed laboratory momentum spectra of the secondaries.

The non-beam rejects require the most difficult correction in our analysis. They arise from two sources: tracks genuinely off-momentum or off-direction, and good beam tracks rejected because of inaccurate measurement. Since beam tracks were not measured on the FSD, the latter effect is larger for those topologies measured more frequently on the Franckensteins. For example, at 860 and 960 MeV/c \( \pi^0 \) events were measured only on the Franckensteins, and in Tables IIIa-b they show a larger
fraction of non-beam rejects than 2- and 3-prong events, which were measured primarily on the FSD. We will make two assumptions: that the scanning criteria for beam tracks were the same for each topology, and that the fraction of 2-prong non-beam rejects due to pion events is small. Then all topologies should have the same fraction of bona-fide non-beam rejects, and we adjust the numbers of non-beam rejects appropriately at 860, 960, and 1200 MeV/c. The adjusted numbers of rejects are given in Tables IIIa-c, and the corresponding corrections in Tables IVa-c.

At 1360 MeV/c the situation is slightly different, since there are no 2-prong, no V^0 events. All topologies were measured almost entirely on the Franckensteins at this momentum, so we expect the same number of "false" rejects due to bad measurement for each topology. The 3-prong and 2-prong with V^0 topologies bear out this expectation; they are uncontaminated by incident pions, and they have the same fraction of non-beam rejects. That there are more four-prong rejects is also expected, since this topology is dominated by pion-induced events, and the pions have a much larger non-beam component. At 1360 MeV/c we make no corrections to the numbers of non-beam rejects.

The unresolved events, unbiased FSD rejects, and immeasurable events are all assumed to be unbiased according to final state, and are distributed within each topology according to the number of accepted events. In the 2-prong and 4-prong topologies they are distributed within the topology according to the number of accepted events. In the 3-prong topology 2-1/2% of the well-measured events are K^+ decays with a \pi^0 giving a Dalitz pair, rather than \tau decays. (Some of the more obvious Dalitz pair events are not recorded by the scanners.) We therefore enter in
Table IV 97-1/2% of the unresolved, unbiased FSD rejected, and immeasurable 3-prongs.

**Beam Contamination**

The contamination of the $K^+$ beam with other particles was small at 1200 MeV/c and lower momenta, but rose rapidly at higher momenta. Partly in anticipation of contamination problems, we measured at 1360 MeV/c only 3-prongs, 4-prongs, and events with associated $\nu^0$'s. The 3-prongs arise only from beam particle decays and are not simulated by incident pions or protons. These contaminants may produce 2-prong events with visible $\nu^0$ decays, as in the reaction $\pi^+p \rightarrow \pi^+K^+\Lambda^0$. Such events would be easily identified from the fit to the neutral decay alone, however. The threshold for this reaction is 1140 MeV/c, and we observed no examples of it at 1200 or 1360 MeV/c. The simplest contaminant reaction producing a neutral $K$ meson, $\pi^+p \rightarrow K^+p\nu^0$, is well below its threshold of 1500 MeV/c incident pion momentum. The 4-prongs are substantially contaminated by incident pions; at 1200 and 1360 MeV/c the majority of them fit only as incident pions. However, the separation of $K^+$ events from $\pi^+$ events by fitting and inspection of ionization is quite adequate for the small number of events observed.

Since the mass separation of the beam improved rapidly with decreasing momentum, of the lower three momenta the contamination problem was most severe at 1200 MeV/c. At that momentum we measured events from half of a roll of film, exposed at the time of the $K^+p$ run but with the beam tuned to transmit pions. These events were measured on the FSD and analyzed by FOG-CLOUDY-FAIR in the same way as in the rest of the experiment.
only incident $K^+$ hypotheses were tried, four-constraint fits were accepted without inspection of ionization, and one-constraint fits were accepted only when compatible with track ionizations. The results of this analysis for the 2-prong events are given in Table VI. We assume that when measured on the Franckenstein the events giving no valid $K^+$ fit would again fail to fit the incident $K^+$ hypotheses and would fit incident $\pi^+$ hypotheses. This gives for the ratio $R$ of spurious accepted $K^+$ fits to events fitting only $\pi^+$ hypotheses,

$$R = \frac{101}{266} = (38\pm5)\%$$

More important than the statistical error in this number are errors due to differences in treatment of the known $\pi^+$ events in the control sample and those from contamination pions in the $K$ beam. One possible source of error is due to more careful scrutiny of track ionization for the known pion events. Another is the fact that the contaminant pions in the $K$ beam were on the average somewhat lower in momentum than the beam particles, most of them having been degraded by passing through the edge of the mass separation slit. The control sample of pions passed through the center of the slit and was not degraded in momentum. Both kinds of pion events were "total beam edited" to the same momentum value, however, and so the "control" sample was in fact not a perfect control. We make the reasonable assumption that the off-momentum pions in the $K$ beam give the same fraction and distribution of spurious $K^+$ fits as the on-momentum control sample.

Using the number of detected pion 2-prong events from Table IVc, we infer the total number of 2-prong pion interactions in our sample to be 257, corresponding to a pion contamination in the beam of 2%. For the
number of spurious accepted \( K^+ \) fits in the 1200 MeV/c sample we find 71±10. Dividing these among the final states according to the numbers of accepted events in the control sample as given in Table VI, we obtain the corrections given in Tables Va-d.

At 860 and 960 MeV/c we assume the same ratio of spurious accepted events to identified pion events, and the same division of the spurious events by final state, as at 1200 MeV/c. This arbitrary assumption is justified by the smallness of the numbers of detected pion events at these momenta. As seen in Tables Va-b, the entire correction at the lower two momenta is always less than 10% of the statistical error.

Detected events due to incident protons are few at all momenta. There is little inelasticity in proton interactions at or below 1200 MeV/c, and a proton-proton elastic scattering event is extremely unlikely to be accepted as a \( K^+p \) inelastic final state, since both secondaries should be recognizable as protons by their heavy ionization. We will make no correction for spurious \( K^+ \) fits due to protons in the beam.

**Scanning Biases**

In order to determine what scanning errors were made, a sample of film was rescanned at each of the lower three momenta. The results are given in Table VII. We do not include loss of elastic scattering events with stopping protons shorter than the length corresponding to elastic scattering with \( \cos \theta_{c.m.} < 0.9 \); this correction is treated separately. In part of the 1360 MeV/c film events of the 2-prong, no \( \nu^0 \) topology were recorded by the scanners, although they were not analyzed further. For this reason events of other topologies accidentally identified as the
<table>
<thead>
<tr>
<th>Reaction</th>
<th>Total number of events, all internal corrections made</th>
<th>Bias corrections</th>
<th>Systematic error</th>
<th>Total number, all corrections made</th>
<th>Cross section, decay normalization (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^+ \rightarrow \pi^+\pi^+\pi^-$</td>
<td>2147±46</td>
<td>0</td>
<td>73±17</td>
<td>--</td>
<td>±47</td>
</tr>
<tr>
<td>$K^+ p \rightarrow K^+ p$</td>
<td>11202±115</td>
<td>-11±4</td>
<td>403±67</td>
<td>316±35</td>
<td>±109</td>
</tr>
<tr>
<td>$\rightarrow K^+ p n^0$</td>
<td>340±19</td>
<td>-1±1</td>
<td>12±2</td>
<td>-</td>
<td>±19</td>
</tr>
<tr>
<td>$\rightarrow K^+ p n$</td>
<td>124±12</td>
<td>-1±1</td>
<td>4±1</td>
<td>-</td>
<td>±11</td>
</tr>
<tr>
<td>$K^+ p \rightarrow K^0 p n^+$, $K^0$ unseen</td>
<td>727±29</td>
<td>-1±1</td>
<td>26±4</td>
<td>-</td>
<td>±27</td>
</tr>
<tr>
<td>$\rightarrow K^0 p n^+$,</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^0 \rightarrow \pi^+\pi^-$</td>
<td>325±17</td>
<td>0</td>
<td>17±7</td>
<td>-</td>
<td>±18</td>
</tr>
<tr>
<td>$\rightarrow \pi^+ p (\pi^+\pi^-)_{K^0 l}$</td>
<td>18±4</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>±4</td>
</tr>
<tr>
<td>$\rightarrow K^0 p^+$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^+ p \rightarrow all final states$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table Va. 860 MeV/c; bias corrections, final numbers of events, and cross sections.
Table Vb. 960 MeV/c; bias corrections, final numbers of events, and cross sections.

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Total number of events, all internal corrections made</th>
<th>Bias corrections</th>
<th>Cross section, decay normalization (mb)</th>
<th>Cross section, counter normalization (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K⁺ → π⁺π⁺π⁻</td>
<td>703±26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K⁺p → K⁺p</td>
<td>3860±64</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>→ K⁺pπ₀</td>
<td>266±17</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>→ K⁺π⁺</td>
<td>100±10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K⁺ → K⁰π⁺⁺,</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K⁰ unseen</td>
<td>542±24</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>→ K⁰π⁺⁺</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K₁ → π⁺π⁻</td>
<td>299±18</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>→ π⁺p(π⁺π⁻)</td>
<td>15±4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K⁺p → K⁰π⁺p</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K⁺p → all final states</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The table lists various reactions with their corresponding total numbers of events, bias corrections, and cross sections. The table headings are properly aligned, and the data is clearly presented.
Table Vc. 1200 MeV/c; bias corrections, final numbers of events, and cross sections.

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Total number of events, all internal corrections made</th>
<th>Bias corrections</th>
<th>Cross section, decay normalization (mb)</th>
<th>Cross section, counter normalization (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Pion contamination</td>
<td>Scanning loss</td>
<td>Short proton loss</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+ \pi^+ \pi^-$</td>
<td>568±24</td>
<td>--</td>
<td>42±11</td>
<td>--</td>
</tr>
<tr>
<td>$K^+_p \rightarrow K^+_p$</td>
<td>3953±70</td>
<td>--</td>
<td>51±10</td>
<td>106±20</td>
</tr>
<tr>
<td>$\rightarrow K^+_p\pi^0$</td>
<td>693±28</td>
<td>--</td>
<td>8±2</td>
<td>19±3</td>
</tr>
<tr>
<td>$\rightarrow K^+_n\pi^+$</td>
<td>220±16</td>
<td>--</td>
<td>7±2</td>
<td>6±1</td>
</tr>
<tr>
<td>$K^+_p \rightarrow K^0\pi^+$,</td>
<td>$K^0$ unseen</td>
<td>1290±38</td>
<td>--</td>
<td>6±2</td>
</tr>
<tr>
<td>$\rightarrow K^0\pi^+$,</td>
<td>$K^0_1 \rightarrow \pi^+\pi^-$</td>
<td>576±24</td>
<td>--</td>
<td>32±12</td>
</tr>
<tr>
<td>$\rightarrow \pi^+p(\pi^+\pi^-)$</td>
<td>$K^0_1$</td>
<td>33±7</td>
<td>--</td>
<td>3±3</td>
</tr>
<tr>
<td>$K^+_p \rightarrow K^0\pi^+$</td>
<td>$K^+_p \rightarrow K^0\pi^+$</td>
<td>1966±66</td>
<td>5.50±0.38</td>
<td>5.10±0.15</td>
</tr>
<tr>
<td>$K^+_p \rightarrow K^+\pi^+\pi^-$</td>
<td>9±3</td>
<td>0</td>
<td>1±1</td>
<td>--</td>
</tr>
<tr>
<td>$K^+_p \rightarrow$ all final states</td>
<td>7091±132</td>
<td>19.84±1.25</td>
<td>18.40±0.12</td>
<td></td>
</tr>
</tbody>
</table>
Table Vd. 1360 MeV/c; bias corrections, final numbers of events, and cross sections.

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Total number of events, all internal corrections made</th>
<th>Bias corrections</th>
<th>Systematic error</th>
<th>Total number, all corrections made</th>
<th>Cross section, decay normalization (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Pion contamination</td>
<td>Scanning loss</td>
<td>Escape loss</td>
<td></td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+ \pi^+ \pi^-$</td>
<td>1344±36</td>
<td>0</td>
<td>123±28</td>
<td>0</td>
<td>±38</td>
</tr>
<tr>
<td>$K^0 \rightarrow K^0 \pi^+$, $K_L^0 \rightarrow \pi^+\pi^-$, $K_L^0 \rightarrow \pi^+\pi^-$</td>
<td>1586±40</td>
<td>0</td>
<td>152±39</td>
<td>15±4</td>
<td>±42</td>
</tr>
<tr>
<td></td>
<td></td>
<td>43±7</td>
<td>0</td>
<td>0</td>
<td>±13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^+ \rightarrow K^0 \pi^+ \pi^0$, $K_L^0 \rightarrow \pi^+\pi^0$, $K_L^0 \rightarrow \pi^+\pi^0$</td>
<td>17±4</td>
<td>0</td>
<td>2±1</td>
<td>0</td>
<td>±4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1±1</td>
<td>0</td>
<td>0</td>
<td>±1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^+ \rightarrow K^0 \pi^+ \pi^0$, $K_L^0 \rightarrow \pi^+\pi^-$, $K_L^0 \rightarrow \pi^+\pi^-$</td>
<td>3±2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>±2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^+ \rightarrow K^0 \pi^+ \pi^0$, $K_L^0 \rightarrow \pi^+\pi^0$, $K_L^0 \rightarrow \pi^+\pi^0$</td>
<td>64±8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>±8</td>
</tr>
</tbody>
</table>

We take $R(K^0 \rightarrow \pi^+\pi^-)/R(K^0 \rightarrow \text{all modes}) = 0.346$, from Ref. 8.
Table VI. Summary of 1200 MeV/c $\pi^+p$ 2-prong interactions analyzed as incident $K^+$ interactions.

<table>
<thead>
<tr>
<th>Description</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Events measured</td>
<td>800</td>
</tr>
<tr>
<td>Events with all tracks well-measured</td>
<td>369</td>
</tr>
<tr>
<td>Non-fitting events</td>
<td>268</td>
</tr>
<tr>
<td>No valid $K^+$ fit</td>
<td>266</td>
</tr>
<tr>
<td>Non-beam track</td>
<td>2</td>
</tr>
<tr>
<td>Well-fitted events [$x^2(4c) &lt; 20$, $x^2(1c) &lt; 7$]</td>
<td>101</td>
</tr>
<tr>
<td>$K^+p \rightarrow K^+p$</td>
<td>72</td>
</tr>
<tr>
<td>$\rightarrow K^0\pi^+p$</td>
<td>8</td>
</tr>
<tr>
<td>$\rightarrow K^+\pi^0p$</td>
<td>11</td>
</tr>
<tr>
<td>$\rightarrow K^+\pi^+n$</td>
<td>10</td>
</tr>
</tbody>
</table>
Table VII. Calculation of scanning efficiencies. The numbers of 2-prong events with no $V^O$ missed do not include elastic scattering events with $\cos \theta_{c.m.} > 0.9$.

<table>
<thead>
<tr>
<th>Nominal beam momentum (MeV/c)</th>
<th>Topology</th>
<th>Total events</th>
<th>Events missed in original scan</th>
<th>% loss in original scan</th>
</tr>
</thead>
<tbody>
<tr>
<td>860</td>
<td>2 prongs, no $V^O$</td>
<td>1001</td>
<td>36</td>
<td>3.6±0.6</td>
</tr>
<tr>
<td></td>
<td>2 prongs with $V^O$</td>
<td>94</td>
<td>5</td>
<td>5.3±2.2</td>
</tr>
<tr>
<td></td>
<td>4 prongs</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3 prongs</td>
<td>611</td>
<td>21</td>
<td>3.4±0.8</td>
</tr>
<tr>
<td>960</td>
<td>2 prongs, no $V^O$</td>
<td>1033</td>
<td>48</td>
<td>4.6±0.7</td>
</tr>
<tr>
<td></td>
<td>2 prongs with $V^O$</td>
<td>140</td>
<td>7</td>
<td>5.0±1.9</td>
</tr>
<tr>
<td></td>
<td>4 prongs</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3 prongs</td>
<td>366</td>
<td>18</td>
<td>4.9±1.2</td>
</tr>
<tr>
<td>1200</td>
<td>2 prongs, no $V^O$</td>
<td>985</td>
<td>27</td>
<td>2.7±0.5</td>
</tr>
<tr>
<td></td>
<td>2 prongs with $V^O$</td>
<td>139</td>
<td>8</td>
<td>5.8±2.0</td>
</tr>
<tr>
<td></td>
<td>4 prongs</td>
<td>13</td>
<td>1</td>
<td>8.0±8.0</td>
</tr>
<tr>
<td></td>
<td>3 prongs</td>
<td>188</td>
<td>14</td>
<td>7.5±2.0</td>
</tr>
<tr>
<td>1360</td>
<td>2 prongs with $V^O$</td>
<td>219</td>
<td>21</td>
<td>9.6±2.1</td>
</tr>
<tr>
<td></td>
<td>4 prongs</td>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3 prongs</td>
<td>164</td>
<td>15</td>
<td>9.2±2.4</td>
</tr>
</tbody>
</table>
2-prong, no \( V^0 \) topology (a common scanner error) were omitted from our sample. Errors of this type are included for 1360 MeV/c in Table VII, contributing to the larger scanning loss at that momentum.

**Short Proton Loss**

Elastic scattering events with short recoil protons are missed by scanners with increasing frequency as the scattering angle decreases. For \( K^+p \) elastic scattering with \( \cos \theta_{\text{c.m.}} = 0.9 \), the recoiling proton has a stopping range of 1.16, 1.70, or 2.9 cm, at 860, 960, or 1200 MeV/c, respectively. Scanning loss for events with larger scattering angles is included in the rescan data. Loss of events with smaller scattering angles can be estimated from inspection of the experimental elastic scattering angular distribution, with no scanning efficiency corrections, to sufficient accuracy for the purpose of determining cross sections. These estimates are given in Tables V-a-c.

**Escape Correction**

The average flight path of a 1000-MeV/c \( K^0_L \) meson before decaying is 5.2 cm. Our fiducial volume includes very few events within 10 cm of the edge of the visible chamber volume; thus, the correction to the numbers of events with visible \( K^0_L \rightarrow \pi^+\pi^- \) decays is small. We have nevertheless made such a correction at 1360 MeV/c for the \( K^0\pi^+p \) final state, amounting to 1%. No correction is necessary at lower momenta, since the lost \( V^0 \) events are still identified and analyzed as members of a no-\( V^0 \) topology.

**Systematic Error**

To account for all sources of systematic error we rather arbitrarily introduce an error of \( \sqrt{N} \), equal to the statistical error, for each channel.
The final corrected numbers of events are given in Tables Va-d, along with cross sections normalized to the number of \( \tau \) decays. The cross sections are calculated from the formula

\[
\sigma = \frac{N_{\text{interaction}}}{N_{\text{decay}}} \times \frac{B}{\eta \cdot c \cdot T} \times \frac{A_H}{\rho_{H_2} N_A},
\]

where

- \( A_H \) (atomic weight of hydrogen) = 1.008,
- \( N_A \) (Avogadro's number) = \( 6.0225 \times 10^{23} \) mole\(^{-1}\),
- \( \rho_{H_2} \) (density of hydrogen in the bubble chamber) = 0.0608 gm/cm\(^3\),
- \( \eta = \frac{p_{\text{beam}}}{M_{K^+}} \),
- \( c = 2.998 \times 10^{10} \) cm/sec,
- \( T \) (\( K^+ \) lifetime) = \( 1.235 \times 10^{-8} \) sec,
- \( B \) (\( K^+ \) branching ratio into the \( \tau \) decay mode) = 0.056\pm0.001.

At 960 and 1200 MeV/c we have also computed cross sections normalized to the total cross section measurements of Cool et al., as privately communicated to us by T. Kycia. The errors on the total cross section measurements are statistical only, and future changes in the cross section values comparable to the statistical error due to refined theoretical corrections cannot be ruled out.

In Table VIII we give a complete summary of \( K^+ p \) cross sections from 600 to 1960 MeV/c. We include all published data in this momentum interval except for the 5-body final states, at 1960 MeV/c only, and final states with two or more missing neutrals. At 1455 and 1960 MeV/c the published cross sections have been scaled by factors of 0.986 and 0.905, respectively, so as to agree with the Cool et al. total cross sections; the errors have
Table VIII. \( K^+p \) cross sections, in mb, in the momentum interval 642-1960 MeV/c.

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>( \sigma_{K^+p} )</th>
<th>( \sigma_{K^+\pi^+p} )</th>
<th>( \sigma_{K^+\pi^+n} )</th>
<th>( \sigma_{K^0\pi^+p} )</th>
<th>( \sigma_{K^0\pi^+n} )</th>
<th>Total</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>642</td>
<td>12.4 ±0.9</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>735</td>
<td>0.13 ±0.01</td>
<td>0.03 ±0.01</td>
<td>0.02 ±0.01</td>
<td>0.18 ±0.01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>785</td>
<td>12.6 ±0.4</td>
<td>0.34 ±0.01</td>
<td>0.11 ±0.01</td>
<td>0.07 ±0.01</td>
<td>0.52 ±0.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>810</td>
<td>13.0 ±0.7</td>
<td>0.64 ±0.08</td>
<td>0.22 ±0.05</td>
<td>0.11 ±0.04</td>
<td>0.95 ±0.16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>865</td>
<td>12.8 ±0.5</td>
<td>1.20 ±0.07</td>
<td>0.38 ±0.02</td>
<td>0.14 ±0.02</td>
<td>1.72 ±0.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>910</td>
<td>2.1 ±0.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>970</td>
<td>12.0 ±0.3</td>
<td>2.63 ±0.13</td>
<td>0.81 ±0.07</td>
<td>0.31 ±0.04</td>
<td>3.75 ±0.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1140</td>
<td>4.6 ±0.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1205</td>
<td>10.9 ±0.2</td>
<td>5.10 ±0.16</td>
<td>1.83 ±0.10</td>
<td>0.57 ±0.06</td>
<td>7.50 ±0.22</td>
<td>0.026</td>
<td></td>
</tr>
<tr>
<td>1365</td>
<td>5.3 ±0.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1450</td>
<td>9.6 ±0.5</td>
<td>4.9 ±0.2</td>
<td>1.9 ±0.1</td>
<td>1.3 ±0.4</td>
<td>8.1 ±0.04</td>
<td>0.15</td>
<td>0.08</td>
</tr>
<tr>
<td>1585</td>
<td>4.8 ±0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1960</td>
<td>6.8 ±0.7</td>
<td>4.2 ±0.3</td>
<td>1.8 ±0.3</td>
<td>1.4 ±0.7</td>
<td>7.4 ±0.2</td>
<td>1.5</td>
<td>1.2</td>
</tr>
</tbody>
</table>

(continued)
not been changed from the published values. We give the interpolated
Cool et al. total cross sections at all tabulated momenta in the interval
covered by that experiment, but the partial cross sections are normalized
to that data only at 960, 1200, 1450, and 1960 MeV/c.

Some of these cross sections are plotted in Figs. 2 and 3. One
cannot avoid concluding from these data that the Cool bump is related
to single-pion-production threshold phenomena. The remainder of this
paper will be devoted to a detailed study of single pion production in
the region of the Cool peak.
Fig. 2. $K^+p$ cross sections in the region of the Cool peak.
Fig. 3. Cross sections for single pion production in the region of the Cool peak.
III. CROSS SECTIONS FOR RESONANCE PRODUCTION

As we have shown in the last section, K⁺p interactions in the momentum region of our experiment are dominated by a few simple processes:

- Elastic scattering,
  
  \[ K^+ p \rightarrow K^+ p \]  

- Single pion production,
  
  \[ K^+ p \rightarrow K^0 \pi^+ p \]  

  \[ \rightarrow K^\pi^0 p \]  

  \[ \rightarrow K^+ \pi^+ n \]  

- Double pion production, mainly
  
  \[ K^+ p \rightarrow K^+ \pi^+ \pi^- p \]  

Single pion production is further simplified by the dominance of two quasi-two-body final states:

\[ K^+ p \rightarrow K N^* (1236) \]

\[ \rightarrow \pi N \]  

\[ K^+ p \rightarrow K^* (891) N \]

\[ \rightarrow K \pi \]  

This is seen most clearly in the Dalitz plots at 1200 and 1360 MeV/c, Fig. 4. The resonance bands are well-defined, and there appears to be remarkably little nonresonant background. In the 860 and 960 MeV/c Dalitz plots, below K⁺ threshold and near the N⁺ threshold, there are no clear resonance bands, since the 116 MeV-wide N⁺ covers the entire Dalitz plots (see Fig. 5). We will show, however, that at these momenta single pion production goes largely through the KN⁺ intermediate state.
Fig. 4. Dalitz plots for the $K\pi N$ final states, at 1200 and 1360 MeV/c.
Fig. 5. Dalitz plots for the $K\pi N$ final states, at 860 and 960 MeV/c.
Separating resonance production from background in the single pion production final states is a difficult problem, requiring in principle a detailed knowledge of the amplitudes for both processes. In this section we will use a very simple model in which we represent the Dalitz plot population as a non-interfering superposition of $N^*$ production, $K^*$ production, and three-body phase space (background). While we do in fact see interference effects not taken into account by this model, it has the virtue of requiring few dynamical assumptions.

We can represent the Dalitz plot density for resonance production alone rather generally as the product of a Breit-Wigner function, a production angular momentum barrier factor, and a function specifying the density distribution within the resonance band along a line of fixed resonance mass. The latter is best represented in terms of the decay angle $\lambda$, and its general form is determined by the spin of the decaying resonance. For the $N\pi$ system, we define $\lambda_K$ as the decay angle, in the $N\pi$ c.m., of the pion with respect to the kaon, as shown in Fig. 6. Along a line of constant $M_{N\pi}^2$, $M_{K\pi}^2$ is proportional to $\cos \lambda_K$, with $\cos \lambda_K = \pm 1$, corresponding to the edges of the Dalitz plot. For the $K\pi$ system we similarly define $\lambda_N$ as the decay angle of the pion with respect to the nucleon, in the $K\pi$ c.m. Since the $N^*$ and $K^*$ decay in $p$-waves, they must have distributions in $\lambda$ of the form

$$W(\cos \lambda) \propto 1 + A \cos^2 \lambda .$$

We will leave the values of $A$ for the two resonances as free parameters to be varied in the fit.

The choice of a particular angular momentum barrier is the feature of this model which most compromises its generality. However, at 1200
Fig. 6. Definition of $\lambda_K$, the pion decay angle in the $N\pi$ c.m. with respect to the kaon direction.
and 1360 MeV/c both resonances are far enough from threshold so that the effect of the angular momentum barrier factor is negligible; and near N* threshold where the effect is important we have a clear indication from the Nπ mass spectrum of what orbital angular momentum is involved. Near threshold the Nπ mass distribution for N* production is strongly dependent on the angular momentum state in which it is produced. If the final-state KN* system is in a relative S wave, the mass spectrum is the product of phase space and the N* Breit-Wigner function. For nonzero orbital angular momentum in the final state, however, the angular momentum barrier factor will have a dependence on outgoing N* momentum favoring high momenta. This tends to suppress production of high-mass Nπ systems. This effect is illustrated in Fig. 7, where we show the Nπ mass distribution at 860 MeV/c and the predictions for phase space and for N* production in S, P, and D waves. We have used an angular momentum barrier function of the form

$$f^J(q') = \frac{q'^2}{(q'^2 + \frac{m^2}{2})^\frac{3}{2}},$$

(9)

where q' is the outgoing kaon momentum in the overall c.m. and where for m we take the rho mass, as suggested by the rho-exchange hypothesis for N* production (see Appendix B). Other values of m over 400 MeV give almost indistinguishable results. Of course, as m is decreased to zero, the P- and D-wave curves approach the S-wave curve. From Fig. 7 we see that the data are inconsistent with any superposition of phase space and S-wave N* production, and thus require the presence of N* production in a higher partial wave. On the basis of the excellent agreement of the 860 MeV/c data with the P-wave predictions as seen in Fig. 7, we will use a P-wave angular momentum barrier factor for the N*. For the K* we
Fig. 7. Experimental $M_{NN}^2$ distribution at 860 MeV/c, and the predictions for pure $N^*$ production in S, P, and D waves.
will use no angular momentum barrier factor, since at none of our momenta is the $K^*$ near enough to its production threshold for this factor to be important.

The Dalitz plot density is given under these assumptions by

$$
\frac{d\sigma}{dM_{N\pi}^2\, dM_{K\pi}^2} = a + b \frac{BW_{K^*}(1 + A_{K^*}\cos^2 \lambda_N)}{M_{K\pi}^2} + c \frac{BW_{N^*}(1 + A_{N^*}\cos^2 \lambda_K)}{q'^2 + m^2};
$$

(10)

here $BW$ is the p-wave Breit-Wigner function given by Eq. (A29) and $q'$ is the outgoing kaon momentum in the overall c.m. The fit is carried out by the maximum likelihood technique, using for each event its values of $M_{N\pi}^2$ and $M_{K\pi}^2$ and the total energy corresponding to the fitted beam momentum for that event. At 860 and 960 MeV/c we assume that the $K^*$ does not contribute, and keep $b = 0$. (With $b = 0$, the fit is completely equivalent, in its determination of the ratio of $N^*$ to background, to fitting only the $N\pi$ mass projection of the Dalitz plot.) The fractions of $N^*$ production, $K^*$ production, and background thus determined are given in Table IX. The errors are statistical only, corresponding to a decrease by one of the logarithm of the likelihood. Also shown are the results of fitting published $M_{N\pi}^2$ distributions for the $K^0\pi^+p$ final state at 735 and 785 MeV/c, using the same assumption of P-wave $N^*$ production and non-interfering phase space background.\textsuperscript{20}

The strongest criticism of the model just discussed is that for the $K^0\pi^+p$ and $K^+\pi^0p$ final states above $K^*$ threshold it ignores $K^*-N^*$ interference. The interference model described in Section VII was devised specifically to account for this effect. There definite amplitudes for
Table IX. Fractions of N* production, K* production, N*-K* interference, and background in the single pion production channel, for the different charge states and as a function of beam momentum.

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>Final state</th>
<th>Data sample</th>
<th>Type of fit</th>
<th>% N*</th>
<th>% K*</th>
<th>% bkgd. % interf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>735&lt;sup&gt;a&lt;/sup&gt;</td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;+p&lt;/sup&gt;</td>
<td>all events</td>
<td>CHISQ</td>
<td>63±25</td>
<td>--</td>
<td>37±25</td>
</tr>
<tr>
<td>785&lt;sup&gt;a&lt;/sup&gt;</td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;+p&lt;/sup&gt;</td>
<td>all events</td>
<td>CHISQ</td>
<td>55±7</td>
<td>--</td>
<td>45±7</td>
</tr>
<tr>
<td></td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0p&lt;/sup&gt;</td>
<td>all events</td>
<td>NO INT</td>
<td>76±6</td>
<td>--</td>
<td>22±6</td>
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<tr>
<td></td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0p&lt;/sup&gt;</td>
<td>all events</td>
<td>PWA</td>
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<td>19±10</td>
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<tr>
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<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0n&lt;/sup&gt;</td>
<td>all events</td>
<td>NO INT</td>
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<td>--</td>
<td>39±11</td>
</tr>
<tr>
<td></td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0n&lt;/sup&gt;</td>
<td>all events</td>
<td>PWA</td>
<td>52±22</td>
<td>--</td>
<td>48±22</td>
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<td>NO INT</td>
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<td>--</td>
<td>44±16</td>
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<tr>
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<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0p&lt;/sup&gt;</td>
<td>all events</td>
<td>PWA</td>
<td>64±18</td>
<td>--</td>
<td>36±18</td>
</tr>
<tr>
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<td>all events</td>
<td>NO INT</td>
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<td>PWA</td>
<td>95±2</td>
<td>--</td>
<td>5±2</td>
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<td>NO INT</td>
<td>56±8</td>
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<td>44±8</td>
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<td></td>
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<td>PWA</td>
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<td>25±2</td>
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<td>all events</td>
<td>INT</td>
<td>57±4</td>
<td>17±2</td>
<td>20±4</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td></td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0p&lt;/sup&gt;</td>
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<td>NO INT</td>
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<td>41±3</td>
<td>12±3</td>
</tr>
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<td></td>
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<td>INT</td>
<td>31±5</td>
<td>33±4</td>
<td>36±5</td>
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<td>all events</td>
<td>NO INT</td>
<td>67±6</td>
<td>--</td>
<td>33±6</td>
</tr>
<tr>
<td></td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0n&lt;/sup&gt;</td>
<td>all events</td>
<td>PWA</td>
<td>62±13</td>
<td>--</td>
<td>38±13</td>
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<td>NO INT</td>
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<td>45±4</td>
<td>25±4</td>
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</tr>
<tr>
<td></td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0p&lt;/sup&gt;</td>
<td>all events</td>
<td>NO INT</td>
<td>60±3</td>
<td>--</td>
<td>4±2</td>
</tr>
<tr>
<td></td>
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<td>all events</td>
<td>NO INT</td>
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<td>36±3</td>
<td>3±2</td>
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<tr>
<td></td>
<td>K&lt;sup&gt;0&lt;/sup&gt;π&lt;sup&gt;0p&lt;/sup&gt;</td>
<td>all events</td>
<td>PWA</td>
<td>59±3</td>
<td>--</td>
<td>5±2</td>
</tr>
</tbody>
</table>

<sup>a</sup> Data from Ref. 11, fit described in text.

CHISQ -- Least-squares fit with non-interfering N* and background
NO INT -- Maximum-likelihood fit with non-interfering N*, K*, and background (see Sec. III)
INT -- Fit to background and interfering N* and K* (see Sec. VII)
PWA -- Partial-wave analysis, including N* and background (see Sec. VI)
N* and K* production are assumed, and the Dalitz plot density is calculated including coherent N* and K* production and a non-interfering phase space background. A least-squares fit to the Dalitz plot distribution determines the amount of N* production, K* production, and background and the N*-K* relative phase. This model has its drawback too, however; the detailed distribution of the interference over the Dalitz plot is highly dependent on the production model assumed for the N* and K*. The results of this fit are shown in Table IX, for 1200 and 1360 MeV/c. Comparing them with the results of the no-interference model, we see that the general effect of neglecting the interference is to overestimate the amounts of N* and K* production, and to underestimate the amount of background.

Both the non-interference hypothesis and the interference model are subject to theoretical uncertainties which may be the dominant source of error in the results of the fits. For this reason it seems advantageous to try to extract the information we need without using the region of the Dalitz plot where the N* and K* bands cross. Using the representation of the Dalitz plot density described in this section, we will first use only events in the \( \cos \lambda_K < 0 \) (low K\( \pi \) mass) half of the Dalitz plot, assuming that effects of K* production are negligible there. This is a relatively mild assumption, since we are neglecting interference only in the tail of one or the other of the resonances. Then N* and background events should be produced in equal numbers in both halves of the Dalitz plot, and so the amount of N* production and background is determined from the fit to the \( \cos \lambda_K < 0 \) half of the Dalitz plot. The excess is attributed to K* production and K*-N* interference. A further
extension of this approach is to use all of the Dalitz plot except the quarter with \( \cos \lambda_K > 0, \cos \lambda_N < 0 \), thus excluding only the quarter of the Dalitz plot including the \( N^*-K^* \) overlap region. Hopefully we still avoid most of the \( K^*-N^* \) interference effects, and so we can infer the amount of \( N^* \) production, \( K^* \) production, and background in the entire Dalitz plot; the remaining excess we attribute to \( N^*-K^* \) interference in the excluded part of the Dalitz plot. The results from fitting the two restricted regions of the Dalitz plot are given in Table IX, for 1200 and 1360 MeV/c. Although less marked, the same differences from the results of the interference fit are present: more \( N^* \) and \( K^* \) production, less background. We prefer the results of the no-interference fit excluding the overlap region, both because of its relative simplicity and because the interference model gives a rather poor fit in the parts of the Dalitz plot away from the resonance bands.

As a final check we list in Table IX the results of the partial wave analysis discussed in Section VI. The partial wave analysis uses only events in the \( \cos \lambda_K < 0 \) half of the Dalitz plot, but uses all of the kinematical information from each event. We see that the PWA results are generally compatible with the no-interference results from the same data sample.

We must now determine from the various fitting results the "best" values and realistic errors. At momenta below 1200 MeV/c and for the \( K^+\pi^+\pi^- \) final state at 1200 MeV/c, \( K^* \) production and its attendant complications are not present. There we will average the fitting results from the no-interference fit and from the partial wave analysis, and for errors take the average and multiply by \( \sqrt{2} \) to account for theoretical uncertainties.
Where the $K^*$ is present we will use the results from the no-interference fit excluding the $N^*-K^*$ overlap region, and multiply all errors by a factor of 2. The amount of $N^*-K^*$ interference is probably not determined reliably by any of the fits discussed, and it is included only in Table IX.

The final cross sections for resonance production, and background are given in Table X. We have included published results at nearby momenta. These values are plotted in Fig. 8, with the exception of those at 1140 and 1450 MeV/c. They have been excluded because at nearby momenta we see strong $N^*-K^*$ interference. In the region where the $N^*-K^*$ overlap region occupies much of the Dalitz plot results obtained with different analysis techniques should not be compared, and to do so might distort the shape of the cross-section curves for resonance production.

**$\kappa(725)$ Production**

There have appeared in the literature at various times unconfirmed reports of a $K\pi$ resonance called the $\kappa$, of mass about 725 MeV and with a width usually consistent with zero, within experimental resolution. A resonance with isospin $I = 1/2$ would not be seen in the $K^+\pi^+\pi^-$ final state, since the $K^+\pi^+$ state has $I_z = 3/2$; an $I = 3/2$ resonance would be produced most strongly in this final state, however. We have looked in all charge states at all of our momenta for a narrow $K\pi$ resonance. We have examined separately events with $N\pi$ mass in the $N^*$ band and outside of it, and events with low and high momentum transfer to the $K\pi$ system. The results are negative.

We can estimate the amount of $\kappa$ production which would pass undetected in our experiment. The number of events per 15 MeV $K\pi$ mass interval in
Table X. Summary of cross sections for $N^*$ production, $K^*$ production, and background.

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>Reaction</th>
<th>$\sigma(KN^*)$ (mb)</th>
<th>$\sigma(K^*N)$ (mb)</th>
<th>$\sigma(\text{bkgd})$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>865</td>
<td>$K^+p \rightarrow K^0\pi^+p$</td>
<td>0.95±0.15</td>
<td>--</td>
<td>0.25±0.14</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K^+\pi^0p$</td>
<td>0.21±0.09</td>
<td>--</td>
<td>0.17±0.09</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K^+\pi^+n$</td>
<td>0.08±0.04</td>
<td>--</td>
<td>0.06±0.04</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K\pi N$</td>
<td>1.24±0.18</td>
<td>--</td>
<td>0.48±0.16</td>
</tr>
<tr>
<td>970</td>
<td>$K^+p \rightarrow K^0\pi^+p$</td>
<td>2.43±0.15</td>
<td>--</td>
<td>0.20±0.09</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K^+\pi^0p$</td>
<td>0.51±0.13</td>
<td>--</td>
<td>0.30±0.12</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow K^+\pi^+n$</td>
<td>0.28±0.06</td>
<td>--</td>
<td>0.03±0.04</td>
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<td>$\rightarrow K\pi N$</td>
<td>3.22±0.21</td>
<td>--</td>
<td>0.53±0.16</td>
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<tr>
<td>1205</td>
<td>$K^+p \rightarrow K^0\pi^+p$</td>
<td>3.21±0.32</td>
<td>0.82±0.21</td>
<td>0.25±0.20</td>
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<td></td>
<td>$\rightarrow K^+\pi^0p$</td>
<td>0.71±0.15</td>
<td>0.55±0.11</td>
<td>0.26±0.15</td>
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<tr>
<td></td>
<td>$\rightarrow K^+\pi^+n$</td>
<td>0.37±0.09</td>
<td>--</td>
<td>0.20±0.08</td>
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<tr>
<td></td>
<td>$\rightarrow K\pi N$</td>
<td>4.29±0.36</td>
<td>1.37±0.24</td>
<td>0.71±0.26</td>
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<tr>
<td>1365</td>
<td>$K^+p \rightarrow K^0\pi^+p$</td>
<td>2.92±0.36</td>
<td>1.91±0.33</td>
<td>0.16±0.21</td>
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<tr>
<td></td>
<td>$\rightarrow K\pi N$</td>
<td>3.9±0.5</td>
<td>2.9±0.5</td>
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</tr>
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</table>

Data from other experiments

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>Reaction</th>
<th>$\sigma$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>735</td>
<td>$K^+p \rightarrow K^0\pi^+p$</td>
<td>0.08±0.05</td>
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<tr>
<td></td>
<td>$\rightarrow K\pi N$</td>
<td>0.11±0.06</td>
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<td>785</td>
<td>$K^+p \rightarrow K^0\pi^+p$</td>
<td>0.19±0.04</td>
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<td>$\rightarrow K\pi N$</td>
<td>0.25±0.09</td>
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<tr>
<td>1140</td>
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<td>3.6±0.5</td>
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<td>$\rightarrow K\pi N$</td>
<td>3.5</td>
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<td>$\rightarrow K\pi N$</td>
<td>2.7</td>
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a We have used $R(\text{all } N^* \text{ charge states})/R(N^{*+} \rightarrow \pi^+ p) = 4/3$
b We have used $R(\text{all } K^* \text{ charge states})/R(K^{*+} \rightarrow K^0\pi^+) = 3/2$
Fig. 8. Cross sections for $N^*$ production, $K^*$ production, and nonresonant background in the single-pion-production final states, in the region of the Cool peak.
the region of 725 MeV is given in Table XI, at each of our momenta and for each charge state measured. Since our mass resolution is about 15 MeV, a narrow $\kappa$ would appear mainly in a single 15-MeV bin. As a crude upper limit on the number of events due to $\kappa$ production we take three times the statistical error on the number of events per bin which we observe experimentally. The cross sections corresponding to these numbers are given in Table XI.
Table XI. Estimated upper limits on the cross section for production of a narrow \(\kappa\) meson.

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>Final state</th>
<th>Background events per 15 MeV K(\pi)-mass bin</th>
<th>Upper limit on (\sigma_{\kappa}) ((\mu b))</th>
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<tbody>
<tr>
<td>860</td>
<td>(K^0\pi^+p)</td>
<td>108</td>
<td>40</td>
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<tr>
<td></td>
<td>(K^+\pi^-p)</td>
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<td>25</td>
</tr>
<tr>
<td></td>
<td>(K^+\pi^-n)</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>960</td>
<td>(K^0\pi^+p)</td>
<td>150</td>
<td>75</td>
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<tr>
<td></td>
<td>(K^+\pi^-p)</td>
<td>22</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>(K^+\pi^-n)</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>1200</td>
<td>(K^0\pi^+p)</td>
<td>115</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>(K^+\pi^-p)</td>
<td>36</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>(K^+\pi^-n)</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>1360</td>
<td>(K^0\pi^+p)</td>
<td>30</td>
<td>60</td>
</tr>
</tbody>
</table>
IV. ANGULAR DISTRIBUTIONS

Four variables are required to specify a three-body final state configuration, in the absence of nucleon polarization measurements. We choose them to be a diparticle effective mass, its production angle in the overall c.m., and a pair of decay angles in the diparticle c.m. Having discussed the mass distributions in the previous section, we will now examine the production and decay angular distributions.

Since our data is dominated by $N^*$ and $K^*$ production, we will emphasize the $N\pi$ and $K\pi$ diparticle systems in this section. The $K^+p$ system is pure $I = 1$, and it follows that the $N^*$ and $K^*$ are produced in reactions 2, 3, and 4 in the ratios 9:2:1 and 2:1:0, respectively. The $K^0\pi^+p$ final state (reaction 2) is thus richest in resonance production, and will be emphasized in the ensuing discussion.

We can estimate the amount of background in the resonance samples in the $K^0\pi^+p$ final state, using the results of Section III. For the $N^*$ the background is $\approx 20\%$ at 860 MeV/c and $\leq 10\%$ at 960, 1200, and 1360 MeV/c. The background under the $K^*$, mainly due to the tail of the $N^*$, is about $40\%$ at 1200 MeV/c and $15\%$ at 1360 MeV/c. Background interference effects may not be negligible, especially with the $K^*$.

In order to avoid repeating the selection criteria for the data samples to be discussed, the mass bands and momentum transfer regions to be used are given in Table XII. In addition to these cuts, in the $K^0\pi^+p$ and $K^+\pi^0p$ final states at 1200 and 1360 MeV/c we must separate the $N^*$ and $K^*$. There we restrict ourselves, when discussing properties of the $N^*$, to the $\cos \lambda_N < 0$ (low $K\pi$ mass) region of the Dalitz plot; and, when discussing the $K^*$, to $\cos \lambda_K < 0$ (high $N\pi$ mass).
Table XII. Mass and momentum transfer intervals used in the analysis of $N^*$ and $K^*$ production, and the experimental peak positions of the $M_{N\pi}^2$ distributions (for $\cos \lambda_K < 0$ at 1200 and 1360 MeV/c).

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>$M_{N\pi}^2$ peak position (BeV)$^2$</th>
<th>$M_{N\pi}^2$ limits (BeV)$^2$</th>
<th>$\Delta_{N\pi}^2$ limits (MeV/c)$^2$</th>
<th>$M_{K\pi}$ limits (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N^*$ band</td>
<td>Low mass</td>
<td>Medium mass</td>
<td>High mass</td>
</tr>
<tr>
<td>860</td>
<td>$(1.170)^2$</td>
<td>no limits</td>
<td></td>
<td></td>
</tr>
<tr>
<td>960</td>
<td>$(1.191)^2$</td>
<td>no limits</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>$(1.212)^2$</td>
<td>1.35-1.65</td>
<td>$\leftarrow$ 1.42</td>
<td>1.42-1.52</td>
</tr>
<tr>
<td>1360</td>
<td>$(1.212)^2$</td>
<td>1.35-1.65</td>
<td>$\leftarrow$ 1.44</td>
<td>1.44-1.52</td>
</tr>
</tbody>
</table>
A. \( N^*(1236) \) Production Angular Distributions

The simplest and most easily interpreted information on the KN* final state is its production angular distribution. We show these distributions, for our four momenta, in Fig. 9. (The values of the squared momentum transfer \( t \) given on the x axis are approximate, assuming a fixed value of the Nn mass.) We note two simple features of the data: near threshold the distributions have a large \( \sin^2 \theta \) component, and, as the total c.m. energy increases, there is an increasing asymmetry toward the forward direction. The second feature continues smoothly into the high energy region, where the reaction is quite peripheral. A more quantitative description of these effects is given by the Legendre coefficients, as defined by

\[
W(\cos \theta) = \sum A_k P_k(\cos \theta) .
\]

(11)

These coefficients are shown as a function of beam momentum in Fig. 10, for our data and for other published data.\(^{11,17} \) We have included coefficients for the \( K^+ \pi^- p \) and \( K^+ n \) final states where available, for completeness. They add little information about \( N^* \) production, however, with differences from the \( K^0 \pi^- p \) final state probably representing background effects.

Let us first consider the even Legendre coefficients. \( A_0 \) is proportional to the cross section for \( N^* \) production, which peaks at about 1150 MeV/c. \( A_2/A_0 \) is large and negative near threshold, corresponding to the \( \sin^2 \theta \) character of the 860 and 960 MeV/c data. (Pure \( \sin^2 \theta \) corresponds to \( A_2/A_0 = -1 \).) The \( \sin^2 \theta \) component requires the presence of \( P \) or higher waves in the final-state KN* system, and it is remarkable
Fig. 9. $N^*$ production angular distributions at 860, 960, 1200, and 1360 MeV/c. The solid and dashed curves are the predictions of the $\rho$-exchange model, with magnetic dipole and Jackson-Pilkuhn coupling, respectively, as discussed in Section V.
$$\frac{d\sigma}{d\Omega}(N^*) = \sum_2 A_l P_l(\cos \theta)$$

Fig. 10. Legendre coefficients for the $N^*$ production angular distribution, as a function of beam momentum.
that it remains strong at the lowest momenta, where a final-state S wave would be favored by the absence of an angular momentum barrier factor. The $A_n/A_0$ coefficient vanishes near threshold, and increases steadily throughout the Cool peak. The even coefficients thus show a large P-wave component near threshold, with a transition to higher waves taking place in the region of the cross-section peak.

The odd Legendre coefficients represent interference between waves of opposite parity, and so should reflect any rapid phase change in a single amplitude. From Fig. 10 we see that both $A_1/A_0$ and $A_3/A_0$ are smooth and monotonic near 1150 MeV/c. This lack of evident phase change will be the crucial piece of evidence against a resonance in our partial wave analysis, in Section VI.

There is one further feature of the $N^*$ production angular distributions which is worth noting, although probably not relevant to the resonance question. This is seen in Fig. 11, where we show the two lowest Legendre coefficients as a function of the $N\pi$ mass. (The mass intervals are given in Table XII.) Considering first the $K\pi^+p$ channel, we see the forward-backward asymmetry ($A_2/A_0$) varying dramatically with $N\pi$ mass, and almost vanishing in the high mass region. The second Legendre coefficient also varies markedly with mass. Such an effect cannot result from $N^*$ production alone, but must result from interference between the $N^*$ and a background wave. Then the relative phase of the $N^*$ and background amplitudes varies with $N\pi$ mass as the phase of the $N^*$ Breit-Wigner varies, giving the effect observed. The background amplitude must be of a rather special sort, however; since in the production angular distribution the decay angles in the $N\pi$ c.m. have been integrated out, the
\[
\frac{d\sigma}{d\Omega}(N^*) = \sum_l A_l P_l(\cos \Theta)
\]

- $860 \text{ MeV/c}$  - $960 \text{ MeV/c}$
- $1200 \text{ MeV/c}$  - $1360 \text{ MeV/c}$

\[
\frac{A_1}{A_0}
\]

\[
\frac{A_2}{A_0}
\]

\[K^+p \rightarrow K^0\pi^+p\]

\[K^+p \rightarrow K^+\pi^0p\]

\[K^+p \rightarrow K^+\pi^+n\]

\[M^2_{N\pi}(\text{BeV})^2\]

Fig. 11. Variation of the first and second Legendre coefficients for $N^*$ production with $N\pi$ mass.
interfering wave must have the \( N\pi \) system in a \( p_{3/2} \) state, as other \( \ell_j \) states are orthogonal to the \( p_{3/2} N^* \). Since the \( K N^* \) final state is almost entirely \( P \)-wave near threshold, the background wave must have components with both even \( K-(N\pi) \) orbital angular momentum, to produce the effect seen in the \( A_1 \) coefficient, and odd orbital angular momentum, to produce the effect in \( A_2 \). Comparison between the three charge states in Fig. 11 provides some additional information. The change in the asymmetry coefficients \( A_1 \) with increasing mass is large and negative for the \( K^0 \pi^+ p \) final state. The effect is much weaker in the other two charge states, and a positive change with increasing mass is suggested in the \( K^+ \pi^0 p \) final state. No model for the background has been suggested which explains these effects.

### B. \( N^*(1236) \) Decay Angular Distributions

We describe the decay of the \( N^* \) in a coordinate system in the \( N\pi \) c.m., with the \( x \) axis along the direction of the incoming proton as seen in the \( N\pi \) c.m., and the \( z \) axis along the direction of the normal to the production plane. The coordinate system and decay angles are illustrated in Fig. 12; the decay angles refer to the nucleon from the \( N^* \) decay.

From symmetry considerations we can predict some properties of the decay angular distribution. First, the entire production and decay process is invariant under the parity transformation, or, equivalently, reflection about the \( N^* \) production plane. From Fig. 12 we see that this corresponds to the transformation

\[
\begin{align*}
\left\{ \cos \alpha, \varphi \right\} & \rightarrow \left\{ \cos \alpha, -\varphi \right\}, \\
(12a)
\end{align*}
\]
Fig. 12. Coordinate system used in describing the $N^*$ decay. The decay angles refer to the direction of the decay nucleon.
or

\[ \left\{ \cos \gamma \right\} \mapsto \left\{ \cos \gamma \right\} . \quad (12b) \]

We will assume this symmetry in our analysis. A second symmetry is the parity inversion of the outgoing nucleon and pion in the \( N\pi \) c.m., corresponding to the transformation

\[ \left\{ \cos \alpha \right\} \rightarrow \left\{ \cos \alpha \right\} , \quad (13a) \]

or

\[ \left\{ \cos \gamma \right\} \rightarrow \left\{ \cos \gamma \right\} . \quad (13b) \]

This is a valid symmetry only if the \( N\pi \) system is in a state of definite parity. Interference between states of opposite parities introduces terms not satisfying this symmetry.

We will parametrize the decay angular distribution with the usual \( N^* \) density matrix elements, \(^{21}\) and add two functions which are odd under parity inversion of the \( N^* \) decay and so represent interference of the \( N^* \) amplitude with a background amplitude. The form of these terms is motivated by their appearance in the partial wave analysis as interference terms between P-wave \( N^* \) production and a totally isotropic background wave. (See Appendix C, Eq. (C10) and (C12).) The general form of the decay angular distribution is then

\[
W(\cos \alpha, \varphi) = \frac{3}{4\pi} \left( \rho_{33} \sin^2 \alpha + \rho_{11} \left( \frac{1}{3} + \cos^2 \alpha \right) \right) \\
- \frac{2}{\sqrt{3}} \Re \rho_{3,-1} \sin^2 \alpha \cos 2\varphi - \frac{2}{\sqrt{3}} \Re \rho_{3,1} \sin 2\alpha \cos \varphi \\
+ R_1 \cos \alpha + R_2 \sin \alpha \cos \varphi \right) ; \quad (14a)
\]

or, in the \((\gamma, \delta)\) system,
\[ W(\cos \gamma, \delta) = \frac{3}{4\pi} \left( \rho_{33}' \sin^2 \gamma + \rho_{11}' \left( \frac{1}{3} + \cos^2 \gamma \right) \right) \]
\[ - \frac{2}{\sqrt{3}} \Re \rho_{33}' \sin^2 \gamma \cos \delta + \frac{2}{\sqrt{3}} \Im \rho_{3,-1}' \sin^2 \gamma \sin \delta \]
\[ + R_1 \sin \gamma \cos \delta + R_2 \sin \gamma \sin \delta \]  

(14b)

where

\[ \rho_{33}' \equiv \frac{1}{4} \left( \frac{3}{2} - 2 \rho_{33} - 2\sqrt{3} \Re \rho_{3,-1} \right) \]
\[ \rho_{11}' \equiv \frac{1}{2} - \rho_{33} \]
\[ \frac{1}{\sqrt{3}} \Re \rho_{3,-1}' \equiv -\frac{1}{4} + \rho_{33} - \frac{1}{\sqrt{3}} \Re \rho_{3,-1} \]
\[ \frac{1}{\sqrt{3}} \Im \rho_{3,-1}' \equiv -\frac{2}{\sqrt{3}} \Re \rho_{31} \]  

(Note that all terms of Eqs. (14a) and (14b) are invariant under the overall parity transformation, Eq. (12), but only the first four terms are even under the N\* c.m. parity transformation, Eq. (13).)

We have determined the density matrix elements and the two interference parameters \( R_1 \) and \( R_2 \) by the maximum likelihood technique. All decay correlations are thus taken into account, so it is immaterial which set of angles is used, \( \alpha \) and \( \phi \) or \( \gamma \) and \( \delta \). However, the one-dimensional angular distributions in \( \cos \alpha \) and \( \phi \) obtained by integrating Eq. (14a) over \( \phi \) and \( \cos \alpha \), respectively, are not equivalent in informational content to the \( \cos \gamma \) and \( \delta \) distributions; \( \Re \rho_{3,1} \) and \( R_2 \) are not measurable from the \( \cos \alpha \) and \( \phi \) distributions, whereas all coefficients are measurable from the \( \cos \gamma \) and \( \delta \) distributions. For this reason we will emphasize the density matrix elements of Eq. (14b) and the \( \cos \gamma \) and \( \delta \) angular distributions.

A complication arises at those momenta where the crossing of the \( K^* \) band obscures part of the \( N^* \) band. In the \( K^0\pi^+p \) and \( K^+\pi^0p \) final states
at 1200 and 1360 MeV/c we have used only N* events with \( \cos \lambda_K < 0 \), corresponding to choosing one hemisphere of decay solid angle in the N\( \pi \) c.m. We are in that case obliged to fold the angular distributions according to Eq. (13) into the ranges \( 0 \leq (\cos \alpha, \cos \theta) \leq 1 \), \( 0 \leq (\phi, \delta) \leq \pi \), and to drop the interference terms from the expansions of Eq. (14). At 860 and 960 MeV/c and in the \( K^+\pi^-n \) final state at 1200 MeV/c the full expansion has been used.

In Fig. 13 we show the density matrix elements of Eq. (14b) for the \( K^0\pi^+p \) final state, as a function of beam momentum, and we give our corresponding \( \cos \gamma \) and \( \delta \) distributions in Figs. 14 and 15, respectively. We have included in Fig. 13 all published values from other experiments in the region from our highest momentum to 3 BeV/c. In Fig. 16 we show the variation of these density matrix elements with momentum transfer; this, along with the production angular distributions, provides in principle a complete description of the N* production process. For the convenience of readers accustomed to the more conventional density matrix elements of Eq. (14a), those are given in Figs. 17 and 18.

The behavior of all density matrix elements, as seen in Fig. 13, is smooth as a function of beam momentum, with no dramatic change near 1150 MeV/c. Thus neither in the production nor in the decay angular distributions do we see evidence for a resonance in the reaction \( K^+p \rightarrow KN^* \).

We have now given a complete description of the production and decay of the N*. For analysis of the process in terms of partial waves, however, it is desirable to consider the decay angular distributions in another coordinate system, identical to the one previously described except that the x axis is taken as the direction of the incident proton as seen in
Fig. 13. $N^*$ density matrix elements of Eq. (14b), as a function of beam momentum; the curves are discussed in Section V.
Figure 11. Distribution in $\cos \theta$ for $N^*$ production, at 860, 960, 1200, and 1360 MeV/c. The curves have the same meaning as in Fig. 9.
Fig. 15. Distributions in $\delta$ for $N^*$ production, at 860, 960, 1200, and 1360 MeV/c. The curves have the same meaning as in Fig. 9.
Fig. 16. Variation of the $N^*$ density matrix elements of Eq. (14b) with momentum transfer to the $N^*$. The predictions of $\rho$ exchange with JP coupling are given by the solid curves; the predictions for ML coupling are zero for all three matrix elements (see Section V).
Fig. 17. $N^*$ density matrix elements of Eq. (14a), as a function of beam momentum.
Fig. 18. Variation of the $N^*$ density matrix elements of Eq. (14a) with momentum transfer to the $N^*$. 

$K^+ p \rightarrow K^0 N^{*++}$

$\rho_{33}$

$\text{Re} \rho_{3,-1}$

$\text{Re} \rho_{31}$

squared momentum transfer (BeV/c)$^2$
the overall c.m. The momenta in the N\(\pi\) c.m. must be obtained through a Lorentz transformation directly from the overall c.m. to the N\(\pi\) c.m. In this system the N\(^*\) at rest will have the same magnetic quantum number as the in-flight N\(^*\) produced in the overall c.m., which simplifies the predictions of a partial wave representation. This coordinate system we refer to as the partial wave analysis (PWA) coordinate system.

The angular distributions in the PWA coordinate system are expanded as before, according to Eq. (14b). The density matrix elements are given as a function of beam momentum in Fig. 19, for all three charge states, and the interference coefficients are given in Fig. 20. The corresponding angular distributions are given in Figs. 14 and 21. (The polar angle \(\gamma\) is the same in both coordinate systems, since they differ only by a rotation about the z axis; the azimuth in the PWA system is denoted by \(\delta'\).) We note that \(R_1\) is always small, consistent with zero, while \(R_2\) is substantial and varies with momentum.

We have already seen that the production angular distributions vary with the N\(\pi\) mass, indicating the presence of an N\(^*\)-background interference. We will now look for a similar effect in the N\(^*\) decay angular distributions. The angular coefficients are plotted as a function of \(m_{N\pi}^2\) in Figs. 22-24 for the three charge states. The only striking effect is in the interference coefficient \(R_2\), which increases with increasing mass in all three charge states. Since the interference terms involve background waves of opposite parity in the N\(\pi\) system, different background waves are involved here from those seen in the production angular distributions. The background must thus include components with the following properties:
Fig. 19. $N^*$ density matrix elements of Eq. (14b) in the PWA coordinate system, as a function of beam momentum.
$K^+ p \rightarrow K N^*$

density matrix elements in the PWA coordinate system

- $K^+ p \rightarrow K^0 \pi^+ p$
- $K^+ p \rightarrow K^+ \pi^0 p$
- $K^+ p \rightarrow K^+ \pi^+ n$

Fig. 20. The interference coefficients $R_1$ and $R_2$ as determined in the PWA coordinate system, as a function of beam momentum.
Fig. 21. Distributions in $\delta'$ for $N^*$ production, at 860, 960, 1200, and 1360 MeV/c. The curves correspond to the density matrix elements and interference coefficients given in the two previous figures.
Fig. 22. The $N^*$ density matrix elements and interference coefficients of Eq. (14b) in the PWA coordinate system, as a function of $N\pi$ mass, for the $K^0\pi_p$ final state.
$K^+ p \rightarrow K^+ N^* \pi^0 p$

density matrix elements in the PWA coordinate system

- $860 \text{ MeV/c}$
- $960 \text{ MeV/c}$
- $1200 \text{ MeV/c}$

Fig. 23. Same as Fig. 22, for the $K^+ p\pi$ final state.
\[ K^+ p \rightarrow K^+ N^{*+} \rightarrow \pi^+ n \]

density matrix elements in the PWA coordinate system

○ 860 MeV/c  ○ 960 MeV/c  □ 1200 MeV/c

Fig. 24. Same as Fig. 22, for the \( K^+ \pi n \) final state.
a) $l_j = P_{3/2}$ for the $N\pi$ system, and even $K^{-}(N\pi)$ orbital angular momentum (variation in $A_1/A_0$);

b) $l_j = P_{3/2}$ for the $N\pi$ system and odd $K^{-}(N\pi)$ orbital angular momentum (variation in $A_2/A_0$);

c) even $l$ for the $N\pi$ system (variation in $R_2$).

We now record one final property of the $N\pi$ angular distributions, an asymmetry in the distributions in $\cos \lambda_K$. As discussed in Section III, $\cos \lambda_K$ is proportional to position on the Dalitz plot along a line of fixed $m^2_{N\pi}$. An asymmetry in this distribution cannot arise from pure $N^*$ production, and probably indicates interference of the $N^*$ with a background wave. This asymmetry is closely related to the presence of the $R_2$ coefficient in the PWA coordinate system, but has the practical advantage that it can be obtained directly from the Dalitz plot. This asymmetry between high and low $K\pi$ mass, defined as $A \equiv (H - L)/(H + L)$, is shown in Fig. 25, for our data and for other low-momentum data.\textsuperscript{11,13} (Final states including the $K^*$ are not considered, since then the asymmetry would be due primarily to the presence of the $K^*$ on the high-$K\pi$ mass side of the Dalitz plot.)

Aside from the aforementioned interference effects, we can summarize the angular distributions as follows: The production angular distributions indicate the presence of large $P$ waves at threshold, with higher waves coming in smoothly with momentum and with all odd Legendre coefficients increasing monotonically with momentum. The decay angular distributions show no marked change with momentum, and deviate only slightly from the magnetic dipole $p$-exchange predictions, $\rho'_{33} = \text{Re} \rho'_{3,-1} = \text{Im} \rho'_{3,-1} = 0$. 
Fig. 25. The Dalitz plot asymmetry coefficient $A$ as a function of beam momentum.
C. $K^*(891)$ Production

We restrict our $K^*$ sample to be in the half of the Dalitz plot away from the $N^*$ ($\cos \lambda_N > 0$), within the $K\pi$ mass band 840-940 MeV. This still does not give complete separation from the $N^*$ at 1200 and 1360 MeV/c, but no more restrictive cuts can be made without biasing the angular distributions in a very complicated way. We have checked for effects due to the presence of the $N^*$ tail by choosing events in a narrow $K\pi$ mass band and comparing them with those in the wider band. Results using the mass band 870-910 MeV are not distinguishable within statistical errors from those given here. We will consider events only from the $K^0\pi^+p$ final state. Besides our own data we include for comparison the data of Bettini et al. at 1450 MeV/c, Bowler et al. at 1585 MeV/c, and S. Goldhaber et al. at 1960 MeV/c.

The production angular distributions are given in Fig. 26, and the corresponding Legendre coefficients in Fig. 27. The variation in all the Legendre coefficients with momentum is remarkably smooth, with the reaction becoming steadily more peripheral with increasing momentum. We will describe the decay of the $K^*$ in a coordinate system similar to that used for the $N^*$ (Fig. 12); the $z$ axis is along the normal to the production plane and the $x$ axis is along the direction of the incoming $K^+$ meson as seen in the $K\pi$ c.m. The $K^*$ decay angular distribution is given in terms of the density matrix elements as

$$W(\cos \alpha, \varphi) = \frac{3}{4\pi} \left[ \rho_{00} \cos^2 \alpha + \rho_{11} \sin^2 \alpha - \rho_{1,-1} \sin^2 \alpha \cos 2\varphi \right.$$  
$$\left. - \sqrt{2} \Re \rho_{10} \sin 2\alpha \cos \varphi \right],$$

(16)
Fig. 26. \( K^* \) production angular distributions at 1200, 1360, 1450, and 1580 MeV/c. The curves are exchange model predictions described in Section V.
\[ \frac{d\sigma}{d\Omega} (K^*) = \sum_{\ell} A_{\ell} P_{\ell}(\cos \Theta) \]

Fig. 27. Legendre coefficients for the \( K^* \) production angular distribution, as a function of beam momentum.
where $\rho_{11} = \frac{1}{2}(1 - \rho_{00})$. In Fig. 28 we show these density matrix elements as a function of momentum, and in Fig. 29 the distributions in $\cos \alpha$ and $\varphi$ are given. The curves correspond to the density matrix elements of Fig. 28. There is evidently little change in the $K^*$ polarization in this momentum region. The implications of these data in terms of exchange models are discussed in Section V.
Fig. 28. Density matrix elements for $K^*$ production, as a function of beam momentum. The curves are exchange model predictions described in Section V.
Fig. 29. $K^*$ decay angular distributions at 1200, 1360, 1455, and 1580 MeV/c. The curves correspond to the density matrix elements given on the previous page.
V. EXCHANGE MODELS

The exchange of a single meson is qualitatively appealing as the dominant process in medium- to high-energy particle interactions. Exchange model calculations have had encouraging successes in predicting the polarization of mesons produced by pion exchange and of N*’s produced by ρ exchange in reaction 2. The ρ-exchange model has failed, however, in predicting production angular distributions and cross sections over a wide range of incident momenta. This is not too surprising, since without additional s-dependent attenuation of the cross section the model violates unitarity. A good part of our data, however, is at energies where the unitarity limit is not important. Exchange model calculations are not usually considered valid near threshold, but the KN channel is unique in the absence of any established s-channel resonances, and may therefore be hoped to have simple t-channel properties.

In this section we will describe the single particle exchange models for K* and N* production in reaction 2 and compare their predictions with our data. The calculations for N* production by ρ exchange are carried out in Appendix B, and for the K* we use the results of Jackson and Pilkuhn.21

A. N*(1236) Production

The meson exchange diagram for N* production is shown in Fig. 30. I-spin conservation at upper and lower vertices requires I = 1 in the t channel, and conservation of angular momentum and parity at the upper vertex requires normal spin-parity: \( J^P = 0^+, 1^-, 2^+, \ldots \). We assume that the \( J^P = 1^- \) ρ meson dominates the cross channel.
Fig. 30. Feynman diagram for $N^*$ production by $\rho$ exchange.
The general amplitude for KN* production by p exchange is discussed
in Appendix B. We will consider here only two specific choices for the
coupling. First, we make the Stodolsky-Sakurai assumption of pure ML
coupling. \(^\text{25}\) This gives the simple matrix element

\[
\mathcal{M}(K^+ p \rightarrow KN^*) = \frac{1}{\pi} \frac{g_1}{\sqrt{4\pi}} \frac{g_2}{\sqrt{4\pi}} \frac{1}{e^2 - m_e^2} \times \left\{ \frac{1}{\sqrt{m_d^2}} \bar{u}(a) u(b) \gamma^\mu \gamma^\nu P \eta e^{\mu\nu\lambda\eta} \right\},
\]

(17)

where \( P = \frac{1}{2}(a + c) \), and the momenta \( a, b, c, d, \) and \( e \) are defined on
the Feynman diagram of Fig. 30. The resulting differential cross section
is

\[
\frac{d\sigma}{d\cos\theta d\Omega_{\text{decay}}} = \frac{q'^2}{32\pi} \frac{g_1^2 g_2^2}{4\pi} \left[ \frac{E_b + m_b}{S} \frac{S}{(e^2 - m_e^2)^2} \right] \left[ 1 + 3\cos^2 \gamma \right].
\]

(18)

The square-bracketed quantity contains the production angle dependence,
and the curly-bracketed expression describes the decay. A zero-width \( N^* \)
is assumed in Eq. (18). In our numerical calculations, however, we have
multiplied by the \( N^* \) Breit-Wigner (Eq. (A29) of Appendix A) and integrated
over the \( N\pi \) mass. The predictions for the ML model are simple:

\[
W(\cos \gamma) = 1 + 3\cos^2 \gamma,
\]

(19a)

\[
W(\theta) = \text{constant},
\]

(19b)

\[
W(\cos \theta) \approx \sin^2 \theta/(1 - b \cos \theta)^2.\]

(19c)

where

\[
b = \frac{2s q'q}{m_\rho^2 + 2(E_b E_b^* - m_\rho^2)}.
\]

(20)

\[
s = sq'^2 q' \sin \theta
\]

(21)

* We neglect the small dependence of \( [E_b + m_b]_{N^*c.m.} \) on \( \cos \theta \).
and $q$ and $q'$ are the c.m. momenta for the incoming and outgoing two-body systems. The expansion parameter $b$ is small near threshold, giving a $\sin^2 \theta$ production angular distribution at low momenta. The total cross section must be obtained by numerical integration of Eq. (18) over $m_{NN}^2$. We note, however, that the resulting cross section rises very rapidly near threshold, due to the $q'^2$ dependence of the p-wave threshold term.

The matrix element of Eq. (17) is probably the best covariant formulation of the magnetic dipole coupling. There is, however, another coupling, proposed by Jackson and Pilkuhn as the correct relativistic generalization of the $M_1$ coupling. The Jackson-Pilkuhn ($J\!P$) coupling reduces to that of Eq. (17) in the "static" limit of $m_b = m_d$, $t = 0$. This coupling is obtained by replacing the curly-bracketed expression in Eq. (17) by

$$\text{SPIN}_{J\!P} = \frac{1}{m_d} \left\{ \bar{u}_\mu (d) i Y^\lambda Y^\nu u(b) \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \r...
Fig. 31. Experimental cross sections for $N^*$ production as a function of beam momentum, and the predictions of $\rho$ exchange with $M_1$ and $J\pi$ couplings.
We also show the ML predictions with the product of coupling constants, Eq. (23), taken to be 450. This gives a good fit near threshold, but the large value for the coupling constants is quite implausible. Unless the SU(3) relation leading to Eq. (23) is badly broken, this result is inconsistent with simple rho exchange. However, the good agreement with the threshold energy dependence of the cross section reaffirms the P-wave nature of the production process.

The production angular distributions predicted by these models are shown in Fig. 9, together with our data. The agreement is fairly good; both models predict largely P-wave production, as already suggested by the N\(\pi\) mass distribution. The JP coupling has an L2 term, multiplied by a coefficient proportional to t, which causes a backward asymmetry near threshold not seen in the data. The predicted forward peaking of the N\(^*\) production is of course due to the presence of the \(\rho\) propagator, and the good agreement with experiment up through 1360 MeV/c confirms that the exchanged particle is a heavy one such as the \(\rho\) meson. It is significant that near threshold no form factor is necessary to fit the data. This is inconsistent with the various form factors proposed to fit higher energy data.\(^{17,19,24}\)

We now examine the decay angular distributions. In terms of the density matrix elements, the magnetic dipole model predicts

\[ \rho_{33}' = \text{Re} \rho_{3,-1}' = \text{Im} \rho_{3,-1}' = 0. \quad (24) \]

The JP predictions are more complicated, and are indicated in Fig. 13. (In calculating the JP density matrix elements we have not averaged over N\(\pi\) mass, but have used the peak masses given in Table XII.)
Fig. 13 we see rather good agreement with the Ml predictions, from threshold to 3 BeV/c. The small discrepancies which do occur are primarily near threshold, where our data lie. In Figs. 14 and 15 the Ml and JP predictions are compared with the experimental cos \( \gamma \) and \( \delta \) distributions. Both models fit the cos \( \gamma \) distribution rather well, but do not reproduce the details of the \( \delta \) distribution. In Fig. 16 the predicted momentum transfer dependence of the density matrix elements is compared with our data; the curves are the JP predictions, and the Ml predictions are all zero. Neither model reproduces the data very well. In Table XIII we give the contributions of the various multipole terms to the density matrix elements. The negative experimental values of \( \text{Im} \rho_{3,-1}' \) indicate the presence of an L2 multipole component, as predicted by the JP model.

We can now draw the following conclusions about the \( \rho \)-exchange models considered:

a) The exchange models predict a rapidly rising cross section for \( N^* \) production which exceeds the unitarity limits on the lower partial waves not far above threshold; this leads to larger cross sections and less forward collimation of the \( N^* \) production than that observed.

b) Near threshold where the unitarity limit is not important the production angular distribution and the observed rate of increase of the cross section with momentum agree rather well with the model's predictions, for pure Ml coupling. The experimental cross section near threshold is much larger than predicted, however, requiring coupling constants much larger than the known
<table>
<thead>
<tr>
<th>Multipole term</th>
<th>$\rho_{33}^3$</th>
<th>Re $\rho_{3,-1}^3$</th>
<th>Im $\rho_{3,-1}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>M_1</td>
<td>^2$</td>
<td>0</td>
</tr>
<tr>
<td>$</td>
<td>E_2</td>
<td>^2$</td>
<td>1/2</td>
</tr>
<tr>
<td>$</td>
<td>L_2</td>
<td>^2$</td>
<td>3/8</td>
</tr>
<tr>
<td>$2 M_1 \cdot E_2$</td>
<td>$\pm 1/2$</td>
<td>0</td>
<td>$\sqrt{3}/8$</td>
</tr>
<tr>
<td>$2 M_1 \cdot L_2$</td>
<td>0</td>
<td>$\mp 1/4$</td>
<td>$\sqrt{3}/8$</td>
</tr>
<tr>
<td>$2 E_2 \cdot L_2$</td>
<td>0</td>
<td>$\mp 1/4$</td>
<td>$\sqrt{3}/8$</td>
</tr>
</tbody>
</table>
p coupling constants. Whether this is a criticism of the p-exchange hypothesis or merely of the extrapolation of the p vertex functions to negative \( t \) is not clear.

c) The \( N^* \) spin-density matrix is well approximated by the magnetic dipole predictions at all energies. This is especially remarkable in view of the evidently large unitarity modifications in the cross section and production angular distributions.

In short, the p-exchange model requires some further theoretical "fixing up" to fit the data. One model which attempts this has been proposed by Gerald Hite.\(^{27}\) He uses the K-matrix formalism to treat the problem of three coupled (quasi-) two-body states: \( K^+p, KN^*, \) and \( K^*N \). Even with a number of approximations, including assuming a zero-width \( N^* \), he succeeds in qualitatively reproducing most features of the \( N^* \) production.

Below about 1 BeV/c his model reduces to the simple exchange model which we have described. The \( N^* \) is produced largely in P waves with a rapidly rising cross section. However, as the large P waves approach their unitary limits, \( \sigma_u(J) = (J + \frac{1}{2})\pi\lambda^2 \), his model correctly predicts the fall-off and subsequent decrease of the cross section. He finds the Cool bump in the total cross section then appearing as a consequence of the bump in the \( N^* \) production cross section. We feel that this calculation removes any necessity to postulate a resonance in the \( K^+p \) channel near the Cool peak.

B. \( K^*(891) \) Production

\( K^* \) production in reaction 2 can proceed through exchange of a pseudoscalar meson \((\pi, \eta)\) or a vector meson \((\rho, \omega, \phi)\). (We neglect the
exchange of higher-spin resonances such as the $A_2$ meson.) As shown below, pseudoscalar and vector exchange result in quite different decay angular distributions for the $K^*$. We use the usual decay angles $\alpha$ and $\varphi$, defined as polar and azimuthal angles with respect to the incident $K^+$ direction, as seen in the $K^*$ c.m. The $\cos \alpha$ distribution for pseudoscalar exchange is pure $\cos^2 \alpha$, corresponding to $\rho_{00} = 1$, while vector exchange gives pure $\sin^2 \alpha$, $\rho_{00} = 0$. From Figs. 28 and 29 it is obvious that in the exchange model interpretation the dominant process is vector exchange, especially near threshold. We can go one step further by using the values of $\rho_{00}$ from Fig. 28 to divide the $K^*$ production cross section into pseudoscalar- and vector-exchange contributions: $\sigma_p/\sigma_v = \rho_{00}/(1 - \rho_{00})$. In Fig. 32 we plot as a function of momentum the $K^*$ production cross section and its particle-exchange components, obtained as described above. (We have minimized the effects of experimental errors by using values of $\sigma$ and $\rho_{00}$ from the hand-drawn interpolating curves in Figs. 28 and 32.)

For the theoretical exchange model predictions we quote here the results of Jackson and Pilkuhn:  

$$\frac{d\sigma_p}{d \cos \theta \, d\Omega_{\text{decay}}} = \frac{2\pi \rho_{00}^2}{\sin^2 \theta} \frac{G^2}{4\pi} \frac{G^2}{4\pi} \frac{1}{4m_K^2} \left\{ (t - (m_K - m_{K^*})^2)(t - (m_{K^*} - m_K)^2) \right\} \times (-t) \left\{ \left| \frac{F_p(t)}{m_p^2} \right|^2 \right\} \left[ \frac{3}{4\pi} \cos^2 \alpha \right]$$

(25)
Fig. 32. Experimental cross section for $K^*$ production as a function of beam momentum, and various theoretical predictions as discussed in Section V.
\[
\frac{d\sigma_V}{d \cos \theta d \Omega_{\text{decay}}} = \frac{\pi q^2 / s q}{4 \pi m_{K^*}^2} \left| \frac{F_V(t)}{m_V^2} \right|^2 \left\{ \frac{(G_V + G_T)^2}{4 \pi} \frac{1}{4} \left( m_K - m_{K^*} \right)^2 - t \right\} \\
\times \left( \left( m_K - m_{K^*} \right)^2 - t \right) (-t) \left[ \frac{3}{8 \pi} \sin^2 \alpha \right] \\
+ 2sq^2 q'^2 \sin^2 \theta \left( \frac{G_V}{4 \pi} + \frac{G_T}{4 \pi} \frac{(-t)}{4m_p^2} \right) \left[ \frac{3}{8 \pi} \sin^2 \alpha \sin^2 \varphi \right].
\]

Here \( q \) and \( q' \) are the incoming and outgoing three-momenta in the overall c.m., and \( \theta \) is the c.m. production angle of the \( K^* \). For pseudoscalar exchange we assume pion exchange, and take \( G^2 / 4 \pi = 15, \ g^2 / 4 \pi = 0.75 \). These values include the Clebsch-Gordon coefficients appropriate to the reaction \( K^+ p \to K^{*+} p \), including both charge states for the \( K^{*+} \) decay.

As the pion form factor we take

\[
F_P(t) = \frac{(\Lambda - m_{\pi}^2)}{(-t)}, (27)
\]

where \( \Lambda = 0.165 \ \text{(BeV/c)}^2 \). This form factor was used successfully by Goldhaber et al.\textsuperscript{28} to predict the momentum transfer distribution and cross sections in the reaction \( K^+ p \to K^{*+} p \) at 1.96 BeV/c. For vector exchange we assume that only the \( \omega \) meson is exchanged. The \( \rho \) meson is excluded because of the apparent absence of vector exchange in charge-exchange production of the \( K^{*0} \) in 2.3 BeV/c \( K^+ p \) interactions.\textsuperscript{29} There is little reason to exclude the \( \phi \), but our conclusions would be affected little by its presence. Following Jackson,\textsuperscript{21} we take for the \( \omega p p \) coupling constants \( G_T = 0 \) and \( G_V = 2.77 \), and \( f^2 / 4 \pi = 4.3 \). For the form factor we choose an exponential form which reasonably well reproduces our data at 1200 MeV/c;

\[
F_V(t) = e^{t/\tau},
\]
with $\tau = 0.8 \text{(BeV/c)}^2$. A stronger form factor, with $\tau = 0.49 \text{(BeV/c)}^2$, was required to fit the CERN 3.0 BeV/c data for this reaction.\textsuperscript{24} We take the view, however, that the higher-energy form factor is a phenomenological representation of absorption effects (as ours may be), and is not applicable near threshold.

We will now compare the data with the predictions of two models; the theoretical model just described, and a "semitheoretical" model obtained by adjusting the exchange model coupling constants at each momentum so as to give the experimentally observed values of $\rho_{00}$ and the cross section. This seems a reasonable modification of the theoretical model, since exchange models have had their greatest successes in predicting angular distributions and their greatest failures in predicting cross sections.

In Fig. 32 we show the cross section contributions predicted by the two calculations. The theoretical curves illustrate a familiar problem encountered in fitting low-energy $K^+$ production; near threshold pion exchange should dominate over vector exchange, whereas the data shows vector exchange dominating. We know of no solution for this problem. It is interesting to note that if we abandon pion exchange completely and multiply the theoretical vector-exchange curve by a factor of 3.8 (dotted curve), the threshold behavior of the cross section is reproduced rather well.

The semitheoretical predictions for the production angular distributions are shown in Fig. 26, along with their pion-exchange and vector-exchange components. (The pion-exchange component is distinguished by its sharp forward peak.) The fit is fairly good. It is furthermore clear that a large pion-exchange component near threshold would be
inconsistent with the data. The two sets of predictions for the density matrix elements are shown in Fig. 28. The theoretical dominance of pion exchange is seen in the large value of $\rho_{00}$ predicted near threshold, which the semitheoretical model is designed to correct. Otherwise both sets of predictions are plausible; detailed agreement would certainly require careful fitting of the form factors and the vector-exchange coupling constants, which we have not attempted.

From the above observations it is tempting to draw a simple picture of $K^*$ production, very similar to that for the $N^*$:

1) Near threshold vector exchange dominates; the coupling constants are a factor of four larger than expected, and only a weak form factor is required. This predicts the cross section and production angular distribution correctly near threshold, and qualitatively reproduces the decay angular distributions.

2) Shortly above threshold unitarity effects become important, first attenuating the cross section and later causing the reaction to become more peripheral than predicted by the Born term.

The role of pion exchange in this reaction is unclear. The production of the $K^{*0}$ in the reaction $K^+ n \to K^{*0} p$ has been attributed largely to pion exchange, implying the presence of some pion exchange in our reaction. It remains a mystery why this does not lead to large pion-exchange contributions near threshold.
VI. PARTIAL WAVE ANALYSIS OF THE KN* FINAL STATE

The precise measurement of the total K⁺p cross section by Cool and coworkers² (see Fig. 2) has raised the important question of whether or not there are resonances in the I = 1 KN system. Such a resonance would have to belong to a 27 representation of SU(3), a representation not required by the other known particles, and in terms of its quark structure would have to be made from at least five quarks. Elastic K⁺p scattering in the region of the Cool bump has not revealed evidence for a resonance,³⁰ but a conclusive analysis is difficult because of the absence of proton polarization information. Furthermore, if the Cool bump is a resonance, it decays mostly into inelastic final states. For a total resonant cross section of 4 mb, as estimated by Cool et al., the implied elasticity is 0.3 for a resonance fed by an incoming J = 1/2 wave, and less for higher-J waves. The KN* final state is a promising candidate for the decay of such a resonance, as its cross section peaks just above threshold, nearly under the Cool bump. In this section we will examine the KN* final state for evidence of resonant behavior.

Ideally in searching for a resonance one should perform a general partial wave fit, using all low order partial waves. Unfortunately, this is not possible; as in K⁺p elastic scattering, we suffer from the absence of data on the final state nucleon polarization. We can, however, do an analysis using a restricted set of waves which is still general enough to reproduce the features of our data. As already noted, the distribution in N* mass and production angle suggest the dominance of P waves near threshold. The magnetic dipole ρ-exchange model reproduces this feature and also fits the angular distributions fairly well. Our
minimal set of partial waves should then include the lowest-order waves of the MI model. To allow for complete freedom in the lower partial waves we include $P_{1/2}$ and $P_{3/2}$ waves, to be varied independently, and the amplitude $M_{l}'$, defined as the MI amplitude with its $P_{1/2}$ and $P_{3/2}$ components subtracted out. This $M_{l}'$ amplitude represents our approximation to all partial waves higher than $P_{3/2}$. These three waves can reproduce most of the features of the $K^{0^+}p$ final state. One feature not explained, however, is the apparent interference with a non-$N^*$ background amplitude. We therefore add the simplest background amplitude, s-wave in all particle pairs, which we call $S_{l}$. (This wave is fed by the incident $P_{1/2}$ state, and has the same $J^P$ as our $Pl$ wave.) This wave interferes with the $N^*$ amplitude to give the observed Dalitz plot asymmetry towards high $K\pi$ mass, or equivalently the negative sin $\delta'$ moment in the $N^*$ decay angular distribution. (These effects were discussed in Section IV.) Finally we include a $S_{3/2}$ $K^{*}N^*$ final state, coming from an incident $D_{3/2}$ $K^+p$ state. This wave is not important in our analysis, but we include it because of its a priori attractiveness as a resonant state: the high inelasticity of the proposed resonance could be explained by the absence of an angular momentum barrier in the $S_{3/2}$ $K^{*}N^*$ final state, as compared with the competing $D_{3/2}$ elastic final state.

The partial waves used in our analysis are given below in Table XIV, with their corresponding angular distributions. We have also included in the table the one remaining low-order partial wave, the $S_{1/2} \rightarrow D_{1/2}$ transition.

Before carrying out a fit, we can ask one very important question; can the bump in the $K^{*}N^*$ cross section arise from a single $J^P$ state?
Table XIV. Low-order partial waves for the reaction $K^+ p \rightarrow KN^*$ and their corresponding distributions in $N^*$ production and decay angles.

<table>
<thead>
<tr>
<th>$K^+ p$ initial state</th>
<th>$KN^*$ final state</th>
<th>$J^P$</th>
<th>Name</th>
<th>Production angular distribution</th>
<th>Decay angular distributions $\cos \gamma$</th>
<th>$d^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{3/2}$</td>
<td>$S_{3/2}$</td>
<td>3/2$^-$</td>
<td>S3</td>
<td>Isotropic</td>
<td>$1 - \frac{3}{5} \cos^2 \gamma$</td>
<td>$1 + \frac{1}{2} \cos 28'$</td>
</tr>
<tr>
<td>$P_{1/2}$</td>
<td>$P_{3/2}$</td>
<td>1/2$^+$</td>
<td>P1</td>
<td>Isotropic</td>
<td>$1 - \frac{3}{5} \cos^2 \gamma$</td>
<td>$1 - \frac{1}{6} \cos 28'$</td>
</tr>
<tr>
<td>$P_{3/2}$</td>
<td>$P_{1/2}$</td>
<td>3/2$^+$</td>
<td>P3</td>
<td>$1 - \frac{4}{5} P_2(\cos \theta)$</td>
<td>$1 + \frac{21}{13} \cos^2 \gamma$</td>
<td>$1 + \frac{11}{30} \cos 28'$</td>
</tr>
<tr>
<td>$S_{1/2}$</td>
<td>$D_{1/2}$</td>
<td>1/2$^-$</td>
<td>D1</td>
<td>Isotropic</td>
<td>$1 - \frac{3}{5} \cos^2 \gamma$</td>
<td>$1 - \frac{1}{6} \cos 28'$</td>
</tr>
<tr>
<td>$P_{1/2}$</td>
<td>$P_{1/2}$</td>
<td>1/2$^+$</td>
<td>S1</td>
<td>Isotropic</td>
<td>Isotropic</td>
<td>Isotropic</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ML'</td>
<td>Isotropic</td>
<td>$1 + 3 \cos^2 \gamma$</td>
<td>Isotropic</td>
</tr>
</tbody>
</table>
If so, this would make a strong case for the existence of a $S = 1$ baryon resonance, decaying predominantly into the $K\bar{N}^*$ final state. In Fig. 33 we compare our data for reaction (2) at 960 MeV/c with the predictions for the various $K\bar{N}^*$ states given in Table XIV. We see that the only pure state consistent with the $\cos \theta$ and $\cos \gamma$ distributions is the $P_{3/2}$ state, while the delta prime distribution rejects this hypothesis. The same features are present in our data at our other three momenta. It is thus clear that a superposition of waves is required to fit the data.

We have attempted to fit our data for reaction 2 separately at each momentum, using five waves; $S_1, S_3, P_1, P_3,$ and $M_1'$. The detailed forms of the amplitudes used and the parametrization of the energy dependence are given in Appendix C. The input information was the one-dimensional distributions in $\cos \theta$, $\cos \gamma$, $\delta'$, and $N\pi$ mass, with the sample selected as described in Section IV. We have also used published data at 1580 MeV/c. No correlations were included and in particular the variation of the angular distributions with $N\pi$ mass was ignored. The $\delta'$ distribution had to be folded at 1200 and 1360 MeV/c, since the $N^*$ sample was taken from only half of the Dalitz plot. The effect in the folded distributions of interference between $N^*$ and background amplitudes is very difficult to calculate, and our analysis assumes that this interference is not present at 1200 MeV/c and higher momenta. There the $S_1$ wave is treated as an incoherent phase-space-like background.

The input data and fitted curves are given in Figs. 34a-e. The fits are quite good except at 1580 MeV/c. There the $M_1'$ amplitude accounts for about 1/3 of the cross section, and more flexibility in the higher waves may be required to fit the data well. The cross section contribution
Fig. 33. \( N^* \) production and decay angular distributions at 960 MeV/c, and the predictions for production in various low-order partial waves.
$$K^+ p \to K^0 \pi^+ p$$

860 MeV/c

$$\chi^2 = 35$$ for 40 d.o.f.

Fig. 34a. 860 MeV/c $N\pi$ mass distribution and $N^*$ production and decay angular distributions used as input to the partial wave analysis of the $KN^*$ final state, and the curves corresponding to the partial wave solutions.
\[ \text{K}^+\text{p} \rightarrow \text{K}^0\pi^+\text{p} \]

960 MeV/c

\( \chi^2 = 64 \text{ for 68 d.o.f.} \)

![Graphs showing mass and angular distributions](image)

Fig. 34b. 960 MeV/c \( \text{N}^* \) mass distribution and \( \text{N}^* \) production and decay angular distributions used as input to the partial wave analysis of the \( \text{K}^+\pi^0 \) final state, and the curves corresponding to the partial wave solutions.
$K^+ p \rightarrow K^0 \pi^+ p$

1200 MeV/c

$\chi^2 = 75$ for 62 d.o.f.

Fig. 34c. 1200 MeV/c $N\pi$ mass distribution and $N^*$ production and decay angular distributions used as input to the partial wave analysis of the $KN^*$ final state, and the curves corresponding to the partial wave solutions.
Fig. 34d. 1360 MeV/c N\pi mass distribution and \n* production and decay angular distributions used as input to the partial wave analysis of the KN* final state, and the curves corresponding to the partial wave solutions.
Fig. 34e. 1580 MeV/c \( \Lambda \kappa \) mass distribution and \( \Lambda^* \) production and decay angular distributions, taken from Ref. 18, used as input to the partial wave analysis of the \( \Lambda \kappa^* \) final state, and the curves corresponding to the partial wave solutions.
and phase for each partial wave is given in Table XV. We have normalized the sum of the cross section contributions to the sum of the $N^*$ production and background cross sections as given in Table X; contributions from $K^*$ production and $N^*-K^*$ interference are thus excluded. The corresponding partial wave amplitudes are displayed in Fig. 35 in the Tripp representation, plotting the amplitudes in the complex plane. The phases of the $K\pi$ amplitudes are of course dependent on the $N\pi$ mass. In fitting the data we have evaluated the phase at the mass corresponding to the peak of the $N\pi$ mass distribution, as given in Table XII, and we use that phase in Fig. 35. One phase is arbitrary, so we take the $M_{11'}$ amplitude to be real, since it represents a $p$-exchange background which should not change phase rapidly. The amplitudes are normalized so that the unit circle on Fig. 35 represents complete absorption of the appropriate incoming wave;

$$
\sigma = \pi\lambda^2 \left[ S_1^2 + 2S_3^2 + P_1^2 + 2P_3^2 + 2M_{11'}^2 \right] \quad (29)
$$

There is an ambiguity in the amplitudes determined by our fit due to the fact that only complex "dot products" of amplitudes appear in the calculations; all phases could be reversed in sign without changing the fit. The finite width of the $N^*$, however, allows us to resolve this ambiguity. If we assume that the background has a fairly constant phase, the $N^*$-background relative phase is a strong function of $N\pi$ mass. We have seen in Fig. 22 that the $R_2$ coefficient varies rapidly with $N\pi$ mass. In the partial wave analysis the $\sin \delta'$ term which $R_2$ multiplies comes largely from the interference term $S_1'(2P_1 + P_3)$. From Fig. 35 this dot product is seen to be negative at 860 and 960 MeV/c. With increasing $N\pi$ mass
Table XV. Partial wave solutions, for the $K^{0}\pi^{+}p$ final state; each double entry gives the cross-section contribution and the phase, relative to the $M_{1}^{'}$ amplitude.

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>$\pi\kappa^2$ (mb)</th>
<th>$S_{1}$</th>
<th>$S_{3}$</th>
<th>$P_{1}$</th>
<th>$P_{3}$</th>
<th>$M_{1}^{'}$</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\chi^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>860</td>
<td>5.5</td>
<td>0.21±0.05</td>
<td>0.01±0.01</td>
<td>0.46±0.05</td>
<td>0.44±0.04</td>
<td>0.08±0.03</td>
<td>1.20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.3±10</td>
<td>120±40</td>
<td>-70±12</td>
<td>-51±9</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>960</td>
<td>4.7</td>
<td>0.12±0.05</td>
<td>0.05±0.04</td>
<td>0.90±0.08</td>
<td>1.30±0.07</td>
<td>0.26±0.06</td>
<td>2.63</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.0±20</td>
<td>175±12</td>
<td>95±6</td>
<td>68±4</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>3.4</td>
<td>0.39±0.10</td>
<td>0.10±0.07</td>
<td>1.10±0.13</td>
<td>1.43±0.11</td>
<td>0.44±0.08</td>
<td>3.46</td>
</tr>
<tr>
<td></td>
<td></td>
<td>--</td>
<td>173±12</td>
<td>84±8</td>
<td>54±5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1360</td>
<td>2.9</td>
<td>0.21±0.06</td>
<td>0.20±0.11</td>
<td>0.43±0.13</td>
<td>1.42±0.15</td>
<td>0.50±0.08</td>
<td>2.76</td>
</tr>
<tr>
<td></td>
<td></td>
<td>--</td>
<td>170±15</td>
<td>89±12</td>
<td>50±5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1580</td>
<td>2.4</td>
<td>0.56±0.07</td>
<td>0.13±0.10</td>
<td>0.02±0.03</td>
<td>1.17±0.15</td>
<td>1.02±0.10</td>
<td>2.90</td>
</tr>
<tr>
<td></td>
<td></td>
<td>--</td>
<td>160±40</td>
<td>42±5</td>
<td>19±6</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 35. Argand plot of the partial wave amplitudes for $N^*$ production.
the Pl and P3 amplitudes would move in a counterclockwise direction in Fig. 35, becoming orthogonal to Sl at a Nπ mass value somewhat above the peak position. From Fig. 22 we see that R₂ is negative at the central Nπ mass values, and vanishes at higher Nπ mass. Thus, in the convention that the N* phase increases with increasing Nπ mass, the sign of the phases in Fig. 35 has been chosen correctly.

In Fig. 35a totally resonant amplitude would start at the center of the plot, traverse a counterclockwise circle with increasing energy, and return to the center. A resonant amplitude superimposed on a constant background would follow a circular trajectory which would not necessarily pass through the center. 960 and 1360 MeV/c are respectively about one half-width below and above the center of the peak in the total cross section as estimated by Cool et al., and span the peak in the N* production cross section. One would thus expect a pure resonant amplitude to go through a phase change of about 90° between 960 and 1360 MeV/c. We see from Fig. 35 that in fact the dominant Pl and P3 amplitudes are nearly stationary in this region, over a range in total c.m. energy of 170 MeV, and the S3 amplitude is always small. (The total c.m. energies corresponding to the five momenta analyzed are: 1730, 1780, 1890, 1960, and 2060 MeV.) Our analysis thus shows no substantial resonant component in any partial wave amplitude in the region of the Cool peak.

A feature of the P3 trajectory in Fig. 35 which must be noted is the large phase change between 1360 and 1580 MeV/c. This could indicate an uncompleted resonant trajectory, somewhat above the Cool peak and with a very large width. More probably it is due to the breakdown of the validity of the simple set of amplitudes that we have chosen. It is a
deficiency of the Ml amplitude that it is less peripheral at higher energies than the data. Our fit at 1580 MeV/c is probably compensating for this deficiency in the Ml' amplitude by maximizing the $2 \text{Re}(P^*_3 M^*_1)$ term, which gives a forward asymmetry.

In the K-matrix treatment of this channel it is the restrictions of unitarity which cause the rapidly rising vector-exchange cross section to turn over at about 1200 MeV/c. It is instructive to compare the cross section for those amplitudes which come from the incident $P_{1/2}$ wave with the maximum inelastic cross section for this wave, $\sigma^2$. At 1200 MeV/c the sum of Sl and Pl cross sections (both come from the incident $P_{1/2}$ wave) is, from Table XV, 1.49 mb, to be compared with $\sigma^2 = 3.4$ mb. If we suppose that the remainder of the single-pion-production cross section has the same partial wave composition as the subsample used in the partial wave analysis, we find $1.49 \text{ mb} \times 7.5/3.46 = 3.2 \text{ mb}$ coming from the incident $P_{1/2}$ wave, very close to the unitarity limit of 3.4 mb. Clearly the Pl wave must be feeling unitarity effects at 1200 MeV/c.

We note that the partial wave solutions at the three lower momenta have Pl and P3 present in about a 1:1 ratio, instead of the 1:5 ratio predicted by the magnetic dipole model. It is this difference that accounts for the main deviations of the decay angular distributions from the magnetic dipole predictions.

To summarize the results of the phase shift analysis, our simple partial wave model gives a good representation of the experimental data from 860 to 1360 MeV/c, throughout the region of the Cool peak. $N^*$ production is dominated by the $P_{1/2}$ and $P_{3/2}$ states, in roughly equal amounts near threshold and in more nearly the magnetic dipole ratio of 1:5 at
higher momenta. No rapid phase variation is seen in either of the dominant amplitudes in the region of the Cool peak. Thus, while a small resonant component in either of the P waves cannot be ruled out, the 1150 MeV/c peak in the N* production cross section is clearly not due primarily to a single resonant amplitude.
VII. K*-N* INTERFERENCE

When two different resonances, the K* and the N*, are produced in the same three-body final state, there are events seen in the bubble chamber which may have passed through either the KN* or the K*N intermediate state, specifically those events in the region of the Dalitz plot where the N* and K* bands cross. There the intensity is not just the sum of squares of the N* and K* production amplitudes, but includes an interference term. Such an effect can be seen in the Dalitz plot for \( K^+p \rightarrow K^0\pi^+p \) at 1200 MeV/c, where there is constructive N*-K* interference. To make a quantitative estimate of this effect we consider the four regions of the Dalitz plot shown in Fig. 36. Region I is in the N*-K* overlap region (\( 1.35 < M_{N\pi}^2 < 1.55 \), \( 0.73 < M_{KN}^2 < 0.83 \)), regions II and III are the mass-conjugate regions to I in the N* and K* bands, respectively, and region IV is mass-conjugate to region III in the N\_c.m. Because of mass conjugation invariance within a resonance band for a pure parity state, we should have, neglecting background and interference, \( N_I = N_{II} + N_{III} \); or, using region IV to subtract possible uniform background, \( N_I + N_{IV} = N_{II} + N_{III} \). Using the numbers from Fig. 36 and converting to cross section, we have \( N_I + N_{IV} - N_{II} - N_{III} = (0.30 \pm 0.06) \text{mb} \). Similarly we get for the K\(^+\)\(\pi^0\)\(p\) final state at 1200 MeV/c, \( (0.10 \pm 0.03) \text{mb} \); and for the K\(^0\)\(\pi^+\)\(p\) final state at 1360 MeV/c, \( (0.24 \pm 0.07) \text{mb} \). The excess in the overlap region is thus always positive, and quite significant in the K\(^0\)\(\pi^+\)\(p\) final state. We attribute this excess to constructive N*-K* interference in the overlap region.

An analysis of this phenomenon has been carried out, largely by George Trilling,\(^31\) and published.\(^32\) In his analysis the N* was assumed.
Fig. 36. Dalitz plots for the $K^0\pi^+p$ final state at 1200 MeV/c, illustrating constructive interference between $K^*$ and $N^*$ production.
to be produced only in $P_{1/2}$ and $P_{3/2}$ waves, in the ratio given by the magnetic dipole model. The $K^*$ was also assumed to be produced in $P$-waves, in the ratio given by vector exchange with $G_V = -G_T$. The density for given values of $M_{N\pi}^2$ and $M_{K\pi}^2$ was calculated using coherent $N^*$ and $K^*$ amplitudes and an incoherent phase space background, averaging over the two remaining kinematical variables. By fitting to the Dalitz plot density the $N^*$ and $K^*$ amplitudes and the amount of background were determined. From these were calculated cross sections for $N^*$ production (with the $K^*$ amplitude turned off), $K^*$ production (with the $N^*$ turned off), $N^*$-$K^*$ interference, and background. These numbers are given in Table XVI along with the phase angle $\phi_0$. $\phi_0 \equiv \phi_{N^*} - \phi_{K^*}$ is the phase angle between the $N^*$ and $K^*$ amplitudes, not including the phase of the complex $N^*$ and $K^*$ propagators. The 1200 MeV/c results in Table XVI are taken from Ref. 32, with the total cross section for each reaction adjusted to agree with our values, given in Table VIII. The 1360 MeV/c results in Table XVI were obtained by carrying out Trilling's analysis on our data at that momentum.

The phase $\phi_0$ is seen to lie always in the region of 30-50 degrees. The $N^*$, having a large width and a small $Q$ value, is quite asymmetric, peaking at about 1212 MeV with a Breit-Wigner phase of about 65°. The additional phase $\phi_0$ thus makes the two resonances interfere most constructively just where the highest-density parts of the bands cross. It also arranges, however, to have the interference between the tails of the two resonances maximally destructive in the low $K\pi$ mass, high $N\pi$ mass quadrant of the Dalitz plot. This interference is not negligible, and in fact depletes that quadrant of the Dalitz plot sufficiently that the fit
Table XVI. Phase angle and cross sections for $K^*-N^*$ interference fits.
The 1200 MeV/c results are taken from Ref. 32.

<table>
<thead>
<tr>
<th>Momentum (MeV/c)</th>
<th>Reaction</th>
<th>$\phi_0$ (degrees)</th>
<th>Cross sections (mb)</th>
<th>N*</th>
<th>K*</th>
<th>Interference</th>
<th>Background</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1200</td>
<td>$K^+ p \rightarrow K^0 \pi^+ p$</td>
<td>40±11</td>
<td></td>
<td>2.9±0.3</td>
<td>0.87±0.14</td>
<td>0.3±0.2</td>
<td>1.0±0.3</td>
<td>5.1±0.3</td>
</tr>
<tr>
<td>1200</td>
<td>$K^+ p \rightarrow K^+ \pi^0 p$</td>
<td>57±17</td>
<td></td>
<td>0.60±0.13</td>
<td>0.59±0.11</td>
<td>-0.05±0.15</td>
<td>0.6±0.1</td>
<td>1.8±0.1</td>
</tr>
<tr>
<td>1360</td>
<td>$K^+ p \rightarrow K^0 \pi^+ p$</td>
<td>29±11</td>
<td></td>
<td>2.4±0.3</td>
<td>1.3±0.2</td>
<td>0.7±0.15</td>
<td>0.9±0.2</td>
<td>5.3±0.3</td>
</tr>
</tbody>
</table>
requires a substantial phase space background. In Table IX we compare the interference fits with several other fitting hypotheses, all of which neglect $N^*-K^*$ interference. The no-interference fit using all of the Dalitz plot but the overlap region agrees reasonably well with the interference fit, except that most of the phase space of the interference fit is divided among the other processes. It is our feeling that the large phase space contribution required by the interference fit is not real, but indicates that the amplitudes assumed for the $K^*$ and $N^*$ are not general enough to give a good fit over the entire Dalitz plot.
VIII. SUMMARY

Single pion production in the $K^+ p$ channel is strongly dominated by the $K N^*$ and $K^* N$ quasi-two-body final states, with very little nonresonant background. Thus, out of necessity, our analysis has emphasized resonance production.

Most of our events can be attributed to the reaction $K^+ p \rightarrow K N^*$. This reaction shows the following features:

a) The production near threshold is largely via $P$ waves, as predicted by the magnetic dipole $p$-exchange model. This is indicated by the $N\pi$ mass distribution, the rate of rise of the cross section, and by the strong $\sin^2 \theta$ component in the production angular distribution.

b) However, the vector-exchange squared coupling constants as determined near threshold are about five times larger than expected on theoretical grounds; and the cross section ratio for the $P_{1/2}$ and $P_{3/2} K N^*$ states, as determined from the decay angular distributions, is about 1 near threshold, instead of 1/5 as predicted by the M1 model.

c) Despite the variation in the $P$-wave amplitudes, the $N^*$ decay angular distributions vary little as a function of momentum, and are always well approximated by the M1 predictions:

$$W(\cos \gamma) = 1 + 3 \cos^2 \gamma, \quad W(\theta) = \text{isotropic}.$$  

The $K^*$ production is similar in many respects to the $N^*$ production.

a) Near threshold the $K^*$ is produced in low partial waves, with a rapidly rising cross section. The production angular distributions are consistent with vector exchange, which is largely $P$-wave near threshold.
b) The vector exchange squared coupling constants as determined near threshold are about four times larger than predicted theoretically.

c) The decay angular distributions are characteristic of vector exchange, with little change from threshold to the high energy region. In fact, the absence of pion exchange near threshold is quite puzzling.

The weakness of vector exchange models is that the predicted cross section rises so rapidly as to violate unitarity. The K-matrix calculation of Hite$^{27}$ has demonstrated that by imposing the restrictions of unitarity on the vector-exchange Born term the peaks in the $N^*$ production and total cross sections can be reproduced, without changing the threshold behavior and without changing significantly the high-energy predictions for the decay angular distributions. We therefore suggest that single-particle exchange is the dominant inelastic process in the $K^+p$ channel from threshold to the high energy region.

Even though particle exchange seems adequate to explain all phenomena observed in the inelastic reactions, the interest in possible $K^+p$ resonances has induced us to study the partial wave structure of the inelastic reactions in more detail. Our partial wave model for the $K^+N^*$ final state has succeeded in fitting our data rather precisely in the region of the Cool bump, with no evidence for a resonance in the dominant $P$-waves. This is probably equivalent to the observation that the Legendre coefficients for the $N^*$ production angular distribution and the density matrix elements are smooth functions of momentum. The latter is true also for
the K*N final state, from its threshold at about the center of the Cool peak to 2 BeV/c. It seems unlikely that a conventional resonance, with a rapid phase variation with total c.m. energy, could pass these tests undetected. We believe that the Cool bump is a consequence of the N* production threshold peak, with several partial waves participating and none of them resonating.

Charles Wohl has studied the K+p elastic scattering$^{33}$ and has found the data in the forward hemisphere to be well approximated by an exponential form, $e^{-t/t_0}$, from high energies down to 1200 MeV/c. Thus in every respect the K+p channel seems to behave as continuously as possible with energy. It should be an ideal channel in which to study nonresonant scattering models, such as the Regge pole model.
FOOTNOTES AND REFERENCES


5. The pictures not included in Table I are either immeasurable by the FSD, largely due to data box failures, or part of other exposures, as follows:

- film at 860, 960, 1200, or 1360 MeV/c, not used for quality reasons: 70 000 pictures
- pion and proton film, for contamination studies: 19 000 "
- \( K^+ p \) at other momenta: 114 000 "
- \( K^+ d \): 172 000 "
6. We have omitted from Tables IV and V events with more than one missing neutral, to wit:

\[
\begin{align*}
960 \text{ MeV/c} & \quad \pi^+ p \to \pi^+ \pi^+ (\text{neutrals}) & 1 \text{ event} \\
1200 \text{ MeV/c} & \quad K^+ p \to K^+ p (\text{neutrals}) & 1 \text{ event} \\
& \quad K^+ p \to K^+ \pi^+ (\text{neutrals}) & 3 \text{ events} \\
& \quad \pi^+ p \to \pi^+ p (\text{neutrals}) & 3 \text{ events} \\
& \quad \pi^+ p \to \pi^+ \pi^+ (\text{neutrals}) & 4 \text{ events}
\end{align*}
\]

7. The density of hydrogen was determined by measuring the stopping length of a muon from the two-body decay of a pion at rest, \( \pi^+ \to \mu^+ + \nu \);


9. The values of the total \( K^+ p \) cross section which we used were communicated privately to us by Dr. T. Kycia on November 15, 1967. They are based on the data of Cool et al., Ref. 2.


11. 735 and 785 MeV/c \( K^+ p \to K\pi N--T. \ A. \ Filippas, V. P. Henri, B. Jongejans, M. Krammer, J. M. Perreau, S. Focardi, A. Minguzzi-Ranzi, L. Monari, G. Saltini, P. Serra, E. Barralet, E. Huffer, and F. Muller, CERN


to September 7, 1966; the unpublished cross section for the reaction \(K^+ p \rightarrow K^+ \pi^- p\) was kindly supplied by Dr. J. A. Kadyk, Lawrence Radiation Laboratory, Berkeley, California.


20. The authors of Ref. 11, using a model quite different from ours, estimate that \(N^*\) production comprises 40% of the \(K^0 \pi^+ p\) final state at 785 MeV/c, rather than our estimate of \((55\pm7)\)%.

Solely for consistency with our results at higher momenta, we will use our result rather than theirs.


26. Other authors have calculated cross sections near threshold using a zero-width $N^*$. This leads to much higher cross sections than when the $N^*$ Breit-Wigner is taken into account. To illustrate this we have calculated the cross section in the zero-width approximation, using for the $N^*$ mass the peak mass values given in Table XII. The $M_1$ cross section is increased by a factor of 6 at 860 MeV/c, and by a factor of 2 at 960 MeV/c. Using the zero-width approximation can thus lead to quite erroneous estimates of the coupling constants.


33. C. G. Wohl, R. W. Bland, M. G. Bowler, J. L. Brown, G. Goldhaber,
APPENDIX A

Conventions and Kinematics

This appendix is intended to include just enough information to make the formulae discussed in the text complete and unambiguous, to the reader who wishes to understand or reproduce the complete calculations.

1. Relativity

We use the four-space metric

\[ g_{\mu \nu} = g^{\mu \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \]

(A1)

so that

\[ A \cdot B = A^\mu B^\nu g_{\mu \nu} \]

\[ = A^0 B^0 - \vec{A} \cdot \vec{B} , \]

(A2)

where the components of a four-momentum vector are defined as

\[ P^\mu = (E, P_x, P_y, P_z) \]

(A3a)

and

\[ P_\mu = P^\nu g_{\nu \mu} = (E, -P_x, -P_y, -P_z) . \]

(A3b)

2. Dirac Equation

We use spinor wave functions \( u \) satisfying

\[ \not{p} u(p) = m u(p) , \quad \bar{u}(p) \not{p} = \bar{u}(p)m , \]

(A4)

where \( \not{p} = \gamma \cdot p \), and \( \bar{u} = u \gamma_0 \). The \( \gamma \)-matrices satisfy

\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu \nu} , \]

(A5)

and \( \gamma_5 \) is defined by
\[ \gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 . \quad (A6) \]

Where a specific representation is needed we use:

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_\perp \\ -\sigma_\perp & 0 \end{pmatrix}, \quad (A7a) \]

and thus

\[ \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \quad (A7b) \]

The matrices \( \sigma_\perp \) are the usual 2 \( \times \) 2 Pauli matrices.

The positive energy spinor wave function is

\[ u(p) = \sqrt{E+m} \begin{pmatrix} \chi \\ \sigma \cdot p \\ \chi \end{pmatrix}, \quad (A8) \]

where \( \chi \) is the two-component spin wave function of the fermion in its own c.m. (The fermion c.m. and the system of observation must be related by a simple boost.) The spinor is normalized to

\[ \bar{u} u = 2m . \quad (A9) \]

3. Kinematics for spin 3/2

We use the Rarita-Schwinger representation for spin 3/2. A projection operator \( \beta_\nu^\mu \) is used to project spin 3/2 and positive energy from the direct product of a four-vector and a Dirac spinor. Thus

\[ U^\mu = N(p) \beta_\nu^\mu (P)_\nu^\nu u(p) , \quad (A10) \]

where the normalization factor

\[ N(p) = \frac{3M}{\sqrt{4\pi} \sqrt{p^\nu p^\nu (E+m)}} \quad (A11) \]

is chosen so that in the particle's c.m.

\[ \bar{U}_\mu U^\mu = 2M \frac{1}{4\pi} . \quad (A12) \]
Here $P$ and $M$ are the momentum and the mass of the spin $3/2$ particle, and $p$, $E$, and $m$ are the momentum, energy, and mass of the spin $1/2$ decay particle. This is the analog of Eq. (A9) for a system with the final state configuration space of (nucleon spin) $\times$ (decay solid angle). Both wave functions are normalized to probability density $2m$ in final state configuration space. In the particle c.m. $U^\mu$ takes the form

$$U^\mu \rightarrow U^\mu = -\sqrt{M/12\pi} (\sigma \cdot p_1 + i(\sigma \times p_1)) \chi ; \quad (A13)$$

i.e.,

$$U^\mu A^\mu = \sqrt{M/12\pi} (\sigma \cdot p + i(\sigma \times p)) \chi \cdot A .$$

The projection operator $P^\mu_\nu$ is given by

$$P^\mu_\nu = \frac{I+M}{2M} \left( \gamma^\mu \gamma_\nu - \frac{1}{3} \gamma^\mu \gamma_\nu - \frac{1}{3} \gamma^\mu \gamma_\nu \right) \frac{2}{3M^2} F^\mu \gamma_\nu , \quad (A14)$$

or, in the particle c.m.,

$$P^\mu_\nu \rightarrow P_{ij}^\mu = -\left( \begin{array}{ccc} \frac{2}{3} \delta_{ij} - \frac{1}{3} 
\epsilon_{ijk} & 0 \\
0 & 0 
\end{array} \right) . \quad (A15)$$

4. Reaction Kinematics

A (differential) cross section is given by

$$\sigma(s, \text{other final state variables}) = (\text{flux}) \times \frac{2 \pi}{q \sqrt{s}} \times \left( \text{phase} \right) . \quad (A16)$$

For, an incident two-body state,

$$\text{flux} = \frac{2 \pi}{q \sqrt{s}} . \quad (A17)$$
where \( q \) is the c.m. incoming three-momentum. \( M \) is the invariant Feynman amplitude, given by the following rules:

For each external line, include a factor of \( 1/(2\pi)^{3/2} \);
for each vertex, include a factor of \( (2\pi)^{4}/i \);
for each internal line, include a factor of \( \frac{1}{(2\pi)^{4}} \frac{\rho_{\alpha\beta}}{p^{2} - m^{2} + i\epsilon} \);

where \( \rho_{\alpha\beta} \) is the appropriate spin projection operator;
multiply by a coupling constant and a spin and angular momentum coupling at each vertex;
and multiply by a "bonus factor" of \( i \).

(A18)

5. Phase Space

Phase space for an \( n \)-body final state is in general

\[
\rho_{n} = \delta(P^{\mu} - \sum_{i=1}^{n} p_{i}^{\mu}) \prod_{i=1}^{n} d^{4} p_{i} \delta(m_{i}^{2} - p_{i}^{2}) . \tag{A19}
\]

We integrate out the individual particle delta functions to put them on the mass shell, giving the familiar factor \( d^{3} p_{i}/2E_{i} \) for each particle, and integrate over four variables to eliminate the overall momentum conservation delta function. This leads to the following phase space factors:

Two-body phase space:

\[
\rho_{2}(\vec{n}) = \frac{\vec{q}}{4\sqrt{s}} , \tag{A20a}
\]
\[
\rho_{2}(\cos \theta) = \frac{\pi \vec{q}}{2\sqrt{s}} , \tag{A20b}
\]
\[
\rho_{2}(t) = \frac{\pi}{4q\sqrt{s}} , \tag{A20c}
\]
where $q$ and $q'$ are overall c.m. incoming and outgoing three-momenta.

Three-body phase space:

$$
\rho_3(\Omega_{\text{prod}}, M^2, \Omega_{\text{decay}}) = \left( \frac{q'}{6\sqrt{s}} \right) (\frac{p}{4M}) .
$$

(A21a)

Note that this is just the product of two-body phase space for the production and decay processes;

$$
\rho_3(E_1, E_2) = \pi^2 \delta_0 .
$$

(A21b)

6. Relationship of $N^*$ Decay Coupling Constant to Decay Width

The matrix element for production and decay of a "stable" (zero-width) $N^*$ can be written

$$
M_{\text{stable}} = \bar{U}^\dagger v_\mu ,
$$

(A22)

where $\bar{U}^\dagger$ is the normalized $N^*$ wave function described in Sec. 3 of this appendix, and $v_\mu$ contains all of the production dynamics. The $N^*$ production rate is

$$
\sigma = (\text{flux}) \times \sum_{\text{decay}} \int d\Omega M_{\text{stable}}^2 \times (\text{production phase space}) .
$$

(A23)

For an unstable $N^*$, the matrix element is

$$
M_{\text{unstable}} = \bar{u}(p) \rho^\mu \frac{\rho^\nu}{F^2 - (M_0 - i \frac{\Gamma}{2})^2} v_\nu \epsilon_{\pi NN^*} \frac{1}{(2\pi)^{3/2}} .
$$

(A24)
The number of final-state particles has increased by 1, hence the factor \(1/(2\pi)^{3/2}\). The corresponding \(N^*\) production rate is

\[
\sigma = (\text{flux}) \times \sum_{\text{proton spins}} \int d\Omega_{\text{decay}} \int dm \frac{1}{|M_{\text{unstable}}|^2}
\times (\text{production space}) \times \frac{\mathbf{p}}{4M}.
\]  

(A25)

We require that in the limit \(\Gamma \to 0\), Eq. (A25) reduce to Eq. (A23). Using the form (A10) for the \(N^*\) wave function, this means that

\[
\lim_{\Gamma \to 0} \int dM^2 \frac{g^2 \pi NN^*}{(2\pi)^3} \frac{p^2}{\mathbf{p}} \frac{1}{[p^2 - (M_0 - i \frac{\Gamma}{2})^2]} = \frac{3M}{4\pi \mathbf{p}} \frac{\mathbf{p}}{(E+m)}.
\]  

(A26)

Since

\[
\lim_{\Gamma \to 0} \int dM^2 \frac{1}{\pi} \frac{\Gamma M_0}{[p^2 - (M_0 - i \frac{\Gamma}{2})^2]} = 1,
\]

we get

\[
\frac{g^2 \pi NN^*}{4\pi} = \frac{6\Gamma M^3}{\mathbf{p}^3 (E+m)}.
\]  

(A28)

7. Breit-Wigner Resonance Formula

The \(N^*(1236)\) and \(K^*(891)\) are both p-wave resonances, and so the differential cross section for production and decay of one of these resonances must include a factor

\[
\text{RES} = 1 \frac{\Gamma M_0}{(M^2 - M_0^2)^2 + (\Gamma M_0)^2},
\]

(A29)

where

\[
\Gamma = \Gamma_0 \frac{q^2}{q_0} \left( \frac{a^2}{q_0^2 + a^2} \right) / \left( \frac{q^2}{q_0^2 + a^2} \right),
\]

\[
\Gamma_0 = \frac{(q_0^2 + a^2)^{3/2}}{a^2 q_0^2}.
\]
$M_o$ and $\Gamma_o$ are the nominal resonance mass and width ($1.236$ and $0.116$ BeV for the $N^*$, $0.891$ and $0.050$ for the $K^*$), $M$ is the $N$ or $K^*$ mass, and $q$ and $q'$ are the decay momenta evaluated at $M$ and $M_o$, respectively. The parameter $\alpha$ was taken to be one pion mass, for both the $N^*$ and the $K^*$.

Equation (A29) is normalized so that

$$\lim_{\Gamma_o \to 0} \int dM^2 \, \text{RES}(M^2) = 1.$$  

It includes the effect of the 2-body resonance decay phase space, in the factor $q/q_o$; if used in an expression multiplied by 3-body phase space including the decay, the factor should be dropped.
APPENDIX B

N* Production by p-Meson Exchange

In this appendix we will carry out the calculations of results used in the text, and will briefly discuss the multipole and partial wave structure of the Ml and Jackson-Pilkuhn coupling.

The Covariant Couplings

The invariant amplitude for the exchange diagram of Fig. 30 is, assuming a stable N*,

\[ M = \frac{1}{(2\pi)^2} \epsilon_{ace} \frac{1}{e^2 - m_p^2} \epsilon_{bde} \times \text{SPIN} \quad (B1) \]

SPIN is the angular momentum coupling function and has the form

\[ \text{SPIN} = \bar{U}^\mu(a) H^\nu \epsilon^i \mu \nu \mu(u(b)) \quad (B2) \]

\( \bar{U}^\mu \) is the N* wave function described in Appendix A, \( H^\nu \) contains the \( p\pi N^* \) coupling information, and \( \epsilon^i \) is the spin vector of the \( p \) meson, after production at the \( KK\rho \) vertex and propagation to the other vertex:

\[ \epsilon^i = [\delta^i_\mu - \epsilon^i \epsilon^\nu / e^2] P^\nu \]

\[ = P^\mu \quad (B3) \]

where \( P \equiv \frac{1}{2}(a + c) \). The second line of (B3) follows only when particles a and c have the same mass; then \( P^\mu \epsilon_\mu = a^2 - c^2 = m_a^2 - m_c^2 = 0 \). This is to a good approximation in our case, since both a and c are K mesons.

There are three ways for a spin 1/2 proton and a spin 1 \( \rho \) meson to combine with their orbital angular momentum to give a spin 3/2 \( N^* \); and so we expect three independent terms in \( H^\mu \). There is a pion produced, so SPIN must be a pseudoscalar, and therefore \( H^\nu \) must be a pseudotensor.
$H^\nu_\mu$ can be built out of the vectors $b_\mu$ and $d_\mu$ and the operators $\gamma_\mu$ and $\gamma_5$.

Remembering that $\overline{U^\mu d_\mu} = \overline{U^\mu} \gamma_\mu = 0$, we can form the three terms

$$H^\nu_\mu = [G_1 \gamma_\mu d^\nu \gamma_5 + G_2 b_\mu b^\nu + \frac{G_2}{m_d^2} b_\mu d^\nu \gamma_5].$$ \hspace{1cm} (B4)

The coupling coefficients $G$ can in general be functions of $t$ and $m_d^2$, allowing quite a variation in the form of the coupling. For pure $Ml$ coupling,

$$H^\nu_\mu(Ml) = [G_\mu \gamma_\mu \gamma_\nu \gamma_5 \gamma_5 - \frac{1}{m_d^2} \gamma_\nu \gamma_\mu d^\nu].$$ \hspace{1cm} (B5)

giving* 

$$\text{SPIN}_{Ml} = \frac{1}{2m_d} \overline{U^\mu}(d_\mu)(b_\nu)b^\nu \eta \epsilon^{\mu \nu \lambda \eta}. \hspace{1cm} (B6)$$

The Jackson-Pilkuhn version is

$$H^\nu_\mu(J-P) = [(m_d + m_b) \gamma_\mu \gamma_\nu \gamma_5 \gamma_5 - \frac{1}{m_d^2} \gamma_\nu \gamma_\mu d^\nu].$$ \hspace{1cm} (B7)

giving

$$\text{SPIN}_{J-P} = \frac{1}{m_d} \overline{U^\mu}(d_\mu)(b_\nu)b^\nu \eta \epsilon^{\mu \nu \lambda \eta} \hspace{1cm} (B8)$$

In the "static limit," $t = 0$, $m_b = m_d$, (B5) reduces to (B7); the first two terms become equal and the third term of (B5) is of higher order in the exchanged particle's momentum, which vanishes.

*In going from (B5) to (B6) we have used the identity

$$\epsilon^{\mu \nu \lambda \eta} = \{\eta^{\mu \nu} \gamma^{\lambda \eta} - \gamma^{\mu \nu} \gamma^{\lambda \eta} + \gamma^{\mu \lambda} \gamma^{\nu \eta} - \gamma^{\mu \eta} \gamma^{\nu \lambda} - \gamma^{\nu \lambda} \gamma^{\lambda \eta} - \gamma^{\nu \eta} \gamma^{\lambda \lambda} - \gamma^{\mu \eta} \gamma^{\nu \lambda} + \gamma^{\mu \lambda} \gamma^{\nu \eta} + \gamma^{\mu \eta} \gamma^{\nu \lambda} - \gamma^{\mu \lambda} \gamma^{\nu \eta} \}.$$
Non-Covariant Reduction

The covariant reduction of \( \frac{1}{2} \sum |M|^2 \) is difficult and the result is not easily interpreted in terms of multipoles or partial waves. We will therefore reduce (B6) and (B8) to the form

\[
\text{SPIN} = \chi_d^+ (A + i \sigma \cdot \vec{B}) \chi_b ,
\]

evaluated in the \( N^* \) c.m. The spin sum is then trivial:

\[
\frac{1}{2} \sum |\text{SPIN}|^2 = A^* A + B^* \cdot B .
\]

M1 Coupling

Using the forms for \( \vec{U}^+ \) and \( u \) given in Appendix A, we get

\[
\text{SPIN}_{M1} = \frac{1}{m_d} \frac{E_b + m_b}{12\pi m_d} \chi_d^+ \left( 2 \hat{e} \times \vec{F} - i \hat{e} \cdot \vec{F} \right) \chi_b .
\]

We now need the relation

\[
S = \left( (c + d) \gamma \right) \lambda^\mu \chi^\nu \chi e^{\mu \nu \lambda \eta} \right)^2 .
\]

We can then evaluate the spin sum for M1 coupling:

\[
\frac{1}{2} \sum |\text{SPIN}_{M1}|^2 = \frac{E_b + m_b}{12\pi m_d} \frac{S}{m_d^2} (1 + 3 \cos^2 \gamma) ,
\]

leading to
\[
\frac{d\sigma}{d\cos \theta_{\text{prod}} d\phi_{\text{decay}}} = \frac{g^2}{3 \Delta} \frac{e_{ac}^2}{4\pi} \frac{e_{bd}^2}{4\pi} \frac{e^2}{m^2} \left[ 1 + \frac{3 \cos^2 \gamma}{2} \right]. \tag{B14}
\]

**Jackson-Pilkuhn Coupling**

The reduction of SPIN$_{JP}$ is more tedious, but straightforward. We obtain

\[
\text{SPIN}_{JP} = \sqrt{\frac{E_b + m_b}{12 \pi m_d}} \left\{ \begin{array}{c}
A \bar{X}^\dagger \cdot \vec{P} i\sigma \cdot \vec{e} \\
+ \bar{X}^\dagger \cdot \vec{e} i\sigma \cdot \vec{P} + B \bar{X}^\dagger \cdot \vec{e} i\sigma \cdot \vec{e} \end{array} \right\}, \tag{B15}
\]

where

\[
\bar{X} = (2\hat{\rho} + i\sigma \times \hat{\rho})\mathbf{d}_1,
\tag{B16a}
\]

\[
A = \frac{m_b + m_d}{E_b + m_b},
\tag{B16b}
\]

and

\[
B = \frac{P_0}{E_b + m_b}.
\tag{B16c}
\]

We now use $\bar{X}^\dagger \cdot \sigma = 0$, $\bar{X}^\dagger \times i\sigma = \bar{X}^\dagger$ to derive

\[
\text{SPIN}_{JP} = \sqrt{\frac{E_b + m_b}{12 \pi m_d}} \left\{ \begin{array}{c}
\frac{1 + A}{2} \bar{X}^\dagger \vec{X} \cdot \vec{e} \times \vec{P} \\
+ \frac{1 - A}{2} \bar{X}^\dagger \vec{X} \cdot i\sigma : [\vec{P} \vec{e} + \vec{e} \vec{P} - \frac{2}{3} \mathbb{I} \vec{P} \cdot \vec{e}] \\
+ B \bar{X}^\dagger \vec{X} \cdot i\sigma : [\vec{e} \cdot \vec{e} - \frac{1}{3} \mathbb{I} \vec{e}^2] \end{array} \right\}. \tag{B17}
\]

Using

\[
\vec{P} = \vec{e} \left( \frac{\vec{e} \cdot \vec{P}}{e^2} \right) + (\vec{e} \times \vec{P}) \times \vec{e} \left( \frac{1}{e^2} \right), \tag{B18}
\]

...
we separate $\text{SPIN}_{JP}$ into its multipole components:

$$\text{SPIN}_{JP} = \frac{\sqrt{E_{b} + m_{b}}}{12\pi m_{d}} \left( \frac{1 + A}{2} \hat{X} \times e \times \vec{P} \right)$$

$$+ \frac{1 - A}{2} \hat{X} \times e \left( i\sigma \frac{1}{e^{2}} \left[ (e \times \vec{P}) \times e \right] \hat{e} \right)$$

$$- \frac{2}{3} \left[ e \times (e \times \vec{P}) \times e \right]$$

$$- \frac{P_{0}^{t}}{E_{b} + m_{b}} \hat{X} \times e \left( i\sigma \frac{1}{e^{2}} \left[ e \times e \right] \right)$$

$$= \sqrt{\frac{E_{b} + m_{b}}{12\pi m_{d}}} (M_{1} + E_{2} + L_{2}) . \quad (B19)$$

The last term in each square bracket contributes nothing, but is retained for elegance. The spin structure of these three terms can be seen by inspection. The first term, $M_{1}$, combines the photon spin $\vec{P}$ and momentum $\vec{e}$, representing the orbital angular momentum, into spin 1. The second term, $E_{2}$, combines the transverse component of $\vec{P}$ with $\vec{e}$ to get a spin 2 tensor (i.e., a 2-index symmetric traceless tensor). The third term, $L_{2}$, makes spin 2 out of $\vec{e}$ and the longitudinal component of $\vec{P}$.

Finally, we reduce this to the form (B9), convenient for calculations.

$$M_{1} = \frac{1 + A}{2} \left( 2\hat{P} \cdot e \times \vec{P} - i\sigma \cdot \hat{P} \times (e \times \vec{P}) \right) , \quad (B20a)$$

$$E_{2} = \frac{1 - A}{2} \frac{1}{e^{2}} \left[ 3i\sigma \cdot (e \times \vec{P}) \times e \hat{P} \cdot e \right. + \left. 3i\sigma \cdot \hat{P} \cdot (e \times \vec{P}) \times e \right] , \quad (B20b)$$

$$L_{2} = \frac{-P_{0}^{t}}{(E_{b} + m_{b})} \frac{1}{e^{2}} \left[ 3i\sigma \cdot \hat{P} \cdot e - i\sigma \cdot \hat{P} e^{2} \right] . \quad (B20c)$$

From Eq. (B20) we can easily do the spin sums, for the direct and interference terms. The coefficients for these terms depend on their
coefficients in the J-P coupling, but the angular distributions are of
course characteristic of the multipoles.

\[ \frac{1}{2} \sum |M_1|^2 = (\vec{e} \times \vec{P})^2 (\frac{1 + A}{2})^2 (1 + \cos^2 \gamma) \]  

\( (B21a) \)

\[ \frac{1}{2} \sum |E2|^2 = (\vec{e} \times \vec{P})^2 9 (\frac{1 - A}{2})^2 \sin^2 \gamma \]  

\( (B21b) \)

\[ \frac{1}{2} \sum |L2|^2 = \left( \frac{P_t}{E_b + m_b} \right)^2 (1 + 3 \cos^2 \alpha) = \left( \frac{P_t}{E_b + m_b} \right)^2 (1 + 3 \sin^2 \gamma \cos^2 \delta) \]  

\( (B21c) \)

\[ \frac{1}{2} \sum |2M1 \cdot E2| = \frac{3}{2} (1 - A^2) (\vec{e} \times \vec{P})^2 (\cos^2 \alpha - \cos^2 \beta) \]  

\[ = \frac{3}{2} (1 - A^2) (\vec{e} \times \vec{P})^2 (\sin^2 \gamma \cos 2\delta) \]  

\( (B21d) \)

\[ \frac{1}{2} \sum |2M1 \cdot L2| = (1 + A) (-\vec{e} \times \vec{P}) \left( \frac{P_t}{E_b + m_b} \right) (-3 \cos \alpha \cos \beta) \]  

\[ = (1 + A) (-\vec{e} \times \vec{P}) \left( \frac{P_t}{E_b + m_b} \right) (-\frac{3}{2} \sin^2 \gamma \sin 2\delta) \]  

\( (B21e) \)

\[ \frac{1}{2} \sum |2E2 \cdot L2| = (1 - A) (-\vec{e} \times \vec{P}) \left( \frac{P_t}{E_b + m_b} \right) (3 \cos \alpha \cos \beta) \]  

\[ = (1 - A) (-\vec{e} \times \vec{P}) \left( \frac{P_t}{E_b + m_b} \right) (\frac{3}{2} \sin^2 \gamma \sin 2\delta) \]  

\( (B21f) \)

The cross section for the J-P coupling is finally obtained:

\[ \frac{1}{2} \sum |\text{SPIN}|^2_{JP} = \frac{E_b + m_d}{12\pi m_d} \frac{1}{2} \sum |M1 + E2 + L2|^2 \]  

\( (B22) \)

and

\[ \frac{d\sigma}{d\cos \theta} d\Omega_{\text{decay}} = \frac{\pi q^4}{4\pi} \frac{g_{ace}^2}{4\pi} \frac{g_{bde}^2}{4\pi} \frac{1}{(e^2 - m_p^2)^2} \times \frac{1}{2} \sum |\text{SPIN}_{JP}|^2 \]  

\( (B23) \)

Note that in the static limit E2 and L2 vanish, A = 1, and the right-
hand side of Eq. (B22) is equal to that of Eq. (B13).
APPENDIX C

Partial Wave Formalism

The following formalism was developed especially for this application, using the techniques developed by Charles Zemach.\(^{34}\) It is non-covariant but relativistically correct, provided that the overall c.m. and the \(N^*\) c.m. are related by a pure velocity transformation, without any rotation.

1. Partial Waves in the Momentum Space Representation

Zemach's approach is to write wave functions as much as possible in terms of the experimentally measured particle momenta, rather than in terms of \(|\ell,m\rangle\) states. This approach is especially advantageous in our application to lower partial waves, as the partial wave structure of meson-exchange amplitudes is more clear in this representation. We therefore construct the tensors \(T_{ij...k',\alpha'}\) with spin 1 indices \(ij...k\) and spin 1/2 index \(\alpha\). A tensor with \(j\) spin 1 indices is generally a linear combination of angular momentum \(j \pm 1/2\). A tensor corresponding to a 2-body state of definite \(\ell\) is built up from \(\ell\) powers of the c.m. momentum \(q\), the Pauli matrices \(\sigma\), and the Pauli spinor \(\chi\). Thus, for the incident \(K^+\pi\) state,

\[
\begin{align*}
T(S_{1/2}) &= \chi_{\alpha} ; & (Cl a) \\
T_{i,\alpha'}(P_{1/2}) &= (\hat{q} - i\sigma \times \hat{q})_i \chi_{\alpha} ; & (Cl b) \\
T_{i,\alpha'}(P_{3/2}) &= (2\hat{q} + i\sigma \times \hat{q})_i \chi_{\alpha} ; & (Cl c) \\
T_{i,\alpha'}(D_{3/2}) &= (\hat{q} \sigma \cdot \hat{q} - \frac{1}{3} \sigma)_i \chi_{\alpha} . & (Cl d)
\end{align*}
\]
To each of the tensors (1a) ... (1d) corresponds another of the same \( L \) but opposite parity. We have chosen the most convenient form in each case, corresponding to parities (+, +, +, -), respectively. In single-pion production the amplitudes must be a pseudoscalar, so they will be combined with outgoing tensors of parity (-, -, -, +). The outgoing tensors are made by combining the \( N^* \) decay momentum and decay proton spinor in the form (1c) to get \( P_3/2 \), then combining with \( L \) powers of the orbital angular momentum. Thus,

\[
V_{1,\alpha}(S_3/2) = (2p + i \sigma \times \hat{p})_1 X_\alpha, \tag{C2a}
\]

\[
V_{1,\alpha}(P_1/2, P_3/2) = [\hat{q}' \times (2\hat{p} + i \sigma \times \hat{p})]_1 X_\alpha, \tag{C2b}
\]

\[
V_{\alpha}(D_1/2) = \sigma \cdot \hat{q}' \sigma' \cdot (2\hat{p} + i \sigma \times \hat{p}) X_\alpha; \tag{C2c}
\]

and, for the nonresonant background wave,

\[
V_{1,\alpha}(S_1/2) = i \sigma_1 X. \tag{C2d}
\]

We have used \( q \) and \( q' \) for the incoming and outgoing c.m. momenta, and \( p \) for the outgoing nucleon momentum in the \( N\pi \) c.m. We will not bother to construct pure \( P_3/2 \) and \( P_1/2 \) amplitudes; the tensor (C2b) contains both, and a pure incoming state will project out the appropriate component.

These tensors combine to give the following partial wave amplitudes (labeled according to the final state):

\[
\mathcal{A}(S3) = X^+(2\hat{p} - i \vec{\sigma} \times \hat{p}) \cdot \hat{q} \ i \sigma \cdot \hat{q} X \tag{C3a}
\]

\[
\mathcal{A}(P1) = X^+(2\hat{p} - i \vec{\sigma} \times \hat{p}) \cdot \hat{q}' \cdot (\hat{q} - i \sigma \times \hat{q}) X \tag{C3b}
\]

\[
\mathcal{A}(P3) = X^+(2\hat{p} - i \sigma \times \hat{p}) \cdot \hat{q}' \cdot (2\hat{q} + i \sigma \times \hat{q}) X \tag{C3c}
\]

\[
\mathcal{A}(D1) = X^+(2\hat{p} - i \sigma \times \hat{p}) \cdot \hat{q}' \ i \sigma \cdot \hat{q}' X \tag{C3d}
\]

\[
\mathcal{A}(S1) = X^+ \ i \sigma \cdot (\hat{q} - i \sigma \times \hat{q}) X. \tag{C3e}
\]
The D1 amplitude is not used in our analysis; we include it here only for completeness. We are also interested in the M1 amplitude,

\[ A(M1) = \frac{\chi^+(2\hat{p} - i \sigma \times \hat{p}) \times \hat{q} \cdot \hat{q} \times}{1 - b \hat{q} \cdot \hat{q}'} \]  

\[ = \chi^+(2\hat{p} - i \sigma \times \hat{p}) \times \hat{q} \cdot \hat{q} \times (1 + b \hat{q} \cdot \hat{q}' + b^2 (\hat{q} \cdot \hat{q}')^2 + \ldots) \]  

(C4)

Here \( b \) is given by

\[ b = \frac{2\hat{q} \cdot \hat{q}'}{M_v^2 + 2(E_k E_{k'} - M_k^2)} \]  

(C5)

where \( M_v \) is the mass of the vector particle exchanged. The first term is pure P-wave, equal to \( A(P1) + A(P3) \), and the second term is a linear combination of \( D_{3/2} \) and \( D_{5/2} \). The third term, however, is a mixture of F and P waves. In our analysis we will use as an approximation to all high partial waves the M1 amplitude with P-waves subtracted out:

\[ A(M1') = A(M1) - (1 + \frac{b^2}{2})(A(P1) + A(P3)) \]  

(C6)

Terms of order \( b^4 \) are neglected in this equation and in our analysis.

2. Calculation of the Matrix Element

The calculation of the final state distributions is lengthy, and I will give just the results. We take as the amplitude for the reaction

\[ A = s1 A(S1) + s3 A(S3) + p1 A(P1) + p3 A(P3) + m1 A(M1') \]  

(C7)

where \( s1, \ldots, m1 \) are undetermined functions of \( s \) and \( M^2(N\pi) \). We use as final-state variables the production angle \( \theta \) of the \( N\pi \) diparticle, the \( N\pi \) mass-squared, and the \( N\pi \) c.m. decay angles \( \gamma \) and \( \delta \). The decay angles are as defined in Fig. 12, with the production plane normal \( \hat{q} \times \hat{q}' \) as the z-axis and as x-axis the direction of the incoming proton.
as seen in the overall c.m. The final-state intensity distribution in terms of these variables is

\[ W(\cos \theta, \Omega_{\text{decay}}, M_{N\pi}^2) = \frac{1}{2} \sum_{\text{initial and final}} |A|^2 \times \text{(phase space)} \times \text{(flux)} \]

\[ = f(\cos \theta, \Omega_{\text{decay}}, M_{N\pi}^2) \times \text{(phase space)} \times \text{(flux)}, \quad (C8) \]

where

\[ \text{phase space} = \frac{q^4 p}{2E_{\text{tot}}^2 M_{N\pi}}, \quad (C9a) \]

\[ \text{flux} = \frac{1}{q\sqrt{s}}, \quad (C9b) \]

and

\[ f(\cos \theta, \Omega_{\text{decay}}, M_{N\pi}^2) = W(\cos^2 \theta(-3P_3 - aM1) \cdot (P_1 + 2P_3 + aM1) \]

\[ + 5 \cos \theta S_3 \cdot (P_1 - P_3) + S_1^2 + \frac{5}{2} S_3^2 + (P_1 + 2P_3 + aM1)^2 \]

\[ + (P_1 - P_3) \cdot (\frac{3}{2} P_1 - \frac{5}{2} P_3 - aM1)) + X(-3 \cos^2 \theta [(P_1 + 2P_3 + aM1)^2 \]

\[ - (P_1 - P_3) \cdot (P_1 + 2P_3 + aM1)] + 3 \cos \theta S_3 \cdot (P_3 - P_1) - \frac{3}{2} S_3^2 \]

\[ + 3(P_1 + 2P_3 + aM1)^2 + 3(P_3 - P_1) \cdot (\frac{3}{2} P_1 + \frac{3}{2} P_3 + aM1)) \]

\[ + Y[3 \sin \theta \cos \theta (P_1 - P_3) \cdot (P_1 + 2P_3 + aM1) + 3 \sin \theta S_3 \cdot (P_1 + 2P_3 + aM1)] \]

\[ + Z[3 \cos^2 \theta (P_1 - P_3) \cdot (P_1 + 2P_3 + aM1) + 3 \cos \theta S_3 \cdot (P_1 - P_3) \]

\[ + \frac{3}{2} S_3^2 - 3(P_1 - P_3) \cdot (P_1 + 2P_3 + aM1) + \frac{3}{2}(P_1 - P_3)^2] \]

\[ + G[4 S_1 \cdot S_3 + 4 \cos \theta S_1 \cdot (P_1 - P_3)] \]

\[ + H 2 \sin \theta S_1 \cdot (2P_1 + P_3 + aM1); \quad (C10) \]

\[ a \equiv \frac{b \cos \theta}{1 - b \cos \theta} - \frac{b^2}{5} \quad (C11) \]
The dot product of two complex numbers is defined as 

\[ Q \cdot R = \Re(Q^*R) \quad \text{(C13)} \]

Perhaps more useful are the various projections of this distribution obtained by integrating over some of the variables. The integration over \( N^* \) decay angles is trivial; the integral over \( \cos \theta \) is somewhat more difficult; and the integral over \( M^2(\text{Nn}) \) can only be done numerically.

To do the integral over \( \cos \theta \) it is helpful to define the following averages over \( \cos \theta \):

\[
A \equiv \langle a \rangle_{\cos \theta} = \left( \frac{1 + b}{2b} + \frac{1 - b}{2b} \right) \cos \theta = \frac{1}{2b} \ln \frac{1 + b}{1 - b} - \left( 1 + \frac{b^2}{5} \right) \quad \text{(C14a)}
\]

\[
A2 \equiv \langle a^2 \rangle_{\cos \theta} = 2 - \frac{b^2}{1 - b^2} - \frac{1}{b} \ln \frac{1 + b}{1 - b} + \frac{b^2}{5}(2 + \frac{b^2}{5}) \quad \text{(C14b)}
\]

\[
SA \equiv \langle \sin \theta \, a \rangle = \frac{\pi}{2b^2} \left( 1 - \sqrt{1 - b^2} \right) - \frac{\pi}{4b} \left( 1 + \frac{b^2}{5} \right) \quad \text{(C14c)}
\]

\[
SCA \equiv \langle \sin \theta \, \cos \theta \, a \rangle = \frac{\pi}{2b^3} \left( 1 - \sqrt{1 - b^2} \right) - \frac{\pi}{4b^2} \quad \text{(C14d)}
\]

\[
S2A \equiv \langle \sin^2 \theta \, a \rangle = -\frac{2}{3}(1 + \frac{b^2}{5}) + \frac{1}{b^2} - \frac{1 - b^2}{2b^3} \ln \frac{1 + b}{1 - b} \quad \text{(C14e)}
\]

\[
S2A2 = \langle \sin^2 \theta \, a^2 \rangle = \frac{1}{15b^2} \left( -15 + \frac{b^4}{5} + \frac{b^2}{25} \right) - \frac{(b^2 - 1)(b^2 + 5)}{5b^3} \ln \frac{1 + b}{1 - b} \quad \text{(C14f)}
\]
We now readily obtain the following projections:

\[
\frac{1}{8\pi^2} f(\cos \theta, M^2) = S_1^2 + 2S_3^2 + 2(Pl - P3)^2 + 2 \sin^2 \theta (Pl + 2P3 + aM1) \cdot (3P3 + aM1) + 4 \cos \theta S_3 \cdot (Pl - P3)
\]  
\[\text{(C15)}\]

\[
\frac{1}{8\pi^2} f(\cos \gamma, m^2) = S_1^2 + \frac{1}{2}(5 - 3 \cos^2 \gamma)(S_3^2 + (Pl - P3)^2) + (1 + 3 \cos^2 \gamma)[2P3 \cdot (Pl + P3) + S_2 A M_1 \cdot (Pl + 5P3) + S_2 A M_2]
\]  
\[\text{(C16)}\]

\[
\frac{1}{8\pi} f(\theta', m^2) = S_1^2 + 2S_3^2 + 2P1^2 + 10P3^2 + 2S_2 A M_1 \cdot (Pl + 5P3) + 2S_2 A M_2^2 + \cos 2\theta' \left[ (- \frac{1}{3}) (Pl - P3) \cdot (Pl + 11P3 + 6S_2 A M_1) + S_3^2 \right] + 2 \sin 2\theta' \left[ S_2 A M_1 \cdot (Pl - P3) + S_2 A S_3 M_1 + \frac{4}{9} S_3 \cdot (Pl + 2P3) \right] + \cos 2\theta' S_1 \cdot S_2 + \sin 2\theta' \left[ S_2 \right] \frac{\sqrt{2}}{4} S_1 \cdot (2P1 + P3) + 2S_2 A S_1 M_1 \right] \]  
\[\text{(C17)}\]

\[
\frac{1}{(4\pi)^2} f(M_{N^*}^2) = S_1^2 + 2S_3^2 + 2P1^2 + 10P3^2 + 2S_2 A M_2^2 \]  
\[\text{(C18)}\]

3. Energy Dependence of the Amplitudes

We parametrize the dependence of the amplitudes on \( s \) and \( M_{N^*}^2 \) (or, equivalently, on incoming and outgoing c.m. momenta) as follows:

\[
S_1 = S_1 \cdot q/\sqrt{q^2 + a^2} \]  
\[\text{(C19a)}\]

\[
S_3 = S_3 \cdot q^2/(q^2 + a^2) \cdot \text{CPROP} \]  
\[\text{(C19b)}\]

\[
P_1 = P_1 \cdot q/\sqrt{q^2 + a^2} \cdot q'/\sqrt{q'^2 + a'^2} \cdot \text{CPROP} \]  
\[\text{(C19c)}\]

\[
P_3 = P_3 \cdot q/\sqrt{q^2 + a^2} \cdot q'/\sqrt{q'^2 + a'^2} \cdot \text{CPROP} \]  
\[\text{(C19d)}\]

\[
M_1 = \frac{M_{N^*}}{2} \sqrt{2S_2 A M_2} \cdot \text{CPROP} \]  
\[\text{(C19e)}\]

\text{CPROP} \text{ is the complex propagator and decay coupling for the } N^* \text{ which when squared gives the usual Breit-Wigner resonance form, Eq. (A30). The}
complex coefficients $\tilde{S}_1$, ..., $\tilde{M}_1$ will be the fitting parameters. The parameter $a$ affects only the dependence of the amplitudes on the total energy, and the results of our fits are insensitive to values of $a'$ over about 200 MeV/c. Our choice of values for $a$ and $a'$ was motivated by the form of $b$, Eq. (C5), which plays the role of a barrier factor in the $\rho$-exchange amplitude. Near threshold $b$ can be rewritten approximately as

$$b \approx \frac{\tilde{a} \cdot \tilde{a}'}{M^2 + \frac{\tilde{a}^2 + \tilde{a'}^2}{2}}.$$  \hspace{1cm} (C20)

Since the $M_1$ amplitude describes our data fairly well, we give our $P$-waves a similar dependence on $g'$ by taking $a = a' = M_\rho \sqrt{2}$ in Eqs. (C19).

To calculate the cross section the following integrals have been done numerically:

$\begin{align*}
S_1 \text{ INT} &\equiv (4\pi)^2 \int dm^2 |S_1/\tilde{S}_1|^2 \times \text{(phase space)} \times \text{(flux)} \hspace{1cm} (C21a) \\
S_3 \text{ INT} &\equiv (4\pi)^2 \int dm^2 |S_3/\tilde{S}_3|^2 \times \text{(phase space)} \times \text{(flux)} \hspace{1cm} (C21b) \\
P_1 \text{ INT} &\equiv (4\pi)^2 \int dm^2 |P_1/\tilde{P}_1|^2 \times \text{(phase space)} \times \text{(flux)} \hspace{1cm} (C21c) \\
P_3 \text{ INT} &\equiv (4\pi)^2 \int dm^2 |P_3/\tilde{P}_3|^2 \times \text{(phase space)} \times \text{(flux)} \hspace{1cm} (C21d) \\
M_1 \text{ INT} &\equiv (4\pi)^2 \int dm^2 |M_1/\tilde{M}_1|^2 \times \text{(phase space)} \times \text{(flux)} \hspace{1cm} (C21e)
\end{align*}$

Then

$$\sigma = |\tilde{S}_1|^2 S_1 \text{ INT} + 2|\tilde{S}_3|^2 S_3 \text{ INT} + 2|\tilde{P}_1|^2 P_1 \text{ INT} + 10|\tilde{P}_3|^2 P_3 \text{ INT} + 2S2A2 \text{ INT} |\tilde{M}_1|^2.$$ \hspace{1cm} (C22)
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