UNIVERSITY OF CALIFORNIA, SAN DIEGO

Hypergraph Independence Numbers

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Alexander Eustis

Committee in charge:
Professor Jacques Verstraëte, Chair
Professor Fan Chung Graham
Professor Ronald Graham
Professor Ramamohan Paturi
Professor Glenn Tesler
Professor Alexander Vardy

2013
The dissertation of Alexander Eustis is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

_________________________________________

_________________________________________

_________________________________________

_________________________________________

_________________________________________

_________________________________________

Chair

University of California, San Diego

2013
DEDICATION

To my parents, for everything.
# TABLE OF CONTENTS

Signature Page .......................................................... iii

Dedication ................................................................. iv

Table of Contents ........................................................ v

Acknowledgements ........................................................ viii

Vita and Publications ..................................................... ix

Abstract of the Dissertation ............................................. x

## Chapter 1  Introduction and History  ............................. 1

1.1 Terminology and Notation .......................................... 1

1.1.1 Formal definition of a hypergraph ............................ 1

1.1.2 Glossary .......................................................... 1

1.1.3 Hypergraph invariants .......................................... 2

1.1.4 Limit notation .................................................... 3

1.1.5 Other notation and conventions ............................... 3

1.2 Independence number in graphs versus hypergraphs .......... 4

1.3 Other motivations .................................................... 6

1.3.1 The Heilbronn Triangle Problem ............................... 6

1.3.2 Algorithmic complexity ......................................... 7

1.4 Generic bounds ....................................................... 8

1.4.1 An elementary bound ............................................ 9

1.4.2 The Caro-Tuza bound .......................................... 9

1.4.3 New improvements to Caro-Tuza ............................. 10

1.5 Small Transversals .................................................. 12

1.6 Log-factor improvement for graphs and hypergraphs .......... 13

1.6.1 Partial Steiner systems ...................................... 14

1.6.2 Our new results for Steiner systems ......................... 15

1.6.3 The lower constant $c(r, l)$ .................................. 17

1.6.4 The Rödl Nibble ............................................. 17

1.7 Overview of the dissertation ...................................... 19

## Chapter 2  Probabilistic Methods  ................................. 20

2.1 Method of Expectation ............................................. 20

2.2 Chernoff bounds ..................................................... 22

2.3 Lóvasz Local Lemma: .............................................. 22
Chapter 3  Overview of independence numbers and Turán Numbers  ...  23
3.1 Basic independence number bounds ........................................... 23
3.1.1 Reduction to \( r \)-uniform, \( d \)-regular ......................... 23
3.1.2 A coloring bound ......................................................... 24
3.2 Independence number of \( G_r(n,p) \) ................................... 25
3.3 Turán Numbers .............................................................. 26
3.3.1 Turán’s Theorem .......................................................... 26
3.3.2 The Tetrahedron .......................................................... 27
3.3.3 Turán densities for \( K_t^r \) .............................................. 28
3.4 Expected weight proof of \( \alpha(H) \geq n \left( \frac{\log d}{d \log \log d} \right) \) for \( K_t \)-free graphs .................................................. 29

Chapter 4  Dual Hypergraphs, matchings, and incidence graphs .......... 34
4.1 The Tutte-Berge formula ..................................................... 34
4.2 The Dual of a Hypergraph .................................................. 35
4.2.1 Multi-hypergraphs ....................................................... 36
4.2.2 The dual of a hypergraph of maximum degree 2 ................. 36
4.3 Independence numbers in the case \( \Delta(H) \leq 2 \) .................... 37
4.3.1 Proof of Theorem 4.5, assuming miss(\( H \)) \( \leq 1 \) ............... 40
4.3.2 Proof of Theorem 4.5: general case .................................. 42
4.3.3 Proof of Theorem 4.4 ................................................... 43
4.4 Hall’s marriage theorem ..................................................... 44
4.5 The Expander-mixing lemma ............................................... 44

Chapter 5  Independent sets in bounded-degree hypergraphs ............ 47
5.1 Two theorems .................................................................... 48
5.1.1 Proof of Theorem 5.1 ..................................................... 49
5.1.2 Proof of theorem 5.2 ..................................................... 51
5.1.3 Avoiding copies of \( T_r \) .................................................. 52
5.2 An improved recursion: the function \( g(d) \) ........................... 54
5.2.1 Proof of Theorem 5.7: preliminary lemmas ........................ 55
5.2.2 Proof of Theorem 5.7 ..................................................... 57
5.3 Further refinements ........................................................... 59
5.3.1 A refinement for \( r = 6 \) .................................................. 60
5.4 Lemmas ............................................................................ 60
5.5 Proof of Theorem 5.12 ........................................................ 64
5.5.1 Assume \( \Delta \leq 2 \). ...................................................... 64
5.5.2 Assume \( \Delta = 3 \). ...................................................... 64
5.5.3 Assume \( \Delta \in 4,5 \). .................................................... 65
5.5.4 Assume \( \Delta \geq 6 \). .................................................... 70
5.6 Numerical comparisons to previously known results .............. 71
5.6.1 The Lonc/Warno Theorem ............................................. 72
5.6.2 Points in \( A_r \) ........................................................... 73
Chapter 6 Basic Projective Geometry
6.1 Projective Planes
   6.1.1 Projective spaces
   6.1.2 Existence results
6.2 Inversive Planes
   6.2.1 Construction
6.3 Eigenvalues
   6.3.1 Projective Plane of order $q$
   6.3.2 Inversive Plane of order $q$

Chapter 7 Algebraic constructions of partial Steiner systems
7.1 The first-moment lower bound for $\alpha(H)$
7.2 Proof of Theorem 1.5, algebraic construction
   7.2.1 Construction
   7.2.2 Proof outline
   7.2.3 Proof of Claim 1
   7.2.4 Bounds on $\alpha$
   7.2.5 Proof of Claim 2
7.3 Concluding remarks
   7.3.1 Complete $(n, r, l)$-system constructions for $l = 2, 3$
   7.3.2 The codegree parameter

Chapter 8 Nibble construction of partial Steiner systems
8.1 Proof of Theorem 1.5
   8.1.1 Construction
   8.1.2 The Regularity conditions
   8.1.3 Proof of Theorem 1.5
   8.1.4 Proof of claim (8.1) part (a)
   8.1.5 Proof of claim (8.1) part (b)
   8.1.6 Proof of claim (8.2)

Appendix A
A.1 List of properties of the function $g$
A.2 Proof of parts (i), (ii), (iii), (iv)
A.3 Proof of part (v)
A.4 The functions $g, h$ from Chapter 6

Bibliography
ACKNOWLEDGEMENTS

Of course, none of this would have been possible without my advisor Jacques Verstraete, for all the time and patience he’s put in. Thanks to Fan Chung Graham, Jason Schweinsberg, and Bruce Driver for discussing problems with me early on. Thanks also to my office buddies Jake, Mary, Andy, and Adrian, for keeping things lively.

I especially need to acknowledge Michael Henning and Anders Yeo, who bounced a ton of ideas back in forth in email correspondence. Anders Yeo has co-authored some of the material in Chapter 5, and this is used with permission. I have been careful to indicate throughout the chapter specifically where he has contributed.

An especially heartfelt thank you to my best friends Steph, Jymmm, and Alan for providing much-needed emotional support during the writing of this thesis. It hasn’t been easy, and I know exactly where I would be without you.
VITA

<table>
<thead>
<tr>
<th>Year</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2006</td>
<td>B. S. in Mathematics, with High Distinction, Harvey Mudd College</td>
</tr>
<tr>
<td>2007-2012</td>
<td>Graduate Teaching Assistant, University of California, San Diego</td>
</tr>
<tr>
<td>2011</td>
<td>Associate Instructor, University of California, San Diego</td>
</tr>
<tr>
<td>2013</td>
<td>Ph. D. in Mathematics, University of California, San Diego</td>
</tr>
</tbody>
</table>
ABSTRACT OF THE DISSERTATION

Hypergraph Independence Numbers

by

Alexander Eustis

Doctor of Philosophy in Mathematics

University of California, San Diego, 2013

Professor Jacques Verstraëte, Chair

An $r$-graph $H$ is a set $V$ of vertices together with a set of $r$-element subsets of $V$ called edges. An independent set in $H$ is a subset $I$ of $V$ with the property that no edge is contained in $I$. The dissertation discusses the general problem of proving the existence or nonexistence of an independent set of a certain size, based on properties of $H$.

The new results presented fall into two categories: combinatorial and probabilistic. On the combinatorial side, in Chapters 4 and 5 we present several theorems which give an independent set of size $\sum_{v \in V} f(d_v)$, where $d_v$ is the number of edges containing $v$ and $f : \{0, 1, 2, \ldots\} \mapsto \mathbb{R}$ is a particular function defined herein. The function $f$ improves previous results of its kind, including [12], [40], [51], and represents the best known bound in several cases.
On the probabilistic side, we give two different randomized constructions for an \((n,r,l)\)-system, which is an \(r\)-graph in which no two edges intersect in \(l\) or more vertices. It is shown that each of these constructions, with high probability, contains no independent set of size

\[
(1 + \epsilon) \left( \frac{l - 1}{r - 1} (r)_l \right)^{1/(r-1)} n^{r-1} (\log n)^{1/r-1}.
\]

This improves the previous bound [44] in which \((1+\epsilon) \left( \frac{l - 1}{r - 1} (r)_l \right)^{1/(r-1)}\) is replaced by an arbitrary constant. This result belongs to a larger class of independence number theorems determined by the average degree of the hypergraph. We conjecture, based on the general theory, that this constant is the best possible.

The first of these constructions is algebraic in nature. The second employs an algorithm known as the “Nibble Method” which has been used widely in the literature. Originally [45] it was used to prove the existence of an \((n,r,l)\)-system having asymptotically the maximum number of edges; our construction additionally provides an upper bound for the size of an independent set in this system.
Chapter 1

Introduction and History

1.1 Terminology and Notation

For reference, we specify here the notation and conventions used throughout this dissertation.

1.1.1 Formal definition of a hypergraph

A hypergraph is a pair $H = (V, E)$ where $V$ is a finite set, called the vertex set, and $E$ is a set of subsets of $V$, called edges. Throughout, we employ the convention where $H$ can represent either the hypergraph itself, or the edge set. That is, $\alpha(H), \Delta(H), \ldots$, refers to the hypergraph, whereas notation such as $|H|, H_1 \subseteq H_2, \ldots$ refers to the edge set. The vertex set will always be denoted by $V$ or $V(H)$. Note that the edge set, together with the number of vertices not contained in any edge (isolated vertices), determines the isomorphism class of $H$.

An $r$-graph, also known as an $r$-uniform hypergraph, is a hypergraph in which every edge has size $r$. We assume the reader is already familiar with 2-graphs, which are just known as “graphs.”

1.1.2 Glossary

Given a hypergraph $H = (V, E)$, we use the following terminology. Most of these terms directly parallel graph theory and should be familiar.
### Hypergraph invariants

For an $r$-graph $H$, the following invariants of $H$ are denoted by the symbols below. These symbols (including $n, m$) will not be used for any other purpose.
When $H$ is understood, we may write $n, m, d,$ etc. rather than $n(H), m(H), d(H)$.

### 1.1.4 Limit notation

Throughout we make use of formulae such as $\alpha = \Omega(nd^{-1/(r-1)})$ which employ standard asymptotic notation. Usually the meaning of such expressions is clear, but if no variable is specified, assume that the notation refers to $n$, the number of vertices. For example, well-known theorem [45] states that there exists an $(n, r, l)$-system with $(1-o(1)){n \choose r}/{l \choose r}$ edges. Formally, this means that for every $\epsilon > 0$ there exists $N$ such that if $n \geq N$, then there exists an $(n, r, l)$-system with at least $(1-\epsilon){n \choose r}/{l \choose r}$ edges.

If $f, g$ are two functions of $n$, then $f \sim g$ means that $\lim_{n \to \infty} f / g = 1$, and $f \lesssim g$ means that $\lim sup_{n \to \infty} f / g \leq 1$.

### 1.1.5 Other notation and conventions

| $[n]$ | the set of integers $\{1, 2, \ldots, n\}$.
| $K_n^r$ | the complete $r$-graph on $n$ vertices.
| $H[X]$ | the induced sub-hypergraph with vertex set $X \subseteq V$.
| $H - X$ | the induced sub-hypergraph with vertex set $V \setminus X$.

(For a difference of sets, we use the symbol \.)

All logs are natural base. Empty sums and products evaluate to 0 and 1, respectively.
1.2 Independence number in graphs versus hypergraphs

Hypergraphs and their independence numbers have been studied since the time of Erdős, as a natural extension of the corresponding problems in graph theory. Typically, a theorem for graphs about independence numbers or chromatic numbers does not have a direct counterpart for $r$-graphs, which has spurred researchers to try and discover what kinds of generalizations do exist. As an example, let us begin with the well-known Brooks’ Theorem and see what happens if we attempt to extend it to $r$-graphs.

**Theorem 1.1** (Brooks). The chromatic number of a graph $G$ satisfies

$$\chi(G) \leq \Delta(G) + 1,$$

with equality if and only if $G$ has a connected component isomorphic to the complete graph $K_{\Delta+1}$, or if $\Delta(G) = 2$ and $G$ has a connected component isomorphic to an odd cycle $C_{2n+1}$.

Note that the equation $\chi \leq \Delta + 1$ is trivial: color the vertices in arbitrary order, giving each vertex a color not already assigned to any neighbor. Hence the real content of Brooks’ Theorem is the characterization of equality. If $r > 2$, Brooks’ Theorem does not extend to $r$-graphs in a useful way. Although $\chi \leq \Delta + 1$ is still trivially true, it is not the case that complete $r$-graphs have higher chromatic number than other $r$-graphs of the same maximum degree. To see this, consider the tetrahedron $K_{4}^{(3)}$; it is 3-regular 3-graph and its chromatic number is 2. Compare this to the Fano Plane, which is also a 3-regular 3-graph shown in the diagram below.
Figure 1: The Fano Plane.

Some inspection shows that the chromatic number of the Fano Plane is 3, which is greater than that of $K_4^{(3)}$. The existence of hypergraphs such as the Fano Plane raises the question: what types of $r$-graphs have the highest chromatic number for a given maximum degree, if not complete graphs?

Next, let us consider Turan’s Theorem for graphs, which can be stated in terms of the independence number as follows:

**Theorem 1.2** (Turán). Among all graphs $G$ on $n$ vertices such that $\alpha(G) \leq k$, the number of edges in $G$ is minimized if and only if $G$ is the graph consisting of $k$ vertex-disjoint cliques whose sizes differ by at most 1.

Once again, there is no analogous statement for $r$-graphs. To construct a counterexample, consider the 4-graph $\tilde{F}$, which is a complement version of the Fano plane $F$ defined as follows: a set of 4 vertices in $\tilde{F}$ is an edge if and only if the complementary three vertices form an edge of $F$. Thus $\tilde{F}$ is a 4-regular 4-graph, and a some inspection shows that $\alpha(\tilde{F}) = 4$. Now let $H_1$ be the 4-graph consisting of three vertex-disjoint copies of $K_3^{(4)}$, and let $H_2$ be the hypergraph consisting of two vertex-disjoint copies of $\tilde{F}$ plus an isolated vertex.

- Both $H_1$ and $H_2$ are 4-graphs with 15 vertices.
- $\alpha(H_1) = \alpha(H_2) = 9$.
- However, $H_2$ has 14 edges while $H_1$ has 15.
This shows that a union of cliques of equal size (in this case $H_1$) does not give the minimum number of edges as in Turán’s Theorem.

This counterexample works because a maximum independent set in $\tilde{F}$ takes up a greater proportion of the vertices than in $K_5^{(4)}$, even though both are 4-regular. That is, we compare the invariant $\alpha(H)/n(H)$ for the two hypergraphs and find that

$$\frac{\alpha}{n}(K_5^{(4)}) = \frac{3}{5} > \frac{4}{7} = \frac{\alpha}{n}(\tilde{F}).$$

Because of this, we were guaranteed to be able to construct the above counterexample by taking an appropriate number of disjoint copies of each (along with one or more isolated vertices). Consequently, our work in this dissertation, and in the literature in general, is focused around describing upper and lower bounds for the invariant $\alpha/n$, in terms of the average or maximum degree. Such bounds will usually be stated in a form such as

$$\alpha \geq n f(r, \Delta). \quad (1.1)$$

### 1.3 Other motivations

A hypergraph is a very general notion, and independence number bounds can be applied to any number of problems which can be phrased in terms of hypergraphs. A good example is the Heilbronn Problem discussed below. We also briefly discuss the algorithmic complexity of computing an independence number, which is a well-studied problem in computer science.

#### 1.3.1 The Heilbronn Triangle Problem

A famous problem that motivated some of the early research in hypergraph independence numbers is the following geometric problem: Consider placing $n$ points into the unit disc, and let $A(n)$ denote the maximum, over all such arrangements, of the area of the smallest triangle formed by three of the $n$ points. That is, to prove $A(n) \geq a$ one must show there exists an arrangement where every triangle has area at least $a$, and to prove $A(n) \leq b$ one must show that every arrangement
contains a triangle of area at most $b$. An easy upper bound is $A(n) \leq 1/(n-2)$ by showing that every arrangement contains at least $n-2$ triangles with disjoint interiors. The current best upper bound $A(n) \leq n^{-8/7} \exp(c\sqrt{\log n})$ is due to Kőmlos, Pintz, and Szemerédi [35].

The current lower bound,

$$A(n) \geq Cn^{-2}\log n,$$

by the same authors [36], disproved the long-standing conjecture of Heilbronn and Erdős that $A(n) \sim n^{-2}$. Their method relies on hypergraph independence numbers and is of motivating interest here. The idea is the following: place down $n^{1+\epsilon}$ points randomly with the uniform distribution, then define the 3-graph $H$ on these points, where $\{x, y, z\} \in H$ if the triangle $xyz$ has area less than $\delta = Cn^{-2}\log n$. The problem then amounts to showing that the independence number of this 3-graph is at least $n$. If so, we can choose an appropriate subset of $n$ points which contains no triangle of area less than $\delta$. Without going into the details of the calculations, it is worth mentioning that the $\log n$ factor in this bound directly corresponds to the $\log \Delta$ factor one can obtain in a hypergraph containing no 2, 3, or 4-cycles (see Section 1.6).

### 1.3.2 Algorithmic complexity

Finding maximal independent sets in graphs and hypergraphs is a well-studied hard problem in the computational sense. The decision problem known as CLIQUE is the following: given a graph $G$ and an integer $k$, determine whether $G$ contains a clique of size $k$. (Of course, this is equivalent to determining if the complement of $G$ has an independent set of size $k$.) CLIQUE is a well-known NP-complete problem. The corresponding optimization problem (stated in terms of independent sets) is, given a graph $G$, return an integer which is never greater than $\alpha(G)$ but is as close to $\alpha(G)$ as possible. However, even the problem of approximating $\alpha(G)$ is NP-hard. In [25] it is shown that if there exists a polynomial-time algorithm which, for any graph $G$ on $n$ vertices, approximates $\alpha(G)$ to within a factor $n^{1-\epsilon}$ for any $\epsilon > 0$, then $ZPP = NP$. (The class $ZPP$ is defined as the
problems solvable by a probabilistic Turing machine — one which is allowed to write a random bit — in expected time $O(n^c)$ for some constant $c$.) Of course, the $r$-uniform hypergraph problem (for $r > 2$) cannot be any easier.

However, the case of bounded-degree graphs is more tractable. A recent result by Halperin [24] gives a polynomial-time algorithm which, given a graph of maximum degree $\Delta$, finds an independent set of size at least $\Theta(\frac{\log \Delta}{\Delta \log \log \Delta} \alpha(G))$, and it is even known that the problem is inapproximable much more closely than this [8].

In Chapters 4 and 5 of this dissertation, our lower bounds of the form $\alpha(H) \geq nf(\Delta)$ are actually polynomial-time algorithms for finding an independent sets of the stated size. These are based on the simple idea of repeatedly deleting a vertex of maximum degree, until there are no edges left in the hypergraph. The largest independent set guaranteed by this method is the bound of Caro and Tuza [12] discussed below. Our results in this dissertation improve this algorithm by deleting vertices more cleverly, but these improvements can be achieved in polynomial time. In particular, we need to be able to find a Tutte-Berge witness set for a maximum matching (see Chapter 4); this is acheived by the well known Edmonds Algorithm [21].

1.4 Generic bounds

We now turn our attention to the problem of finding specific lower bounds for the independence number of a hypergraph, given information about the vertex degrees. Throughout, we shall always assume $r$ is a fixed constant. First we shall discuss generic bounds of the form (1.1) which hold for all values of $n, \Delta,$ and $r$. Once we have a basic understanding of the general case, we shall turn to more specific cases in later chapters, such as specific values of $r$ and $\Delta$, or limiting regimes where $\Delta \to \infty$, or hypergraphs with forbidden substructures.
1.4.1 An elementary bound

For an arbitrary \( r \)-graph \( H \) with average degree \( d \geq 1 \) (i.e. having \( nd/r \) edges), we have the following inequality:

\[
\alpha(H) \geq (1 - 1/r) \frac{n}{d^{1/(r-1)}}.
\]  

(1.2)

This is easy to prove using a probabilistic argument, and we will do so in Chapter 2. Note that \( d \leq \Delta \) for all hypergraphs, so the same inequality is true with \( \Delta \) in place of \( d \). This bound is tight up to the constant \( (1 - 1/r) \), in the following sense: for any prescribed values of \( n, d, r \), there exists an \( r \)-graph \( H \) with \( n \) vertices and average degree at most \( d \), such that

\[
\alpha(H) \leq C_r \frac{n}{d^{1/(r-1)}}.
\]

In particular we may take \( H \) to be a union of cliques; this is Proposition 2.2 in the next chapter and gives \( C_r = \frac{2^{2(r-1)}}{(r-1)!r^{r-1}} \). Thus, although we have seen that complete hypergraphs do not necessarily give an optimal value of \( \alpha/n \), they at least come within a constant factor. We summarize this by saying that for any fixed \( r \), the minimum value of \( \alpha/n \) is of the form \( \Theta(d^{-1/(r-1)}) \). However, determining the optimal value of \( C_r \) is a very difficult problem which is closely related to Turán numbers (see Chapter 3).

1.4.2 The Caro-Tuza bound

A degree sequence bound for \( \alpha(H) \) is a theorem of the form

\[
\alpha(H) \geq \sum_{v \in V} f(d_v),
\]

where \( f : \mathbb{Z}^{\geq 0} \rightarrow \mathbb{R} \) is usually a decreasing, positive-valued function, and \( d_v \) is the degree of \( v \). This type of bound clearly implies \( \alpha(H) \geq n f(\Delta) \), but it is more versatile, giving a better bound if the hypergraph is not regular. The following degree sequence bound was discovered in 1991 by Caro and Tuza [12]:

\[
\alpha(H) \geq \sum_{v \in V} \left( d(v) + \frac{1}{r-1} \right)^{-1}.
\]

(1.3)
Using the gamma-function identity \( \lim_{n \to \infty} \frac{\Gamma(n + \alpha)}{n^\alpha \Gamma(n)} = 1 \), one can show that
\[
\left( d + \frac{1}{r-1} \right)^{-1} \Gamma \left( 1 + \frac{1}{r-1} \right) \sim \frac{\Gamma(1 + \frac{1}{r-1})}{d^{1/(r-1)}}
\]
as \( d \to \infty \). So, if we substitute \( d(v) \leq \Delta \) for each \( v \) this reduces the earlier form (1.2) with a different constant.

The Caro-Tuza theorem is based on the \textit{max-degree deletion algorithm}, which is the following. Given a hypergraph \( H \), pick an arbitrary vertex of maximum degree and delete it, meaning we remove the vertex from the hypergraph along with all edges containing it. If we continue deleting vertices in this manner until there are no edges left, then the remaining vertices form an independent set in \( H \). Caro and Tuza were able to prove the lower bound (1.3) on the size of the independent set obtained in this manner. However, as one might suspect, it is possible to improve this theorem by choosing which vertices to delete more cleverly. In Chapter 5 we will do just that.

### 1.4.3 New improvements to Caro-Tuza

Our first improvement will be to stop the deletion process once the hypergraph has reached maximum degree 2. Finding a maximum independent set in such a hypergraph turns out to be equivalent to finding a maximum matching in a graph. By utilizing the well known Tutte-Berge formula for maximum matchings, we will prove the following:

**Theorem 1.3.** Let \( H \) be an connected \( r \)-graph of maximum degree 2. If \( r \) is even, then \( \alpha(H)/n \) is minimized if and only if \( H \) is isomorphic to the hypergraph \( T_r \) shown below. If \( r \) is odd, then \( \alpha(H)/n \) is minimized if and only if \( H \) is isomorphic to \( T_r^* \) or \( U_r \).
Treating \( \Delta = 2 \) as the base case, we induce the following improvement to (1.3):

**Theorem 1.4.** If \( H \) is an \( r \)-graph of maximum degree \( \Delta \geq 2 \), with \( n \) vertices, then

\[
\frac{\alpha(H)}{n} \geq \left( \frac{\Delta + \frac{1}{r-1}}{\Delta} \right)^{-1} \left( 1 + \frac{r}{2(r-1)(3r-1)}, \quad r \text{ odd} \right)
\left( 1 + \frac{r-2}{6(r-1)^2}, \quad r \text{ even} \right).
\]

The proofs of these two theorems can be found in Chapter 5. Unfortunately, Theorem 1.4 by itself is slightly unfavorable to the most current results, particularly a forthcoming paper by Lonc and Warno [40]. However, we can improve the theorem further by altering the recursion which generates the values \( f(d) \). These improvements are discussed thoroughly in Chapter 5; see Theorems 5.7 and 5.12. The end result is a degree-sequence bound that at least matches the best known bounds for all \( r \), and improves them in the cases \( r = 3 \), and \( r \geq 4 \) even, provided certain particular \( r \)-graphs are excepted from the theorem.
1.5 Small Transversals

We say a subset $T \subset V$ is a *transversal* of a hypergraph $H$ if $T$ intersects every vertex of $H$. Equivalently, it is the complement an independent set. It follows that

$$\tau(H) = n(H) - \alpha(H),$$

where $\tau(H)$ denotes the minimum size of a transversal. Because of this relation, bounds on $\alpha$ and $\tau$ are interchangeable.

For example, a result of Chvátal and McDiarmid [14] states for any 3-graph $H$,

$$\tau \leq (m + n)/4.$$

Using the relations $nd = 3m$ and $\tau = n - \alpha$, this is equivalent to

$$\alpha \geq n \left(\frac{3}{4} - \frac{d}{12}\right).$$

This type of bound is only useful for $d$ in a limited range. Observe that when $d = 3$, the bound $\alpha(H) \geq n/2$ is tight; consider the tetrahedron $K_4^{(3)}$. However if $d \geq 8$, the bound is useless. To get good bounds on a wider range of $d$, we need additional theorems of the form $\tau \leq am + bn$. The aforementioned paper of Chvátal and McDiarmid presents the following problem: describe the set $A_r$ of all points $(a, b)$ where $a \geq 0, b \geq 0$, for which the inequality

$$\tau(H) \leq am(H) + bn(H)$$

holds for every $r$-graph $H$. In general, since $nd = rm$ and $\tau = n - \alpha$, (1.4) is equivalent to the independence number bound

$$\alpha \geq (1 - b - \frac{ad}{r})n.$$

As noted above, such a bound is only useful for values of $d$ in a small range, but the set $A_r$ as a whole provides the best possible independence number bounds over all values of $d$. Unfortunately, not much is known for certain about $A_r$ (except in the case of graphs, $r = 2$). Observe that $A_r$ is an intersection of half-planes, one for each $r$-graph $H$, and therefore it is a convex set. This means it is sufficient
to describe the extreme points. In [14] it is shown that $A_r$ has infinitely many extreme points; however it is conjectured that for all $\epsilon > 0$ there are only finitely many extreme points $(a, b)$ satisfying $a > \epsilon$.

Much of the literature on small transversals amounts to finding particular points $(a, b) \in A_r$ outside the convex hull of previously known points, without any claim to their extremality. Our new results in Chapter 5 can be described in these terms; see Section 5.6 for the relevant numerical comparisons.

### 1.6 Log-factor improvement for graphs and hypergraphs

The second part of this dissertation uses probability theory and deals with the limiting case where $d \to \infty$. Above, we already mentioned the generic bound $\alpha \geq C_r d^{-1/(r-1)}$. Note that in what follows, $C_r, c_r, d_r, etc.$ will represent unspecified constants depending only on $r$, and may be different at each occurrence.

If one imposes a certain amount of structure on $H$, then this bound can be improved (increased) by a factor of $(\log d)^{1/(r-1)}$. The first result along these lines was the influential 1980 theorem of Ajtai, Komlós, Pintz, Spencer, and Szemerédi [4] which states that if $H$ is an $r$-graph of average degree $d$ containing no 2-, 3-, or 4- cycles, then

$$\alpha(H) \geq C_r n \left(\frac{\log d}{d}\right)^{1/(r-1)}.$$  \hspace{1cm} (1.5)

(A $k$-cycle in an $r$-graph can be defined as a set of $k$ edges whose union contains at most $k(r - 1)$ vertices). Other authors were soon able to obtain similar results for other types of graphs. For example, Phelps and Rödl (1986) [41] were the first to show that if $H$ is a 3-graph containing no 2-cycle,

$$\alpha(H) \geq C_3 \sqrt{n \log n}.$$  \hspace{1cm} (1.6)

For this $H$ not hard to see that $\Delta \leq (n - 1)/2$. Note that for any 3-graph $H$ satisfying $d = O(n)$, equation (1.5) reduces to $\alpha(H) \geq C_3 \sqrt{n \log n}$, the same as (1.6).
A simultaneous generalization of these two results was later found by Duke, Lefmann, and Rödl [17] who obtained the bound (1.5) under the weaker assumption that $H$ contains no 2-cycles. The most recent improvement is due to Frieze and Mubayi [19], who show that under the same assumption,

$$\chi(H) \leq \left( \frac{d}{\log d} \right)^{1/(r-1)},$$

which implies an independence number of the same size.

In [19], Frieze and Mubayi state a larger conjecture relating all these results (in fact the same conjecture was stated at one point or another by nearly all these authors): If $F$ is any fixed $r$-graph, and $H$ is an $r$-graph of average degree $d$ not containing $F$ as a subgraph, then

$$\alpha(H) \geq C_F n \left( \frac{\log d}{d} \right)^{1/(r-1)}$$

where $C_F$ is a constant depending only on $F$ (and $r$). Thus “$F$-free for any $F$” is believed to be a strong enough assumption to obtain the log-factor improvement.

Unfortunately the conjecture remains wide open, even for graphs. Alon, Krievelevich, and Sudakov (1999) [5] have shown a partial result: this holds for a $F$-free graphs $G$, if $F$ is a graph which can be made bipartite by deleting one vertex. For example, the conjecture holds if $F$ is a cycle.

The general case would follow if it were proved for the complete graph $F = K_t$ for all $t$. Johannson [28] proved that it almost holds: if $G$ is a $K_t$-free graph,

$$\alpha(H) \geq C_n \frac{\log \Delta}{\Delta \log \log \Delta}.$$  

However, his proof never appeared in a peer-reviewed journal. We include our own proof of this in Chapter 3, which is based on Shearer’s method for triangles. [48]

### 1.6.1 Partial Steiner systems

The assumption that $H$ contains no 2-cycles is the same as saying that the codegree of every pair of vertices is at most one. The name we use for a more general type of hypergraph is an $(n,r,l)$ partial Steiner system, or $(n,r,l)$-system.
for short, which is an $r$-graph on $n$ vertices where the codegree of every $l$-set is at most one. For $l \geq 3$, being an $(n, r, l)$-system is a strictly weaker assumption than having no 2-cycles. Nonetheless, the same log-factor bound (1.5) holds for partial Steiner systems.

The general case was done by Rödl, V. and E. Šinajová (1994) [44]: If $H$ is an $(n, r, l)$-system,

$$\alpha(H) \geq c_r n^{\frac{r-l}{r-1}} (\log n)^{\frac{1}{r-1}}.$$ 

They also proved tightness; i.e. there exist $(n, r, l)$-systems $H$ satisfying

$$\alpha(H) \leq d_r n^{\frac{r-l}{r-1}} (\log n)^{\frac{1}{r-1}}.$$ 

We can view this result as a special case of the Frieze-Mubayi conjecture since an $(n, r, l)$-system is an $r$-graph in which the subgraphs $F_l, F_{l+1}, \ldots, F_{r-1}$ are forbidden, where $F_k$ is the hypergraph consisting of 2 edges intersecting in $k$ vertices. The Frieze-Mubayi conjecture, and the Duke-Lefmann-Rödl theorem in the case $l = 2$, give the same lower bound as this when we take $d = O(n^{l-1})$, the largest average degree for an $(n, r, l)$-system.

### 1.6.2 Our new results for Steiner systems

In Chapters 7 and 8 we explore constructions of partial Steiner systems. Define

$$f(n, r, l) = \min \{ \alpha(H) : H \text{ is an } (n, r, l)\text{-system} \}.$$ 

Recall the result of Rödl and Šinajová [44] which can be stated as follows:

$$c_r n^{\frac{r-l}{r-1}} (\log n)^{\frac{1}{r-1}} \leq f(n, r, l) \leq d_r n^{\frac{r-l}{r-1}} (\log n)^{\frac{1}{r-1}}.$$ 

Though this is a very important theorem, the values of $c_r$ and $d_r$ given were nearly arbitrary. Our contribution is to improve the upper constant $d_r$ to what we believe to be its optimal value. Let us define

$$c(r, l) = \liminf_{n \to \infty} \frac{f(n, r, l)}{n^{\frac{r-l}{r-1}} (\log n)^{\frac{1}{r-1}}},$$ 

$$d(r, l) = \limsup_{n \to \infty} \frac{f(n, r, l)}{n^{\frac{r-l}{r-1}} (\log n)^{\frac{1}{r-1}}}.$$ 

We will prove the following:
Theorem 1.5. For all integers \( r, l \) satisfying \( 2 \leq l < r \), there exists an \((n, r, l)\)-system \( H \) which contains no independent set of size

\[
(1 + o(1)) \left( \frac{l - 1}{r - 1} (r)_l \right)^{1/(r-1)} n^{\frac{r l - l}{r - 1}} (\log n)^{\frac{1}{r-1}}.
\]

In other words,

\[
d(r, l) \leq \left( \frac{l - 1}{r - 1} (r)_l \right)^{1/(r-1)}.
\]

The constant \( (\frac{l - 1}{r - 1} (r)_l)^{1/(r-1)} \) is particularly special because it matches the value of a pure random hypergraph of the same density as a complete \((n, r, l)\) system. We conjecture that this is the value of both \( d(r, l) \) and \( c(r, l) \). See Chapter 7 for details.

For comparison, Rödl and Šinajová’s original paper gave an upper constant of

\[
d(r, l) \leq \left( 4 \left(\frac{2r - l}{r}\right)^2 \frac{l - 1}{r - 1} \frac{r^{2r}}{(r - 1)!} \right)^{\frac{1}{r - 1}}
\]

based on an existence argument using the Lóvasz local lemma. This value is nowhere near the above theorems; for instance it gives \( d(3, 2) \leq \sqrt{11664} \), where Phelps and Rödl [41] had been able to show \( d(3, 2) \leq 4 \), and bound given by Theorem 1.5 is \( d(3, 2) \leq \sqrt{3} \). No attempt to improve the general constant \( d(r, l) \) has been published since then.

We will give two different proofs of this theorem. In Chapter 7 we prove the case \( r \geq 2l - 1 \) using a convenient algebraic construction. The material in this Chapter is from our paper, “On the Independence Number of Steiner Systems”, joint with Jacques Verstraete, which has been reviewed and is due to appear in Combinatorics Probability and Computing.

In Chapter 8 we prove the general case \( 2 \leq l < r \) using the well-known Nibble method. Since this covers the cases left out by the previous paper, we intend to publish this material as well. Although the proof is significantly technical, our analysis of the Nibble method may be of independent interest.
1.6.3 The lower constant $c(r, l)$

The original lower bound of Ajtai-Kőmlos-Pintz-Spencer-Szemerédi [4] for an $r$-graph of girth at least 5, was given in terms of the variable $t = d^{1/(r-1)}$ as

$$\alpha(H) \gtrsim c_r(n/t)(\log t)^{1/(r-1)},$$

with the ad hoc constant $c_r = (0.98/e)10^{-5/(r-1)}$. This constant is based on the analysis of their error terms and could potentially be improved, but not in a significant way. Consequently, the papers that have cited this theorem, including Rödl-Šinajová, have been stated in terms of an unspecified constant. This has been the situation until very recently. Inspired by the method of Shearer [48], Kostochka, Mubayi, and Verstraete [31] have shown that

$$c(r, r-1) \geq \left(\frac{(r-2)!}{(3r-4)2^{r-1}(-\log(1-2^{-r}))}\right)^{1/(r-1)}$$

which is the best known in the case $l = r-1$. This expression is asymptotic to $r/e$ in the limit $r \to \infty$, as is the upper constant $d(r, l)$ from Theorem 1.5. This implies that

$$c(r, r-1) \sim \frac{r}{e} \sim d(r, r-1)$$

as $r \to \infty$, so both (1.7) and Theorem 1.5 are asymptotically optimal in this limit. Unfortunately this does not prove that our value of $d(r, r-1)$ is optimal for any fixed $r$. Further evidence for this claim is given in Chapter 7.

We anticipate that similar improvements could be made to the lower bounds $c(r, l)$ when $l \neq r - 1$. Moreover, it seems intuitively certain that $c(r, l) = d(r, l)$. However, these questions are still open.

1.6.4 The Rödl Nibble

In Chapter 8 we discuss the independence number of a random hypergraph generated by the “Rödl Nibble,” also known as the semirandom method. Originally this method was conceived to resolve the following, known as the Erdős-Hanani conjecture: Fix integers $r > l \geq 2$. Do there exist $(n, r, l)$-systems $H_n$ satisfying $|H_n| \sim \binom{n}{r}/\binom{r}{l}$? Such a sequence is called asymptotically complete because a
complete \((n, r, l)\) system has exactly \(\binom{n}{r}/\binom{n}{l}\) edges, and a partial \((n, r, l)\)-system has at most this number. Rödl’s breakthrough paper [45] constructed asymptotically complete Steiner systems probabilistically, proving the conjecture.

In the context above, the nibble method works as follows (see Chapter 8 for a rigorous description). We start with a hypergraph \(H_0 = K_n^{(r)}\), representing the initial set of “possible” edges. From \(H_0\) we select a random subset \(S_0\), called a “nibble,” representing edges we would like to be in our \((n, r, l)\)-system. Therefore we must delete from \(H_0\) all edges sharing \(l\) or more vertices with any edge in \(S_0\), forming the next iteration \(H_1\). \(S_0\) is chosen small enough to make it unlikely that \(l\)-sets in \(S_0\) have codegree greater than one, but large enough to be able to control the codegrees of \(l\)-sets in \(H_1\). We then select a new random nibble \(S_1\) from \(H_1\) and continue the process as long as necessary. The final \((n, r, l)\)-system is \(S_0^* \cup S_1^* \cup \ldots\), where \(S_i^*\) denotes a version of \(S_i\) in which we have deleted a small number of edges to ensure \(S_i^*\) is itself an \((n, r, l)\)-system.

Nibble methods have been used widely in the hypergraph literature since their development. An early notable example is the 1989 paper by Pippenger and Spencer [42] which deals with covers and matchings of \(H\). In Chapter 3, we will see that minimum covers and maximum matchings are directly related to the independence number of the dual graph. In [42] the main theorem is that for any sequence of hypergraphs in which: (1) the minimum degree is asymptotic to the maximum degree; and (2) the codegree of any vertex pair is asymptotically negligible compared to the vertex degrees; there exists a partition of the edges into covers, almost all of which are almost perfect. (An almost perfect cover is one containing only \((1 + o(1))n/r\) edges.) The covers are generated by nibble method, and their theorem applies, in particular, to the family of complete Steiner triple systems.

The 1997 paper by Alon, Kim, and Spencer [3] takes this a step further by giving a tighter estimate for the size of the \((1 + o(1))\) factor in such a cover. Specifically, they prove that in a \(\Delta\)-regular \((n, r, 2)\)-system, there exists a matching containing all but at most \(c_r n \Delta^{-1/(r-1)}\) of the vertices (or all but \(c_3 n \Delta^{-1/2} \log^{3/2} \Delta\) if \(r = 3\)). Note that a such a matching implies a cover of the same error. The
more recent paper by Alon, Bollobás, Kim, Vu [2] covers a similar situation where the codegrees are allowed to be greater than one (but not large), and where \( r \) is not fixed but increases as \( \Delta \to \infty \); they show that if \( r = o(\log \Delta) \) then an almost perfect cover exists, but this is not necessarily so if \( r \geq 4 \log \Delta \). For a good summary of results along these lines using the Nibble method, see [52].

1.7 Overview of the dissertation

In Chapter 2, we give a light introduction to the probabilistic methods used throughout (with the exception of Chapters 4 and 5). Chapter 3 contains more general information about independence numbers, and a few tools. Chapter 4 provides detailed information about the independence number of an \( r \)-graph of maximum degree 2, which serves as the base case for the inductive methods in Chapter 5. Chapter 5 proves several new lower bounds for \( \alpha(H) \) based on the degree sequence of \( H \), starting with Theorems 4.4 and 1.4 and then introducing several refinements which improve the bounds further. These improved versions represent the best bounds currently known in many cases.

Chapter 6 begins the second part of the dissertation and introduces basic Projective Geometry to be used in the next chapter. Chapter 7 proves Theorem 1.5 in the cases \( r \geq 2l - 1 \) using an algebraic construction, and is based on a paper co-authored with Jacques Verstraete [18]. Chapter 8 proves the theorem a second time using the Nibble method discussed above. This proof covers all the possible cases \( 2 \leq l < r \).
Chapter 2

Probabilistic Methods

In this chapter we list a few probabilistic tools needed in the proofs of our major theorems. The probabilistic method, in short, is the idea of using a random process to generate or analyze a mathematical structure such as a hypergraph, then analyzing the probability that the generated structure would have particular properties. If the probability is greater than zero, then such a structure with those properties exists (although the method is nonconstructive, in general). This approach was pioneered by Paul Erdős and other Hungarian mathematicians, but today has grown into a well-established field that is used worldwide with great success in combinatorics, number theory, information and coding theory, and more. The Probabilistic Method by Alon and Spencer [7] is a standard reference that documents the tools used here.

We assume here that the reader is familiar with the notions of probability space, random variable, and expectation. The following are standard facts from probability theory.

2.1 Method of Expectation

If $X$ is a random variable on a probability space $\Omega$, then there exists $\omega \in \Omega$ such that $X(\omega) \leq E[X]$. This principle is frequently useful for proving that a graph/hypergraph has an independent set of a certain size. Sometimes this method is referred to as the “first moment method” to distinguish it from the “second
moment method,” its counterpart for proving concentration. As an example we give the following proof which is, up to a constant, the best possible lower bound on \( \alpha(H) \) for an arbitrary hypergraph \( H \) of average degree \( d \).

**Proposition 2.1.** If \( H \) is an \( r \)-graph with average degree \( d \geq 1 \), then

\[
\alpha(H) \geq (1 - 1/r) \frac{n}{d^{1/(r-1)}}.
\]

**Proof.** Let \( S \) be a random subset of the vertices where each vertex is included independently with the probability \( p = d^{-1/(r-1)} \). That is to say, for each vertex \( v \), we flip a biased coin whose probability of heads is \( p \), and include the vertex in \( S \) if the coin lands heads. Let \( X \) denote the random variable \( |S| - |H[S]| \).

Since there are \( nd/r \) edges, each of which has probability \( p^r \) of belonging to \( H[S] \), we have

\[
E[X] = np - (nd/r)p^r = \frac{n}{d^{1/(r-1)}} - \frac{nd}{d^{1/(r-1)}} = (1 - 1/r) \frac{n}{d^{1/(r-1)}}.
\]

Therefore there exists a set \( S \) such that \( |S| - |H[S]| \leq (1 - 1/r) \frac{n}{d^{1/(r-1)}} \). From this \( S \), we can delete one vertex for each edge in \( H[S] \), and obtain an independent set of the required size. \( \square \)

For completeness, let us also show that \( Cnd^{-1/(r-1)} \) is the best lower bound possible, in other words, there exist \( r \)-graphs \( H \) with \( n \) vertices and average degree \( d \) satisfying \( \alpha(H) \leq Cnd^{1/(r-1)} \), for nearly all possible values of \( n \) and \( d \). We will now show that in fact, a union of complete hypergraphs can achieve this bound.

**Proposition 2.2.** For any positive integer \( n \) and real number \( D \) satisfying \( 1 \leq D \leq r(n)/n \), there exists a hypergraph \( H \) with \( n \) vertices and average degree at most \( D \), such that

\[
\alpha(H) \leq \frac{2(r-1)n}{((r-1)!D)^{1/(r-1)}}.
\]

**Proof.** Let \( k \) be the largest integer such that \( \frac{(k-1)^{r-1}}{(r-1)!} \leq D \), and let \( H \) be the \( r \)-graph consisting of \( \lfloor n/k \rfloor \) vertex-disjoint copies of the complete hypergraph \( K^{(r)}_k \), plus a complete hypergraph on all the remaining vertices. Since

\[
\frac{(k-1)}{r-1} \leq \frac{(k-1)^{r-1}}{(r-1)!} \leq D,
\]
all the vertices in this graph have degree at most $D$ as required. Since no independent set can contain more than $r - 1$ vertices from each complete graph, and since $k = \lfloor ((r - 1)!D)^{1/(r-1)} \rfloor + 1$, we have

$$\alpha(H) \leq (r - 1)\left\lceil n/k \right\rceil \leq (r - 1)\left(\frac{n}{(r - 1)!D^{1/(r-1)}} + 1\right) \leq \frac{2(r - 1)n}{((r - 1)!D)^{1/(r-1)}}$$

which is the bound required. \hfill \blacksquare

### 2.2 Chernoff bounds

If a random variable $Y$ is the sum of $n$ independent indicators (i.e. random variables that take only the values 0 and 1), and $\mu = E[Y]$, then

$$P(|Y - EY| > t) < 2 \exp\left(-\frac{2t^2}{n}\right),$$

$$P(Y > \mu + t) < \exp[t - \mu \log(1 + t/\mu) - t \log(1 + t/\mu)].$$

Both inequalities are proven in [7].

### 2.3 Lóvasz Local Lemma:

Let $\{A_i\}_{i \in I}$ be a finite set of events. A graph $G$ with vertex set $I$ is a dependency graph for $I$ if for all $i \in I$, $A_i$ is mutually independent of the events $\{A_j: j \text{ not adjacent (nor equal) to } i\}$. The symmetric version of the Local Lemma [7] is as follows:

Suppose $\mathbb{P}(A_i) = p$ for all $i$, and $I$ has a dependency graph of maximum degree $\Delta$. If $ep(\Delta+1) \leq 1$, then with positive probability none of the events occur; in fact

$$\mathbb{P}(\bigcap_i \overline{A_i}) \geq (1 - 1/(\Delta + 1))^{\left|I\right|}. $$
Chapter 3

Overview of independence numbers and Turán Numbers

In this chapter we continue our discussion of the main questions, known bounds, and proof techniques concerning the independence number and related concepts for hypergraphs. We will begin with a straightforward discussion of the simplest type of independence number problem, proving a few basic results for the purpose of context and demonstration.

3.1 Basic independence number bounds

In the previous chapter, we proved (using the first moment method) that if $H$ is an $r$-graph with average degree $d \geq 1$, then
\[
\alpha(H) \geq (1 - 1/r) \frac{n}{d^{1/(r-1)}}.
\]

3.1.1 Reduction to $r$-uniform, $d$-regular

The following proposition shows that if $\alpha/n \geq C_\Delta$ for all $\Delta$-regular $r$-graphs, then in fact the same bound holds for all hypergraphs of maximum degree $\Delta$ and minimum edge size $r$. 

23
Proposition 3.1. Let $H$ be a hypergraph of minimum edge size $r$ and maximum degree at most $\Delta$. Then there is a corresponding $r$-regular, $\Delta$-regular hypergraph $\tilde{H}$ such that $\frac{\alpha(\tilde{H})}{|V(\tilde{H})|} \leq \frac{\alpha(H)}{|V(H)|}$.

Proof. First, form the hypergraph $H_0$ by arbitrarily truncating each edge to size $r$, if there are any larger edges. $H_0$ is an $r$-graph with maximum degree at most $\Delta$, and $\alpha(H_0) \leq \alpha(H)$.

Next, let $\delta$ be the minimum degree of $H_0$. If $\delta = \Delta$ then $H_0 = \tilde{H}$ is the desired hypergraph, so assume $\delta < \Delta$. Form the $r$-graph $H_1$ consisting of $r$ copies of $H_0$, namely $H_0^1, \ldots, H_0^r$, with the following additional edges: for every vertex $v \in H_0$ of degree $\delta$, we create a new edge in $H_1$ consisting of the $r$ different copies of $v$. If $I$ is an independent set in $H_1$, then $I \cap H_0^k$ is an independent set of $H_0^k$ for $k = 1, \ldots, r$, which implies $\alpha(H_1) \leq r\alpha(H_0)$. Note that $H_1$ has minimum degree $\delta + 1$ and maximum degree at most $\Delta$. Repeat this process $\Delta - \delta$ times, to create the $r$-graph $\tilde{H} = H_{\Delta-\delta}$. $\tilde{H}$ is $\Delta$-regular, and by induction we have $\alpha(\tilde{H}) \leq r^{\Delta-\delta}\alpha(H_0) \leq r^{\Delta-\delta}\alpha(H)$. Since $n(\tilde{H}) = r^{\Delta-\delta}n(H)$, the proposition is proved.

Thanks to this proposition, any lower bound for $\alpha(H)$ for a $\Delta$-regular $r$-graph implies the same bound for an $r$-graph of $\Delta$-regular degree, so we can often assume our hypergraphs are $\Delta$-regular.

3.1.2 A coloring bound

Earlier we showed the generic bound $\alpha / n = \Omega(d^{-1/(r-1)})$. Using the Lóvasz Local Lemma we can prove an equivalent result for colorings, namely that $\chi(H) = O(d^{1/(r-1)})$. Note that $\chi(K_n^{(r)}) = \lceil n / (r-1) \rceil$ and $\Delta(K_n^{(r)}) = \binom{n-1}{r-1}$, implying that $\chi(K_n^{(r)}) = \Theta(\Delta^{1/(r-1)})$.

Proposition 3.2. If $H$ is $r$-uniform with maximum degree $d$, then

$$\chi(H) \leq \lceil (e(r-1)\Delta + 1)^{\frac{1}{r-1}} \rceil$$

Proof. Choose a random coloring where colors are selected independently and uniformly from a set of size $C$. Let $A_e$ be the event that $e$ is monochromatic, so
\[ P(A_e) = C^{-(r-1)}. \]

\( A_e \) is mutually independent of all events \( A_f \) where \( f \cap e = \emptyset \), so we have a dependency graph on these events of maximum degree at most \((r-1)d\). Therefore by the Lóvasz Local Lemma, if \( eC^{-(r-1)((r-1)\Delta + 1)} \leq 1 \), then the coloring is proper with nonzero probability; i.e. a \( C \)-coloring exists. Solving for \( C \) we find that
\[
C \geq (e(r-1)\Delta + 1)^{\frac{1}{r-1}}
\]
which proves the proposition.

3.2 Independence number of \( G_r(n, p) \)

Let \( G_r(n, p) \) denote the random hypergraph with vertex set \([n]\), where every \( r \)-set of vertices becomes an edge with probability \( p \) and the choices are independent. This is the hypergraph analogue of the Erdős-Rényi random graph \( G(n, p) \). Since a random graph has "evenly spread" edges, it an example of an \( r \)-graph with low independence number and will be the starting point for many of our constructions.

For any given subset \( U \subseteq [n] \), it is very easy to compute the probability that \( U \) is an independent set in \( G_r(n, p) \); this is just
\[
P(G_r(n, p)[U] = \emptyset) = (1 - p)^{\binom{|U|}{r}}.
\]
Thus, for \(|U| = u\) the expected number of independent sets of size \( u \) is
\[
f(n, p, u) = \binom{n}{u} (1 - p)^{\binom{u}{r}}.
\]
It is perhaps not surprising that if \( u_0 \) denotes the smallest value of \( u \) such that \( f(n, p, u) < 1 \), then
\[
\alpha(G_r(n, p)) \sim u_0
\]
a.a.s. This is nontrivial to prove, but the problem has been solved completely even for the chromatic number. For all integers \( 1 \leq \gamma < r \) one can define the \( \gamma \)-chromatic number of an \( r \)-graph as the minimum number of colors required to color every vertex such that no edge receives more than \( \gamma \) vertices of any color. The case \( \gamma =
$r - 1$ is the usual chromatic number defined in Chapter 1. Shamir [47] determined the asymptotic value of the $\gamma$-chromatic number for the so-called “dense” case of $G_r(n, p)$ where $(n^{r-1}p)^{1/\gamma} \geq n^{1-\epsilon}$ (for some fixed $\epsilon > 0$). All the random graphs considered in this thesis are dense. Krivelevich and Sudakov [38] determined the asymptotic $\gamma$-chromatic number for the remaining values of $p$.

### 3.3 Turán Numbers

In the following sections we give an brief overview of the Turán problem of determining the maximum number of edges in an $r$-graph not containing a copy of the complete $r$-graph $K_t^{(r)}$, and its relationship to the independence number problem of determining the smallest value of $\alpha(H)$ for a hypergraph of maximum degree $\Delta$.

#### 3.3.1 Turán’s Theorem

An early theorem of Mantel (1907) states that the unique triangle-free graph on $n$ vertices with a maximum number of edges is the complete bipartite graph with parts of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. This is a special case of Turán’s Theorem [50], where it is shown that the unique $n$-vertex graph not containing a complete graph of size $r$ is a complete $(r-1)$-partite graph whose part sizes are as equal as possible. This graph, denoted $T_{n,r}$, is referred to as the Turán graph. This result can be stated and proved in the context of the independence number of a graph relative to its degree sequence: the following formulation and proof of Turán’s Theorem is due independently to Caro [11] and Wei [53].

**Theorem 3.3.** Let $G$ be a graph. Then

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v) + 1}$$

with equality possible if and only if $G$ is a vertex-disjoint union of cliques whose sizes are as equal as possible.
Proof. The lower bound on $\alpha(G)$ is proved via a probabilistic method. Let $<$ be a random linear ordering of the vertices of $G$, and let $I$ be the (random) set of vertices $v \in V(G)$ such that $v < u$ for every neighbor $u$ of $v$. Then $I$ is an independent set, and

$$P(v \in I) = \frac{1}{d(v) + 1}.$$  

In particular, $E(|I|)$ is at least the lower bound claimed in the theorem, and therefore there is an ordering $<$ for which the corresponding independent set $I$ is as large as required.

The characterization of equality is given in Alon and Spencer [7]. It is equivalent to Turán’s Theorem, since the complement of $T_{r,n}$ is a disjoint union of cliques.

3.3.2 The Tetrahedron

It is natural to ask if Turán’s Theorem, or even the Caro-Wei Theorem, extends to uniform hypergraphs. The Túran Problem consists in determining the maximum number of edges in an $r$-uniform hypergraph on $n$ vertices which does not contain a complete hypergraph of order $t$. This number is typically denoted $\text{ex}(n, K^r_t)$ for $t > r$. Thus when $r = 2$ this is addressed by Turán’s Theorem. However, there is no single instance $(t, r)$ in which the Túran Problem is solved when $t > r > 2$. Perhaps the most famous question is the tetrahedron question, which is the instance $t = 4$ and $r = 3$, posed famously by Erdős:

**Conjecture 3.4.**

$$\lim_{n \to \infty} \frac{\text{ex}(n, K^3_4)}{\binom{n}{3}} = \frac{5}{9}.$$  

This conjecture remains open, while the best upper bound is $(\sqrt{17} + 2)\binom{n}{3}$, due to Chung and Lu [13]. In a recent development using the notion of flag algebras, Razborov [43] suggests computations that could improve the bound even further, although not to obtain the conjecture in full. On the other hand, Turán gave a fairly natural construction of a $K^3_4$-free family of triples on $n$ points: partition $[n]$ into three sets of roughly equal size, say $X, Y$ and $Z$, and form the hypergraph $H$
consisting of all triples with two points in $X$ and one point in $Y$, two points in $Y$ and one point in $Z$, two points in $Z$ and one point in $X$, and one point in each of $X, Y$ and $Z$. Then as $N \to \infty$,

$$|H| = |X||Y||Z| + \left(\frac{|X|}{2}\right)|Y| + \left(\frac{|Y|}{2}\right)|Z| + \left(\frac{|Z|}{2}\right)|X| \sim \frac{5}{9} \binom{n}{3}. $$

It is straightforward to check that $H$ is $K_3^3$-free.

### 3.3.3 Turán densities for $K_t^r$

Define the *Turán density* to be

$$\pi(K_t^r) = \lim_{n \to \infty} \frac{\text{ex}(n, K_t^r)}{\binom{n}{r}}. $$

The quantity in the limit is a nonincreasing sequence (and therefore the limit exists), due the following averaging argument originally discovered by Katona, Nemetz and Simonovits [29]. Suppose $H$ is a $K_t^{(r)}$-free $r$-graph on $n$ vertices having $\text{ex}(n, K_t^r)$ edges (the maximum number). Then the average degree of $H$ is $d = \text{ex}(n, K_t^r) \cdot r/n$. If we delete a vertex $v$ having degree at most $d$, then the resulting induced subgraph is still $K_t$-free and satisfies

$$\frac{|(H - \{v\})|}{\binom{n-1}{r}} \geq \frac{\text{ex}(n, K_t^r)(1 - r/n)}{\binom{n-1}{r/r}} = \frac{\text{ex}(n, K_t^r)}{\binom{n}{r}}. $$

Conjecture 3.4 states that the tetrahedron has Turán density $5/9$. In many cases, there are many conjectured asymptotically extremal constructions (for example, Kostochka [37] has many non-isomorphic constructions of near-extremal tetrahedron free hypergraphs), and therefore it is very unlikely that an analog of Turán’s Theorem holds for hypergraphs and the structure of extremal $K_t^r$-free hypergraphs seems in general to be very complicated when $t$ is fixed.

Since $\text{ex}(t, K_t^r) = \binom{t}{r} - 1$, the fact that maximal densities are nonincreasing implies that

$$\pi(K_t^r) \leq \left(1 - \frac{1}{\binom{t}{r}}\right).$$
An improvement of this bound was found by de Caen [15], and this remains the best upper bound for hypergraph Turán numbers in general:

$$\pi(K_t^r) \leq 1 - \frac{1}{(t-1)^{r-1}}.$$ 

The highest lower bound was found by a construction of Sidorenko [49], namely

$$\pi(K_t^r) \geq 1 - \left(\frac{r - 1}{t - 1}\right)^{r-1}.$$ 

For more detailed particular results, the reader is referred to the survey of Keevash [30].

### 3.4 Expected weight proof of $\alpha(H) \geq n \left(\frac{\log d}{d \log \log d}\right)$ for $K_t$-free graphs

In Chapter 1 we mentioned Johannson’s proof that if $G$ is a $K_t$-free graph,

$$\alpha(H) \geq Cn \frac{\log d}{d \log \log d}.$$ 

Although this manuscript is referenced in [5], the proof never appeared in print. In this section we present a new proof which is based on the method of Shearer [48]. This could possibly be extended to the hypergraph case if the right weight function were found.

**Proposition 3.5.** Let $G$ be a $K_t$-free graph on $n$ vertices with maximum degree $\Delta$. If $t \geq 4$ then

$$\alpha(G) \geq cn \frac{\log \Delta}{\Delta \log \log \Delta}$$

where $c$ is a constant depending only on $t$. If $t = 3$, then

$$\alpha(H) \geq cn \frac{\log \Delta}{\Delta}.$$ 

**Proof.** For this proof we use the graph notation for neighborhood: if $S \subseteq V$, $\Gamma(S) := \{x : \exists y \in S, \{x, y\} \in E(G)\}$. As usual we write $\Gamma(v)$ rather than $\Gamma(\{v\})$. 

Let $I$ be an independent set in $G$ chosen uniformly at random, and for $v \in V$ define

$$w(v) = \begin{cases} 
\Delta, & v \in I \\
|\Gamma(v) \cap I|, & v \notin I 
\end{cases}$$

Then $\sum_v w(v) = \Delta|I| + (\text{# of edges leaving } I) \leq 2\Delta|I|$. Hence

$$E[\sum_v w(v)] \leq 2\Delta nE[|I|].$$

(3.1)

If $t = 3$, our aim is to prove the lower bound

$$E[w(v)] \geq c \log \Delta$$

(3.2)

for all $v \in V$. If $t \geq 4$, we aim to prove

$$E[w(v)] \geq c \frac{\log \Delta}{\log \log \Delta}.$$  

(3.3)

Combining this with the upper bound above, it follows that

$$\alpha(G) \geq E[|I|] \geq \frac{cn \log \Delta}{2\Delta \log \log \Delta}.$$  

If $t = 3$ we have the same bound without the $\log \log \Delta$ factor.

To prove (3.2) and (3.3), we condition on the random set $I^* := I \setminus \{v\} \setminus \Gamma(v)$. Intuitively, we haven chosen not to “reveal” whether or not $I$ contains $v$ or its neighbors. Given $I^*$, the set $I$ is uniformly random among all independent sets of $G$ containing $I^*$. Let $X = \Gamma(v) \setminus \Gamma(I^*)$. Then one of the following must be true:

1. $I = I^* \cup \{v\}$, or

2. $I = I^* \cup J$, where $J$ is an independent set in the induced sub-hypergraph $G[X]$.

Therefore,

$$E[w(v)|I^*] = \frac{\Delta + \sum J |J|}{1 + \sum J 1},$$

where the sums are taken over independent sets in $G[X]$. Define $N := \sum J 1$, the number of independent sets in $G[X]$, and $\bar{J} = \sum J |J|/N$, the mean size of an independent set in $G[X]$. Then we have

$$E[w(v)|I^*] \geq \frac{\Delta + \sum J |J|}{2N} = \frac{\Delta}{2N} + \bar{J}.$$  

(3.4)
Case 1: $t = 3$. Since $G$ is $K_3$-free, it follows that $X$ is an independent set. Therefore $N = 2^{|X|}$ and $\bar{J} = |X|/2$. We can now prove (3.2) using (3.4), for $c = \frac{1}{2}$ and sufficiently large $\Delta$, as follows. If $|X| \geq \log \Delta$, then

$$E[w(v)|I^*] \geq \bar{J} \geq \frac{1}{2} \log \Delta$$

as desired. On the other hand, if $|X| < \log \Delta$, then

$$E[w(v)|I^*] \geq \frac{\Delta}{2 \cdot 2^{|X|}} = \frac{1}{2} \Delta^{1-\log 2} > \frac{1}{2} \log \Delta.$$  

By the Tower Law for conditional expectation, $E[w(v)] \geq \frac{1}{2} \log \Delta$ which completes the proof for $t = 3$.

Case 2: $t \geq 4$. This case is significantly more difficult because $G[X]$ may be nonempty. First let us take care of the following easy case: if $\bar{J} \geq \frac{|X|}{10}$ (say), then we just use the bound $N \leq 2^{|X|}$ and we get

$$E[w(v)|I^*] \geq \frac{\Delta}{2 \cdot 2^{|X|}} + \frac{|X|}{10}$$

which is always at least $\frac{1}{10} \log \Delta$ regardless of the value of $|X|$, as before.

Otherwise, we use a series of estimates to find an upper bound for $N$ in terms of $\bar{J}$. For any pair of integers $a \geq b \geq 0$ let us define the quantity

$$\mu(a, b) = \frac{\sum_{i=0}^{b} i \binom{a}{i}}{\sum_{i=0}^{b} \binom{a}{i}},$$

which is the mean size of all subsets of $[a]$ whose size does not exceed $b$. It is clear that regardless of the value of $a$, $\mu(a, b) \geq b/2$. Furthermore, if $b$ is any integer such that $N > \sum_{i=0}^{b} \binom{|X|}{i}$, it follows that $\bar{J} > \mu(|X|, b)$. Using this statement in the contrapositive, since $\bar{J} \leq \mu(|X|, \lceil 2\bar{J} \rceil)$ it follows that

$$N \leq \sum_{i=0}^{\lceil 2\bar{J} \rceil} \binom{|X|}{i}.$$  

Next we use the following elementary bound:

**Lemma 3.6.** Let $a, b$ be integers satisfying $0 \leq 4b \leq a$. Then

$$\sum_{i=0}^{b} \binom{a}{i} \leq \frac{3}{2} \binom{a}{b}.$$  

Proof.

\[
\sum_{i=0}^{b} \binom{a}{i} = \binom{a}{b} \left( 1 + \frac{b}{a-b+1} + \frac{b(b-1)}{(a-b+1)(a-b+2)} + \ldots \right) < \binom{a}{b} \left( 1 + \frac{1}{3} + \frac{1}{9} + \ldots \right) = \frac{3}{2} \binom{a}{b}.
\]

As discussed at the beginning of Case 2, we may assume \([2\bar{J}] \leq \lceil |X|/5 \rceil \leq |X|/4\), so the lemma gives

\[ N \leq \frac{3}{2} \left( \frac{|X|}{2\bar{J}} \right). \]

Next, we need only the trivial bound \( \binom{a}{b} \leq a^b = \exp(b \log a) \):

\[ N \leq \frac{3}{2} \exp(2\bar{J} \log |X|) \quad (3.5) \]

Next we want to show that \(|X| \leq J^k\) for some constant \(k\). For this, we need the fact that \(G\) is \(K_t\)-free (we have not used this yet), plus a little Ramsey theory. Because \(G\) is \(K_t\)-free and \(X\) is contained in the neighborhood of \(v\), \(G[X]\) is \(K_{t-1}\)-free. Next, we claim that \(X\) contains no independent set of size \(3\bar{J} \log |X|\). Indeed, if it does, then \(N \geq 2^{3\bar{J} \log |X|} = \exp((3 \log 2)\bar{J} \log |X|)\), contradicting (3.5). Recall Ramsey’s Theorem, which may be stated as follows: If \(G\) is a \(K_a\)-free graph on \(n\) vertices with no independent set of size \(b\), then \(n < R(a,b) \leq \binom{a+b-2}{a-1}\). Applying this to the graph \(G[X]\), we get

\[ |X| < R(t-1, \lceil 3\bar{J} \log |X| \rceil) \leq k(\bar{J} \log |X|)^{t-2}, \]

where \(k\) is a constant depending only on \(t\). Let us simplify this further using \((\log |X|)^{t-2} \leq \sqrt{|X|}\), which gives

\[ |X| < k^2 \bar{J}^{2(t-2)}. \]

Now, (3.5) becomes

\[ N \leq \frac{3}{2} \exp(k_2 \bar{J} \log \bar{J}). \]

Finally, by using this bound in (3.4), we complete the proof by letting \(d = \Delta\), \(y = \bar{J}\) in the following lemma.
Lemma 3.7. Let $k > 0$. There exists a constant $c > 0$ such that for all $d \geq 3$ and $y > 0$,
\[
\frac{d}{\exp(ky \log y)} + y \geq c \frac{\log d}{\log \log d}.
\]

**Proof.** For each $d$ let $y_0(d)$ be the minimizing value. Set $y_0(d) = \beta \frac{\log d}{\log \log d}$, where $\beta$ is an unknown function. The minimum value equals
\[
d \exp\left(-k\beta \frac{\log d}{\log \log d} (\log \beta + \log \log d - \log \log \log d)\right) + \beta \frac{\log d}{\log \log d}.
\]
If $\liminf_{d \to \infty} \beta < 0.9/k$, then on some subsequence this expression is of the order $d \cdot d^{-k\beta} > d^{0.1}$. This is a contradiction since we can do better by setting $\beta = 1.1/k$ (in which case the term on the left goes to zero). Thus the term on the right must be larger than $\frac{0.9}{k} \frac{\log d}{\log \log d}$ for sufficiently large $d$, which proves the claim.  
\[\blacksquare\]
Chapter 4

Dual Hypergraphs, matchings, and incidence graphs

In this chapter we develop various tools related to matchings and incidence graphs, leading up to Theorem 4.5 which is a tight and original theorem for hypergraphs of maximum degree 2. This theorem provides the base case for the general case in Chapter 6.

Let $G$ be a graph (i.e. a 2-graph). Recall that a matching $M$ of $G$ is a set of edges in $G$ which are pairwise disjoint, and a maximum matching is a matching of maximum size. A perfect matching is a matching which covers every vertex, and a near-perfect matching is a matching which covers every vertex but one. Of course, a perfect (near-perfect) matching can only exist if $n(G)$ is even (odd).

4.1 The Tutte-Berge formula

Let us define

$$\text{miss}(G) = \min\{|V(G)| - 2|M| : M \text{ a matching of } G\}.$$  

In other words, miss$(G)$ is the number of vertices not covered by a maximum matching. The well-known Tutte-Berge formula [10] states that

$$\text{miss}(G) = \max\{\text{odd}(G - S) - |S| : S \subseteq V(G)\}, \quad (4.1)$$
where \( \text{odd}(G - S) \) is the number of connected components in the induced subgraph \( G - S \) that have an odd number of vertices. One can quickly see why the “\( \geq \)” direction of this equation should hold, as follows. If \( M \) is a maximum matching of \( G \), \( S \) is an arbitrary subset of \( V \), and \( K \) is an odd-sized component of \( G - S \) which is completely covered by \( M \), then there must be an edge of \( M \) having one endpoint in \( K \) and the other endpoint in \( S \). However, the number of such edges altogether is at most \( |S| \); this means that there are at least \( \text{odd}(G - S) - |S| \) odd components not completely covered by \( M \), and each one contributes at least one vertex to \( \text{miss}(G) \).

The Tutte-Berge formula can be restated as follows: there exists a set \( S \subseteq V \) (possibly empty) such that \( \text{odd}(G - S) - |S| = \text{miss}(G) \). Such a set will be called a Tutte-Berge witness set. We will need the following fact about Tutte-Berge witnesses:

**Proposition 4.1.** Let \( S \) be a Tutte-Berge witness for the graph \( G \). Then every even component of \( G - S \) has a perfect matching, and every odd component of \( G - S \) has a near-perfect matching.

**Proof.** Let \( M \) be a maximum matching for \( G \). Since \( \text{miss}(G) = \text{odd}(G - S) - |S| \), it follows that the vertices not covered by \( M \) belong to distinct odd components of \( G - S \). Moreover, there are exactly \( |S| \) edges having an endpoint in \( S \), and the other endpoints of these edges lead to distinct odd components of \( G - S \). If we delete these edges from \( M \), what remains is a perfect matching for each even component of \( G - S \) and a near-perfect matching for each odd component.

\[\blacksquare\]

### 4.2 The Dual of a Hypergraph

In this section we consider reversing the role of vertices and edges in a hypergraph. To do this in general, however, requires us to consider hypergraphs with multigraphs; that is, two or more edges sharing the same vertex set.
4.2.1 Multi-hypergraphs

Let us define a multi-hypergraph $H$ to be a pair $(G, X)$ where $G$ is a bipartite graph, and $X \subseteq V(G)$ is one of the parts. (That is, all edges in $G$ have one endpoint in $X$ and the other in $X^C$). This graph is referred to as the bipartite adjacency graph for $H$. We identify $V(H) = X$, $E(H) = X^C$, so $X$ represents the vertices of $H$ and $X^C$ represents the edges. If $H$ is a hypergraph, then it is naturally the multi-hypergraph $(G, V(H))$ where $G$ has the vertex set $V(H) \cup E(H)$, and $v \sim e$ in $G$ if and only if $v \in e$ in $H$. However, if $H_1 = (G, X)$ is a multi-hypergraph, there is not necessarily any hypergraph $H$ whose multi-hypergraph is isomorphic to $H_1$. In particular, $H_1$ is not a hypergraph if there exist two edges $e_1, e_2 \in V(H_1)$ which are adjacent (in $G$) to the same set of vertices. In a multi-hypergraph, the degree of a vertex $v$ is its degree in the bipartite adjacency graph. Similarly the size of an edge $e$ is its degree in the bipartite adjacency graph.

If $H = (G, X)$ is a multi-hypergraph, then its dual $H^*$ is the multi-hypergraph $(G, X^C)$. That is, we simply reverse the role of vertices and edges. If $H$ is a hypergraph, we abuse notation and let $H^*$ denote the dual of the multi-hypergraph for $H$. However, this dual is not necessarily a hypergraph.

4.2.2 The dual of a hypergraph of maximum degree 2

Suppose $H$ is a hypergraph with $\Delta(H) = 2$ and no isolated vertices. Our use of the dual construction will, in fact, be limited to this case. Then $H^*$ is a multi-hypergraph in which every edge has size 1 or 2. Thus, we can view $H^*$ as a graph, possibly having size 1 edges (a.k.a. loops) and multi-edges. For the purpose of matchings, however, we will ignore the loops and multi-edges to create the following graph:

**Definition 4.2.** If $H$ is a hypergraph with $\Delta(H) \leq 2$, then define $G(H)$ to be the graph with vertex set $H$, and $e \sim f$ in $G$ if and only if $e \cap f \neq \emptyset$ in $H$. For brevity we write

$$\text{miss}(H) = \text{miss}(G(H)).$$

In terms relating purely to $H$, miss($H$) is the number of edges which do
not intersect $M$, when $M$ is a maximum-size set of pairwise nonadjacent degree 2 vertices. There is a direct relationship between $\alpha(H)$ and $\text{miss}(H)$, which we quantify in the following proposition.

**Lemma 4.3.** Suppose $H$ is an $r$-uniform hypergraph with $n$ vertices and $m$ edges in which all vertices have degree 1 or 2. Then

$$\alpha(H) = n - \frac{m + \text{miss}(H)}{2}.$$ 

**Proof.** Let $M \subseteq V(H)$ be a maximum-size set of pairwise nonadjacent degree 2 vertices (i.e. a maximum matching of $G(H)$). Then $\text{miss}(H) = m - 2|M|$, and clearly $H$ has a transversal of size

$$|M| + \text{miss}(H) = (m - \text{miss}(H))/2 + \text{miss}(H) = (m + \text{miss}(H))/2.$$ 

Therefore

$$\tau(H) \leq (m + \text{miss}(H))/2.$$ 

On the other hand, let $T$ be a minimum transversal of $H$, and let $M \subseteq T$ be a set of nonadjacent degree 2 vertices, of maximum size as a subset of $T$. Consider the subhypergraph $H_1 = \{e \in H : e \cap M = \emptyset\}$. By the maximality of $M$, $H_1$ consists of pairwise disjoint edges. Since $M$ contains no adjacent vertices, $|H_1| = m - 2|M|$. By the minimality of $T$, $|T| = |M| + |H_1| = (m + |H_1|)/2$. Finally, it is clear that $|H_1| \geq \text{miss}(H)$. Therefore

$$\tau(H) = |T| \geq (m + \text{miss}(H))/2.$$ 

Since $\alpha(H) = n - \tau(H)$, the proof is complete.

4.3 Independence numbers in the case $\Delta(H) \leq 2$

In our effort to prove a hypergraph theorem of the form $\alpha(H) \geq \sum_v f(d_v)$, it will be instrumental to first discuss the case where $\Delta(H) \leq 2$. Having discussed dual graphs and the Tutte-Berge formula, we are now ready to prove the following:
Theorem 4.4. Let $H$ be an connected $r$-graph of maximum degree 2. If $r$ is even, then $\alpha(H)/n$ is minimized if and only if $H$ is isomorphic to the hypergraph $T_r$ shown below. If $r$ is odd, then $\alpha(H)/n$ is minimized if and only if $H$ is isomorphic to $T_r^*$ or $U_r$.

\[ \begin{align*}
\diamond &= \text{floor}\left(\frac{r}{2}\right) \text{ vertices} \\
\bullet &= \text{1 vertex} \\
\circ &= \text{edge}
\end{align*} \]

![Figure 3: The hypergraphs $T_r$, $T_r^*$, and $U_r$.](image)

Theorem 4.5. Let $H$ be a connected $r$-graph with $r \geq 3$, $\Delta(H) \leq 2$. For $i = 0, 1, 2$ let $n_i(H)$ denote the number of degree $i$ vertices in $H$. Also let

\[ a_1 = 1 - 1/r, \quad a_2 = \begin{cases} 1 - \frac{4r-2}{(3r-1)r}, & r \text{ odd} \\ 1 - \frac{4(r-1)}{(3r-2)r}, & r \text{ even} \end{cases} \]

If $H \not\cong T_r$, then

\[ \alpha(H) \geq n_0(H) + a_1n_1(H) + a_2n_2(H). \]

Of these two theorems, the former is perhaps more intuitively interesting, but the latter will form the base case for our more general theorems in Chapter
5. $T_r$ is excluded from Theorem 4.5 because doing so allows us to prove a tighter theorem that holds for all other hypergraphs.

First will need the following lemma characterizing $T_r$ and $T^*_r$:

Lemma 4.6. If $r$ is even, $T_r$ is the only 2-regular $r$-graph with exactly 3 edges (up to isomorphism). If $r$ is odd, $T^*_r$ is the only $r$-graph with exactly 3 edges, a single vertex of degree 1, and all other vertices of degree 2.

Proof. Let $H$ be a hypergraph having the above properties, with edge set $\{e_1, e_2, e_3\}$. If $r$ is odd assume that $e_1$ contains the unique vertex of degree 1. Define $e_{ij} = e_i \cap e_j$ for short. Since $e_2$ contains only degree 2 vertices, it is the disjoint union of $e_{12}$ and $e_{23}$, so $r = |e_{12}| + |e_{23}|$. Similarly $r = |e_{13}| + |e_{23}|$. Thus $|e_{12}| = |e_{13}|$. Furthermore, in edge $e_1$ we have $r = \eta + |e_{12}| + |e_{13}|$, where $\eta = 0$ if $r$ is even and $\eta = 1$ if $r$ is odd. Therefore

$$|e_{12}| = |e_{13}| = \frac{r - \eta}{2},$$
$$|e_{23}| = r - |e_{12}| = \frac{r + \eta}{2}.$$

This shows that $H$ has the Venn diagram of $T_r$ or $T^*_r$, completing the proof.

We also require one more lemma which is nothing more than elementary algebra.

Lemma 4.7. Let

$$f(x) = \frac{ax + b}{cx + d}$$

where $a, d \geq 0$ and $c > 0$. If $0 \leq x_1 \leq x_2$, $\frac{a}{c} \geq t$, and $f(x_1) \geq t$, then $f(x_2) \geq t$.

Proof. We may rewrite

$$f(x) = \frac{1}{c} \left( a + \frac{bc - ad}{cx + d} \right).$$

If $bc \leq ad$ then $f$ is increasing (or constant) on the domain $x > -d/c$, so the lemma holds. Otherwise, we have $f(x) > a/c \geq t$ for all $x > -d/c$.

We now turn to the proof of Theorem 4.5. Clearly we may assume that $H$ is connected and contains no vertices of degree 0.
4.3.1 Proof of Theorem 4.5, assuming $\text{miss}(H) \leq 1$

If $\text{miss}(H) \leq 1$, then we must have $\text{miss}(H) = 0$ if $m$ is even, $\text{miss}(H) = 1$ if $m$ is odd. We will extend to the general case in the second part of the proof.

For an arbitrary $r$-graph $k$, let

$$a(H) = n_0(K) + a_1n_1(K) + a_2n_2(K).$$

By Lemma 4.3,

$$\alpha(H) = n - \frac{1}{2}(m + \text{miss}(H)).$$

Since $n = n_1 + n_2$ and $rm = n_1 + 2n_2$, we can rewrite this as

$$\alpha(H) = n_1 + \frac{1}{2}(rm - n_1) - \frac{1}{2}(m + \text{miss}(H))$$

$$= \frac{1}{2}(n_1 + (r - 1)m - \text{miss}(H)). \quad (4.2)$$

Next we rewrite

$$a(H) = a_1n_1 + a_2(rm - n_1)/2 = \frac{1}{2}((2 - \frac{2}{r} - a_2)n_1 + a_2rm).$$

Therefore,

$$\frac{\alpha(H)}{a(H)} = \frac{n_1 + (r - 1)m - \text{miss}(H)}{(2 - \frac{2}{r} - a_2)n_1 + a_2rm}. \quad (4.3)$$

Our aim is to show that $\alpha(H)/a(H) \geq 1$.

**Case 1: $r$ is odd.** First we check that $\alpha(H) = a(H)$ in the following two cases.

If $m = 1$, then $H$ is an isolated edge so $n_1 = r, n_2 = 0$. Thus $a(H) = (1 - \frac{1}{r})r = r - 1 = \alpha(H)$.

If $m = 3$ and $n_1 = 1$, then from (4.2),

$$\alpha(H) = 3(r - 1)/2.$$

Also $n_2 = (rm - n_1)/2 = (3r - 1)/2$, and $a_2$ has been defined precisely so that the following holds:

$$a(H) = 1 - \frac{1}{r} + \frac{(1 - \frac{4r - 2}{r(3r - 1)})(3r - 1)/2}{r - 1} - \frac{2}{2} - \frac{2r - 1}{r}$$

$$= \frac{3r - 1}{2} - 1$$

$$= \frac{3r - 3}{2} = \alpha(H).$$
Next, we claim that
\[ 1 - \frac{2}{r} \leq a_2 \leq 1 - \frac{1}{r}. \] (4.4)

To see this, using the definition of \( a_2 \), the above is equivalent to
\[ 2 \geq \frac{4r - 2}{3r - 1} \geq 1 \]
which is easily verified for all \( r \geq 1 \). Now we can compute the ratios of the coefficients of \( m \) and \( n_1 \) in (4.3). The two inequalities in (4.4) give, respectively,
\[ \frac{1}{2 - \frac{2}{r} - a_2} \geq 1, \quad \frac{r - 1}{a_2 r} \geq 1. \]

If \( m \) is even, then \( \text{miss}(H) = 0 \) so
\[ \frac{\alpha(H)}{f(H)} = \frac{n_1 + (r - 1)m}{(2 - \frac{2}{r} - a_2)n_1 + a_2 rm} \geq \min \left( \frac{1}{2 - \frac{2}{r} - a_2}, \frac{r - 1}{a_2 r} \right) \geq 1. \]
Thus we can assume \( m \) is odd and \( m \geq 3 \). Since \( rm = n_1 + 2n_2 \), \( n_1 \) is odd, and in particular \( n_1 \geq 1 \). Let
\[ f(x, y) = \frac{x + (r - 1)y - 1}{(2 - \frac{2}{r} - a_2)x + a_2 ry}. \]

We checked above \( \alpha(H) = \alpha(H) \) if \( n_1 = 1 \) and \( m = 3 \); this implies \( f(1, 3) = 1 \). Applying Lemma 4.7 with \( x_1 = 1, \ x_2 = n_1, \ t = 1 \) we get \( f(n_1, 3) \geq 1 \). Applying the lemma again in the variable \( y \) gives \( f(n_1, m) \geq 1 \), so we are finished.

**Case 2: \( r \) is even.** Again the case \( m = 1 \) holds. If \( m = 3 \) and \( n_2 = 0 \) then \( H \cong T_r \) by Lemma 4.6, which we are assuming it is not. So let us check the case \( m = 3, \ n_2 = 2 \) (again, \( a_2 \) has been defined so that the following holds):
\[ \alpha(H) = (3r - 2)/2 \]
\[ n_2 = (rm - n_1)/2 = (3r - 2)/2 \]

\[ a(H) = (1 - \frac{1}{r})2 + \left( 1 - \frac{4(r - 1)}{r(3r - 2)} \right) \frac{3r - 2}{2} \]
\[ = \frac{2(r - 1)}{r} + \frac{3r - 2}{2} - \frac{2(r - 1)}{r} \]
\[ = \frac{3r - 2}{2} = \alpha(H). \]
Next, we claim that
\[ 1 - \frac{2}{r} \leq a_2 \leq 1 - \frac{6}{5r} < 1 - \frac{1}{r}. \] (4.5)

By the definition of \( a_2 \) this is equivalent to
\[ 2 \geq \frac{4(r - 1)}{3r - 2} \geq \frac{6}{5} \]
which holds for all \( r \geq 4 \). Again, (4.5) gives
\[ \frac{1}{2 - (2/r) - a_2} \geq 1, \quad \frac{r - 1}{a_2 r} \geq 1 \]
which means that we can reduce to the case \( m \geq 3 \) odd, as before.

Again let us define
\[ f(x, y) = \frac{x + (r - 1)y - 1}{(2 - \frac{2}{r} - a_2)x + a_2 ry}. \]

Suppose \( m = 3 \). Then we have \( n_1 \geq 2 \) (or else \( H = T_r \)). We checked above that \( f(2, 3) = 1 \), so by Lemma 4.7, \( f(n_1, m) = f(n_1, 3) \geq 1 \) also.

Lastly, we check the case \( m \geq 5 \). Since \( a_2 \leq 1 - \frac{6}{5r} \), we have
\[ f(0, 5) = \frac{5r - 6}{5a_2 r} \geq 1. \]

Therefore, by Lemma 4.7, \( f(0, m) \geq 1 \) and \( f(n_1, m) \geq 1 \).

### 4.3.2 Proof of Theorem 4.5: general case

Let \( H \) be a connected \( r \)-graph of maximum degree 2 and let \( S \subseteq H \) be a Tutte-Berge witness set for the dual \( G(H) \). Let \( H_1, H_2, \ldots, H_k \) denote the connected components of \( H \setminus S \). Note that the \( V(H_i) \) partition \( V(H) \). These components correspond directly to the components of \( G(H) - S \), and by Proposition 4.1 we have \( \text{miss}(H_i) \leq 1 \) for \( i = 1, \ldots, k \). Therefore, using Lemma 4.3
\[
\sum_{i=1}^{k} \alpha(H_i) = \sum_{i=1}^{k} \left( n(H_i) - m(H_i) + (m(H_i) \text{ mod } 2) \right)
\]
\[
= n - \frac{m(H) - |S| + \text{odd}(G(H) - S)}{2}
\]
\[
= n - \frac{m(H) + \text{miss}(H)}{2} = \alpha(H).
\]
Next, note that if $v \in H_i$ then $\deg_H(v) \geq \deg_{H_i}(v)$. Since $a_2 \leq a_1$ it follows that $a(H) \leq \sum_{i=1}^{k} a(H_i)$. Lastly, for each $i$ note that $H_i \not\cong T_r$ (otherwise $H = H_i \cong T_r$ since $H$ is connected and $H_i$ only has vertices of degree 2). By the first part of the proof, we can now conclude that

$$\frac{\alpha(H)}{a(H)} \geq \frac{\sum_{i=1}^{k} \alpha(H_i)}{\sum_{i=1}^{k} a(H_i)} \geq \min_{i=1,\ldots,k} \frac{\alpha(H_i)}{a(H_i)} \geq 1.$$

This completes the proof of Theorem 4.5.

### 4.3.3 Proof of Theorem 4.4

Suppose first $r$ is even. Utilizing the notation of the previous theorem, we see that

$$a_2 = 1 - \frac{4(r - 1)}{r(3r - 2)} > 1 - \frac{4}{3r},$$

since this is equivalent to $\frac{r-1}{3r-2} < \frac{1}{3}$ which is true for all $r$. Therefore, if $H$ is a connected $r$-graph of maximum degree at most 2 and different from $T_r$, we have

$$\alpha(H) \geq n_0 + a_1 n_1 + a_2 n_2 \geq n (1 - \frac{4}{3r}).$$

However, if $H \cong T_r$ then it is easy to check that $\alpha(H) = n(1 - \frac{4}{3r})$, which proves that $\alpha(H)/n$ is minimized if and only if $H \cong T_r$.

Now suppose $r$ is odd. For this case we use similar methods as the previous theorem. We claim that $\frac{\alpha}{n} \geq 1 - \frac{4}{3r+1}$ with equality only when $H \cong T_r^*$ or $U_r$.

From Lemma 4.3,

$$\frac{\alpha}{n} = 1 - \frac{m + \text{miss}(H)}{2n}.$$

First let us assume $\text{miss}(H) \leq 1$. Recall $rm = n_1 + 2n_2 = 2n - n_1$. If $m$ is even, then $\text{miss}(H) = 0$ and $n \geq rm/2$, so we get

$$\frac{\alpha}{n} \geq 1 - \frac{1}{r} > 1 - \frac{4}{3r}.$$

Therefore we may assume $m$ is odd. If $m = 1$ we have $\alpha/n = 1 - \frac{1}{r}$. Otherwise $m \geq 3$. Note that $n_1 \geq 1$, so $n \geq (rm + 1)/2$. This gives

$$\frac{\alpha}{n} \geq 1 - \frac{m + 1}{rm + 1} \geq 1 - \frac{4}{3r + 1}.$$
If equality holds, then \( m = 3 \) and \( n_1 = 1 \), which implies \( H \cong T^*_r \) (by Lemma 4.6).

Now for the general case where \( \text{miss}(H) \) is arbitrary. As in the proof of Theorem 4.5, let \( S \) be a Tutte-Berge witness for \( G(H) \) and let \( H_1, \ldots, H_k \) be the connected components of \( H \setminus S \). If \( S = \emptyset \) then \( \text{miss}(H) \leq 1 \) by the Tutte-Berge formula; therefore we may assume \( S \) is nonempty. As before, \( \sum_{i=1}^{k} \alpha(H_i) = \alpha(H) \) and therefore

\[
\frac{\alpha(H)}{n} = \frac{\sum_{i=1}^{k} \alpha(H_i)}{\sum_{i=1}^{k} n(H_i)} \geq \min_{i=1,\ldots,k} \frac{\alpha(H_i)}{n(H_i)} \geq 1 - \frac{4}{3r+1}.
\]

If equality holds, then we must have \( H_i \cong T^*_r \) for all \( i \). If \( e = \{v_1, \ldots, v_r\} \in S \), then each \( v_j \) must be the degree 1 vertex of \( H_i \) for some \( i \). This implies that the connected component of \( e \) is 2-regular and isomorphic to \( U_r \). Since \( H \) is connected, \( H \cong U_r \).

### 4.4 Hall’s marriage theorem

Hall’s marriage theorem is a tool found in many introductory combinatorics tests that we will need later on. The theorem states that if \( G \) is a bipartite graph with parts \( A \) and \( B \), then there exists a matching of \( A \) into \( B \), in other words a matching in \( G \) whose union covers \( A \), if and only if

\[
|N(S)| \geq |S|
\]

for every \( S \subseteq A \). This is clearly a necessary condition, but the theorem states that it is also sufficient.

### 4.5 The Expander-mixing lemma

To complete Chapter 4 we prove one more lemma related to bipartite graphs. Briefly, we use the bipartite graph’s adjacency matrix to show that the number of edges between any two sets is close to its average value. We will need this material for our algebraic construction in Chapter 7.
Lemma 4.8 (Expander-mixing). Suppose $G$ is a bipartite graph with parts $A, B$ such that the vertices in $A$ all have degree $a$ and the vertices in $B$ all have degree $b$. If $X \subseteq A$ and $Y \subseteq B$, then the number of edges from $X$ to $Y$ satisfies

$$|e(X,Y) - \frac{a}{|B|} |X||Y||X||Y| |X||Y| \leq \lambda_3 \sqrt{|X||Y||1 - |X|||A|}(1 - |Y|||B|) \leq \lambda_3 \sqrt{|X||Y|},$$

where $\lambda_3$ is the third-largest eigenvalue, in absolute value, of the adjacency matrix of $G$.

Let $M$ be the adjacency matrix for $G$, which has the form

$$M = \begin{bmatrix} 0 & N \\ N^T & 0 \end{bmatrix}.$$ 

In what follows, for any set of vertices $V$ let $1_V$ denote the column vector having 1s in the positions corresponding to $V$ and 0s elsewhere, and let $K = \begin{bmatrix} 0 & J \\ J^T & 0 \end{bmatrix}$, where $J$ is the $|A| \times |B|$ all-ones matrix. We have

$$e(S,T) = 1_V^T M 1_X,$$

$$|X||Y| = 1_V^T K 1_X,$$

and therefore

$$|e(X,Y) - \frac{a}{|B|} |X||Y||X||Y| |X||Y| = |1_Y^T (M - \frac{a}{|B|} K) 1_X|. \quad (4.6)$$

Define the subspace $W$ by $W^\perp = \text{Span}\{1_A, 1_B\}$. We make the following observations:

1. If $w \in W^\perp$, then $(M - \frac{a}{|B|} K)w = 0$.

   Proof: since $a|A| = b|B|$, it is easy to check that

   $$M 1_A = b 1_B = \frac{a}{|B|} K 1_A,$$

   $$M 1_B = a 1_A = \frac{a}{|B|} K 1_B.$$

2. If $w \in W$, then $Kw = 0$.

   Proof: Every row of $K$ is either $1_A^T$ or $1_B^T$, so this is immediate.
3. If \( w \in W \), then \( Mw \in W \) and \( ||Mw|| \leq \lambda_3||w|| \).

**Proof:** Let \( e_v \) denote the unit vector having a 1 in the position for vertex \( v \) and zeroes elsewhere, and let \( || \cdot ||_1 \) denote the \( L_1 \) vector norm. It is easy to see that \( ||M^2e_v||_1 = ab \) for all \( v \), and therefore \( |\lambda| \leq \sqrt{ab} \) for every eigenvalue \( \lambda \) of \( M \). On the other hand,

\[
M(\sqrt{a}1_A + \sqrt{b}1_B) = b\sqrt{a}1_B + a\sqrt{b}1_A = \sqrt{ab}(\sqrt{a}1_A + \sqrt{b}1_B),
\]

\[
M(\sqrt{a}1_A - \sqrt{b}1_B) = b\sqrt{a}1_B - a\sqrt{b}1_A = -\sqrt{ab}(\sqrt{a}1_A - \sqrt{b}1_B)
\]

which directly shows that the largest two eigenvalues are \( \pm \sqrt{ab} \) and the corresponding eigenvectors lie in \( W^\perp \). Since \( M \) is a symmetric matrix, the remaining eigenvectors span \( W \), which proves the claim.

Now, for any vector \( v \) let \( \overline{v} \) denote the orthogonal projection onto \( W \), so that \( \overline{v} \in W \) and \( v - \overline{v} \in W^\perp \). Using the above observations, we have

\[
1^T_Y(M - \frac{a}{|B|}K)1_X = 1^T_Y(M - \frac{a}{|B|}K)\overline{1_X} = 1^T_YM\overline{1_X} = \overline{1_Y^TM}\overline{1_X}.
\]

Therefore, by equation (4.6), observation 3, and the Cauchy-Schwartz inequality,

\[
|e(X,Y) - \frac{a}{|B|}|X||Y|| \leq \lambda_3||\overline{1_Y}|| ||\overline{1_X}||.
\]

Finally, we compute

\[
\overline{1_X} = 1_X - \frac{1_X \cdot 1_A}{1_A \cdot 1_A}1_A = 1_X - \frac{|X|}{|A|}1_A,
\]

so

\[
||\overline{1_X}|| = \sqrt{|X|(1 - |X|/|A|)^2 + (|A| - |X|)|X|^2/|A|^2}
\]

\[
= \sqrt{|X| - 2\frac{|X|^2}{|A|} + \frac{|X|^3}{|A|^2} - \frac{|X|^3}{|A|^2}} = \sqrt{|X|(1 - |X|/|A|)}.
\]

Similarly, \( ||\overline{1_Y}|| = \sqrt{|Y|(1 - |Y|/|B|)} \) which gives

\[
|e(X,Y) - \frac{a}{|B|}|X||Y|| \leq \lambda_3\sqrt{|X|(1 - |X|/|A|)}\sqrt{|Y|(1 - |Y|/|B|)},
\]

which is the inequality we wanted to prove.
Chapter 5

Independent sets in bounded-degree hypergraphs

This chapter is devoted to proving degree-sequence bounds for $\alpha(H)$, which take the form

$$\alpha(H) \geq \sum_{v \in V(H)} f(\text{deg}(v))$$

for some function $f : \mathbb{Z}^{\geq 0} \mapsto \mathbb{R}$. Henceforth let $f(H)$ denote the sum on the right. In Chapter 4, we proved some degree-sequence bounds for hypergraphs of maximum degree 2. In this chapter, we use inductive methods to extend these results to arbitrary maximum degree.

Our basic method is the following: To find an independent set in an $r$-graph $H$, one can remove maximum-degree vertices one at a time until we are left with a hypergraph $Y$ of maximum degree 2. From the definition of independent set, it is clear that $\alpha(H) \geq \alpha(Y)$. To determine the independence number of $Y$, we will define an invariant $f(H)$ which does not decrease when maximum-degree vertices are removed. Also, using our Theorem 4.5 we can define $f$ so that $\alpha(Y) \geq f(Y)$ when $\Delta(Y) \leq 2$. This gives a lower bound on $\alpha(H)$, since

$$\alpha(H) \geq \alpha(Y) \geq f(Y) \geq f(H).$$

This bound is given formally in Theorem 5.1. This theorem is very similar to a Theorem of Caro and Tuza [12] but improves it by a constant factor by using
$\Delta = 2$ as the base case rather than $\Delta = 0$.

Moreover, when $r$ is even, as a natural consequence of Theorem 4.5, the minimum value of $\alpha(Y)/f(Y)$ is somewhat larger when $Y$ does not contain a connected component isomorphic to $T_r$. We can take advantage of this using the following fact (Proposition 5.6): If $H$ is 3-regular, we can delete degree 3 vertices in such a way that $T_r$ does not arise. This theorem gives a good lower bound for $\alpha(H)/n$ in the case when $\Delta(H) \leq 3$; however it does not give a degree-sequence bound.

In Section 5.2 we demonstrate a way to improve the recurrence relation for $f$. This method, which is based on a collaboration with Anders Yeo (U. of Johannesburg), gives an improved theorem $\alpha(H) \geq g(h)$. After detailing the proof for this, we show how to combine the “avoiding $T_r$” method with the improved recursion, which gives a further improved theorem $\alpha(H) \geq h(H)$. This latter theorem is currently the best known of its kind when $r$ is even and at least 6.

5.1 Two theorems

In this section we state and prove the following theorems, in which the recursion on the function $f$ is originally due to Caro and Tuza [12].

**Theorem 5.1.** As usual $r \geq 3$ is a fixed integer. Let us define

$$f(0) = 1, \quad f(1) = 1 - 1/r,$$

$$f(2) = \begin{cases} 1 - \frac{4r-2}{(3r-1)r}, & r \text{ odd} \\ 1 - \frac{4}{3r}, & r \text{ even} \end{cases}.$$  

Then for each integer $d \geq 3$, define

$$f(d) = f(2) \prod_{k=3}^{d} \frac{k(r-1)}{k(r-1) + 1}.$$  

If $H$ is an $r$-graph, then

$$\alpha(H) \geq \sum_{v \in V(H)} f(d_v).$$
In particular, \( \alpha(H) \geq nf(\Delta) \), and if \( \Delta \geq 2 \) this can be rewritten as follows with a bit of algebra:

\[
\alpha(H)/n \geq \left( \frac{\Delta + \frac{1}{\Delta}}{r+1} \right)^{-1} \left( \frac{1 + \frac{r}{2(r-1)(3r-1)}}{1 + \frac{r-2}{6(r-1)^2}}, \quad r \text{ odd} \right). 
\]

Here, \( \left( \frac{\Delta + \frac{1}{\Delta}}{r+1} \right)^{-1} \) is the previous lower bound in [12].

For the second theorem, we improve the case of \( \Delta = 3, r \text{ even} \).

**Theorem 5.2.** If \( H \) is an \( r \)-graph of maximum degree 3, then

\[
\alpha(H)/n \geq \frac{3(r - 1)^2}{r(3r - 1)}. 
\]

### 5.1.1 Proof of Theorem 5.1

We claim that \( f \) has the following properties:

1. \( f \) is positive and decreasing.

2. The difference function is nonincreasing; for \( d \geq 0 \) we have \( f(d) - f(d + 1) \geq f(d + 1) - f(d + 2) \).

3. For \( d \geq 3 \),

\[
\frac{f(d)}{f(d+1)} = \frac{(r-1)d}{(r-1)d+1}. 
\]

Each of these properties is trivial except (2). If \( d \geq 2 \), note that by property (3), \( \frac{f(d+1)}{f(d)} < \frac{f(d+2)}{f(d+1)} \). In particular \( f(d+2) \) is greater than the third term of the geometric progression determined by \( f(d), f(d+1) \), which is in turn greater than the third term of the arithmetic progression; therefore (2) holds for \( d \geq 2 \).

Lastly we need to check that it holds for \( d = 0, 1 \). For \( d = 0 \) this is equivalent to checking that \( f(2) \geq 1 - \frac{2}{r} \), which is easily done. For \( d = 1 \), we note that \( f(3) = f(2)(1 - \frac{1}{3(r-1)+1}) \), so we have to check that \( f(1) \geq f(2)(1 + \frac{1}{3(r-1)+1}) \), or

\[
f(2) \leq \frac{r - 1}{r} \frac{3r - 2}{3r - 1}. 
\]
When $r$ is odd, we leave the reader to check that this holds with equality. When $r$ is even, it still holds because
\[
1 - \frac{4}{3r} = 1 - \frac{4(r - 1/3)}{3(r - 1/3)r} = 1 - \frac{4r - 4/3}{(3r - 1)r} < 1 - \frac{4r - 2}{(3r - 1)r}.
\]
Therefore $f$ has the properties claimed above.

We now prove that $f(H)$ does not decrease when a maximum-degree vertex is deleted from $H$.

**Proposition 5.3.** Let $H$ be an $r$-graph with $\Delta(H) \geq 3$, and let $w \in V$ be a vertex of maximum degree. Let $H - w$ be the $r$-graph obtained by deleting $w$. Then
\[
f(H - w) \geq f(H).
\]

**Proof.** We use the properties (1) – (3) of $f$ proven above. Let $d_{vw}$ denote the number of edges containing both $v$ and $w$.

\[
\begin{align*}
f(H - w) - f(H) &= \left(\sum_{v \neq w} f(d_v - d_{vw}) - f(d_v)\right) - f(d_w) \\
&\geq \left(\sum_{v \neq w} d_{vw}(f(d_v - 1) - f(d_w))\right) - f(d_w) \quad \text{by (2)} \\
&\geq (r - 1)d_w(f(d_w - 1) - f(d_w)) - f(d_w) \quad \text{by (2)} \\
&= (r - 1)d_w(f(d_w - 1) - f(d_w)) - f(d_w) \\
&= f(d_w - 1)(r - 1)d_w - f(d_w)((r - 1)d_w + 1) \\
&= 0. \quad \text{by (3)}
\end{align*}
\]

Note that the last line requires $d_w = \Delta(H) \geq 3$.

Now we can prove Theorem 5.1. First suppose $\Delta(H) \leq 2$. If $r$ is odd, then by Theorem 4.5, $\alpha(H) \geq \alpha(H) = f(H)$. If $r$ is even, we can check that that $\alpha(T_r)/n(T_r) = 1 - \frac{4}{3r}$. Therefore $\alpha(H) \geq f(H)$ holds even if $H \cong T_r$.

Thus we may assume $\Delta(H) \geq 3$. Let $Y$ be the $r$-graph obtained by repeatedly deleting a vertex of maximum degree, until the maximum degree is at most 2. Since $Y$ is an induced sub-hypergraph of $H$, $\alpha(H) \geq \alpha(Y)$. Since $\Delta(Y) \leq 2$, $\alpha(Y) \geq f(Y)$. By Proposition 5.3, $f(Y) \geq f(H)$. Thus
\[
\alpha(H) \geq \alpha(Y) \geq f(Y) \geq f(H)
\]
so the proof is complete.
5.1.2 Proof of theorem 5.2

In the case where \( r \) is even, Theorem 4.5 gives an improved lower bound if \( Y \) contains no copy of \( T_r \). This is precisely the idea behind the proof of Theorem 5.2. First we need the following lemma, which can be proven using the methods of Chapter 4.

**Lemma 5.4.** Suppose \( r \) is even and let \( H \) be a connected \( r \)-graph of maximum degree 2, not isomorphic to \( T_r \). Then

\[
\frac{\alpha}{3n-m} \geq \frac{r-1}{3r-1}.
\]

**Proof.** Since the methods here are very similar to the proof of Theorem 4.5, we give only an outline. By Lemma 4.3,

\[
\frac{\alpha(H)}{3n-m} = \frac{n - (m + \text{miss}(H))/2}{3n-m}.
\]

First assume that \( H \) has an empty Tutte-Berge witness, so that \( \text{miss}(H) = 1 \) if \( m \) is odd, \( \text{miss}(H) = 0 \) if \( m \) is even. Using simple algebra, the right-hand side of the above equation is minimized when \( H = T_r \); however, the next-smallest value is realized by the hypergraph consisting of a single edge (details omitted). In this case, \( \alpha(H)/(3n-m) = \frac{r-1}{3r-1} \).

If \( H \) has a nonempty Tutte-Berge witness \( S \), then let \( H_1, H_2, \ldots, H_t \) denote the components of \( H \setminus S \). We saw in the proof of Theorem 5.2 that \( \alpha(H) = \sum_i \alpha(H_i) \). Therefore

\[
\frac{\alpha(H)}{n-3m} = \frac{\sum \alpha(H_i)}{\sum n(H_i) - \sum 3m(H_i) - 3|S|} > \frac{\sum \alpha(H_i)}{\sum (n(H_i) - 3m(H_i))} \geq \frac{r-1}{3r-1}.
\]

We also need to define the “blowup” of a graph.

**Definition 5.5.** If \( G = (V,E) \) is a graph (i.e. a 2-graph) and \( k \geq 1 \) is an integer, define \( G^{\times k} \) to be the 2\( k \)-graph whose vertex set is \( V \times [k] \), and whose edge set is \( \{e \times [k] : e \in E\} \).

For example, \( K_3^{\times r/2} \cong T_r \).

The core of the proof Theorem 5.2 is the following proposition:
Proposition 5.6. Suppose $r$ is even, and let $H$ be a 3-regular $r$-graph with no connected component isomorphic to $K_4^{xr/2}$. Then there exists a maximal set $X$ of pairwise nonadjacent vertices in $H$, such that $H - X$ contains no copy of $T_r$.

We postpone the proof of this until the next section. For now, let us use it to prove Theorem 5.2:

**Proof.** Let $H$ be an $r$-graph of maximum degree at most 3. Using Proposition 3.1, we can reduce to the case where $H$ is 3-regular: from this proposition we have $\alpha(H)/|V(H)| \geq \alpha(\tilde{H})/|V(\tilde{H})|$, so if Theorem 5.2 holds for the 3-regular graph $\tilde{H}$, then it holds for $H$. Also, we can reduce to the case where $H$ is connected, because if the theorem holds for each connected component of $H$ then it holds for $H$.

Therefore, let us assume $H$ is a connected 3-regular $r$-graph. If $H \cong K_3^{xr/2}$, then we have

$$\frac{\alpha(H)}{n} = \frac{3r/2 - 2}{3r/2} = \frac{3r - 4}{3r},$$

and by cross-multiplying one can check that $\frac{3r - 4}{3r} > (1 - 1/r)\frac{3r - r}{3r - 1}$ whenever $r > 5/3$, so the theorem holds.

Otherwise, by the above proposition, let $X$ be a maximal set of pairwise nonadjacent vertices such that $Y = H - X$ contains no copy of $T_r$. Note that $\Delta(Y) \leq 2$ since $X$ is maximal. Also note that each time a vertex is deleted from $X$, the invariant $3n - m$ is preserved. Because $Y$ contains no copies of $T_r$, Lemma 5.4 implies that

$$\alpha(H) \geq \alpha(Y) \geq \frac{r - 1}{3r - 1} (3n(Y) - m(Y)) = \frac{r - 1}{3r - 1} (3n(H) - m(H)).$$

Since $H$ is 3-regular, $m(H) = 3n(H)/r$. This gives

$$\alpha(H) \geq \frac{3(r - 1)^2}{r(3r - 1)} n(H).$$

5.1.3 Avoiding copies of $T_r$

In this section we prove Proposition 5.6 which is the key to Theorem 5.2. Assume $H$ is a connected 3-regular $r$-graph not isomorphic to $K_4^{xr/2}$; we must find
a maximal set $X$ of pairwise nonadjacent vertices such that $H - X$ contains no component isomorphic to $T_r$. That is, we want to show that $X$ can be chosen to intersect every copy of $T_r$ present in $H$.

Fortunately for us, copies of $T_r$ can’t intersect very much when $H$ has maximum degree 3. Let $A = \{e_1, e_2, e_3\}$ and $B = \{f_1, f_2, f_3\}$ be sets of 3 edges which induce copies of $T_r$, and assume they have a vertex $v$ in common. That vertex has degree 2 in both $A$ and $B$, but it cannot have degree four in $H$. It follows that $A$ and $B$ have an edge in common. In other words, $A \cup B$ induces a hypergraph isomorphic to $K_{4-r}^{\times r/2}$, where $K_{4-}$ denotes $K_4$ with an edge missing (or equivalently, two triangles sharing an edge). It is easy to see that there cannot be a third copy of $T_r$ attached, without making a $K_4^{\times 2}$.

Therefore, all copies of $T_4$ are contained in groups of one or two (with respect to the relation “shares a vertex”). In each group of two, pick a vertex from the shared edge and put it into $X$. Since each of these vertices has all three of its edges “accounted for,” it is clear that these vertices are not adjacent to each other, nor are they adjacent to any of the remaining copies of $T_r$.

It remains to show that we can choose a vertex from each isolated copy of $T_r$. These vertices might not be adjacent, so we have to be careful. Let $\mathcal{T}$ be the set of all these disjoint $T_r$, where each element of $\mathcal{T}$ is represented with a set of 6 vertices. Let $\hat{E}$ be the set of all edges which intersect at least one $T \in \mathcal{T}$, excluding the edges which actually make up the $T_r$. Since $H$ is 3-regular, every vertex in $\cup \mathcal{T}$ is contained in exactly one edge of $\hat{E}$. In particular, no two of these edges can intersect inside $\cup \mathcal{T}$.

Now let us consider the bipartite graph with parts $\mathcal{T}$ and $\hat{E}$, where of course $T \in \mathcal{T}$ and $e \in \hat{E}$ are adjacent if $T \cap e \neq \emptyset$. In this graph, a matching of $\mathcal{T}$ into $\hat{E}$ is a set of pairwise disjoint edges which covers $\mathcal{T}$. We claim that to choose a system of pairwise nonadjacent representatives for $\mathcal{T}$, it is sufficient to finding a matching of $\mathcal{T}$ into $\hat{E}$. If such a matching is found, then in each $T \in \mathcal{T}$, the representative chosen will be an element of $T \cap e_T$, where $e_T \in \hat{E}$ is the edge matched with $T$. Because we have a matching, no edge contains more than one of these representatives.
To show that a matching exists, we can use Hall’s Marriage Theorem: a matching of $T$ into $\hat{E}$ exists if and only if $|N(S)| \geq |S|$ for all subsets $S \subseteq T$, where $N(S)$ denotes neighborhood of $S$. This is immediate because any set $S$ contains a total of $6|S|$ vertices, and each $e \in \hat{E}$ can cover only 4 of them. Since every vertex is covered, $S$ must have at least $\frac{6|S|}{4} \geq |S|$ neighbors in $\hat{E}$.

This completes the proof of Proposition 5.6, as well as Theorem 5.2.

### 5.2 An improved recursion: the function $g(d)$

Recall the proof of Theorem 5.1: we considered $v \in V(H)$ of degree $\Delta$ and the graph $H' = H - v$. In $H'$, the vertex degrees have been reduced a total of $(r - 1)\Delta$ times; in other words,

$$\sum_{u \neq v} (\deg_H(u) - \deg_{H'}(u)) = (r - 1)\Delta.$$

The worst-case scenario for our function $f$ is when each reduction goes from degree $\Delta$ to degree $\Delta - 1$. This scenario defines the recursion for $f$, namely

$$f(d) = (r - 1)d(f(d - 1) - f(d)).$$

However, what if we could avoid this worst case and choose $v$ so that at least one vertex whose of degree less than $\Delta$ becomes reduced? This idea, pointed out to me through correspondence with Anders Yeo, leads to a function $g$ defined by a new formula:

$$g(d) = (r - 1)d(g(d - 1) - g(d)) - (g(d - 1) - g(d)) + (g(d - 2) - g(d - 1)).$$

When solved for $g(d)$, this yields the recurrence

$$g(d) = g(d - 1) - \frac{2g(d - 1) - g(d - 2)}{(r - 1)d}.$$

There is significant work in verifying that the aforementioned worst case can actually be avoided, except in certain cases when the theorem is true anyway. Below we present an adapted version of Yeo’s proof (with an improvement to the $r$ even case) that this can be accomplished, obtaining the following theorem:
Theorem 5.7. Define $g : \mathbb{Z}^{\geq 0} \mapsto \mathbb{R}$ as follows:

$$
g(0) = 1, \quad g(1) = 1 - 1/r
$$

$$
g(2) = \begin{cases} 
\frac{(r-1)(3r-2)}{r(3r-1)}, & r \text{ odd} \\
\frac{3r-4}{3r}, & r \text{ even}
\end{cases}
$$

$$
g(d) = g(d-1) - \frac{2g(d-1) - g(d-2)}{(r-1)d}, \quad d \geq 3.
$$

If $r > 3$ and $H$ is an $r$-graph, then

$$
\alpha(H) \geq g(H).
$$

5.2.1 Proof of Theorem 5.7: preliminary lemmas

The first step in the proof is to ensure that the new function $g$ is convex:

Proposition 5.8. The function $g$ defined above is positive and decreasing. Moreover, the difference function $\epsilon_g(d) = g(d-1) - g(d)$ is nonincreasing; that is, $g(d-2) - 2g(d-1) + g(d) \geq 0$ for all $d \geq 2$.

This is fairly laborious but routine to prove, so we relegate the proof to Appendix A.

The next lemma and its corollary provide our main tools for proving Theorem 5.7 inductively.

Lemma 5.9. Let $H$ be an $r$-graph of maximum degree $\Delta \geq 3$. Let $Q$ be a strongly independent set in $H$ such that each vertex in $Q$ has degree $\Delta$. Let $H' = H - Q$, and let

$$
k_Q = \sum_{v \sim Q} \min(\Delta - 1, \deg_H(v)) - \deg_{H'}(v),
$$

where $v \sim Q$ denotes that $v$ is adjacent (in $H$) to a vertex of $Q$, but is not a member of $Q$. Then

$$
g(H') - g(H) \geq (k_Q - |Q|)(g(\Delta - 2) - 2g(\Delta - 1) + g(\Delta)).
$$
Proof. Define the following set of ordered pairs, which we call *downsteps*:

\[ D := \{(v, d) : v \in V(H) - Q, \ d \in \mathbb{Z}, \ \deg_{H'}(v) < d \leq \deg_H(v)\}. \]

It is clear from this definition that \(|D| = |Q|(r - 1)\Delta\), and

\[ g(H') - g(H) = -|Q|g(\Delta) + \sum_{(v,d) \in D} (g(d - 1) - g(d)). \]

It is also evident that

\[ k_Q = \{|(v,d) \in D : d \leq \Delta - 1\}|. \]

Lastly, from the definition of \(g\) observe that \((r - 1)\Delta(g(\Delta - 1) - g(\Delta)) = 2g(\Delta - 1) - g(\Delta - 2)\). We now compute as follows:

\[
\begin{align*}
g(H') - g(H) \\
= & -|Q|g(\Delta) + \sum_{(v,d) \in D} (g(d - 1) - g(d)) \\
\geq & -|Q|g(\Delta) + k_Q(g(\Delta - 2) - g(\Delta - 1)) + (|Q|(r - 1)\Delta - k_Q)(g(\Delta - 1) - g(\Delta)) \\
= & -|Q|g(\Delta) + |Q|(r - 1)\Delta(g(\Delta - 1) - g(\Delta)) + k_Q(g(\Delta - 2) - 2g(\Delta - 1) + g(\Delta)) \\
= & -|Q|g(\Delta) + |Q|(2g(\Delta - 1) - g(\Delta - 2)) + k_Q(g(\Delta - 2) - 2g(\Delta - 1) + g(\Delta)) \\
= & (k_Q - |Q|)(g(\Delta - 2) - 2g(\Delta - 1) + g(\Delta)).
\end{align*}
\]

\[ \square \]

**Corollary 5.10.** Let \(Q\) be a strongly independent set of degree \(\Delta \geq 3\) vertices, and \(H' = H - Q\) as above. Suppose there exists a set \(R \subseteq V(H)\) with the following properties:

- \(Q \cap R = \emptyset\)
- For each \(v \in R\), \(v \sim Q\) and \(\deg_{H'}(v) \leq \Delta - 2\)
- \(|R| \geq |Q|\)

Then \(g(H') \geq g(H)\).

**Proof.** With the notation of the previous lemma, it is clear \(k_Q \geq |R|\). Since \(k_Q \geq |R| \geq |Q|\) and \(g(\Delta - 2) - 2g(\Delta - 1) + g(\Delta) \geq 0\), the corollary holds. \[ \square \]
5.2.2 Proof of Theorem 5.7

The proof is by strong induction: let $H$ be an $r$-graph, and assume $\alpha(H') \geq g(H')$ whenever $n(H') < n(H)$. If $H'$ is an induced sub-hypergraph of $H$, then $\alpha(H) \geq \alpha(H')$. If we can show that $g(H') \geq g(H)$ for any such $H'$, it follows that

$$\alpha(H) \geq \alpha(H') \geq g(H') \geq g(H)$$

so $H$ satisfies the theorem. We may also assume that $H$ is connected, since if the theorem holds for every connected component of $H$ then it holds for $H$.

**Case 1:** $\Delta(H) \leq 2$:

If $\Delta \leq 2$ then the theorem follows from Theorem 4.5.

**Case 2:** $H$ is a complete $(n,r,2)$ Steiner system. This implies $H$ is $\Delta$-regular and $n = 1 + (r - 1)\Delta$. Let $e \in H$ be any edge and let $x$ be any vertex not belonging to $e$ (which exists as $\Delta > 1$). Note that for every $y \in e$ there exists an edge containing $x$ and $y$, and all these edges are distinct, implying that $\Delta \geq r$.

Note that we assumed $r > 3$ in the Theorem 5.7. The reason for this is that there exists a $(9,3,2)$ complete Steiner system $H_9$ which is actually a counterexample to the theorem: $\alpha(H_9) = 4$ but $g(H_9) = 9 \left(\frac{43}{56}\right) = 4 + \frac{1}{32}$. However, Anders Yeo has shown that $H_9$ is the only counterexample to the theorem. At the end of this chapter we discuss how to account for $H_9$ and prove a version of the theorem for $r = 3$.

Suppose $\Delta \leq 4$. Since $3 < r \leq \Delta$ we must have $\Delta = r = 4$. If $e$ is any edge in $H$, it is straightforward to see that $V(H) \setminus e$ is an independent set, so we have $\alpha(H) \geq 13 - 4 = 9$, but $g(H) = 13g(4) < 13(.5571) = 7.2423$ so $H$ satisfies the theorem.

We now assume $\Delta \geq 5$. Let $e$ be any edge of $H$ and let $v_1, v_2, v_3$ be distinct vertices in $e$. Let $H' = H - v_1$ and note that $g(H') - g(H) \geq -(g(\Delta-2) - 2g(\Delta-1) + g(\Delta))$ by Lemma 5.9. Next, let $H'' = H' - \{v_2, v_3\}$. Observe that $\Delta(H') = \Delta - 1$, all the hypotheses of Lemma 5.9 hold, and $k_Q = 1 + (r - 1)\Delta - r = (r - 1)(\Delta - 1)$, because if $v \in V(H) - e$ then $\deg_{H'}(v) = \Delta - 3$. Therefore $g(H'') - g(H') \geq ((r - 1)(\Delta - 1) - 2)(g(\Delta - 3) - 2g(\Delta - 2) + g(\Delta - 1))$. Combining these two
equations,
\[ g(H'') - g(H) \geq ((r-1)(\Delta-1)-2)(g(\Delta-3)-2g(\Delta-2)+g(\Delta-1))-(g(\Delta-2)-2g(\Delta-1)+g(\Delta)). \]
By part (v) of Proposition A.1, \( g(H'') \geq g(H) \). As discussed above, this implies
\[ \alpha(H) \geq \alpha(H'') \geq g(H'') \geq g(H) \]
and therefore \( H \) satisfies the theorem.

**Case 3: \( H \) is not regular or not linear.** If \( H \) is not regular, let \( v \) be vertex of degree \( \Delta \) which is adjacent to a vertex \( w \) of degree less than \( \Delta \). If \( H \) is \( \Delta \)-regular but not linear, let \( \{v, w\} \) be a pair of vertices whose codegree is at least two. In either case, the hypotheses of Corollary 5.10 are satisfied with \( Q = \{v\}, R = \{w\} \). Therefore \( g(H') \geq g(H) \) and the theorem holds.

**Case 4: \( H \) is \( \Delta \)-regular and linear.** Let \( G \) be the adjacency graph of \( H \): namely \( V(G) = V(H) \), and \( v \sim w \) in \( G \) if there exists an edge of \( H \) containing \( v \) and \( w \). Note that \( G \) is \( \Delta(r-1) \)-regular and connected. Let \( x_1 \) be any leaf of a spanning tree of \( G \), so \( x_1 \) is not a cut vertex. If every pair of neighbors of \( x_1 \) is adjacent in \( G \), then \( G \) is a complete graph, \( H \) is a complete Steiner system, and we already considered this case.

Therefore let \( x_2, z \) be nonadjacent neighbors of \( x_1 \). Since \( x_1 \) is not a cut vertex, let \( x_2, x_3, \ldots, x_l = z \) be a minimum-length path from \( x_2 \) to \( z \) in \( G - x_1 \). Note that \( l \geq 4 \) and \( x_1, x_2, \ldots, x_l \) is a cycle in \( G \). We now construct sets \( Q, R \) satisfying the conditions of Corollary 5.10. If \( l \) is even, then let \( Q = \{x_2, x_4, \ldots, x_l\}, R = \{x_1, x_3, \ldots, x_{l-1}\} \). By the minimality of \( l \), \( Q \) is independent in \( G \) and therefore strongly independent in \( H \). The other conditions are evident, so we are done.

Therefore we consider the case where \( l \geq 5 \) is odd. Let \( e \) be the edge containing \( x_3 \) and \( x_4 \) in \( H \), let \( y \in e - \{x_3, x_4\} \), and let \( Q' = \{x_2, y, x_5, x_7, \ldots, x_l\} \). If \( Q' \) is strongly independent then let \( Q = Q' \) and \( R = \{x_1, x_3, x_4, x_6, \ldots, x_{l-1}\} \), and we are done since \( |Q| = |R| = (l+1)/2 \). If \( Q' \) is not strongly independent, then by the minimality of \( l \) we have either \( y \sim x_2 \) or \( y \sim x_5 \). If \( y \sim x_2 \) then let \( Q = \{x_2, x_4\}, R = \{x_3, y\} \). If \( y \sim x_5 \) then let \( Q = \{x_3, x_5\}, R = \{x_4, y\} \). In either case the hypotheses of Corollary 5.10 are satisfied.
5.3 Further refinements

If $r = 3$, the 4-regular Steiner system $H_9$ discussed above is an exception to Theorem 5.7. We have $g(H_9) = \alpha(H_9) + \frac{1}{32}$ so the best we can hope for is that $H_9$ is the only exception. Indeed, this is the case and we still have the following theorem:

**Theorem 5.11.** Let $r = 3$ and define $g$ as in Theorem 5.7. If $H$ is a 3-graph then

$$\alpha(H) \geq g(H) - \frac{l}{32},$$

where $l$ denotes the number of connected components of $H$ isomorphic to $H_9$.

The previous proof of Theorem 5.7 will not suffice for this, because of the possibility that $H_9$ can appear in the inductive step when vertices of the hypergraph are deleted. The mechanics for dealing with $H_9$ were contributed by Anders Yeo rather than this dissertation’s author, so the (somewhat lengthy) proof of this theorem has been omitted. Instead, we shall examine the case $r \geq 4$ even, where we can obtain a stronger version of Theorem 5.1 having the hypergraph $T_r$ as its sole exception. A few ideas from the $H_9$ proof have been borrowed and adapted, but the following material is mostly original. However, in the interest of saving space we restrict our attention to $r = 6$ and omit some of the details of the proof. In the near future we intend to publish a joint paper with complete proofs of all these results.

We present the case $r = 6$ here rather than $r = 4$ because the case $r = 4$ already has strong results known. In particular, Thomasse and Yeo [51] proved that if $r = 4$,

$$\tau(H) \leq \frac{5n}{21} + \frac{4m}{21}.$$  

To rewrite this as a degree-sequence bound, note that whenever $\Delta \leq 4$,

$$\alpha(H) \geq n - \tau(H) \geq n - \frac{(5n + 4m)}{21} = (16n - (n_1 + n_2 + n_3 + n_4))/21 = (15n_1 + 14n_2 + 13n_3 + 12n_4)/21.$$  

So we can define a new function $g$ with $g(1) = 15/21$, $g(2) = 14/21$, $g(3) = 13/21$ and $g(4) = 12/21$, and use the same recursion as above to define $g(d)$ for $d \geq 5$. 
This \( g \) gives better results than the function \( h \) below would, if it were defined for \( r = 4 \).

### 5.3.1 A refinement for \( r = 6 \)

**Theorem 5.12.** Define the function \( h : \mathbb{Z}_{\geq 0} \to \mathbb{R} \) as follows.

\[
\begin{align*}
  h(0) &= 1, & h(1) &= 5/6, & h(2) &= 625/798, \\
  h(d) &= h(d-1) + \frac{2h(d-1) - h(d-2)}{5d}, & d &\geq 3.
\end{align*}
\]

Recall the notation \( h(H) = \sum_{v \in V(H)} h(\deg v) \). If \( H \) is a 6-uniform hypergraph, then

\[
\alpha(H) \geq h(H) - \frac{13l}{266},
\]

where \( l \) is the number of connected components in \( H \) isomorphic to \( T_6 \).

The first few values of \( h \) are as follows:

<table>
<thead>
<tr>
<th>( d )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(d) )</td>
<td>1</td>
<td>( \frac{5}{6} )</td>
<td>( \frac{625}{798} \approx .7832 )</td>
<td>( \frac{293}{399} \approx .7343 )</td>
<td>( \approx .7001 )</td>
<td>( \approx .6734 )</td>
<td>( \approx .6519 )</td>
</tr>
</tbody>
</table>

A few remarks: \( h(T_6) - \alpha(T_6) = 13/266 \) as expected from the statement of the theorem. The value \( h(2) = 625/798 \) is tight because of the 6-graph \( H \) consisting of 6 vertex-disjoint copies of \( T_6 \) linked together by a single edge containing one vertex from each copy. For this \( H \),

\[
\alpha(H) = 42, \quad h(H) = 48h(2) + 6h(3) = 48 \cdot \frac{625}{798} + 6 \cdot \frac{293}{399} = 42.
\]

Therefore neither \( h(1) \) nor \( h(2) \) can be increased without introducing additional exceptions to the theorem. Also note that \( h(2) > g(2) = 7/9 \), so this theorem improves Theorem 5.1.

### 5.4 Lemmas

We need the lemmas in this section for the proof of Theorem 5.12.
Lemma 5.13. \( h \) is positive and decreasing. The difference function \( \epsilon_h(d) = h(d-1) - h(d) \) is nonincreasing.

**Proof.** See Appendix A; this follows from Proposition A.1 as before.

For the remainder of the chapter, let us assume \( H \) is a minimal counterexample to Theorem 5.12. That is, we assume that whenever \( H' \) has fewer vertices than \( H \) and has \( l \) components isomorphic to \( T_6 \), \( \alpha(H') \geq h(H') - 13l/266 \). (Note, this method is logically no different than strong induction.)

Lemma 5.14. Let \( H \) be a minimal counterexample to Theorem 5.12. As in Lemma 5.9, let \( Q \) be a subset of \( V(H) \) such that all vertices in \( Q \) have degree \( \Delta \) and \( Q \) is strongly independent, and define \( k_Q \) as before. Let \( l_Q \) denote the number of \( T_6 \) components in \( H - Q \). Then

\[
13l/266 > (h(\Delta - 2) - 2h(\Delta - 1) + h(\Delta))(k_Q - |Q|).
\]

**Proof.** Lemma 5.9 still holds for the function \( h \), since the proof depends only on the convexity and the defining recursion. Let \( H' = H - Q \). If the above inequality does not hold, then

\[
h(H') - h(H) \geq (h(\Delta - 2) - 2h(\Delta - 1) + h(\Delta))(k_Q - |Q|) \geq 13l/266.
\]

Thus

\[
\alpha(H) \geq \alpha(H') \geq h(H') - 13l/266 \geq h(H),
\]

contradicting that \( H \) is a counterexample to the theorem.

The following corollary, which is immediate, will be used frequently without explicit reference.

Corollary 5.15. If \( \deg(v_1) = \Delta \) and \( v_1 \) has a neighbor of degree less than \( \Delta \), then \( H - v_1 \) has a \( T_6 \) component. If \( \deg v_1 = \Delta \) and the set \( \{v_1, v_2\} \) has codegree at least two for some vertex \( v_2 \), then \( H - v_1 \) has a \( T_6 \) component.

**Proof.** In each case take \( Q = \{v_1\} \); then \( k_Q \geq 1 \) and therefore \( l_Q > 0 \).

The next lemma forms the heart of the proof for dealing with copies of \( T_6 \) that may appear in the inductive step.
Lemma 5.16. Suppose $R$ is an induced subgraph of $H$ isomorphic to $t \geq 1$ vertex-disjoint copies of $T_6$. Then there is no set $T \subseteq V(R)$ such that $|T| = 2t$ and every edge of $H$ which intersects $V(R)$ also intersects $T$.

Proof. If $T$ exists, then any independent set of $H' = H - V(R)$ combined with $V(R) \setminus T$ is an independent set of $H$. Therefore

$$\alpha(H) \geq 7t + \alpha(H') \geq 7t + h(H') - 13l/266,$$

where $l \geq 0$ is the number of $T_6$ components in $H'$. Since $H$ is connected, there must be at least $l$ distinct vertices of degree 2 in $H'$ which have degree at least 3 in $H$. Also, there are at least $t$ vertices in $V(R)$ which have degree at least 3 in $H$. Therefore the following holds:

$$h(H') - h(H) \geq (h(2) - h(3))l - (8h(2) + h(3))t = (13/266)l - 7t.$$

These two inequalities imply $\alpha(H) \geq h(H)$, so $H$ does not contradict the theorem.

Definition 5.17. In the hypergraph $T_6$ we say two or more vertices are partners if they are distinct and belong to the same pair of edges. Note that any two vertices of $T_6$ form a transversal unless they are partners.

Definition 5.18. If $R$ is subgraph of $H$ isomorphic to $T_6$, any edge of $H$ which intersects $V(R)$ but is not an edge of $R$ is called a linking edge for $R$.

Corollary 5.19. If $R$ is an induced copy of $T_6$ in $H$ then it has at least two linking edges. If it has exactly two linking edges $e_1, e_2$ then all vertices in the set $V(R) \cap (e_1 \cup e_2)$ are partners.

Proof. If $R$ has only one linking edge $e$ then let $t_1 \in V(R) \cap e$ and choose $t_2 \in V(R)$ any vertex which is not a partner of $t_1$. Then $T = \{t_1, t_2\}$ contradicts the Lemma 5.16. Similarly, if $R$ has two linking edges, it is clear there is a choice of $T$ contradicting the lemma except in the situation described.

Lemma 5.20. There exists at least one vertex $x$ of degree $\Delta$ in $H$ such that $H - x$ has a $T_6$ component.
Proof. By Corollary 5.15, if no such $x$ exists then $H$ is $\Delta$-regular and linear. In particular no subgraph of $H$ is isomorphic to $T_6$, so we can apply the proof of Theorem 5.7 with the function $h$ instead of $g$. The only properties of $g$ used in this proof are the defining recursion, plus those in Proposition A.1, all of which hold for $h$ as well. (The observant reader will note the values of $g$ were used directly in dealing with the case where $H$ is a Steiner system, but only when $r \leq 4$.)

Lemma 5.21. Let $x$ be a vertex of degree $\Delta$, and define $k_x, l_x$ as above with $Q = \{x\}$. Then

$$k_x < 1 + \left(2 + \frac{2h(\Delta - 1) - 2h(\Delta) - 13/266}{h(\Delta - 2) + 2h(\Delta - 1) - h(\Delta)}\right)l_x.$$ 

Proof. Corollary 5.19 gives the following: For every $T_6$ component of $H - x$, there are at least two distinct vertices whose degree is 2 in $H - x$ but greater than 2 in $H$. In the terminology of the proof of Lemma 5.9, there are at least $2l_x$ downsteps of the form $(v, 3)$. It follows (details omitted) that

$$\frac{13l_x}{266} > 2l_x(h(2) - h(3) - (h(\Delta - 1) - h(\Delta))) + (k_x - 1 - 2l_x)(h(\Delta - 2) - 2h(\Delta - 1) + h(\Delta)).$$

Solving for $k_x$ and using the fact that $h(2) - h(3) = 13/266$ gives the desired inequality.

We will make the most use of this inequality specifically when $\Delta = 4, 5$. Substituting the values of $h$ gives the following:

- If $\Delta = 4$, $k_x < 1 + 3.35l_x$.
- If $\Delta = 5$, $k_x < 1 + 2.58l_x$.

We need one last lemma which is a generalization of Corollary 5.19

Corollary 5.22. Suppose $R_1, R_2, \ldots, R_t$ are vertex-disjoint, induced copies of $T_6$. Then the union of the linking edges of the $R_i$ has at least $t + 1$ members.

Proof. Let $\mathcal{R} = \{R_1, \ldots, R_t\}$ and let $L$ be union of the linking edges of the $R_i$. Define the bipartite incidence graph with parts $\mathcal{R}, L$, where $R_i \sim e$ if
$V(R_i) \cap e \neq \emptyset$. Now, if there exists a matching of $L$ into $\mathcal{R}$, then we claim the set $T$ exists, contradicting Lemma 5.16. Indeed, if the component $R_i$ is matched with the edge $e$, then let $T \cap V(R_i) = \{v_1, v_2\}$ where $v_1 \in V(R_i) \cap e$ and $v_2$ is a nonpartner of $v_1$; if $R_i$ does not participate in the matching then let $T \cap V(R_i)$ be an arbitrary transversal of $R_i$.

We now prove the corollary by strong induction on $t$. Assume for a contradiction that $|\mathcal{R}| \geq |L|$. By Hall’s Marriage Theorem, there exists some $S \subseteq L$ such that $|S| > |N(S)|$. If $N(S) = \mathcal{R}$ then we are done, and clearly $|N(S)| > 0$, so $0 < |N(S)| < |\mathcal{R}|$. Let $\mathcal{R}' = \mathcal{R} \setminus N(S)$. Then

$$|\mathcal{R}'| = |\mathcal{R}| - |N(S)| = |L| - |S| \geq |N(\mathcal{R}')|.$$  

This contradicts our inductive hypothesis since $0 < |\mathcal{R}'| < |\mathcal{R}|$. 

\[ \blacksquare \]

### 5.5 Proof of Theorem 5.12

For the proof of this theorem, we continue to assume $H$ is a minimal counterexample and derive a contradiction based on the value of $\Delta$. The most difficult case is $\Delta = 4$; however, the proof for $\Delta = 5$ is very similar so we handle both cases at once. The case $\Delta \geq 6$ essentially follows directly from Lemma 5.21 as we shall see.

#### 5.5.1 Assume $\Delta \leq 2$.

In Theorem 4.5, when $r = 6$ we have $a_1 = 5/6 = h(1), a_2 = 23/24 > h(2)$, so this case follows directly from that theorem.

#### 5.5.2 Assume $\Delta = 3$.

By Lemma 5.20, let $R_1$ be an induced $T_4$ component of $H$. Since $H \neq R_1$, let $x$ be a degree 3 vertex in $V(R_1)$. By Lemma 5.19 (second sentence), $H - x$ has $T_4$ component $R_2$. 

\[ \blacksquare \]
Suppose $R_1$ and $R_2$ are vertex disjoint. In this case, no edge of $R_1$ is a linking edge for $R_2$. Since every linking edge for $R_2$ contains $x$, there is only one linking edge, contradicting Corollary 5.19.

Therefore $R_1$ and $R_2$ are not vertex disjoint. Then they must have an edge in common, or else their shared vertex would have degree 4. This means we know the isomorphism class of the hypergraph $R_1 \cup R_2$. Now let $z$ be a vertex in the shared edge. By considering the fact that $H - z$ also has a $T_4$ component, we conclude that $H \cong K_4^{x^3}$, the blowup of $K_4$ (details omitted). However, $\alpha(K_4^{x^3}) = 9$ and $h(K_4^{x^2}) = 12h(3) < 9$ so $H$ does not contradict the theorem.

5.5.3 Assume $\Delta \in 4, 5$.

As mentioned above, this case is the most difficult, and we need to proceed through some additional claims about the structure of $H$.

**Claim A:** If $x \in X$, then every $T_6$ component of $H - x$ has at least three linking edges.

**Proof.** Suppose $R$ is a $T_6$ component of $H - x$ with exactly two linking edges $e_1, e_2$. By Corollary 5.19, we may assume that $v_1 \in e_1 \cap V(R)$, $e_2 \cap V(R) = \{v_2\}$ where $v_1, v_2$ are partners in $R$. Note that $\deg v_1 = 3$ and consider $H' = H - v_1$. Considering the neighbors of $v_1$, we have (details omitted):

$$h(H') - h(H) \geq -h(3) + (h(1) - h(3)) + (h(0) - h(2)) + 6(h(1) - h(2)) + 5(h(4) - h(5))$$

$$> .015$$

Now, we prove that $H'$ has no $T_6$ component. Indeed, if $R_2$ is a $T_6$ component of $H - v_1$, there are three cases to consider. First suppose $R, R_2$ are vertex-disjoint. Then no edge of $R$ can be a linking edge for $R_2$, implying that $R_2$ has at most one linking edge, contradicting Corollary 5.19.

Next suppose that $R, R_2$ share a common edge $e$. Then we must have $x \in V(R_2)$, or else $R$ is connected to $R_2$ in $H - x$, which is a contradiction. Now, the two edges of $R_2$ besides $e$ are both linking edges for $R$, and $e_1$ is distinct from these. This contradicts our original assumption that $R$ has only two linking edges.
Finally, suppose \( R, R_2 \) have a common vertex \( w \) but no common edge. Then both the edges of \( R_2 \) containing \( w \) are linking edges for \( R \) (since neither is an edge of \( R \)), and so is \( e_1 \). Again this contradicts the assumption that \( R \) has two linking edges.

Now since \( H' \) satisfies Theorem 5.12 and has no \( T_6 \) component,

\[
\alpha(H) \geq \alpha(H') \geq h(H') > h(H) + .015,
\]

meaning \( H \) does not contradict the theorem.

**Claim B:** Suppose \( x \in X, R \) is a \( T_6 \) component of \( H - x \), and \( y \in V(R) \). Then \( \deg_H(y) < \Delta \).

**Proof.** Suppose that \( \deg y = \Delta \). First assume that \( \Delta = 4 \). Note that \( y \in X \), so let \( H' = H - y \) and let \( R_2 \) be a \( T_6 \) component of \( H' \). If \( R_2 \) is vertex disjoint from \( R \) then it has at most two linking edges, contradicting Claim A. This implies \( R_2, R \) share at least one vertex. In particular, \( x \in R_2 \) (otherwise \( R \) is connected to \( R_2 \) in \( H - x \)). Thus \( R_2 \) is unique, i.e. \( l_y = 1 \). By Lemma 5.21, \( k_y \leq 4 \).

Next we find a lower bound for \( k_y \) directly. Suppose \( f \) is any edge which intersects \( R \) but does not belong to \( R \cup R_2 \). Then \( f \) is a linking edge for \( R \) so it contains \( x \); thus it is also a linking edge for \( R_2 \) so it contains \( y \). Now let \( e \) be the edge of \( R \) which does not contain \( y \). From above it follows that each vertex in \( e \) has degree 1 in \( H - y \) if \( e \notin R_2 \), or degree 2 in \( H - y \) if \( e \in R \cap R_2 \). In either case, each vertex of \( e \) contributes 1 to \( k_y \). Additionally, the partners of \( y \) in \( R \), as well as \( x \), contribute 1 to \( k_y \). Thus \( k_y \geq 9 \), so we have a contradiction.

Next we consider the case \( \Delta = 5 \). We make an important observation: Claim B implies \( k_y \geq 3l_y \), whereas Lemma 5.21 implies \( k_y < 1 + 2.58l_y \). One immediately sees that the only positive integer solutions to this system are \( l_y = 1, k_y = 3 \) and \( l_y = 2, k_y = 6 \). Therefore, *every vertex which contributes to \( k_y \) must belong to a \( T_6 \) component of \( H - y \).* In particular the partners of \( y \) in \( R \) belong to \( V(R_2) \) where \( R_2 \) is a \( T_6 \) component of \( H - y \). The argument from the previous paragraph now implies \( k_y \geq 9 \) which is a contradiction.
Definition 5.23. Henceforth let $X$ denote the set of all degree $\Delta$ vertices $x \in V(H)$ such that $H - x$ has a $T_6$ component. By Lemma 5.20, $X$ is nonempty.

Before continuing the proof, we make a few remarks in an attempt to clarify the intuition behind it. Imagine that we try to construct a counterexample to Theorem 5.12, starting with a particular vertex $x \in X$. For the sake of concreteness, let’s say $\Delta = 4$ and $l_x = 1$; i.e. $H - x$ has only one $T_6$ component. Then by Lemma 5.21, $k_x \leq 4$. By Claim A, there are three linking edges to this component, and by Claim B, each edge intersects this component at a vertex of degree 3. This means we have little freedom in choosing the configuration of the four edges containing $x$; they can contain at most one more vertex of degree less than 4, besides the three just accounted for. In particular, the three linking edges contain at least 11 distinct vertices of degree 4, excluding $x$ itself; these are necessarily elements of $X$ (as each is adjacent to a degree 3 vertex). We now have to build in linking edges for each of these in the same way, and intuitively it seems is though the size of $X$ will just keep increasing exponentially and we will be unable to build a finite graph. The construction below makes this idea rigorous by defining a kind of directed graph on $X$ where, in the above example, there would be an arrow from $x$ to each of the 11 vertices mentioned. Our overall goal is to show that every vertex in this directed graph has an outdegree greater than its indegree, which is a contradiction.

Definition 5.24. If $v \in V(H)$, let $L(v)$ denote the set of vertices $x \in X$ such that $v$ belongs to a $T_6$ component of $H - x$. By Claim B, recall that if $L(v) \neq \emptyset$ then $\deg v < \Delta$. Suppose $x, y \in X, x \neq y$. If $e \in H$ contains $x, y$, Define

$$R(x, y, e) = |\{v \in e : x \in L(v)\}|.$$ 

Lastly, define

$$R(x, y) = \sum_{e \supseteq \{x, y\}} R(x, y, e).$$

In terms of the above intuition, think of $R(x, y)$ as the multiplicity of the arrow from $x$ to $y$. 
Claim C: Fix $e \in H$ and a vertex $x \in X \cap e$. Let $A = A(e, x) = \{v \in e : x \in L(v)\}$, and let $B = B(e, x) = \{v \in e : \deg v < \Delta, x \notin L(v)\}$. Then

$$\sum_{y \in X \cap e \setminus \{x\}} R(x, y, e) - R(y, x, e) \geq (5 - |A| - |B|)(|A| - |B|) - \sum_{v \in A}|L(v)| - 1).$$

Proof. Let $y \in X \cap e, y \neq x$. By definition, $R(x, y, e) = |A|$. For $R(y, x, e)$, only the vertices of degree less than $\Delta$ (namely $A \cup B$) can contribute, and $v \in A$ contributes only if $y \in L(v)$. Thus

$$R(x, y, e) - R(y, x, e) \geq |A| - |B| - \sum_{v \in A} 1(y \in L(v)).$$

We note that if $v \in A$, then $L(v) \subseteq e$ (as $e$ is a linking edge for the $T_6$ component of $H - x$ containing $v$). Summing the above over $y \in X \cap e - \{x\}$ gives

$$\sum_{y \in X \cap e \setminus \{x\}} R(x, y, e) - R(y, x, e) \geq |X \cap e - \{x\}|(|A| - |B|) - \sum_{v \in A}|L(v)| - 1).$$

If $e$ contains any vertex of degree less than $\Delta$, then every vertex in $e$ of degree $\Delta$ must belong to $X$ and so $|X \cap e - \{x\}| = 5 - |A| - |B|$. Otherwise, $|A| = |B| = 0$ so in either case

$$|X \cap e - \{x\}|(|A| - |B|) = (5 - |A| - |B|)(|A| - |B|).$$

This proves the desired inequality.

Claim D: Fix $x \in X$. Then

$$\sum_{y \in X, y \neq x} R(x, y) - R(y, x) > 0.$$

Of course, Claim D is the contradiction with proves Theorem 5.12, because it implies $\sum_{x,y} R(x, y) > \sum_{x,y} R(y, x)$.

Proof. Let us assume $\Delta = 4$. Let $e_1, e_2, e_3, e_4$ be the edges containing $x$, and for $i = 1, \ldots, 4$ let $a_i = |A(e_i, x)|, b_i = |B(e_i, x)|$ (using the notation of Claim C). Also define

$$f(a, b) = (5 - a - b)(a - b) = 5(a - b) + b^2 - a^2.$$
By Claim C, we have

$$\sum_{y \neq x} R(x, y) - R(y, x) \geq \sum_{i=1}^{4} f(a_i, b_i) - \sum_{v \in A(e_i, x)} |L(v) - 1|.$$ 

Case 1: First let us assume that $x \in L(v)$ implies $L(v) = \{x\}$, so the sum on the right vanishes. Note that every incidence of a vertex of degree less than $\Delta$ with one of the edges $e_i$ contributes 1 to $k_x$. Therefore, by Lemma 5.21,

$$\sum_{i=1}^{4} a_i + b_i \leq k_x < 1 + 3.35l_x.$$ 

Also note, by Corollary 5.22, $l_x \leq \Delta - 1 = 3$. Next, for $i = 1, \ldots, 4$ define $a_i'$ to be the number of $T_6$ components $R$ of $H - x$ such that $V(R) \cap e_i \neq \emptyset$. Then $a_i' \leq a_i$. Furthermore, Claim B implies that $\sum_{i=1}^{4} a_i' \geq 3l_x$. Subtracting these inequalities gives

$$\sum_{i=1}^{4} (a_i - a_i') + b_i l e [1 + 0.35l_x].$$

Some easy algebra shows that if $0 \leq a \leq 5$, $0 \leq b \leq 5$, then

$$f(a + 1, b) - f(a, b) \geq -4$$

$$f(a, b + 1) - f(a, b) \geq -4.$$ 

Combining these results, we see that

$$\sum_{i=1}^{4} f(a_i, b_i) \geq \sum_{i=1}^{4} f(a_i', 0) - 4[1 + 0.35l_x].$$

Corollary 5.22 implies there are at least $l_x + 1$ indices $i$ such that $a_i' \geq 1$, and furthermore $a_i' \leq l_x \leq 3$ for all $i$. It is easy to check that whenever $1 \leq a \leq 3$, $f(a, 0) \geq 4$. This gives:

$$\sum_{i=1}^{4} f(a_i, b_i) \geq 4(l_x + 1) - 4(1 + 0.35l_x) = 4(1 - 0.35)l_x > 0.$$ 

This proves Case 1.

Otherwise, assume that there is some vertex $x_2 \neq x$ and some vertex $v$ such that $L(v) \supseteq \{x, x_2\}$. This means that there exists some induced subgraph
\( R \cong T_6 \) where every linking edge for \( R \) contains both \( x \) and \( x_2 \). Suppose there are \( m_R \) linking edges; then \( x_2 \) contributes \( m_R - 1 \) to \( k_x \), and this is separate from the contributions from \( a_i \) and \( b_i \) since \( x_2 \) has degree \( \Delta \). Therefore

\[
m_R - 1 + \sum_{i=1}^{4} a_i + b_i \leq k_x < 1 + 3.35 l_x.
\]

As before, this implies

\[
\sum_{i=1}^{4} (a_i - a'_i) + b_i \leq 2 - m_R + \lfloor 0.35 l_x \rfloor.
\]

Since \( m_R \geq 3 \), this has a solution only if \( a'_i = a_i, b_i = 0, m_R = 3, \) and \( l_x = 3 \). Therefore

\[
\sum_{i=1}^{4} f(a_i, b_i) = \sum_{i=1}^{4} f(a'_i, 0) \geq 16.
\]

By Claim C it is therefore sufficient to show \( \sum_{i=1}^{4} \sum_{v \in A(e_i, x)} |L(v) - 1| < 16 \). This sum is at most the number of incidences of the \( e_i \) with vertices of degree less than \( \Delta \), which is 9.

If \( \Delta = 5 \), we can use an easier version of this proof. Since \( k_x < 1 + 2.58 l_x \) and \( k_x \geq 3 l_x \), we only have solutions when \( k_x = 3 l_x \) (as previously mentioned in Claim B). Thus \( a'_i = a_i, b_i = 0 \) and \( |L(v)| = 1 \) whenever \( x \in L(v) \). This is sufficient to show \( R(x, y) - R(y, x) \geq 4(l_x + 1) \geq 8 \).

### 5.5.4 Assume \( \Delta \geq 6 \).

In Lemma 5.21, if \( \Delta \geq 6 \) then

\[
2(h(\Delta - 1) - h(\Delta)) - 13/266 \leq 2(h(5) - h(6)) - 13/266 < 0
\]

which implies \( k_x < 1 + 2 l_x \). However, by Corollary 5.19, \( k_x \geq 2 l_x \) and therefore \( k_x = 2 l_x \). Therefore Claim B is true since each \( T_6 \) has only two linking edges. As in the case \( \Delta = 5 \) just above, this implies that \( a_i = a'_i, b_i = 0, |L(v)| = 1 \), and hence \( R(x, y) - R(y, x) \geq 8 \). This completes the proof of Theorem 5.12.
5.6 Numerical comparisons to previously known results

In this section, we compare the results given by the theorems in this chapter to previously known results. We begin with the following: for given integers \( r \geq 3, d \geq 2 \) let \( a_{r,d} \) denote the minimum value of \( \alpha(H)/n(H) \) where \( H \) ranges over all \( r \)-graphs of average degree at most \( d \). We can obtain bounds on \( a_{r,d} \) from the degree-sequence bounds in this chapter as follows: Suppose \( \alpha(H) \geq \sum_{v \in V(H)} f_r(d_v) \) for all \( r \)-graphs, where \( f_r \) is nonincreasing convex function (i.e. the second differences are nonnegative). Then if the average degree of \( H \) is at most \( d \), \( \alpha(H) \geq \sum_v f_r(d_v) \geq nf_r(d) \) and therefore \( a_{r,d} \geq f_r(d) \). (Note: this is true even when \( d \) is not an integer, if we extend the domain of \( f_r \) to the nonnegative reals via linear interpolation between its points.) Also, any “small transversal” theorem of the form \( \tau(H) \leq am + bn \) implies \( \alpha(H)/n \geq 1 - b - \frac{ad}{r} \) as discussed in Chapter 1, which likewise gives a lower bound for \( a_{r,d} \). This allows us to compare the two types of theorems directly. The following table shows some known lower bounds for \( a_{r,d} \), where the superscripts indicate the source of the bound. All the bounds are rational numbers, but they are shown rounded to four decimal places unless the denominator is small. A box indicates that the value is optimal; there exists a known \( r \)-graph \( H \) of average degree \( d \) such that \( \alpha(H)/n(H) \) equals the number in the box.

Lower Bounds for \( a_{r,d} \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( d = 2 )</th>
<th>( d = 3 )</th>
<th>( d = 4 )</th>
<th>( d = 5 )</th>
<th>( d = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( \frac{3}{5} )</td>
<td>( \frac{1}{2} )</td>
<td>.44792, .49/9</td>
<td>.40832, .40561</td>
<td>.37762, .37501</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{2}{3} )</td>
<td>.61902, .60811</td>
<td>.57143, .55711</td>
<td>.53653, .52921</td>
<td>.50863, .49571</td>
</tr>
<tr>
<td>5</td>
<td>7/120</td>
<td>.729214</td>
<td>.695114</td>
<td>.668714</td>
<td>.647214</td>
</tr>
<tr>
<td>6</td>
<td>7/9</td>
<td>.73435, .72961</td>
<td>.70015, .69561</td>
<td>.67345, .66911</td>
<td>.65184, .64771</td>
</tr>
</tbody>
</table>

The superscripts indicate the following.

0: Chvátal and McDiarmid [14].

1: Theorem 5.25 by Z. Lonc and K. Warno [40], to appear in ***.
2: Theorem 5.7, the case $r = 3$ which requires no connected component isomorphic to $H_9$. (Due to Anders Yeo, as yet unpublished; the extra condition explains why we can obtain a value larger than the boxed value at $r = 4$).

3: See the $r = 4$ comments in Section 5.3. For $d \leq 4$ these values come from the inequality $\tau(H) \leq 5m + 4n$ [51].

4: From Theorem 5.7,

5: From Theorem 5.12, requires no connected component isomorphic to $T_6$.

A few remarks: many of the competing results come from the Lonc/Warno paper [40] which is yet to appear. See the next subsection for the statement of their main theorem. Interestingly, for $r \geq 5$ odd, our results match this paper although they were developed independently. Our results which surpass Lonc/Warno involve making an exception for a specific $r$-graph (either $H_9$ or $T_r$). If we disallow this exception, in each case our methods obtain a bound which matches Lonc/Warno. Prior to this, the known bounds came from [14], citeCaroTuza, [51].

We could also define $b_{r,\Delta}$ to be the minimum value of $\alpha(H)/n(H)$ where $H$ is an $r$-graph of maximum degree at most $\Delta$ (as opposed to average degree). Of course, $d \leq \Delta$ which automatically implies $a_{r,d} \leq b_{r,d}$. For the most part, the best known lower bounds for $b_{r,\Delta}$ are the same as those for in the table, which the following exceptions. If $r$ is odd, Theorem 4.4 implies $b_{r,2} = \frac{3r-3}{3r-1}$, realized at $H = T_r^*$ or $U_r$. In this case, $a_{r,2} < b_{r,2}$ but the exact value of $a_{r,2}$ seems to be unknown with the exception of $r = 3$: the lower bound $a_{3,2} \geq 7/12$ is realized by the 3-graph $H$ which is four disjoint copies of $T_3^*$ and one copy of $K_{4,3}$. If $r$ is even, $a_{r,2} = b_{r,2} = 2$, realized at $H = T_r$. If $\Delta = 3$ and $r$ is even, Theorem 5.2 provides a lower bound for $b_{r,3}$ which is slightly larger than the one in the table.

5.6.1 The Lonc/Warno Theorem

For completeness, we state the main result of the aforementioned paper by Lonc/Warno [40] here. Fix $r \geq 3$. Define the function $e$ as follows. If $r \geq 4$ then,

$$e_1 = \frac{1}{r}, e_2 = \frac{1}{\lfloor 3r/2 \rfloor}$$
\[ e_3 = \begin{cases} \frac{-2}{3r(r-1)}, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \]

\[ e_4 = \begin{cases} \frac{-(r-2)(3r-1)}{36r(r-1)^2}, & r \text{ even} \\ \frac{-1}{4(3r-1)}, & r \text{ odd} \end{cases} \]

Then for \( j \geq 5, \)

\[ e_j = \frac{(j(r-1) - 2r)e_{j-1} + e_{j-2}}{j(r-1)}. \]

This defines \( e_j \) when \( r \geq 4. \) If \( r = 3, \) define

\[ e_1 = \frac{1}{3}, \quad e_2 = -\frac{1}{4}, \quad e_3 = 0, \quad e_4 = -\frac{1}{36}, \quad e_5 = -\frac{1}{60}, \quad e_6 = -\frac{1}{120}, \]

and for \( j \geq 7 \) define \( e_j \) using the same recursion as above.

**Theorem 5.25** (Lonc/Warno). For every \( k = 1, 2, \ldots \) the following holds for every \( r \)-graph \( H: \)

\[ \tau(H) \leq \left( r \sum_{i=1}^{k} e_i \right) m(H) + \left( \sum_{i=1}^{k} (i-1)e_i \right) n(H). \]

**5.6.2 Points in \( A_r \)**

Recall in [14], for each integer \( r \geq 2 \) the convex set \( A_r \) was defined as follows:

\[ A_r = \{(a,b) : \tau(H) \leq am(H) + bn(H) \text{ for every } r-\text{graph } H\}. \]

That is, the space \( A_r \) keeps track of possible independence number theorems. Given a degree sequence bound \( \alpha(H) \geq \sum_v f(d_v) \) where \( f \) is a convex function, we can generate a sequence of points in \( A_r \) as follows. For any given integer \( k \geq 0, \) the graph of \( f \) lies on or above the line defined by \( (k, f(k)) \) and \( (k+1, f(k+1)) \). Therefore, for every \( r \)-graph \( H, \)

\[ \alpha(H) \geq \sum_v f(d_v) \geq \sum_v f(k) + (f(k+1) - f(k))(d_v - k) \]

\[ = (k(f(k) - f(k+1)) + f(k))n(H) + \sum_v (f(k+1) - f(k))d_v \]

\[ = r(f(k+1) - f(k))m(H) + (k(f(k) - f(k+1)) + f(k))n(H). \]
Therefore,

\[ \tau(H) \leq r(f(k) - f(k + 1))m(H) + (1 - f(k) - k(f(k) - f(k + 1)))n(H). \]

Using this we can generate the following list of points in \( A_r \) for small values of \( r \) and \( k \).

\( r = 3: \)

\[ (.25, .25)^0, (.15625, .34375)^*, (.11875, .39375)^*, \]
\[ (.09219, .43802), (.07433, .473743) \in A_3 \]

\( r = 4: \)

\[ (0.13968, 0.28889)^1, (0.11146, 0.32416), (0.09158, 0.35399), \]
\[ (0.07714, 0.37925), (0.06625, 0.40104) \in A_4 \]

\( r = 5: \)

\[ (0.28571, 0.14286)^0, (0.19643, 0.19643), (0.15179, 0.23214), \]
\[ (0.12202, 0.26190), (0.10130, 0.28678) \in A_5 \]

\( r = 6: \)

\[ (0.30075, 0.11654)^*, (0.29323, 0.11905)^*, (0.20564, 0.16284), \]
\[ (0.15979, 0.19341), (0.12936, 0.21877) \in A_6 \]

0: = Previously known [14]

1: = Previously known [51]

*: = Requires that \( H \) have no component isomorphic to \( H_9 \) (\( r = 3 \)) or \( T_r \) (\( r \) even)

Again, if \( r \geq 3 \) is odd these generated points are identical to those of Theorem 5.25 (Lonc/Warno). For \( r = 3 \), creating an exception for \( H_9 \) allows us to surpass this theorem; however, \( H_9 \) is actually an exception only for the points marked with *. A similar situation exists for \( r = 6 \) and \( H_9 \). Our \( r = 4 \) results also improve Theorem 5.25 by making use of \( 21 \tau(H) \leq 4m + 5n \) [40]. The situation is the same as for the table for \( a_{r,d} \) presented earlier.
5.7 Acknowledgement

Most of the material in this chapter from Section 5.2 onward represents joint work with Anders Yeo, University of Johannesburg. The proof of Theorem 5.12 is my own, but was largely inspired by his proof of the case $r = 3$ of Theorem 5.7. This material is still being prepared publication in a joint paper.
Chapter 6

Basic Projective Geometry

In this chapter we define the projective plane and inversive plane from finite projective geometry and state the main theorems about them that will be required in Chapter 7. We will be using these structures as building blocks in our construction for Theorem 1.5.

6.1 Projective Planes

For our purposes we may define a finite projective plane is a hypergraph $H$ having the following properties:

1. for every pair of vertices there is a unique edge containing them both.

2. For every pair of edges $e, f$ in the hypergraph, $|e \cap f| = 1$.

3. $H$ is a regular, uniform hypergraph with at least four vertices and four edges.

A hypergraph satisfying conditions 1 and 2, but not 3, is known as a degenerate projective plane; these fall into easily described categories [1] and are not interesting from our perspective. For example, one could take a hypergraph with only a single edge containing all the vertices. Aside from these degenerate cases, it is well-known that for every finite projective plane there exists an integer $q$, called the order of the projective plane, such that $H$ has the following additional properties:

1. $H$ has $q^2 + q + 1$ vertices and $q^2 + q + 1$ edges.
2. $H$ is $q + 1$-regular and $q + 1$-uniform.

In particular, this means that we can alternatively define a projective plane of order $q > 1$ as a complete $(q^2 + q + 1, q + 1, 2)$-Steiner system in which every pair of edges intersects. Frequently, the edges of a projective plane are referred to geometrically, as *lines*. In this context, a projective plane defines a geometry with some similarities to the Euclidean plane, the main difference being that parallel lines do not exist.

### 6.1.1 Projective spaces

The most well-known type of projective plane is the *projective space* $PG(2, q)$. For an integer $n \geq 1$ and prime power $q$, we can define the $n$-dimensional projective space $PG(n, q)$ as the following hypergraph. Starting with the $n + 1$-dimensional vector space $F_q^{n+1}$ over the finite field $F_q$,

- The vertices (or *points*) of $PG(n, q)$ are the one-dimensional subspaces of $F_q^{n+1}$.
- The edges (or *lines*) of $PG(n, q)$ correspond to the two-dimensional subspaces of $F_q^{n+1}$.
- More generally, a $k$-dimensional hyperplane in $PG(n, q)$ corresponds to a $k + 1$-dimensional subspace of $F_q^{n+1}$.
- The incidence relation among vertices and edges in $PG(n, q)$ (or the containment relations among hyperplanes) are the same as the containment relations among the corresponding subspaces.

From this definition, it is a simple matter to verify that $PG(2, q)$ satisfies the properties 1 through 5 above. Indeed, any two distinct 1-spaces of $F_q^3$ are contained in a unique 2-space, and the intersection of any two distinct 2-spaces is a 1-space.
6.1.2 Existence results

Although $PG(2, q)$ is the only projective plane we intend to use in this paper, there exist projective planes not isomorphic to $PG(2, q)$. For example there there are three non-isomorphic projective planes of order 9 other than $PG(2, 9)$, described in [46]. For $q < 9$ it is well-known that $PG(2, q)$ is the only projective plane that exists, and for $q = 10$ C.W. Lam proved by computer search that no projective plane exists [39]. However, no complete classification of projective planes has been done for any $q > 10$, although there are numerous known examples and partial nonexistence results. It is still unknown if any projective plane of non-prime-power order exists.

6.2 Inversive Planes

Whereas a projective plane is a well-known type of Steiner system with $l = 2$, an inversive plane or circle geometry is a well-known type of Steiner system with $l = 3$. Specifically, it is a hypergraph $H$ satisfying the following axioms:

- Any three vertices are contained in a unique edge.
- If $v, w$ are vertices and $e$ an edge containing $v$ but not $w$, then there is exactly one edge $f$ containing $v$ and $w$ and intersecting $e$ only in $v$.
- There exist four vertices, not all in the same edge.

Again, the third axiom is to eliminate degenerate cases. The edges of an inversive plane are sometimes referred to as circles, since three points determines a unique circle, and axiom 2 describes a circle tangent to another at a given point. The following is a standard theorem concerning inversive planes, proved in [16]

**Theorem 6.1.** If $H$ is an inversive plane, then $H$ is a complete $(q^2 + 1, q + 1, 3)$-Steiner system for some integer $q$, called the order of the inversive plane. In addition, $H$ has $q^3 + q$ edges, and is $(q^2 + q)$-regular.
6.2.1 Construction

An **ovoid** in $\text{PG}(3, q)$ with $q > 2$, is a set of $q^2 + 1$ points, no three of which are collinear. This is the maximum size of such a set [26]. The only known construction of a finite inversive plane comes from taking the intersections of an ovoid in $\text{PG}(3, q)$ with its nontangent planes, as follows.

**Theorem 6.2.** If $O$ is an ovoid in $\text{PG}(3, q)$, then every (projective) plane intersects $O$ in either 1 or $q+1$ points. The hypergraph whose vertex set is $O$ and whose edges are all the $q+1$-sized plane sections of $O$, is an inversive plane.

Dembowski [16] has shown that if $q$ is even, then every inversive plane of order $q$ arises in this way (and hence $q$ is a power of 2). It is not known if this is also true for $q$ odd. There are two only types of ovoids known up to isomorphism: the so-called “elliptic quadric” (for all $q$) and the Suzuki/Tits ovoid (when $q = 2^{2k+1}$), and it is conjectured that these are the only two types.

6.3 Eigenvalues

In this section we compute the eigenvalues for the bipartite incidence graph of either a projective plane or an inversive plane, by counting walks. In Chapter 7 we will need these results along with similar computations, we provide the proofs as a reference.

6.3.1 Projective Plane of order $q$

Let $G$ be the bipartite incidence graph for a projective plane of order $q$, and let $A$ be its adjacency matrix which is of size $2(q^2 + q + 1) \times 2(q^2 + q + 1)$.

**Proposition 6.3.**

$$A^2 = qI + \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$$

where $J$ denotes the $q^2 + q + 1 \times q^2 + q + 1$ all-ones matrix.
Proof. Suppose $v$ and $w$ are vertices in the same part of the bipartite graph (i.e. either two points or two lines of the projective plane). The $(v, w)$-th entry of $A^2$ equals the number of length-2 walks from $v$ to $w$ in the bipartite graph.

Case 1: $v \neq w$. Since any two distinct points are contained on a unique line, or any two lines intersect at a unique point, it is clear that there is exactly one walk of the form $v \sim x \sim w$ and therefore $A^2_{v,w} = 1$.

Case 2: $v = w$. There are exactly $q + 1$ lines containing any point and $q + 1$ points on every line, and therefore there are $q + 1$ walks of the form $v \sim x \sim v$, hence $A^2_{v,v} = q + 1$.

Corollary 6.4. For this adjacency matrix, $\lambda_3 = \sqrt{q}$.

Proof. Recall that an eigenvector $v$ for $\lambda_3$ satisfies

$$\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix} v = 0,$$

so this is immediate.

6.3.2 Inversive Plane of order $q$

Now let $G$ be the bipartite incidence graph for an inversive plane of order $q$, and let $A$ be its adjacency matrix which is of size $(q^2 + 1) + q(q^2 + 1) \times (q^2 + 1) + q(q^2 + 1)$.

Proposition 6.5.

$$A^3 = (q^2 - 1)A + (q^2 + 2q + 1) \begin{bmatrix} 0 & J \\ J^T & 0 \end{bmatrix}$$

where $J$ denotes the $(q^2 + 1) \times q(q^2 + 1)$ all-ones matrix.

Proof. Given a vertex $v$ and circle $C$ of the inversive plane, we count the number of length-3 walks from $v$ to $C$.

Case 1: $v \notin C$. First choose a vertex $v' \in C$; there are $q+1$ choices for this. Then, choose a circle $C'$ containing both $v$ and $v'$. Since $v \neq v'$, and since an inversive
plane is a \((q^2 + 1, q + 1, 3)\) complete Steiner system, there are exactly \(\frac{q^2 - 1}{q - 1} = q + 1\) choices for \(C'\). Thus
\[ A_{v,C}^3 = q^2 + 2q + 1. \]

**Case 2:** \(v \in C\). Choose \(v' \in C\) as above. If \(v' \neq v\) then there are \(q + 1\) choices for \(C'\) as before. If \(v' = v\) then there are exactly \(\frac{q^2(q^2 - 1)}{q(q-1)} = q(q + 1)\) circles containing \(v\). Altogether we have
\[ A_{v,C}^3 = 2q(q + 1) = 2q^2 + 2q. \]

**Corollary 6.6.** For this adjacency matrix, \(\lambda_3 = \sqrt{q^2 - 1}\).
Chapter 7

Algebraic constructions of partial Steiner systems

In this chapter, we will prove theorem 1.5, in the case \( r \geq 2l - 1 \), by giving a construction for an \((n, r, l)\)-system \( H \) which satisfies

\[
\alpha(H) \gtrsim C_r n^{\frac{r-1}{r-1}} (\log n)^{\frac{1}{r-1}}
\]

where \( C_r = \left( \frac{l-1}{r-1} (r)_l \right)^{1/(r-1)} \). We also show that this \( C_r \) is the same as that of a uniformly random hypergraph \( G_r(n, p) \) with the same density as a complete \((n, r, l)\)-system. This theorem is the tightest bound known and we conjecture that is the best possible.

We will begin by proving Lemma 7.1, which uses the first moment method to establish a lower bound on the independence number of a randomly generated graph. Next we detail the construction itself, which makes use of a polynomial system where the graph of each polynomial is used as the vertex set for an asymptotically complete Steiner system. Then we shall check that this construction satisfies the hypotheses of Lemma 7.1. This will involve the machinery of the Expander-Mixing lemma to show that a typical set of vertices in the hypergraph does not have many more edges than its expected value would dictate.
7.1 The first-moment lower bound for $\alpha(H)$

From now on we will use script characters for hypergraphs (i.e. $H$ rather than $H$) in order to avoid confusion with random variables, which have roman characters such as $X$.

In this section we compute an upper bound for $\alpha(H)$ for any random hypergraph $H$, based on the first moment method. We state this as a lemma which will ultimately be used to prove Theorem 1.5.

**Lemma 7.1.** Suppose $\lambda > 0$, and for infinitely many values of $n$, $H_n$ is a random $r$-graph on $n$ vertices with the following property: for all $\beta > 0$, and all $U \subseteq V(H_n)$ of size $|U| = u := \lceil \beta n^{\frac{r-l}{1-(r-1)\lambda}} (\log n)^{\frac{1}{r-1}} \rceil$,

$$-\log P(H_n[U] = \emptyset) \gtrsim \lambda \beta^r n^{\frac{r-l}{1-(r-1)\lambda}} (\log n)^{\frac{1}{r-1}}.$$  

Then, for all $\epsilon > 0$,

$$\alpha(H_n) \leq (1 + \epsilon) \left( \frac{l - 1}{(r - 1)\lambda} \right)^{\frac{1}{r-1}} n^{\frac{r-l}{1-(r-1)\lambda}} (\log n)^{\frac{1}{r-1}}$$  

w.h.p.

**Proof.** Fix $\beta > 0$ and let $I$ denote the number of independent sets of size $u$ in $H_n$ (this is a random variable). Then

$$\log E[I] \lesssim \log \binom{n}{u} - \lambda \beta^r n^{\frac{r-l}{1-(r-1)\lambda}} (\log n)^{\frac{1}{r-1}}.$$  

Using the estimate $\log \binom{n}{u} \sim u \log(n/u)$, the first term is

$$\log \binom{n}{u} \sim u \log(n/u)$$  

$$\sim \beta n^{\frac{r-l}{1-(r-1)\lambda}} (\log n)^{\frac{1}{r-1}} (1 - \frac{r - l}{r - 1}) \log n$$  

$$= \beta \frac{l - 1}{r - 1} n^{\frac{r-l}{1-(r-1)\lambda}} (\log n)^{\frac{1}{r-1}}.$$  

It follows that $\log E[I] \to -\infty$ if $\lambda \beta^r > \beta^{l-\frac{1}{r-1}}$. Therefore if

$$\beta > \left( \frac{l - 1}{\lambda(r - 1)} \right)^{\frac{1}{(r-1)}},$$
then $E[I] \to 0$, which means that with probability approaching 1, $I = 0$ and thus

$$\alpha(H_n) \leq \beta n^{\frac{r-l}{r-1}}(\log n)^{\frac{1}{r-1}}.$$ 

In particular, if $H_n$ is a random $(n, r, l)$-system satisfying the hypotheses of the lemma, then

$$d(r, l) \leq \left(\frac{l - 1}{(r-1)\lambda}\right)^{\frac{1}{r-1}}.$$ 

For comparison, we now compute the independence number of the random hypergraph $G_r(n, p)$, where $p$ is chosen to match the density of complete $(n, r, l)$-system. The purpose of this computation is to show that $\lambda = 1/(r)_l$, which means that the upper bound in Theorem 1.5 is the same as $\alpha(G_r(n, p))$. This is true even though $G_r(n, p)$ is a natural candidate for a hypergraph with low independence number, since the edges are “spread evenly.”

**Lemma 7.2.** Let $p = \frac{\binom{n}{l}}{\binom{n}{r}(r)_l}$, which is the density of edges in a complete $(n, r, l)$-system. Then

$$\alpha(G_r(n, p)) \sim \left(\frac{l - 1}{(r-1)\lambda}\right)^{\frac{1}{r-1}} n^{\frac{r-l}{r-1}}(\log n)^{\frac{1}{r-1}}.$$ 

**Proof.** Following the method of Lemma 1, let $u := \lfloor \beta n^{\frac{r-l}{r-1}}(\log n)^{\frac{1}{r-1}} \rfloor$ and let $U$ be a subset of the vertices of $G_r(n, p)$ with $|U| = u$. Since the edges are chosen independently,

$$-\log P(G_r(n, p)[U = \emptyset]) = -\binom{u}{r} \log(1 - p)$$

$$\sim pu^r r!$$

$$\sim \frac{n^l l!(r - l)!}{r!} \frac{1}{n^r} \frac{1}{r!} [\beta n^{r-l}r - 1(\log n)^{\frac{1}{r-1}}]^r$$

$$= \frac{\beta^r}{(r)_l} n^{l-r+r\left(\frac{r-l}{r-1}\right)}(\log n)^{\frac{1}{r-1}}$$

$$= \frac{\beta^r}{(r)_l} n^{\frac{r-l}{r-1}}(\log n)^{\frac{1}{r-1}}.$$ 

Therefore Lemma 1 holds with $\lambda = 1/(r)_l$, which furnishes an upper bound for $\alpha(G_r(n, p))$ as stated in the lemma. Rather than go through a lower bound
(which requires, say, the second moment method and considerably more work), we refer the reader to the well known result that

$$\alpha(G_r(n,p)) \sim \min\{u \in \mathbb{Z}^+ : g(u) < 1\} \text{ w.h.p.}$$

where $g(u) = \binom{n}{u}(1-p)^r$, the expected number of independent sets of size $u$. (See, for instance, Krievlevich and Sudakov [38].) The proof of Lemma 1 shows that the expected number of independent sets of size $u$ tends to zero if $\beta > \left(\frac{l-1}{(r-1)\lambda}\right)^{1/(r-1)}$, and tends to infinity if $\beta < \left(\frac{l-1}{(r-1)\lambda}\right)^{1/(r-1)}$.

This computation leads to a natural conjecture about the independence number of an $(n,r,l)$-system $\mathcal{H}$. Theorems such as (1.5) suggest that the edge density is what really determines $\alpha(\mathcal{H})$, rather than specific properties such as being a Steiner system. If this is true, then:

**Conjecture 7.3.** For all $r > l > 1$,

$$c(r,l) = d(r,l) = (r)\frac{l - 1}{r - 1}.$$

The upper bound $d(r,l)$ given in Theorem 1.5 matches this conjecture, which is why we believe it is the best possible constant.

### 7.2 Proof of Theorem 1.5, algebraic construction

According to Lemma 7.1, we prove the theorem by constructing an $(n,r,l)$ system via a random process. First we describe the process in detail, then we prove that when $r \geq 2l - 1$ it satisfies the hypotheses of the lemma with $\lambda = 1/(r)_l$. The remaining cases $r < 2l - 1$ do not appear to be amenable to this method, but in Chapter 8 we present a different proof which covers all cases.

#### 7.2.1 Construction

Let $n = q^2$, where $q$ is a prime power and $q > r$. Let $V = \mathbb{F}_q \times \mathbb{F}_q$, where $\mathbb{F}_q$ denotes the finite field of order $q$. If $f$ is a polynomial over $\mathbb{F}_q$, define

$$G_f = \{(x, f(x)) : x \in \mathbb{F}_q\},$$
which is the graph of the polynomial, and define

\[ \mathcal{P} = \{ G_f : \deg(f) \leq l - 1 \}. \]

Since each of these polynomial graphs contain \( q \) points, and no two distinct graphs can have \( l \) points in common, it follows that \( \mathcal{P} \) is a \((q^2, q, l)\)-system. (See [6] for the use of this system in the context of the de Bruijn-Erdős problem). To avoid excessive subscripts we shall henceforth identify each polynomial with its graph, and write \( f \in \mathcal{P} \) whenever \( \deg f \leq l - 1 \).

Next, let \( \mathcal{H}_q \) denote an asymptotically complete \((q, r, l)\)-system; i.e. such that \(|\mathcal{H}_q| \sim \binom{q}{r}/\binom{l}{r}\) as \( q \to \infty \) (c.f. Rödl [45]). For each \( f \in \mathcal{P} \), let \( \pi_f : V(\mathcal{H}_q) \mapsto f \) be a random bijection, chosen with the uniform distribution and independently over \( f \in \mathcal{P} \). We now define the hypergraph \( \mathcal{H} \) with vertex set \( V \), and with the random edge set

\[ \mathcal{H} = \bigcup_{f \in \mathcal{P}} \pi_f(\mathcal{H}_q) \]

where \( \pi_f(\mathcal{H}_q) \) denotes the \( r \)-tuples of the form \( \{ \pi_f(e) : e \in \mathcal{H}_q \} \).

We observe \( \mathcal{H} \) is an \((n, r, l)\)-system, regardless of how the \( \pi_f \) are chosen. Indeed, for any \( l \)-set \( b \subseteq V \), there can be at most one \( f \in \mathcal{P} \) containing \( b \), and for this \( f \) there is at most one \( e \in \mathcal{H}_L \) such that \( b \subseteq \pi_f(e) \).

### 7.2.2 Proof outline

According to Lemma 7.1, to prove Theorem 1.5 it is sufficient to show, for any set \( X \subseteq V \) with \( |X| = x := \lfloor \beta n^{\frac{r-1}{r}} (\log n)^{\frac{1}{r-1}} \rfloor \), that

\[ - \log P(\mathcal{H}[X] = \emptyset) \gtrsim \frac{\beta^r}{(r)!} n^{\frac{r-1}{r}} (\log n)^{\frac{1}{r-1}}. \quad (7.1) \]

Define \( D_f = \pi_f^{-1}(f \cap X) \), so \( \mathcal{H}[X] = \emptyset \) if and only if \( \mathcal{H}_q[D_f] = \emptyset \) for all \( f \). By construction, the events \( \{ \mathcal{H}_q[D_f] = \emptyset \} \) are independent over all \( f \in \mathcal{P} \). This implies that

\[ P(\mathcal{H}[X] = \emptyset) = \prod_{f \in \mathcal{P}} P(\mathcal{H}_q[\pi_f^{-1}(f \cap X)] = \emptyset). \]

Also, if we define \( d_f = |D_f| \) then \( D_f \) has the uniform distribution among all sets of size \( d_f \) in \( V(\mathcal{H}_q) \). Our proof depends on two main claims:
• If $d \leq q^\alpha$ (the exponent $\alpha > 0$ to be given later) and if $D \subseteq V(\mathcal{H}_q)$ denotes a uniformly random set of size $d$, then

$$- \log P(\mathcal{H}_q[D] = \emptyset) \geq (1 - \epsilon) \frac{(d,r)}{(q)_r (l)_r},$$

where $\epsilon \to 0$ as $q \to \infty$, but does not depend on $d$.

• If $\mathcal{P}_0 = \{f \in \mathcal{P} : d_f \leq q^\alpha\}$, then

$$\sum_{f \in \mathcal{P}_0} (d_f)_r \gtrsim q^l (x/q)^r.$$

Following these two claims, we prove inequality (7.1) as follows:

$$- \log P(\mathcal{H}[X] = \emptyset) \geq \sum_{f \in \mathcal{P}_0} - \log P(\mathcal{H}_q[D_f] = \emptyset) \gtrsim \frac{\binom{q}{l}}{(q)_r (l)_r} \sum_{f \in \mathcal{P}_0} (d_f)_r \gtrsim \frac{q^{l-r}}{(r)_l} (q^l (x/q)^r) = \frac{\beta r}{(r)_l} n^{r-1} (\log n)^{r-r}.$$

(We leave it to the reader to verify that the exponent of $n$ is correct in the last step). As noted above, this inequality is sufficient to prove Theorem 1.5. However, as we shall see shortly, the proofs of claims 1 and 2 require the condition $r \geq 2l-1$.

### 7.2.3 Proof of Claim 1

To find an upper bound for the probability that $D$ contains no edge, we can use the principle of inclusion-exclusion to lower-bound the complement event:

$$P(\bigcup_{e \in \mathcal{H}_q} \{e \subseteq D\}) \geq \sum_{e \in \mathcal{H}_q} P(e \subseteq D) - \sum_{e \neq f \in \mathcal{H}_q} P(e \cup f \subseteq D).$$

We’ll break down the second sum according to the variable $k = |e \cap f|$. Because $\mathcal{H}_q$ is a $(q,r,l)$-system, $0 \leq k \leq l - 1$. If we fix a set $b$ of size $k$, observe
that the sets \( \{ e \setminus b : e \in \mathcal{H}_q, b \subseteq e \} \) form a \((q - k; r - k, l - k)\)-system. Therefore the number of edges containing \( b \) is at most \( \binom{q-k}{l-k} / \binom{r-k}{l-k} \), and so the number of pairs of edges whose intersection has size \( k \) is at most

\[
\binom{q}{k} \left( \frac{\binom{q-k}{l-k}}{\binom{r-k}{l-k}} \right) = O(q^{2l-k}).
\]

Finally, if \( A \) is any fixed set of size \( a \leq d \), then

\[
P(A \subseteq D) = \frac{(d)^a}{(q)_a}.
\]

Combining these statements, it follows that

\[
P(\mathcal{H}_q[D] = \emptyset) \leq 1 - |\mathcal{H}_q| \frac{(d)_r}{(q)_r} + \sum_{k=0}^{l-1} O(q^{2l-k} \frac{(d)_{2r-k}}{(q)_{2r-k}}).
\]

Now, we need to show that each term of the sum is asymptotically irrelevant when \( d \) is not too large. Specifically, we want to show

\[
\lim_{q \to \infty} \left( q^{2l-k} \frac{(d)_{2r-k}}{(q)_{2r-k}} \right) / \left( |\mathcal{H}_q| \frac{(d)_r}{(q)_r} \right) = 0
\]

where the convergence is uniform over \( d \leq q^\alpha \). To simplify matters, this ratio is an increasing function of \( d \) so we may assume \( d = q^\alpha \). Recall that \( |\mathcal{H}_q| \sim \frac{q}{\pi^2} = O(q^\ell) \), so this limit is zero provided that

\[
(2l - k) + (2r - k)\alpha - (2r - k) < l + r\alpha - r
\]

which is equivalent to \( r - l > (r - k)\alpha \). If \( 0 < \alpha < 1 - \frac{l}{r} \) then this is true for all \( k \geq 0 \). Consequently, we can ignore the summation and write

\[- \log P(\mathcal{H}_q[D] = \emptyset) \gtrsim |\mathcal{H}_q| \frac{(d)_r}{(q)_r} \sim \frac{q}{\pi^2} \frac{(d)_r}{(q)_r}\]

which is Claim 1.

### 7.2.4 Bounds on \( \alpha \)

We want to make sure \( \alpha \) is large enough that \( d_f \leq q^\alpha \) will be satisfied by almost all \( f \in \mathcal{P} \). This will be true if \( q^\alpha / \overline{d} \to \infty \), where \( \overline{d} \) is the average value of
$d_f$ over $f \in \mathcal{P}$. Since every $v \in V$ is incident with exactly $q^{-1}$ polynomials, we have

$$
\overline{d_f} = \frac{1}{|\mathcal{P}|} \sum_{f \in \mathcal{P}} d_f = q^{-l}(q^{l-1}x) = x/q \sim \beta q^{2(r-1)/r-1}(2 \log q)^{1/2}.
$$

We remark that our method is only going to work when $\overline{d_f} \to \infty$. This is true because the exponent $\frac{2(r-l)}{r-1} - 1 \geq 0$ whenever $r \geq 2l - 1$, which is our hypothesis.

To summarize, if we define

$$
\mathcal{P}_0 = \{ f \in \mathcal{P} : d_L \leq q^\alpha \}
$$

where

$$
\frac{2(r-l)}{r-1} - 1 < \alpha < 1 - \frac{l}{r}, \tag{7.2}
$$

then Claim 1 holds, and also

$$
|\mathcal{P}_0| \sim |\mathcal{P}|.
$$

To check that equation (7.2) is valid: note that left side is equal to $1 - \frac{2l-2}{r-1}$, so it is equivalent to check $\frac{2l-2}{r-1} < \frac{r-1}{r}$. As long as $l \geq 2$ and $r > 0$, the left side is at least 1 and the right side is less than 1.

### 7.2.5 Proof of Claim 2

If we momentarily ignore the restriction to $\mathcal{P}_0$, then claim 2 follows from a convexity argument. For real $t \geq 0$ define

$$
g(t) = \begin{cases} 
0, & 0 \leq t \leq r - 1 \\
t(t-1)\ldots(t-r+1), & t \geq r - 1
\end{cases}
$$

Then $g$ is convex and $g(k) = (k)$, for all integers $k \geq 0$. Therefore

$$
\sum_{f \in \mathcal{P}} (d_f)_r \geq q^l g(\overline{d_f}).
$$

In the previous section we computed that $\overline{d_f} = x/q \to \infty$ which means that $g(\overline{d_f}) \sim (\overline{d_f})^r$, and thus

$$
\sum_{f \in \mathcal{P}} (d_f)_r \gtrsim q^l (x/q)^r.
$$
We need only to prove the same thing for $P_0$, which we already know includes almost all of $P$. Unfortunately, \textit{a priori} the average value of $d_f$ could be significantly smaller on $P_0$ than on $P$.

To prove that in fact the two average values are asymptotically the same, we need to consider the eigenvalues of the incidence graph for $P$, by which we mean the bipartite graph with parts $V$ and $P$, where $v = (x, y)$ is adjacent to $f$ if $f(x) = y$. We will use the bipartite version of the expander-mixing lemma, which was proven in Chapter 4.

**Lemma 7.4.** For the bipartite incidence graph of $P$, $\lambda_3 = q^{l-1}$.

**Proof.** Let $M$ denote the $(q^2 + q^l) \times (q^2 + q^l)$ adjacency matrix for $P$, and let $K = \begin{bmatrix} 0 & J \\ J^T & 0 \end{bmatrix}$, where $J$ is the $q^2 \times q^l$ all-ones matrix. We claim that

\[ M^3 = (q - 1)q^{l-2}K + q^{l-1}M. \]  

(7.3)

To prove this, consider an arbitrary pair $v = (x_0, y_0) \in V, f \in P$, and count the number of length-3 walks from $f$ to $v$. That is, we count the pairs $(v', f')$ such that $f \sim v' \sim f' \sim v$.

First suppose $v \not\sim f$; that is, $f(x_0) \neq y_0$. Then there are $q - 1$ choices for $v'$, namely $(x, f(x))$ where $x \neq x_0$. (If $x = x_0$, then $v$ and $v'$ are distinct points with the same $x$-coordinate, so there is no polynomial $f'$ passing through both). For each of these choices, there are exactly $q^{l-2}$ polynomials $f'$ passing through both $v$ and $v'$, since they have different $x$-coordinates.

On the other hand, if $v \sim f$, then in addition to the above $(q - 1)q^{l-1}$ pairs $(v', f')$, we can also choose $v' = v$. In this case, there are $q^{l-1}$ choices for $f'$, namely all the polynomials that pass through $v$. This proves the matrix equation (7.3).

Now, if $\phi_3$ is an eigenvector corresponding to the third-largest eigenvalue of $M$, it follows from the proof of the expander-mixing lemma that $K\phi_3 = 0$. Therefore

\[ \lambda_3^3 = (q^{l-1})\lambda_3 \]

which means that $\lambda_3 = \pm q^{l-1}$ or $\lambda_3 = 0$. However, $\lambda_3 = 0$ is clearly impossible:
for example the expander-mixing lemma would imply that the number of edges between any single \( v \in V \) and \( f \in \mathcal{P} \) is exactly \( \frac{q}{q^l} \), which is absurd.

Now, we can use the expander-mixing lemma to show that \( \sum_{f \in \mathcal{P}_0} d_f = e(X, \mathcal{P}_0) \) is close to \( \frac{x}{q} \frac{1}{|\mathcal{P}_0|} \) = \( (x/q)|\mathcal{P}_0| \). The error satisfies

\[
\left| \sum_{f \in \mathcal{P}_0} d_f - (x/q)|\mathcal{P}_0| \right| \leq q^{\frac{1}{2} - \frac{1}{2}} \sqrt{x|\mathcal{P}_0|}
\]

In order for \( (x/q)|\mathcal{P}_0| \) to be larger than \( q^{\frac{1}{2}} \sqrt{x|\mathcal{P}_0|} \) by at least some positive power of \( \log q \), we require the exponent of \( q \) on the left to be at least as large as that on the right. A quick computation reveals the necessary condition:

\[
\frac{2(r - l)}{(r - 1)} - 1 + l \geq \frac{l - 1}{2} + \frac{r - l}{r - 1} + \frac{l}{2} \geq \frac{r - l}{r - 1} \geq \frac{1}{2} \geq 2(r - l) \geq r - 1 \geq 2l - 1.
\]

Thus for all \( r \geq 2l - 1 \), we conclude that the error term is asymptotically dominated, and therefore

\[
\frac{1}{|\mathcal{P}_0|} \sum_{f \in \mathcal{P}_0} d_f \sim x/q.
\]

so the convexity argument proves Claim 2. The proof of Theorem 1.5 is now complete, in the case \( r \geq 2l - 1 \).

### 7.3 Concluding remarks

#### 7.3.1 Complete \((n, r, l)\)-system constructions for \( l = 2, 3 \)

The construction of section 7.2 based on the polynomial system \( \mathcal{P} \) gives an asymptotically complete \((n, r, l)\)-system so it suits our purposes in this paper. However, for \( l = 2 \) and \( l = 3 \) there are known constructions which could give a complete system. For \( l = 2 \) we could substitute a projective plane of order \( q \) (i.e. a complete \((q^2 + q + q, q + 1, 2)\)-system) for \( \mathcal{P} \). The necessary eigenvalue computations
are very similar to those of $\mathcal{P}$. If we then choose $\mathcal{H}_{q+1}$ to be a complete $(q+1, r, 2)$-system, the resulting $r$-graph $\mathcal{H}$, after “filling in” each block with $\mathcal{H}_{q+1}$, is a complete $(q^2 + q + 1, r, 2)$-system. However this requires the simultaneous existence of a projective plane of order $q$ and complete Steiner system on $q + 1$ vertices. In the case $r = 3$ it was first shown by Kirkman in 1847 [34] that a complete Steiner $(n, 3, 2)$-system exists if and only if $n \equiv 1$ or $3 \mod 6$. This means that we could choose $q = 2^{2k+1}$, in which case $q + 1 \equiv 3 \mod 6$ so $\mathcal{H}_{q+1}$ exists. However, not much is known about the existence of Steiner systems for $r = 4$ and beyond, so this is a rather difficult unsolved problem.

For $l = 3$, one could consider using an inversive plane of order $q$ which (among other properties) is a complete $(q^2 + 1, q + 1, 3)$-system. For a construction and discussion of inversive planes, see [16]. Again the eigenvalue computations could be repeated for an inversive plane, but theorems concerning the existence of $(n, r, 3)$-systems for infinitely values of $n$ are sparse.

### 7.3.2 The codegree parameter

In the theory of block designs there is often another parameter $d$, representing the maximum number of $r$-blocks containing a given $l$-set. One could in fact define $f(n, d, r, l)$ as the minimum value of $\alpha(\mathcal{H})$ for an $r$-graph $\mathcal{H}$ on $n$ vertices in which the codegree of every $l$-set is at most $d$, and try to determine the growth rate of $f(n, d, r, l)$ for fixed $r, l$ and $d$ bounded in a certain range. In the case $l = r - 1$, Kostochka, Mubayi, and Verstraëte [31] give the bound

$$\alpha(\mathcal{H}) \geq c_r \left( \frac{n}{d} \log \frac{n}{d} \right)^{\frac{r-1}{r-1}} \tag{7.4}$$

for the minimum independence number of an $r$-graph $\mathcal{H}$ on $n$ vertices where every $(r-1)$-set has maximum codegree $d < n/(\log n)^{3(r-1)^2}$. $c_r$ is a constant satisfying $c_r \sim r/e$ as $r \to \infty$. This agrees with the current paper at $d = 1$. Moreover, we can derive an upper bound corresponding to (7.4) if $d$ is a constant by using Lemma 7.1, as follows. If $\mathcal{H}_n$ is a random $(n, r, l)$-system satisfying the hypothesis of Lemma 1, then we can define $\mathcal{H}_n^d$ as the union of $d$ independent copies of $\mathcal{H}_n$. 

Then $\mathcal{H}_n^d$ is an $(n, d, r, l)$ block design as described above, and

$$-\log P(\mathcal{H}_n[X] = \emptyset) \gtrsim n^r r^{r-1} (\log n)^{r-1}.$$  

Thus we get the same conclusion as before with $\lambda$ replaced by $d\lambda$. This has the overall effect of dividing our upper bound by $d^{1/(r-1)}$, which is same overall effect of $d$ in equation (7.4) when $d$ is constant. With more care it should be possible to take $d$ to be a slowly-growing function. However, for the sake of simplicity we have elected not to do this.
Chapter 8

Nibble construction of partial Steiner systems

The so-called “Rödl Nibble” is an algorithm for randomly generating an $(n, r, l)$-system, which is in principle not so different from the intuitive idea of simply building the system one edge at a time — each edge being chosen uniformly at random from among all possible $r$-sets not sharing $l$ vertices with any previously chosen edge. The main difference is that instead of choosing only one of these possible edges, we choose a significant random subset of them (a nibble), then keep as many of these chosen edges as possible. In doing so, we (a) cause each step to have a non-infinitesimal effect on the overall density, and (b) introduce some independence into the problem. Both of these benefits simplify the analysis. In principle the process can continue until there are no more edges left to add, although we will simplify things further by terminating it after the hypergraph has enough edges.

In 1996, V. Rödl and T. Lubos [45] introduced this algorithm and proved that this algorithm can be used to generate an $(n, r, l)$ partial Steiner system with a number of edges asymptotic to $\binom{n}{l}/\binom{r}{l}$, which is the maximum number of edges for such a system. In this chapter, we will prove that the independence number of the hypergraph generated by this algorithm is asymptotically at least as large as that of a pure random graph $G_r(n, p)$ of the same density (namely $p = \frac{\binom{n}{l}}{\binom{r}{l}\binom{r}{l}}$). This is Theorem 1.5 from Chapter 1, and a continuation of our program for finding
Steiner systems with small independence number. We prove Theorem 1.5, but this time for all values $2 \leq l \leq r - 1$, and by using a completely different construction. Additionally, our analysis of the independence number of the Nibble method may be of interest independently from this program.

8.1 Proof of Theorem 1.5

8.1.1 Construction

In this section we define our nibble procedure in full detail. Let $H_0 = \binom{[n]}{r}$, the system all of $r$-tuples on $n$ vertices. If $S \subseteq H_0$, define the neighbor set

$$\mathcal{N}(S) := \{ e \in H_0 : | e \cap f | \geq l \text{ for some } f \in S \}.$$

We say that $S_0, \ldots, S_\tau$ is a nibble sequence if $S_i \subseteq H_i$ for all $0 \leq i \leq \tau$, where $H_i$ is defined inductively by

$$H_{i+1} := H_i \setminus \mathcal{N}(S_i).$$

At this point, we wish to reiterate that we defined $H_{i+1} = H_i \setminus \mathcal{N}(S_i)$ rather than $H_{i+1} = H_i \setminus \mathcal{N}(S^*_i)$. Even when an edge of $S_i$ is not be kept for the final system $H$, we still throw out its neighbors as though it had been kept. The reason for this apparent wastefulness is just to simplify the analysis. It is but one of many instances where we can throw out “lower order terms” with impunity.

We shall select $S_i$ via the $G_r(n,p)$ model, using an arbitrary parameter $\delta > 0$ chosen before the process begins. At stage $i$, we define $S_i \subseteq H_i$ to be a random subset in which each edge of $H_i$ is included independently with probability

$$P(e \in S_i) = p_i := \frac{\delta}{c_i}, \quad c_i := e^{-(l+1)\delta} \binom{n-l}{r-l}.$$ 

As we shall see, $c_i$ is the asymptotic codegree of every $l$-set in $H_i$ whose codegree is not zero. The value of $p_i$ is chosen so that for such an $l$-set, the number of edges in $S_i$ containing that set has approximately a Poisson distribution of parameter $\delta$. 
8.1.2 The Regularity conditions

We will use Lemma 7.1 to prove Theorem 1.5. According to the method of Lemma 7.1, let $U$ be a fixed subset of vertices of size $u = \lfloor \beta n^{\frac{r+1}{r}}(\log n)^{\frac{1}{r-1}} \rfloor$. For any $l$-set $b$, let $C_{i,b}$ denote codegree of $b$ in $H_i$. We’ll say that $H_i$ is regular if the following conditions are met:

(a) For every $(r - 1)$-set $b$ such that $C_{i,b} > 0$,

$$|\frac{C_{i,b}}{c_i} - 1| \leq O(n^{-\gamma})$$

(b) $|H_i[U]| \geq e^{-\binom{i}{r} \delta_i \left(\frac{u}{r}\right)} (1 - O(n^{-\gamma}))$

The exponent $\gamma$ is any positive real number satisfying $1 - 3\gamma > \frac{r - l}{r - 1}$. Throughout this section the constants implicit in the $O(\cdot)$ notation are allowed to depend on $r, l, \delta$, and $i$, but not $n$. We will only keep track of the process through time $\tau = \tau(\delta)$; therefore the maximum value of $i$ does not depend on $n$ either.

8.1.3 Proof of Theorem 1.5

Let $R_i$ denote the event that $H_i$ is regular. In particular, $P(R_0) = 1$. Our proof of Theorem 1.5 is based on the following claims: for each $i = 0, \ldots, \tau$,

$$P(R_{i+1}^C | R_i) = \epsilon_i, \quad -\log \epsilon_i = \Omega(n^{1-3\gamma}). \quad (8.1)$$

$$P(S_i^*[U] = \emptyset | R_i) = q_i, \quad -\log q_i \geq p_ie^{-\binom{i}{r}\delta(i+1)\left(\frac{u}{r}\right)}. \quad (8.2)$$

Given these two claims, we prove Theorem 1.5 as follows. Let $E_i$ denote the event that $S_i^*[X] = \emptyset$.

$$P(E_i \cap E_{i+1} \cap \ldots \cap E_{\tau} | R_i)$$

$$\leq P((E_i \cap R_{i+1}) | R_i) P(E_{i+1} \cap \ldots \cap E_{\tau} | R_{i+1}) + P((E_i \cap R_{i+1}^C) | R_i)$$

$$\leq P(E_i | R_i) P(E_{i+1} \cap \ldots \cap E_{\tau} | R_{i+1}) + P(R_{i+1}^C | R_i)$$

$$\leq q_i P(E_{i+1} \cap \ldots \cap E_{\tau} | R_{i+1}) + \epsilon_i.$$
By induction,
\[ P(E_0 \cap E_1 \cap \ldots \cap E_\tau) \leq q_0(q_1(q_2(\ldots + e_2) + e_1) + e_0 \leq \prod_{i=0}^{\tau} q_i + \sum_{i=0}^{\tau} \epsilon_i. \]

According to Claim 8.2 we have
\[ -\log \prod_{i=0}^{\tau} q_i \geq \binom{\tau}{r} \sum_{i=0}^{\tau} p_i e^{-\binom{i}{r}} \delta(i+1) \]
\[ \sim \frac{\beta^r}{r!} n^{\frac{r(r-1)}{r-1}} (\log n)^{\frac{r}{r-1}} \sum_{i=0}^{\tau} \frac{\delta(r-l)!}{n^{r-l}} e^{\left(\binom{i}{r}-1\right)\delta i} e^{-\binom{i}{r}\delta(i+1)} \]
\[ = \frac{\beta^r}{(r)_l} n^{\frac{r-l}{r-l}} (\log n)^{\frac{r}{r-1}} \delta e^{\binom{i}{r}} \sum_{i=0}^{\tau} e^{-\delta i}. \]

Furthermore, the error term \( \sum_{i=0}^{\tau} \epsilon_i \) is negligible compared to this, because \( 1 - 3\alpha > \frac{1}{r-1} \). Thus, we may apply Lemma 7.1, with
\[ \lambda = \frac{1}{(r)_l} \delta e^{-r\delta} \sum_{i=0}^{\tau} e^{-\delta i}. \]

Of course, \( \delta \) and \( \tau \) are arbitrary parameters, and we are only interested in the best value of \( \lambda \) we can attain. This occurs when \( \delta \to 0 \) while \( \delta \tau \to \infty \), in which case we get \( \lambda \to \frac{1}{(r)_l} \) (using basic calculus). Therefore Lemma 7.1 gives
\[ d(r,l) \leq \left( \frac{l-1}{r-1}(r)_l \right)^{\frac{1}{r-1}} \]
as desired. It now remains only to prove claims 8.1 and 8.2.

### 8.1.4 Proof of claim (8.1) part (a)

Given that the regularity statements (a) and (b) hold in \( \mathcal{H}_i \), we have to show that they can fail in \( \mathcal{H}_{i+1} \) only with exponentially small probability. In this section we prove (a). A remark on notation: for the remainder of this chapter we regard \( \mathcal{H}_i \) to be a fixed (i.e. not random) hypergraph which satisfies condition \( R_i \). This is to avoid the language of conditional expectation, which would only add to the confusion. However, the nibble \( S_i \) and everything depending on it, is of course random.
Let $b$ be an $l$-set of positive codegree in $\mathcal{H}_i$, and let $\mathcal{A}$ be the set of all edges in $\mathcal{H}_i$ containing $b$. We need to control the codegree $C_{i+1,b}$ only on the event that $b$ survives; therefore we may assume $S_i \cap \mathcal{A} = \emptyset$. First, observe that distinct edges $a_1, a_2 \in \mathcal{A}$ could have common neighbors outside of $\mathcal{A}$, implying that they do not survive independently of one another. In order to overcome this difficulty and introduce independence into the problem, our method will be to partition $\mathcal{A}$ into subsets where the number of common neighbors within these subsets is of a lower order than the number of edges neighboring only a single element. Provided we ignore this small number of common neighbors, the edges within each subset do survive independently, allowing us to use the standard Chernoff bounds for controlling sums of independent indicator variables. The details are as follows.

Firstly, we demand a partition of $\mathcal{A}$ such that the edges within each part do not intersect outside $b$. We recognize this as a graph coloring problem, where the graph in question has a maximum degree bounded above by $\Delta = O(n^{r-l-1})$. By the Hajnal-Szemerédi Theorem [23], $\mathcal{A}$ admits a partition into parts of size $n^{1-\gamma}$, since the number of parts required is of order $\Theta(n^{r-l-1+\gamma}) > \Delta$. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_\eta$ denote such a partition.

Now for $j = 1, \ldots, \eta$ let $\mathcal{D}_j$ be the collection of edges having more than one neighbor in $\mathcal{A}_j$:

$$\mathcal{D}_j := \{e \in \mathcal{H}_i \setminus \mathcal{A} : |\mathcal{N}(e) \cap \mathcal{A}_j| \geq 2\}.$$ 

For each $j$, the $a \in \mathcal{A}_j$ now survive independently provided we ignore the set $\mathcal{D}_j$. In other words, the indicators

$$\{I_a := 1(\mathcal{N}(a) \cap (S_i \setminus \mathcal{D}_j) = \emptyset)\}_{a \in \mathcal{A}_j}$$

are independent. Next, let us quantify the amount of error introduced by ignoring $\mathcal{D}_j$: define

$$C_{i+1,b,j} = |\mathcal{A}_j \setminus |\mathcal{N}(S_i \setminus \mathcal{A})|$$

which is the actual number of surviving edges in $\mathcal{A}_j$. Then we claim that

$$\left(\sum_{a \in \mathcal{A}_j} I_a\right) - (r - l + 1)|S_i \cap \mathcal{D}_j| \leq C_{i+1,b,j} \leq \sum_{a \in \mathcal{A}_j} I_a.$$  

(8.3)
The upper bound is obvious, and the lower bound is a consequence of the following fact: no edge in $D_j$ can be a common neighbor of more than $r - l + 1$ distinct edges in $A_j$. This, in turn, is due to the condition that edges in $A_j$ do not intersect outside $b$.

We now aim to show that, as random variables, $\sum_{a \in A_j} I_a$ is concentrated and $|S_i \cap D_j|$ is small, implying that $C_{i+1,b,j}$ is concentrated also. First, we need to estimate $P(I_a = 1)$ by counting neighbors.

**Lemma 8.1.** For each $a \in A_j$,

$$E[I_a] = e^{-((r/l) - 1)\delta}(1 \pm O(n^{-\gamma}))$$

**Proof.** Let

$$d_a = |N(a) \cap (H_i \setminus A \setminus D_j)|,$$

so that $E[I_a] = (1 - p_i)^{d_a}$.

First, according to the regularity condition (a),

$$d_a \leq |N(a) \cap (H_i \setminus A)| \leq \left(\binom{r}{l} - 1\right)c_i(1 + O(n^{-\gamma})),$$

since if we sum up the neighbors of each $l$-set in $a$ except for $b$, we get an overestimate. By similar reasoning, we claim that

$$|N(a) \cap (H_i \setminus A)| \geq \left(\binom{r}{l} - 1\right)c_i(1 - O(n^{-\gamma})),$$

which is valid if we compensate for the overcounting of any edges of $H_i$ which intersect $b$ in more than $l$ vertices. The number of such edges is $O(n^{r-l-1})$, whereas $c_i = \Theta(n^{r-l})$, so by the principle of inclusion-exclusion, this overcount is absorbed into the $(1 - O(n^{-\gamma}))$ error factor.

Of course, $d_a \leq N(a) \cap (H_i \setminus A)$ because we must subtract off the edges in $N(a) \cap D_j$. Using a standard counting argument to upper-bound $|N(a) \cap D_j|$, the highest order term comes from choosing $l - 1$ vertices in $b$, one vertex from $a$, one vertex from some other edge $a' \in A_j$, and $r - l - 1$ more vertices anywhere. It follows that $|N(a) \cap D_j| = O(|A_j|n^{r-l-1}) = O(c_i n^{-\gamma})$. This proves that

$$d_a = \left(\binom{r}{l} - 1\right)c_i(1 \pm O(n^{-\gamma})).$$
It follows that
\[
\mathbb{E}[I_a] = (1 - p_i)^{d_a} \\
= \exp \left[ \left( \binom{r}{l} - 1 \right) c_i \left( 1 \pm O(n^{-\gamma}) \log(1 - \frac{\delta}{c_i}) \right) \right] \\
= e^{-\binom{r}{l-1}\delta \left( 1 \pm O(n^{-\gamma}) \right)}. 
\]

Summing this over \( \mathcal{A}_j \), we get
\[
E \left[ \sum_{a \in \mathcal{A}_j} I_a \right] = \left( e^{-\binom{r}{l-1}\delta} n^{1-\gamma} \right) \left( 1 \pm O(n^{-\gamma}) \right). 
\]  

(8.4)

Next we compute the probability that \( Y_j := \sum_{a \in \mathcal{A}_j} I_a \) deviates from its expectation, by more than the margin of error of the expectation itself. We can use a standard Chernoff bound: whenever \( Y \) is the sum of \( n \) independent indicators, \( P(|Y - EN| > t) < 2 \exp(-2t^2/n) \) [7] (Appendix A). Thus
\[
P(|Y_j - E[Y_j]| > n^{1-2\gamma}) \leq 2 \exp\left( -\frac{2n^{2-4\gamma}}{|\mathcal{A}_j|} \right) = \exp(-\Theta(n^{1-3\gamma})). 
\]  

(8.5)

Next, in equation (8.3) we need to show that \( |\mathcal{S}_i \cap \mathcal{D}_j| \) is small (namely \( O(n^{1-2\alpha}) \)) with high probability. First, note that (by another standard counting argument)
\[
|\mathcal{D}_j| = O(|\mathcal{A}_j|^2 n^{r-l-1}) = O(n^{r-l+1-2\gamma}), \\
E[|\mathcal{S}_i \cap \mathcal{D}_j|] = O(n^{1-2\gamma}).
\]

Now \( Y := |Y_j \cap S_i| \) is also a sum of independent indicators. We have the following Chernoff-type bound [7] (Appendix A):
\[
P(Y > \mu + t) < \exp[t - \mu \log(1 + t/\mu) - t \log(1 + t/\mu)] 
\]
where \( \mu = E[Y] \). Setting \( t = Kn^{1-2\gamma} \), with \( K \) large enough that \( t/\mu \geq 2 \), we get
\[
P(Y > Kn^{1-2\gamma}) < \exp \left( n^{1-2\gamma} - K' n^{1-2\gamma} (\log 3) \right) = \exp \left( -\Theta(n^{1-2\gamma}) \right). 
\]  

(8.6)

Together, the bounds (8.4), (8.5), and (8.6) in equation (8.3), imply that
\[
C_{t+1,b,j} = e^{-\binom{r}{l-1}\delta n^{1-\gamma} \left( 1 \pm O(n^{-\gamma}) \right)},
\]
with probability at least \(1 - \exp[-\Theta(n^{1-3\alpha})]\). Of course, the same bound holds for all \(j\), which implies that

\[
C_{i+1,b} = e^{-\binom{i}{1} - \delta} c_i (1 \pm O(n^{-\gamma})
\]

(or \(C_{i+1,b} = 0\)) with probability at least \(1 - \frac{c_i}{n^{1-\gamma}} \exp(-\Theta(n^{1-3\alpha}))\). Finally, with probability at least \(1 - \binom{n}{l} c_i/n^{1-\gamma} \exp(-\Theta(n^{1-3\alpha}))\) this same bound holds for all \(l\)-sets \(b\). This proves part (a) of (8.1).

### 8.1.5 Proof of claim (8.1) part (b)

The proof of (b) will follow a similar format to that of (a). Recall that \(U\) is an arbitrary vertex set of size \(u = \lceil \beta n^{r-\frac{1}{2}} (\log n)^{\frac{1}{r-1}} \rceil\). If \(y \in \mathcal{H}_i[U]\), then \(y\) could be deleted by an edge having exactly \(l\) vertices in \(U\) — henceforth an outside edge — or by a non-outside edge. We show that \(\mathcal{H}_i[U]\) can be partitioned in such a way that the edges within a given part survive independently, as far as outside edges are concerned. It follows that the number of edges deleted from the outside must be concentrated. Additionally, the non-outside edges are of smaller order and we will show they may be safely ignored.

For \(y \in \mathcal{H}_i[U]\), define

\[
I_y := 1(|e \cap U| > l \text{ for all } e \in \mathcal{N}(y) \cap \mathcal{S}_i), \quad X_{\text{out}} := \sum_{y \in \mathcal{H}_i[U]} I_y.
\]

That is, \(I_y = 1\) indicates that \(y\) is not deleted from the outside, and \(X_{\text{out}}\) is the number of such survivors. Next, let

\[
\mathcal{D} := \{e \in \mathcal{H}_i : |e \cap U| > l\},
\]

the set of non-outside edges. For each \(a \in \mathcal{D}\), we have \(|\mathcal{N}(a) \cap \mathcal{H}_i[U]| \leq \binom{n}{l} (\frac{u}{r-l})\), implying a deterministic bound

\[
X_{\text{out}} - \binom{r}{l} \binom{u}{r-l} |\mathcal{S}_i \cap \mathcal{D}| \leq |\mathcal{H}_{i+1}[U]| \leq X_{\text{out}}. \quad (8.7)
\]

Much as before, our aim is to show that \(X_{\text{out}}\) is concentrated and \(|\mathcal{S}_i \cap \mathcal{D}|\) is sufficiently small.
We can calculate $E[I_y]$ by counting neighbors, as in the proof of Lemma 8.1. Each $l$-set in $y$ has $c_l(1 \pm O(n^{-\gamma}))$ neighbors in $H_i$, and the contribution from non-outside edges is $O(un^{r-l-1})$. Since $u = o(n^{1-\gamma})$, the conclusion is much as before:

$$E[I_y] = e^{-\binom{l}{2}}(1 \pm O(n^{-\gamma})).$$

Next, note that if $y_1, y_2 \in \mathcal{H}_i[U]$ have fewer than $l$ vertices in common, and if $e$ is a common neighbor of $y_1, y_2$, then $e$ must be a non-outside edge. It follows that whenever $\mathcal{Y} \subseteq \mathcal{H}_i[U]$ is a subsystem containing no pair of neighboring edges, then the indicators $\{I_y\}_{y \in \mathcal{Y}}$ are independent. By the Hajnal-Szemerédi Theorem once again, $\mathcal{H}_i[U]$ admits a partition into $\eta$ such subsystems, of size as equal as possible, where $\eta = \binom{l}{2} \binom{n}{r-l-1}$. Let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_\eta$ denote such a partition.

To show concentration of $\mathcal{Y}_j$ we once again we use the Chernoff bound:

$$P(|Y_j - E[Y_j]| > t) < \exp(-2t^2/|Y_j|).$$

Note that $|\mathcal{Y}_j| = \Theta(u')$. To obtain sufficiently small deviations, set $t = n^{-\gamma}|\mathcal{Y}_j| = \Theta(u'n^{-\gamma})$,

which gives

$$P(|Y_j - E[Y_j]| > n^{-\gamma}|\mathcal{Y}_j|) < \exp(\Theta(u'n^{-2\gamma})).$$

The relevant aspect of this probability is the exponent on $n$: $u' = \Omega(n^{\frac{l}{2(r-l)}})$ and this exponent is always at least 1. Therefore, we can sum over all $j$ and conclude that

$$X_{out} = e^{-\binom{l}{2}}|\mathcal{H}_i|(1 \pm O(n^{-\gamma}))$$

with probability at least $1 - \eta \exp(-\Theta(1 - 2\gamma))$.

Also in (8.7) we must consider the random variable $|\mathcal{S}_i \cap \mathcal{D}|$. Since $|\mathcal{D}| \leq \binom{n}{r-l-1}$

$$E[\mathcal{S}_i \cap \mathcal{D}] = p_i|\mathcal{D}| = O(u'^{l+1}n^{-1}).$$

Since $Y := |\mathcal{S}_i \cap \mathcal{D}|$ is the sum of independent indicators,

$$P(Y > \mu + t) < \exp[t - \mu \log(1 + t/\mu) - t \log(1 + t/\mu)].$$
According to (8.7), to obtain a sufficiently small deviation we should set $t = n^{-\alpha}u^\ell$. Note that $\mu = o(t)$, so after discarding lower-order terms we get

$$P(|S_i \cap D| > O(n^{-\gamma}u^\ell)) < \exp \left(-\Theta(n^{-\gamma}u^\ell \log(n))\right).$$

(8.9)

Again, $u^\ell = \Omega(n)$ so this probability is sufficiently small. Together, the bounds (8.8) and (8.9) in equation (8.7) prove part (b) of claim 8.1.

### 8.1.6 Proof of claim (8.2)

Lastly we need to prove the upper bound (8.2) for $P(S^*_i[U] = \emptyset)$. Of course, not all edges in $S_i$ belong to $S^*_i$. However, the maximality of $S^*_i$ gives the following:

**Lemma 8.2.** Suppose there is an edge $y \in S_i[U]$ such that $N(y) \cap S_i \subset H_i[U]$. Then $S^*_i[U] \neq \emptyset$.

**Proof.** By the maximality of $S^*_i$, if $y \notin S^*_i$ then $y$ must have a neighbor $z \in S^*_i$. However, $z \in H_i[U]$ by hypothesis, and therefore $z \in S^*_i[U]$.

Let us define $A = H_i[U] \setminus N(S_i \setminus H_i[U])$. By the above lemma, if $A \cap S_i$ is nonempty, then so is $S^*_i[U]$. If we choose the random set $S_i$ in such a way that the edges not belonging to $H_i[U]$ are revealed first, then this determines $A$, and then each edge in $A$ gives an independent chance to make $S^*_i[U]$ nonempty. We can state this in terms of conditional expectation as follows:

$$P(S^*_i[U] = \emptyset \mid \mathcal{F}) \leq (1 - p_i)|A|,$$

where $\mathcal{F}$ is the $\sigma$-field generated by the random set $S_i \setminus H_i[U]$. Consequently, for any $t$,

$$P(S^*_i[U] = \emptyset) \leq (1 - p_i)^t + P(|A| < t).$$

Note that by definition, $H_{i+1} \subseteq A$, so we set $t = e^{-\delta(i+1)}(u)(1 - O(n^{-\gamma}))$, the bound used in claim (8.1). We compute

$$- \log((1 - p_i)^t) \sim p_it \sim pe^{-\delta(i+1)}(u).$$

(8.10)
which is the bound required for claim (8.2). It remains only to show that the term $P(|A| < t)$ is asymptotically irrelevant compared to this bound. From part (b) we have

$$- \log(P|A| < t) = \Omega(n^{1-2\alpha}),$$

whereas

$$p_i e^{-\left(\delta(i+1) \begin{pmatrix} u \\ r \end{pmatrix}\right)} = \Theta(n^{t-1})(\log n)^{r_t-1}).$$

The exponent of $n$ being smaller, after taking $- \log$, implies that we have the more significant term. This completes the proof of claim (8.2), which was the last thing we needed for the entire proof of Theorem 1.5.
Appendix A

In this appendix we prove the convexity of the functions $g$ and $h$ from Chapter 6. The material in this appendix is adapted from the manuscript “Bounds on Independence and Transversal numbers in Hypergraphs” by Anders Yeo, with certain modifications to improve the generality.

A.1 List of properties of the function $g$

**Proposition A.1.** Let $r \geq 3$ be an integer, and let $g_1, g_2, \ldots$ be a sequence of real numbers such that $g_1, g_2 > 0$ and

$$g_d = g_{d-1} - \frac{2g_{d-1} - g_{d-2}}{(r-1)d}, \quad d \geq 3. \quad (A.1)$$

Also define $\epsilon_d = g_{d-1} - g_d$. If

$$\frac{3r-1}{3r+1} \leq \frac{g_2}{g_1} \leq \frac{3r-2}{3r-1}, \quad (A.2)$$

then:

(i) $g_d > 0$ for all $d \geq 1$

(ii) $\epsilon_d > 0$ for all $d \geq 1$.

(iii) $\epsilon_{d-1} \geq \epsilon_d$ for all $d \geq 3$.

(iv) $\epsilon_{d-1} \leq (1 - 1/d)\epsilon_d$ for all $d \geq 4$.

(v) $((d-1)(r-1) - 2)(\epsilon_{d-2} - \epsilon_{d-1}) \geq \epsilon_{d-1} - \epsilon_d$ for $d \geq 5$. 

105
Recall that in Chapter 6 that we defined

\[ g_0 = 1, \quad g_1 = 1 - 1/r, \quad g_2 = \begin{cases} \frac{(3r-2)(r-1)}{r(3r-1)}, & r \text{ odd} \\ \frac{3r-4}{3r}, & r \text{ even} \end{cases} \]

It is a simple matter to check that (A.2) holds for both the even and odd cases, and that \( \epsilon_1 > \epsilon_2 \).

### A.2 Proof of parts (i), (ii), (iii), (iv)

Define

\[ \lambda_d = \frac{g_d}{g_{d-1}}, \quad d \geq 2. \]

Dividing the recurrence (A.1) by \( g_{d-1} \) yields the following:

\[ \lambda_d = 1 - \frac{2 - \lambda_{d-1}^{-1}}{(r-1)d}. \]  \hspace{1cm} (A.3)

**Lemma A.2.** Let \( \{g_d\} \) and \( \{\lambda_d\} \) be defined as above. For \( d \geq 3 \), we have the following equivalences:

- \( \epsilon_d \leq \epsilon_{d-1} \) if and only if \( \lambda_d \geq 1 - \frac{1}{(r-1)d+1} \).
- \( \epsilon_d \leq (1 - 1/d)(\epsilon_{d-1}) \) if and only if \( \lambda_{d-1} \leq 1 - \frac{1}{(r-1)(d-1)+2} \).

**Proof.** The following steps are reversible, proving the first equivalence:

\[
0 \leq g_d - 2g_{d-1} + g_{d-2} \\
\lambda_d \geq 2 - \lambda_{d-1}^{-1} = (r-1)d(1-\lambda_d) \\
\lambda_d \geq \frac{(r-1)d}{(r-1)d+1} = 1 - \frac{1}{(r-1)d+1}.
\]

The following steps are reversible, proving the second equivalence:

\[
(1 - 1/d)(g_{d-2} - g_{d-1}) \geq g_{d-1} - g_d \\
(1 - 1/d)(\lambda_{d-1}^{-1} - 1) \geq 1 - \lambda_d = \frac{2 - \lambda_{d-1}^{-1}}{(r-1)d} \\
(r-1)(d-1) \geq \frac{2 - \lambda_{d-1}^{-1}}{\lambda_{d-1}^{-1} - 1} \\
= \frac{2\lambda_{d-1} - 1}{1 - \lambda_{d-1}} = \frac{1}{1 - \lambda_{d-1}} - 2 \\
1 - \frac{1}{(r-1)(d-1)+2} \geq \lambda_{d-1}.
\]
Because of the above lemma, the following proves parts (i), (ii), (iii), (iv) of Proposition A.1:

**Lemma A.3.** For all $d \geq 3$,

$$1 - \frac{1}{(r - 1)d + 1} \leq \lambda_d \leq 1 - \frac{1}{(r - 1)d + 2}. \quad (A.4)$$

**Proof.** The proof is by induction. We shall see that the original assumption (A.2) was chosen precisely so that the base case $d = 3$ holds. We verify this as follows: If $\lambda_2 \leq \frac{3r - 2}{3r - 1}$, then

$$\lambda_3 \geq 1 - \frac{2 - \frac{3r - 1}{3r - 2}}{3r - 3} = 1 - \frac{3r - 3}{(3r - 3)(3r - 2)} = 1 - \frac{1}{3(r - 1) + 1}.$$  

Similarly if $\lambda_2 \geq \frac{3r - 1}{3r + 1}$, then

$$\lambda_3 \leq 1 - \frac{2 - \frac{3r + 1}{3r - 1}}{3r - 3} = 1 - \frac{3r - 3}{(3r - 1)(3r - 3)} = 1 - \frac{1}{3(r - 1) + 2}.$$  

Now for the inductive step: Let $d \geq 4$ and assume

$$1 - \frac{1}{(r - 1)(d - 1) + 1} \leq \lambda_{d-1} \leq 1 - \frac{1}{(r - 1)(d - 1) + 2}.$$  

By Lemma A.2, the upper bound implies $\epsilon_d \leq (1 - 1/d) \epsilon_{d-1} \leq \epsilon_{d-1}$. Applying the first part of Lemma A.2 now gives $\lambda_d \geq 1 - \frac{1}{(r - 1)d + 1}$.

To prove the upper bound on $\lambda_d$ we apply our lower bound for $\lambda_{d-1}$ in (A.3) and recall the identity $(1 - \frac{1}{x})^{-1} = 1 + \frac{1}{x - 1}$:

$$\lambda_d = 1 - \frac{2 - \lambda_{d-1}^{-1}}{(r - 1)d} \leq 1 - \frac{2 - (1 + \frac{1}{(r - 1)(d - 1)})}{(r - 1)d} = 1 - \frac{1}{(r - 1)d(1 + \frac{1}{(r - 1)(d - 1) - 1})}$$

$$= 1 - \frac{1}{(r - 1)d + \frac{(r - 1)d}{(r - 1)(d - 1) - 1}}.$$  

In order to prove $\lambda_d \leq 1 - \frac{1}{(r - 1)d + 2}$ it is therefore sufficient to show $\frac{(r - 1)d}{(r - 1)(d - 1) - 1} \leq 2$.

The following computation establishes this:

$$\frac{(r - 1)(d - 1) - 1}{(r - 1)d} = 1 - \frac{1}{d} - \frac{1}{(r - 1)d} \geq 1 - \frac{1}{4} - \frac{1}{12} > \frac{1}{2}.$$  

This completes the inductive step and proves the first four parts of Proposition ??.
A.3 Proof of part (v)

In order to prove that a Steiner system satisfied Theorem 5.7 we also needed property (v) above. If \( d \geq 5 \) then the following holds:

\[
((d-1)(r-1)^2 - 2)\epsilon_d - (d-2)\epsilon_{d-1}
\geq ((d-1)(r-1)^2 - 2)(\frac{\epsilon_d}{d-1} - \epsilon_{d-1})
\]

\[
= ((d-1)(r-1)^2 - 2)(\frac{d-2}{d-1} - 1)\epsilon_{d-1}
\]

\[
= \epsilon_{d-1} \frac{(d-1)(r-1)^2 - 2}{d-2}
\]

\[
\geq \epsilon_{d-1} \frac{2(d-1)^2 - 2}{d-2}
= 2\epsilon_{d-1}
\]

\[
\geq \epsilon_{d-1} - \epsilon_d
\]

A.4 The functions \( g, h \) from Chapter 6

Lastly, we have to check that the hypothesis (A.2) holds for the functions \( g, h \) defined in Chapter 6. If \( r \) is odd, we have

\[
g(2)/g(1) = \frac{(r-1)(3r-2)}{r(3r-1)} \frac{r}{r-1} = \frac{3r-2}{3r-1},
\]

so \( g(2)/g(1) \) is the largest possible value such that \( g \) is convex. (In particular, tracing through the steps of the proof, we see that \( g(1), g(2), g(3) \) is an arithmetic progression when \( r \) is odd).

If \( r \) is even, we have

\[
g(2)/g(1) = \frac{r(3r-4)}{(r-1)3r} = \frac{3r-4}{3r-3}.
\]

By simple algebra,

\[
\frac{3r-1}{3r+1} < \frac{3r-4}{3r-3} < \frac{3r-2}{3r-1}
\]

whenever \( 3 - 12r < -4 + 9r \), or \( r > 3 \). So again (A.2) holds.

Finally, for the function \( h \) defined when \( r = 6 \), we have

\[
\frac{3r-1}{3r+1} = \frac{17}{19} < h(2)/h(1) = \frac{125}{133} < \frac{3r-2}{3r-1} = \frac{16}{17}.
\]
Bibliography


