QUANTUM MECHANICAL TREATMENT OF CLOSE COLLISIONS IN PLASMA TRANSPORT THEORY

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A number of plasma collisional quantities such as the electrical conductivity, the energy loss rate of test charges, and the thermal relaxation time depend on the Coulomb logarithm, \( \ln(x_D/\ell_{\text{min}}) \), which arises from two particle scattering via the Coulomb interaction.\(^1\) The Debye screening length \( \lambda_D \) represents an effective maximum impact parameter. The form of the minimum impact parameter, \( \ell_{\text{min}} \), depends on the temperature of the plasma. At low temperatures \((kT \ll \text{Ryd})\) where a classical calculation is valid, \( \ell_{\text{min}} = r_0 = \beta e^2 \), the classical distance of closest approach \((\beta = 1/kT)\).

At high temperatures \((kT \gg \text{Ryd})\) where the Born approximation may be used, \( \ell_{\text{min}} = \lambda \equiv \frac{n(\beta/2\pi)^{1/2}}{\pi^2} \), the thermal de Broglie wavelength.\(^2\) At intermediate temperatures \((kT \approx \text{Ryd})\) where \( \lambda - \beta e^2 \) neither the classical nor the Born calculation is valid, and a full quantum mechanical calculation is required. In this note we give an approximate quantum mechanical calculation of the Coulomb logarithm which is valid for all temperatures.

For illustration we calculate here the static electrical conductivity of a two component "plasma" where the interaction is a truncated Coulomb potential

\[
V(r) = \begin{cases} 
\frac{-ze^2}{r} & \text{for } r < R \\
0 & \text{for } r \geq R
\end{cases}
\]

(1)

The plasma lengths are chosen to be ordered as

\[
\beta e^2 < R < \frac{1}{\pi^2} \beta < \lambda_D
\]

(2)
so that the plasma is non-degenerate and the ordinary Boltzmann equation may be used to study the collisions. For a real plasma we must add to the Boltzmann collision integral for $V(r)$ in the kinetic equation a Lenard-Balescu collision integral for the long range weak part of the Coulomb potential. A complete calculation with both short and long range effects included will give the electrical conductivity as

$$\sigma = \left( \frac{\ln(R/L_{\text{min}}) + C_1}{(\ln(\lambda_D/R) + C_2)^{-1}} \right) = \left( \frac{\ln(\lambda_D/L_{\text{min}})}{(C_1 + C_2)^{-1}} \right)$$

where $C_1$ and $C_2$ are constants. We consider only the close collision contribution in this note. The quantum mechanical calculation of $\sigma$ using (1) will be made using a phase shift analysis.

The Chapman-Enskog method can be used to obtain solutions to the Boltzmann equation linearized about local thermodynamic equilibrium (LTE) in the presence of an external electric field $E$. Solutions of the form

$$f_{1,e} (E, \mathbf{r}, t) = f_{1,e}^{(o)} (1 + \phi_{1,e}^{(1)})$$

where the $f_{1,e}^{(o)}$ are solutions for LTE and $|\phi| \ll 1$,

can be expanded in a finite series of $\xi$ Sonine polynomials

$$\phi_{1,e}^{(1)} = \sum_{m=0}^{\infty} \frac{C_{1,e}}{(2m+1)!} \left( \frac{\xi}{2} \right)^m S_{3/2}^{(m)}$$

where $C = v/a$ and $a = (2/3m)^{1/2}$ the thermal speed.
We assume that the ions are singly charged. The coefficients $t_m^{(i)}(\xi)$ are obtained from a variational principle. The conductivity $\sigma$ is defined by

$$\sigma = \frac{e}{E} = e \int dy_i \phi^{(i)}_i (y_i) - e \int dy_e \phi^{(e)}_e (y_e)$$

Using the orthogonality property of the Sonine polynomials, it follows that

$$\sigma = (ne)^2 \left\{ \left( \frac{\beta}{2m_i} \right)^{1/2} t^{(i)}(\xi) - \left( \frac{\beta}{2m_e} \right)^{1/2} t^{(e)}(\xi) \right\} \quad (6)$$

We consider here only the simplest case $\xi = 1$, for which it follows from the variational principle that

$$t^{(i)}_o (1) = -\frac{3/(28m_i)}{m^2 (1+\frac{m_i}{m_e}) [C_i, C_e]} \quad \text{and} \quad t^{(e)}_o (1) = +\frac{3/(28m_i)}{m^2 (1+\frac{m_i}{m_e}) [C_i, C_e]}$$

where $[C_i, C_e] = \frac{1}{2n^2} \int dy_i dy_e d\Omega |y_i - y_e|^2$

$$\frac{d\sigma}{d\Omega} f^{(i)}(v_i) f^{(e)}(v_e) (\xi', \xi) (\phi_i', \phi_i) (\phi_e', \phi_e)$$

The primes here refer to post collision quantities.

We now introduce CM and relative velocity variables

$$v = \frac{m_i v_i + m_e v_e}{m_i + m_e}, \quad M = m_i + m_e$$

$$v = v_e - v_i, \quad \mu = \frac{m_i m_e}{(m_i + m_e)}$$
Using a phase shift expansion for the cross section

\[
\frac{d\sigma}{d\Omega} = \left(\frac{n}{2\mu v}\right)^2 \sum_{\ell_1\ell' = 0}^{\infty} (2\ell + 1)(2\ell' + 1) \left(\exp(2i\delta_{\ell}) - 1\right) \left(\exp(2i\delta_{\ell'}) - 1\right)
\]

\[
P_{\ell}(\cos \theta) P_{\ell'}(\cos \theta)
\]

we find that

\[
[C_i, C_e]_{ie} = -\left(\frac{2n\hbar}{M}\right)^{1/2} \left(\frac{\hbar^2}{\mu}\right)^{1/2} \int_{0}^{\infty} dy y^3 e^{-y^2} \sum_{\ell, \ell'} (2\ell + 1)(2\ell' + 1) \int_{-\infty}^{1} dz (1-z) P_{\ell}(z) P_{\ell'}(z)
\]

where \( y = (8u/2)^{1/2} v \). Using the relation

\[
\int_{-\infty}^{1} dz (1-z) P_{\ell}(z) P_{\ell'}(z) = \delta_{\ell, \ell'} \left\{ \frac{2}{2\ell+1} \right\} - \delta_{\ell, \ell-1} \left\{ \frac{2\ell}{2\ell+1}(2\ell-1) \right\}
\]

\[
- \delta_{\ell, \ell+1} \left\{ \frac{2(\ell+1)}{(2\ell+3)(2\ell+1)} \right\}
\]

it follows after some manipulation that

\[
[C_i, C_e]_{ie} = -\left(\frac{2n\hbar}{M}\right)^{1/2} \left(\frac{\hbar^2}{\mu}\right)^{1/2} \int_{0}^{\infty} dy y^3 e^{-y^2} \sum_{\ell=1}^{\infty} \ell \sin^2(\delta_{\ell} - \delta_{\ell-1})
\]

Although the exact phase shifts are readily obtainable, it appears necessary to use approximate expressions for the \( \delta_{\ell} \) in order to continue the calculation. Holdeman and Thaler have examined the asymptotic expression for the phase shifts

\[
\delta_{\ell} = \arg \Gamma(\ell + 1 + \frac{ie^2}{\hbar v}) - \frac{e^2}{\hbar v} \frac{\nu R}{\hbar} + \theta \left(\frac{e^2}{\hbar v^2}/R\right)
\]
This asymptotic form is valid for $\ell < L \equiv [\mu v R/\hbar] = [(R/\lambda) v]^*$. They have shown that from the phase shifts

$$\delta_{\ell} = \begin{cases} \arg \left( \ell + 1 + i \frac{e^2}{\hbar v} \right) - \frac{e^2}{\hbar v} \ln \left( \frac{2 \mu v R}{\hbar} \right) & \text{for } \ell < L \\ 0 & \ell > L \end{cases}$$

(8)

one can recover the Coulomb cross section in closed form when the limit $R \to \infty$ is taken:

$$\lim_{R \to \infty} \frac{d\sigma}{d\Omega} (\theta, \nu) = \frac{e^2}{4 \mu^2 v^4 \sin^2 \theta}$$

The approximate phase shifts (8) will be used here. If the sum over phase shifts is now approximated by an integral

$$\sum_{\ell=1}^{L} \int_{1}^{(R/\lambda) v} d\ell$$

we obtain $[C_i, C_e]_{ie} = - \left( \frac{2 \pi \hbar}{M} \right)^{1/2} (\theta \hbar^2 / \mu) \int_{0}^{\infty} dy \ y^3 \ e^{-y^2} \ i \int_{1}^{(R/\lambda) v} d\ell \ \ell \ \frac{d\ell}{1 + (\frac{\ell \ h}{a})^2}$

since with the phase shifts given by (8) we obtain

$$\sin^2 (\delta_{\ell} - \delta_{\ell-1}) = \frac{1}{1 + (\frac{\ell \ h}{a})^2}$$

where $a \equiv r_o/2\lambda$.

The integration is now readily performed:

$$[C_i, C_e]_{ie} = - \left( \frac{2 \pi \hbar}{M} \right)^{1/2} \left( \frac{2 \hbar^2}{\mu} \right) a^2 \left\{ \exp(-ir_o/2R) \ E_1(-ir_o/2R) + \exp(ir_o/2R) \right\}$$

$$E_1(ir_o/2R) - e^{a^2} E_1(a^2)$$

* $[x]$ is the largest integer less than or equal to $x$ for any real $x$. 
where \( E_1(x) = \int_0^\infty \frac{e^{-t}}{t} \frac{dt}{x} \). If \( R \) is chosen so that \( R \gg r_0 \) then the conductivity is

\[
\sigma = \frac{3}{4r_0} (2\pi \beta \mu)^{-1/2} \left\{ \ln(2R/r_0) - \gamma - \frac{1}{2} \exp(r_0/(2x))^2 E_1((r_0/(2x))^2) \right\}^{-1}
\]  

(9)

where \( \gamma = 0.5772 \) is Euler's constant. The classical limit is given by \( r_0 \gg \kappa (kT \ll \text{Ryd}) \) for which

\[
\sigma^{CL} = \frac{3}{4r_0} (2\pi \beta \mu)^{-1/2} \left\{ \ln(2R/r_0) - \gamma + \theta \left( \frac{\kappa}{r_0} \right)^2 \right\}^{-1}
\]  

(10)

in agreement with the result of Chapman and Cowling\(^6\) who also used only one Sonine polynomial.\(^a\) The Born approximation is given by \( r_0 \ll \kappa \) (\( kT \gg \text{Ryd} \)) for which

\[
\sigma^{BORN} = \frac{3}{4r_0} (2\pi \beta \mu)^{-1/2} \left\{ \ln(R/\kappa) - \frac{\gamma}{2} + \theta \left( \frac{r_0}{\kappa} \right)^2 \right\}^{-1}
\]  

(11)

The quantity in braces in expression (9) contains the essential features of the small impact parameter dependence of the transport properties of a Coulomb plasma. As the temperature increases the "minimum impact parameter" varies continuously from \( r_0 \) to \( \kappa \).

In a later publication a thorough study of transport properties of a plasma will be made using a convergent kinetic equation which includes both short and long range effects. One such convergent kinetic equation which is amenable to a phase shift analysis is a quantum mechanical version of Guernsey's kinetic equation.\(^3\)

\(^a\) The numerical coefficient \( \frac{3}{4} \) differs from the result of Konstantinov and Perel, Soviet Physics, JETP, 14, 944 (1962) and Itikawa, J. of Phys. Soc. Japan, 18, 1499 (1963) who have obtained \( 1.95 \times \frac{3}{4} \). The extra factor of 1.95 arises from the use of more than one Sonine polynomial. In this note we are concerned only with the behavior of the Coulomb logarithm and not with the exact value of the coefficient multiplying it.
Bibliography


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