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Just-Infinite Algebras and An Extension of A Theorem of Herstein

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Cayley A. Pendergrass

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2006
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Chair

University of California, San Diego

2006
To my father- without your encouragement I never would have gotten here.
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PUBLICATIONS

ABSTRACT OF THE DISSERTATION

Just-Infinite Algebras and An Extension of A Theorem of Herstein

by

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Doctor of Philosophy in Mathematics
University of California San Diego, 2006
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We prove that just infinite dimensional $k$–algebras are prime and that just infinite is a Morita invariant. We consider a collection of Herstein’s theorems about simple rings and their related Jordan and Lie algebras and extend these to just infinite rings. In particular, we show that if $A$ is a just infinite associative ring without 2, 3, or 5–torsion, then $[A,A]/(Z \cap [A,A])$ is a just infinite Lie algebra, where $Z$ is the center of $A$. 

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Introduction

Much is known about simple rings. These results often can be extended to direct sums of simple rings. An alternative approach to extending these results is to revise the notion of simplicity by considering a looser restriction on the ideals of a ring. Rather than no proper ideals, a just infinite ring contains, in some sense, few ideals. The same principle can be used in the category of groups, where the term just infinite was first defined.

This dissertation is arranged into three principal sections. The second chapter summarizes the results in the literature. In Chapter 3 we prove, in Theorem 20 that all just infinite algebras are prime and discuss the implications.

Chapter 4 details results of Herstein for simple rings and extends them to just infinite algebras. In particular, we prove in 45 that if \( A \) is a just infinite dimensional associative \( k \)-algebra without 2, 3, or 5 torsion, and \( [A, A] \) is the Lie algebra generated by commutators of \( A \), that \( [A, A]/(Z \cap [A, A]) \) is also just infinite.

Chapter 5 is devoted to further questions about just infinite algebras and potential future research topics. The second section describes methods of constructing examples of just infinite associative algebras and general properties, like primeness, of just infinite algebras. The final section is devoted to extending a well-known
result of Herstein on simple rings to just infinite dimensional algebras.
2 Definitions, Examples, and Previous Work

In this chapter we define just infinite dimensional and related concepts in the realms of groups, Lie algebras, and associative algebras. Following the chronology of the subject, we begin with a discussion of the coclass conjectures for p-groups and Lie algebras - the precursors to just infinite dimensional associative algebras. For example, using epimorphisms from pro-p groups to just infinite groups, Wilson proves an extension to an earlier theorem of Zelmanov regarding pro-p groups having a presentation with few relators [26]. In [22], just infinite Lie algebras are used to examine Lie algebras of finite coclass. The utility of these developments in group theory and Lie theory has recently stimulated an interest in associative just infinite algebras.

After these results is a cataloging of the work that has been done regarding just infinite dimensional associative algebras. Three papers comprise the major developments in this area and we discuss the results of these publications to provide ample background and context for the new results of chapters 3 & 4. In [4], Farkas and Small prove that a finitely generated, semiprimitive, just infinite dimensional algebra over an uncountable field is either primitive or satisfies a polynomial iden-
tity. Reichstein, Rogalski, and Zhang [20] analyze the graded case, defining an algebra to be *projectively simple* if it is a graded just infinite algebra, (that is, homogeneous two-sided ideals have finite codimension). Moreover, they classify a subclass of such algebras within the framework of noncommutative projective geometry.

### 2.1 Work in Groups and Lie Algebras

#### 2.1.1 Groups

Rather than use the nilpotency class of a p-groups as an invariant, Leedham-Green and Newman proposed utilizing the coclass as an invariant to help classify p-groups up to isomorphism. The notion of coclass naturally extends from p-groups to pro-p groups. The five coclass conjectures suggested in [14], which have been proved (see [13] and [21]), describe pro-p groups of finite coclass and also provide asymptotic information on families of finite p-groups of fixed coclass.

Given a $p$-group, $G$, of order $p^n$ and nilpotence class $c$, the coclass of $G$ is defined to be $n - c$. Groups of maximal class are those of coclass 1. The coclass conjectures of Leedham-Green and Newman generalize results regarding groups of maximal coclass. The following are the coclass conjectures of [14] which have now been proved by various mathematicians.

**Theorem 1.**

1. Given $p, r$ there is a positive integer $f$ such that every $p$–group with coclass $r$ has a normal subgroup of class 2 with index at most $p^f$.

2. For every prime $p$ and every positive integer $r$ there is a positive integer $g$ such that every $p$–group with coclass at most $r$ has solvable length at most $p^g$.

3. A group which is residually of $p$–power order and has finite coclass is solvable.
iv. The graph $C(p, r)$ has only finitely many infinite maximal chains.

v. For each $p, r$ there are only finitely many

(a) boundedly solvable maximal infinite chains in $C(p, r)$.

(b) solvable pro-$p$ groups of coclass $r$.

(c) solvable finitely generated groups which are residually of $p$–power order and have coclass $r$.

These conjectures and other related propositions in the context of $p$–groups signaled the potential for classifying $p$–groups using coclass as an invariant; this, in turn, suggested the importance of considering just infinite pro-$p$ groups and Lie algebras.

Definition 2. A pro-$p$ group is just infinite if it is infinite and all non-trivial closed normal subgroups are open; i.e. if every nontrivial normal subgroup has finite index.

In [26] Wilson uses just infinite groups without open abelian normal subgroups and kernels of epimorphisms to such groups to prove the following:

Theorem 3. (Wilson) Let $G$ be a non-pro-cyclic pro-$p$ group having a presentation with few relators. Then there is a closed normal subgroup of a quotient group of $G$ which is a Cartesian product of infinitely many finitely generated infinite pro-$p$ groups, each of which is an epimorphic image of an open normal subgroup of $G$.

Wilson refers to a kernel of an epimorphism from a pro-$p$ group to a just infinite group with no open abelian normal subgroups as a $KJI$ subgroup of $G$.

Definition 4. Given a finitely generated pro-$p$ group, $G$ and a subset $S \subseteq G$, the superclosure of $S$ in $G$, denoted $scl(S)$, is the intersection of all $KJI$ subgroups of $G$ containing $S$. If there are no such $KJI$ subgroups then $scl(S) = G$. 
A normal subgroup is called superclosed if it is equal to its superclosure.

To prove Theorem 3, Wilson first proves that if $G$ is a non-procyclic pro-p group having a presentation with few relators, then $G$ has an infinite, strictly ascending chain of superclosed normal subgroups. This, coupled with the following lemma, immediately implies Theorem 3.

**Lemma 5.** Suppose that the pro-p group $G$ has a strictly ascending chain of superclosed normal subgroups. Then $G$ has closed normal subgroups $L$ and $M$ with $M \leq L$ and with $L/M \cong Cr_{i=1}^{\infty}H_i$, where each $H_i$ is an open normal subgroup of a just infinite quotient of $G$ having no open abelian normal subgroup.

2.1.2 Lie Algebras

Realizing the significance of the coclass conjectures for $p$-groups, Mathieu [18] obtained results regarding the structure of simple narrow Lie algebras and Shalev and Zelmanov developed similar notions for non-simple narrow Lie algebras.

For thoroughness, we note some definitions.

**Definition 6.**

i. A Lie algebra $L$ is said to residually have a property if for all ideals $I \vartriangleleft L$ the quotient Lie algebra $L/I$ has that property. For example, $L$ is said to be residually nilpotent if $L/I$ is nilpotent for all ideals $I$.

ii. The lower central series of a Lie algebra $L$ is defined as $L = L_0, L_1 = [L_0, L], L_2 = [L_1, L], \ldots, L_i = [L_{i-1}, L], \ldots$.

iii. The coclass of a finitely generated and residually finite Lie algebra, $L$, over a field of characteristic 0, is defined by $cc(L) = \sum_{i \geq 1}(dim(L^i/L^{i+1}) - 1)$, where $L^i$ is the lower central series of $L$. Note that the sum, and hence the coclass, may be infinite.

iv. A Lie algebra $L$ is solvable if $L^{(i)} = 0$ for some $i$, where $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. This is an extension of the idea of nilpotence.
A infinite dimensional Lie algebra $L$ is just infinite dimensional if, given any ideal $I < L$, the quotient Lie algebra $L/I$ is finite dimensional over the base field.

As an example, consider a nilpotent $n$-dimensional Lie algebra of class $c$. It’s coclass is then $n - c$. An infinite dimensional Lie algebra $L$ has finite coclass if and only if $\dim(L^i/L^{i+1}) = 1$ for all sufficiently large $i$. Lie algebras of coclass 1 are referred to as Lie algebras of maximal class and have been studied by Vergne ([23],[24],[25]). In her work on nilpotent Lie algebras of dimension $n$ she showed that this variety has an irreducible component of dimension exceeding $n^2$ consisting of Lie algebras of maximal class.

In [22], the authors note, using the following example, that although pro-$p$ groups of finite coclass are always solvable, a graded Lie algebra need not be solvable.

**Example 7.** Let $W$ be the positive part of the Witt algebra; in other words, the subalgebra spanned by $e_i = x^{i+1}/\partial x$ such that $i \geq 1$. Then $[e_i, e_j] = (j - i)e_{i+j}$. Thus $W$ is $\mathbb{N}$-graded with $W_i = \langle e_i \rangle$ and $W = \langle W_1, W_2 \rangle$. For $n \geq 2$ $W^n = \bigoplus_{i \geq n+1} W_i$. Therefore $W/W^2$ is 2-dimensional and $W^i/W^{i+1}$ is 1-dimensional for all $i \geq 2$. Thus $W$ has coclass 1, but $W$ is not solvable.

The authors avoid this example by considering only those $\mathbb{N}$-graded Lie algebras generated by their first components, a set denoted they by $\mathcal{S}$. Although not the main result of the paper, the corollary relevant to this discussion is

**Corollary 8.** Let $L \in \mathcal{S}$ be a Lie algebra of finite coclass. If $L$ is just infinite, then $L \cong M$, where $M$ is a linear space over the base field $F$ with basis $x, y_i (i \geq 1)$. The product on $M$ is given by $[y_i, x] = y_{i+1}$ and $[y_i, y_j] = 0$ for $i, j \geq 1$.
2.2 Just Infinite Dimensional Associative Algebras

We begin this section with a definition for associative algebras analogous to the conditions for groups and for Lie algebras.

**Definition 9.** An associative $k$-algebra is called *just infinite dimensional*, or *just infinite* for short, if $\dim_k A = \infty$ and each of its non-zero two-sided ideals has finite codimension.

Examples of just infinite algebras abound. Given a field $k$, the ring of polynomials in one variable, $k[x]$, the ring of Laurent polynomials in one variable, $k[x, x^{-1}]$, and the ring of power series in one variable, $k[[x]]$, are just infinite. Beyond these, we note another, perhaps less familiar, example attributed to Farkas and Small [4]. Let $S$ denote the Golod-Shaverevich example [7]. $S$ is a finitely generated, infinite dimensional $k$-algebra which is nil, but not nilpotent. Since $S$ is finitely generated, we can use Zorn’s Lemma to find an ideal $H$ of $S$ maximal with respect to the property that $\dim_k (S/H) = \infty$. Set $A = S/H$. If we have an ideal $I \triangleleft A$, then $I$ corresponds to an ideal, $J$, of $S$ which contains $H$. Thus, by our choice of $H$, $\dim_k (S/J) < \infty$ and so $\dim_k (A/I) < \infty$. This shows that $A$ is just infinite.

Bartholdi constructs another class of just infinite rings, called branch algebras, which he derives from examples of groups acting on trees [3]. Using this construction he proves the existence of just infinite algebras over any field, $k$, which have a subalgebra of finite codimension isomorphic to $M_2(k)$, have Gelfand-Kirillov dimension 2, satisfy no polynomial identity, contain a transcendental invertible element, and are semi-primitive if $k$ has characteristic not 2.

As discussed further in section 2.2.2, the authors of [20] consider algebras related to projectively simple algebras for which projectively simple carries over. Beyond their results, additional examples of just infinite rings can be created by combining or extending known just infinite algebras, by e.g. localization, finite
module extensions, etc.

Throughout this paper $M_n(A)$ will denote the ring of $n \times n$ matrices with entries from $A$. Note that if $A$ is a $k$–algebra, so too is $M_n(A)$.

**Lemma 10.** Let $A$ be a just infinite algebra. Then

i. $M_n(A)$ is just infinite for all $n \in \mathbb{N}$.

ii. If $e \in A$ is idempotent, that is, if $e^2 = e$, then $eAe$ is just infinite.

**Proof.**

i. Any ideal of $M_n(A)$ is of the form $J = M_n(I)$ for some $I \triangleleft A$, so any factor ring is of the form $M_n(A/I)$ and has dimension $n^2 \cdot \dim(A/I) < \infty$.

ii. For any ideal $J \triangleleft eAe$ we have $J = eAJAe$. Consider the natural map, $\phi : eAe \to A/\text{AJA}$. The kernel of this map is all elements of the form $eae \in AJA$ so $eAJAe = \ker(\phi)$. Then we have an embedding $eAe/J = eAe/eAJAe \hookrightarrow A/\text{AJA}$ and, since $\dim_k(A/\text{AJA}) < \infty$, so too must $\dim_k(eAe/J)$.

This lemma demonstrates, in particular, that being just infinite is a Morita invariant property.

### 2.2.1 Results of Farkas and Small

The principal result in [4] is the following:

**Theorem 11.** (Farkas, Small) Assume that $k$ is an uncountable field. If $A$ is an affine, semi-primitive, just infinite dimensional $k$-algebra then either $A$ is (left) primitive or $A$ satisfies a polynomial identity.
Recall that in a non-commutative ring an ideal $I \triangleleft A$ is prime if for any $J_1, J_2 \triangleleft A$ such that $J_1J_2 \subseteq I$ we have that either $J_1 \subseteq I$ or $J_2 \subseteq I$. An ring is itself prime if $(0)$ is a prime ideal. Also, a ring is semi-prime if it does not contain non-zero nilpotent ideals. Also, $A$ is primitive if it has a faithful simple module and semiprimitive if it is the subdirect product of primitive rings. Equivalently, an algebra is semiprimitive if its Jacobson radical, the intersection of all maximal left ideals, is zero.

Before returning to the proof of Theorem 11, we note the following preliminary result.

**Lemma 12.** If $A$ is a finite dimensional algebra then $A$ contains only finitely many distinct primitive ideals.

**Proof.** Suppose not. Then there is an infinite set, $\{P_i\}$, of distinct primitive ideals. Consider the chain of ideals, $P_1 \supseteq P_1 \cap P_2 \supseteq P_1 \cap \ldots \cap P_n \supseteq \ldots$. Because $A$ is finite dimensional and hence satisfies the descending chain condition, this chain must stabilize and so for some $n$, $P_1 \cap \ldots \cap P_n = P_1 \cap \ldots \cap P_n \cap P_{n+1} \subseteq P_{n+1}$. In a finite dimensional ring primitive implies both prime and maximal. Because $P_{n+1}$ must be prime, there exists some $i \leq n$ such that $P_i \subseteq P_{n+1}$. As $P_i$ is maximal, $P_i = P_{n+1}$, which gives us our contradiction. □

We now prove Theorem 11.

**Proof.** Suppose that neither conclusion holds. Because $A$ is just infinite, each primitive ideal of $A$ has finite codimension. Given a positive integer, $n$, call $\{P_i\}$ the set of primitive ideals of codimension $n$. Using Lemma 12 and the correspondence between ideals of $A$ containing $\cap_i P_i$ and ideals of $A/(\cap_i P_i)$, $\{P_i\}$ has only finitely many elements. Hence $A$ has only countably many primitive ideals.

Given $a \in A$, the image of $a - \lambda$ in $A/P$ is invertible for all but finitely many $\lambda \in k$. Because there are countably many primitive ideals in $A$ and, for each primitive ideal $P$ only finitely many $\lambda \in k$ such that $a - \lambda$ is not invertible modulo
there are countably many \( \lambda \) such that \( a - \lambda \) is not invertible for some \( P \). Let \( C \subseteq k \) be the countable set of \( k \) of all \( \lambda \) for which \( a - \lambda \) is not invertible modulo some primitive ideal.

Let \( \lambda \not\in C \). Then \( a - \lambda \) itself has a left inverse in \( A \). Suppose not. Then \( A(a - \lambda) \) lies in some maximal left ideal \( M \). If \( P \) is the annihilator of \( A/M \) then \( P \) is primitive. Consequently, the image of \( a - \lambda \cdot 1 \) in \( A/P \) lies in a maximal left ideal of \( A/P \), contradicting its invertibility modulo \( P \).

Thus \( a - \lambda \cdot 1 \) has a left inverse for uncountably many scalars \( \lambda \in k \). Amitsur’s linear independence trick [1] implies that \( a \) is algebraic over \( k \). Summarizing, \( A \) is an algebraic algebra over \( k \). Since \( A \) is semiprimitive, it has no nonzero nil left ideals. Hence every nonzero left ideal of \( A \) contains a nontrivial idempotent.

Let \( T_i(A) \) be the ideal generated by the set of specializations in \( A \), all polynomial identities of \( n \times n \) matrices over \( k \). Then \( A/T_i(A) \) is a polynomial identity algebra. If one of these ideals is zero then \( A \) satisfies a polynomial identity. If none are zero, then this collection satisfies the requirement of [5] because each nonzero ideal of \( A \) has finite codimension. This shows that \( A \) is either primitive or PI.

\[ \square \]

### 2.2.2 Results of Reichstein, Rogalski, and Zhang

An algebra \( A \) is said to be \( \mathbb{N} \)-graded if \( A = \bigoplus_{i \geq 0} A_i \) with \( 1 \in A_0 \) and \( A_iA_j \subseteq A_{i+j} \) for all \( i, j \geq 0 \). The classic example of an \( \mathbb{N} \)-graded algebra is a polynomial ring in one variable, \( A = k[x] \), where \( A_i = \{ \text{polynomials of degree } i \} \). If \( A_0 = k \) the algebra is called connected graded. \( A \) is left (right) Noetherian if it satisfies the ascending chain condition on left (right) ideals. Thus in a left Noetherian ring any increasing chain of left ideals, \( I_1 \subseteq I_2 \subseteq \ldots \) stabilizes. A ring is Noetherian if it is both left and right Noetherian.

Similar to the notion of just infinite is the idea of projectively simple. An infinite dimensional \( \mathbb{N} \)-graded \( k \)-algebra \( A \) is projectively simple if \( \dim_k(A/I) < \infty \) for all non-zero two-sided graded ideals \( I \triangleleft A \). In [20] the authors classify a subclass
of projectively simple rings as twisted homogeneous coordinate rings.

**Theorem 13.** (Reichstein, Rogalski, Zhang) Let \( k \) be an algebraically closed field and let \( A \) be a projectively simple Noetherian \( k \)-algebra. Suppose that \( A \) is strongly Noetherian, generated in degree 1, and has a point module. Then there is an injective homomorphism \( A \hookrightarrow B \) of graded algebras such that \( \dim_k B/A < \infty \) and \( B = B(X, \mathcal{L}, \sigma) \) is a projectively simple twisted homogeneous coordinate ring for some smooth projective variety \( X \) with a wild automorphism \( \sigma \) and a \( \sigma \)-ample line bundle \( \mathcal{L} \).

Besides this characterization, Reichstein, Rogalski, and Zhang describe more general properties of projectively simple rings and their modules and describe conditions under which the property carries over to related rings. The remainder of this section closely follows the first part of [20].

**Lemma 14.**

i. (Small) Let \( A \) be finitely generated, connected graded, and infinite dimensional over \( k \). Then there is a graded ideal \( J \subset A \) such that \( A/J \) is projectively simple.

ii. (Reichstein, Rogalski, Zhang) Let \( A \) be a Noetherian projectively simple algebra over an uncountable field \( k \). If \( GKdim A > 1 \) then \( A \) is primitive.

The authors also relate the significance of \( A \) being projectively simple to bi-modules of \( A \). Given a graded, Noetherian ring \( A \) and graded right \( A \)-modules \( M \) and \( N \), \( Hom_A(M, N) \) is the group of module homomorphisms preserving the gradings. Also, let \( N(n) \) denote the \( n \)th degree shift of the module \( N \) and \( Hom_A(M, N) = \oplus_{n \in \mathbb{Z}} Hom_A(M, N(n)) \). Note that if \( M \) is finitely generated, this is just the group of all module homomorphisms \( M \rightarrow N \).

Call a graded right module \( M \) torsion if for every \( x \in M \) there is an \( n \) such that \( xA_{\geq n} = 0 \). A Noetherian module is torsion if and only if it is finite dimensional.
over $k$. If, for every $x \in M$, the right annihilator of $x$, $r(x)$, is an essential right ideal of $A$, then call the module Goldie torsion.

**Lemma 15.** Let $A$ be a Noetherian projectively simple ring. Then

i. Let $B$ be any graded ring. If $M$ is a Noetherian $(B, A)$–bimodule such that $M_A$ is Goldie torsion, then $M$ is finite dimensional and thus torsion over $A$.

ii. Let $M$ and $N$ be Noetherian graded $(A, A)$–bimodules such that $M_A$ and $N_A$ are not torsion. Then $\dim_k \text{Hom}_A(M_A, N_A) = \infty$.

Of particular relevance to our work are the lemmas of [20] noting conditions under which the property of projectively simple passes from a ring to a related ring. We use $Kdim(A)$ to denote the Krull dimension.

**Lemma 16.** Let $A$ be a Noetherian graded ring. Then

i. if $B$ is a graded subring of $A$ such that $\dim_k A/B < \infty$, then $A$ is projectively simple if and only if $B$ is.

ii. if $B$ is a projectively simple graded subring of a graded Goldie prime ring $A$ such that $A_B$ is finitely generated, then $A$ is projectively simple.

iii. suppose $A$ is projectively simple and $B = A^{(n)}$ is the $n$th Veronese subring $\bigoplus_{i=0}^{\infty} A_{ni}$ for some $n \geq 2$. If $B$ is prime and $A_B$ (or $B_A$) is finitely generated, then $B$ is projectively simple.

iv. Let $K$ be a field extension of $k$, with $k$ algebraically closed. Then $A$ is a projectively simple $k$–algebra if and only if $A \otimes_k K$ is a projectively simple $K$–algebra.

**Proof.**

i. Suppose that $A$ is not projectively simple. Then there is some ideal of $A$, $0 \neq I \triangleleft A$, such that $\dim_k A/I = \infty$. Then $J = I \cap B$ is a nonzero ideal of $B$ with $\dim_k B/J = \infty$ and so $B$ is not projectively simple.
To prove the converse, suppose that \( A \) is projectively simple and let \( 0 \neq J \) be an ideal of \( B \). Because \( A \) is projectively simple, \( \operatorname{dim}_k A / (A_{\geq n}J A_{\geq n}) < \infty \). Now \( A_{\geq n}J A_{\geq n} \subseteq BJB = J \) for some \( n \), so \( \operatorname{dim}_k B / J < \infty \) and hence \( B \) is projectively simple.

ii. Let \( I \) be a nonzero graded ideal of \( A \). Since \( A \) is Goldie prime, \( I \) contains a homogenous regular element of \( A \) and thus \( K \dim(A/I)_B < K \dim A_B = K \dim B \). Hence the map \( B \to A/I \) cannot be injective. Thus \( B \cap I \neq 0 \) and \( B/(B \cap I) \) is finite dimensional because \( B \) is projectively simple. Now \( A/I \) is finitely generated over \( B/(B \cap I) \), so it is also finite dimensional.

iii. Consider \( B \) as a subring of \( A \) which is zero except in degrees which are multiples of \( n \). By [2], \( B \) is Noetherian. Let \( J \) be any nonzero right ideal of \( A \). Suppose \( J \cap B = 0 \). Then given a homogeneous element \( 0 \neq x \in J \), \( x^n \in J \cap B = 0 \). Thus \( J \) is a right nil ideal and, since \( A \) is Noetherian, \( J \) is a nilpotent ideal [19]. As \( A \) is projectively simple it is prime by 20, so \( J = 0 \). This contradicts our choice of \( J \), so, in fact, \( J \cap B \neq 0 \).

Now let \( I \) be a nonzero graded ideal of \( B \). Since \( B \) is prime, \( I \) contains a homogeneous regular element, \( x \). If \( J = r_A(x) \), then \( r_B(x) = J \cap B = 0 \) since \( x \) is regular in \( B \); thus \( J = 0 \) and \( x \) is regular in \( A \). Then \( A/I A \) is a Noetherian \((B, A)\)-bimodule which is Goldie torsion as a right \( A \)-module.

By 15, \( \operatorname{dim}_k (A/I A) < \infty \). Finally, since \( I = IA \cap B \), \( \operatorname{dim}_k B / I < \infty \) implies that \( B \) is projectively simple.

iv. Suppose that \( A \) is not projectively simple. Then there exists an ideal, \( I \triangleleft A \) such that \( \operatorname{dim}_k A / I = \infty \). Then \( I \otimes_k K \) is a nonzero ideal of \( A \otimes_k K \) with infinite codimension (over \( K \)), so \( A \otimes_k K \) is not projectively simple.

Conversely, assume \( A \otimes_k K \) is not projectively simple. Then \( A \otimes_k K \) contains a nonzero ideal, \( I \) of infinite codimension. Without loss of generality, assume that \( I = (g) \) is a principal ideal generated by a homogeneous element, \( g \in \)
\( A \otimes_k K \). Write \( g = \sum_{i=0}^{q} b_i x_i \), where \( 0 \neq b_i \in K \) and \( x_i \in A \). Choose \( q \) as small as possible so that \( x_0, x_1, ..., x_q \) are linearly independent over \( k \). Replacing \( g \) by \( b_0^{-1} g \) we can assume \( b_0 = 1 \). Let \( R = k[b_1, ..., b_q] \subset K \) be the commutative affine \( k \)-algebra generated by the \( \{b_i\} \). Let \( J \) be the ideal of \( A \otimes_k R \) generated by \( g \). Then \( (A \otimes_k R/J) \otimes_R K \cong A \otimes_k K/I \). Necessarily the degree \( n \) part \( (A \otimes_k R/J)_n \) is not \( R \)-torsion for all \( n \) in an infinite subset \( \Omega \subseteq \mathbb{Z} \). Now let \( \theta : R \rightarrow k \) be any \( k \)-algebra homomorphism (the Nullstellensatz guarantees the existence of a homomorphism because \( k \) is algebraically closed). Extend \( \theta \) to a map \( \theta : A \otimes_k R \rightarrow A \otimes_k k = A \). Then letting \( k \) be an \( R \)-module via \( \theta \), we have \( A' = (A \otimes_k R/J) \otimes_R k \cong A/(\theta(J)) \). For each \( n \in \Omega \), \( A' \neq 0 \) because \( (A \otimes_k R/J)_n \) is not \( R \)-torsion. Hence \( A' \) has infinite dimension over \( k \). Then \( \theta(g) = x_0 + \sum_{i=1}^{q} \theta(b_i)x_i \) is nonzero in \( \theta(J) \) since \( \{x_0, ..., x_q\} \) is linearly independent over \( k \). Therefore \( A \) is not projectively simple.

\( \square \)
Primeness

Fix a field, $k$ and let $A$ be an associative $k$–algebra. Then tensor products are taken over $k$. We will often identify $(A, A)$-bimodules with left modules over $A \otimes A^{op}$, where $A^{op}$ denotes the opposite ring of $A$.

Lemma 17. Given $A$ an associative algebra, the following statements are equivalent:

i. $A$ is prime

ii. For any left ideals $I, J \triangleleft_{l} A$, $IJ = (0)$ implies either $I = (0)$ or $J = (0)$.

iii. For any right ideals $I, J \triangleleft_{r} A$, $IJ = (0)$ implies either $I = (0)$ or $J = (0)$.

iv. For any ideals $I, J \triangleleft A$, $IJ = (0)$ implies either $I = (0)$ or $J = (0)$.

v. $A$ is commutative and for any $a, b \in A$, $ab = 0$ implies either $a = 0$ or $b = 0$.

These equivalences are standard and the proofs can be found in an introductory graduate algebra text [12].

In [4] the authors prove that any affine infinite dimensional $k$-algebra $A$ contains a prime ideal $I$ for which $A/I$ is just infinite. It follows that all affine just infinite algebras are prime. As an amusing aside, we suggest the following alternative proof.
Lemma 18. If $A$ is a finitely generated just infinite algebra over $k$, then $A$ is prime.

Proof. Given two non-zero ideals $I, J \triangleleft A$, Lewin [17] describes a $k$-algebra embedding

$$A/IJ \hookrightarrow S := \begin{pmatrix} A/I & M \\ 0 & A/J \end{pmatrix}$$

where $M$ is an $(A/I, A/J)$-bimodule. Because $A$ is just infinite, $A/I$ and $A/J$ are both finite dimensional algebras, and since $A$ is an affine $k$-algebra, we may take $M$ to be finitely generated as an $(A/I, A/J)$-bimodule. It then follows that $S$ is a finite dimensional algebra, and so $\dim_k(A/IJ) < \infty$. As $\dim_k A = \infty$, $IJ \neq (0)$, from which we conclude that $A$ is prime.

Of course, there are many non-affine just infinite algebras, perhaps the simplest example being $k[[x]]$. These infinitely generated just infinite algebras are also prime. The general case, though, requires a different argument. First we need the following:

Lemma 19. If $A$ is a just infinite dimensional $k$-algebra, then $A$ is semiprime.

Proof. Because $A$ is just infinite it satisfies the ascending chain condition on two-sided ideals. Thus if $I \triangleleft A$ is a two-sided ideal then $I$ is finitely generated as an $(A, A)$-bimodule, and thus also as a left $A \otimes A^{\text{op}}$-module.

Suppose $0 \neq I \triangleleft A$ with $I^2 = 0$. Then, since $I$ is contained in both the left and right annihilator of $I$, $I$ is also finitely generated as an $(A/I, A/I)$-bimodule. This implies that $I$ is finitely generated as a left $(A/I) \otimes (A/I)^{\text{op}}$-module. But $A/I$ and $(A/I)^{\text{op}}$ are finite dimensional over $k$, so $\dim_k(A/I) \otimes (A/I)^{\text{op}} < \infty$. As $I$ is a finitely generated module over a finite dimensional algebra, we see that $\dim_k I < \infty$.

As both $\dim_k(A/I) < \infty$ and $\dim_k(I) < \infty$, we conclude that $A$ is finite dimensional over $k$, which is a contradiction. \qed
With this lemma in hand, we can now prove the following:

**Theorem 20.** If $A$ is a just infinite $k$-algebra, then $A$ is prime.

*Proof.* Suppose $0 \neq I, J \triangleleft A$ are ideals of $A$ with $IJ = 0$. By Lemma 19, $A$ is semiprime and so $JI = 0$ as well. We then have $J \subseteq \text{ann}(I)$, the two-sided annihilator of $I$, so $I$ is finitely generated as a left $(A/J) \otimes (A/J)^{\text{op}}$-module. Because $A$ is just infinite, $\dim_k (A/J) \otimes (A/J)^{\text{op}} < \infty$, and so $\dim_k (I) < \infty$. As in Lemma 19, this shows that $A$ is finite dimensional, which is a contradiction. □

Theorem 20 is particularly satisfying because primality is such a well studied and fundamental property in ring theory. As such, knowing that any just infinite algebra is prime suggests numerous additional conditions satisfied by just infinite algebras. These conditions are tools both for identifying just infinite algebras and for classifying them. We note two corollaries to Theorem 20. First we give a weaker version of the definition of just infinite for affine algebras. We then we classify just infinite algebras satisfying a polynomial identity.

**Corollary 21.** If $A$ is a finitely generated infinite dimensional algebra such that for any non-zero prime ideal $P$ of $A$, $\dim_k A/P < \infty$, then $A$ is just infinite.

*Proof.* Let $A$ be an affine infinite dimensional algebra with $A/P$ finite dimensional for any prime ideal $0 \neq P$ of $A$. Were $A$ not just infinite, it would have a non-zero ideal of infinite codimension. Because $A$ is finitely generated, we can apply Zorn’s Lemma to find an ideal, $M \triangleleft A$, maximal with respect to the property that $A/M$ is infinite dimensional. This implies $A/M$ is just infinite and hence, by Theorem 20, prime, which in turn implies that $M$ is a prime ideal of $A$. By assumption, $M$ prime in $A$ means that $\dim_k (A/M) < \infty$, a contradiction. □

**Corollary 22.** If $A$ is a just infinite $k$-algebra satisfying a polynomial identity, then $A$ is a finite module over its Noetherian center, $Z$. Moreover, $Z$ is also just infinite.
Proof. By a theorem of Formanek [6], if $A$ is a prime algebra satisfying a polynomial identity whose center, $Z$ is Noetherian, then $A$ is a finite module over $Z$. So for the first part we need only show that $Z$ is Noetherian.

Let $I$ be a nonzero ideal of $Z$ and fix a nonzero $z \in I$. It is easy to see that $zZ = zA \cap Z$ and that this implies that we have an inclusion $Z/zZ \hookrightarrow A/zA$; thus, $Z/zZ$ is finite dimensional. Since $I/zZ \subseteq Z/zZ$, we see that $I/zZ$ is also finite dimensional. In particular, $I/zZ$ is a finitely generated $Z$-module and since $zZ$ is obviously finitely generated as well, we see that $I$ is finitely generated over $Z$.

Now let $0 \neq I \triangleleft Z$, and fix a nonzero element $z \in I$. Then $zZ \subseteq I$ and hence $Z/zZ$ maps surjectively onto $Z/I$. As before, we have an inclusion $Z/zZ \hookrightarrow A/zA$ and we conclude that $\dim_k Z/zZ < \infty$ and thus also that $\dim_k Z/I < \infty$. This proves the center is also a just infinite $k$-algebra. \hfill \Box

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Extending Herstein’s Theorems of Simple Rings

As just infinite is an extension of the idea of simplicity, a natural starting point for conjectures about just infinite rings is theorems about simplicity. In particular, in this section we consider two theorems of I.N. Herstein relating simplicity in an associative algebra to simplicity in a corresponding Jordan or Lie algebra. For the convenience of the reader, we duplicate the relevant results from [8], [9], [10], and [11].

4.1 The Theorems of Herstein

By the Jordan ring of an associative ring $R$, denoted $R^+$, we mean the additive group $R$ together with multiplication $\ast$. For $x, y \in R$, $x \ast y = xy + yx$, where $xy$ and $yx$ are the products under the standard associative multiplication of $R$. The Lie ring of $R$, $R^-$, is defined similarly with product the commutator $[\cdot, \cdot]$ given by $[x, y] = xy - yx$. A Jordan ideal is an ideal of the Jordan ring of $R$; in other words, an additive subgroup of $R$ closed under the operation $\ast$. Similarly, a Lie ideal of $R$ is an ideal of $R^-$. By $[R, R]$ we mean the subalgebra of $R^-$ generated by all
elements of the form \([x, y]\) for some \(x, y \in R\).

For aesthetic reasons, in the case of repeated commutators we will omit the internal braces; thus \([[x, y], z]\) will instead be written \([x, y, z]\) and, more generally, 
\([x_1, x_2, \ldots, x_n]\) will denote the commutator \(\underbrace{[\ldots[ [x_1, x_2], x_3], \ldots], x_n]}_{n-1}\).

The following sequence of lemmas is due to Herstein [8] and leads to the proof of the

**Theorem 23. (Herstein)** If \(A\) is a simple ring of characteristic not equal to 2 then

i. \(A^+\) is a simple Jordan ring.

ii. \([A, A]/(Z \cap [A, A])\) is a simple Lie ring, where \(Z\) is the center of \(A\).

We first focus our attention on Theorem 23 part i. To show that inside Jordan ideals of \(A\) sit associative ideals of \(A\), we consider elements lying in a Jordan ideal.

**Lemma 24.** If \(U\) is a Jordan ideal of \(A\) and \(x, y \in U\) then, for any \(a \in A\),

\([x \ast y, a] = (xy + yx)a - a(xy + yx) \in U\).

**Proof.** Because \(x \in U\) and \(U\) is an ideal of \(R^+\), \(x \ast [a, y] \in U\) for all \(a \in A\). Note that, with some clever manipulations,

\(x \ast [a, y] = [x, a \ast y] + [a, x \ast y]\). \hfill (4.1)

Since \(y \in U\), the first part of the right side of 4.1 is also contained in \(U\); in conjunction with the expression on the left belonging to \(U\) this demonstrates that \([a, x \ast y]\), and thus \([x \ast y, a]\), is in \(U\). \qed

Again, considering elements contained in a Jordan ideal \(U\) of \(A\), let \(x \in U\). Suppose that \(xa - ax \in U\) for all \(a \in A\). Because \(U\) is a Jordan ideal of \(A\), \(xa + ax \in U\) for all \(a \in A\). Adding these elements, \(2xa \in U\). Since \(A\) is not of characteristic 2, \(2A = A\) so \(xa \in U\) for all \(a \in A\). However, \(xa \in U\) implies that, for all \(b \in A\), \((xa) \ast b \in U\).
Now $x(ab) \in U$ by the argument above, which then implies that $bxa \in U$ for any $a, b \in A$. Thus $AxA \subseteq U$; because $A$ is simple, either $A = AxA$ or $x = 0$. If $U \subseteq A$, then it must be the latter case. This is stated precisely in

**Lemma 25.** If $A$ is a simple ring of characteristic not 2 and $U$ is a Jordan ideal of $A$ with $U \neq A$, then if $x \in U$ satisfies $[x, a] \in U$ for all $a \in A$, then $x = 0$.

With these lemmas, we are equipped to prove Theorem 23 part i.

**Theorem 26.** (Herstein) If $A$ is a simple ring, not of characteristic 2, then $A^+$ is a simple Jordan ring.

**Proof.** Suppose that $A$ is a ring with characteristic different from 2 and that $U \neq A$ is a Jordan ideal of $A$. By Lemma 24, for any $x, y \in U$ and any $a \in A$, $[x*y, a] \in U$.

Since $x*y$ is an element of $U$, by Lemma 25 $x*y = 0$. In particular, if $y = x$ we have $2x^2 = 0$ and so $x^2 = 0$ for any $x \in U$. If $x \in U$, then for all $a \in A$, $x*a \in U$ so, by the above, letting $y = x*a$, $x*(x*a) = 0$. Since $x^2 = 0$ this simplifies to $2xax = 0$ and thus to $xax = 0$ for any $a \in A$. This shows $xA$ to be a nilpotent right ideal of $A$, which, given that $A$ is a simple ring, implies $x = 0$. Thus $U = (0)$ and $A^+$ is a simple Jordan ring. \qed

Before developing the analogous theorem for $[A, A]$, we summarize results related results on the structure of the Lie ring, $A^-.$

**Lemma 27.** Let $A$ be a ring with no non-zero nilpotent ideals such that $2a = 0$ implies $a = 0$. Suppose that $U \neq (0)$ is both a Lie ideal and a subring of $A$. Then either $U \subset Z$, the center of $A$, or $U$ contains a non-zero (associative) ideal of $A$.

**Proof.** First assume $Z \neq A$, that is, $U$, as a ring, is not commutative. Then for some $x, y \in U$, $[x, y] \neq 0$. For any $a \in A$, $[x, y]a + y[x, a] = [x, ya] \in U$. $y[x, a] \in U$ because both $y$ and $[x, a]$ are elements of $U$ and $U$ is a subring of $A$. Thus $[x, y]a \in U$ for any $a \in A$ and so $[x, y]A \subseteq U$. Now given $a, b \in A$, the commutator $[[x, y]a, b] \in U$ and, because $([x, y]a)b = [x, y](ab) \in U$, we have
b[x, y]a ∈ U for all a, b ∈ A. Thus the ideal A[x, y]A ⊂ U. If A[x, y]A = (0) then certainly (A[x, y]A)^2 = (0), so the Lie ideal U contains a proper ideal of A as desired.

Now consider the case where U is commutative. We want to prove U ⊂ Z. Given x ∈ U and a ∈ A, [x, a] ∈ U so [x, a] commutes with x. Given any b ∈ A, substitute ab for a, so that x[x, ab] = [x, ab]x. Expanding this using the identity [x, ab] = [x, a]b + a[x, b] and noting that x commutes with [x, a]b + a[x, b], [x, a], and [x, b] gives that 2[x, a][x, b] = 0 and thus that [x, a][x, b] = 0 for all a, b ∈ A. Now setting b = ab we have [x, ab][u, a] = 0. Then,

\[ [x, ab][x, a] = [x, ab][x, a] - a[x, b][x, a] \] (4.2)
\[ = ([x, ab] + a[b, x]) [x, a] \] (4.3)
\[ = [x, a]b[x, a]. \] (4.4)

Because 4.4 holds for all b ∈ A, [x, a]A[x, a] = (0). Since A is semi-prime, [x, a] = 0 from which we conclude that x ∈ Z.

The immediate corollary to this is that if A is a simple ring (with characteristic not 2) then any Lie ideal, U, that is also a subring of A must either be A itself or contained in the center of A. To apply this result to any ideal of A^−, define the set S(U) = \{a ∈ A : [x, A] ⊂ U\} for each Lie ideal U ⊲ A^−. The relevance of S(U) is suggested in the following:

**Lemma 28.** For any ring A, if U is a Lie ideal of A then S(U), defined as above, is both a Lie ideal and a subalgebra of A.

**Proof.** [S(U), A] ⊂ U by definition and U ⊂ S(U) since U is a Lie ideal of A, so S(U) is a Lie ideal of A.

To show S(U) is a subring of A we must demonstrate that, for r, s ∈ S(U), rs ∈ S(U). Let r, s ∈ S(U), a ∈ A. Then [rs, a] = [r, sa] + [s, ar] and, because r, s ∈ S(U), [r, sa] + [s, ar] ∈ U. Therefore [rs, A] ⊂ U and rs ∈ S(U) as claimed.

□
The next theorem is similar to Theorem 23 part ii. Because any element in the center of an algebra $A$ is an annihilator of $A^-$, we would rather consider $A^-/Z$ than $A^-$ itself. With this in mind, the following theorem shows that Lie ideals of an algebra are either large or, if we expect to look at only $A$ module its center, not very important.

**Theorem 29.** (Herstein) Let $A$ be a simple ring of characteristic not equal to 2 and let $U$ be a Lie ideal of $A$. Then either $U \subset Z$, the center of $A$, or $[A, A] \subset U$.

**Proof.** $S(U)$ is both a subring and Lie ideal of $A$ by Lemma 28, so, using the immediate implication of Theorem 27 mentioned above, either $S(U) \subset Z$ or $S(U) = A$. If $S(U) = A$ then, by the definition of $S(U)$, $[A, A] \subset U$. Conversely, if $S(U) \subset Z$, then, as $U \subset S(U)$, $U \subset Z$.

Herstein then proves the only exceptions to the conclusion of Theorem 29 are those rings, $R$, such that $R$ is a 4–dimensional simple algebra over its center, which is a field of characteristic 2.

We now return to proving Theorem 23 part ii. Although similar to Theorem 29, we must show that ideals of $[A, A]$ are contained in the center of $A$, rather than considering ideals of $A^-$. In order to develop the relationship between ideals $U \triangleleft [A, A]$, $S(U)$, and $[A, A]$ itself, we need the following technical lemma.

**Lemma 30.** If $a, b \in A$, where $A$ is an associative algebra and $[\cdot, \cdot]$ and $\cdot \ast \cdot$ are defined as before, then

i. $ab = 1/2[a, b] + 1/2a \ast b$.

ii. $[a, b \ast c] = [a, b] \ast c + [a, c] \ast b$

**Proof.**
With this, we can prove a few properties of \( S = S(U) \).

**Lemma 31.** Let \( U \) be a Lie ideal of \([A, A]\). Then

i. \( S \) is a subalgebra of \( A \).

ii. \([U, U] \in S \).

iii. \([U, S(U)] \subset S(U) \).

**Proof.**

i. Let \( r, s \in S \). Then

\[
[r + s, A] = (r + s)A - A(r + s) = rA - Ar + sA - As \quad (4.12)
\]

\[
= [r, A] + [s, A] \subseteq U \quad (4.13)
\]

as both \([r, A]\) and \([s, A]\) are subsets of \( U \). Thus \( r + s \in S \).
To prove $S$ is closed under multiplication consider $[rs, A] = rsA - Ars$. Using the technical lemma proved above, Lemma 30 part i,

$$rsA - Ars = \{1/2[r, s] + 1/2(r * s)\} A - A\{1/2[r, s] + 1/2(r * s)\}. \quad (4.14)$$

Noting that $[r, s]A = r[s, A] + [rA, s]$, we reduce the above to

$$rsA - Ars = \{1/2[r, s] + 1/2(r * s)\} A - A\{1/2[r, s] + 1/2(r * s)\}.$$ \quad (4.15)

In the last line, the first group of terms is in $U$ because $r, s \in S$. Turning to the second part of 4.20,

$$[(r * s), A] = [r, s * A] + [s, r * A]. \quad (4.21)$$

Because $r \in S$, $[r, s * A] \subseteq U$. Similarly we find that the second term, and hence $[(r * s), A]$ itself, is contained in $U$. Thus, returning to the expression 4.20, we see that $[rs, A] \subseteq U$ and so $rs \in S$.

It is easily shown that 0, 1 $\in S$ and that, for $r \in S$, $-r \in S$, so $S$ is in fact a subalgebra of $A$.

tii. Let $x, y \in U$ and $a \in A$. Then, by the Jacobi identity, $[[x, y], a] = [x, [y, a]] + [y, [a, x]]$. As $x \in U, [y, a] \in [A, A]$, and $U$ is a Lie ideal of $[A, A]$, $[x, [y, a]] \in U$. Similarly, $[y, [a, x]] \in U$ and so $[[U, U], A] \subseteq U$ and hence we have $[U, U] \subseteq S$ as desired.
iii. \([U, S(U)] \subset U\) by the definition of \(S(U)\). Thus \([U, S(U), U] \subset [U, U] \subset U\), proving the statement.

We now turn to the structure of \([A, A]\), following [9]. Given a Lie ideal, \(U \triangleleft A^\circ\), denote by \(U^{[i]}\) a member of the descending chain \(U^{[1]} \supseteq U^{[2]} \supseteq U^{[n]} \supseteq \ldots\) defined recursively by \(U^{[1]} = U, U^{[n]} = [U^{[n-1]}, U^{[n-1]}]\). Although similar, note that this is not the same as the lower central series whose elements are denoted \(U^{(n)}\).

**Theorem 32.** (Herstein) If \(A\) is a simple ring of characteristic not 2, and if \(U\) is a Lie ideal of \([A, A]\) with \(U \neq [A, A]\), then

\[
U^{[4]} = [[[U, U], [U, U]], [[U, U], [U, U]]] = (0) \tag{4.22}
\]

*Proof.** \(S = S(U)\) is a subring of \(A\). Since \([U, U] \subset S\), either \(S = (0)\) and the theorem is true, or else \(S \neq (0)\). Let \(s_1, s_2 \in S\) and let \(a \in A\). By the definition of \(S\), \([s_2, a] \in U\), and so, if \(c = [s_1, [s_2, a]] \in [U, S], m \in S\) by Lemma 31. Since \(S\) is a subring of \(A\), \(s_1m \in S\) and we have

\[
s_1m = s_1[s_1, [s_2, a]] = s_1([s_1s_2, a] - [s_1, a]s_2) - ([s_1s_2, a] - [s_1, a]s_2)s_1 \tag{4.24}
\]

is in \(S\). Since \(s_1s_2\) is an element of \(S\),

\[
[s_1s_2, a] \in U \tag{4.25}
\]

and so

\[
[s_1, [s_1s_2, a]] \in [U, S] \subset S. \tag{4.26}
\]

Thus we have

\[
n = s_1[a, s_1]s_2 - [a, s_1]s_2s_1 \in S, and also \tag{4.27}
\]

\[
n = [s_1, [a, s_1]]s_2 + [a, s_1][s_1, s_2]. \tag{4.28}
\]
The first term on the right side of equation 4.28 is contained in \([U, S]S\). Since \([U, S] \subset S\) and \(S\) is a subring of \(A\), we have that the first term in the expression for \(n\) in 4.28 is in \(S\). Since \(n \in S\), we see that

\[
[a, s_1][s_1, s_2] \in S
\]  

(4.29)

for all \(s_1, s_2 \in S\) and all \(a \in A\). Now, if \(s_3 \in S\), because \(S\) is a subring of \(A\) and by 4.29, we have that \(S\) contains

\[
[a, s_1][s_1, s_2]s_3 = [a, s_1][s_1, s_2s_3] + [a, s_1]s_2[s_3, s_1].
\]  

(4.31)

Since \(s_2s_3 \in S\), the first term in 4.31 is in \(S\) by 4.29. Thus the second term of 4.31 is also in \(S\) for all \(s_1, s_2, s_3 \in S\) and all \(a \in A\). By replacing \(a\) by \(as_3\) in 4.29 we obtain that \(S\) contains

\[
[s_1, as_3][s_1, s_2]
\]  

(4.32)

\[
= [s_1, a]s_3[s_1, s_2] + a[s_1, s_3][s_1, s_2].
\]  

(4.33)

Using 4.31, we note that the first term of 4.33 is in \(S\), so that

\[
a[s_1, s_3][s_1, s_2] \in S
\]  

(4.34)

for all \(s_1, s_2, s_3 \in S\) and all \(a \in A\). With similar manipulations we can show that

\[
[s_1, s_3][s_1, s_2]a \in S
\]  

(4.35)

\[
\Rightarrow A[s_1, s_3][s_1, s_2]A \subset S.
\]  

(4.36)

Because \(A\) is simple, either \(A \subset S\) or \([s_1, s_3][s_1, s_2] = 0\) for all \(s_1, s_2, s_3 \in S\).

If \(S = A\), the definition of \(S\) implies that \(U\) is a Lie ideal of \(A\). Then by 29, either \(U \subset Z\), the center of \(A\), or \([A, A] \subset U\). The first case implies that \([U, U] = (0)\), in which event the theorem is true. The second case contradicts the assumption that \(U \subsetneq [A, A]\).
Suppose, then, that \([s_1, s_3][s_1, s_2] = 0\) for all \(s_1, s_2, s_3 \in S\). Replacing \(s_1\) by \(r_1 + r_2\) gives

\[
[s_3, r_1][s_2, r_2] = -[s_3, r_2][s_2, r_1] \tag{4.37}
\]

and repeatedly appealing to 4.37 we see that

\[
[s_3, r_1][s_2, r_2] = [r_1, s_3][r_2, s_2] \tag{4.38}
\]

\[
= -[r_1, s_2][r_2, s_3] \tag{4.39}
\]

\[
= -[s_2, r_1][s_3, r_2] \tag{4.40}
\]

\[
= [s_2, r_2][s_3, r_1] \tag{4.41}
\]

As this equality holds for all \(r_1, r_2, s_2, s_3 \in S\), \([S, S], [S, S] = (0)\). By Lemma 30 we know \([U, U] \subset S\) and thus \([U, U], [U, U] = (0)\) as desired.

Still working towards the proof of Theorem 23 part ii, we prove that if \(U\) is a Lie ideal of the algebra \([A, A]\) and not in the center of \(A\), that its left annihilator is \((0)\).

**Proposition 33.** If \(A\) is a simple ring and if \(U\) is a Lie ideal of \([A, A]\) such that \(U\) is not in the center of \(A\), then given \(a \in A\), \(aU = (0)\) if and only if \(a = 0\).

**Proof.** Let \(\lambda = \{a \in A : aU = (0)\}\). Clearly \(\lambda\) is a left ideal of \(A\). Now suppose that \(w \in \lambda\), \(a \in [A, A]\), and \(x \in U\). Then \([w, a]x = w[a, x] + [wx, a] = 0\) since both \(wx = 0\) and \(w[a, x] = 0\). Thus we have that \([\lambda, [A, A]] \subseteq \lambda\); because \(\lambda\) is a left ideal of \(A\), \([A, A] \lambda \subseteq \lambda\) so we obtain that \(\lambda[A, A] \subset \lambda\).

Suppose \(\lambda \neq (0)\). Then \(\lambda + [\lambda, A] \subseteq \lambda + [A, A]\). Choose any \(w \in \lambda\) and \(a \in A\). Then \(wa = aw - [a, w] \in \lambda + [\lambda, A]\) so that \(\lambda A \subseteq \lambda + [A, A]\). Since \(\lambda\) is a left ideal of \(A\) and since \(A\) is simple, we must have \(A = \lambda A \subseteq \lambda + [A, A]\). Then \(\lambda A = \lambda^2 + \lambda[A, A] \subseteq \lambda\). Thus \(\lambda \neq (0)\) is a two sided ideal of \(A\), a simple ring, and so \(\lambda = A\). This forces \(AU = (0)\) which implies \(U = (0)\) for our contradiction. Hence \(\lambda = (0)\) which proves the proposition. \(\square\)
This proposition sheds light on $U$ if it is not contained in the center of $A$. We will later use this to show that such a $U$ must be all of $[A, A]$. However, for now we prove a lemma that will force the other condition of our duality.

**Lemma 34.** If $A$ is a semi-prime ring and if $z \in A$ commutes with every element of $[A, A]$, then $z$ is in $Z$, the center of $A$.

*Proof.* As $z$ commutes with every element of $[A, A]$, $z[z, ab] = [z, ab]z$. Following the same procedure as in the second half of the proof of Lemma 27, we find that $[z, a][z, b] = 0$ for all $a, b \in A$ and that $[z, a]A[z, b] = (0)$. $A$ is semi-prime so $[z, b] = 0$ and so $z \in Z$. \qed

Using the previous results, we build up to proving a Lie ideal, $U$, of $[A, A]$ is in the center of $A$ by first showing it's in the center if $[U, U] = (0)$.

**Theorem 35.** (Herstein) If $A$ is a simple ring of characteristic not equal to $2$, and if $U$ is a Lie ideal of $[A, A]$ such that $[U, U] = (0)$, then $U \subseteq Z$.

*Proof.* Let $x \in U$ and $a, b \in A$. Since $U$ is a Lie ideal of $[A, A]$,

$$[x, [a, b]] \in U \quad (4.42)$$

and, since we assume $[U, U] = (0)$, we have

$$[x, [a, b]]x = x[x, [a, b]]. \quad (4.43)$$

In identity 4.43 replace $b$ by $cy$, where $y \in U$. Since $[a, cy] = [a, c]y + c[a, y]$, it follows that

$$[x, [a, c]y + c[a, y]]x = x[x, [a, c]y + c[a, y]] \quad (4.44)$$

Note that because $x, y \in U$ and $[U, U] = (0)$, $xy = yx$. Then more manipulations demonstrate that

$$[x, [a, c]y + c[a, y]] = [x, [a, c]y] + [x, c[a, y]] \quad (4.45)$$

$$= [x, [a, c]]y + [x, c[a, y]]. \quad (4.46)$$
Thus 4.43 and 4.44 yield

\[ [x, c[a, y]] x = x \{ x c[a, y] - c[a, y] x \}. \] (4.47)

Now suppose \( a \in [A, A] \). Then \([a, y] \in U\) and so \([a, y]\) commutes with \( x \). This reduces 4.47 to

\[ [x, c][a, y] x = x [x, c][a, y]. \] (4.48)

Then \([x, c, x][a, y] = 0\) holds for all \( a \in [A, A] \) and \( x \in U \) so that \([x, c, x][A, A, U] = (0)\). But \([A, A, U]\) is a Lie ideal of \([A, A]\), so if \([A, A, U] \neq (0)\) Proposition 33 dictates that \([x, c, x] = 0\) for all \( x \in U \) and \( c \in A \). If, on the other hand, \([A, A, U] = (0)\), \([x, c, x] \in [A, A, U] = (0)\). Thus we have that

\[ [x, c, x] = 0 \text{ for all } x \in U \text{ and } c \in A. \] (4.49)

Replacing \( c \) by \( c^2 \) in the last identity and noting that \([x, c^2] = [x, c] x\) we obtain that \(2[x, c^2] = 0\) for all \( x \in U \) and all \( c \in A \). Because \( a \) does not have 2–torsion, this implies that \([x, c^2] = 0\). Applying two results of Levitzki, [16] and [15], we can conclude that \([x, c] = 0\). By Lemma 34, this proves \( x \in Z \) for all \( x \in U \), and thus we attain the desired result that \( U \subset Z \).

Now, rather than assume \([U, U] = (0)\), we ask that \([U, U, A] = 0\). We are moving closer to requiring only that \([U, U] = (0)\), at which point we can apply Theorem 32.

**Lemma 36.** Let \( A \) be a simple ring of characteristic neither 2 nor 3. Suppose \( U \) is a Lie ideal of \([A, A]\) such that \([U, U] \subset Z\). Then \( U \subset Z\).

**Proof.** If \([U, U] = (0)\), then, by Theorem 35 we are done.

Assume, then, that \([U, U] \neq (0)\). There exist some \( x, y \in U \) such that \( 0 \neq [x, y] \in Z \). Being nonzero and in the center of \( A \) implies that \([x, y]\) is a unit in \( Z \) so the proof of Theorem 2 in [11] (except for the case of characteristic 3) carries over in its entirety up to the point where it is established that \([U, U] = (0)\). Since we now that \([U, U] = (0)\), we have, by Theorem 35 that \( U \subset Z \) as desired. \( \square \)
Finally, we are able to prove that a proper Lie ideal of \([A, A]\) is contained in the center of \(A\).

**Theorem 37.** (Herstein) If \(A\) is a simple ring of characteristic neither 2 nor 3, then any proper Lie ideal of \([A, A]\) is contained in the center of \(A\).

**Proof.** Let \(U\) be a proper Lie ideal of \([A, A]\). By Lemma 32, \(U[^4] = [U[^3], U[^3]] = 0\). Then by Theorem 35, it follows that \(U[^3] = [U[^2], U[^2]] \subset Z\). Utilizing Lemma 36, we have that \(U[^2] = [U, U] \subset Z\). And thus, by another appeal to Lemma 36, we have that \(U \subset Z\).

The following corollary (which is Theorem 23 part ii) completes the proof of Theorem 23.

**Corollary 38.** If \(A\) is a simple ring then \([A, A]/(Z \cap [A, A])\) is also simple.

### 4.2 Extending Lie Theorem to Just Infinite Algebras

In [10] Herstein proves Theorem 26 as a corollary to a more general result. Rather than focus on simple rings, we can use the same techniques to prove the following:

**Theorem 39.** (Herstein) Let \(A\) be a ring of characteristic not equal to 2. Suppose that \(A\) has no non-zero nilpotent ideals, i.e., \(A\) is a semi-prime ring. Then any non-zero Jordan ideal of \(A\) contains a non-zero associative ideal of \(A\).

**Proof.** Let \(U \neq 0\) be an ideal of \(A^+\) and let \(x, y \in U\). By Lemma 24, for any \(a \in A\), \(aw - wa \in U\) where \(w = x * y\). However, since \(w \in U\), \(aw + wa \in U\). Adding, \(2aw \in U\) for all \(a\), and thus for \(b \in A\), \((2aw)b + b(2aw) \in U\). Since \(2baw \in U\), so too is \(2awb \in U\). This holds for all \(a, b \in A\) and thus \(2AwA \in U\). Now \(2AwA\) is an ideal of \(A\), which completes the proof unless \(2AwA = (0)\). By assumption,
$2AwA = (0)$ implies that $AwA = (0)$ and thus that $(Aw)^2 = (0)$. $A$ was assumed to be semi-prime which forces $xy + yx = w = 0$.

Let $0 \neq x \in U$; then for $a \in A$, $x^a \in U$ and hence $x(xa + ax) + (xa + ax)x = 0$. Simplifying this expression, $x^2a + ax^2 + 2xax = 0$. For $x \in U$, $0 = xx + xx = 2x^2$ whence $x^2 = 0$. The relation $x^2a + ax^2 + 2xax = 0$ then reduces to $2xax = 0$ which implies $xax = 0$ for all $a \in A$. Thus $xAx = (0)$; however, $xA \neq (0)$ is a nilpotent right ideal of $A$, contrary to assumption. Thus $U$ contains a non-zero ideal of $A$.

From this, and Theorem 20 which proved all just infinite algebras are prime, we see the immediate implication for just infinite rings.

**Corollary 40.** Let $A$ be a just infinite algebra over a field $k$ with $\text{char}(A) \neq 2$. Then $A^+$ is also a just infinite algebra over $k$.

Herstein’s analogous result for Lie algebras does not extend as easily. Let $A$ be a $k$-algebra of characteristic not equal to 2 and let $U$ be a Lie ideal of $[A, A]$. Recall the set $S(U) = \{a \in A | [a, A] \subseteq U\}$ defined for the simple case.

An immediate consequence of Lemma 31 is

**Corollary 41.**

i. $[S^{[k]}, [A, A]] \subseteq S^{[k]}$

ii. $[S^{[k+1]}, A] \subseteq [[S^{[k]}, S^{[k]}], A] \subseteq [S^{[k]}, [A, A]] \subseteq S^{[k]}$

**Lemma 42.** $S^{[3]}A \subseteq S$

**Proof.** By i,

$$S^{[3]} \subseteq [S^{[3]}, A] + S^{[3]} * A \subseteq S^{[2]} + S^{[3]} * A.$$ (4.50)

Using ii,

$$S^{[3]}A = [S^{[2]}, S^{[2]}] * A \subseteq [S^{[2]}, S^{[2]} * A] + [S^{[2]}, A] * S^{[2]}$$ (4.51)

By ii and 31 the right hand side of this inclusion, and thus also the left, is contained in $S$.  

\[\square\]
Using an identical argument on the other side, we can conclude that \( AS^{[3]} \subseteq S \), too. These together demonstrate that the ideal generated by \( S^{[d]} \) in \( A \), \( A[S^{[3]}, S^{[3]}]A \) is also contained in \( S \).

**Lemma 43.** \( id_A(S^{[d]}) \subseteq S \).

We need next an analogue of Lemma 36.

**Lemma 44.** Let \( A \) be a semiprime ring without \( 2, 3, \) or \( 5 \) torsion. Let \( U \) be an abelian subgroup of \( A \) with \( [U, [A, A]] \subseteq U \) and also \( [U, U] \subseteq Z \), where \( Z \) is the center of \( A \). Then \( U \subseteq Z \).

**Proof.** Let \( a, b, c \in A \). Then by a straightforward induction

\[
[ab, c, \ldots, c] = \sum_{i \leq n} \binom{n}{i} [a, c, \ldots, c][b, c, \ldots, c].
\]

First, suppose that

\[
[A, c, \ldots, c] = (0)
\]

and \( A \) does not have \( \left( \begin{array}{c} 2n \\ n+1 \end{array} \right) \) -torsion. Then for all \( a \in A \),

\[
[a^2, c, \ldots, c] = \left( \begin{array}{c} 2n \\ n \end{array} \right) [a, c, \ldots, c]^2 + \sum_{\max\{i, j\} \geq n+1} [a, c, \ldots, c][x, c, \ldots, c].
\]

Thus for any \( a \in A \),

\[
[x, c, \ldots, c]^2 = 0.
\]

Consider, now, \( x \in U \).

\[
[A, x, x, x, x] \subseteq [A, A, x, x, x] \subseteq [U, U, U] \subseteq [Z, U] = (0).
\]

Because

\[
[a, x, x, x] \in [A, A, x, x] \subseteq [U, U] \subseteq Z
\]
and, by the comment above,

\[
[a, x, \ldots, x]^2 = 0 \tag{4.58}
\]

for all \( a \in A \), we see that \( ([a, x, x]A)^2 = 0 \). As \( A \) is a semiprime algebra, this implies that the ideal itself is zero. Thus for all \( a \in A \), we have that \( [a, x, x, x] = (0) \) which, in turn, implies that \( [a, x, x] = (0) \) and hence we have \( [A, x, x, x] = (0) \). Iterating the strategy with \( [A, x, x] = (0) \), we have \( [a, x, x] = 0 \) for all \( a \in A \) and \( [A, A, x, x] = (0) \).

Now let \( x \) be a nilpotent element of \( U \), with index of nilpotence \( n \). That is, \( x^n = 0, x^{n-1} \neq 0 \). Then for \( a \in [A, A] \),

\[
xax = \frac{1}{2}([x, a, x] - x^2a - ax^2) = -\frac{1}{2}(x^2a + ax^2) = 0. \tag{4.59}
\]

Thus \( x^{n-1}[A, A]x^{n-1} = 0 \). Then given \( a \in A \),

\[
\begin{align*}
x^{n-1}[A, A]x^{n-1} \ni x^{n-1}[ax^{n-1}, a]x^{n-1} &= x^{n-1}(ax^{n-1}a - a^2x^{n-1})x^{n-1} \tag{4.60} \\
&= x^{n-1}ax^{n-1}ax^{n-1} = 0
\end{align*}
\]

which implies that \( (x^{n-1}A)^3 = 0 \). Because \( A \) is semiprime, \( x^{n-1}A = (0) \) and thus \( x^{n-1} = 0 \). This contradicts the assumption that \( x \) has index of nilpotence \( n \), so \( U \) must not contain any nonzero nilpotent elements.

We know that, for all \( a \in A \), \( [a, x, x]^2 = 0 \). This, together with \( U \) not containing nonzero nilpotent elements, implies that \( [A, x, x] = (0) \). This, coupled with the beginning of the proof, shows that \( [a, x]^2 = 0 \) for all \( a \in A \). Now, if \( a \in [A, A] \) then \( [a, x] \in U \) and \( U \) has no nonzero nilpotent elements; hence \( [a, x] = 0 \). Thus \( [A, A, x] = 0 \) and \( [A, A, U] = 0 \). By Lemma 34, this proves \( U \subset Z \).

\[\square\]

**Theorem 45.** If \( A \) is a just infinite dimensional algebra without 2, 3, or 5-torsion, then \( [A, A]/(Z[A, A]) \) is also just infinite.

**Proof.** Let \( U \) be a nonzero Lie ideal of \( [A, A] \). Suppose \( S^{[4]} \neq (0) \). Then by 43, \( S \) is of finite codimension in \( A \), implying that \( A = V + S \) and \( \dim V < \infty \).
\[ [A, A] = [S + V, S + V] \subseteq [S, A] + [V, V] \subseteq U + [V, V]. \] As \( \text{dim} V < \infty \), \( U \) is of finite codimension in \( A \).

Now suppose \( S^{[4]} = (0) \). \( A \) is just infinite so, by Theorem 20, \( A \) is prime and applying Lemma 44 to \( S^{[3]} \) shows \( S^{[3]} \subseteq Z \). Again, using Lemma 44 on \( S^{[2]} \), we get \( S^{[2]} \subseteq Z \). Repeating once more with \( S \) gives \( S \subseteq Z \). However, by Lemma 31, we know that \( [U, U] \subseteq S \subseteq Z \) and so by Lemma 44 we have \( U \subseteq Z \). \hfill \Box
5

More Questions

Primeness is a fundamental property in rings; another is primitivity. In this direction, Farkas and Small [4] demonstrated that an affine, just infinite, semi-primitive algebra over an uncountable field is either primitive or else satisfies a polynomial identity. The hypothesis of semi-primitivity is necessary, by the example of the just infinite image of the Golod-Shavarevich example, but it is unknown whether the other hypotheses can be relaxed.

Through a standard manipulation it can be shown that if $A$ is a just infinite algebra, then each ideal of $A$ is just infinite. However, it is not true that subalgebras of $A$ inherit the property. Consider, for example, the matrix ring, $M_2(k)$ and its subalgebra

$$S := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subseteq M_2(k).$$

Because

$$I := \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \leq_r S$$

is a nilpotent right ideal of $S$, $S$ cannot be just infinite by Lemma 19. For a subalgebra of a just infinite algebra to inherit the property, Theorem 20 shows that it is necessary that the subalgebra be prime. However, this is not sufficient, as a (rather complicated) example of Wilson demonstrates [27].
Another strategy to create new algebras from a given $k$–algebra, $A$, is to extend the scalars by tensoring $A$ with a field extension, $k \subset K$. This is particularly convenient if the original field is not algebraically closed or perhaps has only countably many elements. This new algebra, $A \otimes_k K$, has many properties in common with $A$. As reproduced in Lemma 16 part iv, in [20] the authors prove that $A$ is projectively simple if and only if $A \otimes K$ is also projectively simple. However, the stability of the just infinite property under extension of scalars is not known for the general, in particular infinitely generated, case.
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