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Fringe Fields for the N Channel Permanent Magnet Array

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Fringe Fields for the N Channel Permanent Magnet Array

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Analytical expressions are obtained for fringe field multipoles of an N channel permanent magnet quadrupole array [1]. See figure 1 below. It is assumed that the system of magnetic wedges starts at some transverse (x, y) plane located at z=0, and it continues to a magnet length z=\ell, where it stops. The iron yoke continues to z = \pm \infty, but it will be shown that only a small overhang is actually required to maintain the quadrupole and translational symmetries. Recall the 2-d solution for the magnetic potential \( H = \nabla \phi \):

\[
\phi_2 = A \left[ (x-x_i)^2 - (y-y_i)^2 \right],
\]

where \( A = -M_0/4b \), \( M_0 \) is the remnant field of the wedges, and \((x_i, y_i)\) are the coordinates for the center of box (i). Boxes have dimensions 2b x 2b and alternate between vacuum fill (for beams) and magnetic wedge fill. The 2-d system looks like a portion of an infinite transverse lattice with periodicity lengthy = 4b in both the x and y directions. For the magnetic potential \( \phi \), the periodicity length is 2b.
3-d Magnetic potential for a semi-infinite system

More generally we have a 3-d situation where the potential $\phi$ is a function of $z$ as well as $(x, y)$. It still satisfies

$$\nabla^2 \phi = -\nabla \cdot M,$$

$$\nabla^2 = \nabla_{\perp}^2 + \frac{\partial^2}{\partial z^2},$$

where $M = B - H$ is the local density of magnetization.

For now we take $M$, which is still transverse, to turn on at $z=0$ and continue to $z=+\infty$.

Hence

$$\nabla \cdot M = \Theta(z) \nabla_{\perp} \cdot M_{\perp}.$$

Where $\Theta(z)$ is the unit step function and $\nabla_{\perp} \cdot M_{\perp}$ is the source of the 2-d potential $(\phi_2)$ given above. The truncated 2-d potential $\Theta(z) \phi_2$ is clearly not a valid 3-d solution. However, we can add homogeneous 3-d potentials in the zones $z<0$ and $z>0$ to obtain the desired 3-d solution. Generally the homogeneous 3-d solutions consist of a sum of terms of the form

$$\left[ \cos \left( \frac{n\pi x}{b} \right) \right] \cdot \left[ \cos \left( \frac{m\pi y}{b} \right) \right] \cdot e^{\pm \sqrt{n^2 + m^2} \frac{\pi z}{b}}.$$

These terms preserve the periodicity in $x$ and $y$ exhibited by the 2-d solution. In fact, since $\phi_2$ is a pure quadrupole potential in each $2b \times 2b$ box, we only use homogeneous combinations of the form
\[ \left[ \cos \left( \frac{n\pi x}{b} \right) - \cos \left( \frac{n\pi y}{b} \right) \right] \pm \frac{n\pi z}{b}. \]

Since \( M \) is transverse, \( \partial \phi / \partial z \) must be continuous at \( z = 0 \), and we obtain

\[ z < 0: \quad \phi = -\sum_{1}^{\infty} F_n \left[ \cos \left( \frac{n\pi x}{b} \right) - \cos \left( \frac{n\pi y}{b} \right) \right] e^{-\frac{n\pi z}{b}}, \]

\[ z > 0: \quad \phi = \phi_2 + \sum_{1}^{\infty} F_n \left[ \cos \left( \frac{n\pi x}{b} \right) - \cos \left( \frac{n\pi y}{b} \right) \right] e^{\frac{-n\pi z}{b}}. \]

Here \( \{F_n\} \) is a set of coefficients to be determined (below).

We must also have \( \phi \) continuous at \( z = 0 \); this gives

\[ \sum_{1}^{\infty} F_n \left[ \cos \left( \frac{n\pi x}{b} \right) - \cos \left( \frac{n\pi y}{b} \right) \right] = -\frac{1}{2} \phi_2. \]

We may restrict attention to a single \( 2b \times 2b \) box centered at \( x = y = 0 \); substituting for \( \phi_2 \) we get (inside the box)

\[ \sum_{1}^{\infty} F_n \left( \cos \frac{n\pi x}{b} - \cos \frac{n\pi y}{b} \right) = -\frac{A}{2} (x^2 - y^2). \]

To evaluate \( F_n \) we set \( y = 0 \); an elementary Fourier series on the interval 
- \( b < x < +b \) remains:

\[ \sum_{1}^{\infty} F_n \left( \cos \frac{n\pi x}{b} - 1 \right) = -\frac{A x^2}{2}. \]
It follows that

$$
\sum_{n=1}^{\infty} F_n = \frac{\Delta b^2}{6},
$$

and the individual coefficients are

$$
F_n = \frac{1}{b} \int_{-b}^{b} \left( - \frac{Ax^2}{2} \right) \cos \left( \frac{n\pi x}{b} \right) = \frac{-2b^2 A}{n^2 \pi^2} \cos (n\pi) = \frac{2b^2 A}{\pi^2} \cdot \left\{ \frac{1}{1^2}, \frac{-1}{2^2}, \frac{1}{3^2}, \frac{-1}{4^2}, \ldots \right\}.
$$

**Evaluation of the quadrupole amplitude**

Note that

$$
\left( \cos \frac{n\pi x}{b} - \cos \frac{n\pi y}{b} \right) = \frac{-\pi^2}{2b^2} n^2 \left( x^2 - y^2 \right) + \frac{\pi^4}{24b^4} n^4 \left( x^4 - y^4 \right) - \frac{\pi^6}{720b^6} n^6 \left( x^6 - y^6 \right) + \ldots.
$$

So in the box centered at \( x = y = 0 \), the potential has the simple form

$$
\phi = f(z) \left( x^2 - y^2 \right) + g(z) \left( x^4 - y^4 \right) + h(z) \left( x^6 - y^6 \right) + \ldots.
$$

This is true for \( z > 0 \) as well as \( z < 0 \) since \( \phi_2 \sim \Theta(z) \left( x^2 - y^2 \right) \) and can be absorbed into \( f(z) \). For other boxes \((x-x_i)\) and \((y-y_i)\) must be inserted in this expression in place of \( x \) and \( y \).

To obtain the quadrupole amplitude \( f(z) \) we gather terms of \( \phi \) proportional to \( (x^2-y^2) \); for \( z < 0 \)
\[ f(z) = \sum_{n=1}^{\infty} F_n \left( \frac{\pi^2 n^2}{2b^2} \right) e^{\frac{\pi n z}{b}} = \frac{A}{1 + e^{-\pi z/b}} = \frac{A}{2} \left( 1 + \tanh \frac{\pi z}{2b} \right). \]

The same formula is found for \( z > 0 \) by direct summation (or by invoking analyticity).

To obtain the field for a magnet of finite length \( \ell \) we add a displaced semi-infinite solution with reverse \( M \) to get

\[ f = \frac{A}{2} \left[ \tanh \frac{\pi z}{2b} - \tanh \frac{\pi (z - \ell)}{2b} \right]. \]

Note that for the semi-infinite magnet the quadrupole amplitude has a "Fermi-Dirac" distribution form with amplitude \( A/2 \) at \( z = 0 \). We expect a profile of this general shape from elementary considerations. Outside the magnet \( \phi \) falls off very rapidly with \( z \), so it should actually be sufficient to continue the iron yoke to only about \( \Delta z \approx 2b \) beyond the magnet ends; note the very low value of the quadrupole amplitude at this distance from the magnet:

\[ f(z = -2b) = (.00186) \text{ A}. \]

Other fringe field components

So far we have evaluated the lowest order quadrupole term:

\[ \phi_{\text{quad}} = f(z) \left( x^2 - y^2 \right) = f r^2 \cos 2\theta, \]
with the expressions for $f(z)$ derived in the previous section. For other fringe field features we will treat only the semi-infinite magnet, so

$$f(z) = \frac{A}{2} \left( 1 + \tanh \frac{\pi z}{2b} \right).$$

A first result is a formula for $H_z$ in lowest order:

$$H_z = \frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial z} r^2 \cos 2\Theta$$

$$= \left( \frac{A}{2} \right) \left( \frac{\pi}{2b} \right) \left( 1 - \tanh^2 \frac{\pi z}{2b} \right) r^2 \cos 2\Theta.$$

This field component has magnitude similar to the transverse quadrupole components near $z = 0$. Higher order terms of $\phi$ can be extracted from the general expansion in the same way as was done for quadrupole component, however a short cut is available. We simply plug the expansion

$$\phi = f(x^2 - y^2) + g(x^4 - y^4) + h(x^6 - y^6) + \ldots$$

directly into $\nabla^2 \phi = 0$ and equate coefficients of $(x^m - y^m)$:

$$0 = \nabla^2 \phi = f''(x^2 - y^2) + g''(x^4 - y^4) + h''(x^6 - y^6) + \ldots$$

$$+ 12g(x^2 - y^2) + 30h(x^4 - y^4) + \ldots.$$
We have immediately

\[ g = \frac{-f''}{12}, \]

\[ h = \frac{-g''}{30} = \frac{f'''}{360}, \]

and so forth. Hence

\[ \phi = f(x^2 - y^2) - \frac{f''}{12} (x^4 - y^4) + \frac{f'''}{360} (x^6 - y^6) - \ldots. \]

After a little algebra this expansion may be cast in the form

\[ \phi = fr^2 \cos 2\theta - \frac{f''}{12} r^4 \cos 2\theta \]

\[ + \frac{f'''}{360} \left( \frac{15}{16} r^6 \cos 2\theta + \frac{1}{16} r^6 \cos 6\theta \right) + \ldots, \]

displaying directly the allowed terms through sixth order (quadrupole, pseudo-octopole, pseudo dodecapole, and dodecapole).

Returning to the semi-infinite quadrupole amplitude

\[ f(z) = \frac{A}{2} \left( 1 + \tanh \frac{\pi z}{2b} \right), \]
we find by successive differentiations

\[ f' = \frac{A}{2} \left( \frac{\pi}{2b} \right) \left( 1 - \tanh^2 \frac{\pi z}{2b} \right), \]

\[ f'' = \frac{A}{2} \left( \frac{\pi}{2b} \right)^2 (-2 \tanh (1 - \tanh^2)), \]

\[ f''' = \frac{A}{2} \left( \frac{\pi}{2b} \right)^3 (-2 + 6 \tanh^2) \left( 1 - \tanh^2 \right), \]

\[ f'''' = \frac{A}{2} \left( \frac{\pi}{2b} \right)^4 (16 \tanh - 24 \tanh^3) \left( 1 - \tanh^2 \right), \]

etc.

The pseudo octopole potential is

\[ \phi_{PO} = \frac{-f''}{12} r^4 \cos 2\theta \]

\[ = \frac{A}{12} \left( \frac{\pi}{2b} \right)^2 \left( r^4 \cos 2\theta \right) \left( \tanh \frac{\pi z}{2b} \right) \left( 1 - \tanh^2 \frac{\pi z}{2b} \right). \]

There is a very small dodecapole potential

\[ \phi_{DOD} = \frac{f''''}{(360) \cdot (16)} r^6 \cos 6\theta \]

\[ = \frac{A}{11520} \left( \frac{\pi}{2b} \right)^4 \left( r^6 \cos 6\theta \right) (16 \tanh - 24 \tanh^3) \left( 1 - \tanh^2 \right). \]
Comparison of Multipole Fields

We calculate the relative strengths of the various multipoles near \( z = 0 \).
Specifically, \( B_r = \frac{\partial \phi}{\partial r} \) is computed and normalized to the 2-d maximum = \(-M_0/2\):

\[
\left( \frac{B_r}{-M_0/2} \right) = \left[ 1 + \tanh \left( \frac{\pi z}{2b} \right) \right] \left( \frac{r}{b} \right) \cos 2\theta
\]

\[
+ \left[ \frac{\pi^2}{24} \tanh \left( 1 - \tanh^2 \right) \right] \left( \frac{r}{b} \right)^3 \cos 2\theta
\]

\[
+ \left[ \frac{\pi^4}{240} \left( \tanh - \frac{3}{2} \tanh^3 \right) \left( 1 - \tanh^2 \right) \right] \left( \frac{r}{b} \right)^5 \left[ \frac{15}{16} \cos 2\theta + \frac{1}{16} \cos 6\theta \right]
\]

\[\text{(quadrupole)}\]

\[\text{(pseudo octopole)}\]

\[\text{(pseudo dodecapole + dodecapole)}\]
Values of the bracketed functions of z in the multipole field comparison are given in table 1. Note the antisymmetry of 3rd and 5th orders.

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<th>3rd order</th>
<th>5th order</th>
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Table 1

Relative strength of components of \( B_r \) at \( r = b, \phi = 0 \).
Reference

1. E. Lee and M. Vella, LBL-38430, "Perfect 2-d Quadrupole Fields from Permanent Magnets".