UNIVERSITY OF CALIFORNIA, SAN DIEGO

Noncoherent Detection of Pulse-Position Modulation with Correlated Gaussian
Interference in a Slowly Fading Two-Path Channel

A dissertation submitted in partial satisfaction of the
requirements for the degree
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by

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2006
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2006
To my family, friends and wife,

who are so important in my life.
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LIST OF ACRONYMS

PPM  Pulse Position Modulation
CDMA  Code Division Multiple Access
PMD  Polarization Mode Dispersion
ULP  Ultrashort Light Pulse
MAI  Multiple Access Interference
SMF  Single Mode Fiber
DGD  Differential Group Delay
ISI  Inter-Symbol Interference
CD  Chromatic Dispersion
AWGN  Additive White Gaussian Noise
SNR  Signal to Noise Ratio
SIR  Signal-to-Interference Ratio
i.i.d.  independent, identically distributed
pdf  probability density function
cdf  cumulative distribution function
DOF  degrees of freedom
2-D  two-dimensional
1-D  one-dimensional
DPSK  Differential Phase Shift Keying
FSK  Frequency Shift Keying
ADM  Add/Drop Multiplexer
LAN  Local Area Network
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PUBLICATIONS


Noncoherent Detection of Pulse-Position Modulation with Correlated Gaussian
Interference in a Slowly Fading Two-Path Channel

by

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Professor Laurence Milstein, Co-Chair
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In this work, a general analysis is developed for the noncoherent detection of non-overlapping
PPM with correlated Gaussian interference, transmitted through a slowly fading two-path channel.
Subject to a few general assumptions about the channel and the correlation time of the
interference, it is shown that the pairwise error probabilities separate into three cases of interest.
In Part I of this dissertation, closed-form solutions are derived for the first two cases, and for the
third case under certain limiting conditions. In Part II, of the dissertation, general solutions for
this third case are computed numerically by inverting a characteristic function, and are shown to
agree with the solutions for the limiting cases in Part I.

As a specific example, in Part III we investigate the performance of a novel modulation
scheme employing Ultrashort Light Pulses (ULPs), modulated with Pulse Position Modulation
(PPM) for $M$-ary data transmission in conjunction with asynchronous time-spread CDMA, transmitted through a Single Mode Fiber (SMF) channel with PMD. Considering the first-order effect of the PMD, the throughput and capacity degradation are derived in terms of physical system parameters.
Chapter 1

Introduction

In this work, we develop a general analysis for the noncoherent detection of non-overlapping Pulse Position Modulation (PPM) with correlated Gaussian interference, transmitted through a slowly fading two-path channel. As a specific example, we investigate the performance of a novel modulation scheme employing Ultrashort Light Pulses (ULPs), modulated with PPM for $M$-ary data transmission in conjunction with time-spread Code Division Multiple Access (CDMA) for asynchronous multiple-access, transmitted through a Single Mode Fiber (SMF) channel with Polarization Mode Dispersion (PMD).

The analysis proceeds in three parts. After defining both the statistical model and certain simplifying assumptions, it is shown that the $(M - 1)$ pairwise error probabilities separate into three cases of interest, conditioned on several key parameters. The first two cases are solved in closed form, while the third case is has analytical or quasi-analytical solutions only in certain limiting cases. An analytical solution is obtained in the limits of either independent, identically distributed (i.i.d.) or completely correlated random variables. When the random variables are independent, but not identically distributed, the error probability can still be obtained numerically
by evaluating the sum of several integrals with finite-limits.

For partial correlation, the third case has no closed-form solution. In the second part, a general numerical solution for the third case is obtained from the characteristic function, which is derived using a general Hermitian quadratic form. The pairwise error probability for this case must then be unconditioned numerically over a two-dimensional (2-D) plane as a function of the pulse-shape and the interference autocorrelation function. This is accomplished using a 2-D recursive bisection technique.

The third part investigates a practical example of the analysis developed in the first two parts. First, the optical PPM/CDMA system and the SMF PMD channel are defined in terms of physical parameters required to analyze the performance. The statistical model is shown to apply to this system under practical conditions. The Multiple Access Interference (MAI) is proven to be an asymptotically stationary colored Gaussian process, and the autocorrelation function of the interference is determined by the autocorrelation of the pulse-shape. Using the analytical results and the numerical techniques developed in the first two sections, the total error probability is determined as a function of the relevant system parameters. Finally, for a maximum desired error probability, the capacity and throughput degradation are derived in terms of the physical system parameters.
Part I

Analytical Results
In this first part, we develop a general analysis for the noncoherent detection of non-overlapping PPM with correlated Gaussian interference, transmitted through a stationary two-path channel. In part two, we ultimately derive the union bound of the total error probability, as a function of the Signal-to-Interference Ratio (SIR), and the constellation size, \( M \), given the the pulse shape, the interference autocorrelation function, and the distribution of the differential time delay between the two paths. For a given maximum desired error rate, this imposes a maximum allowable constellation size, \( M \), and thus constrains the throughput.

In Chapter 2, we develop detailed definitions and assumptions for the signal, interference, channel, and receiver. Subject to a few general assumptions about the channel and the correlation time of the interference, it is shown that the pairwise error probabilities separate into three cases of interest, conditioned on two or three parameters. The union bound of the total error probability is then defined as a sum of the three cases of the pairwise error probability, conditioned on the pulse-position and the differential delay. It is proven in Chapter 3 that when the conditioning of these parameters is removed, the union bound can be expressed in terms of the three cases of the average error probability, weighted by a “Probability of Interference” parameter, \( P_{\text{int}}(M) \) that is a function of the constellation size, \( M \).

The three cases of the average error probability are the the primary contributions of this dissertation, and are analyzed in detail in Chapters 5 - 10 The average error probability for Case I is solved in closed form in Chapter 5. It is then show in Chapter 5 that Case I is an upper bound for Case II, and thus the average error rate for Case I upper bounds the average error rate for Case II. The third case is tractible only in certain limiting cases. An analytical solution for the third case is obtained in the limits of completely correlated and i.i.d. random variables in Chapters 7 and 8, respectively. When the random variables are independent, but not identically distributed,
it is shown in Chapter 8 that the error probability can still be obtained numerically by evaluating the sum of several integrals with finite-limits.

For arbitrary correlation, it appears that there is not an analytical solution for Case III. In the second part, we describe numerical techniques to evaluate the average error probability for Case III. In Chapter 9, the conditional characteristic function is derived, and a Beaulieu series expansion is then used to obtain the conditional error probability for Case III. The average error probability is obtained numerically using a recursive bisection that is described in Chapter 10, and several examples are given which demonstrate that this case has a power-law dependence.

Finally, in Chapter 11, the results for the three cases of the average error probability and the interference probability are applied to obtain a throughput constraint for a given maximum error rate.
Chapter 2

Analytical Model

2.1 Overview

![Diagram of PPM signal]

Figure 2.1: Transmitted PPM Signal, $s(t)$.

PPM is an M-ary modulation scheme that transmits $\log_2(M)$ bits of information every $T_{rep}$ seconds by shifting a pulse $p(t)$, which has peak power $P_o$ and one-sided equivalent width of $T_p$ seconds, to one of $M$ time slots separated by $T_{ps}$ seconds. Figure 2.1 depicts a typical PPM signal, $s(t)$, where the pulse is transmitted in the $m^{th}$ time slot. To prevent Inter-Symbol Interference (ISI), a guard time of $T_{guard}$ seconds, chosen to exceed the maximum delay spread of the channel, is required after the transmission of each symbol.
**Equivalent Width** We define $W\{f(x)\}$ to be the one-sided equivalent width of an even (i.e., symmetric) function $f(x)$:

$$W\{f(x)\} \equiv \int_{0}^{\infty} \frac{\|f(x)\|^2 \, dx}{\|f(0)\|^2},$$

which is analogous to the definition of a one-sided noise equivalent bandwidth [30, Eq. 5.96]. Furthermore, let $x(t) \leftrightarrow X(\omega)$ denote a Fourier transform pair such that $W\{x(t)\}$ is the equivalent one-sided time-width of $x(t)$ in seconds and $W\{X(\omega)\}/\pi$ is the equivalent two-sided bandwidth of $x(t)$ in Hertz (Hz).

**Notation** We shall follow the notational convention of denoting random variables with a bold font, and conditional values of a random variable with a regular font. Matrices are denoted using a bold, non-italic capital font.

**System Block Diagram** Figure 2.2 depicts the block diagram of the system. The signal, $s(t)$, and the interference, $\eta(t)'$, propagate through a two-path channel, $\hat{h}(t)$, that is assumed to be stationary over the symbol duration, $T_{sym}$, with a random differential delay, $\tau_d$. The receiver front-end consists of a square-law detector with a pre-detection filter to reject out-of-band interference. The PPM detector synchronizes to the strongest path, forms $M$ intensity samples spaced by $T_{ps}$ seconds, and then chooses the largest intensity value as the most likely transmitted...
Overview of the Analysis  Detailed definitions and assumptions for the signal, interference, channel, and receiver are developed in Section 2.2. In Section 2.2.5, the union bound on the error probability is shown to be a sum of $M-1$ pairwise error probabilities. In Section 2.3, the pairwise error are shown to separate into one of three cases, depending on the transmitted pulse-position and the differential delay. These three cases of the error probability are the the primary contributions of this dissertation, and are analyzed in detail in Chapters 5 - 10 In the last section of this chapter, 2.5, the union bound of the total error probability is defined as a sum of the three cases of the pairwise error probability, conditioned on the pulse-position and the differential delay.

2.2 Definitions and Assumptions

2.2.1 Transmitted Signal

Pulse  $p_u(x)$ is a real, symmetric, non-increasing unit pulse (about the origin) that has unity peak power (i.e., $P_u(0) = 1$), a (single-sided) equivalent width of $T_p \equiv W\{p_u(t)\}$ seconds, and an equivalent bandwidth of $B_p \equiv W\{P_u(\omega)\}/\pi$ Hz. The unmodulated pulse has peak power $P_o$, and is defined as

$$p(t) \equiv \sqrt{P_o}p_u(t)$$  \hspace{1cm} (2.1)

PPM Signal  The transmitted signal $s(t)$ is modulated by shifting the pulse $p(t)$ to one of $M$ time slots that are separated by $T_{ps}$seconds,

$$s(t) = s(t') \equiv \sqrt{P_o}p_u(t - mT_{ps}),$$ \hspace{1cm} (2.2)
where the transmitted pulse-position, $m$, is a discrete random variable from the equiprobable set $[0 \ldots M - 1]$. The pre-detection filter is designed to reject out-of-band interference without distorting the signal, such that the filtered signal $s(t) = s(t)'$ is unchanged. Figure 2.1 illustrates a transmitted symbol, and the relationships among various system parameters.

**Non-overlapping Condition**  We assume that $T_{ps} > T_p$ so that $p(\pm T_{ps}) \approx 0$, and the transmitted pulse is confined to a single time slot (e.g., non-overlapping PPM).

**Negligible ISI Condition**  We require the symbol length, $T_{sym} \equiv MT_{ps}$, to be less than the repetition time, $T_{rep}$, so that there is a guard time of $T_{guard} \equiv T_{rep} - T_{sym}$ seconds. This guard time is assumed to be sufficiently larger than the maximum anticipated value of the differential delay between the two paths, $\tau_d$, such that the probability of Inter-Symbol Interference (ISI) is negligible.

### 2.2.2 Interference

**Interference-Limited Assumption**  We consider an interference-limited channel, modeled as a colored Gaussian noise process, where the interference dominates all other potential sources of noise such that the performance can be characterized by the Signal-to-Interference Ratio (SIR). This is justified for many practical multiple access systems. For example, in CDMA systems with perfect power control, the SIR depends on the number of users and processing gain, but not on the signal power since it is the same for every user. Therefore, systems of this type are generally interference limited whenever the signal power and the number of users are sufficiently large.
**Interference Process**  We define $\eta(t)'$ as a zero-mean, stationary, circularly symmetric complex Gaussian noise process. The pre-detection filter rejects out-of-band interference such that the filtered interference process, $\eta(t)$, has the same bandwidth as the signal. The autocorrelation function of this filtered interference process is defined as $R(t)$, and is equal to the autocovariance $C(t)$ since the interference has zero mean. The correlation time is the (one-sided) equivalent width of the autocovariance [17, Eq. 10-49], which is defined as $T_R \equiv W\{R(t)\}$ seconds. The variance of the real and imaginary parts of $\eta(t)$ is defined as $\sigma^2$, such that

$$\text{Var}\{\eta(t)\} = E\{\eta(t)\eta^*(t)\} = C(0) = R(0)$$

$$= E\left\{\text{Re}\{\eta(t)\}^2\right\} + E\left\{\text{Im}\{\eta(t)\}^2\right\} = 2\sigma^2.$$  \hspace{1cm} (2.3)

The correlation coefficient is the normalized autocovariance, defined as [17, Eq. 10-46]

$$\rho(t) \equiv \frac{C(t)}{C(0)} = \frac{R(t)}{2\sigma^2}. \hspace{1cm} (2.4)$$

Lastly, the signal-to-interference ratio is

$$\text{SIR} \equiv \frac{P_o}{\text{Var}\{\eta(t)\}} = \frac{P_o}{2\sigma^2}. \hspace{1cm} (2.5)$$

This definition follows the standard convention of measuring the SIR prior to the square-law device. Note that for optical communications systems, this corresponds to the SIR of the optical signal, rather than the electrical signal measured after photodetection.

**Sufficient Spacing Condition**  We assume that $T_{ps} > T_R$, so that the pulse separation exceeds the correlation time of the interference. This ensures that $R(\pm T_{ps}) \approx 0$, so that samples of $\eta(t)$ separated by at least $T_{ps}$ seconds are uncorrelated, and thus independent, since they are jointly Gaussian.
2.2.3 Channel

Co-propagating Interference Assumption We assume that the signal, $s(t)'$, and the interference, $\eta(t)'$, co-propagate through the same channel, $\hat{h}(t)$. This situation can arise at the edge of an asynchronous CDMA network with unidirectional half-duplex links, such as the system illustrated in Figure 2.3. In this scenario, a point-to-multipoint link is used to connect an asynchronous Add/Drop Multiplexer (ADM) in the network core to a Local Area Network (LAN) connected by a passive coupler (splitter/combiner) or linear bus at the network edge. Note that is not a star-coupler configuration, but rather an asymmetric configuration with a $1 : J$ splitter to broadcast the incoming link (i.e., point-to-multipoint link), and a $J : 1$ combiner for the outgoing link (i.e., multipoint-to-point link). The only difference between the passive coupler and linear bus configurations is that the passive coupler uses one centralized splitter/combiner junction to serve all $J$ nodes, while the linear bus has of a series of $J$ splitter/combiner junctions, one per node.

In this passive splitter/combiner configuration, the routing, network control and signal re-
generation occur at the ADMs in the network core. The $k^{th}$ ADM selectively drops into the $k^{th}$ LAN only the signals desired by the $J_k$ nodes in the LAN, broadcasting this aggregate signal to all nodes via the forward point-to-multipoint link. For the reverse multipoint-to-point link, the aggregated transmissions from the LAN are added into the core network traffic by the ADM. Thus, for any particular user in LAN $k$, the aggregate received signal consists of one desired signal, and $J_k - 1$ interfering signals. For example, suppose user $i$ in LAN 1 transmits to user $j$ in LAN 2, and vice-versa. The transmission from user $i$ generates Multiple Access Interference (MAI) only for the other $(J_2 - 1)$ users in in LAN 2, and the transmission from user $j$ generates MAI only for the $(J_1 - 1)$ users in LAN 1.

If the propagation distance $L_k$ for the common span connecting the $k^{th}$ ADM to LAN $k$ is significantly greater than the intra-LAN propagation distances $\ell_{k,i}$ between terminals, then the propagation length is approximately equal to $L_k$ for all $I$ terminals on LAN $k$. In this event, the channel transfer function for all $I$ terminals in LAN $k$ can be approximated by the transfer function of the common span, $\hat{h}_k(t)$.

For example, consider the two LAN configurations depicted in Figure 2.3. In the linear bus configuration of LAN 1, $L_1$ is the propagation distance between the ADM and the junction for the first node, and $\ell_{1,i}$ is the distance between the junction of node $(i - 1)$ and node $i$. If $L_1$ is much greater than the sum of the distances between adjacent splitters, then the total propagation distance for any of the terminals is approximately $L_1$. That is, $L_1 + \sum \ell_{1,i} \approx L_1$ if $L_1 \gg \sum \ell_{1,i}$. By similar reasoning, if the coupler-to-terminal propagation distance in LAN 2 is $\ell_{2,j}$ km for the $i^{th}$ terminal, then $L_2 + \ell_{2,j} \approx L_2$ for all terminals if $L_2 \gg \max \{L_{2,j}\}$, and all coupler-to-terminal links can be neglected. When these approximations are valid, and all terminals in the LAN have approximately the same propagation length, $L_k$, then the channel transfer function is
approximately that of the common span, $\hat{h}_k(t)$.

**Orthogonal Two-Path Channel** The signal $s(t)'$ and the interference $\eta(t)'$ both propagate through the same channel, with the orthogonal two-path transfer function

$$\hat{h}(t) \equiv \sqrt{\alpha} \delta(t - \tau_p) \hat{a}_\parallel + \sqrt{(1 - \alpha)} \delta(t - \tau_p \mp \tau_d) \hat{a}_\perp,$$  \hspace{1cm} (2.6)

which is slowly-varying, such that the three random variables $\alpha, \tau_p$ and $\tau_d$ are assumed to have values $0 \leq \alpha \leq 1$, $\tau_p > 0$ and $\tau_d > 0$ that are constant over the duration of a symbol. The vectors $\hat{a}_\parallel$ and $\hat{a}_\perp$ represent orthonormal vectors, such as orthogonal states of polarization. The random variable $\tau_p$ is the propagation delay common to both paths, and is removed by proper synchronization, and the random variable $\tau_d$ is the magnitude of the Differential Group Delay (DGD) between the two paths. The probability density functions (pdfs) and cumulative distribution functions (cdfs) of $\alpha$ and $\tau_d$ are denoted by $f_\alpha(\alpha)$, $F_\alpha(\alpha)$, $f_{\tau_d}(\tau_d)$ and $F_{\tau_d}(\tau_d)$, respectively. For compactness, we often denote the compliment $\overline{\alpha} \equiv (1 - \alpha)$.

**Quantized Differential Delay** In terms of the pulse separation, $T_{ps}$, the magnitude of the random differential delay, $\tau_d$, has the quantized representation

$$\tau_d \equiv T_{ps} (\Delta + \tau_\delta)$$  \hspace{1cm} (2.7)

where the discrete random variable $\Delta$ is a non-negative integer, and the continuous random variable $\tau_\delta$ is a remainder in the interval $[0,1)$. The probabilities of $\Delta$ are

$$Pr \{ \Delta = \Delta \} = [F_{\tau_d}((\Delta + 1)T_{ps}) - F_{\tau_d}(\Delta T_{ps})]$$  \hspace{1cm} (2.8)

**Uniform Remainder Condition** We assume that $T_{ps}$ is sufficiently small such that the remainder delay, $\tau_\delta$, and its compliment, $\overline{\tau_\delta} \equiv 1 - \tau_\delta$, are approximately uniform random variables
in the interval \([0,1]\), and that \(\tau \equiv \tau_\delta T_{ps}\) and \(\nu \equiv \nu_\delta T_{ps}\) are approximately uniform random variables in the interval \([0,T_{ps})\) seconds. This is a reasonable approximation when the maximum quantization error can be neglected. That is, for some maximum desired quantization error \(\epsilon_{\text{quant}}\), we can choose \(T_{ps}\) sufficiently small so that the maximum possible quantization error is less than \(\epsilon_{\text{quant}}\):

\[
\max_{k=1,\ldots,\infty} \left| f_{\tau_d}(kT_{ps}) - f_{\tau_d}((k-1)T_{ps}) \right| \approx \max_{t=0,\ldots,\infty} \frac{d}{dt} f_{\tau_d}(t) \left| T_{ps} \leq \epsilon_{\text{quant}}. \right.
\number{2.9}

**Sufficient Delay Condition** We also assume that \(T_{ps}\) is sufficiently small, such that the quantized differential delay, \(\Delta > 0\). In other words, we can neglect the probability that the differential delay is less than the pulse separation, \(F_{\tau_d}(T_{ps}) \approx 0\), and thus \(\tau_d > T_{ps}\) and \(\Delta \neq 0\). In conjunction with the sufficient spacing condition, which ensures that interference samples spaced by at least \(T_{ps}\) seconds are independent, this condition guarantees that the interference in the two paths is also independent at the same instant.

### 2.2.4 Receiver Front-End

The transmitted signal \(s(t)'\), interference process \(\eta(t)'\), and channel transfer function \(\hat{h}(t)\) in Figure 2.2 were defined in the preceeding section. The transmitted signal is

\[
x(t)' \equiv s(t)' + \eta(t)',
\number{2.10}
\]

and the received signal is

\[
\hat{y}(t) \equiv x(t)' \ast \hat{h}(t) = [s(t)' + \eta(t)'] \ast \hat{h}(t).
\number{2.11}
\]
**Pre-detection Filter**  The pre-detection filter is assumed to reject out-of-band interference without affecting the signal, such that \( s(t) \rightarrow s(t) \) is not changed (2.2), while \( \eta(t) \rightarrow \eta(t) \) is a filtered interference process with the same bandwidth as the signal, as described in Section 2.2.2. Furthermore, we assume that the filter is symmetric such that the real and imaginary components of the interference process remain uncorrelated.

Therefore, the output of the pre-detection filter is

\[
\hat{y}(t) \equiv [s(t) + \eta(t)] \star \hat{h}(t) 
\]

\[
\equiv x(t) \star \hat{h}(t).
\]

(2.12)

The square-law device then forms the output intensity, \( y_I(t) \), from the filtered received signal,

\[
y_I(t) \equiv \| \hat{y}(t) \|^2 
\]

\[
= \hat{y}(t) \cdot \hat{y}(t)^* 
\]

\[
= \alpha \|s(t - \tau_p) + \eta(t - \tau_p)\|^2 + (1 - \alpha) \|s(t - \tau_p \mp \tau_d) + \eta(t - \tau_p \mp \tau_d)\|^2,
\]

where the orthonormal vector dot-product relations \( \hat{a} \parallel \cdot \hat{a} \perp = 0 \), \( \hat{a} \parallel \cdot \hat{a} \parallel = 1 \), and \( \hat{a} \perp \cdot \hat{a} \perp = 1 \) have been used. Thus, the output intensity is the sum of the intensities from the two paths, and each path has chi-square statistics, since it is the magnitude squared of signal plus noise. Note that this channel is linear in the output intensity,

\[
y_I(t) = \|s(t - \tau_p) + \eta(t - \tau_p)\|^2 \ast [\alpha \delta(t - \tau_p) + (1 - \alpha) \delta(t - \tau_p \mp \tau_d)]
\]

\[
\equiv x_I(t) \ast h_I(t).
\]

(2.14)

However, this is only true because of the orthogonality of the vectors, \( \hat{a} \parallel \cdot \hat{a} \perp = 0 \). If the two paths are not orthogonal, cross-terms are generated that have zero-mean, but cannot be removed by filtering because they occupy the same bandwidth as the orthogonal intensity sum (2.14). Therefore, a channel with non-orthogonal paths produces different statistics for the output intensity, and the analysis developed in this work for orthogonal paths is not applicable.
2.2.5 PPM Detector

**Strongest Path Synchronization** We assume that the receiver maintains perfect synchronization to the strongest path, such that the common propagation delay, $\tau_p$, is removed, and $0 \leq (1 - \alpha) \leq \alpha$, and $\frac{1}{2} \leq \alpha \leq 1$. Thus, the synchronization correctly aligns the local time-reference with the start of each symbol in the stronger path, while the weaker path creates a scaled image of the same signal that is advanced or delayed by the differential group delay, $\tau_d$.

The sign of the differential delay, $\tau_d$, is taken relative to the strongest path, and is assumed to be positive or negative equiprobably, i.e., the weaker path arrives before or after the stronger path with equal probability. Whenever the differential delay $\tau_d$ or the quantized differential delay $\Delta$ are expressed with a random sign, (i.e. $\pm$ or $\mp$) the upper signs are used when the stronger path arrives first, and the lower signs are used if the weaker path arrives first.

**Multipath Image** The strongest path synchronization ensures that the stronger pulse arrives in the correct time slot, $m$, with zero differential delay, and the weaker pulse arrives $\pm \tau_d = \pm (\Delta T_{ps} + \tau)$ seconds later, in between time slots $m \pm \Delta$ and $m \pm (\Delta + 1)$. We will often refer to this secondary, time-shifted pulse from the weaker path as the *multipath image*.

**Synchronization for $\alpha = \frac{1}{2}$** In the zero-probability event that $\alpha = \frac{1}{2}$, the two-paths have equal intensity and it does not matter which path is chosen for synchronization. In general, it is still possible to communicate because the multipath image arrives in between time slots $m \pm \Delta$ and $m \pm (\Delta + 1)$. However, the zero probability event that $\alpha = \frac{1}{2}$ and $\tau = 0$ is particularly problematic, even at infinite SIR, because the multipath image aligns exactly with pulse-position $m \pm \Delta$. The error rate is $\frac{1}{2}$ in this event, because the intensity samples $I_m$ and $I_{m\pm\Delta}$ are equal,
and it is impossible to determine which pulse-position corresponds to the transmitted symbol (pulse-position $m$), and which corresponds to the multipath image (pulse-position $m \pm \Delta$).

**Test Statistic**  After synchronization to the strongest path, the output intensity of the receiver front-end, $y_I(t)$, is sampled instantaneously every $T_{ps}$ seconds, so that $I_k = y_I(kT_{ps})$ with $k \in [0 \ldots (M - 1)]$. Applying the definitions of the PPM signal (2.2) and the quantized delay (2.7),

$$I_k = \alpha \left\| \sqrt{P_o}p_u \left( [k - m]T_{ps} \right) + \eta \left( kT_{ps} \right) \right\|^2 + \bar{\alpha} \left\| \sqrt{P_o}p_u \left( [k - m = (\Delta + \tau_\delta)]T_{ps} \right) + \eta \left( [k = (\Delta + \tau_\delta)]T_{ps} \right) \right\|^2$$

$$\equiv I_{k,1} + I_{k,2},$$

where we have defined $I_{k,1}$ and $I_{k,2}$ as the intensity contributions from the stronger and weaker paths, respectively. Both $I_{k,1}$ and $I_{k,2}$ have chi-square statistics, characterized by covariance and non-centrality parameters that are derived in Section 2.3.2. The position with the largest intensity sample, $\tilde{m}$, is chosen as the most likely symbol, i.e., $I_{\tilde{m}} = \max\{I_k\}$, producing an error when $\tilde{m} \neq m$. The union bound for the symbol error probability is the sum of $(M - 1)$ pairwise error probabilities,

$$P_e(SIR) \leq \sum_{\substack{k=0 \\kneq m}}^{M-1} Pr \{ I_m < I_k \}.$$  

(2.16)

### 2.3 Pairwise Error Probabilities

In the absence of multipath (i.e., $\alpha = 1$), the non-overlapping and sufficient spacing conditions ensure that all of the intensity samples are independent, and the transmitted signal $s(t)$ is zero everywhere except $I_m$, the intensity sample corresponding to the transmitted pulse-position. Each intensity sample $I_k = I_{k,1} + I_{k,2}$ (2.15) is the sum of intensity samples from the stronger
and weaker paths (at the same instant of time). The example in Section 2.3.1 illustrates that the summation destroys the mutual independence of certain samples, and generates a multipath image that yields a non-zero contribution to the intensity samples $m \pm \Delta$ and $m \pm \Delta + 1$.

From (2.16), the $k^{\text{th}}$ pairwise error probability is formed by comparing $I_k$, the intensity at the $k^{\text{th}}$ time slot, to $I_m$, the intensity at the $m^{\text{th}}$ time slot, which corresponds to the correct transmitted pulse position. Consequently, $I_m$ is the point of reference for the $M - 1$ pairwise error probabilities, and the correlation of any other intensity pairs (e.g., between $I_k$ and $I_n$, where \( \{k, n\} \neq m \)) has no bearing on the test-statistic.

It will be shown in Section 2.3.2 that the statistics of the interference samples which form the intensity samples $I_{k,1}$ and $I_{k,2}$ separate into three cases, depending on the time slot index, $k$, the transmitted pulse position, $m$, and the quantized differential delay, $\Delta$. Consequently, the conditional pairwise error, which is defined in Section 2.3.2, must be conditioned on the values of the four random variables $m$, $\Delta$, $\alpha$ and $\tau$.

In Section 2.4, it is shown that the unconditioning over the random variables $\alpha$ and $\tau$ can be handled separately from the unconditioning over the random variables $m$ and $\Delta$ by defining three separate cases of the pairwise error probability. These three cases are conditioned only on $\alpha$ and $\tau$, and the union bound of the total error probability is then defined in terms of the average error probability for each case.

### 2.3.1 Illustration of the Intensity Statistics

To better understand how the three cases of the pairwise error arise, it is useful to first illustrate how the multipath and the correlation affect the $M$ intensity samples and the test statistic. The following section refers to a full-page figure that appears at the end of this chapter.
In the example of Figure 2.4, the equivalent correlation width, $T_R$, equals the equivalent pulse width, $T_p$. The weaker path is offset from the stronger path by the differential delay, $\tau_d = (2 + \frac{1}{2}) T_{ps}$. The intensity samples are generated every $T_{ps}$ seconds, synchronized to the time slots of the stronger path. These time instants are denoted by the dotted lines. The hatched boxes illustrate the correlation width $\pm T_R$ of the point centered in the interval, i.e., all of the points in a hatched interval are correlated to the sample at the midpoint. Note that the pulse separation, $T_{ps}$, exceeds the correlation width, $T_R$, as required by the sufficient spacing condition, and that $\Delta = 2$, so the sufficient delay condition is also satisfied. The different hatching angles are used to denote the mutual independence of the intervals. The “default” condition for the intensity samples $k \neq m \pm \Delta, m \pm (\Delta + 1)$ is that the intensity samples $I_{k,1}$ and $I_{k,2}$ have no contribution from either the transmitted pulse or the multipath image, and are independent from both $I_{m,1}$ and $I_{m,2}$. We refer to this “default” condition as “Case I”. The points marked by A, B, and C correspond to $I_{m,2}$, $I_{m+\Delta,1}$ and $I_{m+\Delta+1,2}$, respectively, and are important exceptions to the “default” condition, as described below.

**Point A**  The sample in the weaker path denoted by the letter A corresponds to $I_{m,2}$, the contribution to $I_m$ from the weaker path. The intensity samples from the stronger path in time slots $(m - \Delta) = (m - 2)$ and $(m - \Delta - 1) = (m - 3)$, $I_{m-\Delta,1}$ and $I_{m-\Delta-1,1}$, fall within the correlation width of A, and are therefore correlated to $I_{m,2}$. Therefore, $I_{m-\Delta}$ and $I_{m-\Delta-1}$ are both correlated to $I_m$. We refer to the pairwise error probabilities for $k = m - \Delta$ and $k = m - \Delta - 1$ as “Case II”.

**Point B**  The sample in the weaker path denoted by the letter B corresponds to $I_{m+\Delta,2}$, the contribution to $I_{m+\Delta}$ from the weaker path. This sample is correlated to $I_{m,1}$, the contribution
to $I_m$ from the first path. Therefore, the intensities $I_m$ and $I_{m+\Delta}$ are correlated. Furthermore, the multipath image in the weaker path yields a non-zero contribution to the intensity $I_{m+\Delta,2}$.

Note that point B is independent of point A (the contribution $I_{m,2}$ in the second path), since the separation of A and B exceeds the correlation width.

**Point C** The sample in the weaker path denoted by the letter C corresponds to $I_{m+\Delta+1,2}$, the contribution to $I_{m+\Delta+1}$ from the second path. Like point B, point C is also correlated to $I_{m,1}$, the intensity sample at $I_m$ from the first path, and thus $I_m$ and $I_{m+\Delta+1}$ are correlated. The multipath image also yields a non-zero contribution to $I_{m+\Delta+1}$. We refer to the pairwise error probabilities for $k = m + \Delta$ and $k = m + \Delta + 1$ as “Case III”. This is the most difficult case, because $I_m$ and $I_k$ both have non-zero contributions from the signal $s(t)$, and are correlated.

**2.3.2 Intensity Statistics**

**Covariance of Gaussian Interference Samples**

We define $[N_1 \ldots N_4]$ as the samples of the interference process $\eta(t)$ contributing to the intensity (2.15) at the correct time slot, $I_m$, and at one of the other $(M-1)$ incorrect time slots, $I_k$:

\[
N_1 \equiv \eta(m T_{ps}) \quad N_2 \equiv \eta([m \mp (\Delta + \tau_\delta)] T_{ps}) \\
N_3 \equiv \eta(k T_{ps}) \quad N_4 \equiv \eta([k \mp (\Delta + \tau_\delta)] T_{ps}).
\]

(2.17)

Recall that the autocorrelation and autocovariance of $\eta(t)$ are equal because the interference is zero-mean, and must be symmetric since the interference is circularly-symmetric (i.e., the real and imaginary components are independent and statistically identical). Using the definitions of the variance (2.3) and correlation coefficient (2.4) of $\eta(t)$, the correlation coefficient and
covariance of $N_i$ and $N_j$ are defined as

$$\rho_{i,j} = \frac{E\left\{ N_i N_j^* \right\}}{2\sigma^2} = \rho(D_{i,j}) \quad (2.18)$$

$$C_{i,j} = E\left\{ N_i N_j^* \right\} = 2\sigma^2 \rho_{i,j}$$

where $D_{i,j}$ is the differential time lag between $N_i$ and $N_j$, and is a function of the time slot index, $k$, conditioned on the transmitted pulse-position, $m$, the quantized differential delay, $\Delta$, and the remainder delay, $\tau$. Given these parameters, the differential time lags between the four interference samples are expressed in a matrix as follows:

$$D = \begin{bmatrix} 0 & \pm(\Delta+\tau) & m-k & (m-k)\pm(\Delta+\tau) \\ \mp(\Delta+\tau) & 0 & (m-k)\mp(\Delta+\tau) & m-k \\ -(m-k) & -(m-k)\pm(\Delta+\tau) & 0 & \pm(\Delta+\tau) \\ -(m-k)\mp(\Delta+\tau) & -(m-k) & \mp(\Delta+\tau) & 0 \end{bmatrix}_{T_{ps}} \quad (2.19)$$

Note that the magnitude of $m - k \geq 1$ because $k \neq m$, and $\Delta + \tau \geq 1$ because $\Delta > 0$ from the sufficient delay condition. Therefore, from the sufficient spacing condition, $R([m-k]T_{ps}) = 0$ so $\rho_{1,3} = \rho_{2,4} = \rho_{3,1} = \rho_{4,2} = 0$, and $R([\Delta + \tau]T_{ps}) = 0$ so $\rho_{1,2} = \rho_{2,1} = \rho_{3,4} = \rho_{4,3} = 0$. Therefore, the covariance matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & \rho_{1,4} \\ 0 & 1 & \rho_{2,3} & 0 \\ 0 & \rho_{2,3} & 1 & 0 \\ \rho_{1,4} & 0 & 0 & 1 \end{bmatrix}_{2\sigma^2} \quad (2.20)$$

where, recalling that $\tau \equiv \tau_{\delta}T_{ps}$ and $\bar{\tau} \equiv (1 - \tau_{\delta})T_{ps}$, we have defined

$$\rho_{1,4} = \begin{cases} \rho(\tau) & \text{if } k = m \pm \Delta \\ \rho(\bar{\tau}) & \text{if } k = m \pm (\Delta + 1) \\ 0 & \text{otherwise} \end{cases} \quad (2.21)$$
and
\[
\rho_{2,3} = \begin{cases} 
\rho(\tau) & \text{if } k = m \mp \Delta \\
\rho(\bar{\tau}) & \text{if } k = m \mp (\Delta - 1) \\
0 & \text{otherwise.}
\end{cases}
\] (2.22)

Thus, the interference samples are mutually independent whenever \( k \) is at least a full time slot away from the multipath image. This image arrives in between time slots \( m \pm \Delta \) and \( m \pm (\Delta + 1) \).

The three cases of the pairwise error probability arise from the piecewise definitions of \( \rho_{1,4} \) and \( \rho_{2,3} \), since \( \rho_{1,4} \neq 0 \) if \( k = m \pm \Delta \) or \( k = m \pm (\Delta + 1) \), and \( \rho_{2,3} \neq 0 \) if \( k = m \mp \Delta \) or \( k = m \mp (\Delta - 1) \). None of the other elements in the covariance matrix (2.20) depend on \( k \).

**Notation for Chi-Square Variates**

As mentioned previously, the intensity samples of each path, \( I_{k,1} \) and \( I_{k,2} \), have chi-squared statistics, to each intensity \( I_k \) is the sum of chi-square variates. For simplicity, chi-square variates with \( 2L \) degrees of freedom, non-centrality parameter \( a^2 \), and component noise variance \( \sigma^2 \) shall be denoted as \( \chi^2(L, a^2, \sigma^2) \), omitting \( L \) if \( L = 1 \) (i.e., non-central chi-square with 2 degrees of freedom), and both \( L \) and \( a^2 \) if \( L = 1 \) and \( a^2 = 0 \) (i.e., central chi-square with 2 degrees of freedom).

**Noncentrality Scaling Parameter**

First combining the definitions of the four interference samples (2.17) and the intensity samples (2.15), and then applying the above notation for chi-square variates, the intensity sample

\[
\rho(\tau) \quad \text{if } k = m \mp \Delta \\
\rho(\bar{\tau}) \quad \text{if } k = m \mp (\Delta - 1) \\
0 \quad \text{otherwise.}
\]
from the correct time slot, $I_m$, and one of the other $(M - 1)$ incorrect time slots, $I_k$, is

\[
I_m = \alpha \left( \sqrt{P_o + N_1} \right)^2 + \bar{\alpha} \left( N_2 \right)^2
\]

(2.23)

\[
I_k = \alpha \left( N_3 \right)^2 + \bar{\alpha} \left( \sqrt{P_o \gamma} + N_4 \right)^2
\]

(2.25)

\[
= \chi^2 (\alpha P_o, \alpha \sigma^2) + \chi^2 \left( \bar{\alpha} \sigma^2 \right)
\]

(2.24)

\[
= \chi^2 (\alpha \sigma^2) + \chi^2 \left( \bar{\alpha} P_o \gamma^2, \bar{\alpha} \sigma^2 \right)
\]

(2.26)

respectively. Therefore, the non-centrality parameter has a maximum value of $\sqrt{P_o}$, and we define the non-centrality scaling parameter, $\gamma$, is

\[
\gamma = \begin{cases} 
  p_u(\tau) & \text{if } k = m \pm \Delta, \\
  p_u(\bar{\tau}) & \text{if } k = m \pm (\Delta + 1), \\
  0 & \text{otherwise}, 
\end{cases}
\]

(2.27)

which corresponds to the same piecewise conditions as the correlation coefficient $\rho_{1,4}$.

**Conditional Pairwise Error Probability**

Grouping the correlated variates on the same sides of the inequality, the conditional pairwise error probability is

\[
\mathcal{P}(I_m < I_k) = P_k \left( SIR | \Delta, m, \alpha, \tau \right)
\]

\[
= \mathcal{P} \left\{ \alpha \left( \sqrt{P_o + N_1} \right)^2 - \bar{\alpha} \left( \sqrt{P_o \gamma} + N_4 \right)^2 < \alpha \left( N_3 \right)^2 - \bar{\alpha} \left( N_2 \right)^2 \right\}
\]

\[
= \mathcal{P} \left\{ \chi^2 (\alpha P_o, \alpha \sigma^2) - \chi^2 \left( \bar{\alpha} \gamma P_o, \bar{\alpha} \sigma^2 \right) \right\}
\]

(2.28)

where the corresponding correlation coefficients of the interference samples are listed underneath each term. Note that the parameters $\gamma, \rho_{1,2}$ and $\rho_{3,4}$ are implicitly conditioned on the
values of the three random variables $\Delta, m$ and $\tau$, as defined in (2.27), (2.21), and (2.22), respectively. Thus, the intensities $I_m$ and $I_k$ are sums of conditional chi-square variates, and the conditional pairwise error probabilities $\Pr \{ I_m - I_k < 0 \}$ involve differences of conditionally correlated non-central chi-square variates. Solving the pairwise error probabilities given these assumptions is the primary contribution of this dissertation.

### 2.4 Three Cases of the Pairwise Error Probability

#### 2.4.1 Conditional Pairwise Error Probability

The conditional pairwise error probability (2.28) is a function of the non-centrality scaling parameter, $\gamma$ (2.27), and the correlation coefficients, $\rho_{1,4}$ (2.21) and $\rho_{2,3}$ (2.22). We have noted that these parameters separate into one of three cases as a function of the sample index $k$, conditioned on the transmitted symbol $m$, the quantized delay $\Delta$, and the remainder delay $\tau$. Therefore, the general conditional error separates into one of three specific cases, and for simplicity, we define the following shorthand notation:

\[
P_k(\text{SIR} | m, \Delta, \alpha, \tau) \equiv \begin{cases} 
P_{III}(\text{SIR} | \alpha = \alpha, \rho = R(\tau), \gamma = p_u(\tau)) & \text{if } k = m \pm \Delta \\
P_{III}(\text{SIR} | \alpha = \alpha, \rho = R(\tau), \gamma = p_u(\tau)) & \text{if } k = m \pm (\Delta + 1) \\
P_{II}(\text{SIR} | \alpha = \alpha, \rho = R(\tau)) & \text{if } k = m \mp \Delta \\
P_{II}(\text{SIR} | \alpha = \alpha, \rho = R(\tau)) & \text{if } k = m \mp (\Delta + 1) \\
P_I(\text{SIR} | \alpha = \alpha) & \text{otherwise,} \end{cases}
\]
where the three cases of the conditional pairwise error probability are $P_I$, $P_{II}$ and $P_{III}$. These are defined as:

\[
P_I(SIR | \alpha) \equiv P_r \left\{ \alpha \| \sqrt{P_o + N_1} \|^2 - \tilde{\alpha} \| N_4 \|^2 < \alpha \| N_3 \|^2 - \tilde{\alpha} \| N_2 \|^2 \right\}
\]

\[
= P_r \left\{ \chi^2(\alpha P_o, \alpha \sigma^2) - \chi^2(\tilde{\alpha} \sigma^2) < \chi^2(\alpha \sigma^2) - \chi^2(\tilde{\alpha} \sigma^2) \right\},
\]

(2.30)

\[
P_{II}(SIR | \alpha, \rho) \equiv P_r \left\{ \alpha \| \sqrt{P_o + N_1} \|^2 - \tilde{\alpha} \| N_4 \|^2 < \alpha \| N_3 \|^2 - \tilde{\alpha} \| N_2 \|^2 \right\}
\]

\[
= P_r \left\{ \chi^2(\alpha P_o, \alpha \sigma^2) - \chi^2(\tilde{\alpha} \sigma^2) < \chi^2(\alpha \sigma^2) - \chi^2(\tilde{\alpha} \sigma^2) \right\},
\]

(2.31)

\[
P_{III}(SIR | \alpha, \rho, \gamma) \equiv P_r \left\{ \alpha \| \sqrt{P_o + N_1} \|^2 - \tilde{\alpha} \| \sqrt{P_o \gamma + N_4} \|^2 < \alpha \| N_3 \|^2 - \tilde{\alpha} \| N_2 \|^2 \right\}
\]

\[
= P_r \left\{ \chi^2(\alpha P_o, \alpha \sigma^2) - \chi^2(\tilde{\alpha} \sigma^2) < \chi^2(\alpha \sigma^2) - \chi^2(\tilde{\alpha} \sigma^2) \right\}.
\]

(2.32)

Note that Case I is obtained in the limit of $\rho \to 0$ in Case II, and in the limit of both $\rho \to 0$ and $\gamma \to 0$ in Case III. Since the correlation and non-centrality parameters for Cases II and III are both deterministic functions of the random remainder delay, $\tau$, these cases can also be defined as:

\[
P_{II}(SIR | \alpha, \tau) \equiv P_{II}(SIR | \alpha, \rho = R(\tau))
\]

\[
P_{III}(SIR | \alpha, \tau) \equiv P_{III}(SIR | \alpha, \rho = R(\tau), \gamma = p_u(\tau)).
\]

(2.33)
2.4.2 Average Pairwise Error Probabilities

Recalling that $\tau$ and $\bar{\tau}$ are both uniform over $[0, T_{ps})$ seconds, (i.e., they are identically distributed) and that $\alpha$ is independent of $\tau$ and $\bar{\tau}$, then

$$f_{\alpha,\tau}(\alpha, \bar{\tau}) = f_{\alpha,\tau}(\alpha, \tau) = f_{\alpha}(\alpha)f_{\tau}(\tau) = \frac{2}{T_{ps}}. \quad (2.34)$$

Therefore, the average pairwise error probability for each case is

$$P_{III}(SIR) = \frac{2}{T_{ps}} \int_0^{T_{ps}} \int_{\frac{1}{2}}^1 P_{III}(SIR | \alpha, \tau) \, d\alpha \, d\tau$$

$$P_{II}(SIR) \equiv \frac{2}{T_{ps}} \int_0^{T_{ps}} \int_{\frac{1}{2}}^1 P_{II}(SIR | \alpha, \tau) \, d\alpha \, d\tau \quad (2.35)$$

$$P_{I}(SIR) = 2 \int_{\frac{1}{2}}^1 P_{I}(SIR | \alpha) \, d\alpha. \quad (2.36)$$

Since $f_{\alpha,\tau}(\alpha, \bar{\tau}) = f_{\alpha,\tau}(\alpha, \tau)$, the average error probability for Case II is the same for $k = m \mp \Delta$ and $k = m \mp (\Delta + 1)$, and the average error probability for Case III is the same for $k = m \pm \Delta$ and $k = m \pm (\Delta + 1)$. Therefore, the pairwise error probability for the $k$th time-slot, conditioned on the values of $m$ and $\Delta$, is

$$P_k(SIR|m, \Delta) = \int_0^{T_{ps}} \int_{\frac{1}{2}}^1 P_k(SIR|m, \Delta, \alpha, \tau) \, f_{\alpha,\tau}(\alpha, \tau) \, d\alpha \, d\tau \quad (2.37)$$

$$= \begin{cases} P_{III}(SIR) & \text{if } k = \{m \pm \Delta, m \pm (\Delta + 1)\} \\ P_{II}(SIR) & \text{if } k = \{m \mp \Delta, m \mp (\Delta + 1)\} \\ P_I(SIR) & \text{otherwise.} \end{cases}$$

This separation of the $M - 1$ pairwise error probabilities into the three cases for time-slots $k = [0 \ldots M - 1], k \neq m$ as a function of the transmitted symbol, $m = [0 \ldots (M - 1)]$ and the magnitude of the differential delay, $\Delta = [1 \ldots \infty]$ is the most fundamental aspect of this analysis, and will be illustrated for several specific examples in Section 3.2.
Key Results for the Three Cases

The analysis of these three cases is developed in Chapters 5 - 10, with Chapters 9 and 10 devoted to numerical techniques that must be used to obtain results for Case III. In Theorem 6.1.1 and Corollary 5.1.4, the average error probability for Cases I and II are shown to have the exponentially decreasing upper bound

\[
P_{II} (\text{SIR}) \leq P_I (\text{SIR}) \leq P_I \left( \text{SIR} \left| \alpha = \frac{1}{2} \right. \right)
= \frac{1}{2} \exp \left( - \frac{\text{SIR}}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{\text{SIR}}{2} \right) \right]
\]

(2.38)

Using a recursive numerical integration algorithm, it is shown in Chapter 10 that at high SIR, the average error probability for Case III has an approximate power-law dependence

\[
P_{III} (\text{SIR} \gg 1) \approx \Lambda \frac{T_p}{T_{ps}} \text{SIR}^{-\xi},
\]

(2.39)

where the constants \( \Lambda \) and \( \xi \) depend on the pulse shape, \( p_u(t) \), and the interference autocorrelation function, \( R(t) \).

2.5 Union Bound of the Total Error Probability

The union bound for the total error probability is obtained by forming the sum of the \((M - 1)\) pairwise error probabilities, assuming that \( \Delta = \Delta \) and \( m = m \), and then unconditioning each variable. We denote

\[
P_e (\text{SIR}, M) \quad \text{as the union bound of the unconditional error probability},
\]

\[
P_e (\text{SIR}, M | \Delta) \quad \text{as the union bound of the error probability, given } \Delta = \Delta,
\]

\[
P_e (\text{SIR}, M | \Delta, m) \quad \text{as the union bound of the error probability, given } \Delta = \Delta \text{ and } m = m,
\]
These three probabilities are defined as follows:

\[
P_e(SIR, M) \equiv \sum_{\Delta=0}^{\infty} P_e(SIR, M|\Delta) \Pr\{\Delta = \Delta\}
\]

\[
P_e(SIR, M|\Delta) \equiv \sum_{m=0}^{M-1} \frac{1}{M} P_e(SIR, M=\Delta, m=m)
\]

\[
P_e(SIR, M|\Delta, m) \leq \sum_{k=0}^{M-1} P_k(SIR|\Delta = \Delta, m = m)
\]

\[
= \sum_{k=0}^{M-1} \left\{ \begin{array}{ll}
P_{II}(SIR) & \text{if } k = \{m \pm \Delta, m \pm (\Delta + 1)\} \\
P_{III}(SIR) & \text{if } k = \{m \mp \Delta, m \mp (\Delta + 1)\} \\
P_I(SIR) & \text{otherwise.}
\end{array} \right.
\]

The first equation expresses the unconditioning over the distribution of \(\Delta\), which is the quantized representation of the differential delay, \(\tau_d\) (as a multiple of the time-slot width, \(T_{ps}\)). The second equation is the unconditioning over the distribution of \(m\), which is uniform in \([0 \ldots M - 1]\), and the third equation states the union bound in terms of the \(M - 1\) pairwise error probabilities, conditioned on \(\Delta\) and \(m\), as given in (2.36).

**Probability of Interference**

Depending on the values of the random variables \(m\) and \(\Delta\), it is in general possible to have \(m \pm \Delta < 0\) or \(m \pm \Delta \geq M\) (and possibly both) such that Cases II or III (and possibly both) are not one of the \(k = [0 \ldots M - 1], k \neq m\) pairwise error probabilities in the union bound of the total error probability. Given that the average error probability for Case III has a power-law dependence (2.39), while the average error for Case I decreases exponentially (2.38), it is extremely fortunate that Case III does not always affect the test statistic.

The probability that Cases II and III do not affect the test statistic is denoted as \(P_{\text{int}}(M) \equiv 1 - P_{\text{int}}(M)\), where \(P_{\text{int}}(M)\) is defined as the average “Probability of Interference”. It will be
proven in Chapter 3 that
\[
P_{\text{int}}(M) \approx \left(1 - \frac{1}{2M}\right) F_{\tau_d} ((M - 1) T_{ps}) - \frac{1}{MT_{ps}} \int_0^{MT_{ps}} z f_{\tau_d}(z) \, dz, \tag{2.41}
\]
which is defined in terms of the pdf and cdf of the differential delay, $\tau_d$. Using this definition, it is proven in Theorem 3.1.3 that for $M \geq 4$, the union bound for the total error probability is
\[
P_e(SIR, M) \leq P_I(SIR) \left[ (M - 5) + 4P_{\text{int}}(M) \right] + 2 \left[ P_{II}(SIR) + P_{III}(SIR) \right] P_{\text{int}}(M). \tag{2.42}
\]
Given that Case I is an upper bound for Case II, which is proven in Chapter 6,
\[
P_e(SIR, M) \leq \begin{cases} 
(M - 1) P_I(SIR) & \text{if } P_{\text{int}}(M) = 0 \\
(M - 3) P_I(SIR) + 2P_{III}(SIR) & \text{if } P_{\text{int}}(M) = 1
\end{cases}. \tag{2.43}
\]

**Throughput Constraint from Multipath Interference**

The average probability of multipath interference, $P_{\text{int}}(M)$, is a function of the constellation size $M$. As $M \to \infty$, $P_{\text{int}}(M) \to 1$. At high SIR, $P_{III}(SIR) \gg P_I(SIR)$, since the pairwise error probability for Case III has a power-law dependence, while the average error probability of Cases I and II decrease exponentially. Thus, we can see from (2.43) that $P_{\text{int}}(M)$ will have a dominant effect on the total error probability in the high SIR regime. It is shown in Chapter 11 that a maximum error rate requirement constrains the maximum value of the interference probability $P_{\text{int}}(M)$, thereby constraining $M$ and limiting the throughput.
Figure 2.4: Illustration of the intensity statistics with $\tau_d = (2 + \frac{1}{2}) T_{ps}$. 
Chapter 3

Total Error Probability

3.1 Overview

The random variables \( m \) and \( \Delta \) govern how the three cases combine to form the total error probability. These are discrete random variables, with \( m \in [0 \ldots M-1] \) denoting the transmitted pulse-position and \( \Delta \in [1 \ldots \infty] \) denoting the magnitude of the quantized differential delay. Recall that Case III affects time-slots \( k = \{m \pm \Delta, m \pm (\Delta + 1)\} \) and Case II affects time-slots \( k = \{m \mp \Delta, m \mp (\Delta + 1)\} \) (see (2.36)), where the sign of the delay is positive or negative with equal probability, and always opposite for Cases II and III.

Since these four time-slots do not always correspond to one of the \( k = [0 \ldots M - 1] \) time-slots for possible transmitted pulse-positions, Cases II and III do not always affect the test statistic. In fact, most of the time-slots correspond to the “default” case (i.e., Case I), which occurs for at least \( (M - 5) \) of the time-slots, up to all \( (M - 1) \) time-slots whenever the multipath is sufficiently large. In particular, as stated earlier, if either \( m \pm \Delta < 0 \) or \( m \pm \Delta \geq M \) (i.e. \( m < \Delta \) or \( m \geq M - \Delta \)), then the multipath interference falls outside of the detection window.
\( k = [0 \ldots M - 1] \) and Cases II and III will not affect the test statistic. This is guaranteed whenever \( \Delta \geq M \), such that the differential delay, \( \tau_d \), exceeds the symbol length, \( T_{\text{sym}} = M T_{\text{ps}} \).

The average interference probability, \( P_{\text{int}}(M) \), is governed by the statistics of the differential delay. This average interference probability, \( P_{\text{int}}(M) \) has a profound effect on the overall performance, constraining the constellation size, \( M \), for a given error rate requirement and differential delay distribution. Note that since we are only interested in transmitting an integer number of bits, the constellation size, \( M \), is always a power of two. The binary case is trivial, since there is only one pairwise error probability, but requires separate treatment from the general \( M \)-ary case.

The unconditional error probability in Theorem 3.1.3 is obtained from the conditional error probabilities, which are summarized in the following lemmas. Lemma 3.1.2 is proven in Section 3.4 and Lemma 3.1.1 is proven in Section 3.3.

**Lemma 3.1.1.** The union bound of the error probability, conditioned on both \( \Delta \) and \( m \), has three different piecewise expressions, depending on the value of the delay, \( \Delta \), with respect to the symbol length, \( M \).

**Regime 1:** Delays within the first half of the symbol length:
For $0 < \Delta < \frac{M}{2}$,

$$P_e (SIR, M | \Delta, m) \leq \begin{cases} 
2P_{II} (SIR) + (M - 3)P_I (SIR) & \text{if } m = [0 \ldots (\Delta - 1)] \\
2P_{II} (SIR) + P_I (SIR) + (M - 4)P_I (SIR) & \text{if } m = \Delta \\
2P_{II} (SIR) + 2P_{II} (SIR) + (M - 5)P_I (SIR) & \text{if } m = [(\Delta + 1) \ldots (M - \Delta - 2)] \\
P_{III} (SIR) + 2P_{II} (SIR) + (M - 4)P_I (SIR) & \text{if } m = M - \Delta - 1 \\
2P_{II} (SIR) + (M - 3)P_I (SIR) & \text{if } m = [(M - \Delta) \ldots (M - 1)].
\end{cases}$$

(3.1)

**Regime 2:** Delays within the second half of the symbol length:

For $\frac{M}{2} \leq \Delta < M$,

$$P_e (SIR, M | \Delta, m) \leq \begin{cases} 
2P_{III} (SIR) + (M - 3)P_I (SIR) & \text{if } m = [0 \ldots (M - \Delta - 2)] \\
P_{III} (SIR) + (M - 2)P_I (SIR) & \text{if } m = M - \Delta - 1 \\
(M - 1)P_I (SIR) & \text{if } m = [(M - \Delta) \ldots (\Delta - 1)] \\
P_{II} (SIR) + (M - 2)P_I (SIR) & \text{if } m = \Delta \\
2P_{II} (SIR) + (M - 3)P_I (SIR) & \text{if } m = [(\Delta + 1) \ldots (M - 1)].
\end{cases}$$

(3.2)

**Regime 3:** Delays exceeding the symbol length:

If $\Delta \geq M$, then

$$P_e (SIR, M | \Delta, m) \leq (M - 1)P_I (SIR).$$

(3.3)

Note that for $M = 4$, some of these series are non-ascending and yield empty sets because there are only four possible values of $m$, but five different subsets. Specifically, these empty subsets are $m = [(\Delta + 1) \ldots (M - \Delta - 2)]$ for $\Delta = 1$, $m = [(M - \Delta) \ldots (\Delta - 1)]$ for $\Delta = 2$, and
both \( m = [0 \ldots (M - \Delta - 2)] \) and \( m = [(\Delta + 1) \ldots (M - 1)] \) for \( \Delta = 3 \). Since these sets are empty, they can be excluded without any loss of generality.

**Lemma 3.1.2.** The union bound of the error probability, conditioned on \( \Delta \), is

\[
P_e(SIR, M | \Delta) \leq \begin{cases} 
  P_I(SIR) \left[ (M - 3) - \frac{2}{M}(M - 1 - 2\Delta) \right] & \text{if } \Delta < M \\
  (M - 1)P_I(SIR) & \text{if } \Delta \geq M
\end{cases}
\]

\[
= P_I(SIR) \left[ (M - 5) + 4\overline{P_{\text{int}}}(M | \Delta = \Delta) \right] + \frac{2}{M} \left[ P_{II}(SIR) + P_{III}(SIR) \right] \overline{P_{\text{int}}}(M | \Delta = \Delta)
\]

where \( \overline{P_{\text{int}}}(M | \Delta) = 1 - P_{\text{int}}(M | \Delta) \) is the probability of interference, conditioned on \( \Delta \), which is defined as

\[
P_{\text{int}}(M | \Delta) \equiv \left[ 1 - \left( \min\left(\Delta, \frac{M - 1}{2} \right) + \frac{1}{2} \right) \right].
\]

The only difference between this result and the unconditioned error probability (3.6) in Theorem 3.1.3 is the unconditioning of \( P_{\text{int}}(M | \Delta) \).

Using the results of the previous two Lemmas, the following Theorem 3.1.3 is proven in Section 3.5. In Section 11.0.1, this theorem is used, in conjunction with the results of the analysis for Cases I-III, to obtain a succinct expression for the union bound of the total error probability in the high SIR regime.

**Theorem 3.1.3.** The total error probability differs for \( M \)-ary PPM and binary PPM, as given below:

**M-ary PPM** For \( M \geq 4 \), the union bound for the total error probability is

\[
P_e(SIR, M) \leq P_I(SIR) \left[ (M - 5) + 4\overline{P_{\text{int}}}(M) \right] + 2 \left[ P_{II}(SIR) + P_{III}(SIR) \right] \overline{P_{\text{int}}}(M),
\]

(3.6)
where $\bar{P}_{\text{int}}(M) \equiv 1 - P_{\text{int}}(M)$, and $P_{\text{int}}(M)$ is the unconditioned probability that the multipath affects the test statistic, defined in terms of the pdf and cdf of the differential delay distribution as

$$P_{\text{int}}(M) \equiv \sum_{\Delta = 1}^{M-1} P_{\text{int}}(M|\Delta = \Delta) \Pr\{\Delta = \Delta\}$$

$$= \sum_{\Delta = 1}^{M-1} \left(1 - \frac{\Delta + \frac{1}{2}}{M}\right) \left[F_{\tau_d}((\Delta + 1)T_{ps}) - F_{\tau_d}(\Delta T_{ps})\right]$$

$$\approx \left(1 - \frac{1}{2M}\right) F_{\tau_d}(MT_{ps}) - \frac{1}{MT_{ps}} \int_0^{MT_{ps}} z f_{\tau_d}(z) \, dz. \quad (3.7)$$

**Binary PPM** For $M = 2$, the unconditional error probability is

$$P_e(SIR, M = 2) \leq P_I(SIR) P_{\text{int}}(M = 2) + \frac{1}{2} P_{\text{int}}(M = 2) \left[P_{II}(SIR) + P_{III}(SIR)\right], \quad (3.8)$$

and the average interference probability, $P_{\text{int}}(M)$ is

$$P_{\text{int}}(M = 2) = F_{\tau_d}(2T_{ps}) - F_{\tau_d}(T_{ps}). \quad (3.9)$$

### 3.2 Separation of the Pairwise Error Probability into Cases I-III

We begin by illustrating a few specific examples of how the $M - 1$ pairwise error probabilities, conditioned on the values of $\Delta$ and $m$, separate into Cases I-III, as defined in (2.40). In all of the figures in this section, the columns represent the $k^{\text{th}}$ time-slot, and the rows represent a symbol transmitted using the $m = [0 \ldots M - 1]^{\text{th}}$ pulse-position. The white (blank) time-slots correspond to Case I, the shaded time-slots (with upwards or downwards hatching) correspond to either Case II or III, and the $m^{\text{th}}$ time-slot is black to indicate $k \neq m$, i.e., the point of reference for the $M - 1$ pairwise error probabilities. To illustrate the trends of how the random quantities $m \pm \Delta$ and $m \pm (\Delta + 1)$ in (2.40) affect the test statistic over a range of different values for
\( m \) and \( \Delta \), the time-slots corresponding to the conditioned values of \( m \pm \Delta \) and \( m \pm (\Delta + 1) \) in Figures 3.1-3.6 are labeled without substituting the conditioned values of \( m \) and \( \Delta \).

If the sign of the differential delay is positive, the time-slots with downwards hatching correspond to Case III, and the time-slots with upwards hatching correspond to Case II. The situation is reversed if the sign of the differential delay is negative. It will be shown that the sign of the differential delay is irrelevant, since the \( M \) pulse-positions are equiprobable, and the separation of the pairwise error probabilities into Cases II and III is always symmetric with opposing signs (2.40). Therefore, we assume that the sign is positive without any loss of generality. Furthermore, we may also neglect the event that \( \Delta = 0 \) due to the sufficient delay condition, and begin with \( \Delta = 1 \).

It has been stated in Lemma 3.1.1 that the conditional error \( P_e(SIR, M | \Delta, m) \) separates into the three regimes depending on the value of \( \Delta \); \( \Delta = [1 \ldots (M/2 - 1)] \), \( \Delta = [M/2 \ldots (M-1)] \), and \( \Delta = [M \ldots \infty] \). To illustrate this behavior, we provide examples for \( \Delta = \{1, 2, M/2 - 1, M/2, M - 2, M - 1\} \), which define the boundaries of the three regimes. Note that for \( M = 4 \), \((M/2 - 1) = 1 \) and \( M/2 = M - 2 = 2 \), so only \( \Delta = \{1, M - 2, M - 1\} \) and \( m = \{0, 1, M - 2, M - 1\} \) are applicable.

### 3.2.1 \( \Delta = 1 \)

Given \( \Delta = 1 \), Figure 3.1 illustrates how the \( M - 1 \) conditional pairwise error probabilities (2.40) separate into the three cases, for the following values of the transmitted pulse position, \( m \).

\( m = 0 \): Since \( m - \Delta = -1 \) and \( m - \Delta - 1 = -2 \) are not one of the \([0 \ldots M - 1]\) possible pulse locations, Case II does not affect the test statistic. Case III affects the test statistic for
time-slots $m + \Delta = 1$ and $m + \Delta + 1 = 2$. The remaining $(M - 3)$ time-slots correspond to Case I.

$m = 1$: Since $m - \Delta = 0$ but $m - \Delta - 1 = -1$, then Case II affects only one time-slot ($k = 0$), while Case III still affects two time-slots ($k = 2, 3$). Case I affects the remaining $(M - 4)$ time-slots.

$m = 2$: Both $m - \Delta = 1$ and $m - \Delta - 1 = 0$ are in $[0 \ldots M - 1]$, so Case II and Case III each affect two time-slots ($k = 0, 1$ and $k = 3, 4$, respectively). Case I affects the remaining $(M - 5)$ time-slots.

$m = \{2 \ldots (M - 3)\}$: Throughout this entire range, both $m \pm \Delta$ and $m \pm (\Delta + 1)$ are in $[0 \ldots M - 1]$, so Case II and Case III each affect two time-slots, and Case I affects the
remaining \((M - 5)\) time-slots.

**\(m = M - 2\):** We now reach the boundary where \(m + \Delta + 1 = M\) exceeds \(M - 1\), and Case III affects only the last time-slot \((k = M - 1)\). Case II still affects two time-slots \((k = M - 4, M - 3)\), and Case I affects the remaining \((M - 4)\) time-slots.

**\(m = M - 1\):** Finally, both \(m + \Delta = M\) and \(m + \Delta + 1 = M + 1\) exceed \(M - 1\), eliminating Case III from the test statistic. Case II still affects two time-slots \((k = M - 3, M - 2)\), and Case I affects the remaining \((M - 3)\) time-slots.

Note that at least two of the time-slots are always affected by either Case II or Case III, since \(m + \Delta\) or \(m - \Delta\) (and possibly both) are always included in the detection window.

### 3.2.2 \(\Delta = 2\)

Given \(\Delta = 2\), and \(M > 4\), Figure 3.2 illustrates how the \(M - 1\) conditional pairwise error probabilities (2.40) separate into the three cases, for the following values of the transmitted pulse position, \(m\).

**\(m = \{0, 1\}\):** Case II does not affect the test statistic until \(m - \Delta = 0\), i.e., until \(m = \Delta = 2\). Thus, for the first two possible pulse-positions \((m = 0, 1)\), Case III affects the test statistic at two time-slots, and Case I affects the remaining \(M - 3\) time-slots.

**\(m = 2\):** Since \(m - \Delta = 0\) but \(m - \Delta - 1 = -1\), then Case II only affects the first time-slot, while Case III still affects two time-slots, and Case I affects the remaining \((M - 4)\) time-slots.

**\(m = 3\):** Both \(m - \Delta = 1\) and \(m - \Delta - 1 = 0\) are in \([0 \ldots M - 1]\). Therefore Case II and Case III each affect two time-slots, and Case I affects the remaining \((M - 5)\) time-slots.
Figure 3.2: Pairwise Error Probabilities, given $\Delta = 2$ and $m$.

$m = \{3 \ldots (M - 4)\}$: Throughout this entire range, both $m \pm \Delta = 1$ and $m \pm (\Delta + 1)$ are in $[0 \ldots M - 1]$, so Case II and Case III each affect two time-slots, and Case I affects the remaining $(M - 5)$ time-slots.

$m = (M - 3)$: We now reach the boundary where $m + \Delta + 1 = M$ exceeds $M - 1$, and Case III only affects time-slot $k = (M - 1)$. Case II still affects two time-slots, and Case I affects the remaining $(M - 4)$ time-slots.
\( m = \{(M-2), (M-1)\}: \) Finally, both \( m + \Delta = M \) and \( m + \Delta + 1 = M + 1 \) exceed \( M - 1 \), eliminating Case III from the test statistic. Case II still affects two time-slots, and Case I affects the remaining \((M-3)\) time-slots.

Again, at least two of the time-slots are always affected by either Case II or Case III, since \( m + \Delta \) or \( m - \Delta \) (and possibly both) are always included in the detection window.

### 3.2.3 \( \Delta = \frac{M}{2} - 1 \)

Given \( \Delta = \frac{M}{2} - 1 \) and \( M > 4 \), Figure 3.3 illustrates how the \( M - 1 \) conditional pairwise error probabilities (2.40) separate into the three cases, for the following values of the transmitted pulse position, \( m \).

![Figure 3.3: Pairwise Error Probabilities, given \( \Delta = \frac{M}{2} - 1 \) and \( m \).](image)

\( m = \{0 \ldots \frac{M}{2} - 2\} \): Case II does not affect the test statistic until \( m - \Delta = 0 \), i.e., until \( m = \Delta = \frac{M}{2} - 1 \). Thus, for the first \( \frac{M}{2} - 2 \) pulse-positions, Case III affects the test
statistic at two time-slots, and Case I affects the remaining \( M - 3 \) time-slots.

\( m = \frac{M}{2} - 1 \): Since \( m - \Delta = 0 \) but \( m - \Delta - 1 = -1 \), then Case II only affects the first time-slot, while Case III affects the last two time-slots, and Case I affects the remaining \( (M - 4) \) time-slots. Note that it is not possible to have two time-slots with Case II and also two time-slots with Case III for any \( m \), as the leading value of \( m + \Delta + 1 \) is excluded from the detection window whenever the trailing value \( m - \Delta - 1 \) is included.

\( m = \frac{M}{2} \): Both \( m - \Delta = 1 \) and \( m - \Delta - 1 = 0 \) are in \([0 \ldots M - 1]\), so Case II affects the first two time-slots, but \( m + \Delta = M - 1 \) and \( m + \Delta + 1 = M \), so Case III only affects the last time-slot. Thus Case I affects the remaining \( (M - 4) \) time-slots.

\( m = \frac{M}{2} + 1 \): Both \( m + \Delta = 2 \) and \( m + (\Delta + 1) = 1 \) are in \([0 \ldots M - 1]\), so Case II affects two time-slots, while both \( m + \Delta = M \) and \( m + \Delta + 1 = M + 1 \) exceed \( M - 1 \), so Case III does not affect the test statistic. Case I affects the remaining \( (M - 3) \) time-slots.

\( m = \{ \frac{M}{2} + 1 \ldots (M - 1) \} \): Throughout this entire range, Case II affects two-time slots, \( (k = (m - \frac{M}{2}), (m - \frac{M}{2} + 1)) \), while Case III is eliminated from the test statistic, and Case I affects the remaining \( (M - 3) \) time-slots.

This is largest value of \( \Delta \) that must have either Case II or Case III (and possibly both) affecting two time-slots, since \( m + \Delta \) or \( m - \Delta \) (and possibly both) must be included in the detection window. It will be shown in the next example that this no longer guaranteed for \( \Delta = \frac{M}{2} \).

### 3.2.4 \( \Delta = \frac{M}{2} \)

Given \( \Delta = \frac{M}{2} \), and \( M > 4 \), Figure 3.4 illustrates how the \( M - 1 \) conditional pairwise error probabilities (2.40) separate into the three cases, for the following values of the transmitted
pulse position, $m$. With the delay exceeding half the symbol length, it is no longer possible to have both Cases II and III affect the test statistic. In other words, Cases II and III are mutually exclusive because the delay is sufficiently large that $m + \Delta$ is excluded from the detection window whenever $m - \Delta$ is included, i.e., either $m + \Delta \geq M$ or $m - \Delta < 0$.

$m = \{0 \ldots (\frac{M}{2} - 2)\}$: Case II does not affect the test statistic until $m - \Delta = 0$, i.e., until $m = \Delta = \frac{M}{2}$. Thus the $\frac{M}{2}$ symbols using the first $\frac{M}{2}$ pulse-positions are not affected by Case II. Since both $m + \Delta$ and $m + \Delta + 1 \leq (M - 1)$, Case III affects the test statistic at two time-slots, which correspond to the last two time-slots when $m = \frac{M}{2} - 2$. Case I affects the remaining $M - 3$ time-slots.

$m = \frac{M}{2} - 1$: Since $m - \Delta = -1$ and $m - \Delta - 1 = -2$, Case II still does not affect the test statistic. On the other hand, $m + \Delta = M - 1$ and $m + \Delta + 1 = M$, so Case III only affects the last time-slot, and Case I affects the remaining $(M - 2)$ time-slots. This is the
first example so far where there are not at least two time-slots affected by either Case II or Case III.

\( m = \frac{M}{2} \): Since \( m - \Delta = 0 \) and \( m - \Delta - 1 = -1 \), Case II affects the first time-slot, but
\( m + \Delta = M \) and \( m + \Delta + 1 = M + 1 \), so Case III no longer affects the test statistic. Thus Case I affects the remaining \( (M - 2) \) time-slots. This is another example where Cases II or III do not affect at least two time-slots.

\( m = \{ \frac{M}{2} + 1 \ldots (M - 1) \} \): Throughout this range, \( m - \Delta \) and \( m - \Delta - 1 \) are in \( [0 \ldots (M - 1)] \), while \( m + \Delta \) and \( m + \Delta + 1 \) are not. Thus Case II affects two time-slots, and Case I affects the remaining \( (M - 3) \) time-slots.

In addition to having Cases II and III mutually exclusive, this marks the beginning of the regime \( \Delta = [\frac{M}{2} \ldots M - 1] \), where Cases II or III are no longer guaranteed to affect at least two time-slots. It will be shown in the next two examples, for \( \Delta = M - 2 \) and \( \Delta = M - 1 \), that there are several possible pulse-positions where neither \( m + \Delta \) nor \( m - \Delta \) are in the detection window, and only Case I affects the test statistic.

3.2.5 \( \Delta = M - 2 \)

Given \( \Delta = M - 2 \), Figure 3.5 illustrates how the \( M - 1 \) conditional pairwise error probabilities (2.40) separate into the three cases, for the following values of the transmitted pulse position, \( m \).

\( m = 0 \): Case II does not affect the test statistic until \( m - \Delta = 0 \), i.e., until \( m = \Delta = M - 2 \).

Thus there is no Case II for the first \( (M - 3) \) possible pulse-positions. Since \( m + \Delta = (M - 2) \) and \( m + \Delta + 1 = (M - 1) \), Case III affects the last two time-slots, and Case I
Figure 3.5: Pairwise Error Probabilities, given $\Delta = M - 2$ and $m$.

affects the remaining $(M - 3)$ time-slots.

$m = 1$: Case II still does not affect the test statistic, but now $m + \Delta + 1 = M$, so Case III only affects the last time-slot, and Case I affects the remaining $(M - 2)$ time-slots.

$m = \{2 \ldots (M - 3)\}$: Throughout this range, $m \pm \Delta$ and $m \pm (\Delta + 1)$ are outside of the range $[0 \ldots (M - 1)]$, so Cases II and III do not affect the test statistic. Therefore, all of the $(M - 1)$ time-slots have Case I.

$m = (M - 2)$: This is the boundary where $m - \Delta = 0$, but $m - \Delta - 1 = -1$, so Case II only affects the first time-slot, and Case I affects the remaining $(M - 2)$ time-slots.

$m = (M - 1)$: Case II now affects the first and second time-slots, since $m - \Delta = 1$ and $m - \Delta - 1 = 0$. Thus, Case II affects two time-slots, and Case I affects the remaining $(M - 3)$ time-slots.
Compared to $\Delta = 1$, from Figures 3.2 and 3.5, we can see that the behavior exhibits a certain symmetry. The first and last possible pulse-positions ($m = 0$ and $m = M - 1$) each have two time-slots affected by either Case II or III. Furthermore, both $\Delta = 1$ and $\Delta = M - 1$ have Cases II or III affecting only one time-slot given pulse-positions $m = 1$ and $m = M - 2$.

However, given $\Delta = 1$, Cases II or III (and possibly both) must affect two time-slots, whereas Cases II and III are mutually exclusive given $\Delta = M - 2$. Given $\Delta = 1$, all of the remaining pulse-positions have Cases II and III affecting two time-slots, whereas given $\Delta = M - 2$, these pulse-positions do not have Cases II or III affecting any of the time-slots.

### 3.2.6 $\Delta = M - 1$

Given $\Delta = M - 1$, Figure 3.6 illustrates how the $M - 1$ conditional pairwise error probabilities (2.40) separate into the three cases, for the following values of the transmitted pulse position, $m$. This is the largest possible value of $\Delta$ that can still produce either Case II or III, since the multipath exceeds the symbol length for any $\Delta \geq M$. 

![Figure 3.6: Pairwise Error Probabilities, given $\Delta = M - 1$ and $m$.](image)
Case II does not affect the test statistic until \( m - \Delta = 0 \), i.e., until \( m = \Delta = M - 1 \).

Thus there is no Case II for the first \( (M - 2) \) possible pulse-positions. Since \( m + \Delta = (M - 1) \) and \( m + \Delta + 1 = M \), Case III only affects the last time-slot, and Case I affects the remaining \( M - 2 \) time-slots.

Throughout this range, \( m \pm \Delta \) and \( m \pm (\Delta + 1) \) are outside of the range \([0 \ldots (M - 1)]\), so Cases II and III do not affect the test statistic. Therefore, all of the \( (M - 1) \) time-slots have Case I.

Case II now only affects the first time-slot, since \( m - \Delta = 0 \), but \( m + \Delta + 1 = -1 \), and thus Case I affects the remaining \( (M - 2) \) time-slots.

### 3.3 Union Bound of the Error Probability, Conditioned on \( \Delta \) and \( m \)

Since the union bound is the sum of the \( M - 1 \) pairwise conditional error probabilities, only the total number of time-slots corresponding to each case needs to be determined. The previous examples have illustrated how, for a given value of \( \Delta \), there are different subsets of possible pulse-positions \( m = [0 \ldots M - 1] \) where the three cases each affect the same number of time-slots.

**Proof of Lemma 3.1.1.** We have observed that: for \( \Delta < \frac{M}{2} \), Cases II or III (and possibly both) must affect at least one of the pairwise error probabilities; for \( \frac{M}{2} \leq \Delta < M \), Cases II and III are mutually exclusive, but must affect at least one of the pairwise error probabilities; for \( \Delta \geq M \), it is not possible for Case II or III to affect any of the pairwise error probabilities.
Therefore, the breakdown of the \( M - 1 \) conditional pairwise error probabilities into Cases I-III can be summarized by creating three distinct regimes for a given value of \( \Delta: \Delta = \left[1 \ldots \frac{M}{2} - 1\right], \Delta = \left[\frac{M}{2} \ldots M - 1\right], \) and \( \Delta = \left[M \ldots \infty\right]. \)

### 3.3.1 Pairwise Error, given \( \Delta \in \left[1 \ldots \frac{M}{2} - 1\right] \)

In the first regime, where the delay is within half the symbol length, the \( (M-1) \) conditional pairwise error probabilities, \( P_e(SIR, M|\Delta, m) \) can be summarized by five subsets of \( m: \)

1. The first \( \Delta \) pulse-positions \( (m = \left[0 \ldots \Delta - 1\right]) \) are not affected by Case II, but Case III affects two time-slots.

2. There is 1 pulse-position \( (m = \Delta) \) where Case II affects only one time-slot, but Case III affects two time-slots.

3. The intermediate \( M - 2\Delta - 2 \) pulse-positions \( (m = \left[\Delta + 1 \ldots M - \Delta - 2\right]) \) have both Cases II and III affecting two time-slots.

4. There is 1 pulse-position \( (m = M - \Delta - 1) \) where Case III affects only one time-slot, but Case II affects two time-slots.

5. The last \( \Delta \) pulse-positions \( (m = \left[M - \Delta \ldots M - 1\right]) \) are not affected by Case III, but Case II affects two time-slots.
Therefore, the union bound of the error probability, conditioned on $\Delta$ and $m$ is

$$P_e(SIR, M | \Delta, m) \leq \begin{cases} 
2P_{III}(SIR) + (M - 3)P_I(SIR) & \text{if } m = [0 \ldots (\Delta - 1)] \\
2P_{III}(SIR) + P_{II}(SIR) + (M - 4)P_I(SIR) & \text{if } m = \Delta \\
2P_{III}(SIR) + 2P_{II}(SIR) + (M - 5)P_I(SIR) & \text{if } m = [(\Delta + 1) \ldots (M - \Delta - 2)] \\
P_{III}(SIR) + 2P_{II}(SIR) + (M - 4)P_I(SIR) & \text{if } m = M - \Delta - 1 \\
2P_{II}(SIR) + (M - 3)P_I(SIR) & \text{if } m = [(M - \Delta) \ldots (M - 1)] 
\end{cases}$$

(3.10)

Note that there is perfect symmetry between Cases II and III, and the sign of the differential delay has no effect on the test statistic in this regime.

### 3.3.2 Pairwise Error, given $\Delta \in [\frac{M}{2} \ldots M - 1]$

In this second regime, the (M-1) conditional pairwise error probabilities are also summarized by five subsets of $m$, but they have a different symmetry since Cases II and III are mutually exclusive:

1. The first $M - \Delta - 1$ ($m = [0 \ldots M - \Delta - 2]$) pulse-positions are not affected by Case II, but Case III affects two time-slots.

2. There is one pulse-position ($m = M - \Delta - 1$) where Case II affects only one time-slot, and Case III does not affect any time-slots.

3. The intermediate $2\Delta - M$ pulse-positions ($m = [\Delta + 1 \ldots M - 1]$) are not affected by Case II or Case III.
4. There is one pulse-position \((m = \Delta)\) where Case III affects only one time-slot, and Case II does not affect any time-slots.

5. The last \(M - \Delta - 1\) \((m = [\Delta + 1 \ldots M - 1])\) pulse-positions are not affected by Case III, but Case II affects two time-slots.

Therefore, the union bound of the error probability, conditioned on \(\Delta\) and \(m\), is

\[
P_e(SIR, M | \Delta, m) \leq \begin{cases} 
2P_{III}(SIR) + (M - 3)P_I(SIR) & \text{if } m = [0 \ldots (M - \Delta - 2)] \\
\begin{align*} 
P_{III}(SIR) + (M - 2)P_I(SIR) & \text{if } m = M - \Delta - 1 \\
(M - 1)P_I(SIR) & \text{if } m = [(M - \Delta) \ldots (\Delta - 1)] \\
\end{align*}
\end{cases} \quad (3.11)
\]

Once again, there is perfect symmetry between Cases II and III, and thus the sign of the differential delay is also irrelevant in this regime.

### 3.3.3 Pairwise Error, given \(\Delta \geq M\)

This regime is trivial, since it guarantees that only Case I can affect the test statistic. Therefore, for any \(m\) and any \(\Delta \geq M\), the pairwise error probabilities, conditioned on \(\Delta\) and \(m\), are all

\[
P_k(SIR | \Delta, m) = P_I(SIR), \quad (3.12)
\]

and the union bound of the error probability, conditioned on \(\Delta\) and \(m\), is

\[
P_e(SIR, M | \Delta, m) \leq (M - 1)P_I(SIR). \quad (3.13)
\]
Since the sign of the differential delay also has no effect in this final regime, we have justified the earlier claim that the sign of the delay does not effect the test statistic.

\[\square\]

### 3.4 Union Bound of the Error Probability, Conditioned on $\Delta$

**Proof of Lemma 3.1.2.** We have derived the union bound of the error probability, conditioned on $\Delta$ and $m$, which was shown to separate into three regimes, depending on $\Delta$. These results are summarized in Lemma 3.1.1. To obtain the union bound, conditioned on $\Delta$, we average the union bound, conditioned on both $\Delta$ and $m$, over the $M$ equiprobable pulse-positions, $m = [0 \ldots M - 1]$. Since all of the $M$ pulse-positions are equiprobable, it does not matter which values of $m$ correspond to each subset, only the number of pulse-positions in each subset.

The results in Lemma 3.1.1 are summarized in the following tables, 3.1-3.3, which relate the number of pulse-positions with the same separation of the $M - 1$ pairwise error probabilities into Cases I-III for each regime of $\Delta$:

In the first regime of $0 < \Delta \leq \frac{M}{2} - 1$, the union bound of the error probability, conditioned on $\Delta$, is

\[
P_e(SIR, M | \Delta) \leq \frac{\Delta}{M} [2P_{II}(SIR) + (M - 3)P_I(SIR)]
\]

\[
+ \frac{\Delta}{M} [2P_{III}(SIR) + (M - 3)P_I(SIR)]
\]

\[
+ \frac{1}{M} [2P_{II}(SIR) + P_{II}(SIR) + (M - 4)P_I(SIR)]
\]

\[
+ \frac{1}{M} [2P_{I}(SIR) + P_{III}(SIR) + (M - 4)P_I(SIR)]
\]

\[
+ \frac{M - 2\Delta - 2}{M} (2P_{II}(SIR) + 2P_{II}(SIR) + (M - 5)P_I(SIR))
\]

\[
= P_I(SIR) \left[ (M - 3) - \frac{2}{M}(M - 1 - 2\Delta) \right]
\]

\[
+ (P_{II}(SIR) + P_{III}(SIR)) \frac{1}{M} [2M - 1 - 2\Delta],
\]

(3.14)
where the last line is obtained by first expressing \((M - 4) = (M - 3) + 1\) and \((M - 5) = (M - 3) + 2\), and then collecting the coefficients for each case.

Table 3.1: Regime 1: \(\Delta = [1 \ldots \frac{M}{2} - 1]\).

<table>
<thead>
<tr>
<th># of pulse-positions</th>
<th>Case I</th>
<th>Case II</th>
<th>Case III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta)</td>
<td>(M-3)</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>(M-4)</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(M - 2\Delta - 2)</td>
<td>(M-5)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>(M-4)</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(\Delta)</td>
<td>(M-3)</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.2: Regime 2: \(\Delta = [\frac{M}{2} \ldots M - 1]\).

<table>
<thead>
<tr>
<th># of pulse-positions</th>
<th>Case I</th>
<th>Case II</th>
<th>Case III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M - \Delta - 1)</td>
<td>(M-3)</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>(M-2)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(2\Delta - M)</td>
<td>(M-1)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>(M-2)</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(M - \Delta - 1)</td>
<td>(M-3)</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.3: Regime 3: \(\Delta \geq M\).

<table>
<thead>
<tr>
<th># of pulse-positions</th>
<th>Case I</th>
<th>Case II</th>
<th>Case III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M)</td>
<td>(M-1)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
In the second regime of $\frac{M}{2} \leq \Delta < M$, the union bound of the error probability, conditioned on $\Delta$, is

\[
P_e(SIR, M | \Delta) \leq \frac{M - 1 - \Delta}{M} [2P_{II}(SIR) + (M - 3)P_I(SIR)] + \frac{1}{M} [P_{III}(SIR) + (M - 2)P_I(SIR)] + \frac{2\Delta - M}{M} (M - 1)P_I(SIR)
\]

\[
= P_I(SIR) \left[ (M - 3) - \frac{2}{M} (M - 1 - 2\Delta) \right] + (P_{II}(SIR) + P_{III}(SIR)) \frac{1}{M} [2M - 1 - 2\Delta],
\]

where the last line is obtained by first expressing $(M - 1) = (M - 3) + 2$ and $(M - 2) = (M - 3) + 1$, and then collecting the coefficients for each case. This is the same result as (3.14) in the first regime of $0 < \Delta < \frac{M}{2}$. Note that the sign of $(M - 1 - 2\Delta)$ is positive for $\Delta < \frac{M}{2}$, and negative for $\Delta \geq \frac{M}{2}$, while the sign of $(2M - 2\Delta - 1)$ is always positive for $\Delta < M$.

In the third regime of $\Delta \geq M$, the union bound of the error probability, conditioned on $\Delta$, is

\[
P_e(SIR, M | \Delta) \leq (M - 1)P_I(SIR).
\]

Note that this result can be obtained by replacing $(M - 1 - 2\Delta)$ with $-M$ and $(2M - 2\Delta - 1)$ with 0 in (3.14). Since $(M - 1 - 2\Delta) = -M$ if $\Delta = M - \frac{1}{2}$ and $(2M - 2\Delta - 1) = 0$ if $\Delta = M - \frac{1}{2}$, then (3.16), which applies in the regime of $\Delta > M - 1$, is obtained from (3.14), which applies for $\Delta \leq M - 1$, by substituting $\Delta = M - \frac{1}{2}$ whenever $\Delta > M - 1$.

Therefore, we can generalize (3.14) to apply for all three regimes by incorporating a non-linear threshold function, such as $\min(x, y)$, to replace $\Delta$ with $M - \frac{1}{2}$ whenever $\Delta > M - 1$. 
That is,

\[ P_e(SIR, M | \Delta) = P_I(SIR) \left[ (M - 3) - \frac{2}{M} \left( M - 1 - 2 \min(\Delta, M - \frac{1}{2}) \right) \right] \]

\[ + (P_{II}(SIR) + P_{III}(SIR)) \frac{1}{M} \left[ 2M - 1 - 2 \min(\Delta, M - \frac{1}{2}) \right] \]

\[ = P_I(SIR) \left[ (M - 5) + 4 \left( \frac{\min(\Delta, M - \frac{1}{2}) + \frac{1}{2}}{M} \right) \right] \]

\[ + 2 (P_{II}(SIR) + P_{III}(SIR)) \left[ 1 - \left( \frac{\min(\Delta, M - \frac{1}{2}) + \frac{1}{2}}{M} \right) \right] \]  

(3.17)

Defining

\[ P_{int}(M | \Delta) \equiv \left[ 1 - \left( \frac{\min(\Delta, M - \frac{1}{2}) + \frac{1}{2}}{M} \right) \right] \]  

(3.18)

as the probability that the multipath interferes with the test statistic, conditioned on \( \Delta \), we obtain

the desired result:

\[ P_e(SIR, M | \Delta) = \]  

(3.19)

\[ P_I(SIR) \left[ (M - 5) + 4 (1 - P_{int}(M | \Delta)) \right] + 2 (P_{II}(SIR) + P_{III}(SIR)) P_{int}(M | \Delta). \]

To obtain the total error probability, we must uncondition the probability of interference over the distribution of \( \Delta \).

\[ \square \]

### 3.5 Total Error Probability

**Proof of Theorem 3.1.3.** We first prove the result for \( M \geq 4 \), and then treat the case of \( M = 2 \) separately.

#### 3.5.1 \( M \)-ary Case: \( M \geq 4 \)

For \( M \geq 4 \), the union bound of the total error probability is

\[ P_e(SIR, M) = P_I(SIR) \left[ (M - 5) + 4P_{int}(M) \right] + 2 (P_{II}(SIR) + P_{III}(SIR)) P_{int}, \]  

(3.20)
where \( P_{\text{int}}(M) \equiv 1 - P_{\text{int}}(M) \), and \( P_{\text{int}}(M) \) is the total interference probability, which must be obtained by unconditioning \( P_{\text{int}}(M|\Delta) \) over the distribution of the quantized delay, \( \Delta \). From Lemma 3.1.2, this is

\[
P_{\text{int}}(M) \equiv \sum_{\Delta=1}^{\infty} P_{\text{int}}(M|\Delta) \mathbb{P}_r \{ \Delta = \Delta \}
\]

\[
= \sum_{\Delta=1}^{\infty} \left[ 1 - \left( \frac{\min(\Delta, M - \frac{1}{2}) + \frac{1}{2}}{M} \right) \right] \mathbb{P}_r \{ \Delta T_{ps} \leq \tau_d < (\Delta + 1)T_{ps} \} \quad (3.21)
\]

\[
= \sum_{\Delta=1}^{M-1} \left( \frac{\Delta + \frac{1}{2}}{M} \right) \left[ F_{\tau_d} \left( (\Delta + 1)T_{ps} \right) - F_{\tau_d} \left( \Delta T_{ps} \right) \right].
\]

We now develop a series of approximations that are valid for small \( T_{ps} \). The above result is exact, but must be calculated numerically. The approximation developed in the remainder of this section yields a simple analytical solution for some distributions, such as the Maxwellian distribution.

The quantity \( [ F_{\tau_d} \left( (\Delta + 1)T_{ps} \right) - F_{\tau_d} \left( \Delta T_{ps} \right) ] \) is the area obtained by integrating the pdf of the differential delay, \( \tau_d \), over the interval \( (\Delta T_{ps}, (\Delta + 1)T_{ps}) \). Recall that the uniform remainder condition states that \( T_{ps} \) is small such that the pdf of \( \tau_d \) is approximately constant throughout this interval. Therefore, this area is approximately equal to the width of the interval, \( T_{ps} \), times the value of the pdf at either of the endpoints of the interval, which are \( f_{\tau_d} (\Delta T_{ps}) \) and \( f_{\tau_d} ((\Delta + 1)T_{ps}) \). Thus,

\[
P_{\text{int}}(M) \approx \sum_{\Delta=1}^{M-1} \left( 1 - \frac{\Delta + \frac{1}{2}}{M} \right) \left[ T_{ps} f_{\tau_d} (\Delta T_{ps}) \right]
\]

\[
\approx \int_0^M \left( 1 - \frac{1}{2M} - \frac{t}{M} \right) T_{ps} f_{\tau_d} (t T_{ps}) \, dt
\]

\[
= \int_0^{MT_{ps}} \left( 1 - \frac{1}{2M} - \frac{t}{MT_{ps}} \right) f_{\tau_d} (t) \, dt
\]

\[
= \left( 1 - \frac{1}{2M} \right) F_{\tau_d} (MT_{ps}) - \frac{1}{MT_{ps}} \int_0^{MT_{ps}} t f_{\tau_d} (t) \, dt.
\]

The second approximation, where the sum is replaced the integral, is exact in the limit of
$T_{ps} \to 0$, and is a reasonable approximation when $T_{ps}$ is small, and the differences between the continuous quantized differential delay distributions can be neglected. This is reasonable whenever the uniform remainder condition is satisfied. The limits of integration extend from $T_{ps}$ to $MT_{ps}$, which is the interval of $\tau_d$ corresponding to $\Delta = [1 \ldots M - 1]$. However, $F_{\tau_d}(T_{ps}) \approx 0$ from the sufficient delay condition, and there is effectively zero area from 0 to $T_{ps}$, so we can approximate the limits of integration as from 0 to $MT_{ps}$.

### 3.5.2 Binary Case: $M = 2$

For the binary case of $M = 2$, it is straightforward to obtain the total error probability. Since there is only one pulse-position where $k \neq m$, Cases II and III can only affect the test statistic given $\Delta = 1$, so $P_{\text{int}}$ is simply

$$P_{\text{int}}(M = 2) = Pr\{\Delta = 1\}$$

$$= [F_{\tau_d}(2T_{ps}) - F_{\tau_d}(T_{ps})]$$

$$\approx T_{ps}f_{\tau_d}(T_{ps}).$$

(3.23)

Given that Case II or Case III applies, Case III applies if the sign of the differential delay is positive, while Case II applies if it is negative. Since the sign is positive or negative with equal probability, the total error probability is

$$P_e(SIR, M = 2) = P_{\text{int}}(M = 2)P_I(SIR) + \frac{1}{2}P_{\text{int}}(M = 2) [P_{II}(SIR) + P_{III}(SIR)]$$

(3.24)
Chapter 4

Independent, Identically Distributed Limits

It has been shown that the intensity samples are expressed as sums conditionally correlated, non-central chi-square variates with 2 degrees of freedom (2.17). In certain limiting cases, it will be shown that the three cases of the pairwise error probability can be expressed as the pairwise error between independent, identically distributed (i.i.d.) chi-square variates, with generally unequal degrees of freedom. In this chapter, we summarize some key properties of independent chi-square variates, and then define a general i.i.d. pairwise error probability.

Then, in Section 4.3, we present a known result for the probability that a non-central chi-square variate with $2L_2$ degrees of freedom exceeds an independent non-central chi-square variate with $2L_1$ degrees of freedom. However, this known result yields an indeterminate result when either of the non-centrality parameters is zero (i.e. central chi-square). In order to accommodate this important limiting case, we derive a closed-form expression in Section 4.4 for the probability that a central chi-square variate with $2L_2$ degrees of freedom exceeds a non-central
chi-square variate with $2L_1$ degrees of freedom.

4.1 Chi-square Distributions

In this section, we summarize some of the key properties of chi-square distributions.

4.1.1 Chi-square Variate with $2L$ Degrees of Freedom

Let $N_i, i \in [1 \ldots L]$ denote $L$ arbitrary samples of the interference $\eta(t)$, which are zero-mean, circularly symmetric complex Gaussian with correlation coefficients $\rho_{i,j}$ and covariance matrix $C$. If $C = I 2\sigma^2$, then the $N_i$ are mutually independent and the random variable $\| a + \sum_{i=1}^{L} N_i \|^2$ has a non-central chi-squared distribution with $2L$ degrees of freedom, non-centrality parameter $a^2$ and component noise variance $\sigma^2$.

From [23, Eqs. (2.44), (2.45)], the pdf and cdf for the random variable $\chi^2(L, a^2, \sigma^2)$ are

$$f_{\chi^2(L, a^2, \sigma^2)}(z) = \frac{1}{2\sigma^2} \left(\frac{z}{a^2}\right)^{\frac{L-1}{2}} \exp\left(-\frac{z + a^2}{2\sigma^2}\right) I_{L-1}\left(\sqrt{\frac{a^2z}{\sigma^4}}\right), \quad z \geq 0,$$

and

$$F_{\chi^2(L, a^2, \sigma^2)}(z) = 1 - Q_L\left(\sqrt{\frac{a^2}{\sigma^2}}, \sqrt{\frac{z}{\sigma^2}}\right), \quad z \geq 0,$$

where $I_n(z)$ is the modified Bessel function of the first kind, and $Q_L(A, B)$ is the generalized Marcum-Q function, defined as [24, Eq. 4.33]

$$Q_L(A, B) = \frac{1}{AL^{-1}} \int_{B}^{\infty} x^L \exp\left(-\frac{x^2 + A^2}{2}\right) I_{L-1}(Ax) \, dx. \quad (4.1)$$

With $a^2 = 0$, we obtain a central chi-square distribution, and the pdf and cdf simplify to [23, Eqs. (2.32), (2.33)]

$$f_{\chi^2(L, 0, \sigma^2)}(z) = \frac{1}{2\sigma^2(L-1)!} \left(\frac{z}{2\sigma^2}\right)^{\frac{L-1}{2}} \exp\left(-\frac{z}{2\sigma^2}\right), \quad z \geq 0 \quad (4.2)$$
and

\[ F_{\chi^2(L,0,\sigma^2)}(z) = 1 - \exp \left( -\frac{z}{2\sigma^2} \right) \sum_{\ell=0}^{L-1} \frac{1}{\ell!} \left( \frac{z}{2\sigma^2} \right)^\ell, \quad z \geq 0. \] (4.3)

### 4.1.2 Scaled Chi-Square Variates

Let \( X = A\chi^2(L, a^2, \sigma^2) \). Then the non-centrality and variance parameters change directly with the scaling factor. That is,

\[
X = \left\| a_1 + \sum_{i=1}^{L} N_i \right\|^2 = \left\| \sqrt{A} a_1 + \sum_{i=1}^{L} \sqrt{A} N_i \right\|^2 = \chi^2(L, Aa^2, A\sigma^2) \equiv \chi^2(L, a^2_1, \sigma^2_1),
\] (4.4)

with \( a^2_1 \equiv Aa^2 \) and \( \sigma^2_1 \equiv A\sigma^2 \) defined as scaled non-centrality and variance parameters.

### 4.1.3 Sum of Two Independent Chi-Square Variates

Let \( X_1 = \chi^2(L_1, a^2_1, \sigma^2_1) \) and \( X_2 = \chi^2(L_2, a^2_2, \sigma^2_2) \) denote independent noncentral chi-square variates. From [23, Eq. 5.63], the pdf of the sum \( X = X_1 + X_2 \) is represented as a doubly-infinite sum of modified Bessel functions of the first kind,

\[
f_{X_1+X_2}(z) = \frac{1}{2\sigma^2_1} \left( \frac{\sigma^2_1}{\sigma^2_2} \right)^{L_2} \left( \frac{z}{a_1^2} \right)^{L_1+L_2-1} \exp \left( -\frac{z}{2\sigma^2_1} \right) \exp \left[ -\frac{1}{2} \left( \frac{a^2_1}{\sigma^2_1} + \frac{a^2_2}{\sigma^2_2} \right) \right] \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(L_2 + m + n)}{m!n!\Gamma(L_2 + n)} \left( \sqrt{\frac{z a^2_2 \sigma^2_1}{2a_1^2 \sigma^2_2}} \right)^n \left( \sqrt{\frac{\sigma^2_2 - \sigma^2_1}{2a_1^2 \sigma^2_2}} \right)^m I_{L_1+L_2-1+m+n} \left( \sqrt{\frac{z a^2_2}{\sigma^2_1}} \right),
\] (4.8)

for \( z \geq 0 \), where \( \Gamma(z) \) is the complete Gamma function. There are no restrictions on \( \sigma^2_1 \) and \( \sigma^2_2 \), but when \( \sigma^2_1 > \sigma^2_2 \) the difference \( \sigma^2_2 - \sigma^2_1 \) is negative, forming an alternating series in \( m \).
4.1.4 Sum of Two Independent, Identically Distributed Chi-Square Variates

We now assume \( \sigma_1^2 = \sigma_2^2 = \sigma^2 \), such that \( X_1 \) and \( X_2 \) are identically distributed in the sense that the component noise variance in every degree of freedom equals \( \sigma^2 \). It is proven in Appendix A that, under this condition, the doubly-infinite sum (4.8) collapses, and \( X = X_1 + X_2 \) is a chi-square variate with \( 2(L_1 + L_2) \) degrees of freedom, component noise variance \( \sigma^2 \) and non-centrality parameter \( a_1^2 + a_2^2 \). That is,

\[
X_1 + X_2 = \chi^2 (L_1, a_1^2, \sigma^2) + \chi^2 (L_2, a_2^2, \sigma^2)
\]

or equivalently, in terms of \( L_1 + L_2 \) independent interference samples,

\[
X_1 + X_2 = \left[ a_1 + \sum_{i=1}^{L_1} N_i \right]^2 + \left[ a_2 + \sum_{i=1}^{L_2} N_{L_1+i} \right]^2
\]

\[
\rightarrow X = \left[ \sqrt{a_1^2 + a_2^2 + \sum_{i=1}^{L_1+L_2} N_i} \right]^2.
\]

(4.10)

In general, the non-centrality parameters can be zero to include the limiting case of central chi-square variates.

4.1.5 Sum of \( L \) Independent, Identically Distributed Chi-Square Variates

Now we consider the sum of \( L \) independent, identically interference samples with two degrees of freedom. The mutual independence makes the sum of squares and the square of sums equivalent, so that, by definition, \( X \) is a chi-square variate with \( 2L \) degrees of freedom,

\[
X = \sum_{i=1}^{L} \|a_i + N_i\|^2 = \left[ a + \sum_{i=1}^{L} N_i \right]^2
\]

\[
= \sum_{i=1}^{L} \chi^2 (1, a_i^2, \sigma^2) = \chi^2 (L, a^2, \sigma^2),
\]

(4.11)
with a non-centrality parameter \( a^2 \) defined as the sum of the non-centrality parameters, \( a^2 \equiv \sum_{i=1}^{L} a_i^2 \) [23, 1.14].

### 4.2 Definition of the I.I.D. Pairwise Error Probability

In this section, we define a notation for an independent, identically distributed (i.i.d.) pairwise error probability, i.e., the probability that a non-central chi-square variate \( X \) with \( 2L_1 \) degrees of freedom exceeds an independent non-central chi-square variate \( Y \) with \( 2L_1 \) degrees of freedom:

\[
P_{\text{id}} (L_1, L_2; A, B; \mu_1^2, \mu_2^2) \equiv P_r \{ X < Y \} = P_r \{ X - Y < 0 \} = P_r \{ D < 0 \}.
\]

\( X \) and \( Y \) are defined as sums of squares of \( L \equiv L_1 + L_2 \) independent, identically distributed (i.i.d.) zero-mean, circularly symmetric, complex Gaussian interference samples, \( N_i, i \in [1 \ldots L] \), plus the non-centrality parameters, \( a_k (k \in [1 \ldots L_1]) \) and \( b_k, (k \in [1 \ldots L_2]) \):

\[
X - Y = A \sum_{k=1}^{L_1} \| a_k + N_k \|^2 - B \sum_{k=1}^{L_2} \| b_k + N_{L_1+k} \|^2 \tag{4.12}
\]

\[
= A \sum_{k=1}^{L_1} \chi^2 (a_k^2, \sigma^2) - B \sum_{k=1}^{L_2} \chi^2 (b_k^2, \sigma^2) \tag{4.13}
\]

\[
= A \left\| a + \sum_{k=1}^{L_1} N_k \right\|^2 - B \left\| b + \sum_{k=1}^{L_2} N_{L_1+k} \right\|^2 \tag{4.14}
\]

\[
= A \chi^2 (L_1, a^2, \sigma^2) - B \chi^2 (L_2, b^2, \sigma^2). \tag{4.15}
\]
Thus, the non-centrality parameters for $X$ and $Y$ are defined as

$$a^2 \equiv \sum_{k=0}^{L_1} a_k^2 \equiv \mu_1^2 2\sigma^2$$

$$b^2 \equiv \sum_{k=0}^{L_2} b_k^2 \equiv \mu_2^2 2\sigma^2,$$  

and are related to the SIR parameters for $X$ and $Y$ by

$$\mu_1^2 = \frac{a^2}{2\sigma^2}$$

$$\mu_2^2 = \frac{b^2}{2\sigma^2}. $$

(4.16)

The equivalence between the sum of the squares (4.12), (4.13), and the square of the sums (4.14), (4.15) is due to the fact that the $L$ interference samples are all independent with equal variance in the real and imaginary parts (4.11). Furthermore, since the variance per degree of freedom is defined as $\sigma^2$ the non-centrality parameters $a^2$ and $b^2$ can be obtained implicitly from the SIR parameters $\mu_1$ and $\mu_2$ (4.16), which are provided as arguments.

Therefore, the statistics of $X$ and $Y$ are completely determined by the degrees of freedom $2L_1$ and $2L_2$, the weights $A$ and $B$, the non-centrality parameters $a^2$ and $b^2$, and the noise variance per degree of freedom. It will be shown that the error probability is then determined by the degrees of freedom $2L_1$ and $2L_2$, the weights $A$ and $B$, and the SIRs $\mu_1^2$ and $\mu_2^2$.

This is the same quadratic form considered in [25, Eq. 1], which differs from the classic quadratic form investigated in [20, Appendix B], [24, Appendix 9A] and in [25] by extending the results to a more general case where the degrees of freedom, $L_1$ and $L_2$, can differ (subject, however, to the more restrictive condition that the underlying Gaussian distributions are mutually i.i.d.). With $L_1 = L_2 = L$, the classic result [20, Eq. B-21] is obtained. With $L_1 = L_2 = L$ and $A = B$, a particularly succinct result in terms of the generalized Marcum-Q function is derived in [24, Appendix 9A]. All of these derivations are obtained using contour integration, and it is
noted in [24] that this approach can only yield a closed-form if the variances of the $X_k$ and $Y_k$ are independent of $k$.

For $\mu_2^2 = 0$, when $Y$ is a central chi-square variate with $2L_2$ degrees of freedom (DOF), the error probabilities in [25, Eq. 1],[20, Eq. B-21] result in indeterminate forms. The restrictive case of $L_1 = L_2 = L$ and $A = B$ has been analyzed for $L$ channel noncoherent equal gain combining of binary signals with antipodal (e.g. Differential Phase Shift Keying (DPSK)) or orthogonal constellations (e.g. Frequency Shift Keying (FSK)) in [20, Sec. 12.1.1] and [3]. A summary of these results is presented in [24, Sec. 9.4.1.2], but the equivalence between the two expressions is not explicitly shown or referenced.

In Section 4.4, we derive a general error probability when $\mu_2^2 = 0$, without restrictions on $L_1, L_2, A$ or $B$. This result also applies for $\mu_1^2 = 0$ without taking a limit. The proof is based upon direct integration of the pdf of $D = X - Y$, rather than a contour integration of the characteristic function, as in [20]. The result is shown to reduce to [3, Eq. 32] under the same restrictive conditions when $L_1 = L_2 = L$ and $A = B$. 
4.3 Non-central $\chi^2 < \text{Non-central } \chi^2$

In the general case when both $\mu_1^2 > 0$ and $\mu_2^2 > 0$ and both $X$ and $Y$ are non-central chi-square, the i.i.d. error probability is [25, Eq. 4-7]

$$P_{id} (L_1, L_2; A, B; \mu_1^2, \mu_2^2) =$$

$$1 - Q_1(a, b) + \exp \left( -\frac{\nu_1^2 + \nu_2^2}{2} \right) I_0 (\nu_1^2, \nu_2^2)$$

$$- \exp \left( -\frac{\nu_1^2 + \nu_2^2}{2} \right) I_0 (\nu_1^2, \nu_2^2) (1 + \eta)^{- (L_1 + L_2 - 1)} \sum_{k=0}^{L_1-1} (L_1 + L_2 - 1) \eta^k$$

$$+ \exp \left( -\frac{\nu_1^2 + \nu_2^2}{2} \right) (1 + \eta)^{- (L_1 + L_2 - 1)}$$

$$\times \left[ \sum_{n=1}^{L_2-1} I_n (\nu_1^2, \nu_2^2) \sum_{k=0}^{L_1-1-n} (L_1 + L_2 - 1) \eta^{L_1 + L_2 - 1 - k} \left( \frac{\nu_1}{\nu_2} \right)^n \right]$$

where

$$\nu_1^2 = \frac{A}{A + B} \sum_{k=1}^{L_1} \frac{a_k^2}{\sigma^2} = \frac{2A}{A + B} \mu_1^2$$

$$\nu_2^2 = \frac{A}{A + B} \sum_{k=1}^{L_2} \frac{b_k^2}{\sigma^2} = \frac{2B}{A + B} \mu_2^2$$

$$\eta = B \frac{A}{A + B}$$

Substituting $L_1 = L_2 = L$ into this expression immediately yields the classic result of Proakis [20, App. B] that he originally derived in [19]. However, note that the above parameter definitions are only equivalent to those of [20] under the restriction of a zero cross-term coefficient and zero cross-correlation (e.g., $C = 0$ and $\| \mu_{xy} \|^2 = \| \mu_{yx} \|^2 = 0$ in the notation of [20, App. B]).

Simon has also shown [24, App. 9A] that imposing the restriction of $A = B$ in the Proakis
result yields an expression in terms of the generalized Marcum-Q function [24, Eq 9.103],

\[ P_{\text{iid}}(L, L; A, A; \mu_1^2, \mu_2^2) \equiv P_{\text{iid}}(L; \mu_1^2, \mu_2^2) \]
\[ = \frac{1}{2} + \frac{1}{2^{2L-1}} \sum_{\ell=1}^{L} \left( \frac{2L-1}{L-\ell} \right) \left[ Q_\ell(\mu_1, \mu_2) - Q_\ell(\mu_2, \mu_1) \right], \quad (4.19) \]

### 4.3.1 Case III I.I.D. Limit

It will be shown that for Case III, in the limits of \( \rho = 0 \) and \( \alpha = \frac{1}{2} \), the error probability reduces to an i.i.d. form with \( L_1 = L_2 = 2, A = \alpha = B = (1 - \alpha) = \frac{1}{2} \) and \( \mu_1^2 = SIR, \mu_2^2 = \gamma^2 SIR \). Applying the above result (4.19), this is

\[ \lim_{\rho \to 0, \alpha \to \frac{1}{2}} P_{\text{III}}(SIR|\alpha, \rho, \gamma) \to P_{\text{iid}}(2; SIR, \gamma^2 SIR) \]
\[ = \frac{1}{2} + \frac{3}{8} \left[ Q_1(\sqrt{SIR}, \gamma \sqrt{SIR}) - Q_1(\gamma \sqrt{SIR}, \sqrt{SIR}) \right] \]
\[ + \frac{1}{8} \left[ Q_2(\sqrt{SIR}, \gamma \sqrt{SIR}) - Q_2(\gamma \sqrt{SIR}, \sqrt{SIR}) \right]. \quad (4.20) \]

### 4.4 Non-central \( \chi^2 < \) Central \( \chi^2 \)

**Theorem 4.4.1.** For the case of \( \mu_2^2 = 0 \), so that \( Y \) reduces to a central chi-square variate, the i.i.d. error probability, \( P_{\text{iid}}(L_1, L_2, A, B, \mu_1^2, \mu_2^2) \), which shall be denoted as \( P_{\text{iid}}(L_1, L_2; A, B; \mu_1^2) \), is given by

\[ P_{\text{iid}}(L_1, L_2; A, B; \mu_1^2) = \]
\[ \left( \frac{B}{A + B} \right)^{L_1} \exp \left( -\frac{A}{A + B} \mu_1^2 \right) \sum_{k=0}^{L_2-1} \left( \frac{A}{A + B} \right)^{\ell} \mathcal{L}^{(L_1-1)}_{\ell} \left( -\frac{B}{A + B} \mu_1^2 \right), \quad (4.21) \]

where

\[ \mathcal{L}^{(N)}_{\ell}(-x) \equiv \sum_{k=0}^{\ell} \binom{\ell + N}{\ell - k} \frac{x^k}{k!} \quad (4.22) \]
is the generalized Laguerre polynomial [23, A.24].

**Corollary 4.4.2.** With $A = \alpha$, $B = 1 - \alpha$, and $\mu_1^2 = \text{SIR}$, the i.i.d. error probability simplifies to

$$P_{\text{iid}}(L_1, L_2; \alpha, (1 - \alpha); \text{SIR}) = (1 - \alpha)^{L_1} \exp \left( -\alpha \text{SIR} \right) \sum_{\ell=0}^{L_2-1} \alpha^{\ell} \mathcal{L}_{\ell}^{(L_1-1)} \left( -[1 - \alpha] \text{SIR} \right)$$

which shall be denoted as $P_{\text{iid}}(L_1, L_2; \alpha; \text{SIR})$.

**Corollary 4.4.3.** With $\mu_1^2 = \text{SIR}$ and $B = A$, we have $\frac{A}{A+B} = \frac{B}{A+B} = \frac{1}{2}$, so the error probability is independent of $A$, and simplifies to

$$P_{\text{ iid}}(L_1, L_2; A, A; \text{SIR}) = \left( \frac{1}{2} \right)^{L_1} \exp \left( -\frac{1}{2} \text{SIR} \right) \sum_{\ell=0}^{L_2-1} \left( \frac{1}{2} \right)^{\ell} \mathcal{L}_{\ell}^{(L_1-1)} \left( -\frac{\text{SIR}}{2} \right)$$

which shall be denoted as $P_{\text{ iid}}(L_1, L_2; \text{SIR})$.

**Proof of Theorem 4.4.1.** Let $D$ denote the difference $D = X - Y$, such that the error probability is

$$P_{\text{ iid}}(L_1, L_2; A, B; \mu_1^2) = \Pr \{ D < 0 \} = F_D(0) = \int_{-\infty}^{0} f_D(z) \, dz,$$

where $f_D(z)$ and $F_D(z)$ denote the pdf and cdf of $D$, respectively. From (4.15), $X$ and $Y$ are

$$D = X - Y = A \chi^2(L_1, a^2, \sigma^2) - B \chi^2(L_2, b^2, \sigma^2)$$

$$= \chi^2(L_1, Aa^2, A\sigma^2) - \chi^2(L_2, 0, B\sigma^2)$$

$$\equiv \chi^2(L_1, \nu^2, \sigma_1^2) - \chi^2(L_2, 0, \sigma_2^2),$$

which is the difference of an independent non-central chi-square variate with $2L_1$ degrees of freedom, non-centrality parameter $\nu^2 \equiv Aa^2$ and variance $\sigma_1^2 \equiv A\sigma^2$, and a central chi-square
where the definitions \(4.26\) of \(\sigma^2\) functions, therefore, the error probability can be expressed as a finite sum of confluent hypergeometric function \([7, Eq. 3.351.3]\), and reduces to a factorial since the argument is an integer, where \(1F_1 (a; b; x)\) is the confluent hypergeometric function (also known as Kummer's hypergeometric function).

Reference [23] notes that \(F_D(z)\) has a closed-form expression only for \(L_2 = 1\). For the error probability, however, we are only interested in \(F_D(0)\):

\[
F_D(0) = \left( \frac{B}{A + B} \right)^{L_1} \exp (- \mu_1^2) \sum_{\ell=0}^{L_2-1} \frac{(L_1 + \ell - 1)!}{\ell!(L_2 - \ell - 1)!(L_1 - 1)!} \left( \frac{A}{A + B} \right)^\ell \times 1F_1 \left( L_1 + \ell; L_1; \frac{\nu^2 \sigma^2_2}{2 \sigma^2_1 \sigma^2_2} \right),
\]

where the definitions \(4.26\) of \(\sigma_1^2 = A \sigma^2, \sigma_2^2 = B \sigma^2\) and \(\frac{\nu^2 \sigma^2_2}{2 \sigma^2_1} = \mu_1^2\) have been used.

Making the substitution \(u = -\frac{z}{2 \sigma^2_2}, dz = -2 \sigma^2_2 \, du\), the integral is the complete gamma function \([7, Eq. 3.351.3]\), and reduces to a factorial since the argument is an integer,

\[
\frac{1}{2 \sigma^2_2} \int_{-\infty}^{0} \left( -\frac{z}{2 \sigma^2_2} \right)^{L_2-\ell-1} \exp \left( \frac{z}{2 \sigma^2_2} \right) \, dz = \int_{0}^{\infty} u^{L_2-\ell-1} \exp (-u) \, du = \Gamma (L_2 - \ell) = (L_2 - \ell - 1)!
\]

Therefore, the error probability can be expressed as a finite sum of confluent hypergeometric functions,

\[
P_{\text{ld}}(L_1, L_2; A, B; \mu_1^2, 0) = \left( \frac{B}{A + B} \right)^{L_1} \exp (- \mu_1^2) \times \sum_{\ell=0}^{L_2-1} \frac{(L_1 + \ell - 1)!}{\ell!(L_1 - 1)!} \left( \frac{A}{A + B} \right)^\ell 1F_1 \left( L_1 + \ell; L_1; \left( \frac{B}{A + B} \right) \mu_1^2 \right).
\]

variates with \(2L_2\) degrees of freedom and variance \(\sigma_2^2 \equiv B \sigma^2\). From [23, Eq. 4.38], for \(z < 0\) the pdf of \(D\) is

\[
f_D(z) = \frac{1}{2 \sigma^2_2} \left( \frac{\sigma^2_2}{\sigma^2_1 + \sigma^2_2} \right)^{L_1} \exp \left( \frac{z}{2 \sigma^2_2} \right) \exp (- \nu^2) \sum_{\ell=0}^{L_2-1} \frac{(L_1 + \ell - 1)!}{\ell!(L_2 - \ell - 1)!(L_1 - 1)!} \left( \frac{B}{A + B} \right)^\ell \times \left( -\frac{z}{2 \sigma^2_2} \right)^{L_2-\ell-1} \left( \frac{\sigma^2_1}{\sigma^2_1 + \sigma^2_2} \right)^\ell 1F_1 \left( L_1 + \ell; L_1; \frac{\nu^2 \sigma^2_2}{2 \sigma^2_1 \sigma^2_2} \right),
\]

(4.27)
We note that under the restrictions of $L_1 = L_2 = L$ and $A = B$, as in [3], we obtain

\[
P_{\text{iid}}(L, L; A, A; \mu_1^2) = \left(\frac{1}{2}\right)^L \exp\left(-\mu_1^2\right) \sum_{\ell=0}^{L-1} \frac{(L + \ell - 1)!}{\ell!(L-1)!} \left(\frac{1}{2}\right)^\ell 1F_1\left(L + \ell; \mu_1^2\right)
\]

\[
= \exp\left(-\mu_1^2\right) \frac{L-1}{2^L \Gamma(L)} \sum_{\ell=0}^{L-1} \frac{\Gamma(L + \ell)}{2^\ell \Gamma(\ell + 1)} 1F_1\left(L + \ell; \frac{\mu_1^2}{2}\right)
\]

which agrees with [3, Eq. 32],[24, Eq. 9.105] for orthogonal signals.

It is proven in Appendix B that for integers $m > 0$ and $\ell \geq 0,$

\[
1F_1(m + \ell; m; x) = \exp(x) \sum_{k=0}^{\ell} \binom{\ell}{k} \frac{x^k}{(m)_k},
\]

where $(m)_k \equiv \frac{(m + k - 1)!}{(m-1)!}$ is the Pochhammer symbol and $\binom{\ell}{k} \equiv \frac{\ell!}{k!(\ell-k)!}$ is the binomial coefficient. After applying this identity to (4.30), by using (4.22) the error probability can then be simplified to an exponential times a sum of $L_2$ Laguerre polynomials in the SIR that have a maximum degree of $L_2$,

\[
P_{\text{iid}}(L_1, L_2; A, B; \mu_1^2)
\]

\[
= \left(\frac{B}{A+B}\right)^{L_1} \exp\left(-\frac{A \mu_1^2}{A+B}\right) \sum_{\ell=0}^{L_2-1} \frac{A}{A+B} \sum_{k=0}^{\ell} \frac{(L_1 + \ell - 1)!}{\ell!(\ell-k)!} \left[\frac{B}{A+B}\right]^k
\]

\[
= \left(\frac{B}{A+B}\right)^{L_1} \exp\left(-\frac{A \mu_1^2}{A+B}\right) \sum_{\ell=0}^{L_2-1} \frac{A}{A+B} \sum_{k=0}^{\ell} \frac{(L_1 + L_2 - 1)!}{\ell!(\ell-k)!} \frac{1}{k!} \left[\frac{B}{A+B}\right]^k
\]

\[
= \left(\frac{B}{A+B}\right)^{L_1} \exp\left(-\frac{A \mu_1^2}{A+B}\right) \sum_{\ell=0}^{L_2-1} \frac{A}{A+B} \sum_{k=0}^{\ell} \binom{L_1-1}{\ell} \left[-\frac{B}{A+B}\right]^k,
\]

which is the desired result (4.21).

\[\square\]
4.4.1 Unfaded Error Probability

With $\alpha = 1$, the secondary path is eliminated, and the error probability for any of the three cases is

$$P_r \left\{ \| \sqrt{P_o + N_1} \|^2 < \| N_3 \|^2 \right\}.$$  \hfill (4.34)

Note that $\rho_{1,3} = 0$ is guaranteed for all three cases due to the sufficient delay and sufficient spacing conditions. These two variates are i.i.d. with $L_1 = L_2 = 1$, $A = B = 1$ and $\mu_1^2 = SIR$. Therefore, from Corollary 4.4.3 the error probability is

$$P_{\text{id}} (1, 1; SIR) = \frac{1}{2} \exp \left( -\frac{1}{2} SIR \right),$$  \hfill (4.35)

which we shall denote as $P_{\text{UF}} (SIR)$, the unfaded error probability. This is the standard error rate for the non-coherent detection of binary orthogonal signals (e.g., Binary FSK) in the Additive White Gaussian Noise (AWGN) channel [20, Eq. 5.4-48].

4.4.2 Case I Upper Bound

With $\alpha = 1/2$, the Case I error probability is

$$P_I (SIR | \alpha = 1/2) = P_r \left\{ \frac{1}{2} \left( \| \sqrt{P_o + N_1} \|^2 + \| N_2 \|^2 \right) < \frac{1}{2} \left( \| N_3 \|^2 + \| N_4 \|^2 \right) \right\}.$$  \hfill (4.36)

The four variates are i.i.d. with $L_1 = L_2 = 2$, $A = B$, $a^2 = P_o$ and $\mu_1^2 = SIR$. Thus, from Corollary 4.4.3, the error probability is

$$P_{\text{id}} (2, 2; SIR) = \exp \left( -\frac{1}{2} SIR \right) \left[ \frac{1}{2} + \frac{1}{16} SIR \right].$$  \hfill (4.37)

It will be shown in Section 5.2.1 that this error probability is obtained for Case I in the limit of $\alpha \to \frac{1}{2}$, and it will be shown in Section 5.3 that this is the upper bound for this case. That is, it
will be shown that

\[ P_I(SIR | \alpha) \leq \lim_{\alpha \to \frac{2}{3}} P_I(SIR | \alpha) \to P_{iid}(2, 2; SIR). \]  \hspace{1cm} (4.38)

### 4.4.3 Case II Lower Bound

With \( \alpha = 2/3 \), and \( \rho_{2,3} = 1 \), then \( N_2 = N_3 \), and the Case II error probability is

\[
P_{II}(SIR | \alpha = 2/3, \rho = 1) = \Pr\left\{ \frac{2}{3} \left\| \sqrt{P_o} + N_1 \right\|^2 + \frac{1}{3} \left\| N_2 \right\|^2 < \frac{2}{3} \left\| N_3 \right\|^2 + \frac{1}{3} \left\| N_4 \right\|^2 \right\}.
\]

Subtracting \( \frac{1}{3} \left\| N_2 \right\|^2 \) from both sides,

\[
P_{II}(SIR | \alpha = 2/3, \rho = 1) = \Pr\left\{ \frac{2}{3} \left\| \sqrt{P_o} + N_1 \right\|^2 < \frac{1}{3} \left( \left\| N_2 \right\|^2 + \left\| N_4 \right\|^2 \right) \right\}.
\]

The remaining three variates are i.i.d. with \( L_1 = 1, L_2 = 2, A = \alpha = \frac{2}{3}, B = (1 - \alpha) = 1/3, \) and \( \mu_2^2 = SIR \). Note that the total number of degrees of freedom has been reduced by two, and \( L_1 \neq L_2 \). Using Corollary 4.4.2, the error probability is

\[
P_{iid}(1, 2; 2/3; SIR) = \exp\left( -\frac{2}{3} SIR \right) \left[ \frac{5}{9} + \frac{2}{27} SIR \right]. \]  \hspace{1cm} (4.39)

It will be shown in Section 6.2.1 that this error probability is obtained for Case II with \( \rho = 1 \), in the limit of \( \alpha \to \frac{2}{3} \). That is, it will be shown that

\[
\lim_{\alpha \to \frac{2}{3}} P_{II}(SIR | \alpha, \rho = 1) \to P_{iid}(1, 2; 2/3; SIR). \]  \hspace{1cm} (4.40)

This result cannot be obtained using the previous known results from the literature. The classic Proakis result [20, App C.] is not applicable because \( L_1 \neq L_2 \), and 4.19 is indeterminate because \( \mu_2^2 = 0 \).
Chapter 5

Case I Performance Analysis

5.1 Summary

In this chapter, we derive several closed form results for the Case I pairwise error probability. First, the error probability conditioned on $\alpha$ is derived in the proof of Theorem 5.1.1, and it is shown that the solution in the limit of $\alpha = \frac{1}{2}$ agrees with the i.i.d. result presented in Chapter 4. In Corollary 5.1.2, it is then proven that this i.i.d. limit is the upper bound. Then, the average error probability is derived in the proof of Theorem 5.1.3. Ultimately, it is shown in the proof of Corollary 5.1.4 that the average error probability for Case I is upper bounded at high SIR by

$$P_I(SIR \gg 1) < \frac{1}{2} \exp \left( - \frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right]. \quad (5.1)$$

Theorem 5.1.1. The Case I error probability, conditioned on $\alpha$, is

$$P_I(SIR|\alpha) = \begin{cases} \frac{1}{2(2\alpha-1)} \left[ \alpha^2 \exp \left( - \frac{SIR}{2} \right) - (1 - \alpha)^2 \exp \left( - SIR \alpha \right) \right] & \text{if } \frac{1}{2} < \alpha \leq 1, \\ \frac{1}{2} \exp \left( - \frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \left( \frac{SIR}{2} \right) \right] & \text{if } \alpha = \frac{1}{2}, \end{cases} \quad (5.2)$$

with $P_I(SIR|\alpha = 1) = P_{UF}(SIR)$, the unfaded error probability, and $P_I(SIR|\alpha = \frac{1}{2}) =$
Corollary 5.1.2. Over the interval $\alpha \in \left[ \frac{1}{2}, 1 \right]$, for sufficiently large $SIR$ the derivative with respect to $\alpha$ is negative,

$$\frac{d}{d\alpha} P_l(SIR|\alpha) < 0.$$  \hfill (5.3)

Therefore, the error monotonically decreases with $\alpha$ increasing from $\frac{1}{2}$ to 1, and is bounded by

$$\frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \leq P_l(SIR|\alpha) \leq \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \left( \frac{SIR}{2} \right) \right]. \hfill (5.4)$$

Theorem 5.1.3. If $\alpha$ is uniformly distributed over the interval $\left[ \frac{1}{2}, 1 \right]$, then the average error
Case I: Average Error and Asymptotic Bounds

Asymptotic Bounds

Average

$\alpha = 1$ Lower Bound

$\alpha = 1/2$ Upper Bound

Figure 5.2: Case 1 Average Performance and Asymptotic Bounds

The probability is

$$P_I(SIR) \equiv 2 \int_{1/2}^{1} P_I(SIR|\alpha) \, d\alpha =$$

$$\frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left\{ \frac{1}{4} \left[ \frac{5}{2} + C + \log \left( \frac{SIR}{2} \right) \right] + \Gamma \left( 0, \frac{SIR}{2} \right) \right\} \quad (5.5)$$

$$+ \left( 1 - \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \right) \frac{SIR^{-1}}{SIR} - \left( 1 - \exp \left( -\frac{SIR}{2} \right) \right) \frac{SIR^{-2}}{SIR^2},$$

which converges asymptotically to

$$\lim_{SIR \to \infty} P_I(SIR) = \frac{1}{8} \exp \left( -\frac{SIR}{2} \right) \left[ \frac{5}{2} + C + \log \left( \frac{SIR}{2} \right) \right], \quad (5.6)$$

where $C = 0.5772157 \ldots$ is Euler’s constant, and $\Gamma(a, x)$ is the incomplete gamma function.

Corollary 5.1.4. The high $SIR$ asymptote for the average error is bounded by

$$P_I(SIR \gg 1) > \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ \frac{3}{4} + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right],$$

$$P_I(SIR \gg 1) < \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right]. \quad (5.7)$$
From Corollary 5.1.2, the conditional error is bounded by

$$\frac{1}{2} \exp \left( - \frac{SIR}{2} \right) \leq P_l(SIR | \alpha) \leq \frac{1}{2} \exp \left( - \frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \left( \frac{SIR}{2} \right) \right].$$

(5.8)

Thus, on average, the error rate increases by a factor of $1 + \frac{1}{4} \log(SIR/2)$, while for the worst case, it increases by a factor of $1 + \frac{1}{4} (SIR/2)$.

### 5.2 Conditional Error Probability

*Proof of Theorem 5.1.1.* As defined in (2.30), the conditional error probability for Case I is

$$P_l(SIR | \alpha) = Pr \{ X < Y \},$$

(5.9)

where

$$X = \alpha \left\| \sqrt{P_o + N_1} \right\|^2 - (1 - \alpha) \left\| N_4 \right\|^2$$

(5.10)

$$Y = \alpha \left\| N_3 \right\|^2 - (1 - \alpha) \left\| N_2 \right\|^2$$

(5.12)

$$X = \chi^2 (\alpha \sigma^2) - \chi^2 ((1 - \alpha) \sigma^2) \left. \right|_{\rho_{1,4}=0}$$

(5.11)

$$Y = \chi^2 (\alpha \sigma^2) - \chi^2 ((1 - \alpha) \sigma^2) \left. \right|_{\rho_{2,3}=0}.$$  

(5.13)

The random variable $X$ is the difference of a non-central chi-square and a central chi-square random variable, each with two degrees of freedom, and $Y$ is the difference of two central chi-square random variables, each with two degrees of freedom. The random variables $X$ and $Y$ are both conditioned on $\alpha$, and the error probability can be expressed in terms of the probability density function (pdf) and cumulative distribution function (cdf) of $X$ and $Y$ as

$$P_l(SIR | \alpha) = 1 - \int_{-\infty}^{\infty} F_{Y|\alpha}(z) f_{X|\alpha}(z) dz,$$

(5.14)
where \( f_{X|\alpha}(z) \) is the conditional pdf of \( X \) and \( F_{Y|\alpha}(z) \) is the conditional cdf of \( Y \). From [23, Eq. 4.32 and 4.5], with \( \sigma_1^2 = \alpha \sigma^2 \), \( \sigma_2^2 = (1 - \alpha) \sigma^2 \) and \( a_1^2 = P_o \alpha \), these are

\[
f_{X|\alpha}(z) = \begin{cases} \frac{1}{2\sigma^2} \exp\left(\frac{z}{2(1-\alpha)\sigma^2}\right) \exp\left(-\frac{P_o \alpha}{2\sigma^2}\right) & z < 0 \\ \frac{1}{2\sigma^2} \exp\left(\frac{z}{2(1-\alpha)\sigma^2}\right) \exp\left(-\frac{P_o \alpha}{2\sigma^2}\right) Q_1\left(\sqrt{\frac{P_o (1-\alpha)}{\sigma^2}}, \sqrt{\frac{z}{\alpha(1-\alpha)\sigma^2}}\right) & z \geq 0 \end{cases}
\]

and

\[
F_{Y|\alpha}(z) = \begin{cases} (1 - \alpha) \exp\left(\frac{z}{2(1-\alpha)\sigma^2}\right) & z < 0 \\ 1 - \alpha \exp\left(-\frac{z}{2\alpha\sigma^2}\right) & z \geq 0 \end{cases}
\]

with \( Q_1(A, B) \) denoting the first-order Marcum-Q function.

Defining

\[
F_{Y|\alpha}^+(z) \equiv \alpha \exp\left(-\frac{z}{2\alpha\sigma^2}\right),
\]

we can express the error probability as the sum of three integrals,

\[
P_I(SIR|\alpha) = 1 - \int_{-\infty}^{0} F_{Y|\alpha}(z < 0) f_{X|\alpha}(z < 0) \, dz - \int_{0}^{\infty} [1 - F_{Y|\alpha}^+(z)] f_{X|\alpha}(z \geq 0) \, dz \\
\equiv \Phi_1 + \Phi_2 + \Phi_3
\]

where

\[
\Phi_1 \equiv -\int_{-\infty}^{0} F_{Y|\alpha}(z) f_{X|\alpha}(z) \, dz \\
\Phi_2 \equiv 1 - \int_{0}^{\infty} f_{X|\alpha}(z) \, dz \\
\Phi_3 \equiv \int_{0}^{\infty} F_{Y|\alpha}^+(z) f_{X|\alpha}(z) \, dz.
\]

The first integral is

\[
\Phi_1 = -\frac{(1 - \alpha)}{2\sigma^2} \exp\left(-\frac{P_o \alpha}{2\sigma^2}\right) \int_{-\infty}^{0} \exp\left(\frac{z}{(1 - \alpha)\sigma^2}\right) \, dz \\
= -\frac{(1 - \alpha)^2}{2} \exp\left(-\frac{P_o \alpha}{2\sigma^2}\right).
\]
The second integral is

$$\Phi_2 = 1 - [1 - F_X|\alpha](0) = F_X|\alpha(0)$$

$$= (1 - \alpha) \exp \left( - \frac{P_o \alpha}{2\sigma^2} \right),$$

where $F_X|\alpha(0)$ is obtained from [23, Eq. 4.33] with $\sigma_1^2 = \alpha \sigma^2$, $\sigma_2^2 = (1 - \alpha) \sigma^2$ and $a_1^2 = P_o \alpha$.

Using (5.17), the last integral is

$$\Phi_3 = \frac{\alpha}{2\sigma^2} \exp \left( - \frac{P_o \alpha}{2\sigma^2} \right) \int_0^\infty \exp \left( \frac{z(2\alpha - 1)}{2\alpha(1 - \alpha)\sigma^2} \right) Q_1 \left( \sqrt{\frac{P_o (1 - \alpha)}{\sigma^2}}, \sqrt{\frac{z}{\alpha(1 - \alpha)\sigma^2}} \right) \, dz$$

$$= \frac{\alpha}{\sigma^2} \exp \left( - \frac{P_o \alpha}{2\sigma^2} \right) \int_0^\infty u \exp \left( \frac{u^2(2\alpha - 1)}{2\alpha(1 - \alpha)\sigma^2} \right) Q_1 \left( \sqrt{\frac{P_o (1 - \alpha)}{\sigma^2}}, u \sqrt{\frac{1}{\alpha(1 - \alpha)\sigma^2}} \right) \, du,$$

where we have used the substitution $u = \sqrt{z} \rightarrow dz = 2 \, u \, du$.

From [23, Eq. B.24],

$$\int_0^\infty x \exp \left( \frac{p^2 x^2}{2} \right) Q_1 (b, ax) \, dx = \frac{1}{p^2} \left[ \frac{a^2}{a^2 - p^2} \exp \left( \frac{p^2 b^2}{2(a^2 - p^2)} \right) - 1 \right]$$

for $p < a$. With $p = \sqrt{\frac{2\alpha - 1}{a(1 - \alpha)\sigma^2}}$, $a = \sqrt{\frac{1}{a(1 - \alpha)\sigma^2}}$, and $b = \sqrt{\frac{P_o (1 - \alpha)}{\sigma^2}}$, we are guaranteed that $p < a$ for $1/2 \leq \alpha < 1$, yielding the solution to $\Phi_3$,

$$\Phi_3 = \frac{\alpha^2(1 - \alpha)}{2\alpha - 1} \exp \left( - \frac{P_o \alpha}{2\sigma^2} \right) \left[ \frac{1}{2(1 - \alpha)} \exp \left( \frac{P_o (2\alpha - 1)}{4\sigma^2} \right) - 1 \right].$$

Recalling that the signal-to-interference ratio is $SIR = \frac{P_o}{2\sigma^2}$, the sum of the three integrals after grouping like terms is

$$P_I(SIR|\alpha) = \frac{\alpha^2}{2(2\alpha - 1)} \exp (-SIR \alpha) \exp \left( \frac{SIR \, 2\alpha - 1}{2} \right) \exp (-SIR \alpha) \left[ - \frac{(1 - \alpha)^2}{2(2\alpha - 1)} \right],$$

where

$$\left[ (1 - \alpha) - \frac{(1 - \alpha)^2}{2} - \frac{\alpha^2(1 - \alpha)}{2\alpha - 1} \right] = - \frac{(1 - \alpha)^2}{2(2\alpha - 1)}$$

(5.30)
has been used. Combining the exponentials,

\[ \exp(-\text{SIR} \alpha) \exp \left( \text{SIR} \frac{2\alpha - 1}{2} \right) = \exp \left( -\frac{\text{SIR}}{2} \right), \]  

(5.31)

we obtain

\[ P_\text{I} \left( \frac{\text{SIR}}{\alpha} \right) = \frac{1}{2(2\alpha - 1)} \left[ \alpha^2 \exp \left( -\frac{\text{SIR}}{2} \right) - (1 - \alpha)^2 \exp \left( -\text{SIR} \alpha \right) \right], \]  

(5.32)

which is non-singular except in the limit of \( \alpha = \frac{1}{2} \). At \( \alpha = 1 \), the error probability equals \( P_\text{I} \left( \text{SIR} \mid \alpha = 1 \right) = \exp \left( -\frac{\text{SIR}}{2} \right)/2 \), in agreement with the unfaded case (4.35) derived previously.

### 5.2.1 \( \alpha \to \frac{1}{2} \) Limit

To evaluate the error probability in the limit of \( \alpha = 1/2 \), we apply L’Hopital’s rule

\[
\lim_{\alpha \to \frac{1}{2}} P_\text{I} \left( \frac{\text{SIR}}{\alpha} \right) = \frac{d}{d\alpha} \left[ \alpha^2 \exp \left( -\frac{\text{SIR}}{2} \right) - (1 - \alpha)^2 \exp \left( -\text{SIR} \alpha \right) \right]_{\alpha = \frac{1}{2}} \\
= \frac{d}{d\alpha} \left[ 2\alpha \exp \left( -\frac{\text{SIR}}{2} \right) + 2(1 - \alpha) \exp \left( -\text{SIR} \alpha \right) \right]_{\alpha = \frac{1}{2}} \\
= \frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \left( 1 + \frac{\text{SIR}}{8} \right),
\]  

(5.33)

which agrees with the previously derived result from the i.i.d. limit (4.37).
5.2.2 Piecewise Definition

Combining the limit at $\alpha = \frac{1}{2}$ into a piecewise function to remove the singularity, we obtain the desired result:

$$P_I(SIR | \alpha) = \begin{cases} \frac{1}{2(2\alpha-1)} \left[ \alpha^2 \exp(-SIR / 2) - (1 - \alpha)^2 \exp(-SIR \alpha) \right] & \text{if } \frac{1}{2} < \alpha \leq 1, \\ \frac{1}{2} \exp(-\frac{SIR}{2}) \left[ 1 + \frac{1}{4} \left( \frac{SIR}{2} \right) \right] & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

(5.34)

5.3 Upper and Lower Bounds

Proof of Corollary 5.1.2. The Case I error rate is conditioned on $\alpha$, while the $SIR$ is a deterministic parameter requiring no averaging. We wish to show that the derivative of the conditional error is negative throughout the interval $\alpha \in \left[ \frac{1}{2}, 1 \right]$. This ensures that the error decreases monotonically as $\alpha$ increases from $\frac{1}{2}$ to 1.

Making the substitution $u = \alpha - \frac{1}{2}$ to shift the interval to origin, the conditional error rate
is

\[ P_l(SIR|\alpha) = \frac{1}{2(2\alpha-1)} \left[ \alpha^2 \exp(-SIR/2) - (1-\alpha)^2 \exp(-SIR\alpha) \right] \]

\[ \rightarrow P_l(SIR, u) = \frac{1}{4u} \left[ (u + \frac{1}{2})^2 \exp(-SIR/2) - (u - \frac{1}{2})^2 \exp(-SIR/2 - SIR u) \right] \]

\[ = \frac{1}{4} \exp\left( -\frac{SIR}{2} \right) \frac{1}{u} \left[ (u + \frac{1}{2})^2 - (u - \frac{1}{2})^2 \exp(-SIR u) \right] \]

\[ = \frac{1}{4} \exp\left( -\frac{SIR}{2} \right) \frac{1}{u} \left[ (u^2 + u + \frac{1}{4}) - (u^2 - u + \frac{1}{4}) \exp(-SIR u) \right] \]

\[ = \frac{1}{4} \exp\left( -\frac{SIR}{2} \right) \left[ u(1 - e^{-SIR u}) + (1 + e^{-SIR u}) + \frac{1}{4u}(1 - e^{-SIR u}) \right] \]

\[ = \frac{1}{4} \exp\left( -\frac{SIR}{2} \right) \left[ 1 + u + e^{-SIR u} - u e^{-SIR u} + \frac{1}{4u} - \frac{e^{-SIR u}}{4u} \right] \]

\[ \equiv \frac{1}{4} \exp\left( -\frac{SIR}{2} \right) G(SIR, u), \quad (5.35) \]

where we have defined

\[ G(SIR, u) \equiv \left[ 1 + u + e^{-SIR u} - u e^{-SIR u} + \frac{1}{4u} - \frac{e^{-SIR u}}{4u} \right], \quad (5.36) \]

so that

\[ P_l(SIR, u)' = \frac{\delta}{\delta u} P_l(SIR|\alpha) = \frac{1}{4} \exp\left( -\frac{SIR}{2} \right) G'(SIR, u) \quad (5.37) \]

and \( P_l(SIR|\alpha)' < 0 \) if and only if \( G'(SIR, u) < 0 \). Thus, by proving that the derivative of \( G \) is negative throughout the interval \( u \in (0, \frac{1}{2}) \), then \( \alpha = \frac{1}{2} \) and \( \alpha = 1 \) correspond to the upper and lower bounds of \( P_l(SIR|\alpha) \), respectively.
The derivative of $G$ is

$$G'(\text{SIR}, u)$$

$$= \frac{\delta}{\delta u} \left[ 1 + u + e^{-\text{SIR}u} - u e^{-\text{SIR}u} + \frac{1}{4u} - \frac{e^{-\text{SIR}u}}{4u} \right]$$

$$= \left[ 0 + 1 - \text{SIR} e^{-\text{SIR}u} - (e^{-\text{SIR}u} - \text{SIR} e^{-\text{SIR}u}) - \frac{1}{4u^2} + \frac{e^{-\text{SIR}u}}{4} \left( \frac{\text{SIR}}{u} + \frac{1}{u^2} \right) \right]$$

$$= \left[ -\frac{1}{4}(1 - e^{-\text{SIR}u}) u^{-2} + \frac{\text{SIR} e^{-\text{SIR}u}}{4} u^{-1} + (1 - (1 + \text{SIR}) e^{-\text{SIR}u}) + \text{SIR} e^{-\text{SIR}u} u \right].$$

(5.38)

Since the closed interval excludes the endpoint $u = 0$, with sufficient SIR $\gg 0$, 

$$\exp (-\text{SIR} u) \to 0$$

and 

$$(\text{SIR} \exp (-\text{SIR} u)) \to 0,$$

so

$$\lim_{\text{SIR} \to \infty} \frac{G'(\text{SIR}, u)}{u \neq 0} = -\frac{1}{4} u^{-2} + 1 < 0 \quad \text{if } u < \frac{1}{2}. \quad (5.39)$$

Therefore, with sufficient SIR, $G'(\text{SIR}, u) < 0$ over the closed interval $0 < u < \frac{1}{2}$.

Because the derivative is negative, the error probability decreases monotonically as $\alpha$ increases from $\frac{1}{2}$. Therefore, the i.i.d. limit with $\alpha = \frac{1}{2}$ is the upper bound for Case I, and the unfaded limit with $\alpha = 1$ is the lower bound for Case I:

$$P_I(\text{SIR} | \alpha = 1) \leq P_I(\text{SIR} | \alpha) \leq P_I(\text{SIR}, \alpha = \frac{1}{2}). \quad (5.40)$$

Therefore,

$$\frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \leq P_I(\text{SIR} | \alpha) \leq \frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \left[ 1 + \frac{1}{4} \left( \frac{\text{SIR}}{2} \right) \right]. \quad (5.41)$$

with equality at the endpoints of $\alpha = 1$ and $\alpha = \frac{1}{2}$, respectively.
5.4 Average Error Probability

Proof of Theorem 5.1.3. From Theorem 5.1.1, the Case I error probability conditioned on $\alpha$ is

$$P_i(SIR|\alpha) = \frac{1}{2(2\alpha - 1)} \left[ \alpha^2 \exp\left(-\frac{SIR}{2}\right) - (1 - \alpha)^2 \exp\left(-\frac{SIR}{2}\right) \right],$$

(5.42)

over the interval $\alpha = \left[\frac{1}{2}, 1\right]$.

Therefore, since $\alpha$ is uniformly distributed over this interval, the average error probability is

$$P_i(SIR) = 2 \int_{\frac{1}{2}}^{1} P_i(SIR|\alpha) \, d\alpha$$

(5.43)

Making the substitution $u = \alpha - \frac{1}{2}$, $d\alpha = du$, to shift the region of integration to the origin,

$$P_i(SIR) = \int_{0}^{\frac{1}{2}} \frac{1}{2u} \left[ (u + \frac{1}{2})^2 \exp\left(-\frac{SIR}{2}\right) - (u - \frac{1}{2})^2 \exp\left(-\frac{SIR}{2} - SIR u\right) \right] \, du$$

$$= \frac{1}{2} \exp\left(-\frac{SIR}{2}\right) \int_{0}^{\frac{1}{2}} \frac{1}{u} \left[ (u + \frac{1}{2})^2 - (u - \frac{1}{2})^2 \exp\left(-SIR u\right) \right] \, du$$

$$= \frac{1}{2} \exp\left(-\frac{SIR}{2}\right) \int_{0}^{\frac{1}{2}} \frac{1}{u} \left[ (u^2 + u + \frac{1}{4}) - (u^2 - u + \frac{1}{4}) \exp(-SIR u) \right] \, du$$

$$= \frac{1}{2} \exp\left(-\frac{SIR}{2}\right) \left[ u(1 - e^{-SIR u}) + (1 + e^{-SIR u}) + \frac{1}{4u}(1 - e^{-SIR u}) \right] \, du$$

$$\equiv \frac{1}{2} \exp\left(-\frac{SIR}{2}\right) [\Phi_1 + \Phi_2]$$

(5.44)

where we have defined the two integrals

$$\Phi_1 \equiv \int_{0}^{\frac{1}{2}} \left[ 1 + u + e^{-SIR u} - u e^{-SIR u} \right] \, du$$

(5.45)

$$\Phi_2 \equiv \frac{1}{4} \int_{0}^{\frac{1}{2}} \frac{1 - e^{-SIR u}}{u} \, du.$$  

(5.46)
The first integral is
\[
\Phi_1 = \left[ u + \frac{1}{2} u^2 \frac{1}{\text{SIR}} e^{-\text{SIR} u} + \left( \frac{u}{\text{SIR}} + \frac{1}{\text{SIR}^2} \right) e^{-\text{SIR} u} \right]_0^\frac{1}{2} 
\]
\[
= \frac{1}{2} + \frac{1}{8} + \frac{1 - e^{-\text{SIR}/2}}{\text{SIR}} + \left( \frac{1}{2 \text{SIR}} + \frac{1}{\text{SIR}^2} \right) e^{-\text{SIR}/2} \frac{1}{\text{SIR}^2} 
\]
\[
= \frac{5}{8} + \left( 1 - \frac{1}{2} e^{-\text{SIR}/2} \right) \text{SIR}^{-1} - (1 - e^{-\text{SIR}/2}) \text{SIR}^{-2}. 
\]

Making the substitution \( v = \text{SIR} u \) so that \( u = \frac{v}{\text{SIR}} \) and \( du = \frac{dv}{\text{SIR}} \), the second integral is
\[
\Phi_2 = \frac{1}{4} \int_0^\frac{1}{2} \frac{1 - e^{-\text{SIR} u}}{u} \, du 
\]
\[
= \frac{1}{4} \int_0^{\text{SIR} \frac{1}{2}} \frac{1 - e^{-v}}{v} \, dv. 
\]

From [8, Eq. 8.212]
\[
\int_0^x \frac{1 - e^{-t}}{t} \, dt = C + \log x - \text{Ei}(-x), 
\]
where \( C \) is Euler’s constant (also known as the Euler-Mascheroni constant) defined as the limit [28, 12.1]
\[
C \equiv \lim_{m \to \infty} \left\{ \sum_{k=1}^m \frac{1}{k} - \log m \right\} = 0.5772157 \ldots, 
\]
and \( \text{Ei}(x) \) is the exponential-integral function, defined as [8, Eq. 8.211]
\[
\text{Ei}(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} \, dt \quad \text{if } x < 0. 
\]

Therefore, with \( x = -\frac{\text{SIR}}{2} \), which is negative since the SIR is positive,
\[
- \text{Ei} \left( -\frac{\text{SIR}}{2} \right) = \int_{\text{SIR} \frac{1}{2}}^{\infty} \frac{e^{-t}}{t} \, dt 
\]
which is special case of the incomplete gamma function, [8, Eq. 8.350]
\[
\Gamma(a, x) \equiv \int_x^{\infty} t^{a-1} e^{-t} \, dt 
\]
with \( a = 0 \), and thus
\[
- \operatorname{Ei}
\left(- \frac{\text{SIR}}{2}\right) = \Gamma
\left(0, \frac{\text{SIR}}{2}\right).
\]
The representation \( \Gamma
\left(0, \frac{\text{SIR}}{2}\right) \) is preferable to
\(- \operatorname{Ei}
\left(- \frac{\text{SIR}}{2}\right) \), in the sense that it more clearly denotes a positive-valued function that decays asymptotically to zero with \( \text{SIR} \to \infty \).

Therefore, the second integral is
\[
\Phi_2 = \frac{1}{4} \left[ C + \log \left( \frac{\text{SIR}}{2} \right) + \Gamma
\left(0, \frac{\text{SIR}}{2}\right) \right].
\] (5.54)

Combining this with the first integral (5.47), the average error probability is
\[
P_I
\left(\text{SIR}\right) = \frac{1}{2} \exp
\left(- \frac{\text{SIR}}{2}\right) \left\{ \frac{5}{8} + \left(1 - \frac{1}{2} e^{- \frac{\text{SIR}}{2}}\right) \text{SIR}^{-1} - \left(1 - e^{- \frac{\text{SIR}}{2}}\right) \text{SIR}^{-2} \right\}
\
+ \frac{1}{4} \left[ C + \log \left( \frac{\text{SIR}}{2} \right) + \Gamma
\left(0, \frac{\text{SIR}}{2}\right) \right] \right\}
\]
\[
= \frac{1}{2} \exp
\left(- \frac{\text{SIR}}{2}\right) \left\{ \frac{1}{4} \left[ \frac{5}{2} + C + \log \left( \frac{\text{SIR}}{2} \right) + \Gamma
\left(0, \frac{\text{SIR}}{2}\right) \right] \right\}
\
+ \left(1 - \frac{1}{2} e^{- \frac{\text{SIR}}{2}}\right) \text{SIR}^{-1} - \left(1 - e^{- \frac{\text{SIR}}{2}}\right) \text{SIR}^{-2} \right\}
\] (5.55)

\[\square\]

### 5.5 Asymptotic Bounds

**Proof of Corollary 5.1.4.** Since
\[
\lim_{\text{SIR} \to \infty} \left\{ \left(1 - \frac{1}{2} e^{- \frac{\text{SIR}}{2}}\right) \text{SIR}^{-1} \right\} = 0
\]
\[
\lim_{\text{SIR} \to \infty} \left\{ \left(1 - e^{- \frac{\text{SIR}}{2}}\right) \text{SIR}^{-2} \right\} = 0
\]
\[
\lim_{\text{SIR} \to \infty} \left\{ \Gamma
\left(0, \text{SIR}/2\right) \right\} = 0,
\]
then, at high SIR
\[
\Phi_1 \to \frac{5}{8},
\]
\[
\Phi_2 \to \frac{1}{4} \left[ C + \log \left( \frac{\text{SIR}}{2} \right) \right],
\] (5.56)
and the average error probability asymptotically converges to

\[ P_I(SIR \gg 1) \to \frac{1}{8} \exp \left( -\frac{SIR}{2} \right) \left[ \frac{5}{2} + C + \log \left( \frac{SIR}{2} \right) \right]. \]  

(5.57)

Since \( \frac{5}{2} + C = 3.0772 \ldots \), then

\[ 3 < \frac{5}{2} + C < 4, \]  

(5.58)

and the asymptote is therefore bounded by

\[ P_I(SIR \gg 1) > \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ \frac{3}{4} + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right] \]  

(5.59)

\[ P_I(SIR \gg 1) < \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right]. \]
Chapter 6

Case II Performance Analysis

6.1 Summary

Theorem 6.1.1. In the limit of complete correlation with $\rho = 1$, the Case II error probability is

$$P_{II}(SIR | \alpha, \rho = 1) = \begin{cases} 
\frac{5}{9} + SIR \frac{2}{27} \exp(-SIR \frac{2}{3}) & \text{if } \alpha = \frac{2}{3} \\
\frac{(2\alpha-1)^2}{(3\alpha-2)(3\alpha-1)} \exp\left(-SIR \frac{\alpha}{3\alpha-1}\right) - \frac{(1-\alpha)^2}{(3\alpha-2)} \exp(-SIR \alpha) & \text{if } \frac{1}{2} \leq \alpha \leq 1, \alpha \neq \frac{2}{3}
\end{cases}$$

(6.1)

6.2 $\rho = 1$ Correlated Limit

Proof of Theorem 6.1.1. As defined in (2.31), the error probability for Case II is

$$P_{II}(SIR | \alpha, \rho) \equiv \mathbb{P}r \{X < Y\}$$

(6.2)
where

\[
X = \alpha \left\| \sqrt{P_o} + N_1 \right\|^2 - (1 - \alpha) \left\| N_4 \right\|^2
\]

\[
= \frac{\chi^2(\alpha P_o, \alpha \sigma^2) - \chi^2((1 - \alpha) \sigma^2)}{\rho_{1,4} = 0}
\]

\[
Y = \alpha \left\| N_3 \right\|^2 - (1 - \alpha) \left\| N_2 \right\|^2
\]

\[
= \frac{\chi^2(\alpha \sigma^2) - \chi^2((1 - \alpha) \sigma^2)}{\rho_{2,3} = \rho}
\]

In the limit of \( \rho_{2,3} = 1 \), \( N_2 = N_3 \), and \( Y \) reduces to

\[
Y \rightarrow (2\alpha - 1) \left\| N_3 \right\|^2
\]

\[
= \chi^2((2\alpha - 1) \sigma^2)
\]

The random variable \( X \) is the difference of a non-central and a central chi-square random variable, each with two degrees of freedom, and in this limit of complete corelation, \( Y \) is a central chi-square random variable with two degrees of freedom. Both random variables \( X \) and \( Y \) are conditioned on \( \alpha \), and the error probability in this limit in can be expressed as

\[
\lim_{\rho \rightarrow 1} P_{II}(SIR | \alpha, \rho) \rightarrow P_{II}(SIR | \alpha, \rho = 1)
\]

\[
= 1 - \int_{-\infty}^{\infty} F_{Y|\alpha}(z) f_{X|\alpha}(z) dz,
\]

where \( F_{Y|\alpha}(z) \) is the conditional cdf of \( Y \) and \( f_{X|\alpha}(z) \) is the conditional pdf of \( X \). The random variable \( X \) is identical to Case I, and thus has the same pdf:

\[
f_{X|\alpha}(z) = \begin{cases} 
\frac{1}{2\sigma^2} \exp \left( \frac{z}{2(1-\alpha)\sigma^2} \right) \exp \left( -\frac{P_o \alpha}{2\sigma^2} \right) & z < 0 \\
\frac{1}{2\sigma^2} \exp \left( \frac{z}{2(1-\alpha)\sigma^2} \right) \exp \left( -\frac{P_o \alpha}{2\sigma^2} \right) Q_1 \left( \sqrt{\frac{P_o(1-\alpha)}{\sigma^2}}, \sqrt{\frac{z}{\alpha(1-\alpha)\sigma^2}} \right) & z \geq 0
\end{cases}
\]

\[
(6.10)
\]

and the same cdf at \( z = 0 \):

\[
F_{X|\alpha}(0) = (1 - \alpha) \exp \left( -\frac{P_o \alpha}{2\sigma^2} \right).
\]

\[
(6.11)
\]
Recalling the definition for the cdf of a central chi-square variate with \(2L\) degrees of freedom (4.3), the conditional cdf of \(Y\) is

\[
F_{Y|\alpha}(z) = F_{\chi^2(2(\alpha - 1)\sigma^2)} = 1 - \exp\left(-\frac{z}{2(2\alpha - 1)\sigma^2}\right) z \geq 0. 
\] (6.12)

Similar to Case I, we define

\[
F^+_{Y|\alpha}(z) \equiv \exp\left(-\frac{z}{2(2\alpha - 1)\sigma^2}\right), 
\] (6.13)

and express the error probability as the sum of two integrals,

\[
P_{II}(SIR |\alpha, \rho = 1) = 1 - \int_0^\infty [1 - F^+_{Y|\alpha}(z)] f_X(z \geq 0) \, dz \quad (6.14)
\]
\[
\equiv \Phi_1 + \Phi_2 \quad (6.15)
\]

where

\[
\Phi_1 \equiv 1 - \int_0^\infty f_X(z) \, dz \quad (6.16)
\]
\[
\Phi_2 \equiv \int_0^\infty F^+_{Y|\alpha}(z) f_X(z) \, dz. \quad (6.17)
\]

As in Case I, from Equation (6.11) the first integral is

\[
\Phi_1 = 1 - [1 - F_X(0)] = (1 - \alpha) \exp\left(-\frac{P_o\alpha}{2\sigma^2}\right), \quad (6.18)
\]
\[
= \left(1 - \alpha\right) \exp\left(-\frac{P_o\alpha}{2\sigma^2}\right), \quad (6.19)
\]

while the second integral is

\[
\Phi_2 = \frac{1}{2\sigma^2} \exp\left(-\frac{P_o\alpha}{2\sigma^2}\right) \int_0^\infty \exp\left(-\frac{z}{2(2\alpha - 1)\sigma^2}\right) \exp\left(-\frac{z}{2(1 - \alpha)\sigma^2}\right) \, dz
\]
\[
= \frac{1}{\sigma^2} \exp\left(-\frac{P_o\alpha}{2\sigma^2}\right) \int_0^\infty u \exp\left(-\frac{u^2(3\alpha - 2)}{2(1 - \alpha)(2\alpha - 1)\sigma^2}\right) \, dz
\]
\[
Q_1\left(\frac{P_o(1 - \alpha)}{\sigma^2}, \frac{1}{\alpha(1 - \alpha)\sigma^2}\right) dz, \quad (6.20)
\]
using the same substitution \( u = \sqrt{z} \rightarrow dz = 2u\, du \) as in Case I.

Note that the sign of the exponential in the integrand depends on the value of \((3\alpha - 2)\).

Since \((3\alpha - 2)\) is positive for \(\alpha > \frac{2}{3}\), zero for \(\alpha = \frac{2}{3}\) and negative for \(\alpha < \frac{2}{3}\), a different solution is required for each case [23, Eqs. B.14, B.22, B.24]:

\[
\int_{0}^{\infty} x\, Q_1(b, ax)\, dx = \frac{1}{a^2} \left[ 1 + \frac{b^2}{2} \right] \tag{6.21}
\]

\[
\int_{0}^{\infty} x\, \exp\left(\frac{-x^2p^2}{2}\right) Q_1(b, ax)\, dx = -\frac{1}{p^2} \left[ 1 - \frac{a^2}{a^2 - p^2} \exp\left( \frac{p^2b^2}{2(a^2 - p^2)} \right) \right] \tag{6.22}
\]

\[
\int_{0}^{\infty} x\, \exp\left(\frac{x^2p^2}{2}\right) Q_1(b, ax)\, dx = +\frac{1}{p^2} \left[ 1 - \frac{a^2}{a^2 + p^2} \exp\left( -\frac{p^2b^2}{2(a^2 + p^2)} \right) \right] \tag{6.23}
\]

It will be shown that the solutions for the positive and negative cases are equivalent and form a general solution with a removable discontinuity at the point \(\alpha = \frac{2}{3}\). It will also be shown that \(\alpha = \frac{2}{3}\) is a critical point and locally minimum over the interval \(\alpha \in \left(\frac{1}{2}, 1\right)\).

### 6.2.1 Solution for \((3\alpha - 2) \neq 0\)

**Positive Case**

When \((3\alpha - 2)\) is positive, for \(\frac{2}{3} < \alpha \leq 1\), the integral \(\Phi_2\) (6.20) has the same form as Equation (6.22), with

\[
p = \sqrt{\frac{3\alpha - 2}{(1 - \alpha)(2\alpha - 1)\sigma^2}}, \quad a = \sqrt{\frac{1}{\alpha(1 - \alpha)\sigma^2}}, \quad b = \sqrt{\frac{P_{\sigma}(1 - \alpha)}{\sigma^2}}, \tag{6.24}
\]

and

\[
\frac{1}{a^2 - p^2} = \frac{\alpha(2\alpha - 1)\sigma^2}{3\alpha - 1}. \tag{6.25}
\]
Note that this solution is restricted for \( a^2 - p^2 = \frac{3\alpha - 1}{\alpha(2\alpha - 1)\sigma^2} > 0 \), which is guaranteed, given that \( 3\alpha - 2 \) is positive. Substituting these values into the RHS of Equation (6.22) yields

\[
\Phi_2 = \exp(-SIR\alpha) \left[ \frac{(2\alpha - 1)^2}{(3\alpha - 2)(3\alpha - 1)} \exp\left(SIR\frac{\alpha(3\alpha - 2)}{(3\alpha - 1)}\right) - \frac{(1 - \alpha)(2\alpha - 1)}{(3\alpha - 2)} \right] \quad (6.26)
\]

**Negative Case**

When \( 3\alpha - 2 \) is negative, for \( \frac{1}{2} \leq \alpha < \frac{2}{3} \), the integral \( \Phi_2 \) (6.20) has the same form as (6.23), with

\[
\Phi_2 = \frac{1}{\sigma^2} \exp\left(-\frac{P_o\alpha}{2\sigma^2}\right) \int_0^\infty u \exp\left(-\frac{u^2(2 - 3\alpha)}{2(1 - \alpha)(2\alpha - 1)\sigma^2}\right) Q_1\left(\sqrt{\frac{P_o(1 - \alpha)}{\sigma^2}}, u\sqrt{\frac{1}{\alpha(1 - \alpha)\sigma^2}}\right) dz, \quad (6.27)
\]

where

\[
p = \sqrt{\frac{2 - 3\alpha}{(1 - \alpha)(2\alpha - 1)\sigma^2}}, \quad a = \sqrt{\frac{1}{\alpha(1 - \alpha)\sigma^2}}, \quad b = \sqrt{\frac{P_o(1 - \alpha)}{\sigma^2}}, \quad (6.28)
\]

and

\[
\frac{1}{a^2 + p^2} = \frac{\alpha(2\alpha - 1)\sigma^2}{3\alpha - 1}. \quad (6.29)
\]

Substituting these values into the RHS of Equation (6.23) yields

\[
\Phi_2 = \exp(-SIR\alpha) \left[ -\frac{(2\alpha - 1)^2}{(2 - 3\alpha)(3\alpha - 1)} \exp\left(-SIR\frac{\alpha(2 - 3\alpha)}{(3\alpha - 1)}\right) + \frac{(1 - \alpha)(2\alpha - 1)}{(2 - 3\alpha)} \right] \\
= \exp(-SIR\alpha) \left[ \frac{(2\alpha - 1)^2}{(3\alpha - 2)(3\alpha - 1)} \exp\left(SIR\frac{\alpha(3\alpha - 2)}{(3\alpha - 1)}\right) - \frac{(1 - \alpha)(2\alpha - 1)}{(3\alpha - 2)} \right] \quad (6.30)
\]

which is identical to the positive case in Equation (6.26).
Adding equations (6.19) and (6.26), the error probability for $\alpha \neq \frac{2}{3}$ is

$$
P_{II} \left( SIR \mid \alpha \neq \frac{2}{3}, \rho = 1 \right) 
= \frac{(2\alpha - 1)^2}{(3\alpha - 2)(3\alpha - 1)} \exp \left( SIR \alpha \left[ \frac{(3\alpha - 2)}{(3\alpha - 1)} - 1 \right] \right) 
+ \left[ (1 - \alpha) - \frac{(1 - \alpha)(2\alpha - 1)}{(3\alpha - 2)} \right] \exp (- SIR \alpha) 
= \frac{(2\alpha - 1)^2}{(3\alpha - 2)(3\alpha - 1)} \exp \left( - SIR \frac{\alpha}{(3\alpha - 1)} \right) 
- \frac{(1 - \alpha)^2}{(3\alpha - 2)} \exp (- SIR \alpha).$$

(6.31)

(6.32)

Note that this solution is singular at the excluded point $\alpha = \frac{2}{3}$. This discontinuity can be removed by creating a piecewise definition provided that the value at $\alpha = \frac{2}{3}$ equals the limit at this point.

### 6.2.2 Solution for \((3\alpha - 2) = 0\)

At $\alpha = \frac{2}{3}$, we obtain

$$
\Phi_2 = \frac{1}{\sigma^2} \exp \left( - \frac{P_o}{3\sigma^2} \right) \int_{0}^{\infty} u Q_1 \left( \sqrt{\frac{P_o}{3\sigma^2}}, u \sqrt{\frac{9}{2\sigma^2}} \right) du.
$$

(6.33)

From [23, Eq. B.14],

$$
\int_{0}^{\infty} xQ_1 (b, ax) \, dx = \left( 1 + \frac{b^2}{2a^2} \right) \frac{1}{a^2}.
$$

(6.34)

Thus with $a = \sqrt{\frac{9}{2\sigma^2}}$ and $b = \sqrt{\frac{P_o}{3\sigma^2}}$, we obtain

$$
\Phi_2 = \frac{2}{9} \exp \left( - \frac{P_o}{3\sigma^2} \right) \left[ 1 + \frac{P_o}{6\sigma^2} \right].
$$

(6.35)

For $\alpha = \frac{2}{3}$, Equation (6.19) is

$$
\Phi_1 = \frac{1}{3} \exp \left( - \frac{P_o}{3\sigma^2} \right).
$$

(6.36)

Recalling that the signal to interference ratio is $SIR = \frac{P_o}{2\sigma^2}$, the error probability at $\alpha = \frac{2}{3}$ is

$$
P_{II} \left( SIR \mid \alpha = \frac{2}{3}, \rho = 1 \right) = \exp \left( - SIR \frac{2}{3} \right) \left[ \frac{5}{9} + SIR \frac{2}{27} \right],
$$

(6.37)
which agrees with the i.i.d. error probability derived previously (4.39). Next, it will be shown that this result is also attained in the limit of $\alpha = \frac{2}{3}$ in Equation (6.32), which proves that it is a removable discontinuity.

### 6.2.3 $\alpha \to \frac{2}{3}$ Limit

Expressing the error probability (6.32) as the difference of two functions of $\alpha$ over a common denominator,

$$P_{II}\left(\text{SIR} \mid \alpha \neq \frac{2}{3}, \rho = 1\right) \equiv \frac{N_1(\alpha) - N_2(\alpha)}{(3\alpha - 2)(3\alpha - 1)}$$

with

$$N_1(\alpha) \equiv (2\alpha - 1)^2 \exp \left(-\frac{\alpha}{3\alpha - 1}\right)$$

and

$$N_2(\alpha) \equiv (1 - \alpha)^2(3\alpha - 1) \exp (- \text{SIR} \alpha),$$

by applying L’Hopital’s rule, the limit at $\alpha = \frac{2}{3}$ is

$$\lim_{\alpha \to \frac{2}{3}} P_{II}(\text{SIR} \mid \alpha, \rho = 1) \equiv \frac{\frac{d}{d\alpha} \left(\frac{N_1'(\alpha) - N_2'(\alpha)}{(3\alpha - 2)(3\alpha - 1)}\right)}{\alpha = \frac{2}{3}}$$

$$= \frac{1}{3} \left[ N_1' \left(\frac{2}{3}\right) - N_2' \left(\frac{2}{3}\right) \right].$$

The derivatives of $N_1$ and $N_2$ are

$$N_1'(\alpha) = \exp \left(-\text{SIR} \cdot \frac{\alpha}{3\alpha - 1}\right) \left[4(2\alpha - 1) + \text{SIR} \frac{(2\alpha - 1)^2}{(3\alpha - 1)^2}\right]$$

$$N_2'(\alpha) = \exp (- \text{SIR} \alpha) \left[3(1 - \alpha)^2 - 2(1 - \alpha)(3\alpha - 1) - \text{SIR} (1 - \alpha)^2(3\alpha - 1)\right].$$

Evaluated at $\alpha = \frac{2}{3}$, the derivatives are

$$N_1 \left(\frac{2}{3}\right)' = \exp \left(-\text{SIR} \frac{2}{3}\right) \left[\frac{4}{3} + \text{SIR} \frac{3}{9}\right]$$

$$N_2 \left(\frac{2}{3}\right)' = -\exp \left(-\text{SIR} \frac{2}{3}\right) \left[\frac{1}{3} + \text{SIR} \frac{3}{9}\right].$$
Substituting these values into (6.41), the limit is

$$\lim_{\alpha \to \frac{2}{3}} \frac{\text{P}_{II}(SIR | \alpha, \rho = 1)}{\exp \left( -\frac{SIR}{2} \right)} = \exp \left( -\frac{SIR}{2} \right) \left[ \frac{5}{9} + \frac{2}{27} SIR \right],$$

(6.44)

which agrees with the explicit solution (6.37). Therefore, the discontinuity in (6.32) can be removed.

### 6.2.4 Piecewise Solution

Combining (6.32) and (6.37) into a piecewise continuous function, we obtain

$$\frac{\text{P}_{II}(SIR | \alpha, \rho = 1)}{\exp \left( -\frac{SIR}{2} \right)} = \begin{cases} F_0(\alpha) & \text{if } \alpha = \frac{2}{3}, \\ F_1(\alpha) - F_2(\alpha) & \text{for } \frac{1}{2} \leq \alpha \leq 1, \alpha \neq \frac{2}{3}, \end{cases}$$

(6.45)

where

$$F_0(\alpha) \equiv \exp \left( -\frac{SIR}{2} \right) \left[ \frac{5}{9} + \frac{2}{27} SIR \right],$$

(6.46)

$$F_1(\alpha) \equiv \exp \left( -\frac{SIR}{2} \right) \frac{(2\alpha - 1)^2}{(3\alpha - 2)(3\alpha - 1)},$$

(6.47)

$$F_2(\alpha) \equiv \exp \left( -\frac{SIR}{2} \right) \frac{(1 - \alpha)^2}{(3\alpha - 2)}.$$

(6.48)

At both the endpoints of the interval, $\alpha = \frac{1}{2}$ and $\alpha = 1$, the unfaded error probability, $\exp(-SIR/2)/2$ is obtained.

With $\alpha = 1$, the unfaded error probability appears in the first term, while the second (negative) term is zero. For $\alpha = \frac{1}{2}$, the roles are reversed, and the first term is zero, while the sign of the second term is now positive since $3\alpha - 2$ is negative for $\alpha < \frac{2}{3}$. Thus with $\alpha = 1$, the performance equals the unfaded case, and the negative sign in the second term decreases the error probability as $\alpha$ decreases from 1, approaching 2/3. Decreasing $\alpha$ below 2/3 reverses the sign of the second term, thereby increasing the error probability back to the unfaded performance in the limit of $\alpha = 1/2$. 
Thus in the limit of complete correlation, the unfaded error probability at $\alpha = \frac{1}{2}$ and $\alpha = 1$ is in fact the upper bound, and the lower bound occurs at $\alpha = \frac{2}{3}$, as derived previously. Furthermore, in the limit of complete independence, with $\rho = 0$, Case II reduces to Case I and thus shares the same upper bound derived in the previous section using L'Hopital's rule at $\alpha = 1/2$. □
Chapter 7

Case III, $\rho = 1$, Correlated Limit

7.1 Summary

In this chapter, we derive several results for Case III, in the limit of complete correlation, with $\rho = 1$. There is a solution for general $(\alpha, \gamma)$, in terms of a complimentary Marcum Q function, plus a difference of two integrals. However, it does not appear to yield a closed-form expression for all values of $(\alpha, \gamma)$. The point $\alpha = \frac{1}{2}$ produces a singularity, and must be handled separately. Furthermore, the second integral requires numerical evaluation over a finite $(0, 1)$ interval when $2/3 < \alpha \leq 1$.

However, there is a simple closed-form solution for all $\alpha$ in the worst case of $\tau = 0$, where $\rho = 1$ and $\gamma = 1$. When averaged over the uniform distribution of $\alpha$, a closed-form solution is obtained that is an upper bound for the Case III average error probability.
Theorem 7.1.1. In the limit of $\rho = 1$ and $\alpha = \frac{1}{2}$, the Case III error probability is

\[
P_{III}(SIR \mid \alpha = \frac{1}{2}, \rho = 1, \gamma) = \frac{1}{2} \text{erfc} \left( \sqrt{\text{sir}} \frac{1 + \gamma}{2} \right)
\]

(7.1)

\[
+ \frac{1}{4} \exp \left( -\gamma SIR 2[1 - \gamma] \right) \text{erfc} \left( \sqrt{\text{sir}} \frac{1 - 3\gamma}{2} \right)
\]

(7.2)

\[
- \frac{1}{4} \exp \left( SIR 2[1 - \gamma] \right) \text{erfc} \left( \sqrt{\text{sir}} \frac{3 - \gamma}{2} \right).
\]

(7.3)

For $\gamma = 1$, the error probability is

\[
P_{III}(SIR \mid \alpha = \frac{1}{2}, \rho = 1, \gamma = 1) = \frac{1}{2}.
\]

(7.4)

and for $\gamma = 0$, the error probability is upper bounded by

\[
P_{III}(SIR \mid \alpha = \frac{1}{2}, \rho = 1, \gamma = 0) \leq \frac{4}{3\sqrt{\pi SIR}} \exp \left( -\frac{SIR}{4} \right).
\]

(7.5)

Theorem 7.1.2. In the worst case of $\tau = 0$, $\gamma = p_u(0) = 1$ and the Case III error probability is

\[
P_{III}(SIR \mid \alpha, \tau = 0) = P_{III}(SIR \mid \alpha, \rho = 1, \gamma = 1)
\]

\[
= \frac{\alpha^2}{3\alpha - 1} \exp \left( -SIR \right) \frac{2\alpha - 1}{3\alpha - 1}.
\]

(7.6)

Theorem 7.1.3. The average error probability for Case III is upper bounded by the worst case of $\tau = 0$, where $\gamma = p_u(0) = 1$ and, since $\alpha$ is uniform over $[\frac{1}{2}, 1]$,

\[
P_{III}(SIR) < \int_{\frac{1}{2}}^{1} P_{III}(SIR \mid \alpha = \alpha, \tau = 0) f_\alpha(\alpha) d\alpha
\]

\[
= \frac{2}{27} \exp \left( -\frac{2}{3} SIR \right) \left[ \left\{ Ei \left( \frac{2}{3} SIR \right) - Ei \left( \frac{1}{6} SIR \right) \right\} \left( 1 + \frac{2}{3} SIR + \frac{1}{18} SIR^2 \right) \right.
\]

\[
- \exp \left( \frac{2}{3} SIR \right) \left( \frac{9}{8} + \frac{1}{12} SIR \right) + \exp \left( \frac{1}{6} SIR \right) \left( 6 + \frac{1}{3} SIR \right) \right].
\]

(7.7)

Theorem 7.1.4. In the limit of complete correlation, the Case III error probability for general $(\alpha, \gamma)$, except for $\alpha = \frac{1}{2}$, is

\[
\lim_{\rho \to 1} P_{III}(SIR \mid \alpha, \rho, \gamma) = 1 - Q_1 \left( \sqrt{\frac{\nu^2}{\sigma_x^2}}, \sqrt{\frac{\kappa}{\sigma_x^2}} \right) + \alpha \phi_1 - (1 - \alpha) \phi_2,
\]

(7.8)
Figure 7.1: Error Probability, Case 3 Correlated Limit, 15dB SIR

where

$$\phi_1 = \frac{\alpha}{3\alpha - 1} \exp \left( -\Lambda \frac{2\alpha - 1}{3\alpha - 1} \left[ \mu^2 + (3\alpha - 1)(1 - \alpha)(1 - \gamma)^2 \right] \right)$$

$$\phi_1 \times Q_1 \left( \mu \sqrt{\frac{2\Lambda \alpha}{3\alpha - 1}}, (1 - \gamma) \sqrt{2\Lambda(1 - \alpha)(3\alpha - 1)} \right)$$

(7.9)

in terms of the scaled SIR parameter, $\Lambda \equiv \frac{\text{SIR}}{(2\alpha - 1)^2} = \frac{P_o}{2(2\alpha - 1)^2\sigma^2}$ with $\mu \equiv \alpha - (1 - \alpha)\gamma$, and $\phi_2$ has a piecewise definition that is given below. Note that the point $\alpha = \frac{1}{2}$ must be excluded, due to the singularity in $\Lambda$. The error probability for this limit is given by Theorem 7.1.1.

The piecewise definition for $\phi_2$ is:

$$\phi_2 = \frac{1 - \alpha}{2 - 3\alpha} \exp \left( +\Lambda \frac{2\alpha - 1}{2 - 3\alpha} \left[ \mu^2 - \alpha(2 - 3\alpha)(1 - \gamma^2) \right] \right)$$

$$\phi_2 \times \left[ 1 - Q_1 \left( \mu \sqrt{\frac{2\Lambda(1 - \alpha)}{2 - 3\alpha}}, (1 - \gamma)\sqrt{2\Lambda\alpha(2 - 3\alpha)} \right) \right]$$

(7.10)
for $\frac{1}{2} < \alpha < \frac{2}{3},$

$$\phi_2 = \frac{\sqrt{2}(1-\gamma)}{(2-\gamma)} \exp \left( -\frac{\Lambda}{9} [3\gamma^2 - 8\gamma + 6] \right) I_1 \left( \frac{\Lambda \sqrt{8}}{9} [\gamma^2 - 3\gamma + 2] \right) \quad (7.11)$$

for $\alpha = \frac{2}{3}$, and

$$\phi_2 = \Lambda\alpha(1-\alpha)(1-\gamma)^2 \exp \left( -\Lambda \left[ \mu^2 + \alpha(2\alpha - 1)(1-\gamma)^2 \right] \right)$$

$$\times \int_0^1 \exp \left( -\Lambda\alpha(2-3\alpha)(1-\gamma)^2 u \right) I_0 \left( 2\Lambda\mu(1-\gamma)\sqrt{\alpha(1-\alpha)u} \right) \, du. \quad (7.12)$$

for $\frac{2}{3} < \alpha \leq 1$. This last expression must be evaluated numerically, but is easily calculated due to the finite limits. Furthermore, this finite-limit representation is valid for all $\alpha > \frac{1}{2}$, and is thus preferable for the purposes of numerical evaluation.

### 7.2 $\alpha = \frac{1}{2}$, Gaussian Limit

**Proof of Theorem 7.1.1.** As defined in (2.32), the error probability for Case III is

$$P_{III}(SIR|\alpha, \rho, \gamma) \equiv \mathcal{P}_r \{ X < Y \} \quad (7.13)$$

where

$$X = \alpha \left\| \sqrt{P_o} + N_1 \right\|^2 - (1-\alpha) \left\| \sqrt{P_o^\gamma} + N_4 \right\|^2 \quad (7.14)$$

$$Y = \chi^2 \left( \frac{\alpha \sigma^2}{\rho_1} \right) - \chi^2 \left( \frac{(1-\alpha) \sigma^2 \rho_1}{\rho_2} \right) \quad (7.15)$$

$$= \chi^2 \left( \alpha \sigma^2 \right) - \chi^2 \left( (1-\alpha) \sigma^2 \right) . \quad (7.16)$$

With $\alpha = 1/2$, and in the limit of complete correlation with $\rho = 1$, $N_4 = N_1$, and we obtain

$$X = \frac{1}{2} \left[ \left\| \sqrt{P_o} + N_1 \right\|^2 - \left\| \sqrt{P_o^\gamma} + N_1 \right\|^2 \right] \quad (7.18)$$

$$Y = \frac{1}{2} \left[ \left\| N_3 \right\|^2 - \left\| N_2 \right\|^2 \right] . \quad (7.19)$$
Since $\Pr \{ X < Y \} = \Pr \{ 2X < 2Y \}$, the leading factors of 1/2 can be removed:

\[
X = \|\sqrt{P_o + N_1}\|^2 - \|\sqrt{P_o + \gamma N_1}\|^2 \quad (7.20)
\]
\[
Y = \|N_3\|^2 - \|N_2\|^2. \quad (7.21)
\]

Note that neither $X$ or $Y$ are conditioned on $\alpha$, but $X$ is conditioned on $\gamma$. Therefore, in the limit of $\rho \to 1$ with $\alpha = \frac{1}{2}$, the Case III error probability is expressed in terms of the conditional pdf and cdf of $X | \gamma$ and $Y$,

\[
\lim_{\rho \to 1} P_{III}(SIR \mid \alpha = \frac{1}{2}, \rho, \gamma) = \Pr \{ X < Y \} \quad (7.22)
\]

As in Case I, the random variable $Y$ is the difference of two independent central chi-square random variables, each with two degrees of freedom. From [23] (Equation 4.5, with $\sigma_1^2 = \sigma_2^2 = \sigma^2$), the cdf of $Y$ is

\[
F_Y(z) = \begin{cases} 
\frac{1}{2} \exp\left(\frac{-z}{2\sigma^2}\right) & z < 0 \\
1 - \frac{1}{2} \exp\left(\frac{-z}{2\sigma^2}\right) & z \geq 0.
\end{cases} \quad (7.23)
\]

The random variable $X$ is the difference of two completely correlated non-central chi-squared random variables with the same variance, but different means. It is proven in Appendix C that $X$ is Gaussian with mean and variance

\[
\bar{X} = P_o(1 - \gamma)(1 + \gamma), \quad (7.24)
\]
\[
\sigma_X^2 = 4(1 - \gamma)^2 P_o \sigma^2 = \kappa \sigma^2. \quad (7.25)
\]
From [23] (Equations 1.1-1.4) the pdf and cdf are

\[ f_{X|\gamma}(z) = \frac{1}{\sqrt{2\pi} \sigma_X^2} \exp \left( -\frac{(z - \bar{X})^2}{2\sigma_X^2} \right) \]  \hspace{1cm} (7.27)

\[ F_{X|\gamma}(z) = 1 - Q \left( \frac{z - \bar{X}}{\sigma_X^2} \right) \]  \hspace{1cm} (7.28)

\[ = 1 - \frac{1}{2} \text{erfc} \left( \frac{z - \bar{X}}{\sqrt{2} \sigma_X^2} \right) \]  \hspace{1cm} (7.29)

with \( Q(z) \) denoting the Gaussian Q function and \( \text{erfc}(z) \) denoting the complimentary error function. Upon substituting the definitions of the mean and variance, the cdf of \( X \) at \( z = 0 \), in terms of the complimentary error function, is

\[ F_{X|\gamma}(0) = 1 - \frac{1}{2} \text{erfc} \left( \frac{-\bar{X}}{\sqrt{2} \sigma_X^2} \right) \]  \hspace{1cm} (7.30)

\[ = 1 - \frac{1}{2} \text{erfc} \left( -\sqrt{\frac{P_0}{8\sigma^2}} (1 + \gamma) \right) \]  \hspace{1cm} (7.31)

\[ = \frac{1}{2} \text{erfc} \left( \sqrt{\text{SIR} \frac{1 + \gamma}{2}} \right). \]  \hspace{1cm} (7.32)

The last equation is obtained using the anti-symmetric property of the error function, \( \text{erf}(-z) = -\text{erf}(z) \) to yield the identity \( 1 - \frac{1}{2} \text{erfc}(-z) = \frac{1}{2} \text{erfc}(z) \), and the definition of the Signal-to-Interference Ratio, \( \text{SIR} = \frac{P_0}{2\sigma^2} \).

The error probability is

\[ P_{III} \left( \text{SIR} \right| \alpha = \frac{1}{2}, \rho = 1, \gamma \right) \]

\[ = 1 - \frac{1}{2} \int_{-\infty}^{0} \exp \left( \frac{-z}{2\sigma^2} \right) f_{X|\gamma}(z) \, dz - \int_{0}^{\infty} f_{X|\gamma}(z) + \frac{1}{2} \int_{0}^{\infty} \exp \left( -\frac{z}{2\sigma^2} \right) f_{X|\gamma}(z) \, dz \]

\[ = 1 - \left[ 1 - F_{X|\gamma}(0) \right] + \frac{1}{2} \left( \int_{0}^{\infty} \exp \left( -\frac{z}{2\sigma^2} \right) f_{X|\gamma}(z) \, dz - \int_{-\infty}^{0} \exp \left( \frac{z}{2\sigma^2} \right) f_{X|\gamma}(z) \, dz \right) \]

\[ \equiv F_{X|\gamma}(0) + \frac{1}{2} \left( \Phi_1 - \Phi_2 \right). \]  \hspace{1cm} (7.33)
Using \( \sigma^2 = \sigma^2_X / \kappa \) to obtain a common denominator of \( 2\sigma^2_X \), the first integral is

\[
\Phi_1 = \frac{1}{\sqrt{2\pi \sigma^2_X}} \int_0^\infty \exp \left( -\frac{z}{2\sigma^2} - \frac{(z - X)^2}{2\sigma^2_X} \right) \, dz
\]

(7.34)

\[
= \frac{1}{\sqrt{2\pi \sigma^2_X}} \int_0^\infty \exp \left( -\frac{1}{2\sigma^2_X} \left[ z^2 + (\kappa - 2X)z + X^2 \right] \right) \, dz,
\]

(7.35)

and the second integral is

\[
\Phi_2 = \frac{1}{\sqrt{2\pi \sigma^2_X}} \int_{-\infty}^0 \exp \left( \frac{z}{2\sigma^2} - \frac{(z - X)^2}{2\sigma^2_X} \right) \, dz
\]

(7.36)

\[
= \frac{1}{\sqrt{2\pi \sigma^2_X}} \int_0^\infty \exp \left( -\frac{u}{2\sigma^2} - \frac{(u + X)^2}{2\sigma^2_X} \right) \, du
\]

(7.37)

\[
= \frac{1}{\sqrt{2\pi \sigma^2_X}} \int_0^\infty \exp \left( -\frac{1}{2\sigma^2_X} \left[ u^2 + (\kappa + 2X)z + X^2 \right] \right) \, du.
\]

(7.38)

where we have used the substitution \( u = -z \) and \( dz = -du \) such that \( z = -u \) and \( (z - X)^2 = (-u - X)^2 = (u + X)^2 \). It is proven in Appendix C.1 that integrals of this form have the solution

\[
\int_0^\infty \exp \left( -[Az^2 + Bz + C] \right) \, dz = \sqrt{\frac{\pi}{4A}} \exp \left( \frac{B^2 - 4AC}{4A} \right) \text{erfc} \left( \frac{B}{\sqrt{4A}} \right).
\]

(7.39)

### 7.2.1 Integral \( \Phi_1 \)

For the first integral, we have

\[
A = \frac{1}{2\sigma_X}, \quad B = \frac{\kappa - 2X}{2\sigma^2_X}, \quad C = \frac{X^2}{2\sigma^2_X};
\]

\[
B^2 - 4AC = \frac{\kappa^2 - 4\kappa X + 4X^2}{4\sigma^4_X} - \frac{4X^2}{4\sigma^4_X} = \frac{\kappa(\kappa - 4X)}{4\sigma^4_X}.
\]

Substituting these definitions into (7.39), we obtain

\[
\Phi_1 = \frac{\sqrt{2\pi \sigma^2_X}}{\sqrt{8\pi \sigma^2_X}} \exp \left( \frac{\kappa(\kappa - 4X)}{8\sigma^2_X} \right) \text{erfc} \left( \frac{\kappa - 2X}{\sqrt{8\sigma^2_X}} \right)
\]

(7.40)

\[
= \frac{1}{2} \exp \left( \frac{\kappa - 4X}{8\sigma^2} \right) \text{erfc} \left( \frac{\sqrt{\kappa - 2X}}{\sqrt{8\sigma^2_X}} \right).
\]

(7.41)
The last equation was obtained using $\sigma_X^2 = \kappa \sigma^2$. Substituting the definitions of $\kappa$ and $X$,

$$\sqrt{\kappa} = \sqrt{P_0} \frac{2(1 - \gamma)}{\kappa \sigma^2}$$

$$\frac{X}{\sqrt{\kappa}} = \sqrt{P_0} \left( \frac{1 + \gamma}{2} \right)$$

$$\frac{\sqrt{\kappa} - 2\left( \frac{X}{\sqrt{\kappa}} \right)}{\sqrt{8\sigma^2}} = \sqrt{\frac{P_0}{8\sigma^2}} \left[ 2(1 - \gamma) - (1 + \gamma) \right]$$

$$= \sqrt{\text{SIR}} \left( \frac{1 - 3\gamma}{2} \right)$$

$$\frac{\kappa + 4X}{8\sigma^2} = 4P_0 \left( \frac{1 - \gamma)^2 - (1 + \gamma)(1 - \gamma)}{8\sigma^2} \right)$$

$$= \text{SIR} (1 - \gamma) \left[ (1 - \gamma) - (1 + \gamma) \right]$$

$$= -\text{SIR} 2\gamma (1 - \gamma).$$

Thus, in terms of the SIR, the first integral is

$$\Phi_1 = \frac{1}{2} \exp \left( -\text{SIR} 2\gamma [1 - \gamma] \right) \text{erfc} \left( \sqrt{\text{SIR}} \frac{1 - 3\gamma}{2} \right). \quad (7.42)$$

### 7.2.2 Integral $\Phi_2$

The only difference from the last integral is that $B = \frac{\kappa + 4X}{2\sigma_X^2}$ instead of $B = \frac{\kappa - 4X}{2\sigma_X^2}$. Thus

$$B^2 - 4AC = \frac{\kappa^2 + 4\kappa X + 4X^2}{4\sigma_X^4} - \frac{4X^2}{4\sigma_X^4} = \frac{\kappa(\kappa + 4X)}{4\sigma_X^4}. \quad (7.39)$$

Substituting into (7.39), we obtain

$$\Phi_2 = \frac{1}{2} \exp \left( \frac{\kappa + 4X}{8\sigma^2} \right) \text{erfc} \left( \frac{\sqrt{\kappa} + 2\left( \frac{X}{\sqrt{\kappa}} \right)}{\sqrt{8\sigma^2}} \right). \quad (7.43)$$

Substituting the definitions of $\kappa$ and $X$,

$$\frac{\sqrt{\kappa} + 2\left( \frac{X}{\sqrt{\kappa}} \right)}{\sqrt{8\sigma^2}} = \sqrt{\frac{P_0}{8\sigma^2}} \left[ 2(1 - \gamma) + (1 + \gamma) \right]$$

$$= \sqrt{\text{SIR}} \left( \frac{3 - \gamma}{2} \right)$$

$$\frac{\kappa + 4X}{8\sigma^2} = \frac{4P_0}{8\sigma^2} \left[ (1 - \gamma)^2 + (1 + \gamma)(1 - \gamma) \right]$$

$$= \text{SIR} (1 - \gamma) \left[ (1 - \gamma) + (1 + \gamma) \right]$$

$$= \text{SIR} 2(1 - \gamma).$$
In terms of the SIR, the second integral is

$$\Phi_2 = \frac{1}{2} \exp \left( SIR \frac{2[1 - \gamma]}{2} \right) \text{erfc} \left( \sqrt{\text{SIR}} \frac{3 - \gamma}{2} \right). \quad (7.44)$$

### 7.2.3 Error Probability

Combining equations (7.32), (7.42) and (7.44) into (7.33), we finally obtain

$$P_{III} \left( SIR \bigg| \alpha = \frac{1}{2}, \rho = 1, \gamma \right) = \frac{1}{2} \text{erfc} \left( \sqrt{\text{SIR}} \frac{1 + \gamma}{2} \right)$$

$$+ \frac{1}{4} \exp \left( -\gamma SIR \frac{2[1 - \gamma]}{2} \right) \text{erfc} \left( \sqrt{\text{SIR}} \frac{1 - 3\gamma}{2} \right) \quad (7.45)$$

$$- \frac{1}{4} \exp \left( SIR \frac{2[1 - \gamma]}{2} \right) \text{erfc} \left( \sqrt{\text{SIR}} \frac{3 - \gamma}{2} \right).$$

An example for 15dB SIR is plotted in Figure 8.1. In the worst case of $\gamma = 1$, we obtain

$$P_{III} \left( SIR \bigg| \alpha = \frac{1}{2}, \rho = 1, \gamma = 1 \right) = \frac{1}{2} \text{erfc} \left( \sqrt{\text{SIR}} \right) + \frac{1}{4} \text{erfc} \left( -\sqrt{\text{SIR}} \right) - \frac{1}{4} \text{erfc} \left( \sqrt{\text{SIR}} \right)$$

$$= \frac{1}{4} \left[ \text{erfc} \left( \sqrt{\text{SIR}} \right) + \text{erfc} \left( -\sqrt{\text{SIR}} \right) \right]$$

$$= \frac{1}{4} \left[ 1 - \text{erf} \left( \sqrt{\text{SIR}} \right) + 1 - \text{erf} \left( -\sqrt{\text{SIR}} \right) \right]$$

$$= \frac{1}{4} \left[ 2 - \text{erf} \left( \sqrt{\text{SIR}} \right) + \text{erf} \left( \sqrt{\text{SIR}} \right) \right]$$

$$= \frac{1}{2} \quad (7.46)$$

as expected. It is evident in this example of Figure 8.1 that the error probability rapidly decreases with $\gamma \to \frac{1}{2}$, obtaining the minimum value near this point. The exact location of this minima depends on the SIR and cannot be easily obtained. The error probability then slowly increases up to $\gamma = 0$. The upper dashed line in 8.1 is an upper bound for $\gamma = 0$, which is derived below.
7.2.4 Upper Bound for $\gamma = 0$

At $\gamma = 0$, the error probability is

$$P_{III}(SIR \mid \alpha = \frac{1}{2}, \rho = 1, \gamma = 0) = \frac{3}{4} \text{erfc} \left( \sqrt{\frac{SIR}{4}} \right) - \frac{1}{4} \exp \left( 2 SIR \right) \text{erfc} \left( \sqrt{\frac{9 SIR}{4}} \right).$$

(7.47)

Although this is an exact result, it still somewhat cumbersome, and a simple exponential-type bound is more useful. Using the upper bound [23, Eq. C.14]

$$Q \left( \sqrt{x} \right) \leq \exp \left( -\frac{x}{2} \right)$$

and the relation [23, Eq. 1.4]

$$Q \left( \sqrt{x} \right) = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{x}{2}} \right)$$

$$\rightarrow \text{erfc} \left( \sqrt{x} \right) = 2Q \left( \sqrt{2x} \right),$$

we obtain an upper bound for the complimentary error function,

$$\text{erfc} \left( \sqrt{x} \right) \leq \frac{\exp(-x)}{\sqrt{\pi x}}.$$ 

Applying this to (7.47),

$$P_{III}(SIR \mid \alpha = \frac{1}{2}, \rho = 1, \gamma = 0)$$

$$\leq \frac{3}{4} \sqrt{\frac{4}{\pi SIR}} \exp \left( -\frac{SIR}{4} \right) - \frac{1}{4} \sqrt{\frac{4}{9\pi SIR}} \exp \left( -\frac{9}{4} SIR + 2 SIR \right)$$

$$= \frac{3}{2\sqrt{\pi SIR}} \exp \left( -\frac{SIR}{4} \right) - \frac{1}{6\sqrt{\pi SIR}} \exp \left( -\frac{SIR}{4} \right)$$

$$= \frac{4}{3\sqrt{\pi SIR}} \exp \left( -\frac{SIR}{4} \right)$$

(7.48)
7.3 \( \gamma = 1 \), Non-central \( \chi^2 \) Limit

We wish to prove that in the worst case of complete correlation when \( \tau = 0 \), the Case III error probability conditioned on \( \alpha \) is

\[
P_{\text{III}} (SIR | \alpha, \tau = 0) = \frac{\alpha^2}{3\alpha - 1} \exp \left( - \frac{SIR}{3\alpha - 1} \right). \tag{7.49}
\]

Proof of Theorem 7.1.2. As defined in (2.32), the error probability for Case III is

\[
P_{\text{III}} (SIR | \alpha, \rho, \gamma) \equiv P_r \{ X < Y \} \tag{7.50}
\]

where

\[
X = \alpha \| \sqrt{P_0 + N_1} \|^2 - (1 - \alpha) \| \sqrt{P_0 \gamma + N_4} \|^2 \tag{7.51}
\]

\[
= \chi^2 \left( \alpha P_0, \alpha \sigma^2 \right) - \chi^2 \left( (1 - \alpha) P_0 \gamma^2 \sigma^2 \right)_{\rho = 1, \rho \neq \rho} \tag{7.52}
\]

\[
Y = \alpha \| N_3 \|^2 - (1 - \alpha) \| N_2 \|^2 \tag{7.53}
\]

\[
= \chi^2 \left( \alpha \sigma^2 \right) - \chi^2 \left( (1 - \alpha) \sigma^2 \right)_{\rho = 1, \rho \neq \rho} \tag{7.54}
\]

Note that \( Y \) is conditioned on \( \alpha \). With \( \tau = 0 \), both \( \rho = R(\tau) = R(0) = 1 \) and \( \gamma = R(\tau) = R(0) = 1 \). Therefore, in the limit of \( \rho = 1 \), \( N_1 = N_4 \) as in the previous section. With \( \gamma = 1 \), we obtain

\[
X = \alpha \| \sqrt{P_0 + N_1} \|^2 - (1 - \alpha) \| \sqrt{P_0 + N_1} \|^2 \tag{7.55}
\]

which is a non-central chi-square variate with two degrees of freedom that is also conditioned on \( \alpha \). Note that the subtraction of two differently scaled replicas of the same random variable, \( N_1 \), to produce a different scaling parameter was only possible because \( \gamma = 1 \), and the non-centrality parameters are the same. Recall that for the case of \( \alpha = \frac{1}{2} \) with general \( \gamma \), the random variable \( X \) was Gaussian, conditioned on \( \gamma \), and \( Y \) was not conditioned.
We express the error probability in terms of the conditional pdf and cdf of $X$ and $Y$,

$$P_{III} (SIR | \alpha, \rho = 1, \gamma = 1) = 1 - \int_{-\infty}^{\infty} F_{Y|\alpha}(z) f_{X|\alpha}(z) dz. \quad (7.56)$$

Since $X$ is a non-central chi-square random variable with two degrees of freedom, conditioned on $\alpha$, the conditional pdf is [23]

$$f_{X|\alpha}(z) = \frac{1}{2\sigma_1^2} \exp \left( -\frac{z + a^2}{2\sigma_1^2} \right) I_0 \left( \frac{a^2 z}{\sigma_1^2} \right) = \frac{1}{(2\alpha - 1)2\sigma^2} \exp \left( -z + \frac{(2\alpha - 1)P_o}{(2\alpha - 1)2\sigma^2} \right) I_0 \left( \sqrt{\frac{P_o z}{(2\alpha - 1)\sigma^4}} \right) \quad (7.57)$$

for $z \geq 0$, with $\sigma_1^2 = (2\alpha - 1)\sigma^2$ and $a^2 = (2\alpha - 1)P_o$, and $Y$ is the difference of independent, central chi-square random variables with two degrees of freedom and cdf [23]

$$F_{Y|\alpha}(z) = 1 - \frac{\sigma_2^3}{\sigma_2^2 + \sigma_3^2} \exp \left( -\frac{z}{2\sigma_3^2} \right) \quad (7.58)$$

$$= 1 - \alpha \exp \left( -\frac{z}{2\alpha\sigma^2} \right) \quad (7.59)$$

for $z \geq 0$, with $\sigma_2 = (1 - \alpha)\sigma^2$ and $\sigma_3^2 = \alpha\sigma^2$.

Thus, the error probability is

$$P_{III} (SIR | \alpha, \rho = 1, \gamma = 1)$$

$$= 1 - \int_{0}^{\infty} f_{X|\alpha}(z) F_{Y|\alpha}(z) \, dz$$

$$= \int_{0}^{\infty} \left[ \alpha \exp \left( -\frac{z}{2\alpha\sigma^2} \right) \right]$$

$$\times \left[ \frac{1}{(2\alpha - 1)2\sigma^2} \exp \left( -\frac{z}{(2\alpha - 1)2\sigma^2} \right) \exp \left( -\frac{P_o}{2\sigma^2} \right) I_0 \left( \sqrt{\frac{P_o z}{(2\alpha - 1)\sigma^4}} \right) \right] \, dz \quad (7.60)$$

$$= \frac{\alpha}{(2\alpha - 1)2\sigma^2} \exp \left( -\frac{P_o}{2\sigma^2} \right) \int_{0}^{\infty} \exp \left( -\frac{z}{\alpha(2\alpha - 1)2\sigma^2} \right) I_0 \left( \sqrt{\frac{P_o z}{(2\alpha - 1)\sigma^4}} \right) \, dz$$

$$= \frac{\alpha}{(2\alpha - 1)2\sigma^2} \exp \left( -\frac{P_o}{2\sigma^2} \right) \int_{0}^{\infty} \exp (-cz) I_0 \left( \sqrt{bz} \right) \, dz$$

with $c \equiv \frac{3\alpha - 1}{2\sigma^2\alpha(2\alpha - 1)}$, $b \equiv \frac{P_o}{(2\alpha - 1)\sigma^4}$.
Now let \( u \equiv \sqrt{z} \), with \( dz = 2udu \). We can express the last integral as

\[
2 \int_{0}^{\infty} u \exp\left(-cu^2\right) I_0\left(\sqrt{bu}\right) \, du. \tag{7.61}
\]

From the definition of the modified Bessel function, \( I_n(x) = j^{-n} J_n(jx) \), where \( j \equiv \sqrt{-1} \) and \( J_n(x) \) is the Bessel function of the first kind [1]. Thus, the previous integral can be rewritten in terms of the (unmodified) Bessel function as

\[
2 \int_{0}^{\infty} u \exp\left(-cu^2\right) J_o\left(\sqrt{bju}\right) \, du. \tag{7.62}
\]

From [7, Eq. 6.631.4],

\[
2 \int_{0}^{\infty} u \exp\left(-cu^2\right) J_0\left(\sqrt{bju}\right) \, du = \frac{1}{c} \exp\left(\frac{b}{4c}\right). \tag{7.63}
\]

Substituting the defined values for \( b \) and \( c \) and cancelling common factors, the error probability simplifies to

\[
P_{III}(\text{SIR} \mid \alpha, \tau = 0) = P_{III}(\text{SIR} \mid \alpha, \rho = 1, \gamma = 1) \tag{7.64}
= \frac{\alpha^2}{3\alpha - 1} \exp\left(-\frac{P_o}{2\sigma^2} \left[1 - \frac{\alpha}{3\alpha - 1}\right]\right) \tag{7.65}
= \frac{\alpha^2}{3\alpha - 1} \exp\left(-\text{SIR} \frac{2\alpha - 1}{3\alpha - 1}\right). \tag{7.66}
\]

7.4 Case III Unconditioned Upper Bound

Given that \( \tau = 0 \), we wish to show that when unconditioned on \( \alpha \),

\[
P_{III}(\text{SIR} \mid \tau = 0) = \int_{1/2}^{1} P_{III}(\text{SIR} \mid \alpha = \alpha, \tau = 0) \, f_{\alpha}(\alpha) \, d\alpha
= \frac{2}{27} \exp\left(-\frac{2}{3} \text{SIR}\right) \left[\left\{ \text{Ei}\left(\frac{2}{3} \text{SIR}\right) - \text{Ei}\left(\frac{1}{6} \text{SIR}\right)\right\} \left(1 + \frac{2}{3} \text{SIR} + \frac{1}{18} \text{SIR}^2\right)
- \exp\left(\frac{2}{3} \text{SIR}\right) \left(\frac{9}{8} + \frac{1}{12} \text{SIR}\right) + \exp\left(\frac{1}{6} \text{SIR}\right) \left(6 + \frac{1}{3} \text{SIR}\right)\right], \tag{7.67}
\]
where the exponential-integral, \( \text{Ei}(x) \), is defined as [8, Eq. 8.211]

\[
\text{Ei}(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} \, dt \quad \text{if } x < 0.
\] (7.68)

**Proof of Theorem 7.1.3.** Since \( \alpha \) is uniform over \( (\frac{1}{2}, 1) \), the error probability given \( \tau = 0 \), unconditioned over \( \alpha \) is

\[
P_{III}(\text{SIR} | \tau = 0) = 2 \int_{1/2}^{1} P_{III}(\text{SIR} | \alpha = \alpha, \tau = 0) \, d\alpha
\]

\[
= 2 \int_{1/2}^{1} \frac{\alpha^2}{3\alpha - 1} \exp \left( - SIR \frac{2\alpha - 1}{3\alpha - 1} \right) \, d\alpha.
\] (7.69)

where we have applied the previous result from Theorem 7.1.2. By partial fraction expansion,

\[
\frac{2\alpha - 1}{3\alpha - 1} = \frac{2}{3} - \frac{1}{3} \frac{1}{3\alpha - 1}.
\] (7.70)

Thus, the unconditioned error probability is

\[
P_{III}(\text{SIR} | \tau = 0) = 2 \int_{1/2}^{1} \frac{\alpha^2}{3\alpha - 1} \exp \left( - \frac{2}{3} SIR \right) \exp \left( \frac{1}{3} SIR \frac{1}{3\alpha - 1} \right) \, d\alpha.
\] (7.71)

Let \( u \equiv \frac{1}{3\alpha - 1} \) with \( d\alpha = -\frac{1}{3u^2} \, du \). Then \( \alpha = \frac{1}{3} \left(1 + \frac{1}{u}\right) \) and \( \frac{\alpha^2}{3\alpha - 1} = \frac{u}{9} \left(1 + \frac{1}{u}\right)^2 \), yielding

\[
= \frac{2}{27} \exp \left( - \frac{2}{3} SIR \right) \int_{1/2}^{2} \frac{1}{u} \left(1 + \frac{1}{u}\right)^2 \exp \left( \frac{1}{3} SIR u \right) \, du
\] (7.72)

\[
= \frac{2}{27} \exp \left( - \frac{2}{3} SIR \right) \int_{1/2}^{2} \left(\frac{1}{u^2} + \frac{1}{u^3}\right) \exp \left( \frac{1}{3} SIR u \right) \, du.
\] (7.73)

(7.74)

From [7], Equations 2.325

\[
\int \frac{1}{u} \exp (au) \, du = \text{Ei} (au)
\] (7.75)

\[
\int \frac{1}{u^2} \exp (au) \, du = a \text{Ei} (au) - \frac{1}{u} \exp (au)
\] (7.76)

\[
\int \frac{1}{u^3} \exp (au) \, du = \frac{a^2}{2} \text{Ei} (au) - \exp (au) \left(\frac{a}{2u} + \frac{1}{2u^2}\right)
\] (7.77)
Thus, with $a \equiv \frac{1}{3} \text{SIR}$, we obtain

\[
P_{\text{III}}(\text{SIR} | \tau = 0) = \frac{2}{27} \exp \left( -\frac{2}{3} \text{SIR} \right) \left[ \text{Ei} \left( \frac{u}{3} \text{SIR} \right) \left( 1 + \frac{2}{3} \text{SIR} + \frac{1}{18} \text{SIR}^2 \right) \right. \\
- \exp \left( \frac{u}{3} \text{SIR} \right) \left( \frac{2}{u} + \frac{\text{SIR}}{6u} + \frac{1}{2u^2} \right) \left] \right|_{u=2}^{u=1/2}.
\] (7.78)

\[= \frac{2}{27} \exp \left( -\frac{2}{3} \text{SIR} \right) \left[ \left\{ \text{Ei} \left( \frac{2}{3} \text{SIR} \right) - \text{Ei} \left( \frac{1}{6} \text{SIR} \right) \right\} \left( 1 + \frac{2}{3} \text{SIR} + \frac{1}{18} \text{SIR}^2 \right) \\
- \exp \left( \frac{2}{3} \text{SIR} \right) \left( \frac{9}{8} + \frac{1}{12} \text{SIR} \right) + \exp \left( \frac{1}{6} \text{SIR} \right) \left( 6 + \frac{1}{3} \text{SIR} \right) \right].
\]

\[\Box\]

7.5 Case III, $\rho = 1$, General $(\alpha, \gamma)$

Proof of Theorem 7.1.4. As defined in (2.32), the error probability for Case III is

\[
P_{\text{III}}(\text{SIR} | \alpha, \rho, \gamma) \equiv \mathbb{P} \{ X < Y \} \]

(7.79)

where

\[
X = \alpha \left\| \sqrt{P_o + N_1} \right\|^2 - (1 - \alpha) \left\| \sqrt{P_o \gamma + N_4} \right\|^2
\] (7.80)

\[
Y = \alpha \left\| N_3 \right\|^2 - (1 - \alpha) \left\| N_2 \right\|^2
\] (7.82)

In the limit of $\rho = 1$, $N_4 = N_1$ and we obtain

\[
X = \alpha \left\| \sqrt{P_o + N_1} \right\|^2 - (1 - \alpha) \left\| \sqrt{P_o \gamma + N_1} \right\|^2.
\]
In this general case, the random variable \( X \) is conditioned on both \( \alpha \) and \( \gamma \), and \( Y \) is conditioned on \( \alpha \). As in the previous cases, we can express the error probability in terms of the conditional pdf and conditional cdf of \( X \) and \( Y \),

\[
P_{\text{III}}(SIR \mid \alpha, \rho = 1, \gamma) = 1 - \int_{-\infty}^{\infty} F_{Y\mid\alpha}(z) f_{X\mid\alpha,\gamma}(z) \, dz. \tag{7.84}
\]

As in Case I, the random variable \( Y \) is the difference of two independent central chi-square random variables with two degrees of freedom. From [23, Eq. (4.5)], with \( \sigma_1^2 = \alpha \sigma^2 \), and \( \sigma_2^2 = (1 - \alpha) \sigma^2 \), the cdf of \( Y \) is

\[
F_{Y\mid\alpha}(z) = \begin{cases} 
(1 - \alpha) \exp\left(\frac{-z^2}{2(1-\alpha)\sigma^2}\right) & z < 0 \\
1 - \alpha \exp\left(-\frac{z^2}{2\alpha \sigma^2}\right) & z \geq 0.
\end{cases} \tag{7.85}
\]

In the limiting case of \( \alpha = (1 - \alpha) = \frac{1}{2} \), it was shown that the random variable \( X \) is Gaussian. In general, with \( \alpha \neq (1 - \alpha) \), it is proven in Appendix D that the random variable \( X \), which is the difference of completely correlated non-central chi-square random variables with different means and weights, has a shifted non-central chi-square distribution,

\[
f_{X\mid\alpha,\gamma}(z) = \frac{U(z + \kappa)}{2\sigma_x^2} \exp\left(-\frac{(z + \kappa) + \nu^2}{2\sigma_x^2}\right) I_0 \left(\frac{\sqrt{(z + \kappa)\nu^2}}{\sigma_x^2}\right) \tag{7.86}
\]

where \( \sigma_x^2 = (2\alpha - 1)\sigma^2 \), \( \nu^2 = P_o \frac{[\alpha - (1-\alpha)\gamma]^2}{2\alpha - 1} \), \( \kappa = P_o \frac{\alpha(1-\alpha)(1-\gamma)^2}{2\alpha - 1} \), and \( U(x) \) is the unit step function. Thus

\[
P_{\text{III}}(SIR \mid \alpha, \rho = 1, \gamma) = 1 - \int_{-\kappa}^{\infty} F_{Y\mid\alpha}(z) f_{X\mid\alpha,\gamma}(z) \, dz, \tag{7.87}
\]
and, since $\kappa \geq 0$, the interval of integration separates into two regions,

$$P_{III}(\text{SIR} | \alpha, \rho = 1, \gamma)$$

$$= 1 - \left[ \int_{-\kappa}^{0} F_Y|\alpha(z) f_X|\alpha,\gamma(z) \, dz + \int_{0}^{\infty} F_Y|\alpha(z) f_x(z) \, dz \right]$$

$$= 1 - \left[ (1 - \alpha) \int_{-\kappa}^{0} f_X|\alpha,\gamma(z) \exp \left( -\frac{z}{2(1 - \alpha)} \right) \, dz \int_{0}^{\infty} f_X|\alpha,\gamma(z) \, dz 
- \alpha \int_{0}^{\infty} f_x(z) \exp \left( -\frac{z}{2\alpha \sigma^2} \right) \, dz \right]$$

$$= 1 - \left[ (1 - \alpha) \int_{0}^{\kappa} f_X|\alpha,\gamma(z - \kappa) \exp \left( -\frac{z - \kappa}{2(1 - \alpha)} \right) \, dz + \int_{\kappa}^{\infty} f_X|\alpha,\gamma(z - \kappa) \, dz 
- \alpha \int_{\kappa}^{\infty} f_X|\alpha,\gamma(z - \kappa) \exp \left( -\frac{z - \kappa}{2\alpha \sigma^2} \right) \, dz \right]$$

$$= 1 - \left[ (1 - \alpha) \int_{0}^{\kappa} f_X|\alpha,\gamma(z) \exp \left( -\frac{z - \kappa}{2\alpha \sigma^2} \right) \, dz \right]$$

(7.88)

where we have shifted the regions of integration by $\kappa$. Let $f_X(z) \equiv f_X|\alpha,\gamma(z - \kappa)$ denote the standard non-central chi-squared distribution that arises from shifting the pdf of $X$ back to the origin, (i.e. $\chi \equiv X + \kappa$). Then, we can express the error probability as

$$P_{III}(\text{SIR} | \alpha, \rho = 1, \gamma)$$

$$= 1 - \left[ (1 - \alpha) \int_{0}^{\kappa} f_X|\alpha,\gamma(z) \exp \left( -\frac{z}{2(1 - \alpha)} \right) \, dz 
+ \int_{\kappa}^{\infty} f_X|\alpha,\gamma(z) \, dz - \alpha \int_{\kappa}^{\infty} f_X|\alpha,\gamma(z) \exp \left( -\frac{z - \kappa}{2\alpha \sigma^2} \right) \, dz \right]$$

$$= F_X(\kappa) + \alpha \int_{\kappa}^{\infty} f_X|\alpha,\gamma(z) \exp \left( -\frac{z - \kappa}{2\alpha \sigma^2} \right) \, dz - (1 - \alpha) \int_{0}^{\kappa} f_X|\alpha,\gamma(z) \exp \left( -\frac{z}{2(1 - \alpha)} \right) \, dz$$

$$= 1 - Q_1 \left( \sqrt{\frac{\nu^2}{\sigma_x^2}}, \sqrt{\frac{\kappa}{\sigma_x^2}} \right) + \alpha \phi_1 - (1 - \alpha) \phi_2,$$

(7.89)

since $\chi$ is a non-central chi-square r.v. with non-centrality parameter $\nu^2$, component noise variance $\sigma_x^2$ and two degrees of freedom [23, Eq. (2.45)], and the cdf is evaluated at $\kappa$. The two integrals $\phi_1$ and $\phi_2$ in (7.89) are defined as

$$\phi_1 \equiv \int_{\kappa}^{\infty} f_X|\alpha,\gamma(z) \exp \left( -\frac{z - \kappa}{2\sigma_1^2} \right) \, dz$$

(7.90)
with $\sigma_1^2 = \alpha \sigma^2$, and

$$
\phi_2 = \int_0^\kappa f_X(z) \exp \left( \frac{z - \kappa}{2\sigma^2} \right) \, dz
$$

(7.91)

with $\sigma_2^2 = (1 - \alpha)\sigma^2$, and the first-order Marcum-Q function is defined as [23, Eq. (2.20)]

$$
Q_1(A, B) \equiv \int_B^\infty x \exp \left( - \frac{x^2 + A^2}{2} \right) I_0(Ax).
$$

(7.92)

It will be shown that the integrals $\phi_1$ and $\phi_2$ can also be expressed in terms of the first-order Marcum-Q function.

### 7.5.1 Integral $\phi_1$

Turning our attention to the first integral, and using the substitution $u = \sqrt{z}$, such that $z = u^2$ and $dz = 2udu$, we have

$$
\phi_1 = 2 \int_{\sqrt{\kappa}}^\infty uf_X(u^2) \exp \left( - \frac{u^2 - \kappa}{2\sigma_1^2} \right) \, du
$$

(7.93)

$$
= \frac{1}{\sigma_1^2} \int_{\sqrt{\kappa}}^\infty u \exp \left( - \frac{u^2(\sigma_x^2 + \sigma_1^2) + \nu^2\sigma_1^2 - \kappa\sigma_x^2}{2\sigma_x^2\sigma_1^2} \right) I_0 \left( \frac{u\nu}{\sigma_x^2} \right) \, du.
$$

(7.94)

Now, we make the substitution $q^2 = u^2 \left[ \frac{\sigma_x^2 + \sigma_1^2}{\sigma_x^2\sigma_1^2} \right] = u^2p^2$, i.e. $q = up$ with $p \equiv \sqrt{\frac{\sigma_x^2 + \sigma_1^2}{\sigma_x^2\sigma_1^2}}$, such that $u = \frac{q}{p}$ and $du = \frac{dq}{p}$. Note that $p$ must be real since the variances are non-negative.

Therefore,

$$
\phi_1 = \frac{1}{p^2\sigma_x^2} \int_{p\sqrt{\kappa}}^\infty q \exp \left( - \frac{q^2}{2} - \frac{\nu^2\sigma_1^2 - \kappa\sigma_x^2}{2\sigma_x^2\sigma_1^2} \right) I_0 \left( \frac{\nu q}{p\sigma_x^2} \right) \, dq.
$$

(7.95)
Note that \( p^2 \sigma_x^2 = \frac{\sigma_x^2 + \sigma_1^2}{\sigma_1^2} \). Now, let \( A = \frac{\nu}{\sigma_x^2} \), so that \( A^2 = \frac{\nu^2 \sigma_x^2}{\sigma_x^2 + \sigma_1^2} \) and we can express \( \phi_1 \) in terms of the first-order Marcum-Q function as

\[
\phi_1 = \frac{\sigma_1^2}{\sigma_x^2 + \sigma_1^2} \exp\left(-\frac{\nu^2 \sigma_1^2}{2 \sigma_x^2 \sigma_1^2}\right) \exp\left(\frac{A^2}{2}\right) \int_0^\infty q \exp\left(-\frac{q^2 + A^2}{2}\right) I_0(Aq) \, dq
\]

\[
= \frac{\sigma_1^2}{\sigma_x^2 + \sigma_1^2} \exp\left(-\frac{[\nu^2 \sigma_1^2 - \kappa \sigma_x^2]}{2 \sigma_x^2 \sigma_1^2}\left(\sigma_x^2 + \sigma_1^2\right) - \nu^2 \sigma_1^4\right) Q_1(A, p \sqrt{\kappa})
\]

\[
= \frac{\sigma_1^2}{\sigma_x^2 + \sigma_1^2} \exp\left(-\frac{\nu^2}{2 \sigma_x^2} + \frac{\kappa}{2 \sigma_1^2}\right) Q_1\left(\sqrt{\frac{\nu^2 \sigma_1^2}{\sigma_x^2 \sigma_1^2}}, \sqrt{\frac{\kappa \sigma_x^2}{\sigma_x^2 \sigma_1^2}}\right)
\]

\[
= \frac{1}{\beta_1} \exp\left(-\frac{\nu^2}{2 \sigma_x^2} + \frac{\kappa}{2 \sigma_1^2}\right) Q_1\left(\sqrt{\frac{\nu^2 \sigma_x^2}{\sigma_x^2 \sigma_1^2}}, \sqrt{\frac{\kappa}{\sigma_1^2}}\right),
\]

(7.96)

with \( \sigma_x^2 \equiv \sigma_x^2 + \sigma_1^2 \) and \( \beta_1 \equiv \frac{\sigma_x^2 + \sigma_1^2}{\sigma_1^2} \).

To express this in terms of \( \Lambda \equiv \frac{SIR}{(2\alpha - 1)^2} = \frac{P_0}{2(2\alpha - 1)^2 \sigma_x^2} \), we substitute for \( \nu^2, \kappa, \sigma_x^2 \) and \( \sigma_1^2 \) and then simplify to obtain

\[
\phi_1 = \frac{\alpha}{3\alpha - 1} \exp\left(-\Lambda \frac{2\alpha - 1}{3\alpha - 1} \left[\mu^2 + (3\alpha - 1)(1 - \alpha)(1 - \gamma)^2\right]\right)
\]

\[
\times Q_1\left(\sqrt{\frac{2\Lambda \alpha}{3\alpha - 1}}, (1 - \gamma) \sqrt{2\Lambda(1 - \alpha)(3\alpha - 1)}\right)
\]

(7.97)

with \( \mu = \alpha - (1 - \alpha) \gamma \).

**7.5.2 Integral \( \phi_2 \)**

Similar to the previous integral, we now wish to express \( \phi_2 \) in terms of the Marcum-Q function. We begin with

\[
\phi_2 = \int_0^{\kappa} f_\chi(z) \exp\left(-\frac{z - \kappa}{2 \sigma_2^2}\right) dz,
\]

(7.98)

where the non-central chi-square pdf for \( \chi \equiv X + \kappa \) is

\[
f_\chi(z) = \frac{U(z)}{2 \sigma_2^2} \exp\left(-\frac{z + \nu^2}{2 \sigma_2^2}\right) I_0\left(\sqrt{\frac{z \nu^2}{\sigma_2^2}}\right),
\]

(7.99)
We define the difference of the variances as \( \sigma_{\text{diff}} \) originally defined for the shifted pdf (7.86). Therefore, \( \sigma_{\text{diff}} = \sigma_2^2 - \sigma_x^2 \) which a standard non-central chi-square distribution with the same parameters \( u \) and \( \nu \).

Now, defining \( A = \sigma_2^2 - \sigma_x^2 \), and make the substitution \( q^2 = u^2p^2 \) such that \( u^2 = \frac{q^2}{p^2} \), \( u = \frac{q}{p} \), \( du = \frac{dq}{p} \), so that

\[
\phi_2 = \frac{1}{\beta_2} \exp\left(-\left[\frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2}\right]\right) \int_0^{\sqrt{p^2 \kappa}} q \exp\left(-\frac{q^2}{2}\right) I_0 \left(\frac{\sqrt{p^2 \kappa}}{\beta_2 \sigma_x^2}\right) dq.
\]

We now introduce \( eta_2 \equiv \frac{\sigma_{\text{diff}}^2}{\sigma_x^2} \) and \( p^2 \equiv \frac{\beta_2}{\sigma_x^2} \), and make the substitution \( q^2 = u^2p^2 \) such that \( u^2 = \frac{q^2}{p^2} \), \( u = \frac{q}{p} \), \( du = \frac{dq}{p} \), so that

\[
\phi_2 = \frac{1}{\beta_2} \exp\left(-\left[\frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2}\right]\right) \int_0^{\sqrt{p^2 \kappa}} q \exp\left(-\frac{q^2}{2}\right) I_0 \left(\frac{\sqrt{p^2 \kappa}}{\beta_2 \sigma_x^2}\right) dq.
\]

Now, defining \( A^2 \equiv \frac{\nu^2}{\beta_2 \sigma_x^2} \),

\[
\phi_2 = \frac{1}{\beta_2} \exp\left(-\left[\frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2}\right]\right) \int_0^{\sqrt{p^2 \kappa}} q \exp\left(-\frac{q^2 + A^2}{2}\right) I_0 \left(Aq\right) dq
\]

\[
\phi_2 = \frac{1}{\beta_2} \exp\left(-\left[\frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2}\right]\right) \left[1 - Q_1 \left(\sqrt{\frac{\nu^2}{\beta_2 \sigma_x^2}}, \sqrt{\frac{\kappa}{\beta_2 \sigma_x^2}}\right)\right]
\]

The Interval \( \frac{1}{2} < \alpha < \frac{2}{3} \):

Since \( \sigma_{\text{diff}}^2 > 0 \) is positive in this case, the integral is well-behaved since the sign of the exponential in the integral (7.100) is negative. Using the substitution \( u = \sqrt{z} \),

\[
\phi_2 = \frac{1}{\sigma_x^2} \exp\left(-\left[\frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2}\right]\right) \int_0^{\sqrt{\kappa}} \exp\left(-\frac{u^2 \sigma_{\text{diff}}^2}{2\sigma_x^2 \sigma_2^2}\right) I_0 \left(\frac{\sqrt{z \nu^2}}{\sigma_x^2}\right) du.
\]
which follows from the definition of the first-order Marcum Q-Function (7.92). Recognizing that \( \beta_2 - 1 = -\frac{\sigma_2^2}{\sigma_x^2} \), we finally obtain
\[
\phi_2 = \frac{1}{\beta_2} \exp \left( + \frac{\nu^2}{2\sigma_{\text{diff}}^2} - \frac{\kappa}{2\sigma_x^2} \right) \left[ 1 - Q_1 \left( \sqrt{\frac{\nu^2}{\sigma_x^2}} \beta_2, \sqrt{\frac{\kappa}{\sigma_x^2}} \beta_2 \right) \right].
\]
(7.104)

In terms of \( \Lambda = \frac{\text{SIR}}{(2\alpha - 1)^2} \), again substituting for \( \nu^2, \kappa, \sigma_x^2 \) and then simplifying, we obtain
\[
\phi_2 = \frac{1 - \alpha}{2 - 3\alpha} \exp \left( + \Lambda \frac{2\alpha - 1}{2 - 3\alpha} \left[ \mu^2 - \alpha(2 - 3\alpha)(1 - \gamma^2) \right] \right) \times \left[ 1 - Q_1 \left( \mu \sqrt{\frac{2\Lambda(1 - \alpha)}{2 - 3\alpha}}, (1 - \gamma)\sqrt{2\Lambda(2 - 3\alpha)} \right) \right]
\]
(7.105)
again with \( \mu = \alpha - (1 - \alpha)\gamma \).

The Point \( \alpha = \frac{2}{3} \):

From (7.101), substituting \( \sigma_{\text{diff}}^2 = 0 \), we have
\[
\phi_2 = \frac{1}{\sigma_x^2} \exp \left( - \left[ \frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2} \right] \right) \int_0^{\sqrt{\kappa}} u I_0 \left( u \frac{\sqrt{\nu^2}}{\sigma_x^2} \right) \, du.
\]
(7.106)

From [8, Eq. (6.561.7)],
\[
\int_0^1 x I_0 \left( ax \right) \, dx = \frac{1}{a} I_1 \left( a \right).
\]
(7.107)

Making the substitution \( p = \frac{u}{\sqrt{\kappa}} \) such that \( u = \sqrt{\kappa} p \) and \( du = \sqrt{\kappa} \, dp \), \( \phi_2 \) can be expressed as
\[
\phi_2 = \frac{\kappa}{\sigma_x^2} \exp \left( - \left[ \frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2} \right] \right) \int_0^1 p I_0 \left( p \frac{\sqrt{\nu^2}}{\sigma_x^2} \right) \, dp
\]
(7.108)
with \( a = \frac{\sqrt{\nu^2}}{\sigma_x^2} \). Therefore,
\[
\phi_2 = \frac{\sqrt{\kappa}}{\nu^2} \exp \left( - \left[ \frac{\nu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2} \right] \right) I_1 \left( \frac{\sqrt{\nu^2}}{\sigma_x^2} \right).
\]
(7.109)

After substituting the definitions for \( \kappa, \nu^2, \sigma_x^2 \), and \( \sigma_2^2 \) and simplifying, we obtain
\[
\phi_2 = \frac{\sqrt{2}(1 - \gamma)}{(2 - \gamma)} \exp \left( - \frac{\text{SIR}}{(2\alpha - 1)^2} \left[ (\alpha - (1 - \alpha)\gamma)^2 + \alpha(2\alpha - 1)(1 - \gamma)^2 \right] \right) \times I_1 \left( \frac{\text{SIR}}{(2\alpha - 1)^2} 2(1 - \gamma)\sqrt{\alpha(1 - \alpha)} \left[ \alpha - (1 - \alpha)\gamma \right] \right).
\]
(7.110)
Substituting $\alpha = \frac{2}{3}$ and simplifying in terms of $\Lambda = \frac{SIR}{(2\alpha - 1)^2} = 9SIR$, we finally obtain

$$\phi_2 = \frac{\sqrt{2}(1 - \gamma)}{(2 - \gamma)} \exp \left( - \frac{\Lambda}{9} \left[ 3\gamma^2 - 8\gamma + 6 \right] \right) I_1 \left( \frac{\Lambda \sqrt{8}}{9} \left[ \gamma^2 - 3\gamma + 2 \right] \right). \quad (7.111)$$

**The Interval $\frac{2}{3} < \alpha \leq 1$:**

When $\sigma_{\text{diff}}^2$ is negative, the sign of the exponential in the integral (7.100) is positive, and the error cannot be represented in terms of the Marcum-Q function with real arguments. This is simply a consequence of the algebraic manipulations required to obtain arguments in the form of the Marcum-Q function.

Since $\phi_2$ has finite limits, for the purposes of numerical evaluation it is preferable to use the original Equation (7.100), which is well behaved for all $\alpha > \frac{1}{2}$ and any $\gamma$, and does not need to be evaluated piecewise. Recall that the limit of $\alpha = 1/2$ was treated in Theorem 7.1.1. A succinct representation for $\phi_2$ with a range over $(0, 1)$ is presented in the next section.

**Numerical Evaluation of $\phi_2$**

Beginning with (7.100), let $u = \frac{z}{\kappa}$ such that $z = u\kappa$ and $dz = \kappa du$, and

$$\phi_2 = \frac{\kappa}{2\sigma_x^2} \exp \left( - \left[ \frac{\mu^2}{2\sigma_x^2} + \frac{\kappa}{2\sigma_x^2} \right] \right) \int_0^1 \exp \left( -u \frac{\kappa \sigma_{\text{diff}}^2}{2\sigma_x^2 \sigma_\gamma^2} \right) I_0 \left( \frac{\sqrt{\kappa \mu^2 u}}{\sigma_\gamma^2} \right) du. \quad (7.112)$$

Then, in terms of the scaled $SIR$, $\Lambda = \frac{SIR}{(2\alpha - 1)^2}$, and $\mu = \alpha - (1 - \alpha)\gamma$, $\phi_2$ is

$$\phi_2 = \Lambda \alpha (1 - \alpha)(1 - \gamma)^2 \exp \left( -\Lambda \left[ \mu^2 + \alpha(2\alpha - 1)(1 - \gamma)^2 \right] \right) \times \int_0^1 \exp \left( -\Lambda \alpha (2 - 3\alpha)(1 - \gamma)^2 u \right) I_0 \left( 2\Lambda \mu (1 - \gamma) \sqrt{\alpha(1 - \alpha)u} \right) du. \quad (7.113)$$

Since this finite limit representation has no singularities except for $\alpha = \frac{1}{2}$, it is a convenient form for numerical evaluation.
Chapter 8

Case III, $\rho = 0$, Independent Limit

8.1 Summary

Theorem 8.1.1. Using (4.18), the i.i.d. error probability for Case III is

$$P_{III} \left( SIR \mid \alpha = \frac{1}{2}, \rho = 0, \gamma \right) = 1 - Q_1 \left( \sqrt{SIR}, \gamma \sqrt{SIR} \right) + \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ I_0 (\gamma SIR) + I_1 (\gamma SIR) \left( \frac{1 - \gamma^2}{4\gamma} \right) \right].$$

In the limit of $\gamma = 0$, this is

$$\lim_{\gamma \to 0} P_{III} \left( SIR \mid \alpha = \frac{1}{2}, \rho = 0, \gamma \right) = \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{SIR}{8} \right].$$

which agrees with Case I, as it must since Case I is obtained from Case III in the limit of $\alpha \to \frac{1}{2}$ and $\rho \to 0$.

Corollary 8.1.2. Since $A = B = \frac{1}{2}$, the form (4.19) is also applicable, yielding an i.i.d. error
probability in terms of the Marcum-Q function, given by

\[
\lim_{\rho \to 0} \lim_{\alpha \to \frac{1}{2}} = \frac{1}{2} + \frac{3}{8} \left[ Q_1 \left( \sqrt{SIR}, \gamma \sqrt{SIR} \right) - Q_1 \left( \gamma \sqrt{SIR}, \sqrt{SIR} \right) \right]
\]

\[
+ \frac{1}{8} \left[ Q_2 \left( \sqrt{SIR}, \gamma \sqrt{SIR} \right) - Q_2 \left( \gamma \sqrt{SIR}, \sqrt{SIR} \right) \right]
\]

(8.3)

Figure 8.1: Comparison of Independent and Correlated Limits for Case III with \( \alpha = \frac{1}{2}, 15\text{dB SIR} \)

Figure 8.1 compares the correlated and independent limits for the Case III error probability in the limit of \( \alpha = \frac{1}{2} \), where there is a solution for general \( \gamma \). It also compares these curves to the Case I lower bound (unfaded bound), the Case I upper bound, and the asymptotic approximation for the \( \gamma = 0, \rho = 1 \) limit. We can see that the correlation generally improves the performance substantially for most \( \gamma \), except when \( \gamma \) is small. In this particular example, the cross-over point where the error probability in the correlated limit exceeds the independent limit occurs near \( \gamma = 0.15 \).
8.2 \( \alpha = \frac{1}{2} \), I.I.D. Limit

As defined in (2.32), the error probability for Case III is

\[
P_{III}(SIR \mid \alpha, \rho, \gamma) \equiv \mathcal{P}r \{ X < Y \} \tag{8.4}
\]

where

\[
X = \alpha \left\| \sqrt{P_o} + N_1 \right\|^2 - (1 - \alpha) \left\| \sqrt{P_o \gamma} + N_4 \right\|^2 \tag{8.5}
\]

\[
Y = \alpha \| N_3 \|^2 - (1 - \alpha) \| N_2 \|^2 \tag{8.7}
\]

In the limit of \( \rho = 0 \), the four noise samples \( N_i \) are mutually independent, zero-mean, circularly-symmetric complex Gaussian random variables, with variance of the real and imaginary parts equal to \( \sigma^2 \). Rearranging the terms so that \( X \) and \( Y \) are i.i.d. sums,

\[
X = \frac{1}{2} \left[ \left\| \sqrt{P_o} + N_1 \right\|^2 + \| N_2 \|^2 \right] \tag{8.9}
\]

\[
Y = \frac{1}{2} \left[ \| N_3 \|^2 + \left\| \sqrt{P_o \gamma} + N_4 \right\|^2 \right] \tag{8.10}
\]

Since \( \alpha = 1/2 \), we can express the difference \( X - Y \) as an i.i.d. quadratic form,

\[
X - Y = \left[ \alpha \left\| \sqrt{P_o} + N_1 \right\|^2 + (1 - \alpha) \| N_2 \|^2 \right] \tag{8.9}
\]

\[
- \left[ \alpha \| N_3 \|^2 + (1 - \alpha) \left\| \sqrt{P_o \gamma} + N_4 \right\|^2 \right] \tag{8.10}
\]

\[
= A \sum_{k=1}^{L_1} \| X_k \|^2 + B \sum_{k=1}^{L_2} \| Y_k \|^2 \tag{8.11}
\]
with $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $L_1 = L_2 = 2$, means $X_1 = \sqrt{P_o}$, $X_2 = 0$, $Y_1 = 0$, $Y_2 = \gamma \sqrt{P_o}$, and variances

$$\mu_{xx} = \frac{1}{2} E \left\{ \| X_k - \overline{X}_k \|^2 \right\} = \sigma^2$$

$$\mu_{yy} = \frac{1}{2} E \left\{ \| Y_k - \overline{Y}_k \|^2 \right\} = \sigma^2$$

that do not depend on $k$. Note that this is only possible when $A = \alpha = (1 - \alpha)$ and $B = -\alpha = -(1 - \alpha)$, that is, if and only if $\alpha = \frac{1}{2}$.

This is the same quadratic form considered in [25, Eq. 1], with $L_1 = L_2 = 2$. The derivation is a generalization of [20, Appendix B], providing a solution even when $L_1$ and $L_2$ are not necessarily equal (which reduces to the same result when $L_1 = L_2$). It is noted in [25] that a closed-form solution for the error probability exists if and only if the variances of the $X_k$ and $Y_k$ are independent of $k$. Therefore, a closed-form solution for the independent limit is possible if and only if $\alpha = \frac{1}{2}$. This closed-form solution is [25, Eq. 4-7]

$$1 - P_e = Q_1(a, b) - \exp \left( -\frac{a^2 + b^2}{2} \right) I_0 (ab)$$

$$+ \exp \left( -\frac{a^2 + b^2}{2} \right) I_0 (ab) (1 + \eta)^{-(L_1+L_2-1)} \sum_{k=0}^{L_1-1} (L_1+L_2-1) \eta^k$$

$$+ \exp \left( -\frac{a^2 + b^2}{2} \right) (1 + \eta)^{-(L_1+L_2-1)} \left[ \sum_{n=1}^{L_1-1} I_n (ab) \sum_{k=0}^{L_1-1-n} (L_1+L_2-1) \eta^k \left( \frac{b}{a} \right)^n \right.$$ 

$$- \sum_{n=1}^{\bar{L}_2-1} I_n (ab) \sum_{k=0}^{L_1+L_2-1} \eta^{L_1+L_2-1-k} \left( \frac{a}{b} \right)^n \left] \right. ,$$

(8.12)
where

\[
\begin{align*}
  a & \equiv \sqrt{\sum_{k=1}^{L_1} A \|X_k\|^2 / A\mu_{xx} - B\mu_{yy}} = \sqrt{SIR} \\
  b & \equiv \sqrt{-\sum_{k=1}^{L_2} B \|Y_k\|^2 / A\mu_{xx} - B\mu_{yy}} = \gamma \sqrt{SIR} \\
  \eta & \equiv -B\mu_{yy} / A\mu_{xx} = 1.
\end{align*}
\]

Upon expanding out the sums with \( L_1 = L_2 = 2 \), substituting for \( a \), \( b \), and \( \eta \), and then simplifying, we obtain

\[
1 - P_e = Q_1(a, b) - \exp \left( -\frac{a^2 + b^2}{2} \right) I_0(ab) + \exp \left( -\frac{a^2 + b^2}{2} \right) I_0(ab) \frac{1 + 3\eta}{(1 + \eta)^3} \\
+ \exp \left( -\frac{a^2 + b^2}{2} \right)(1 + \eta)^{-3} \left[ I_1(ab) \left( \frac{b}{a} \right) - I_1(ab) \eta^3 \left( \frac{a}{b} \right) \right] \\
= Q_1 \left( \sqrt{SIR}, \gamma \sqrt{SIR} \right) - \exp \left( -SIR \frac{1 + \gamma^2}{2} \right) I_0(\gamma SIR) \\
+ \frac{4}{8} \exp \left( -SIR \frac{1 + \gamma^2}{2} \right) I_0(\gamma SIR) + \frac{1}{8} \exp \left( -SIR \frac{1 + \gamma^2}{2} \right) I_1(\gamma SIR) \left( \gamma - \frac{1}{\gamma} \right) \\
= Q_1 \left( \sqrt{SIR}, \gamma \sqrt{SIR} \right) \\
- \frac{1}{2} \exp \left( -SIR \frac{1 + \gamma^2}{2} \right) \left[ I_0(\gamma SIR) + I_1(\gamma SIR) \left( \frac{1 - \gamma^2}{4\gamma} \right) \right].
\]

At last, we obtain our desired result,

\[
P_{III}(SIR \mid \alpha = \frac{1}{2}, \rho = 0, \gamma) \\
= 1 - Q_1 \left( \sqrt{SIR}, \gamma \sqrt{SIR} \right) \quad (8.14)
\]

\[
+ \frac{1}{2} \exp \left( -SIR \frac{1 + \gamma^2}{2} \right) \left[ I_0(\gamma SIR) + I_1(\gamma SIR) \left( \frac{1 - \gamma^2}{4\gamma} \right) \right].
\]

8.2.1 Representation in terms of Marcum-Q Functions

It was just shown that in the limits of \( \rho = 0 \) and \( \alpha = \frac{1}{2} \), the Case III error probability reduces to an i.i.d. quadratic form (8.11), with \( L_1 = L_2 = 2 \), \( A = \alpha = B = (1 - \alpha) = \frac{1}{2} \) and
\[ \mu_1^2 = SIR, \mu_2^2 = \gamma^2 SIR. \]

Applying these parameters to (4.19), the error probability is expressed in terms of the Marcum-Q function as

\[ \lim_{\rho \to 0} P_{III}(SIR | \alpha, \rho, \gamma) \to P_{iid}(2; SIR, \gamma^2 SIR) \]
\[ = \frac{1}{2} + \frac{1}{8} \sum_{\ell=1}^{2} \left( \frac{3}{2} - \ell \right) \left[ Q_{\ell}\left(\sqrt{SIR}, \gamma\sqrt{SIR}\right) - Q_{\ell}\left(\gamma\sqrt{SIR}, \sqrt{SIR}\right) \right] \]
\[ = \frac{1}{2} + \frac{3}{8} \left[ Q_1\left(\sqrt{SIR}, \gamma\sqrt{SIR}\right) - Q_1\left(\gamma\sqrt{SIR}, \sqrt{SIR}\right) \right] \]
\[ + \frac{1}{8} \left[ Q_2\left(\sqrt{SIR}, \gamma\sqrt{SIR}\right) - Q_2\left(\gamma\sqrt{SIR}, \sqrt{SIR}\right) \right]. \] (8.15)

### 8.2.2 Limit of $\alpha = \frac{1}{2}$ and $\gamma = 1$

With $\gamma = 1$ and $\alpha = \frac{1}{2}$, the multipath image and the signal have equal power, and the multipath image always produces an error probability of $\frac{1}{2}$. Substituting $\gamma = 1$ into (8.14),

\[ P_{III}\left(SIR | \alpha = \frac{1}{2}, \rho = 0, \gamma = 1\right) = 1 - Q_1\left(\sqrt{SIR}, \sqrt{SIR}\right) + \frac{1}{2} \exp (- SIR) I_0\left(SIR\right) \]

From [24, Eq. 4.17],

\[ Q_1\left(\sqrt{SIR}, \sqrt{SIR}\right) = \frac{1}{2} + \frac{1}{2} \exp (- SIR) I_0\left(SIR\right) \]

and thus $P_{III}\left(SIR | \alpha = \frac{1}{2}, \rho = 0, \gamma = 1\right) = \frac{1}{2}$ for $\alpha = \frac{1}{2}$ and $\gamma = 1$, as expected.

### 8.2.3 Limit of $\alpha = \frac{1}{2}$ and $\gamma = 0$

In the limit of $\rho = 0$ and $\gamma = 0$, Case III reduces to Case I. Thus, in the limit of $\gamma = 0$, the solution of the Case III error probability for $\rho = 0$ and $\alpha = \frac{1}{2}$ must agree with the Case I solution for $\alpha = \frac{1}{2}$. The series expansion of the modified Bessel function of the first kind is [1, 9.6.10] is given by

\[ I_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left( \frac{z}{2} \right)^{2k+n}. \] (8.16)
Thus, the values at zero are

\[ I_n(0) = \sum_{k=0}^{\infty} 0^{2k+n} = 0^n \]

\[ = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{otherwise,} 
\end{cases} \] (8.17)

Taking the limit of \( \gamma \to 0 \) for (8.14), the last term is indeterminate, since \( I_1(0) = 0 \), while \( I_0(0) = 1 \) and \( Q_1(x, 0) = 1 \) [24, Eq.4.23]:

\[ \lim_{\gamma \to 0} P_e = 1 - Q_1\left(\sqrt{\text{SIR}}, 0\right) + \frac{1}{2} \exp\left(-\frac{\text{SIR}}{2}\right) \left[ I_0(0) + \lim_{\gamma \to 0} \left\{ I_1(\gamma \text{SIR}) \frac{1-\gamma^2}{4\gamma} \right\} \right] \]

\[ = \frac{1}{2} \exp\left(-\frac{\text{SIR}}{2}\right) \left[ 1 + \lim_{\gamma \to 0} \left\{ I_1(\gamma \text{SIR}) \left(\frac{1-\gamma^2}{4\gamma}\right) \right\} \right]. \] (8.18)

Using L’Hopital’s rule, we obtain

\[ \lim_{\gamma \to 0} \left\{ I_1(\gamma \text{SIR}) \left(\frac{1-\gamma^2}{4\gamma}\right) \right\} \]

\[ = \left[ \frac{d}{d\gamma} \left\{ I_1(\gamma \text{SIR}) \left(1-\gamma^2\right) \right\} \right]_{\gamma=0} \]

\[ = \frac{1}{4} \left[ I_1(\gamma \text{SIR}) \frac{d}{d\gamma} \left(1-\gamma^2\right) + \frac{d}{d\gamma} \left\{ I_1(\gamma \text{SIR}) \right\} \left(1-\gamma^2\right) \right]_{\gamma=0}. \] (8.19)

From [1, Eq. 9.6.29],

\[ \frac{d}{dz} I_1(z) = \frac{1}{2} \left[ I_0(z) + I_2(z) \right], \]

therefore,

\[ \frac{d}{d\gamma} I_1(\gamma \text{SIR}) = \frac{1}{2} \left[ I_0(\gamma \text{SIR}) + I_2(\gamma \text{SIR}) \right] \frac{d}{d\gamma} \{ \gamma \text{SIR} \} \]

\[ = \frac{\text{SIR}}{2} \left[ I_0(\gamma \text{SIR}) + I_2(\gamma \text{SIR}) \right]. \] (8.20)
Noting that \( \frac{d}{d\gamma} \left\{ 1 - \gamma^2 \right\} = -2\gamma \), and substituting (8.20) into (8.19), we obtain

\[
\lim_{\gamma \to 0} \left\{ I_1 (\gamma \text{SIR}) \left( \frac{1 - \gamma^2}{4\gamma} \right) \right\} = \frac{1}{4} \left[ -2\gamma I_1 (\gamma \text{SIR}) + \frac{\text{SIR} \left(1 - \gamma^2\right)}{2} \left[ I_0 (\gamma \text{SIR}) + I_2 (\gamma \text{SIR}) \right] \right]_{\gamma=0} = \frac{\text{SIR}}{8} \tag{8.21}
\]

Substituting (8.21) into (8.18), we finally obtain

\[
\lim_{\gamma \to 0} P_e = \frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \left[ 1 + \frac{\text{SIR}}{8} \right] \tag{8.22}
\]

which agrees with both the Case I result with \( \alpha = \frac{1}{2} \) and the corresponding i.i.d limit.

### 8.3 Case III, \( \rho = 0 \), General \((\alpha, \gamma)\)

As defined in (2.32), the error probability for Case III is

\[
P_{\text{III}} (\text{SIR} | \alpha, \rho, \gamma) \equiv \Pr \{ X < Y \} \tag{8.23}
\]

where

\[
X = \alpha \left\| \sqrt{P_0} + N_1 \right\|^2 - (1 - \alpha) \left\| \sqrt{P_0\gamma} + N_4 \right\|^2 \tag{8.24}
\]

\[
Y = \alpha \| N_3 \|^2 - (1 - \alpha) \| N_2 \|^2 \tag{8.26}
\]

In the limit of \( \rho = 0 \), the random variables \( X \) and \( Y \) are differences of a non-central chi-square variate and an independent central chi-square variate, with \( X \) conditioned on \( \alpha \), and \( Y \)
is conditioned on both $\alpha$ and $\gamma$. Since all four intensities are mutually independent, the error probability can be solved using the conditional pdf and conditional cdf of $X$ and $Y$,

$$P_{III} \left( SIR \mid \alpha, \rho = 0, \gamma \right) = 1 - \int_{-\infty}^{\infty} F_{Y|\alpha,\gamma}(z) f_{X|\alpha}(z) \, dz. \quad (8.28)$$

From [23, Eq. 4.32] with $\sigma_1^2 = \sigma_2^2 = \alpha \sigma^2$ and $a_2 = \alpha P_o$, the pdf of $X$ is

$$f_{X|\alpha}(z) = \begin{cases} 
\frac{1}{4\alpha \sigma^2} \exp \left( \frac{z}{2\alpha \sigma^2} \right) \exp \left( -\frac{P_o}{4\sigma^2} \right) & \text{if } z < 0, \\
\frac{1}{4\alpha \sigma^2} \exp \left( \frac{z}{2\alpha \sigma^2} \right) \exp \left( -\frac{P_o}{4\sigma^2} \right) Q_1 \left( \sqrt{\frac{P_o}{2\sigma^2}}, \sqrt{\frac{2z}{\alpha \sigma^2}} \right) & \text{if } z \geq 0,
\end{cases} \quad (8.29)$$

and from [23, Eq. 4.33] with $\sigma_1^2 = \sigma_2^2 = (1 - \alpha) \sigma^2$ and $a_2 = (1 - \alpha) \gamma^2 P_o$, the cdf of $Y$ is

$$F_{Y|\alpha,\gamma}(z) = \begin{cases} 
\frac{1}{2} \exp \left( \frac{z}{2(1 - \alpha) \sigma^2} \right) \exp \left( -\frac{P_o \gamma^2}{4\sigma^2} \right) & \text{if } z < 0, \\
1 - Q_1 \left( \sqrt{\frac{P_o \gamma^2}{2\sigma^2}}, \sqrt{\frac{2z}{(1 - \alpha) \sigma^2}} \right) + \frac{1}{2} \exp \left( \frac{z}{2(1 - \alpha) \sigma^2} \right) \exp \left( -\frac{P_o \gamma^2}{4\sigma^2} \right) Q_1 \left( \sqrt{\frac{P_o \gamma^2}{2\sigma^2}}, \sqrt{\frac{2z}{(1 - \alpha) \sigma^2}} \right) & \text{if } z \geq 0,
\end{cases} \quad (8.30)$$

Therefore, the error probability is

$$P_{III} \left( SIR \mid \alpha, \rho = 0, \gamma \right) = 1 - \int_{-\infty}^{\infty} F_{Y|\alpha,\gamma}(z) f_{X|\alpha}(z) \, dz - \int_{0}^{\infty} \left[ 1 - F_{Y,1}^+(z) + F_{Y,2}^+(z) \right] f_{X|\alpha}(z) \, dz$$

$$= 1 - \int_{0}^{\infty} f_{X|\alpha}(z) \, dz - \int_{-\infty}^{0} F_{X|\alpha,\gamma}(z) f_{X|\alpha}(z) \, dz + \int_{0}^{\infty} F_{Y,1}^+(z) f_{X|\alpha}(z) \, dz$$

$$- \int_{0}^{\infty} F_{Y,2}^+(z) f_{X|\alpha}(z) \, dz \equiv F_{X|\alpha}(0) - \Phi_0 + \Phi_1 - \Phi_2, \quad (8.32)$$
where the integrals $\Phi_n$ will be evaluated in the following sections. From [23, Eq. 4.33] with $\sigma_1^2 = \sigma_2^2 = \alpha \sigma^2, a^2 = \alpha P_0$ and $z = 0$, the first term is simply

$$F_{X|\alpha}(0) = \frac{1}{2} \exp \left( - \frac{SIR}{2} \right).$$ \hfill (8.33)

**Integral $\Phi_0$**

Combining the definitions (8.29) and (8.31) with (8.32),

$$\Phi_0 = \frac{1}{8 \alpha \sigma^2} \exp \left( - \frac{SIR}{2} \left[ 1 + \gamma^2 \right] \right) \int_{-\infty}^{0} \exp \left( \frac{z}{2 \alpha (1 - \alpha) \sigma^2} \right) dz = \frac{(1 - \alpha)}{4} \exp \left( - \frac{SIR}{2} \left[ 1 + \gamma^2 \right] \right) \int_{-\infty}^{0} \exp (-u) du,$$

$$= \frac{(1 - \alpha)}{4} \exp \left( - \frac{SIR}{2} \left[ 1 + \gamma^2 \right] \right), \hfill (8.34)$$

using the substitution $u = \frac{z}{2 \alpha (1 - \alpha) \sigma^2}$.

Applying this result and (8.33) to (8.32), the error probability is

$$P_{\text{err}} (SIR | \alpha, \rho = 0, \gamma) = \frac{1}{2} \exp \left( - \frac{SIR}{2} \right) \left[ 1 - \frac{1 - \alpha}{2} \exp \left( - \frac{SIR \gamma^2}{2} \right) \right] + \Phi_{\text{diff}}. \hfill (8.35)$$

where $\Phi_{\text{diff}} \equiv \Phi_1 - \Phi_2$.

**Integrals $\Phi_1$ and $\Phi_2$**

Once again, combining the definitions (8.29) and (8.31) with (8.32),

$$\Phi_1 = \frac{C_1}{2} \exp (-A_2) \int_{0}^{\infty} \exp (C_1 z) Q_1 \left( \sqrt{2A_1}, \sqrt{2B_{1,1}z} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2B_{2}z} \right) dz,$$

$$\Phi_2 = \frac{C_1}{4} \exp (-[A_1 + A_2]) \int_{0}^{\infty} \exp (C_2 z) Q_1 \left( \sqrt{2A_1}, \sqrt{2B_{1,2}z} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2B_{2}z} \right) dz,$$

$$\hfill (8.36)$$
where we have defined

\[ A_1 \equiv \frac{\text{SIR}_1 \gamma^2}{2}, \quad A_2 \equiv \frac{\text{SIR}_2}{2}, \]

\[ B_{1,1} \equiv \frac{1}{2(1-\alpha)\sigma^2}, \quad B_2 \equiv \frac{1}{\alpha \sigma^2}, \]

\[ B_{1,2} \equiv \frac{1}{(1-\alpha)\sigma^2}, \]

\[ C_1 \equiv \frac{B_2}{2} = \frac{1}{2\alpha \sigma^2}, \quad C_2 \equiv \frac{B_{1,2} + B_2}{2} = \frac{1}{2\alpha(1-\alpha)\sigma^2}. \] (8.37)

Therefore, the integral on the RHS of \( \Phi_1 \) and \( \Phi_2 \) has the generic form

\[
\Lambda (A_1, B_{1,1}, A_2, B_2, C) = \int_0^\infty \exp \left( Cz \right) Q_1 \left( \sqrt{2A_1}, \sqrt{2B_{1,1}z} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2B_2}z \right) dz,
\] (8.38)

which appears to have no closed-form solution. In Subsection (8.3.1), we derive a doubly-infinite series representation, which is shown to reduce to Case I in the limit of no multipath (\( \gamma = 0 \)). For general values of \( (\alpha, \gamma) \) however, the convergence of this series is slow, especially in the high SIR regime. For the purposes of efficient numerical evaluation, in Subsection (8.3.2) we use the well-known alternative form [24] of the Marcum-Q function to derive an expression in terms of a finite number of finite-limit integrals.

### 8.3.1 Series Representation

Using Dillard’s recursive representation of the Marcum-Q function [4],

\[
Q_1 \left( \sqrt{A}, \sqrt{B} \right) = \exp \left( -\frac{A}{2} \right) \exp \left( -\frac{B}{2} \right) \sum_{n=0}^\infty \frac{(A/2)^n}{n!} \sum_{k=0}^n \frac{(B/2)^k}{k!}
\] (8.39)

we can represent the product of Marcum-Q functions as the doubly-infinite series

\[
Q_1 \left( \sqrt{2A_1}, \sqrt{2B_{1,1}z} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2B_2}z \right) = \exp \left( -[A_1 + A_2] - [B_1 + B_2]z \right) \sum_{k=0}^\infty \sum_{\ell=0}^\infty \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{A_1^k A_2^\ell B_{1,1}^m B_2^n}{k! \ell! m! n!} z^{m+n}. \] (8.40)
Now, substituting this into (8.38), after exchanging the order of summation and integration, we obtain

\[
\Lambda (A_1, B_1, A_2, B_2, C) = \quad (8.41)
\]

\[
\exp \left( -[A_1 + A_2] \right) \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{\ell} \frac{A_1^k A_2^\ell B_1^m B_2^n}{k! \ell! m! n!} \int_0^\infty z^{m+n} \exp \left( -[B_1 + B_2 - C]z \right) dz.
\]

Recognizing the integral on the RHS as the complete gamma function, which will reduce to a factorial function since \(m\) and \(n\) are integers \([1, Eqs. 6.1.1 and 6.1.5]\), we obtain

\[
\int_0^\infty z^{m+n} \exp \left( -[B_1 + B_2 - C]z \right) dz = \frac{\Gamma(m + n + 1)}{[B_1 + B_2 - C]^{m+n+1}} = \frac{(m + n)!}{[B_1 + B_2 - C]^{m+n+1}},
\]

and therefore

\[
\Lambda (A_1, B_1, A_2, B_2, C) = \quad (8.42)
\]

\[
\exp \left( -[A_1 + A_2] \right) \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{\ell} \frac{A_1^k A_2^\ell B_1^m B_2^n}{k! \ell! m! n!} \frac{(m + n)!}{[B_1 + B_2 - C]^{m+n+1}}.
\]

Applying this result first to \(\Phi_1\), we substitute \(C_1\) for \(C\) and \(B_{1,1}\) for \(B_1\), as defined in (8.37).

Using these definitions, \((B_1 + B_2 - C) \rightarrow (B_{1,1} + B_2 - C_1) = \frac{1}{2\alpha(1-\alpha)\sigma^2} = C_2\). Similarly then, for \(\Phi_2\), in the above series expansion we substitute \(C_2\) for \(C\) and \(B_{1,2}\) for \(B_1\). With these definitions, \((B_1 + B_2 - C) \rightarrow (B_{1,2} + B_2 - C_2) = \frac{1}{2\alpha(1-\alpha)\sigma^2} = C_2\). Therefore,

\[
\Phi_1 = \frac{C_1}{2C_2} \exp \left( -[A_1 + 2A_2] \right) \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{\ell} \frac{A_1^k A_2^\ell B_1^m B_2^n (m + n)!}{C_2^{m+n} k! \ell! m! n!}, \quad (8.44)
\]

\[
\Phi_2 = \frac{C_1}{4C_2} \exp \left( -[2A_1 + 2A_2] \right) \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{\ell} \frac{A_1^k A_2^\ell B_1^m B_2^n (m + n)!}{C_2^{m+n} k! \ell! m! n!}. \quad (8.45)
\]
Substituting the parameter definitions from (8.37) and simplifying,

\[
\Phi_1 = \frac{(1 - \alpha)}{2} \exp \left( -SIR \left[ 1 + \frac{\gamma^2}{2} \right] \right) \\
\times \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{k! \ell! m! n!} \left( \frac{SIR \gamma^2}{2} \right)^k \left( \frac{SIR}{2} \right)^\ell \left( \alpha \right)^m \left( 2(1 - \alpha) \right)^n, \tag{8.46}
\]

\[
\Phi_2 = \frac{(1 - \alpha)}{4} \exp \left( -SIR \left[ 1 + \gamma^2 \right] \right) \\
\times \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{k! \ell! m! n!} \left( \frac{SIR \gamma^2}{2} \right)^k \left( \frac{SIR}{2} \right)^\ell \left( 2\alpha \right)^m \left( 2(1 - \alpha) \right)^n. \tag{8.47}
\]

Note that the \((m+n)!\) term in the numerator makes further simplification extremely difficult. If it were not there, we could express both \(\Phi_1\) and \(\Phi_2\) as the product of two first-order Marcum-Q functions by using the same relation as in (8.40). The error probability (8.32) is based on the difference \(\Phi_{\text{diff}} \equiv \Phi_1 - \Phi_2\), which must always be less than \(\frac{1}{2}\), since the error probability \(P_{\text{III}}(SIR | \alpha, \rho, \gamma) < \frac{1}{2}\). We can express this difference as

\[
\Phi_{\text{diff}} = \frac{(1 - \alpha)}{2} \exp \left( -SIR \left[ 1 + \frac{\gamma^2}{2} \right] \right) \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{k! \ell! m! n!} \left( \frac{SIR \gamma^2}{2} \right)^k \left( \frac{SIR}{2} \right)^\ell \left( 2\alpha \right)^m \left( 2(1 - \alpha) \right)^n \left[ 1 - \frac{1}{2m} - \frac{1}{2} \exp \left( -\frac{SIR \gamma^2}{2} \right) \right]. \tag{8.48}
\]

**Limit of \(\gamma = 0\)**

In the limit of \(\gamma = 0\), there is no multipath, and since all of the interference samples are uncorrelated, the performance must be identical with Case I. With \(\gamma = 0\) in the previous expression, the \(m\) and \(k\) summations collapse, since \(k = 0\) is the only non-zero term, forcing \(m = 0\) as well. As a result, \(\frac{(m+n)!}{m!n!} = \frac{n!}{n!} = 1\) and thus

\[
\lim_{\gamma \to 0} \Phi_{\text{diff}} = \frac{(1 - \alpha)}{2} \exp \left( -SIR \right) \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\ell!} \left( \frac{SIR}{2} \right)^\ell \left( 2(1 - \alpha) \right)^n \left[ 1 - \frac{1}{2} \right] \\
= \frac{(1 - \alpha)}{4} \exp \left( -SIR \right) \sum_{\ell=0}^{\infty} \frac{\left( SIR / 2 \right)^\ell}{\ell!} \sum_{n=0}^{\infty} \left( 2(1 - \alpha) \right)^n. \tag{8.49}
\]
First, using the following identity, from [9, Table 9.2],
\[
\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x},
\]  
(8.50)
and then recognizing the series expansion of the exponential function,
\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp (n),
\]  
(8.51)
we will reduce the double-summation on the RHS of (8.49):
\[
\sum_{\ell=0}^{\infty} \left( \frac{\text{SIR}}{2} \right)^{\ell} \sum_{n=0}^{\infty} \frac{2(1 - \alpha)^{\ell}}{\ell!} = \sum_{\ell=0}^{\infty} \left( \frac{\text{SIR}}{2} \right)^{\ell} \left( \frac{1 - [2(1 - \alpha)]^{\ell+1}}{1 - 2(1 - \alpha)} \right)
\]  
\[
= \frac{1}{2\alpha - 1} \left[ \sum_{\ell=0}^{\infty} \frac{\text{SIR}^{\ell}}{\ell!} - 2(1 - \alpha) \sum_{\ell=0}^{\infty} \frac{(1 - \alpha) \text{SIR}^{\ell}}{\ell!} \right]
\]  
\[
= \frac{1}{2\alpha - 1} \left[ \exp \left( \frac{-\text{SIR}}{2} \right) - 2(1 - \alpha) \exp \left( (1 - \alpha) \text{SIR} \right) \right].
\]  
(8.52)
Inserting (8.52) into (8.49) and distributing the factor of \( \exp (-\text{SIR}) \),
\[
\lim_{\gamma \to 0} \Phi_{\text{diff}} = \frac{(1 - \alpha)}{4(2\alpha - 1)} \left[ \exp \left( -\frac{\text{SIR}}{2} \right) - 2(1 - \alpha) \exp \left( -\text{SIR} \left[ 1 - (1 - \alpha) \right] \right) \right]
\]  
\[
= \frac{(1 - \alpha)}{4(2\alpha - 1)} \exp \left( -\frac{\text{SIR}}{2} \right) - \frac{(1 - \alpha)^2}{2(2\alpha - 1)} \exp \left( -\alpha \text{SIR} \right).
\]  
(8.53)
Therefore, from (8.35) the error probability in this limit of \( \gamma = 0 \) is
\[
\lim_{\gamma \to 0} P_{\text{III}} (\text{SIR} | \alpha, \rho = 0, \gamma)
\]  
\[
= \frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \left[ 1 - \frac{1 - \alpha}{2} \right] + \lim_{\gamma \to 0} \Phi_{\text{diff}}
\]  
\[
= \frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \left[ 1 - \frac{1 - \alpha}{2} + \frac{(1 - \alpha)}{2(2\alpha - 1)} \right] - \frac{(1 - \alpha)^2}{2(2\alpha - 1)} \exp \left( -\alpha \text{SIR} \right),
\]  
where
\[
\left[ 1 - \frac{1 - \alpha}{2} + \frac{(1 - \alpha)}{2(2\alpha - 1)} \right] = \frac{\alpha^2}{2(2\alpha - 1)},
\]  
(8.54)
and thus
\[
\lim_{\gamma \to 0} P_{III}(SIR | \alpha, \rho = 0, \gamma) = \frac{\alpha^2}{2(2\alpha - 1)} \exp \left( -\frac{SIR}{2} \right) - \frac{(1 - \alpha)^2}{2(2\alpha - 1)} \exp (-\alpha SIR), \quad (8.55)
\]
which agrees with the result for Case I given in Theorem 5.1.1.

### 8.3.2 Representation in terms of Finite Limit Integrals

It is proven in Appendix E that for all \(0 \leq z \leq \infty\),
\[
Q_1 \left( \sqrt{2A_n}, \sqrt{2B_n}z \right) = \begin{cases} 
\phi_1 (A_n, B_n, z) & \text{if } z > \xi_n, \\
1 + \phi_2 (A_n, B_n, z) & \text{if } z < \xi_n,
\end{cases} \quad (8.56)
\]
with \(\xi_n \equiv \frac{A_n}{B_n}\) and
\[
\phi_1 (A_n, B_n, z) \equiv \frac{1}{\pi} \int_0^\pi F(\xi_n, \theta, z) \exp [-H(A_n, B_n, \theta, z)] \, d\theta, \quad (8.57)
\]
\[
\phi_2 (A_n, B_n, z) \equiv \frac{1}{\pi} \int_0^\pi G_n(\theta, z) \exp [-H(A_n, B_n, \theta, z)] \, d\theta, \quad (8.58)
\]
where
\[
F(\xi_n, \theta, z) \equiv \frac{1 \pm \sqrt{\frac{\xi_n}{z}} \cos \theta}{1 \pm 2 \sqrt{\frac{\xi_n}{z}} \cos \theta + \frac{\xi_n}{z}}, \quad (8.59)
\]
\[
G(\xi_n, \theta, z) \equiv \frac{\frac{z}{\xi_n} \pm \sqrt{\frac{\xi_n}{z}} \cos \theta}{1 \pm 2 \sqrt{\frac{\xi_n}{z}} \cos \theta + \frac{\xi_n}{z}}, \quad (8.60)
\]
and
\[
H(A_n, B_n, \theta, z) \equiv A_n + B_n z \pm 2 \sqrt{A_n B_n z} \cos \theta. \quad (8.61)
\]
General Form

We can use this result to express integrals of the following form as the sum of seven integrals with finite limits that can be evaluated numerically:

\[ \Lambda(A_1, B_1, A_2, B_2) \]

\[ \equiv \int_0^\infty \exp(z)Q_1\left(\sqrt{2A_1}, \sqrt{2B_1}z\right) Q_1\left(\sqrt{2A_2}, \sqrt{2B_2}z\right) dz \quad (8.62) \]

\[ = \int_0^{\xi_1} \exp(z) \left[1 + \phi_2(A_1, B_1, z)\right] \left[1 + \phi_2(A_2, B_2, z)\right] dz \]

\[ + \int_{\xi_1}^{\xi_2} \exp(z) \phi_1(A_1, B_1, z) \left[1 + \phi_2(A_2, B_2, z)\right] dz \]

\[ + \int_{\xi_2}^{\infty} \exp(z) \phi_1(A_1, B_1, z) \phi_1(A_2, B_2, z) dz \quad (8.63) \]

\[ \equiv \sum_{k=1}^7 \psi_k, \]

where \( \xi_1 \equiv \frac{A_1}{B_1} \leq \xi_2 \equiv \frac{A_2}{B_2} \). We have required that \( \xi_1 \leq \xi_2 \) so that the three regions of integration in \( z \) are as follows:

\[ 0 \leq z < \xi_1 \leq \xi_2 \]

\[ \xi_1 < z < \xi_2 \]

\[ \xi_1 \leq \xi_2 < z \leq \infty \]

The regions are properly ordered, ensuring that the correct finite-limit representation from Appendix E is taken for each Marcum-Q function, in accordance with the piecewise definition for whether \( z < \xi_n \) or \( z > \xi_n \). Therefore, if \( \xi_2 < \xi_1 \), then we must interchange the argument pairs \((A_1, B_1) \leftrightarrow (A_2, B_2)\) to ensure that the condition \( \xi_1 \leq \xi_2 \) is satisfied.

Expanding the products in (8.63) and then substituting the definitions for \( \phi_1 \) and \( \phi_2 \), the
seven integrals $\psi_k$ are

$$\psi_1 \equiv \int_0^{\xi_1} \exp(z) \, dz$$

$$= \exp(\xi_1) - 1,$$

$$\psi_2 \equiv \int_0^{\xi_1} \exp(z) \phi_2(A_1, B_1, z) \, dz$$

$$= \frac{1}{\pi} \int_0^{\xi_1} \int_0^{\pi} G(\xi_1, \theta, z) \exp(-H(A_1, B_1, \theta, z) + z) \, d\theta \, dz,$$

$$\psi_3 \equiv \int_0^{\xi_1} \exp(z) \phi_2(A_2, B_2, z) \, dz$$

$$= \frac{1}{\pi} \int_0^{\xi_1} \int_0^{\pi} G(\xi_2, \theta, z) \exp(-H(A_2, B_2, \theta, z) + z) \, d\theta \, dz,$$

$$\psi_4 \equiv \int_0^{\xi_1} \exp(z) \phi_2(A_1, B_1, z) \phi_2(A_2, B_2, z) \, dz$$

$$= \frac{1}{\pi^2} \int_0^{\xi_1} \int_0^{\pi} \int_0^{\pi} G(\xi_1, \theta, z) G(\xi_2, \phi, z)$$

$$\times \exp(-H(A_1, B_1, \theta, z) - H(A_2, B_2, \phi, z) + z) \, d\phi \, d\theta \, dz,$$

$$\psi_5 \equiv \int_{\xi_1}^{\xi_2} \exp(z) \phi_1(A_1, B_1, z) \, dz$$

$$= \frac{1}{\pi} \int_{\xi_1}^{\xi_2} \int_0^{\pi} F(\xi_1, \theta, z) \exp(-H(A_1, B_1, \theta, z) + z) \, d\theta \, dz,$$

$$\psi_6 \equiv \int_{\xi_1}^{\xi_2} \exp(z) \phi_1(A_1, B_1, z) \phi_2(A_2, B_2, z) \, dz$$

$$= \frac{1}{\pi^2} \int_{\xi_1}^{\xi_2} \int_0^{\pi} \int_0^{\pi} F(\xi_1, \theta, z) G(\xi_2, \phi, z)$$

$$\times \exp(-H(A_1, B_1, \theta, z) - H(A_2, B_2, \phi, z) + z) \, d\phi \, d\theta \, dz,$$

$$\psi_7 \equiv \int_{\xi_2}^{\infty} \exp(z) \phi_1(A_1, B_1, z) \phi_1(A_2, B_2, z) \, dz$$

$$= \int_0^{\xi_2^{-1}} \frac{1}{u^2} \exp(u^{-1}) \phi_1(A_1, B_1, u^{-1}) \phi_1(A_2, B_2, u^{-1}) \, du$$

$$= \frac{1}{\pi^2} \int_0^{\xi_2^{-1}} \int_0^{\pi} \int_0^{\pi} \frac{1}{u^2} F(\xi_1, \theta, u^{-1}) F(\xi_2, \phi, u^{-1})$$

$$\times \exp(-H(A_1, B_2, \theta, u^{-1}) - H(A_2, B_2, \phi, u^{-1}) + u^{-1}) \, d\phi \, d\theta \, du.$$ 

(8.64)
In the last, integral we made the substitution $u = \frac{1}{z}$, $dz = -\frac{1}{u^2}du$, so that all of the limits are finite.

**Integral $\Phi_1$**

We will now rewrite $\phi_1$, the first integral of interest from (8.36), in terms of the general form for the finite limit integrals (8.62),

$$
\Phi_1 = \frac{C_1}{2} \exp (-A_2) \int_0^\infty \exp (C_1 z) Q_1 \left( \sqrt{2A_1}, \sqrt{2B_{1,1}z} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2B_2z} \right) \; dz
$$

$$
= \frac{1}{2} \exp (-A_2) \int_0^\infty \exp (u) Q_1 \left( \sqrt{2A_1}, \sqrt{2B_{1,1}u/C_1} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2B_2u/C_1} \right) \; du
$$

$$
= \frac{1}{2} \exp (-A_2) \Lambda \left( A_1, B_{1,1}/C_1, A_2, B_2/C_1 \right)
$$

$$
= \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \Lambda \left( \frac{SIR \gamma^2}{2}, \frac{\alpha}{1-\alpha}, \frac{SIR}{2}, 2 \right),
$$

(8.65)

where we have made the substitution $u = C_1 z$, $dz = \frac{1}{C_1} du$, expressed the result in terms of the general form (8.62), and substituted the parameter definitions from (8.37).

Using the parameter definitions (8.37), the argument ratios are

$$
\xi_1 \equiv \frac{A_1}{B_1} = \frac{A_1C_1}{B_{1,1}} = \frac{SIR \gamma^2 (1-\alpha)}{2\alpha}, \quad \xi_2 \equiv \frac{A_2}{B_2} = \frac{A_2C_1}{B_2} = \frac{SIR}{4},
$$

(8.66)

and thus $\xi_1 < \xi_2$ holds only if $\gamma^2 \leq \frac{\alpha}{2(1-\alpha)}$, which is only guaranteed for $\alpha > \frac{2}{3}$. Therefore, we must make the piecewise definition:

$$
\Phi_1 = \begin{cases} 
\frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \Lambda \left( \frac{SIR \gamma^2}{2}, \frac{\alpha}{1-\alpha}, \frac{SIR}{2}, 2 \right) & \text{if } \gamma \leq \sqrt{\frac{\alpha}{2(1-\alpha)}}, \\
\frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \Lambda \left( \frac{SIR}{2}, 2, \frac{SIR \gamma^2}{2}, \frac{\alpha}{1-\alpha} \right) & \text{if } \gamma > \sqrt{\frac{\alpha}{2(1-\alpha)}}.
\end{cases}
$$

(8.67)

Given $\alpha$ and $\gamma$, the above parameters are substituted into each of the seven integrals (8.64) and calculated numerically to solve the general form $\Lambda$ on the RHS of (8.67).
Integral $\Phi_2$

Similarly, the second integral of interest from (8.36), in terms of the general form for the finite limit integrals (8.62) is

$$\Phi_2 = \frac{C_1}{4} \exp(-[A_1 + A_2]) \int_0^\infty \exp(C_2 z) Q_1 \left( \sqrt{2A_1}, \sqrt{2B_{1,2}z} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2B_2z} \right) \, dz$$

$$= \frac{C_1}{4C_2} \exp(-[A_1 + A_2]) \int_0^\infty \exp(u) Q_1 \left( \sqrt{2A_1}, \sqrt{2 \frac{B_{1,2}}{C_2} u} \right) Q_1 \left( \sqrt{2A_2}, \sqrt{2 \frac{B_2}{C_2} u} \right) \, du$$

$$= \frac{C_1}{4C_2} \exp(-[A_1 + A_2]) \Lambda \left( A_1, \frac{B_{1,2}}{C_2}, A_2, \frac{B_2}{C_2} \right)$$

$$= \frac{(1 - \alpha)}{4} \exp \left( -\frac{\text{SIR}}{2} \left[ 1 + \gamma^2 \right] \right) \Lambda \left( \frac{\text{SIR} \gamma^2}{2}, 2\alpha, \frac{\text{SIR}}{2}, 2(1 - \alpha) \right) , \quad (8.68)$$

where we have made the substitution $u = C_2 z$, $dz = \frac{1}{C_2} \, du$, expressed the result in terms of the general form (8.62), and substituted the parameter definitions from (8.37).

Using the parameter definitions (8.37), the argument ratios are

$$\xi_1 \equiv \frac{A_1}{B_1} = \frac{A_1 C_2}{B_{1,2}} = \frac{\text{SIR} \gamma^2}{4\alpha}, \quad \xi_2 \equiv \frac{A_2}{B_2} = \frac{A_2 C_2}{B_2} = \frac{\text{SIR}}{4(1 - \alpha)} , \quad (8.69)$$

and thus $\xi_1 \leq \xi_2$ if $\gamma^2 \leq \frac{\alpha}{1 - \alpha}$, which is guaranteed since $\frac{\alpha}{1 - \alpha} \geq 1$ and $0 \leq \gamma \leq 1$. Thus, given $\alpha$ and $\gamma$, the above parameters are substituted into each of the seven integrals (8.64) and calculated numerically to solve the general form $\Lambda$ on the RHS of (8.68).
Part II

Numerical Results
Chapter 9

Characteristic Functions

9.1 General Hermitian Quadratic Form

In this chapter, we derive $\psi_{D_k}(\omega)$, the characteristic function of $D_k$ by using the general Hermitian quadratic form as described in [23]. Let the superscript $H$ denote the conjugate transpose operation. A Hermitian symmetric matrix is any matrix $Q$ where $Q = Q^H$. The definition of the intensity difference $D_k$ can be easily expressed as the product $D_k = V^HQV$, of a Hermitian symmetric matrix $Q$ and a vector of random variables $V$:

$$D_k \equiv I_m - I_k$$

$$D_k = \begin{bmatrix} \sqrt{\mathcal{P}_o} + N_1^* \\ N_2^* \\ N_3^* \\ \sqrt{\mathcal{P}_o} + N_4^* \end{bmatrix}^T \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -(1 - \alpha) \end{bmatrix} \begin{bmatrix} \sqrt{\mathcal{P}_o} + N_1 \\ N_2 \\ N_3 \\ \sqrt{\mathcal{P}_o} + N_4 \end{bmatrix}$$

$$\equiv V^HQV \quad (9.1)$$
where $\gamma \equiv p_u(\tau)$ is the noncentrality scaling parameter. Since $Q$ is diagonal and real, it is clearly Hermitian symmetric. Furthermore, it is evident that $|V_\ell|^2 = V_\ell V_\ell^*$ for each element $\ell = 1 \ldots 4$ of the random variable vector $V$. The vector of random variables $V$ has mean $\bar{V}$:

$$
\bar{V} = \begin{bmatrix}
\sqrt{P_o} \\
0 \\
0 \\
\sqrt{P_o \gamma}
\end{bmatrix}
$$

(9.2)

and covariance matrix $C$ defined in (2.20).

The vectors $V$ and $\bar{V}$ are simplified for our three cases of interest by recalling that

$$
\gamma = \begin{cases}
p_u(\tau) & \text{if } k = m + \Delta \\
p_u(\bar{\tau}) & \text{if } k = m + \Delta + 1 \\
0 & \text{otherwise}
\end{cases}
$$

(9.3)

Let $I$ denote the identity matrix. From [23], the characteristic function of $D_k$ is given by

$$
\psi_{D_k}(\omega) = \det\{(I - j\omega CQ)^{-1}\} \exp\left(-\bar{V}^H C^{-1} [I - (I - j\omega CQ)^{-1}] \bar{V}\right)
$$

(9.4)

provided that:

- $C$ is nonsingular,

- the covariance of the real and imaginary parts are the same for any pair of variables in the vector $V$,

- the covariance between the real and imaginary parts are anti-symmetric.

The first condition has to be validated for the three cases of interest. The second condition is guaranteed because $C(t)$, the covariance of the complex Gaussian interference process $\eta(t)$, is
real and thus Hermitian symmetric. The third condition is ensured because the real and imaginary parts of $\eta(t)$ are uncorrelated and the covariance between them is zero, due to our requirement that the pre-detection filter is symmetric.

As in [23], the argument of the exponential can be diagonalized by using the transformation $F = U_2^{-1}NU_1^{-1}\bar{V}$, where $U_1$ is the normalized eigenvector matrix of $C^{-1}$, $N^2 = U_1^T C^{-1} U_1$, and $U_2$ is the normalized eigenvector matrix of $N^{-1}U_1^HQU_1N^{-1}$. The characteristic function of $D_k$ can then be expressed as

$$\psi_{D_k}(\omega) = \prod_{\ell=1}^{2n} \frac{1}{1 - j\omega \xi_\ell} \exp \left( \frac{j\omega \xi_\ell |f_\ell|^2}{1 - j\omega \xi_\ell} \right)$$

(9.5)

where $\xi_\ell$ are the eigenvalues of $CQ$ and $f_\ell$ are the elements of the transformed mean vector $F$.

### 9.2 Characteristic Function Derivations

Appendix 9.1 contains a generalized derivation of $\psi_{D_k}$, the characteristic function of $D_k$ using the general Hermitian quadratic form. The intensity difference $D_k$ can be expressed as a product of a Hermitian symmetric matrix $Q$ and a vector $V$ comprised of four random variables that have different statistics for the three cases of interest.

To find the characteristic function for each case, we begin by defining the mean vector $\bar{V}$ and the covariance matrix $C$. After solving for the four eigenvalues $\xi_\ell$ of $CQ$ and finding the transformed mean vector $F$, the characteristic function is given by Equation 9.5.
**Case I:** \( k \neq \{m \pm \Delta, m \pm (\Delta + 1)\} \)

For all \( k \neq \{m \pm \Delta, m \pm (\Delta + 1)\} \),

\[
\bar{V} = \begin{bmatrix}
\sqrt{P_o} \\
0 \\
0 \\
0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\( 2\sigma^2 \). \hspace{1cm} (9.6)

Since \( C = I \ 2\sigma^2 \), then \( CQ = Q \ 2\sigma^2 \) so the eigenvalue matrix is simply \( \xi = Q \ 2\sigma^2 \) and the transformed mean vector is simply \( F = \bar{V} \frac{1}{\sqrt{2\sigma^2}}: \)

\[
\xi = \begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & 1 - \alpha & 0 & 0 \\
0 & 0 & -\alpha & 0 \\
0 & 0 & 0 & -1 + \alpha
\end{bmatrix} \quad 2\sigma^2 \quad F = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \sqrt{SIR}. \hspace{1cm} (9.7)
\]

Substituting these values into Equation 9.5, we obtain the characteristic function for Case I:

\[
\psi_1(\omega, SIR|\alpha) = \exp\left(\frac{\alpha P \omega^2}{\sigma^2} \frac{1}{1 - \alpha \omega^2} \right) \frac{\exp\left(\frac{\alpha P \omega^2}{\sigma^2} \frac{1}{1 - \alpha \omega^2} \right)}{(1 + \alpha \omega^2)(1 - \alpha \omega^2)(1 + (1 - \alpha)\omega^2)(1 - (1 - \alpha)\omega^2)}
\]

\[
= \exp\left(\frac{\alpha P \omega^2}{\sigma^2} \frac{1}{1 - \alpha \omega^2} \right) \frac{(1 + (\alpha \omega^2)^2)(1 + ((1 - \alpha)\omega^2)^2)}{(1 + (\alpha \omega^2)^2)(1 + ((1 - \alpha)\omega^2)^2)} \hspace{1cm} (9.8)
\]

which is singular at \( \alpha = 1 \).
Case II: $k = m - \Delta$ or $k = m - (\Delta + 1)$

For $k = m - \Delta$, $\rho = R[\tau]$, while for $k = m - (\Delta + 1)$, $\rho = R[1 - \tau]$.

\[
\tilde{V} = \begin{bmatrix} \sqrt{P_0} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \rho & 0 \\ 0 & \rho & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} 2\sigma^2.
\] (9.9)

Therefore,

\[
CQ = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 - \alpha & -\alpha \rho & 0 \\ 0 & \rho(1 - \alpha) & -\alpha & 0 \\ 0 & 0 & 0 & -1 + \alpha \end{bmatrix}
\] (9.10)

It can be shown that the eigenvalue matrix and mean vector are

\[
\xi = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \nu^+ & 0 & 0 \\ 0 & 0 & \nu^- & 0 \\ 0 & 0 & 0 & -1 + \alpha \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 2\sigma^2, \quad F = \begin{bmatrix} 0 \\ 0 \sqrt{\text{SIR}} \end{bmatrix} \end{bmatrix}
\] (9.11)

where $\nu^\pm \equiv \frac{1}{2} \left( 1 - 2\alpha \pm \sqrt{1 - 4\alpha(1 - \alpha)\rho^2} \right)$. Note that if $\rho = 0$, then Case II and Case I are equivalent. It is easily verified that $\nu^+ = 1 - \alpha$ and $\nu^- = -\alpha$, yielding the same eigenvalues as Case I. Substituting these values into Equation (9.5), we obtain the characteristic function for Case II:

\[
\psi_{II}(\omega, \text{SIR}|\alpha, \rho) = \frac{\exp \left( \frac{\alpha P}{\sigma^2} \frac{j\omega \sigma^2}{1 - \alpha j\omega \sigma^2} \right)}{(1 + (1 - \alpha) j\omega \sigma^2)(1 - \alpha j\omega \sigma^2)(1 - \nu^+ j\omega \sigma^2)(1 - \nu^- j\omega \sigma^2)}
\] (9.12)

which is singular for $\alpha = 1$, and is equivalent to Case I when $\rho = 0$, as expected.
Case III: $k = m + \Delta$ or $k = m + \Delta + 1$

For $k = m - \Delta$, $\rho = R[\tau]$ and $\gamma = p_u(\tau T_{ps})$, while for $k = m - (\Delta + 1)$, $\rho = R[1 - \tau]$ and $\gamma = p_u((1 - \tau)T_{ps})$. Therefore,

\[
\mathbf{V} = \begin{bmatrix} \sqrt{P_o} \\ 0 \\ 0 \\ \sqrt{P_o \gamma} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \rho \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \rho & 0 & 0 & 1 \end{bmatrix} 2\sigma^2 \quad (9.13)
\]

and,

\[
\mathbf{CQ} = \begin{bmatrix} \alpha & 0 & 0 & -\rho(1 - \alpha) \\ 0 & 1 - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ \rho \alpha & 0 & 0 & -1 + \alpha \end{bmatrix}. \quad (9.14)
\]

It can be shown that the eigenvalue matrix and mean vector are

\[
\mathbf{\xi} = \begin{bmatrix} -\nu^- & 0 & 0 & 0 \\ 0 & 1 - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\nu^+ \end{bmatrix} 2\sigma^2 \mathbf{F} = \begin{bmatrix} \eta^+ \\ 0 \\ \frac{\sqrt{3RI}}{\kappa} \\ \eta^- \end{bmatrix} \quad (9.15)
\]

where

\[
\eta^\pm = \sqrt{(\mu^\pm)^2 + 1 - \rho^2} \left[ (1 + \gamma)\sqrt{1 - p} - (\mu^\pm(1 - \gamma))\sqrt{1 + p} \right], \quad (9.16)
\]

\[
\kappa = 2\sqrt{2(1 - \rho^2)(\mu^+ - \mu^-)}, \quad (9.17)
\]

and

\[
\mu^\pm \equiv \rho(2\alpha - 1) \pm \sqrt{1 - 4\alpha(1 - \alpha)\rho^2}. \quad (9.18)
\]
Substituting these values into Equation (9.5), we obtain the characteristic function for Case III:

\[
\psi_{III}(\omega, \text{SIR} | \alpha, \rho, \gamma) = \exp\left(\frac{P\sigma^2}{\sigma^2 - (2\alpha - 1)j\omega \sigma^2} - (1 - \alpha)(1 - \rho^2)j\omega \sigma^2\right)  \\
\frac{1 - [1 - \alpha(1 - \alpha)(2 + \rho^2)](j\omega \sigma^2)^2 + \alpha(1 - \alpha)\rho^2(j\omega \sigma^2)^3 + \alpha^2(1 - \alpha)^2(1 - \rho^2)(j\omega \sigma^2)^4}{1 - [1 - \alpha(1 - \alpha)(2 + \rho^2)](j\omega \sigma^2)^2 + \alpha(1 - \alpha)\rho^2(j\omega \sigma^2)^3 + \alpha^2(1 - \alpha)^2(1 - \rho^2)(j\omega \sigma^2)^4}.
\]

(9.19)

9.3 Numerical Inversion of the Characteristic Function

The Beaulieu series [26] is a well-known technique for obtaining accurate estimates of the error rate \( \mathcal{P} \{ D = X - Y < 0 \} \) from the characteristic function of \( D \). This difference, \( D \), is generally infinite in range but is “effectively time-limited” in the sense that the pdf and cdf can be reconstructed from a finite number of samples with negligible aliasing if the sample spacing is sufficiently small. The spacing can generally be made sufficiently small, and the number of points can be made sufficiently large, such that the error arising from aliasing is smaller than the finite machine precision. To compute the error using \( N \) samples over a truncated range of \( R \) (i.e., \( D = X - Y \in [a, b], R = b - a \)),

\[
P_{III}(\text{SIR} | \alpha, \rho, \gamma) = \frac{1}{2} - \sum_{k=0}^{N-1} \frac{\psi_{III}(\omega, \text{SIR} | \alpha, \gamma, \rho)}{j\omega_k R}
\]

(9.20)

where \( k = 0 \ldots N - 1 \) and \( \omega_k = (2k + 1 - N)\pi/R \).

9.3.1 Example: Case III Correlated Limit

Figure 7.1 is an example of the error probability at 15dB SIR over the \((\alpha, \gamma)\) plane, and Figure 9.1 illustrates the relative error between the analytical results derived in this section, and the results from the analytically derived characteristic function. Both are evaluated numerically and compared on a logscale. The results show an excellent agreement between these two dif-
Figure 9.1: Relative Error, Case 3 Correlated Limit, 15dB SIR

Different methods, with error probabilities spanning seven orders of magnitude and a relative error consistently within 1%.
Chapter 10

2-D Recursive Bisection

10.1 Background

Using the Beaulieu series to invert the characteristic function, the Case III conditional error probability can be computed for given values of the SIR, $\alpha$ and $\tau$ with an absolute error on the order of the machine precision. Since we do not have an analytic expression for the conditional error, we must compute the average error probability for a given SIR from a finite number of samples in the $(\alpha, \tau)$ plane.

This computation is complicated by the fact that the error probability at the point $(\frac{1}{2}, 0)$ is $\frac{1}{2}$, even at infinite SIR. On the other hand, it is extremely advantageous that we are computing a definite integral with finite intervals $\alpha = (\frac{1}{2}, 1)$ and $\tau = (0, T_{ps})$. By recursively subdividing these intervals until a convergence criterion is satisfied, the average error rate can be effectively upper and lower bounded within a specified tolerance.

After describing the algorithm, we apply it to several examples where the interference correlation function equals the autocorrelation of the pulse-shape function. It is shown in Section
13.2 that this situation arises in spread-time, spectrally-encoded CDMA systems. At high SIR, the Case III average error probability is shown to have a power-law dependence on the SIR. That is,

\[ P_{III}(SIR \gg 1) \approx \Lambda \frac{T_p}{T_{ps}} SIR^{-\xi}, \]

(10.1)

for some constants \( \Lambda \) and \( \xi \) that depend on the pulse-shape. Since the average is computed numerically, there does not appear to be an analytical solution for these constants.

### 10.2 Evaluation of a Definite Integral using Riemann Sums

The definite integral of a bounded function \( f(x, y) \) can be expressed as the limit of the Riemann sum, [6, 4.14],

\[
\int_a^b \int_c^d f(x, y) \, dy \, dx = \lim_{\|R\| \to 0} \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} f(u_k, v_\ell) \Delta x_k \Delta y_\ell,
\]

(10.2)

where \( R \) is any partition of the intervals \([a, b] \) and \([c, d] \) into \( N_x \) and \( N_y \) sub-intervals of length \( \Delta x_k \) and \( \Delta y_\ell \), and \( u_k \) and \( v_\ell \) are any points from the \( k^{th} \) sub-interval, \([x_{k-1}, x_k] \) and \( \ell^{th} \) sub-interval, \([y_{\ell-1}, y_\ell] \), respectively. If \( f(x, y) > 0 \), then every term in the sum is positive, and we may interpret the definite integral as the sum of the volumes of \( N_x \times N_y \) cuboids. Note that for our application, \( f(x, y) \) is a probability and thus is bounded by \([0, 1] \).

Let the points \( u_k \) and \( v_\ell \) be chosen for a given partition such that \( U_{k,\ell} \) and \( L_{k,\ell} \) are the upper and lower bounds of \( f(x, y) \) over the sub-interval \((k, \ell) \), \( S_{k,\ell} \) is the exact value of the definite integral of \( f(x, y) \) over this sub-interval, and \( A_{k,\ell} \approx S_{k,\ell} \) is an approximation for the definite integral such that \( L_{k,\ell} \leq A_{k,\ell} \leq U_{k,\ell} \). Then

\[
U \equiv \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} U_{k,\ell} \Delta x_k \Delta y_\ell
\]

(10.3)
is a Riemann sum that upper bounds the definite integral in (10.2),

$$L \equiv \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} L_{k,\ell} \Delta x_k \Delta y_\ell$$  \hspace{1cm} (10.4)$$

is a Riemann sum that lower bounds the definite integral,

$$S \equiv \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} S_{k,\ell} \hspace{1cm} (10.5)$$

$$S_{k,\ell} \equiv \int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} f(x, y) \, dy \, dx$$

is a Riemann sum equal to the exact value of the definite integral, and

$$A \equiv \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} A_{k,\ell} \hspace{1cm} (10.6)$$

is a Riemann sum approximates the definite integral [28, Section 4.1]. The exact and approximate values of the Riemann sum are related by

$$S = \int_a^b \int_c^d f(x, y) \, dx \, dy = \lim_{\|R\| \to 0} A. \hspace{1cm} (10.7)$$

Both the exact and approximate Riemann sums are bounded by the upper and lower Riemann sums,

$$L \leq S \leq U \hspace{1cm} (10.8)$$

$$L \leq A \leq U$$

These upper and lower bounds are also known as Darboux sums. The total absolute difference (error) between the exact and approximate Riemann sums is defined as

$$D \equiv \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} D_{k,\ell} \equiv |A_{k,\ell} - S_{k,\ell}| \hspace{1cm} (10.9)$$

where $D_{k,\ell}$ is the absolute difference of each sub-interval.

Since $A_{k,\ell}$ and $S_{k,\ell}$ are lower bounded by $L_{k,\ell} \Delta x_k \Delta y_\ell$ and upper bounded by $U_{k,\ell} \Delta x_k \Delta y_\ell$, the absolute difference of each sub-interval, $D_{k,\ell}$ is upper bounded by the absolute error term

$$D_{k,\ell} \leq E_{k,\ell} \equiv (U_{k,\ell} - L_{k,\ell}) \Delta x_k \Delta y_\ell. \hspace{1cm} (10.10)$$
Therefore, the total absolute difference between the approximate and exact Riemann sums, $D$, is upper bounded by the absolute error between the upper and lower bounds,

$$D \leq E \equiv \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} E_{k,\ell} \equiv \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} (U_{k,\ell} - L_{k,\ell}) \Delta x_k \Delta y_\ell = U - L,$$

and the approximation $A$ is guaranteed to converge to the definite integral $S$ if the maximum absolute error $E$ vanishes since $D \leq E$.

A refinement of the partition $R \rightarrow R'$ created by further sub-dividing the intervals $[x_{k-1}, x_k]$ and $[y_{\ell-1}, y_\ell]$ is also a Riemann sum, and the upper and lower Riemann sums of this new partition are [28, Section 4.1]

$$\int_a^b \int_c^d f(x, y) \, dx \, dy \left\{ \begin{array}{ll}
\geq \sum_{k=1}^{N'_x} \sum_{\ell=1}^{N'_y} L'_k,\ell \Delta x'_k \Delta y'_\ell, & \geq \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} L_{k,\ell} \Delta x_k \Delta y_\ell, \\
\leq \sum_{k=1}^{N'_x} \sum_{\ell=1}^{N'_y} U'_k,\ell \Delta x'_k \Delta y'_\ell, & \leq \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} U_{k,\ell} \Delta x_k \Delta y_\ell,
\end{array} \right.$$

where there are now $N'_x > N_x$ and $N'_y > N_y$ total sub-intervals, $[x'_{k-1}, x'_k]$ and $[y'_{\ell-1}, y'_\ell]$, of length $\Delta x'_k$ and $\Delta y'_\ell$, respectively. In other words, the refined upper and lower bounds $L \leq L' \leq S \leq U' \leq U$ are at least as tight as the original bounds, and the maximum absolute error $E' = (U' - L') \leq E = (U - L)$. Intuitively, the Riemann sum converges as the number of points grows sufficiently large, but the precise convergence criterion must be stated carefully.

One possible condition for the convergence of the approximate Riemann sum $A$ to the definite integral $S$ is that the maximum absolute error is within a specified tolerance,

$$D = \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} |A_{k,\ell} - S_{k,\ell}| \leq \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} (U_{k,\ell} - L_{k,\ell}) \Delta x_k \Delta y_\ell = E < \epsilon,$$

for some $\epsilon > 0$ [28, Section 4.1]. Given an $\epsilon$, the most straightforward way to accomplish this is to create a uniform partition of $N_x = N$ and $N_y = N$ equally spaced points, and increase $N$ until the desired convergence is obtained.
However, this is a very cumbersome approach, potentially requiring an enormous number of points where the integrand $f(x, y)$ must be determined. For our application, the integrand must be computed, using the Beaulieu series expansion, for every given value of the SIR, $\alpha$, and $\tau$. To obtain a result that is accurate within the level of the machine precision, several seconds of computation are typically required for each point, which is undesirable (or even prohibitive) if millions of points are required for convergence.

When the function is highly non-uniform over the integration interval, it can be advantageous to use a non-uniform partition. In the high SIR regime for the Case III conditional error probability, there is an extreme concentration of probability mass near the $(\alpha_0, \tau_0) = (\frac{1}{2}, 0)$ endpoint, where $P_e = \frac{1}{2}$, and diminishes exponentially away from this point. For a fixed number of points, a non-uniform partition using many small sub-intervals concentrated in the neighborhood of $(\alpha_0, \tau_0) = (\frac{1}{2}, 0)$ converges much more rapidly than a uniform partition, which has most of the terms effectively equal to zero.

### 10.2.1 Uniform Convergence Criterion

For a given total convergence criterion, $\epsilon$, and a fixed number of points, $N = N_x \times N_y$, we create a non-uniform partition such that the upper and lower bounds for each term in (10.13) satisfies the same convergence criterion $\epsilon' \equiv \frac{\epsilon}{N}$. That is, the total absolute difference of the approximate Riemann sum $A$ is less than the total convergence criterion, $\epsilon$,

$$\sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} D_{k,\ell} \leq \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} E_{k,\ell} \leq \sum_{k=1}^{N_x} \sum_{\ell=1}^{N_y} \epsilon' = N\epsilon' = \epsilon$$ (10.14)

if the absolute error $D_{k,\ell} \leq E_{k,\ell} < \epsilon'$ in all $N = N_x \times N_y$ sub-intervals. Therefore, if

$$E_{k,\ell} = (U_{k,\ell} - L_{k,\ell}) \Delta x_k \Delta y_\ell < \epsilon' \quad \forall k = [1 \ldots N_x] \text{ and } \ell = [1 \ldots N_y],$$ (10.15)
then the total absolute error of the approximate Riemann sum $A$ is guaranteed to converge within a tolerance of

$$D \leq E < Ne' = \epsilon.$$  

(10.16)

Creating a non-uniform partition to ensure the same convergence criterion for each sub-interval employs a similar philosophy to maximum-entropy source coding. The recursive bisection algorithm developed in this chapter is an efficient method to find a non-uniform partition such that the error is approximately equal for each interval. The central idea behind the recursive bisection algorithm is to continue dividing the sub-intervals that have not yet converged, such that every single sub-interval has an error bounded by $e' = \frac{\epsilon}{N}$, and the approximate Riemann sum, $A$, converges within $\epsilon = Ne'$.

### 10.2.2 Implementation Issues and Assumptions

Suppose that the integrand is a function of the Signal-to-Interference Ratio, such that

$$f(x, y) = P(SIR \mid x, y) \; f_x(x) \; f_y(y),$$

and the Riemann sum $A$ is to be calculated for a range of SIR values, such as $SIR_i = (SIR_{\text{min}} + i\Delta) \text{dB}$ for $i = [1 \ldots I]$, with an initial value of $SIR_{\text{min}} \text{dB}$ an increment of $\Delta\text{dB}$.

If a fixed absolute convergence criterion $e'$ is used, then the relative error grows progressively worse as the SIR increases, since the exact value of the integral $S$ decreases monotonically with increasing SIR, and the relative error is inversely proportional to $S$.

Therefore, it is typically more desirable to calculate $A$ so that the relative error is approximately constant for all SIR values. Given certain conditions, this can be accomplished using the techniques described in the remainder of this section. Although these conditions cannot be explicitly guaranteed, we also present a simple procedure for verifying the convergence.
Relative Error  The maximum relative error $\overline{E}$ is defined as the maximum absolute error, $E$, normalized by $S$, the exact value of the the Riemann sum. That is,

$$\overline{E} \equiv \frac{E}{S} = \frac{U - L}{S}$$  \hspace{1cm} (10.17)

Similarly, the total relative error $\overline{D}$ is defined as

$$\overline{D} \equiv \frac{D}{S} = \sum_{k=1}^{N_y} \sum_{\ell=1}^{N_x} \frac{|A_{k,\ell} - S_{k,\ell}|}{S_{k,\ell}}$$  \hspace{1cm} (10.18)

All of the quantities in the uniform convergence criterion (10.16) can be normalized by $S$ to express the uniform convergence criterion in terms of the relative error $\overline{D}$ and the maximum relative error $\overline{E}$,

$$\frac{D}{S} \leq \frac{E}{S} \leq \frac{U - L}{S} < \frac{Ne'}{S} = \frac{\epsilon}{S}$$

$$\equiv \overline{D} \leq \overline{E} < \frac{U - L}{S} < N\epsilon'_{\text{rel}} = \epsilon_{\text{rel}}$$  \hspace{1cm} (10.19)

where we have defined the relative convergence criterion $\epsilon'_{\text{rel}} \equiv \epsilon' / S$ and the total relative convergence criterion $\epsilon_{\text{rel}} \equiv \epsilon / S$. Note that this is a theoretical result obtained using the value of $S$, which is not known apriori.

Relative Convergence Criterion  In practice, the relative convergence criterion $\overline{E} < \epsilon_{\text{rel}}$ must be based on an approximation $\tilde{S} \approx S$ to obtain the convergence criterion for each sub-interval, $\epsilon' = \tilde{S}\epsilon'_{\text{rel}}$, from the fixed relative convergence criterion, $\epsilon'_{\text{rel}}$. Applying this to the uniform convergence criterion (10.15), if

$$E_{k,\ell} < \tilde{S}\epsilon'_{\text{rel}} = \frac{\tilde{S}}{S}\epsilon' \approx \epsilon'$$  \hspace{1cm} (10.20)

for all $N$ sub-intervals, then the total error is bounded by

$$E < \tilde{S}N\epsilon'_{\text{rel}} = \tilde{S}\epsilon_{\text{rel}} \approx \epsilon,$$  \hspace{1cm} (10.21)
and the total relative error is
\[ \bar{E} < \frac{\tilde{S}}{S} \epsilon_{rel} \approx \epsilon_{rel}. \] (10.22)

The accuracy of this approximation clearly depends on the ratio \( \tilde{S}/S \). When this ratio is approximately constant for each SIR, it can be compensated by choosing a smaller relative convergence criterion, \( \epsilon'_{rel} \).

**Small Increment Condition**  In practice, when the SIR increment \( \Delta \) is sufficiently small, the approximate value \( \tilde{S} \) can be obtained using the approximate Reimann sum \( A \) from the previous SIR step, \( \tilde{S} = A_{prev} \). If \( S \) decreases linearly for each step of the SIR, then the ratio \( \tilde{S}/S \approx P(SIR - \Delta)/P(SIR) \) is constant, and directly proportional to \( \Delta \). Therefore, by choosing \( \Delta \) sufficiently small, the maximum relative error
\[ \bar{E} \equiv \frac{E}{S} < \frac{\tilde{S}}{S} N \epsilon'_{rel} \approx \epsilon_{rel} \] (10.23)
remains approximately constant for each SIR point evaluated. The starting value \( SIR_{min} \) is chosen sufficiently small that \( S \approx \frac{1}{2} \) (e.g., 1dB).

**Sufficient Smoothness Condition**  The ability to form \( L \leq A \leq U \) as upper and lower bounds for the approximate Riemann sum \( A \), and therefore bound the absolute error \( D \leq E < N \epsilon' \) such that \( A \) converges to the exact value \( S \) within \( \epsilon = N \epsilon' \), is fundamentally predicated on the assumption that the extrema of the integrand \( f(x, y) \) are known over all \( N \) sub-intervals. However, the value of the integrand \( f(x, y) \) is only known at the \((N_x + 1)(N_y + 1)\) corner points bounding the \( N \) sub-intervals. Therefore, we require the integrand to be “sufficiently smooth” with respect to the partition such that the extrema of \( f(x, y) \) over each sub-interval occur at one of the four corner points. Less-well behaved functions with local extrema throughout the region
of integration can be handled by manually subdividing the interval at known critical points, and solving a separate Riemann sum for each sub-interval, or by forcing a minimum recursion depth, which imposes a constraint on the maximum sub-interval length.

**Convergence of the Total Relative Error** Given a fixed relative convergence criterion, $\epsilon'_{\text{rel}}$, the uniform convergence in each sub-interval is bounded by the criterion (10.20), using the approximation $\tilde{S} = A_{\text{prev}}$. However, the convergence for a fixed total relative error $\epsilon_{\text{rel}}$ is not guaranteed. In terms of the specified convergence criterion $\epsilon'_{\text{rel}}$, the total relative error is bounded by

$$E < A_{\text{prev}}^N \epsilon'_{\text{rel}} \approx \frac{P(SIR - \Delta)}{P(SIR)} N \epsilon'_{\text{rel}},$$

(10.24)

where the approximation is exact if the approximate Riemann sum from the previous SIR value has converged to the exact value. Given that $P(SIR)$ decreases linearly, and that $N$ is approximately constant in the high SIR regime, then the RHS of (10.24) is a constant. Furthermore, if the relative error $D \ll E$ for the first SIR value $SIR_{\text{min}}$, then the convergence of $A_{\text{prev}} \rightarrow P(SIR_{\text{min}})$ is also guaranteed, then $A_{\text{prev}} \rightarrow P(SIR)$ will continue to converge for every successive SIR point. However, these conditions cannot be guaranteed, and if $N$ grows without bound as a function of the SIR, then the total relative error also grows with the SIR and never stabilizes. This divergence is compounded for successive calculations by the fact that the approximation $\tilde{S} = P(SIR - \Delta) \approx A_{\text{prev}}$ grows increasingly inaccurate, but this is a second-order effect.

**Verification of Convergence** The recursive bisection algorithm, which is defined for one-dimensionals (1-Ds) in Section 10.3.3 and for 2-Ds in Section 10.4, calculates an approximate Riemann sum $A$ over a range of SIR values using these assumptions and techniques to maintain
an approximately constant total relative error for each SIR value, given a fixed $\epsilon_{rel}$. A simple approximation that can be used to verify that $\overline{E}$ remains constant, without knowledge of the exact value $S$, is

$$\overline{E} = \frac{U - L}{S} \approx \frac{U - L}{L} = \frac{U}{L} - 1 \propto \frac{U}{L}. \quad (10.25)$$

Thus, the total relative error is approximately constant if the ratio of the upper to lower bounds is also constant. This is easily verified graphically by plotting the upper and lower bounds. If they diverge, then the algorithm must be run with smaller values of $\epsilon'_{rel}$ until the ratio of the upper and lower bounds is constant.

### 10.3 Example in One-Dimension

Before defining the 2-D recursive bisection algorithm, we consider a 1-D example of a decaying exponential function. This will illustrate the utility of using a non-uniform partition with uniform convergence compared to partitions with uniform or non-uniform geometric spacing.

Consider the one dimensional function

$$P(\text{SIR} | x) = \frac{1}{2} \exp(-\text{SIR} \, x), \quad (10.26)$$

where $x$ is a conditional parameter that is uniform in the interval $[0, 1]$, such that the unconditioned error rate is

$$P(\text{SIR}) = \int_0^1 P(\text{SIR} | x) \, dx = \frac{1}{2 \, \text{SIR}} \left(1 - \exp(-\text{SIR})\right) \approx \frac{1}{2 \, \text{SIR}}. \quad (10.27)$$

This has a similar power-law dependence to the results obtained for Case III (10.1), so it provides insight into the algorithm.
The exact 1-D Riemann sum of the unconditioned error probability is

$$P(SIR) = S \equiv \sum_{k=1}^{N} S_k$$

(10.28)

for any partition of $x = [0, 1]$ into $N$ sub-intervals, where

$$S_k \equiv \int_{x_{k-1}}^{x_k} P(SIR | x = u_k) \Delta x_k,$$

(10.29)

for some point $u_k$ in the $k^{th}$ sub-interval $[x_{k-1}, x_k]$.

Since $P(SIR | x)$ is a monotonically decreasing function, the minimum over every interval is the right endpoint, and the maximum over every interval is at the left endpoint. Thus, $U_k = P(SIR | x_{k-1})$, $L_k = P(SIR | x_k)$, and the upper and lower Riemann sums are

$$U \equiv \sum_{k=1}^{N} P(SIR | x_{k-1}) \Delta x_k$$

$$L \equiv \sum_{k=1}^{N} P(SIR | x_k) \Delta x_k.$$  

(10.30)

Therefore, the maximum error terms are $E_k = (U_k - L_k) \Delta x_k$ and the maximum total error is

$$E \equiv \sum_{k=1}^{N} E_k \equiv \sum_{k=1}^{N} (U_k - L_k) \Delta x_k$$

(10.31)  

$$\equiv \sum_{k=1}^{N} \left[ P(SIR | x_{k-1}) - P(SIR | x_k) \right] \Delta x_k.$$

A reasonable and straightforward method to calculate the approximate Riemann sum $A$, which only requires the evaluation of the endpoints, is to use the average (or trapezoidal) Riemann sum. This is defined as $A_k \equiv \left( \frac{U_k + L_k}{2} \right) \Delta x_k$ so that

$$A \equiv \sum_{k=1}^{N} A_k \equiv \sum_{k=1}^{N} \left( \frac{U_k + L_k}{2} \right) \Delta x_k$$

(10.32)  

$$= \sum_{k=1}^{N} \left( \frac{P(SIR | x_{k-1}) + P(SIR | x_k)}{2} \right) \Delta x_k,$$

which is bounded by $U$ and $L$. Note that each term $A_k \equiv \left( \frac{U_k + L_k}{2} \right) \Delta x_k$ has an area that is equal to a rectangle with the height equal to the average value, or a trapezoid created by
linear interpolation, which is illustrated in figure 10.2.b). This implies that simply using the average value from the endpoints is equivalent to the more cumbersome process of first linearly interpolating, and then integrating each interval. For the 2-D case, it is proven in Appendix I that the same relationship holds for bilinear interpolation. That is, a cuboid with the height equal to the average of the four corner points is equivalent to the integral over the 2-D function created using bilinear interpolation over the region.

For the purposes of comparison, we derive the maximum absolute error, $E$, and maximum relative error, $\bar{E}$, for two specific examples; first, for a uniform partition, and then for a geometrically spaced partition. For a fixed number of points, $N$, these partitions illustrate the difficulty of obtaining accurate estimates for the average error rate in the high SIR regime. A 1-D implementation of the recursive bisection is then introduced, and compared to uniform and geometric partitions with the same number of points. This example clearly demonstrates the efficiency of using a uniform convergence criterion for each sub-interval.

![Figure 10.1: Uniform and Geometric Partitions.](image-url)
10.3.1 Uniform Partition

For a uniform (linear) partition, \( \Delta x_k = \frac{1}{N} \) for all \( k \), and the \( k \)th sub-interval is simply \([\frac{k-1}{N}, \frac{k}{N}]\). An example with \( N = 4 \) sub-intervals is illustrated in Figure 10.1.a). The upper, lower, and approximate Riemann sums are

\[
U_{\text{lin}} = \frac{1}{N} \sum_{k=0}^{N-1} P\left( \text{SIR} \mid x = \frac{k}{N} \right)
\]

\[
L_{\text{lin}} = \frac{1}{N} \sum_{k=1}^{N} P\left( \text{SIR} \mid x = \frac{k}{N} \right)
\]

\[
A_{\text{lin}} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{2} \left[ P\left( \text{SIR} \mid x = \frac{k-1}{N} \right) + P\left( \text{SIR} \mid x = \frac{k}{N} \right) \right],
\]

and thus the absolute error for the approximate Riemann sum, \( A_{\text{lin}} \) is upper bounded by

\[
E_{\text{lin}} = U_{\text{lin}} - L_{\text{lin}} = \frac{1}{N} \left[ \sum_{k=0}^{N-1} P\left( \text{SIR} \mid x = \frac{k}{N} \right) - \sum_{k=1}^{N} P\left( \text{SIR} \mid x = \frac{k}{N} \right) \right]
\]

\[
= \frac{1}{N} \left[ P\left( \text{SIR} \mid x = 0 \right) - P\left( \text{SIR} \mid x = 1 \right) \right]
\]

\[
= \frac{1}{2N} \left[ 1 - \exp\left( -\text{SIR} \right) \right].
\]

Note that this is a telescoping sum, where all of the interior terms cancel. Figure 10.1.a) provides a graphical interpretation of the maximum absolute error. The shaded areas correspond the \( N \) error terms, \( E_k \), which form the total maximum error \( E \). Graphically, we can see that by shifting all four shaded areas into the first column, this column is filled except for a very small piece at the bottom that corresponds to the lower bound from the last column. Using the exact value of the integral from (10.27), the maximum relative error is

\[
\overline{E}_{\text{lin}} \equiv \frac{E_{\text{lin}}}{P(\text{SIR})} = \frac{\text{SIR}}{N},
\]

Therefore, with the SIR increasing linearly on a decibel scale, \( N \) must increase exponentially to maintain a given relative error.
10.3.2 Geometric Non-uniform Partition

We now consider a non-uniform partition where the length of the sub-intervals decreases geometrically. That is, for some base $B > 1$, the $k$th subinterval is $\left[\left(\frac{1}{B}\right)^{N+1-k}, \left(\frac{1}{B}\right)^{N-k}\right]$, except for the first sub-interval, which is $[0, \left(\frac{1}{B}\right)^{N-1}]$. An example with base $B = 2$ and $N = 4$ subdivisions is shown in Figure 10.1.b). The upper, lower, and approximate Riemann sums are

$$U_{geo} = P(SIR|x = 0)\frac{1}{B^N} + \sum_{k=2}^{N} P\left(SIR|x = \frac{1}{B}^{N+1-k}\right)\left[\frac{1}{B^{N-k}} - \frac{1}{B^{N+1-k}}\right]$$

$$= \frac{1}{2B^{N-1}} + B\sum_{k=1}^{N-1} P\left(SIR|x = \frac{1}{B}^{N-k}\right)\left[\frac{1}{B^{N-k}} - \frac{1}{B^{N+1-k}}\right]$$

$$L_{geo} = \sum_{k=1}^{N} P\left(SIR|x = \frac{1}{B}^{N-k}\right)\left[\frac{1}{B^{N-k}} - \frac{1}{B^{N+1-k}}\right]$$

$$= \exp\left(-\frac{SIR}{2}\right)\left(1 - \frac{1}{B}\right) + \sum_{k=1}^{N-1} P\left(SIR|x = \frac{1}{B}^{N-k}\right)\left[\frac{1}{B^{N-k}} - \frac{1}{B^{N+1-k}}\right]$$

$$A_{geo} = \frac{1}{2}\left[P(SIR|x = 0) + P\left(SIR|x = \frac{1}{B}^{N-1}\right)\right]\frac{1}{B^N}$$

$$+ \frac{1}{2}\sum_{k=2}^{N} \left[P\left(SIR|x = \frac{1}{B}^{N-k}\right) + P\left(SIR|x = \frac{1}{B}^{N+1-k}\right)\right]\left[\frac{1}{B^{N-k}} - \frac{1}{B^{N+1-k}}\right]$$

(10.36)

where we have separated the endpoints from the interior sum to form a telescoping series for the total absolute error, $E_{geo} = U_{geo} - L_{geo}$. The shaded regions in Figure 10.1.b) are an example of the error terms. However, the sum in the upper bound is scaled by $B$, so a telescoping series is not obtained, unlike the uniform partition. Intuitively, this arises because the value of the function at every point (except the endpoints) is the upper bound for the segment to the right, and the lower bound for the segment to the left. In the uniform partition, all segments have the same width, so these terms cancel, but with geometric spacing, the width of every segment to the right of a point (the upper bound) is $B$ times greater than the width of the segment on the left (the lower bound).
We will now show that, for a given base $B$, this sum creates an asymptotic limit on the maximum error for large $N$. The maximum absolute error for the approximate Riemann sum, $A_{\text{geo}}$, is the difference of the upper an lower Riemann sums over the partition,

$$E_{\text{geo}} = U_{\text{geo}} - L_{\text{geo}}$$

$$= \left( \frac{1}{2B^{N-1}} \right) - \exp \left( -SIR \right) \frac{B - 1}{2B} + \frac{(B - 1)}{2} \sum_{k=1}^{N-1} \exp \left( - \frac{SIR}{B^k} \right) \left( \frac{B - 1}{B^k+1} \right)$$

$$= \left( \frac{1}{2B^{N-1}} \right) - \exp \left( -SIR \right) \frac{B - 1}{2B} + \frac{(B - 1)^2}{2B} \sum_{k=1}^{N-1} \frac{1}{B^k} \exp \left( - \frac{SIR}{B^k} \right)$$

$$\approx \frac{1}{2} \left[ \left( \frac{1}{B} \right)^{N-1} + (B - 1)^2 \exp \left( - \frac{SIR}{B^{N-1}} \right) \right]$$.

(10.37)

where we have approximated the sum as an integral,

$$\sum_{k=1}^{N-1} \frac{1}{B^k} \exp \left( - \frac{SIR}{B^k} \right) \approx \int_{k=1}^{N-1} \frac{1}{B^k} \exp \left( - \frac{SIR}{B^k} \right) dk$$

$$= \frac{1}{SIR \log(B)} \left[ \exp \left( - \frac{SIR}{B^{N-1}} \right) - \exp \left( - \frac{SIR}{B} \right) \right]$$

(10.38)

and neglected the terms $\exp \left( -SIR \right)$ and $\exp \left( -SIR/B \right)$. All of these approximations are valid for high SIR. Therefore, the maximum relative error is approximately

$$\overline{E}_{\text{geo}} = \frac{E_{\text{geo}}}{P(SIR)} \approx \frac{SIR}{B^{N-1}} + \frac{(B - 1)^2}{B \log(B)} \exp \left( - \frac{SIR}{B^{N-1}} \right).$$

(10.39)

If $N$ and $B$ are sufficiently large that $B^{N-1} \gg SIR$ and $\frac{SIR}{B^{N-1}} \approx 0$, then the relative error is asymptotically equal to

$$\overline{E}_{\text{geo}} \rightarrow \frac{(B - 1)^2}{B \log(B)},$$

(10.40)

which is a limiting value that is only a function of the base, $B$. The exponential convergence in $N$ provides a clear advantage over the uniform partition, although the relative error, $\overline{E}_{\text{geo}}$. 
cannot be made arbitrarily small for a fixed \( B \). Given a relative error criterion \( \epsilon_{\text{rel}} \), there is always some value of \( B \) sufficiently close to one such that the asymptotic value of the relative error, \( \frac{(B-1)^2}{B \log(B)} < \epsilon_{\text{rel}} \). The number of points, \( N \), can then be increased until \( B^{N-1} \gg \text{SIR} \) and the relative error (10.39) converges to the asymptotic value (10.40). Conversely, if \( N \) and \( B \) are fixed, then for sufficiently low SIR the relative error equals the asymptotic value. But as the SIR increases, at some point \( \text{SIR} \gg B^{N-1} \), and the relative error is dominated by \( \frac{\text{SIR}}{B^{N-1}} \), which behaves the same as the uniform partition, increasing exponentially as the SIR is increased linearly (on a dB scale).

10.3.3 1-Dimensional Recursive Bisection

We will now demonstrate the advantage of bounding every error term \( E_k = (U_k - L_k) \Delta x_k \) by using a 1-D recursive bisection algorithm. The 1-D recursive bisection method used in this example is summarized in Algorithm 1 of Appendix I, and illustrated in Figure 10.2. The

Figure 10.2: One-Dimensional Recursive Bisection Algorithm.

this example is summarized in Algorithm 1 of Appendix I, and illustrated in Figure 10.2. The
“RecursiveBisection” function returns the Upper, Approximate, and Lower Riemann sums,

\[ L_{rb} \leq A_{rb} \approx \int_{x_{min}}^{x_{max}} P(SIR \mid x) f_x(x) \, dx \leq U_{rb} \]  \hspace{1cm} (10.41)

for a given SIR, absolute convergence tolerance \( \epsilon' \), conditional error probability function \( P(SIR \mid x) \), and pdf \( f_x(x) \), with range \((x_{min}, x_{max})\). As noted in Section 10.2.2, the fundamental assumption behind the validity of \( U \) and \( L \) as upper and lower bounds requires the integrand \( P(SIR \mid x) f_x(x) \) to be “sufficiently smooth”. This condition certainly applies to the monotonically decreasing exponential integrand considered for this example.

The “AverageErrorRates” function computes the Upper, Approximate and Lower Riemann sums using the “RecursiveBisection” function for the SIR values \( SIR = [1, 1+ \Delta, \ldots, 1+I\Delta] \) dB, given a fixed relative error criterion, \( \epsilon'_r \). Since the intended application of the algorithm is to find the average error rate of a function that includes a point \( P(SIR \mid x = 0) = \frac{1}{2} \) for any SIR, the initial average must be close to \( \frac{1}{2} \) at low SIR (e.g., 1 dB), and decrease slowly for small increments of the SIR. Therefore, the small increment condition discussed in Section 10.2.2 is applicable for \( \Delta \approx 1\) dB.

The maximum absolute error of the approximate Riemann sum \( A_{rb} \) is

\[ D_{rb} \leq E_{rb} = U - L < \epsilon = N \epsilon' = N P(SIR) \epsilon'_{rel} \]  \hspace{1cm} (10.42)

and the maximum relative error is

\[ E_{rb} = \frac{E_{rb}}{S} < \frac{N \tilde{S}}{S} \epsilon'_{rel} \approx \epsilon_{rel} \]  \hspace{1cm} (10.43)

where \( \tilde{S} = A_{prev} \) is the initial estimate of the Riemann sum, taken as the approximate Riemann sum at the previous step of the SIR. The ratio of the upper and lower bounds is proportional to the relative error \( E_{rb} \), and the convergence within the fixed relative error criterion \( \epsilon_{rel} \) can be
checked by plotting the upper and lower Riemann sums, $U$ and $L$, and verifying that their ratio remains approximately constant.

The total absolute difference, $D$ (10.9) between the approximate Riemann sum $A$ and the definite integral $S$ is generally much smaller than the maximum absolute error $E = U - L$. In practice, it cannot be determined since the value of the definite integral is unknown. Another common criterion for convergence is to iterate by using successively smaller values of $\epsilon_{\text{rel}}$ until the approximate Riemann sum $A$ “does not change”, i.e., the difference between the new iteration and the previous iteration is sufficiently small.

### 10.3.4 Comparison of the Partitioning Techniques

The recursive partitions were computed in 1dB increments up to 40dB using the 1-D recursive bisection algorithm for a convergence criterion of $\epsilon_{\text{rel}} = \frac{1}{1200}$, taken relative to the exact value of the average error rate, such that $e' = \frac{P(SIR)}{1200}$. For an equal comparison at each value of the SIR, the uniform and geometric partitions use the same number of subdivisions, $N$, as determined by the recursive bisection algorithm.

Figure 10.3 illustrates an example of the upper, lower, and average Riemann sums for one uniform and three different non-uniform partitions. The recursive partition (d.) and the $B = 2$ geometric partition (c.) clearly maintain a constant ratio between the upper and lower Riemann sums across 40dB of SIR, indicating that the total relative error is approximately constant. On the other hand, for the uniform (a.) and the $B = 21/20$ geometric partition (b.) the upper and lower Riemann sums diverge at high SIR, and the approximate Riemann sum reaches a floor around $10^{-3}$.

Figure 10.4 compares the total relative error, $\overline{E} \equiv \frac{E}{P(SIR)}$ to the analytical results derived
Figure 10.3: Upper, Lower and Approximate Riemann Sums: a.) Linear Partition, b.) Geometric Partition, \( B = \frac{21}{20} \), c.) Geometric Partition, \( B = 2 \), d.) Recursive Bisection, \( \epsilon_{rel} = \frac{1}{1200} \).

for each partition:

a.) \( E_{lin} \) \( (10.35) \), which yields an exact value.

b.) \( E_{geo} \) \( (10.39) \) with \( B = 21/20 \), which yields a rather poor approximation until the SIR is large (25dB), due to the small value of \( B \).

c.) \( E_{geo} \) \( (10.39) \) with \( B = 2 \), which yields an approximate value with excellent agreement due to the large value of \( B \).

d.) \( E_{rb} \approx \epsilon_{rel}/2 \).
While $\epsilon_{\text{rel}}$ is the strict upper bound for $E_{rb}$, as stated in (10.43), $\epsilon_{\text{rel}}/2$ is an excellent approximation because the error terms oscillate between the maximum value of $\epsilon'$ and a minimum value of $\epsilon' / 4$. This is discussed in more depth in the discussion of figures 10.6 - 10.8.

The relative error between the approximate Reimann sum, $A$, and the exact Riemann sum, $S$, is $\overline{D}$. This is illustrated in Figure 10.5 for the same four partitioning techniques. The behavior is essentially the same as for the total relative error, $E_{\text{rel}}$, in Figure 10.4 with different scaling, i.e. the relative error of the average Riemann sum, $\overline{T}$, is upper bounded by, and directly proportional to, the total relative error of the upper and lower Riemann sums, $\overline{E}$. For the recursive partition (d.), the relative error $\overline{T}$ is about two orders of magnitude less than the relative error $\overline{E}$. For the geometric partition with $B = 2$, (c.), $\overline{T}$ is about one order of magnitude less than $\overline{E}$. 

Figure 10.4: Total Relative Error, $\frac{E}{P(SIR)}$: a.) Linear Partition, b.) Geometric Partition, $B = \frac{21}{20}$, c.) Geometric Partition, $B = 2$, d.) Recursive Bisection, $\epsilon_{\text{rel}} = \frac{1}{1200}$. 

The diagram shows the relative error of upper and lower bounds as a function of SIR (dB).
In order to better understand the total error $E$, it is useful to plot the error terms for each sub-interval, $E_k$. Figures 10.6, 10.7 and 10.8 show the error terms, $E_k$, as a function of the interval index, $k = [1 \ldots N]$, for 5dB, 20dB and 40dB, respectively. The total absolute error $E$ is directly proportional to the total relative error $E = \frac{E}{P(SIR)}$ in Figure 10.4 at the same SIR. In general, the total error $E$ for any given SIR tends to be dominated by the largest value of $E_k$ when the variance of the $E_k$ is large. It will be shown that, independent of the SIR, the recursive bisection algorithm effectively constrains the error to oscillate in the range $\epsilon/4 < E_k < \epsilon'$, such that the peak value is $\epsilon'$, and the variance is also constrained. None of the other partition techniques guarantee such tight constraints on the error terms for arbitrary SIR, and the following examples demonstrate that the the total error $E$ is dominated by the largest value of $E_k$. 

Figure 10.5: Relative Error of the Approximate Riemann Sum, $\overline{D}$. 
At 5dB (Figure 10.6), all of the partitions have approximately the same peak value except for the geometric partition with $B = 2$, which is greater by about two orders of magnitude. Accordingly, the total relative errors at 5dB in Figure 10.4 are approximately equal, except for the geometric partition with $B = 2$, which is substantially larger.

At 20dB (Figure 10.7), the peak values in all the partitions are approximately equal except for the recursive partition, which has a peak value that is about three orders of magnitude smaller. We see this same relationship in the total relative errors at 20dB in Figure 10.4.

At 40dB (Figure 10.8), the linear and the $B = 21/20$ geometric partitions have an extremely large peak value that is two orders of magnitude greater than the peak for the $B = 2$ geometric partition, and at least four orders of magnitude greater than the peak recursive parti-
Figure 10.7: Error terms, $E_k$, of the Riemann Sum at 20dB SIR.

This is also consistent with the relative error at 40dB in Figure 10.4.

**Constant Peak-to-Average Ratio of $E_k$ for Recursive Bisection**

In figures 10.6, 10.7 and 10.8 it is evident that $E_k$ in the recursive partition essentially oscillates between $\epsilon'/4$ and $\epsilon'$, with an average of approximately $\epsilon'/2$. This is understood by following the $E_k$ from right to left. Recall that the integrand is $\exp(-SIRx)/2$ with $x = [0, 1]$, which decreases monotonically from $1/2$ at the left endpoint, to $\exp(-SIR)/2$ at the right endpoint. The algorithm must successively bisect the left-most sub-interval, which has the left endpoint equal to $1/2$, $n$ times until the maximum error, $[1 - \exp(SIR/2^n)]/2^{(n+1)} < \epsilon'$. Therefore, the right-most $(N - n)$ sub-intervals are the same as the geometric partition, with $B = 2$ except
Error Terms of the Riemann Sum, 40dB SIR

Figure 10.8: Error terms, $E_k$, of the Riemann Sum at 40dB SIR.

Figure 10.9: Reduction of the error by $\frac{1}{4}$
at low SIR (5dB, for instance) where \[ \frac{\exp(SIR/2) - \exp(SIR))}{4} > \epsilon', \] and the right-most sub-interval must also be bisected. This behavior is clearly evident by comparing the recursive and \( B = 2 \) geometric partitions at 20dB and 40dB, where the right-most error terms are so much smaller than \( \epsilon' \) that they do not appear in the range of plotted values.

The error terms of the recursive and \( B = 2 \) geometric partitions differ after \( n \) successive bisections, as this marks a transition where both the left sub-interval, \([0, 2^{-n}]\) and the right sub-interval, \([2^{-n}, 2^{1-n}]\) must be bisected. In this “steady-state” region, the recursive bisection algorithm effectively adapts the sub-interval width so that it remains constant until the error exceeds \( \epsilon' \), whereupon the sub-interval width is halved, and the error is reduced by approximately \( 1/4 \). This reduction by \( 1/4 \) occurs because the decrease of the integrand between the left and right end-points is approximately linear (given that the width of the interval is already small), and the height of the midpoint is approximately equal to average of the end-points. This is illustrated in Figure 10.9. The sub-interval width is then kept constant until it the error exceeds \( \epsilon \), and the sub-interval width must be halved again. In this manner, the error terms in the “steady-state” region (the leftmost \( N - n \) sub-intervals) oscillates between \( \epsilon'/4 \) and \( \epsilon' \), with an average of approximately \( \epsilon'/2 \). We conclude that the peak-to-average ratio of the \( E_k \) is approximately two for the recursive bisection technique.

Furthermore, this peak-to-average effectively remains constant for all SIR, since the only difference in the error terms at high SIR is the larger value of \( n \), the number of bisections required to reach the “steady-state” regime. In effect, the length of this “steady-state” regime, \( N - n \), remains essentially constant beyond moderate values of the SIR (about 10dB), and \( N \), the total number of sub-intervals, increases only to compensate for the larger required value of \( n \). In Figures 10.7 and 10.8, \( N - n \approx 100 \) with \( n \approx 3 \) at 20dB and \( n \approx 10 \) at 40dB. The SIR at 5dB
is sufficiently low that \( n = 0 \), and all error-terms are “steady-state”, i.e., every sub-interval must be bisected at least once.

### 10.4 2-D Recursive Bisection Algorithm

The 2-D recursive bisection method is summarized in Algorithm 2 of Appendix I, and illustrated in Figure 10.10. The “RecursiveBisection” function returns the Upper, Average, and Lower Riemann sums,

\[
L \leq A \approx \int_{y_{\min}}^{y_{\max}} \int_{x_{\min}}^{x_{\max}} P(SIR \mid x, y) f_x(x) f_y(y) \, dx \, dy \leq U,
\]

(10.44)

for a given SIR, conditional error probability function \( P(SIR \mid x) \), and pdfs \( f_x(x) \) and \( f_y(y) \), which ranges over \((x_{\min}, x_{\max})\) and \((y_{\min}, y_{\max})\). It is proven in Appendix I that the volume of the cuboid with height equal to the average of the four corner points is equivalent to the 2-D integral of the bilinear interpolation function. This is analogous to the Riemann sums in one dimension, where the area of a rectangle with height equal to the average of the two endpoints is equivalent to the 1-D integral of the linear interpolation function (i.e. the trapezoidal area).

As in the one-dimensional case, we require the integrand \( f(x, y) = P(SIR \mid x, y) f_x(x) f_y(y) \) to be “sufficiently smooth” such that the extrema over every sub-interval are at the endpoints. We also require that that \( \Delta dB \) be sufficiently small that the average error probability decreases very slightly and \( P(SIR \, dB) \approx P(SIR \pm \Delta dB) \). The “AverageErrorRates” function computes the Upper, Approximate and Lower Riemann sums for the \( N \) values \( SIR = [1, 1+\Delta, \ldots 1+N\Delta] \) dB. SIR, and the convergence of the integral throughout the range of SIR values can be verified by plotting the Upper and Lower Riemann sums.
10.4.1 Case III Average Error Probability

To compute the Case III average probability (2.35) we use the 2-D recursive bisection algorithm (10.44), where \( x \rightarrow \alpha, \ y \rightarrow \tau, \ (x_{\text{min}}, x_{\text{max}}) \rightarrow (\frac{1}{2}, 1), \ (y_{\text{min}}, y_{\text{max}}) \rightarrow (0, T_{\text{ps}}), \) and \( P(\text{SIR} \mid x, y) \rightarrow P_{\text{III}}(\text{SIR} \mid \alpha, \rho = R(\tau), \gamma = p_u(\tau)), \) which is evaluated by using the Bealieu series. The fact that \( \alpha \) and \( \tau \) are both uniform simplifies the algorithm somewhat, as the product of the parameter distributions \( f_x(x) f_y(y) \) in (10.44) simplifies to a constant (of two).

For each SIR, the recursion is initialized with

\[
\begin{align*}
P_{00} & = P_{\text{III}}(\text{SIR} \mid \alpha = \alpha_0, \tau = \tau_0) = \frac{1}{2} \\
P_{10} & = P_{\text{III}}(\text{SIR} \mid \alpha = \alpha_1, \tau = \tau_0) = \frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \\
P_{01} & = P_{\text{III}}(\text{SIR} \mid \alpha = \alpha_0, \tau = \tau_1) \leq \frac{4}{3\sqrt{\pi} \text{SIR}} \exp \left( -\frac{\text{SIR}}{4} \right) \\
P_{11} & = P_{\text{III}}(\text{SIR} \mid \alpha = \alpha_1, \tau = \tau_1) = \frac{1}{2} \exp \left( -\frac{\text{SIR}}{2} \right).
\end{align*}
\]

The inequality for the point \((\alpha, \tau) = (\frac{1}{2}, 1)\) is from Equation (7.48), corresponding to the limit of complete correlation. This clearly indicates the exponentially decaying nature of the error probability, as \( \alpha \) and \( \tau \) move away from the origin.
Figure 10.11 demonstrates how the recursion is performed for a square pulse shape at 35dB SIR, which is plotted in figure 10.12. The recursive algorithm clearly places the majority of the points in the vicinity of $\alpha = \frac{1}{2}$, where the error is about 15 orders of magnitude greater than the noise floor from the machine precision.

Figure 10.11: Bisection Quadrants for a Square Pulse, 35dB SIR.
10.5 Numerical Examples

The four pulse shapes considered all have unity two-sided equivalent width:

1. Square Pulse: \( p_u(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{if } |t| > \frac{1}{2} \end{cases} \)

2. Gaussian Pulse: \( p_u(t) = \exp(-t^2\pi/2) \)

3. Triangular pulse: \( p_u(t) = \begin{cases} 1 - \frac{2}{3}|t| & \text{if } |t| \leq \frac{3}{2} \\ 0 & \text{if } |t| > \frac{3}{2} \end{cases} \)

4. Exponential Pulse: \( p_u(t) = \exp(-|t|) \)

The timescale has been normalized by the pulse width \( T_p \), such that \( p_u(t) \) has unity equivalent width, and the pulse separation is a multiple of \( T_p \), \( N_{ps} \equiv T_{ps}/T_p \).
Figure 10.13: Case III Average Error, $\rho = 1$ Correlated Limit: Square, Gaussian, Triangle and Exponential Pulses.

### 10.5.1 $\rho = 1$ Correlated Limit

Figure 10.13 illustrates the Case III average error for several pulse shapes, $\gamma = p_u(\tau)$, where the autocorrelation function of the interference is $\rho = R(\tau) = 1$.

Table 10.1: Approximate Power-Law Constants for the Case III Average Error, $\rho = 1$ Correlated Limit

<table>
<thead>
<tr>
<th>Pulse Shape</th>
<th>Approximate Slope ($-\xi$)</th>
<th>Approximate y-Intercept ($\Lambda$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>-2</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>-2</td>
<td>$\frac{1}{8}$</td>
</tr>
</tbody>
</table>
Figure 10.14: Case III Average Error, $\rho = 0$ Independent Limit: Triangle and Exponential Pulses.

### 10.5.2 $\rho = 0$ Independent Limit

Table 10.2: Approximate Power-Law Constants for the Case III Average Error, $\rho = 0$ Independent Limit

<table>
<thead>
<tr>
<th>Pulse Shape</th>
<th>Approximate Slope ($\xi$)</th>
<th>Approximate $y$-Intercept ($\Lambda$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$-\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Triangle</td>
<td>-1</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>-1</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

Figure 10.14 illustrates the Case III average error for several pulse shapes, $\gamma = p_a(\tau)$,
Figure 10.15: Case III Average Error, $\rho = R_p(\tau)$: Square, Gaussian, Triangle and Exponential Pulses.

where the autocorrelation function of the interference is $\rho = R(\tau) = 0$.

### 10.5.3 Case III, General Unconditioned Error Probability

Figure 10.15 illustrates the Case III average error for several pulse shapes, $\gamma = p_u(\tau)$, where the autocorrelation function of the interference equals the autocorrelation of the pulse,

$$\rho = R_p(\tau) \equiv p_u(t) * p_u(t + \tau).$$
Table 10.3: Approximate Power-Law Constants for the Case III Average Error

<table>
<thead>
<tr>
<th>Pulse Shape</th>
<th>Slope</th>
<th>y-Intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$-\frac{3}{2}$</td>
<td>$\frac{9}{25}$</td>
</tr>
<tr>
<td>Triangle</td>
<td>-2</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>-2</td>
<td>$\frac{1}{8}$</td>
</tr>
</tbody>
</table>
Chapter 11

Approximate Total Error Probability

**Corollary 11.0.1.** Since Cases I and II are upper bounded (5.5) for uniform $\alpha$, and $P_{\text{int}}(M) \leq 1$, the union bound (3.6) is upper bounded by

$$P_e(SIR, M) \leq \frac{M - 1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right] + 2P_{\text{III}}(SIR)P_{\text{int}}(M). \quad (11.1)$$

For any distribution of $\alpha$, Cases I and II are still upper bounded by (5.1.2), and the union bound (3.6) is upper bounded by

$$P_e(SIR, M) \leq \frac{M - 1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \left( \frac{SIR}{2} \right) \right] + 2P_{\text{III}}(SIR)P_{\text{int}}(M). \quad (11.2)$$

Recalling the power-law dependence of the third case, $P_{\text{III}}(SIR \gg 1) \propto SIR^{-\xi}$ for some pulse-shape dependent constant $\xi$ (10.1), we note that the error probability is dominated by the second term, $P_e(SIR, M) \approx P_{\text{int}}(M)P_{\text{III}}(SIR) \propto \frac{P_{\text{int}}(M)}{SIR^{\xi}}$, whenever $P_{\text{int}}(M) \gg \exp \left( -\frac{SIR}{2} \right) \left( 1 + \frac{SIR}{2} \right)$ (provided $\xi$ is not large). This is virtually guaranteed in the high SIR regime, and thus the interference probability, $P_{\text{int}}(M)$, has a dominant effect in this regime. The interference probability would have a less dominant effect only if it were possible to find a pulse shape function yielding a large value of the power-law exponent, $\xi$. However, in
the examples of Chapter 10, \( \frac{1}{2} \leq \xi \leq 2 \), yielding the largest value for triangular and exponential pulse-shapes. Although it seems unlikely that any given pulse-shape could yield a substantially larger exponent, this has not been explicitly proven, since there is no apparent analytical solution for the power-law exponent.

*Proof of Corollary 11.0.1.* From Theorem 3.1.3, the total error probability is

\[
P_e (SIR, M) = P_I (SIR) \left[ (M - 5) + 4P_{int}(M) \right] + 2 (P_{II} (SIR) + P_{III} (SIR)) P_{int}(M).
\]

(11.3)

It was proven in Chapter 6 that Case I is an upper bound for Case II, achieving equality in the limit of independent variates. Furthermore, it was shown in Corollary 5.1.2 that Case I is upper bounded by the i.i.d. error probability with \( L_1 = L_2 = 2 \) degrees of freedom. Therefore,

\[
P_{II} (SIR | \alpha, \rho) \leq P_I (SIR | \alpha) \leq P_{iid} (2; 2; SIR)
\]

(11.4)

which must also bound the unconditioned error probabilities \( P_{II} (SIR) \) and \( P_I (SIR) \). Note that equality is obtained between Cases II and I with \( \rho = 1 \), and between Case I and the i.i.d. limit with \( \alpha = 1/2 \). Therefore, \( P_{II} (SIR) < P_I (SIR) < P_{iid} (2; 2; SIR) \), and the total error probability is bounded by

\[
P_e (SIR, M) \leq P_{iid} (2; 2; SIR) [(M - 5) + 4(1 - P_{int}(M))]
\]

\[
+ P_{iid} (2; 2; SIR) 2P_{int}(M) + P_{III} (SIR) 2P_{int}(M)
\]

(11.5)

\[
= P_{iid} (2; 2; SIR) [(M - 1) - 2P_{int}(M)] + P_{III} (SIR) 2P_{int}(M)
\]

\[
\leq P_{iid} (2; 2; SIR) (M - 1) + P_{III} (SIR) 2P_{int}(M)
\]

For uniform \( \alpha \), we can replace \( P_{iid} (2; 2; SIR) \) with the tighter upper bound from Corollary
5.1.4,

\[ P_H(SIR) \leq P_I(SIR) \leq \frac{1}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right]. \]  \hspace{1cm} (11.6)
Part III

The Performance of Optical Time-Spread PPM/CDMA with First-Order Polarization Mode Dispersion (PMD)
A novel optical modulation scheme employing Ultrashort Light Pulse (ULP)s modulated with Pulse Position Modulation (PPM) in conjunction with time-spread Code Division Multiple Access (CDMA) has recently been investigated in a two part paper by Kim, et.al. [15] and [16]. In the first paper, the performance of the PPM/CDMA scheme is analyzed considering only the effect of Multiple Access Interference (MAI), which determines the error floor at high Signal to Noise Ratio (SNR) [15]. In the second paper, various extensions to the basic modulation scheme were proposed and their performance improvement was analyzed [16]. Since MAI was the only source of degradation considered in these works, the impact of propagation through an optical fiber has not been addressed.

PPM is inherently sensitive to timing jitter and multipath interference, thus any phenomena that produce these effects in the channel are likely to be the predominant sources of performance degradation. In a SMF, the inherent random birefringence causes the group delay of the two polarization modes to differ, producing Polarization Mode Dispersion (PMD) and multipath interference. Whereas Chromatic Dispersion (CD) is essentially a deterministic function of the propagation length and the CD coefficient, PMD is a stochastic function of the propagation length and the PMD coefficient.

The first-order effect of PMD is the difference in arrival time of the two polarization modes, which is called the Differential Group Delay (DGD). The second order effect, which is related to the slope of the DGD, is the wavelength dependence that produces stochastic spreading, in a manner similar to CD.

Assuming that the chromatic dispersion has been compensated, and that the higher-order effects of PMD are negligible, it has been shown [14] that the SMF channel can be modeled as two paths with a Maxwellian distributed differential delay. The mean and variance are a function
of the propagation distance and a physical constant associated with the mean birefringence of the fiber. Modern manufacturing techniques produce fibers with very low birefringence, yielding a mean delay in the range of 1ps to 10ps for propagation distances between 100km and 10,000km. In the same range of propagation lengths, older fibers can exhibit a mean differential delay in the range of 10ps to 100ps.

Since the ULP PPM/CDMA technique employs sub-picosecond ultrashort pulses, the PMD is likely to have a significant effect on the performance. First, assuming a polarization insensitive receiver, the statistics of the intensity samples at the M possible pulse positions are derived. The performance is then analyzed as a function of key system parameters, such as SIR, propagation distance, number of users, pulse shape, the symbol duration and the pulse separation. The uncorrected performance is likely to motivate the need for polarization sensitive processing and/or electronic processing.
Chapter 12

System Description

System Parameters  The fundamental physical parameters affecting the system performance are:

\[ T_p \]  Ultrashort Pulse Equivalent Width (\( \approx 100 \text{ fs} \))

\[ P_o \]  Ultrashort Pulse Power
Ω Spectral Chip Equivalent Bandwidth ($\approx 10$ GHz)

$T_{\text{rep}}$ Symbol Duration ($\approx 10$ ns)

$T_{\text{ps}}$ PPM Pulse-slot Separation ($\approx 500$ fs)

$M$ Number of PPM timeslots ($\approx 100$)

$J$ Number of interfering users ($\approx 1000$)

$L$ Fiber Length ($\approx 1000$ km)

$D_p$ First-order PMD Coefficient ($\approx 100$ fs/$\sqrt{\text{km}}$)

$T_d$ Mean Differential Group Delay ($\approx 1 - 3$ ps)

12.1 System Overview

We begin by summarizing the modulation and demodulation process for the desired user, denoted $i = 0$, illustrated in Figure 12.1. Key definitions and properties of these signals, adopted from the original work in [15], are summarized in the remainder of this section.

1. Every $T_s$ seconds, a mode-locked laser produces an ultrashort pulse $p(t)$ with peak power $P_o$, and equivalent width $T_p$.

2. $p(t)$ is pulse-position modulated to the $m^{th}$ of M timeslots that are separated by $T_{\text{ps}}$ seconds, producing $p(t - mT_{\text{ps}})$.

3. $p(t - mT_{\text{ps}})$ is then time-spread using a unique signature sequence of rectangular chips, $C_o(\omega)$, that is multiplied with the signal in the spectral domain to produce $y_o(t - mT_{\text{ps}})$. 
4. J other users combine asynchronously to produce Multiple Access Interference (MAI).

For large $J$, $N_s$, and $N_{\text{eff}}$, it will be shown in Section 13.2 that the MAI asymptotically forms a stationary, zero-mean, complex Gaussian process, $n(t)$. This is the only source of noise considered (e.g. there is no thermal noise or shot noise).

5. The aggregate signal $y_o(t - mT_{\text{ps}} - \lambda_o) + n(t)$ is transmitted through $L$ km of fiber with first-order PMD coefficient $D_p \text{ ps}/\sqrt{\text{km}}$ and impulse response $\hat{h}(t)$.

6. The decoding spectral filter uses the same spectral code, $C_o$, and multiplies the received signal in the spectral domain to recover the shifted ultrashort pulse $p(t - mT_{\text{ps}} - \tau_p)$ from the time-spread pulse $y_o(t - mT_{\text{ps}} - \lambda_o)$;

7. $\tau_p$ is the random propagation delay, removed by proper synchronization.

8. The intensity $\left| \left[ p(t - mT_{\text{ps}}) + n(t) \right] * \hat{h}(t) \right|^2$ is sampled at $t = [0, T_{\text{ps}}, ..., (M - 1)T_{\text{ps}}]$.

9. The timeslot $\tilde{m}$ with the largest intensity is selected as the most likely pulse position. When $\tilde{m} \neq m$, a symbol error is made.

12.1.1 Ultrashort Pulse

Every $T_{\text{rep}}$ seconds, a mode-locked laser produces an ultrashort light pulse that has equivalent width $W \{ p(t) \} = T_p$ seconds, a time-bandwidth product of $N_{TB} \{ p(t) \}$, and peak power $p(0)^2 = P_o$. In terms of the notation of the generalized model defined in Chapter 2, the bandpass representation of the ULP is

$$p_{bp}(t) = \text{Re} \left\{ \sqrt{P_o} u \left( \frac{t}{T_p} \right) \exp(j\omega_0 t) \right\},$$

(12.1)
where \( \omega_o \approx 100 \) THz is the optical carrier frequency. The two-sided equivalent bandwidth, \( W \{ P(\omega) \} / \pi = B_p \ll f_o \). Thus, there is no loss in generality to perform all derivations at baseband, using the narrowband, low-pass equivalent representation for the ULP

\[
p(t) = \sqrt{P_o} p_u \left( \frac{t}{T_p} \right),
\]

which is equivalent to (2.1).

**Pulse Autocorrelation** The autocorrelation of \( p_u(x) \) is defined as

\[
R_p(x_o) \equiv \frac{1}{2} p_u(x_o) * p_u(-x_o),
\]

which is normalized with \( R_p(0) = 1 \) since \( R_p(0) = 2 \int_0^\infty p_u(x)^2 \, dx = 2 \).

**Time-Bandwidth Product** Since the energy in the Fourier domain is conserved by Parseval’s relation

\[
\int_{-\infty}^{\infty} \| f(x) \|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \| F(\chi) \|^2 \, d\chi.
\]

and the two-sided equivalent bandwidth is defined as

\[
W \{ F(\chi) \} / \pi \equiv \frac{\int_{-\infty}^{\infty} \| F(\chi) \|^2 \, d\chi}{\pi F(0)^2}.
\]

By definition, the D.C. coefficient is \( F(0) = 2 \int_0^\infty f(x) \, dx \), so the time-bandwidth product is

\[
N_{TB} \{ f(x) \} = \frac{\left( \int_0^\infty \| f(x) \|^2 \, dx \right) \left( \int_0^\infty \| F(\chi) \|^2 \, d\chi \)}{\pi f(0)^2 F(0)^2}
\]

\[
= \frac{\left[ \int_0^\infty \| f(x) \|^2 \, dx \right]^2}{\| f(0) \|^2 \left[ \int_0^\infty f(x) \, dx \right]^2}
\]

Since, by definition, \( p_u(0) = 1 \) and \( \int_0^\infty p_u(x)^2 \, dx \), then

\[
N_{TB} \{ p_u(x) \} = \left[ \frac{1}{\int_0^\infty f(x) \, dx} \right]^2
\]

The time-bandwidth product is conserved such that if the equivalent width of \( f(x) = a \), the time-scaled function \( f \left( \frac{x}{a} \right) \) must have unity equivalent width and bandwidth equal to the time-bandwidth product.
12.1.2 Normalized Timescale

Since the shortest temporal parameter of interest is the mode-locked pulse width $T_p$, it is convenient to scale the temporal variable $t$ by $T_p$ seconds to obtain the dimensionless temporal variable $x \equiv \frac{t}{T_p}$, and the corresponding dimensionless spectral variable $\chi \equiv \omega T_p$. With this normalization, the ultrashort pulse is represented as

$$p(x) = \sqrt{P_o}p_u(x) \longleftrightarrow P(\chi) = \sqrt{P_o}P_u(\chi)$$

(12.5)

where $p_u(x)$ is a real unit pulse function with both unity peak power and unity equivalent width. Thus the bandwidth is $N_{TB}\{p(x)\}$.

Normalizing the fundamental physical temporal parameters by $T_p$, we obtain the following dimensionless ratios:

$$N_{\text{eff}} \equiv \frac{1}{N_{TB}} \text{ Effective Processing Gain (\approx 1000)}$$

$$N_s \equiv \frac{T_{\text{rep}}}{T_p} \text{ Normalized Symbol Duration (\approx 100,000)}$$

$$N_{ps} \equiv \frac{T_{ps}}{T_p} \text{ Normalized Pulse Separation (\approx 5)}$$

$$\overline{N_d} \equiv \frac{T_d}{T_p} \text{ Normalized Mean Differential Delay (\approx 30)}$$

12.1.3 Time-Spread Pulse

The properties of the time-spread pulses are determined by the effective processing gain, $N_{\text{eff}}$, the ultrashort pulse waveform, $p_u(x)$ and the spectral chip waveform, $\text{rect}(\chi)$. These waveforms are defined to have unity peak power, unity energy, unity equivalent width, and a bandwidth equal to the time-bandwidth product.
**Spectral Chip**  The unit rectangular spectral chip is defined as

\[
\text{rect}(\chi) = \begin{cases} 
1 & |\chi| \leq 1/2 \\
0 & \text{otherwise}
\end{cases}
\]

where \(\text{sinc}(x) \leftrightarrow \text{rect}(\chi)\) are a Fourier transform pair with \(\text{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}\). Note that \(W_B\{\text{sinc}(x)\} = 1, W_T\{\text{sinc}(x)\} = 1, \) and \(N_{TB}\{\text{sinc}(x)\} = 1\).

**Spreading Sequence**  After \(p(x)\) is pulse-position modulated to one of \(M\) timeslots, it is then time-spread by multiplying by a unique pseudo-random sequence of \(\pm 1\) valued chips in the spectral domain. The encoding spectral filter of the \(i\)th user is defined as

\[
C_i(\chi) = \sum_{n=-\infty}^{\infty} c_n^{(i)} \text{rect}\left(\frac{N_{\text{eff}}\chi}{2\pi} - n\right)
\]

\(N_{\text{eff}}\) is the effective processing gain, defined as the dimensionless ratio \(\frac{1}{\Omega \tau}\).

The signature sequence of the \(i\)th user is denoted as \(c_n^{(i)}\), which takes values \(\pm 1\) equiprobably. The signature sequences are uncorrelated such that \(E\{c_n^{(i)} c_n^{(k)}\} = 0 \) if \(i \neq k\) or \(n \neq \ell\). We assume the signature sequence \(c_n^{(0)}\) for the desired user is deterministic, while the signature sequences of the \(J\) interfering users are considered random.

**Statistical Properties**

The unshifted, time-spread pulse of the \(i\)th user is given by

\[
Y_i(\chi) = P(\chi)C_i(\chi)
\]

\[
= \sqrt{P_o P_u(\chi)} \sum_{n=-\infty}^{\infty} c_n^{(i)} \text{rect}\left(\frac{N_{\text{eff}}\chi}{2\pi} - n\right)
\]

The inverse Fourier transform, \(y_i(x)\), is derived in [15]. It is proven in Appendix C of [15] that as \(N_{\text{eff}} \to \infty\), \(y_i(x)\) converges to a nonstationary complex Gaussian process with zero mean and
time-varying variance of the real and imaginary parts given by

$$\sigma_i^2(x) = \frac{P_o}{2N_{\text{eff}}} \sin^2 \left( \frac{x}{N_{\text{eff}}} \right)$$  \hspace{1cm} (12.9)

For $N_{\text{eff}} \to \infty$, it is shown in Appendix F that the autocorrelation of $y_i(x)$ is

$$R_y(x, x_o) = \frac{P_o}{N_{\text{eff}}} \sin^2 \left( \frac{x}{N_{\text{eff}}} \right) R_p(x_o)$$  \hspace{1cm} (12.10)

$$= 2\sigma_i^2(x) R_p(x_o).$$  \hspace{1cm} (12.11)

Note that since the variance is defined for the independent real and imaginary parts, $\text{Var} \{ y_i(x) \} = \text{Var} \{ \text{Re} \{ y_i(x) \} \} + \text{Var} \{ \text{Im} \{ y_i(x) \} \} = 2\sigma_i^2(x)$.

### 12.2 Channel Model

We address the effect of multipath arising from first-order PMD in a SMF of length $L$ km with chromatic dispersion compensation. The lowpass equivalent channel can then be modeled as the two path channel defined in 2.2.3 with a Maxwellian distributed differential delay [14]:

$$\hat{h}(t) = \sqrt{\alpha} \delta(t) \hat{a}_\parallel + \sqrt{1 - \alpha} \delta(t + \tau_d) \hat{a}_\perp$$  \hspace{1cm} (12.12)

where the vectors $\hat{a}_\parallel$ and $\hat{a}_\perp$ represent orthonormal states of polarization, the random variable $\alpha$ is uniformly distributed in $(\frac{1}{2}, 1)$, and the random differential delay $\tau_d$ is Maxwellian distributed with mean $\overline{\tau_d} = E \{ \tau_d \} = D_p \sqrt{L/\tau}$ and variance $\sigma_d^2 = \frac{\pi}{8} \overline{\tau_d^2} = \frac{D_p^2 L \pi}{8\tau}$. Recall that the equiprobable sign and the guarantee that $\alpha \geq (1 - \alpha)$ are result from the strongest path synchronization condition 2.2.5 The pdf and cdf are [23, 14]

$$f_{\tau_d}(\tau) = \sqrt{\frac{2}{\pi}} \left( \frac{\tau}{\overline{\tau_d}} \right)^2 \frac{1}{\sigma_d} \exp \left( -\frac{\tau^2}{2\sigma_d^2} \right)$$  \hspace{1cm} (12.13)

$$F_{\tau_d}(\tau) = 1 - 2Q \left( \frac{\tau}{\sigma_d} \right) - \frac{2\tau}{\sqrt{2\pi}\sigma_d} \exp \left( -\frac{\tau^2}{2\sigma_d^2} \right)$$  \hspace{1cm} (12.14)
where the $Q(x)$ function is the standard right-tail probability of a zero-mean, unit variance, Gaussian random variable, defined by

$$Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp(-y^2/2) dy$$  \hspace{1cm} (12.15)$$

The Maxwellian distribution is equivalent to a $\chi^2$-distribution with 3 degrees of freedom, which arises due to the fact that the polarization state is asymptotically a random walk in 3 dimensions.
Chapter 13

Multiple Access Interference Statistics

13.1 Nonstationary Multiple Access Interference

Consider a point-to-multipoint link where there are J asynchronous users coupled into the fiber at the same point, such as the network illustrated in Figure 2.3. Let $\lambda_i$ denote the normalized delay of the $i^{th}$ user, incorporating both the asynchrony and the pulse position modulation relative to the desired user. As in [15] (using the appropriate changes in notation), the MAI resulting from combining the signals of the J interfering users can be expressed as

$$y(x; \lambda) \equiv \sum_{i=1}^{J} y_i(x - \lambda_i)$$  \hspace{1cm} (13.1)

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_J)$ is the set of normalized delays $\lambda_i$, which are assumed to be uniformly distributed over the interval $(0, N_s)$.

For a given $x$ and a set of normalized delays $\lambda$, the MAI $y(x; \lambda)$ is a nonstationary Gaussian
random variable with zero mean and variance of the real and imaginary parts given by [15]

\[ \sigma^2(x; \lambda_i) = \sum_{i=1}^{J} \sigma^2_i(x - \lambda_i) \] (13.2)

\[ = \sum_{i=1}^{J} \frac{P_o}{2N_{\text{eff}}} \text{sinc}^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right) \] (13.3)

The signals of any two different users, \( y_i(x - \lambda_i) \) and \( y_k(x - \lambda_k) \) are mutually uncorrelated (for all \( i \neq k \) and any \( \lambda \)). Thus the autocorrelation of the MAI is the sum of the autocorrelation function for each user, time-shifted by \( \lambda_i \)

\[ R_y(x, x_o, \lambda) = \sum_{i=1}^{J} R_y(x - \lambda_i, x_o) \] (13.4)

\[ = R_p(x_o) \sum_{i=1}^{J} 2\sigma^2_i(x - \lambda_i) \] (13.5)

\[ = R_p(x_o)2\sigma^2(x; \lambda). \] (13.6)

### 13.2 Asymptotic Stationarity

We now consider the effect of letting \( N_{\text{eff}} \to \infty \). Since the equivalent width of each user’s spread pulse, \( y_i(x) \), equals the processing gain, the number of spread pulses that overlap at any point in time increases with \( N_{\text{eff}} \). The power of each interferer is simultaneously reduced by the same factor, so the total interference power over the symbol interval is left unchanged, but is distributed more uniformly (less impulsively) throughout the symbol interval. As a practical limit, the processing gain cannot exceed the symbol length, as this would cause the spread pulse to overlap adjacent symbols, and therefore we require that \( N_{\text{eff}} \leq N_s \).
It is proven in Appendix G that, for $J \to \infty$ and $N_s \to \infty$,

$$E \{ \sigma^2(x; \lambda) \} = E \left\{ \frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \text{sinc}^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right) \right\} = \frac{P_o J}{2N_s} \quad (13.7)$$

$$Var \{ \sigma^2(x; \lambda) \} = Var \left\{ \frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \text{sinc}^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right) \right\} = \frac{2}{3} \frac{P_o P_o J}{N_{\text{eff}} 2N_s} \quad (13.8)$$

throughout the interval $(0, N_s)$. With $J$ and $N_s$ increasing together so that $J/N_s$ is constant, the mean $E \{ \sigma^2(x; \lambda) \}$ remains constant while the variance $Var \{ \sigma^2(x; \lambda) \}$ approaches zero asymptotically with $N_{\text{eff}} \to N_s \to \infty$. Thus $\sigma^2(x; \lambda)$ converges to the mean value throughout the interval $(0, N_s)$.

The proof in Appendix G uses the fact that $\frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \text{sinc}^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right)$ asymptotically forms a shot process with delays $\lambda_i$ uniformly distributed over $(0, N_s)$, since the time-ordered set of delays becomes a Poisson point process asymptotically as the number of points, $J$, and the length of the interval, $N_s$, grow large [17]. Thus, in the limit of a Poisson point process, the sum of $J$ time-shifted sinc-squared functions forms a shot process, with mean and variance determined by Campbell’s theorem [17].

### 13.2.1 Variance and Autocorrelation

Using Equations (13.7,13.8), the variance of Equation (13.3) becomes asymptotically

$$\sigma^2(x; \lambda) = \sum_{i=1}^{J} \frac{P_o}{2N_{\text{eff}}} \text{sinc}^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right) \quad (13.9)$$

$$\longrightarrow \sigma^2 = \frac{P_o J}{2N_s} \quad (13.10)$$

the autocorrelation (eq. 13.5) becomes

$$R_y(x, x_o, \lambda) = R_p(x_o) 2\sigma^2(x; \lambda) \quad (13.11)$$

$$\longrightarrow R_y(x_o) = R_p(x_o) 2\sigma^2 \quad (13.12)$$
and the correlation coefficient is

$$\frac{R_y(x_o)}{2\sigma^2} = R_p(x_o). \quad (13.13)$$

In conclusion, it has been shown that $y(x; \lambda)$, the aggregate transmitted lowpass equivalent signal from all other users, is an asymptotically stationary Gaussian noise process as the number of users $J$ and the symbol length $N_s$ grow large with $N_{\text{eff}} \to N_s$.

To simplify the notation in the analysis section, we define this stationary lowpass equivalent noise process as

$$n(x) \equiv y(x; \lambda) \quad (13.14)$$

such that the lowpass equivalent transmitted signal can be represented as

$$t(x) = \sum_{i=0}^{J} y_i(x - \lambda_i)$$

$$= y_o(x - mN_{ps}) + n(x) \quad (13.15)$$

$$T(\chi) = \sum_{i=0}^{J} Y_i(\chi) \exp(-j\chi \lambda_i)$$

$$= Y_o(\chi) \exp(-j\chi mN_{ps}) + N(\chi). \quad (13.16)$$

Throughout the rest of the work, we will assume that the conditions for asymptotic stationarity are satisfied.

### 13.3 Matched Filter Output

We now derive the statistics of the signal after transmission through the SMF and processing with the matched spectral filter. After propagating $L$ meters through a SMF with PMD coefficient $D_p$, the low-pass equivalent received signal is

$$\hat{r}(x) = t(x - \lambda_o) \ast \hat{h}(x) \quad (13.17)$$
where \( \hat{h}(x) \) is the low-pass equivalent transfer function from Equation (12.12). The random delay \( \lambda_o \) is removed by proper synchronization so that \( \lambda_o = 0 \). In the spectral domain, this yields

\[
\hat{R}(\chi) = T(\chi)\hat{H}(\chi)
\]

\[
= \hat{H}(\chi) \left[ Y_o(\chi) \exp(-j\chi m N_{ps}) + N(\chi) \right].
\]

The received signal is then processed with a matched filter in the spectral domain to de-spread the desired user’s signal and recover the shifted ultrashort pulse:

\[
\hat{Z}(\chi) = \hat{R}(\chi)C_o^*(\chi)
\]

\[
= \hat{H}(\chi)C_o^*(\chi) \left[ Y_o(\chi) \exp(-j\chi m N_{ps}) + N(\chi) \right].
\]

We decompose the matched filter output into the desired signal and interference: \( \hat{Z}(\chi) = \hat{Z}_o(\chi) + \hat{Z}_i(\chi) \). Focusing first on the signal component, we obtain

\[
\hat{Z}_o(\chi) = \hat{H}(\chi)Y_o(\chi)C_o^*(\chi) \exp(-j\chi m N_{ps})
\]

\[
= \hat{H}(\chi)P(\chi)C_o(\chi)C_o^*(\chi) \exp(-j\chi m N_{ps})
\]

\[
= \hat{H}(\chi)P(\chi) \exp(-j\chi m N_{ps})
\]

where \( C_o(\chi)C_o^*(\chi) = 1 \). We have thus recovered the shifted ultrashort pulse, corrupted by multipath:

\[
\hat{z}_o(x) = \hat{h}(x) \ast p(x - m N_{ps}).
\]

Similarly, for the interference component, we obtain the matched filter output

\[
\hat{Z}_i(\chi) = \hat{H}(\chi)N(\chi)C_o^*(\chi)
\]

\[
= \hat{H}(\chi)\tilde{N}(\chi)
\]
where $\tilde{N}(\chi)$ is statistically identical to the original Gaussian process $N(\chi)$ since $C_0(\chi)$ is deterministic while all other sequences $C_i(\chi)$ are random and mutually uncorrelated. This result is explicitly proven in Appendix H. Hereafter, we will drop the tilde over the interference term since it is statistically identical to the original definition in the previous section. Thus the interference component is stationary Gaussian noise influenced by the channel multipath:

$$\hat{z}_i(x) = \hat{h}(x) \ast n(x)$$ (13.29)

Combining the signal and interference components, we obtain

$$\hat{z}(x) = \hat{h}(x) \ast [p(x - mN_{ps}) + n(x)]$$ (13.30)

$$= [\sqrt{\alpha} \delta(x) \hat{a}_{||} + \sqrt{1 - \alpha} \delta(x - N_d) \hat{a}_{\perp}] \ast [p(x - mN_{ps}) + n(x)]$$ (13.31)

$$= \sqrt{\alpha}[p(x - mN_{ps}) + n(x)]\hat{a}_{||} + \sqrt{1 - \alpha}[p(x - mN_{ps} - N_d) + n(x - N_d)]\hat{a}_{\perp}.$$ (13.32)

Therefore, the detected instantaneous intensity, $I(x)$, is

$$I(x) = \hat{z}(x) \cdot \hat{z}^*(x)$$

$$= \alpha \left| \sqrt{P_o}p_u(x - mN_{ps}) + n(x) \right|^2$$

$$+ (1 - \alpha) \left| \sqrt{P_o}p_u(x - mN_{ps} - N_d) + n(x - N_d) \right|^2,$$ (13.33)

which is the sum of the intensities of the two polarization modes.
Chapter 14

Performance Evaluation

Since the PPM/CDMA system has been shown to satisfy the conditions of the analysis developed in Chapter 2, the total error probability is given by Theorem 3.1.3. This theorem gives the union bound on the total error probability as a function of the average pairwise error probabilities of the three cases, and the interference probability, which is a function of the constellation size, $M$, and is governed by the distribution of the differential delay. In this chapter, we combine these results to derive the union bound of the total error probability in the high SIR regime, as a function of the system parameters defined in Section 12.

**Asymptotically Stationary MAI** Provided that the processing gain and number of users is large, it was proven in Appendix G that the complex Gaussian interference process is asymptotically stationary, with the autocorrelation function equal to the pulse-shape autocorrelation,

$$ R(t_o) = R_p(t_o) \equiv p_a(t) \ast p_a(t + t_o). $$

(14.1)

The variance of the stationary interference process was then shown to be $\frac{P_o J}{2N_c}$, as shown in (13.10). Therefore, the signal-to-interference ratio is directly related to the normalized symbol
length, \( N_s \), and inversely related to the number of users, \( J \), and is given by

\[
\text{SIR} = \frac{P_o}{2\sigma^2} = \frac{N_s}{J},
\]

(14.2)

which is independent of the transmitted power, \( P_o \), provided that the received signal power for all \( J \) users is equal. (i.e., there is no near-far effect). Recall that \( N_s \) is essentially the maximum effective processing gain, since a processing gain that exceeds the symbol length would cause the pulses to spread into adjacent symbol intervals.

**Upper Bound for Cases I and II** Since Cases I and II are upper bounded (5.5) for uniform \( \alpha \), and \( \overline{P_{\text{int}}(M)} \leq 1 \), the union bound (3.6) is upper bounded by

\[
P_e(\text{SIR}) \leq \frac{M - 1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{\text{SIR}}{2} \right) \right] + 2\overline{P_{\text{III}}(\text{SIR})} \overline{P_{\text{int}}(M)}.
\]

(14.3)

For any distribution of \( \alpha \), Cases I and II are still upper bounded by Corollary (5.1.2), and the union bound (3.6) is upper bounded by

\[
P_e(\text{SIR}) \leq \frac{M - 1}{2} \exp \left( -\frac{\text{SIR}}{2} \right) \left[ 1 + \frac{1}{4} \left( \frac{\text{SIR}}{2} \right) \right] + 2\overline{P_{\text{III}}(\text{SIR})} \overline{P_{\text{int}}(M)}.
\]

(14.4)

These bounds are extremely useful at high SIR, as they indicate that the total error probability is proportional to that of the unfaded system, \( \exp \left( -\frac{\text{SIR}}{2} \right) \), plus degradation equal to the product of the Case III average error rate and the total interference probability.

14.0.1 Case III Average Error Probability

It was shown in Chapter 10 that the recursive bisection algorithm can be used to efficiently compute the average error probability for Case III, given the pulse shape function, \( p_u(t) \) and the interference autocorrelation function, \( R(t) \). It was shown for a few different examples that, at
high SIR, the error probability has a power-law dependence on the SIR (10.1),

$$P_{III}(SIR \gg 0) \approx \frac{\Lambda T_p}{T_{ps}} SIR^{-\xi},$$  \hspace{1cm} (14.5)

where the constants $\xi$ and $\Lambda$ are determined by estimating the slope and y-intercept of the error curve, respectively. To estimate the parameter $\Lambda$ using the y-intercept, the pulse separation should be chosen as $T_{ps} = T_p$, such that the ratio $T_p/T_{ps} = 1$ and $P_{III}(SIR \gg 0) \approx \Lambda SIR^{-\xi}$.

Figure 14.1 illustrates the Case III average error rate with $\gamma = p_u(\tau)$ and $\rho = R_p(\tau)$ for a Gaussian pulse, $p_u(t) = \exp(-t^2\pi/2)$, and a triangular pulse, $p_u(t) = 1 - |\frac{2}{3}t|$ (where $|t| \leq 3/2$). The timescale has been normalized by the pulse width $T_p$, such that $p_u(t)$ has unity equivalent width, and the pulse separation is a multiple of $T_p$, $N_{ps} \equiv T_{ps}/T_p$. The family of curves is generated by successively doubling $N_{ps}$. Note that at high SIR, the error decreases by a factor of two for either pulse shape when the pulse separation is doubled, shifting the curve down by 3dB. The Gaussian pulse has an error rate of approximately $\approx \frac{9}{25} \frac{T_p}{T_{ps}} SIR^{-\frac{7}{2}}$, and the triangle pulse has an error rate of approximately $\frac{1}{5} \frac{T_p}{T_{ps}} SIR^{-2}$. In Chapter 10, we investigated both square and double-sided exponential pulse shapes. In addition to investigating the Case III average error probability, in which $R(t) = R_p(t)$, we also calculated the average error probabilities in the limits of completely independent and completely correlated variates. The approximate power-law constants for the high SIR regime are summarized in Tables 10.1 - 10.3.

### 14.0.2 Probability of Interference

The interference probability derived in Theorem 3.1.3 is determined by the constellation size, $M$, the pulse separation, $T_{ps}$, and the distribution of the differential delay, $\tau_d$. For a Maxwellian distributed differential delay with mean $\overline{\tau_d}$, we will show that it is possible to approximately characterize the interference probability by a power-law dependence on the ratio
Figure 14.1: Case III Average Error Probability for Gaussian and Triangular Pulses.

\( MT_{ps}/\tau_d \). The mean delay, \( \tau_d \), is assumed to be given, and is related directly to the square root of the propagation distance, \( K \), and a first-order PMD coefficient, \( D_p \). Under these conditions, the interference probability is minimized by making the product \( MT_{ps} \) as small as possible. However, we also wish to maximize \( M \) in order to increase the throughput. Given a maximum permissible error probability, this will be shown to severely constrain the throughput.

Figure 14.2 illustrates the interference probability for Maxwellian differential delay, as a function of \( \log_2 \left( \frac{MT_{ps}}{\tau_d} \right) \). For small \( P_{\text{int}}(M) \), this curve changes linearly on a log-log scale, so it varies inversely with the argument, \( \frac{MT_{ps}}{\tau_d} \). In this regime, \( P_{\text{int}}(M) \) increases by a factor of 8 for each doubling of the argument. Therefore, in the low interference probability regime, where the mean delay \( \tau_d \gg MT_{ps} \), the interference probability associated with the Maxwellian distribution
Figure 14.2: Probability of Interference with Maxwellian DGD

is also closely approximated by a power-law dependence,

\[ P_{\text{int}}(M) \approx \begin{cases} 
\frac{4}{5} \left( \frac{MT_{ps}}{\bar{\tau_d}} \right)^3 & \text{if } MT_{ps} > \bar{\tau_d}, \\
1 & \text{if } MT_{ps} \leq \bar{\tau_d}. 
\end{cases} \]  

(14.6)

14.0.3 Total Error Probability

At high SIR, the Case III average error rate dominates Cases I and III, since the average error for these cases has an upper bound that decreases exponentially with the SIR (5.1.3), while the Case III average decreases only with the inverse of the SIR, raised to a small exponent (14.5). Therefore, at high SIR, the error probability is dominated by Case III, provided that the interference probability is non-zero. Since the sufficient delay condition must be satisfied,
i.e., the differential delay must exceed the pulse separation, then the probability of interference cannot be zero. The union bound of the total error probability in Theorem 3.1.3 is further upper bounded by the result in Corollary 11.0.1,

\[
P_e(SIR, M) \leq \frac{(M - 1)}{2} \exp \left( -\frac{SIR}{2} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{SIR}{2} \right) \right] + 2P_{\text{int}}(M)P_{\text{III}}(SIR) = P_{e,0}(M, SIR) + P_{e,1}(M, SIR).
\]

(14.7)

To express the first term, \(P_{e,0}(M, SIR)\), in terms of system parameters, we substitute the definition of the SIR (14.2),

\[
P_{e,0}(M, J, N_s) = \frac{(M - 1)}{2} \exp \left( -\frac{N_s}{2J} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{N_s}{2J} \right) \right].
\]

(14.8)

To express the second term, \(P_{e,1}(M, SIR)\), in terms of system parameters, we first apply the power-law approximations for the Case III average error (10.1) and the interference probability (14.6), and then substitute the definition of the SIR (14.2):

\[
2P_{\text{int}}(M)P_{\text{III}}(SIR) \approx 2 \left[ \Lambda \frac{T_p}{T_{ps}} SIR^{-\xi} \right] \left[ \frac{4}{5} \left( \frac{MT_{ps}}{\tau_d} \right)^3 \right] \]
\[
= \frac{8\Lambda T_p}{5T_{ps}} \left( \frac{J}{N_s} \right)^{\xi} \left( \frac{MT_{ps}}{\tau_d} \right)^3
\]
\[
= \frac{2\Lambda T_p^2}{5T_p^2} \left( \frac{J}{N_s} \right)^{\xi} \left( \frac{M}{N_d} \right)^3,
\]

(14.9)

where the definition of the normalized mean delay, \(N_d = \tau_d/T_p\), has been used. This is valid in the regime of \(J \ll N_s\) and \(MT_{ps} \ll \tau_d\). Similarly, for \(MT_{ps} \gg \tau_d\), in the regime where \(P_{\text{int}} \to 1\),

\[
2P_{\text{int}}(M)P_{\text{III}}(SIR) \approx 2 \left[ \Lambda \frac{T_p}{T_{ps}} \left( \frac{J}{N_s} \right)^{-\xi} \right].
\]

(14.10)

Note that the total error probability is proportional to \(T_{ps}^2\), and thus the pulse separation should be chosen as small as possible to minimize the total error. In order to satisfy both the
sufficient spacing and the non-overlapping conditions, $T_{ps} \geq \max(T_R, T_p)$. Since the autocorrelation of $p_u(t)$ must be broader than the width of $p_u(t)$, then $T_R > T_p$, and thus the smallest possible value of the pulse separation is to choose $T_{ps} = T_R$. Defining the normalized correlation width $N_R = \frac{T_R}{T_p} = \frac{W(R_p(t))}{W(p_u(t))}$, we finally obtain

$$P_{e,1}(M, J, \xi, \Lambda, N_s, \overline{N_d}, N_R) \approx \begin{cases} 8\Lambda N_{R}^2 \left(\frac{J}{N_s}\right)^\xi & \text{if } M \ll \frac{\overline{N_d}}{N_R} \\ \frac{2\Lambda}{N_R} \left(\frac{J}{N_s}\right)^{-\xi} & \text{if } M \gg \frac{\overline{N_d}}{N_R} \end{cases} \quad (14.11)$$

Expressed in terms of the number of bits per symbol, $M = 2^B$, so the second factor for $M \ll \frac{\overline{N_d}}{N_R}$ is $\left(\frac{2^{3B}}{N_d}\right)$. Therefore, an increase of one bit per symbol can incur a substantial performance penalty. However, (14.11) also indicates that this effect completely vanishes as the normalized mean delay, $\overline{N_d} \to \infty$. This seems counter-intuitive, because it violates the negligible ISI condition.

**Violation of Negligible ISI Condition** This counter-intuitive behavior arises because at some point, the mean delay is on the order of the symbol length, i.e., $\overline{N_d} \approx N_s$, and the ISI can no longer be neglected. Since the multipath image will arrive in the adjacent symbol, the ISI can be accommodated by including the detection interval of the next symbol in the calculation for the interference probability. Given that $\overline{N_d}$ is proportional to the square root of the propagation length, this is unlikely to be a problem in practical implementations of this system, as $N_s \approx 10^6$, while the delay is in the range $\overline{N_d} \approx (10, 10^3)$. It has been noted previously that the negligible ISI condition also constrains the effective processing gain $N_{eff} \leq N_s$ so that the pulses do not spread into adjacent symbols.
Approximate Total Error  Combining the two terms, \( P_{e,1} \) and \( P_{e,2} \), the total error probability in the high SIR regime (i.e., \( J \ll N_s \)) and low \( P_{\text{int}} \) regime (i.e., \( M \ll \overline{N}_d/N_R \)) is approximately

\[
P_e(M, J, \xi, \Lambda, N_s, \overline{N}_d, N_R) \\approx P_{e,0}(M, J, N_s) + P_{e,1}(M, J, \xi, \Lambda, N_s, \overline{N}_d, N_R) \\approx \frac{(M - 1)}{2} \exp \left( -\frac{N_s}{2J} \right) \left[ 1 + \frac{1}{4} \log \left( \frac{N_s}{2J} \right) \right] + \frac{8\Lambda N_R^2}{5} \left( \frac{J}{N_s} \right)^{\xi} \left( \frac{M}{\overline{N}_d} \right)^3
\]

(14.12)
and as $P_{\text{int}} \to 1$ with $M > \overline{N_d}/N_R$,

\[
P_e \left( M, J, \xi, \Lambda, N_s, \overline{N_d}, N_R \right) \\
\approx \frac{2\Lambda}{N_R} \left( \frac{J}{N_s} \right) ^\xi.
\]  
(14.13)

Therefore, for arbitrary $M$, the error probability is approximately the minimum of the two expressions:

\[
P_e \left( M, J, \xi, \Lambda, N_s, \overline{N_d}, N_R \right) \\
\approx \frac{2\Lambda}{N_R} \left( \frac{J}{N_s} \right) ^\xi \min \left\{ 1, \frac{4}{5} \left( \frac{MN_R}{N_d} \right)^3 \right\}.
\]  
(14.14)

Two examples for square and exponential pulses are plotted in Figures 14.3 and 14.4, com-
paring the original union bound of Theorem 3.1.3 (the solid curves) to the above approximation (the dashed lines) (14.12). The normalized mean delay is fixed, and the performance is plotted as a function of the SIR for several different values of $B$, where $M = 2^B$. From Figure 14.2, it is evident that the power-law approximation for the interference probability is accurate until $MN_{ps} \geq N_d$, and then the interference probability is asymptotically equal to one. When the interference probability is one, the total error probability is approximately equal to twice the Case III average error probability (14.13).

Since $N_{ps} = 1$ for this example, $2^B \approx N_d$ is the value of $M$ where the interference probability saturates. Once these asymptotic limits are reached, $M$ can then be increased without sacrificing the performance. The slope of the asymptote for $M \to \infty$ is equal to $\xi$, but the y-intercept is a function of multiple parameters. For example, with $N_d = 1000$, the interference probability saturates for $B > 10$, since $2^{10} \approx 1000$. Similarly, with $N_d = 100$, the upper asymptote is reached for $B > 7$ since $2^7 \approx 100$. Finally, we note the same behavior for $B > 5$ with $N_d = 20$, since $2^4 < 20 < 2^5$.

14.1 Capacity and Throughput Constraints

In the high SIR we can apply the approximate error rates given above in (14.12), (14.13), and (14.14) to obtain the following results for the throughput and the capacity, given a maximum error rate, $P_{\text{max}}$. The following constraints are derived by rearranging the terms in (14.12) to solve for each constraint, given the other values.

If $J$ is sufficiently small that

$$\frac{2\Lambda}{N_R} \left( \frac{J}{N_s} \right)^{\xi} < P_{\text{max}},$$

(14.15)
then there is no constraint on $M$, except that $MT_{ps} < T_{\text{rep}}$ If this is not the case, however, then the value of $M$ is constrained, by the mean of the differential delay,

$$M \leq \mathcal{N}_d \left( \frac{N_s}{J} \right) ^{\frac{3}{4}} \sqrt[3]{\frac{5 P_{\text{max}}}{8 \Lambda N^2_R}}. \quad (14.16)$$

However, the maximum number of users, given the other parameters and a maximum error rate, is always constrained,

$$J \leq N_s \sqrt[4]{\frac{P_{\text{max}} N^2_R}{2 \Lambda}} \min \left\{ 1, \sqrt[4]{\frac{5}{4} \left( \frac{\mathcal{N}_d}{MN_R} \right)^{\frac{3}{2}}} \right\} \quad (14.17)$$

Lastly, the minimum value of the normalized symbol length that satisfies the error rate requirement, given values of $M$ and $J$, is

$$N_s \geq J \left( \frac{M}{N_d} \right) ^{\frac{3}{4}} \sqrt[4]{\frac{8 \Lambda N^2 R}{5 P_{\text{max}}}}. \quad (14.18)$$

The throughput for any particular user in the low $P_{\text{int}}$ regime is constrained,

$$T \equiv \frac{\log_2 M}{T_{\text{rep}}} \text{ bits/sec}$$

$$\leq \frac{1}{T_{\text{rep}}} \log_2 \left( \mathcal{N}_d \left( \frac{N_s}{J} \right) ^{\frac{3}{4}} \sqrt[3]{\frac{5 P_{\text{max}}}{8 \Lambda N^2_R}} \right) \text{ bits/sec}, \quad (14.19)$$

and therefore, the aggregate throughput is $J$ times greater. The aggregate capacity is

$$C \equiv \frac{JT}{B_p} = \frac{J \log_2(M)}{T_{\text{rep}} \left( \frac{N_{\text{rep}}}{T_{\text{rep}}} \right)} = \frac{J \log_2(M)}{N_s N_{TB}} \text{ bits/sec/Hz}, \quad (14.20)$$

where we have used the definitions of the time-bandwidth product, $N_{TB} = T_p B_p$, and the normalized symbol duration, $N_s = T_{\text{rep}}/T_p$. Substituting the minimum required value of $N_s$, the aggregate capacity is

$$C \leq \mathcal{N}_d \frac{\frac{3}{4} \log_2 M}{M^{\frac{3}{4}}} \sqrt[4]{\frac{5 P_{\text{max}}}{8 \Lambda N^2 R}} \text{ bits/sec/Hz}. \quad (14.21)$$
Chapter 15

Conclusions

We have seen that the general model developed in Chapter 2 is applicable to the analysis of performance degradation in the optical PPM/CDMA system arising from PMD-induced multipath. The assumptions of the model were shown to apply in the regime where the differential delay is greater than the PPM time-slot spacing $T_{ps}$, but less than the symbol repetition interval $T_{rep}$. Therefore, it is a violation of these assumptions to have the average differential delay tend to either zero or infinity. The analysis can be extended for longer delays by incorporating the detection interval of a larger number of adjacent symbols into the calculation for the interference probability. However, an analysis in the short delay regime, where the mean differential delay is much smaller than the pulse separation, would require a completely different set of assumptions.

The Maxwellian distribution, which provides a model for the stochastic Differential Group Delay (DGD) arising from PMD, produces a probability of interference that increases exponentially with the ratio of the symbol length, $MT_{ps}$, divided by the mean delay, $\tau_d$, until $MT_{ps} \approx \tau_d$ and $P_{int} \approx 1$. When the interference probability saturates to unity, the total error probability is equal to the Case III average error probability, which has a power-law dependence...
on the SIR. In the high SIR regime, simple relationships were obtained for the resulting degradation in throughput and capacity, constraining both the constellation size $M$ and the number of users $J$, given a maximum error rate, and three constants determined by the pulse-shape.

The single most important parameter in these equations is the power-law exponent $\xi$, which is determined from the approximate slope of the Case III average error probability at high SIR. This average error probability was computed using the Beaulieu series to numerically obtain the conditional error probability from the closed-form expression of the conditional characteristic function, and then averaged using the recursive bisection technique. It was necessary to use these techniques to obtain results for partial correlated interference, which appear to be otherwise intractible.

Comparing the limiting cases of complete correlation and total independence, which were derived analytically, it is clear that the correlation has a profound effect on improving the system performance. For instance, the triangle and exponential pulse shapes have $\xi = 2$ in the correlated limit, but only $\xi = 1$ in the independent limit. The slope of the Case III average error coincides with the correlated limit for these two pulse shapes, as the correlation function of the pulse shapes are very broad, and provide near perfect correlation when the non-centrality parameter, $\gamma$, is also large.

The Case III average is essentially dominated by the worst case value of $\alpha = 1/2$, which was solved in closed-form for both independent and correlated limits, conditioned on the non-centrality parameter $\gamma$. As the non-centrality parameter approaches one half, the error probability for the correlated limit diminishes quite rapidly (concave up), and rather slowly in the independent limit (concave down), as shown in Figure 8.1.

A similar improvement was seen in the correlated limit of Case II, compared to the indep-
dent limit of Case I. In this correlated limit, the performance is, in fact, better than the unfaded case. This limit was derived by noting that the remaining three independent interference samples are i.i.d., yielding a quadratic form with an equal number of degrees of freedom in the two sums. The error probability for unequal degrees of freedom involving non-central and central chi-square variates appears to not have been previously analyzed in the literature, and a new closed-form expression for this more general quadratic form was derived.

The key assumption of the model which generates such strong correlation of the interference is the co-propagating channel condition. If this restriction is removed, such that the differential delay of each user is independent, but not identical as we have assumed, then the autocorrelation function of the received interference is dispersed. Therefore, in this scenario of independent channels, the system performance is expected to tend to the independent limit, which was shown to have somewhat smaller values of $\xi$, and thus higher error rates. Therefore, the independent limits analyzed in this dissertation should provide a reasonable upper bound if co-propagating interference is not assumed.

In conclusion, it has been shown that a channel with even one interfering orthogonal path can substantially degrade the throughput and capacity of PPM systems. Therefore, equalization and/or diversity schemes should be investigated if PPM is to be considered for transmission over links with a delay spread much greater than the pulse separation.
Appendix A

Distribution of the Sum of Independent Chi-Square Variates

Theorem A.0.1. Let $X_1 = \chi^2(L_1, a_1^2, \sigma_1^2)$ and $X_2 = \chi^2(L_2, a_2^2, \sigma_2^2)$ denote independent noncentral chi-square variates with $2L_1$ and $2L_2$ degrees of freedom, respectively. If and only if $\sigma_1^2 = \sigma_2^2 = \sigma^2$, then the sum $X \equiv X_1 + X_2$ simplifies to a noncentral chi-square variate with $2L \equiv 2(L_1 + L_2)$ degrees of freedom and noncentrality parameter $a^2 \equiv a_1^2 + a_2^2$, i.e., the sum of the noncentrality parameters. That is,

$$X_1 + X_2 = \chi^2(L_1, a_1^2, \sigma_1^2) + \chi^2(L_2, a_2^2, \sigma_2^2)$$

$$\rightarrow X = \chi^2(L_1 + L_2, a_1^2 + a_2^2, \sigma^2)$$

$$\equiv \chi^2(L, a^2, \sigma^2), \quad (A.1)$$

if and only if $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

Corollary A.0.2. Theorem A.0.1 also applies for any $L_1 = L_2 \equiv N$ such that $X_1$ and $X_2$ both have $N$ degrees of freedom and $X$ has $2(L_1 + L_2) = 2N$ degrees of freedom.
Proof. From [23, Eq. 5.62], the pdf of the sum of two independent non-central chi-square random variables \(X_1 = \chi^2(L_1, a_1^2, \sigma_1^2)\) and \(X_2 = \chi^2(L_2, a_2^2, \sigma_2^2)\), i.e., with \(2L_1\) and \(2L_2\) degrees of freedom, component noise variances \(\sigma_1^2 = \sigma_2^2 = \sigma^2\) and non-centrality parameters \(a_1^2\) and \(a_2^2\), is

\[
f_{X_1+X_2}(z) =
\frac{1}{2\sigma_1^2} \left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{L_2} \left(\frac{z}{a_1^2}\right)^{\frac{L_1+L_2-1}{2}} \exp\left(-\frac{z}{2\sigma_1^2}\right) \exp\left[-\frac{1}{2} \left(\frac{a_1^2}{\sigma_1^2} + \frac{a_2^2}{\sigma_2^2}\right)\right]
\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(L_2 + m + n)}{m! n! \Gamma(L_2 + n)} \left(\frac{\sqrt{2}a_2^2 \sigma_1^2}{a_1^2}\right)^n \left(\frac{\sqrt{z}(\sigma_2^2 - \sigma_1^2)}{2a_1 \sigma_2^2}\right)^m I_{L_1+L_2-1+m+n} \left(\frac{\sqrt{z}a_1}{\sigma_1}\right),
\]

if \(z \geq 0\), where \(\Gamma(z)\) is the complete Gamma function. Note that although there are no restrictions on \(\sigma_1^2\) and \(\sigma_2^2\), the infinite summation in \(m\) forms an alternating series when \(\sigma_1^2 > \sigma_2^2\).

In the limit of \(\sigma_1^2 = \sigma_2^2 = \sigma^2\), the ratio \((\sigma_1^2/\sigma_2^2) \to 1\), and the difference \((\sigma_2^2 - \sigma_1^2)^m \to 0^m\). This forces \(m = 0\), collapsing the double summation into a single sum:

\[
\lim_{\sigma_1^2 \to \sigma^2} \lim_{\sigma_2^2 \to \sigma^2} f_{X_1+X_2}(z) \equiv f_X(z) =
\frac{1}{2\sigma^2} \left(\frac{z}{a_1^2}\right)^{K} \exp\left(-\frac{z + a_1^2 + a_2^2}{2\sigma^2}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\sqrt{2}a_2^2 \sigma_1^2}{2a_1 \sigma_2^2}\right)^n I_{K+n} \left(\frac{\sqrt{z}a_1}{\sigma^2}\right), \quad z \geq 0,
\]

with \(L \equiv L_1 + L_2\) and \(K \equiv L - 1\). Note that this elimination of the second summation is possible if and only if the variances are equal.

An extension of the multiplication theorem for Bessel functions of the first kind in [27, 55.2-15] to the modified Bessel function of the first kind \(I_K(\lambda u)\), with no restrictions on \(\lambda\) [1, 9.6.51], is

\[
I_K(\lambda u) = \lambda^K \sum_{n=0}^{\infty} \frac{(\lambda^2 - 1)^n \left(\frac{1}{2}u\right)^n}{n!} I_{K+n} (u).
\]
Substituting \( u = \left( \frac{\sqrt{z} a_1}{\sigma^2} \right) \) and dividing both sides by \( \lambda^K \),

\[
\lambda^{-K} I_K \left( \lambda \frac{\sqrt{z} a_1}{\sigma^2} \right) = \sum_{n=0}^{\infty} \frac{\left( \lambda^2 - 1 \right)^n}{n!} \left( \frac{\sqrt{z} a_1}{2 \sigma^2} \right)^n I_{K+n} \left( \frac{\sqrt{z} a_1}{\sigma^2} \right).
\]

(A.5)

Comparing the infinite summation in (A.3) to the right-hand side of (A.5), we can see that they are equivalent if

\[
\left( \lambda^2 - 1 \right) \frac{a_1}{2 \sigma^2} = \frac{a_2^2}{2 a_1 \sigma^2} \quad \text{(A.6)}
\]

\[
\lambda = \frac{a_1^2 + a_2^2}{\lambda^2 - 1} \quad \text{(A.7)}
\]

Substituting \( \lambda \) into (A.5), we obtain the identity

\[
\left( \frac{a_1^2}{a_1^2 + a_2^2} \right)^{\frac{K}{2}} I_K \left( \frac{\sqrt{z} (a_1^2 + a_2^2)}{\sigma^2} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\sqrt{z} a_1^2}{2 a_1 \sigma^2} \right)^n I_{K+n} \left( \frac{\sqrt{z} a_1}{\sigma^2} \right),
\]

(A.9)

which can then be used to eliminate the infinite summation in (A.3). The resulting pdf in the limit of equal variance is therefore

\[
f_X(z) = \frac{1}{2\sigma^2} \left( \frac{z}{a_1^2 + a_2^2} \right)^{\frac{L}{2}} \exp \left( -\frac{z + a_1^2 + a_2^2}{2 \sigma^2} \right) I_K \left( \frac{\sqrt{z} (a_1^2 + a_2^2)}{\sigma^2} \right)
\]

\[
= \frac{1}{2\sigma^2} \left( \frac{z}{a^2} \right)^{\frac{L-1}{2}} \exp \left( -\frac{z + a^2}{2 \sigma^2} \right) I_{L-1} \left( \frac{\sqrt{z} a^2}{\sigma^2} \right),
\]

(A.10)

which we recognize as a non-central chi-square pdf [23, Eq. 2.44] with \( 2L = 2(L_1 + L_2) \) degrees of freedom and non-centrality parameter \( a^2 \equiv a_1^2 + a_2^2 \), i.e., the sum of the non-centrality parameters.
If we substitute $L_1 = L_2 = N/2$ into (A.2), the resulting pdf is

$$
f_{X_1+X_2}(z) = \frac{1}{2\sigma_1^2} \left( \frac{\sigma_1}{\sigma_2} \right)^N \left( \frac{z}{\sigma_1^2} \right)^{N-1} \exp \left( -\frac{z}{2\sigma_1^2} \right) \exp \left( -\frac{1}{2} \left( \frac{a_1^2}{\sigma_1^2} + \frac{a_2^2}{\sigma_2^2} \right) \right)
$$

\[\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma (N/2 + m + n) \left( \sqrt{z} a_2^2 \sigma_1^2 \right)^n \left( \sqrt{z} (\sigma_2^2 - \sigma_1^2) \right)^m \Gamma (N-1+m+n)}{m!n!\Gamma (N/2 + n)} \left( \frac{\sqrt{z} \sigma_2^2}{2a_1 \sigma_1^2} \right) I_{N-1+m+n} \left( \sqrt{z} a_1 \sigma_1^2 \right), \]

if $z \geq 0$, which is equivalent to the definition of the pdf for the sum of two independent non-central chi-square variates with $N$ degrees of freedom [23, 5.60]. Thus, the derivation also applies to a sum of chi-square variates with $N$ degrees of freedom, resulting in a non-central chi-square pdf with $2L = 2(L_1 + L_2) = 2N$ degrees of freedom.
Appendix B

Proof of Hypergeometric Function Identity

Lemma B.0.3. It will be proven by induction that for integers \( m > 0 \) and \( \ell \geq 0 \),

\[
_1 F_1 (m + \ell; m; x) = \exp (x) \sum_{k=0}^{\ell} \binom{\ell}{k} \frac{x^k}{(m)_k},
\]

(B.1)

where \((m)_k \equiv \frac{(m+k-1)!}{(m-1)!} = m(m+1) \ldots (m+k-1)\) is the Pochhammer symbol and \(\binom{\ell}{k} \equiv \frac{\ell!}{k!(\ell-k)!}\) is the binomial coefficient.

Proof. From [1, Eq. 13.6.12, 13.4.8, 13.4.10], the confluent hypergeometric function has the following properties:

\[
_1 F_1 (a; a; x) = \exp (x)
\]

(B.2)

\[
_1 F_1' (a; b; x) = \frac{a}{b} _1 F_1 (a+1; b+1; x)
\]

(B.3)

\[
_1 F_1 (a+1; b; x) = _1 F_1 (a; b; x) + \frac{x}{a} _1 F_1' (a; b; x) = _1 F_1 (a; b; x) + \frac{x}{b} _1 F_1 (a+1; b+1; x)
\]

(B.4)

(B.5)
for arbitrary natural numbers $a$ and $b$.

We begin with $\ell = 1$. Using the above identities,

$$1 F_1 (m + 1; m; x) = 1 F_1 (m; m; x) + \frac{x}{m} 1 F_1 (m + 1; m + 1; x)$$

(B.6)

$$= \exp(x) \left[ 1 + \frac{x}{m} \right]$$

(B.7)

$$= \exp(x) \left[ \sum_{k=0}^{\infty} \frac{1}{k} \frac{x^k}{(m)_k} \right].$$

(B.8)

Note that since $m$ is an integer greater than 0, the Gamma function $\Gamma(m + 1) = m!$ and the Pochhammer symbol becomes $(m)_k \equiv \frac{(m+k-1)!}{(m-1)!}$.

Moving on to $\ell = 2$, we use the identities and the result from $1 F_1 (m + 1; m; x)$ to obtain

$$1 F_1 (m + 2; m; x) = 1 F_1 (m + 1; m; x) + \frac{x}{m} 1 F_1 (m + 2; m + 1; x)$$

(B.9)

$$= 1 F_1 (m + 1; m; x) + \frac{x}{m} 1 F_1 (n + 1; n; x) \quad \text{where } n = m + 1$$

$$= \exp(x) \left[ \sum_{k=0}^{\infty} \frac{1}{k} \frac{x^k}{(m)_k} + \frac{x}{m} \sum_{k=0}^{\infty} \frac{1}{k} \frac{x^k}{(m+1)_k} \right]$$

$$= \exp(x) \left[ \sum_{k=0}^{\infty} \frac{1}{k} \frac{x^k}{(m)_k} + \sum_{k=1}^{\infty} \frac{1}{k-1} \frac{x^k}{(m)_k} \right]$$

$$= \exp(x) \left[ \sum_{k=0}^{\infty} \frac{2}{k} \frac{x^k}{(m)_k} \right].$$

The fourth line follows because $m((m+1)_k = (m)_{k+1}$, and the last equation is a consequence of the identity $(n+1)_k \equiv (n)_k + (n)_{k-1}$, which is the mathematical formulation of Pascal’s triangle [1, Eq. 3.1.4].
Repeating this process $\ell$ times, we finally obtain

\[ _1F_1 (m + \ell; m; x) = _1F_1 (m + \ell - 1; m; x) + \frac{x}{m} _1F_1 (m + \ell; m + 1; x) \]

\[ = _1F_1 (m + \ell - 1; m; x) + \frac{x}{m} _1F_1 (n + \ell - 1; n; x) \quad \text{where } n = m + 1 \]

\[ = \exp(x) \left[ \sum_{k=0}^{\ell-1} \binom{\ell - 1}{k} \frac{x^k}{(m)_k} + \frac{x}{m} \sum_{k=0}^{\ell-1} \binom{\ell - 1}{k} \frac{x^k}{(m + 1)_k} \right] \]

\[ = \exp(x) \left[ \sum_{k=0}^{\ell-1} \binom{\ell - 1}{k} \frac{x^k}{(m)_k} + \sum_{k=1}^{\ell} \binom{\ell - 1}{k - 1} \frac{x^k}{(m)_k} \right] \]

\[ = \exp(x) \left[ \sum_{k=0}^{\ell} \binom{\ell}{k} \frac{x^k}{(m)_k} \right]. \quad (B.10) \]
Appendix C

Distribution of the Difference of Completely Correlated Non-central Chi-square Random Variables with Equal Variances and Different Means

Theorem C.0.4. Given

1. A circularly symmetric complex Gaussian random variable $\mathbf{N} \equiv N_r + jN_i$ that has
   - Zero mean, $E\{N_r\} = E\{N_i\} = 0$
   - Equal variance of the real and imaginary parts, $Var\{N_r\} = Var\{N_i\} = \sigma^2$.

2. Two complex means $\mu_1$ and $\mu_2$, that have
   - Rectangular Form: $\mu_1 \equiv A_r + jA_i$, $\mu_2 \equiv B_r + jB_i$
\[ X \equiv \| U \|^2 - \| V \|^2, \]

which is a difference of completely correlated non-central chi-square random variables with equal weights but different means,

\[ U \equiv (\mu_1 + N) \]
\[ V \equiv (\mu_2 + N) \]

is Gaussian with mean and variance

\[
E \{X\} = \|\mu_1\|^2 - \|\mu_2\|^2
\]
\[
Var \{X\} = 4\sigma^2 \left[ (\|\mu_1\| - \|\mu_2\|)^2 + 2\|\mu_1\| \|\mu_2\| (1 - \cos \phi) \right],
\]

respectively.

**Corollary C.0.5.** When the means are related by a positive real constant, such as

\[
\mu_2 = \gamma \mu_1
\]
\[
= \gamma (A_r + jA_i)
\]
for some positive real gamma, then the mean and variance simplify to

\[
E \{ X \} = (1 - \gamma^2) \| \mu_1 \|^2 \\
= (1 - \gamma)(1 + \gamma) \| \mu_1 \|^2
\]

\[
Var \{ X \} = 4\sigma^2 [(1 - \gamma) \| \mu_1 \|]^2 \\
= 4\sigma^2 (1 - \gamma)^2 \| \mu_1 \|^2,
\]

respectively.

Proof of Theorem C.0.4. Figures C.1 and C.2 illustrate the vector sums \( U = (\mu_1 + N) \) and \( V = (\mu_2 + N) \) in the complex plane. The mean vectors \( \mu_1 \) and \( \mu_2 \) are deterministic, while the noise vector \( N \) is random, but identical in both cases. We wish to find the difference of the squared lengths of these two vectors. We begin by finding the squared length of these vectors,
\[ \|U\|^2 \text{ and } \|V\|^2. \] The squared length of \( U = (\mu_1 + N) \) is

\[ \|U\|^2 = \|\mu_1 + N\|^2 \]
\[ = \Re \{ \mu_1 + N \}^2 + \Im \{ \mu_1 + N \}^2 \]
\[ = (A_r + N_r)^2 + (A_i + N_i)^2 \]
\[ = (A_r^2 + A_i^2) + (N_r^2 + N_i^2) + 2 (A_r N_r + A_i N_i) \]
\[ = \|\mu_1\|^2 + \|N\|^2 + 2 (A_r N_r + A_i N_i) \]

And similarly (by simply interchanging \( \mu_2 \) for \( \mu_1 \) and \( B \) for \( A \)), the squared length of \( V = (\mu_2 + N) \) is

\[ \|V\|^2 = \|\mu_2\|^2 + \|N\|^2 + 2 (B_r N_r + B_i N_i) \]

Thus, the difference of these squared lengths is

\[ X = \|U\|^2 - \|V\|^2 \]

\[ = \|\mu_1\|^2 - \|\mu_2\|^2 + 2 (A_r N_r + A_i N_i) - [B_r N_r + B_i N_i] \]
\[ = \|\mu_1\|^2 - \|\mu_2\|^2 + 2 N_r (A_r - B_r) + 2 N_i (A_i - B_i) \]

(C.3)

Note that the noise squared terms have cancelled, leaving the sum of a constant plus two independent, (but differently scaled) zero-mean Gaussian random variables. Therefore, \( X \) must also be Gaussian.

The mean of \( X \) is

\[ E \{ X \} = E \left\{ \|\mu_1\|^2 - \|\mu_2\|^2 \right\} + E \{ 2 N_r (A_r - B_r) \} + E \{ 2 N_i (A_i - B_i) \} \]
\[ = \|\mu_1\|^2 - \|\mu_2\|^2 \]

(C.5)

since \( \|\mu_1\|^2 - \|\mu_2\|^2 \) is a constant, and \( N_r \) and \( N_i \) both have zero-mean. Similarly, since the variance of the constant is zero, and the variances of \( N_r \) and \( N_i \) are both \( \sigma^2 \), the variance of \( X \)
is

\[ \text{Var} \{ X \} = \text{Var} \left\{ \| \mu_1 \|^2 - \| \mu_2 \|^2 \right\} + \text{Var} \left\{ 2N_r (A_r - B_r) \right\} + \text{Var} \left\{ 2N_i (A_i - B_i) \right\} \]

\[ = 4 (A_r - B_r)^2 \text{Var} \{ N_r \} + 4 (A_i - B_i)^2 \text{Var} \{ N_i \} \]

\[ = 4\sigma^2 \left[ (A_r - B_r)^2 + (A_i - B_i)^2 \right] \quad \text{(C.6)} \]

\[ = 4\sigma^2 \left[ (A_r^2 + A_i^2) + (B_r^2 + B_i^2) - 2 (A_r B_r + A_i B_i) \right] \]

\[ = 4\sigma^2 \left[ \| \mu_1 \|^2 + \| \mu_2 \|^2 - 2 (A_r B_r + A_i B_i) \right] \quad \text{(C.7)} \]

Expressing this last term in polar form, using the substitutions \( A_r = \| \mu_1 \| \cos \theta_1, \quad A_i = \| \mu_1 \| \sin \theta_1, \)

\( B_r = \| \mu_2 \| \cos \theta_2, \quad B_i = \| \mu_2 \| \sin \theta_2, \)

\[ = 2 (A_r B_r + A_i B_i) = 2 \| \mu_1 \| \| \mu_2 \| (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \quad \text{(C.10)} \]

\[ = 2 \| \mu_1 \| \| \mu_2 \| \cos (\theta_1 - \theta_2) \quad \text{(C.11)} \]

\[ = 2 \| \mu_1 \| \| \mu_2 \| \cos \phi. \quad \text{(C.12)} \]

The second line follows from the trigonometric identity \( \cos (A) \cos (B) \pm \sin (A) \sin (B) = \cos (A \pm B), \) and the third line follows from the definition of \( \phi \) as the difference of the angles.

We now complete the square by noting that

\[ (\| \mu_1 \| - \| \mu_2 \|)^2 = \| \mu_1 \|^2 + \| \mu_2 \|^2 - 2 \| \mu_1 \| \| \mu_2 \| \]

and thus

\[ \text{Var} \{ X \} = 4\sigma^2 \left[ \| \mu_1 \|^2 + \| \mu_2 \|^2 - 2 \| \mu_1 \| \| \mu_2 \| \cos \phi \right] \]

\[ = 4\sigma^2 \left[ \left( \| \mu_1 \|^2 + \| \mu_2 \|^2 - 2 \| \mu_1 \| \| \mu_2 \| \right) + 2 \| \mu_1 \| \| \mu_2 \| - 2 \| \mu_1 \| \| \mu_2 \| \cos \phi \right] \]

\[ = 4\sigma^2 \left[ (\| \mu_1 \| - \| \mu_2 \|)^2 + 2 \| \mu_1 \| \| \mu_2 \| (1 - \cos \phi) \right] \]

\[ \square \]
Proof of Corollary C.0.5. When the means are related by a positive real constant, such as

\[ \mu_2 = \gamma \mu_1 \]

\[ = \gamma (A_r + jA_i) \]

for some positive real gamma, then \( \| \mu_2 \| = \gamma \| \mu_1 \|, \| \mu_2 \|^2 = \gamma^2 \| \mu_1 \|^2, \theta_2 = \arctan \left( \frac{\gamma A_i}{\gamma A_r} \right) = \theta_1 \), and thus

\[ \| \mu_1 \|^2 - \| \mu_2 \|^2 = (1 - \gamma^2) \| \mu_1 \|^2 \]

\[ \| \mu_1 \| - \| \mu_2 \| = (1 - \gamma) \| \mu_1 \| \]

\[ \phi \equiv \theta_1 - \theta_2 = 0. \]

Therefore, the mean and variance in this case simplify to

\[ E \{ X \} = (1 - \gamma^2) \| \mu_1 \|^2 \]

\[ = (1 - \gamma)(1 + \gamma) \| \mu_1 \|^2 \]

\[ Var \{ X \} = 4\sigma^2 [(1 - \gamma) \| \mu_1 \|]^2 \]

\[ = 4\sigma^2 (1 - \gamma)^2 \| \mu_1 \|^2. \]

\[ \square \]

C.1 Integrals of an Exponential With a Quadratic Argument

Theorem C.1.1. Integrals of the form

\[ \int_0^\infty \exp \left( - [Ax^2 + Bx + C] \right) \, dx \] (C.13)

have the solution

\[ \sqrt{\frac{\pi}{4A}} \exp \left( \frac{B^2 - 4AC}{4A} \right) \text{erfc} \left( \frac{B}{\sqrt{4A}} \right) \] (C.14)
provided that $\text{Re} \{A\} > 0$.

Proof. We begin by pulling out the constant term,

$$\exp(-C) \int_0^\infty \exp\left(-[Ax^2 + Bx]\right) dx.$$  \hfill (C.15)

From [7], (Eq. 3.462), the integral of the form

$$\int_0^\infty \exp\left(-[Ax^2 + Bx]\right) dx$$  \hfill (C.16)

has the solution

$$\frac{1}{\sqrt{2A}} \exp\left(\frac{B^2}{8A}\right) D_{-1}\left(\frac{B}{\sqrt{2A}}\right)$$  \hfill (C.17)

provided that $\text{Re} \{A\} > 0$, where $D_n(z)$ denotes Whittaker’s Parabolic Cylinder Function. For $n = -1$, this can be expressed in terms of the complimentary error function [7] (Eq. 9.254) as

$$D_{-1}(z) = \sqrt{\frac{\pi}{2}} \exp\left(\frac{z^2}{4}\right) \text{erfc}\left(\frac{z}{\sqrt{2}}\right)$$  \hfill (C.18)

and thus

$$D_{-1}\left(\frac{B}{\sqrt{2A}}\right) = \sqrt{\frac{\pi}{2}} \exp\left(\frac{B^2}{8A}\right) \text{erfc}\left(\frac{B}{\sqrt{4A}}\right)$$  \hfill (C.19)

Combining (C.15), (C.17) and (C.19), we obtain

$$\sqrt{\frac{\pi}{4A}} \exp\left(\frac{B^2 - 4AC}{4A}\right) \text{erfc}\left(\frac{B}{\sqrt{4A}}\right)$$  \hfill (C.20)

$\square$
Appendix D

Distribution of the Difference of
Completely Correlated Non-central
Chi-square Random Variables with
Different Weights and Different Means

Theorem D.0.2. Given:

1. A circularly symmetric complex Gaussian random variable $N \equiv N_r + jN_i$ that has
   - Zero mean, $E\{N_r\} = E\{N_i\} = 0$
   - Equal variance of the real and imaginary parts, $Var\{N_r\} = Var\{N_i\} = \sigma^2$.

2. Two complex means $\mu_1$ and $\mu_2$, that have
• Rectangular Form: \( \mu_1 \equiv \mu_{1r} + j\mu_{1i}, \mu_2 \equiv \mu_{2r} + j\mu_{2i} \)
  \(- \{\mu_{1r}, \mu_{2r}\} \) and \( \{\mu_{1i}, \mu_{2i}\} \) are arbitrary real coefficients

• Polar Form: \( \mu_k = \|\mu_k\|(\cos \theta_k + j\sin \theta_k), k = \{1, 2\} \)
  \(- \{\theta_1, \theta_2\} \) are arbitrary angles with difference \( \phi \equiv (\theta_1 - \theta_2) \).

3. Two different positive real weights, \( \beta < \alpha \leq 1 \).

We wish to prove that the random variable

\[ Z \equiv \|U\|^2 - \|V\|^2, \]

which is a difference of completely correlated non-central chi-square random variables with different means and different weights,

\[ U \equiv \sqrt{\alpha} (\mu_1 + N) \]

\[ V \equiv \sqrt{\beta} (\mu_2 + N) \]

has a non-central chi-square distribution with 2 degrees of freedom. The non-centrality parameter and variance are

\[ \nu^2 \equiv \frac{B^2}{A} \]

\[ \sigma_z^2 \equiv A\sigma^2, \]

respectively, and the pdf is shifted by

\[ \kappa \equiv C - \frac{B^2}{A} \]
where the coefficients $A$, $B$ and $C$ are

\[
A \equiv (\alpha - \beta) \\
B^2 \equiv \|\alpha \mu_1 - \beta \mu_2\|^2 \\
= (\alpha \|\mu_1\| - \beta \|\mu_2\|)^2 + 2\alpha\beta \|\mu_1\| \|\mu_2\| (1 - \cos \phi) \\
C \equiv \alpha \|\mu_1\|^2 - \beta \|\mu_2\|^2
\]

**Proof of Theorem D.0.2.** Figures D.1 and D.2 illustrate the vector sums $U = \sqrt{\alpha} (\mu_1 + N)$ and $V = \sqrt{\beta} (\mu_2 + N)$ in the complex plane. The mean vectors $\mu_1$ and $\mu_2$ are deterministic, while the noise vector $N$ is random, but *identical* in both cases. We wish to find the difference of the squared lengths of these two vectors. We previously analyzed the limiting case in which $\alpha = \beta = \frac{1}{2}$, and proved that the distribution of the difference $Z = \|U\|^2 - \|V\|^2$ is Gaussian because the noise squared terms cancel out, leaving a sum of two independent Gaussian random variables (from the real and imaginary components). It will be shown that when $\alpha \neq \beta$, the noise squared terms do not cancel, but by completing the square we obtain the sum of two independent
non-central chi-square random variables (also from the real and imaginary components). The constant term required to complete the square simply shifts the distribution.

We begin by finding the squared length of the vectors $U$ and $V$.

$$\|U\|^2 = \alpha \|\mu_1 + N\|^2$$

$$= \alpha \left[ \Re \{\mu_1 + N\}^2 + \Im \{\mu_1 + N\}^2 \right]$$

$$= \alpha \left[ (\mu_{1r} + N_r)^2 + (\mu_{1i} + N_i)^2 \right]$$

$$= \alpha \left[ (\mu_{1r}^2 + \mu_{1i}^2) + (N_r^2 + N_i^2) + 2 (\mu_{1r} N_r + \mu_{1i} N_i) \right]$$

Similarly (by simply interchanging $\mu_2$ for $\mu_1$ and $\beta$ for $\alpha$), the squared length of $V$ is

$$\|V\|^2 = \beta \left[ (\mu_{2r}^2 + \mu_{2i}^2) + (N_r^2 + N_i^2) + 2 (\mu_{2r} N_r + \mu_{2i} N_i) \right]$$

Collecting terms in the real and imaginary parts of $N$, the difference of these squared lengths is

$$Z = \|U\|^2 - \|V\|^2$$

$$= (\alpha - \beta) N_r^2 + 2 (\alpha \mu_{1r} - \beta \mu_{2r}) N_r + (\alpha \mu_{1r}^2 - \beta \mu_{2r}^2)$$

$$+ (\alpha - \beta) N_i^2 + 2 (\alpha \mu_{1i} - \beta \mu_{2i}) N_i + (\alpha \mu_{1i}^2 - \beta \mu_{2i}^2)$$

$$\equiv (A N_r^2 + 2B_r N_r + C_r) + (A N_i^2 + 2B_i N_i + C_i)$$
with the coefficients defined as

\[ A \equiv \alpha - \beta \]
\[ B_r \equiv \alpha \mu_{1r} - \beta \mu_{2r} \]
\[ B_i \equiv \alpha \mu_{1i} - \beta \mu_{2i} \]
\[ C_r \equiv \alpha \mu_{1r}^2 - \beta \mu_{2r}^2 \]
\[ C_i \equiv \alpha \mu_{1i}^2 - \beta \mu_{2i}^2. \]

Note that since \( \beta < \alpha \leq 1 \), \( 0 < A \leq 1 \). Completing the square, we obtain

\[
Z = A \left( N_r^2 + \frac{2B_r}{A} N_r \right) + C_r + A \left( N_i^2 + \frac{2B_i}{A} N_i \right) + C_i
\]

\[= A \left[ \left( N_r + \frac{B_r}{A} \right)^2 + \left( N_i + \frac{B_i}{A} \right)^2 \right] + \left( C_r - \frac{B_r^2}{A} \right) + \left( C_i - \frac{B_i^2}{A} \right) \]

\[\equiv A (W_r + W_i) + (\kappa_i + \kappa_r) \]

\[\equiv AW + \kappa, \] (D.7)

where \( W_r \equiv (N_r + \frac{B_r}{A}) \) and \( W_i \equiv (N_i + \frac{B_i}{A}) \) are independent non-central chi-square random variables with one degree of freedom, with different non-centrality parameters \( a_r^2 = \frac{B_r^2}{A^2} \) and \( a_i^2 = \frac{B_i^2}{A^2} \), and equal noise variances since \( Var \{N_r\} = Var \{N_i\} = \sigma^2 \). Therefore, from Corollary A.0.2 in Appendix A, the sum \( W = W_i + W_r \) is non-central chi square with two degrees of freedom, with a noncentrality parameter equal to the sum of the noncentrality parameters, \( a^2 = a_r^2 + a_i^2 \). Since \( Z = AW + \kappa \), the pdf of the random variable \( Z \) is just a scaled and shifted version of the pdf of \( W \), where \( \kappa \equiv \kappa_r + \kappa_i \), with \( \kappa_r = C_r - \frac{B_r^2}{A} \) and \( \kappa_i = C_i - \frac{B_i^2}{A} \).
We can combine the real and imaginary coefficients into the parameters $B$ and $C$ such that

\[
B^2 \equiv B_r^2 + B_i^2 \\
\quad = (\alpha \mu_1 - \beta \mu_2)^2 + (\alpha \mu_1 - \beta \mu_2)^2 \\
\quad = \|\alpha \mu_1 - \beta \mu_2\|^2 \\
\quad = \alpha^2 \|\mu_1\|^2 [\cos^2 \theta_1 + \sin^2 \theta_1] + \beta^2 \|\mu_2\|^2 [\cos^2 \theta_2 + \sin^2 \theta_2] \\
\quad - 2\alpha\beta \|\mu_1\| \|\mu_2\| [\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2] \\
\quad = (\alpha \|\mu_1\| - \beta \|\mu_2\|)^2 + 2\alpha\beta \|\mu_1\| \|\mu_2\| (1 - \cos \phi) \\
C \equiv C_r + C_i \\
\quad = (\alpha \mu_1 - \beta \mu_2) + (\alpha \mu_1 - \beta \mu_2) \\
\quad = \alpha \|\mu_1\|^2 - \beta \|\mu_2\|^2 \\
\kappa = C - \frac{B^2}{A} \\
\quad = \frac{(\alpha - \beta) (\alpha \|\mu_1\|^2 - \beta \|\mu_2\|^2) - [\alpha^2 \|\mu_1\|^2 + \beta^2 \|\mu_2\|^2 - 2\alpha\beta \|\mu_1\| \|\mu_2\|]}{\alpha - \beta} \\
\quad = -\frac{\alpha\beta}{\alpha + \beta} \left[\|\mu_1\|^2 + \|\mu_2\|^2 - 2 \|\mu_1\| \|\mu_2\| \cos \phi\right] \\
\quad = -\frac{\alpha\beta}{\alpha - \beta} \|\mu_1 - \mu_2\|^2
\]

Therefore, $a^2 = a_r^2 + a_i^2 = \frac{B_r^2 + B_i^2}{A^2}$ and $B^2 \equiv B_r^2 + B_i^2$, so $a^2 = \frac{B^2}{A^2}$, and the pdf of $W$ is

\[
f_W(w) = \frac{U(w)}{2\sigma^2} \exp \left(-\frac{A^2w + B^2}{2A^2\sigma^2}\right) I_0 \left(\frac{\sqrt{wB^2}}{A\sigma^2}\right).
\]
Since $Z = Aw + \kappa$, from [17, Eq. 5.6],

$$f_Z(z) = \frac{1}{|A|} f_W \left( \frac{z - \kappa}{A} \right) \quad (D.10)$$

$$= \frac{1}{2A\sigma^2} \exp \left( - \frac{A(z - \kappa) + B^2}{2A^2\sigma^2} \right) I_0 \left( \frac{\sqrt{(z - \kappa)B^2}}{A\sigma^2} \right) U \left( \frac{z - \kappa}{A} \right) \quad (D.11)$$

$$= \frac{1}{2A\sigma^2} \exp \left( - \frac{(z - \kappa) + B^2}{2A\sigma^2} \right) I_0 \left( \frac{\sqrt{(z - \kappa)B^2}}{A\sigma^2} \right) U \left( \frac{z - \kappa}{A} \right). \quad (D.12)$$

Noting that the unit step function is scale invariant, e.g., $U(Ax) = U(x)$ (for all positive $A$), and defining the variance $\sigma_z^2 \equiv A\sigma^2$, and non-centrality parameter $\nu^2 \equiv \frac{B^2}{A}$, we obtain

$$f_Z(z) = \frac{U(z - \kappa)}{2\sigma_z^2} \exp \left( - \frac{(z - \kappa) + \nu^2}{2\sigma_z^2} \right) I_0 \left( \frac{\sqrt{(z - \kappa)\nu^2}}{\sigma_z^2} \right) \quad (D.13)$$

which we recognize as a non-central chi-square pdf with two degrees of freedom that is shifted by $\kappa = C - \frac{B^2}{A}$.

Another possible representation for $Z$ is as a sum of two shifted non-central chi-square variates arising from the real and imaginary components. Rearranging the terms in (D.7), we can define $Z = Z_r + Z_i$, where $Z_r \equiv AW_r + \kappa_r$ and $Z_i \equiv AW_i + \kappa_i$ are shifted independent non-central chi-square variates with one degree of freedom, equal variances $Var \{Z_r\} = Var \{Z_i\} = A\sigma^2$, and noncentrality parameters of $\frac{B^2}{A}$ and $\frac{B^2}{A}$, respectively. Since the variances are equal, $Z$ has a shifted non-central chi-square distribution with two degrees of freedom, variance $A\sigma^2$, a non-centrality parameter equal to the sum of the non-centrality parameters, 

$$\nu^2 = \frac{B^2 + B^2}{A^2} = \frac{B^2}{A},$$

and a total shift of $\kappa = \kappa_r + \kappa_i = (C_r + C_i) - \left( \frac{B^2 + B^2}{A^2} \right) = C - \frac{B^2}{A}$. \hfill $\square$

**Corollary D.0.3.** When the means are related by a real scaling factor, such as

$$\mu_2 = \gamma \mu_1$$
for some positive real gamma, then the parameters of the distribution simplify to

\[ \nu^2 = \| \mu_1 \|^2 \frac{(\alpha - \gamma \beta)^2}{\alpha - \beta} \]

\[ \kappa = -\| \mu_1 \|^2 \frac{\alpha \beta (1 - \gamma)^2}{\alpha - \beta} \]

Proof. \qed

D.1 Example

Figure D.3 compares the analytical result for the pdfs of

\[ Z = \alpha \left( \sqrt{P_o + (N_r + jN_i)} \right)^2 - (1 - \alpha) \left( \sqrt{P_o \gamma (\cos \phi + j \sin \phi) + (N_r + jN_i)} \right)^2 \]

\[ Z_r = \alpha \Re \left( \sqrt{P_o + N} \right)^2 - (1 - \alpha) \Re \left( \sqrt{P_o \gamma (\cos \phi + j \sin \phi) + N} \right)^2 \]

\[ = \alpha \left( \sqrt{P_o + N_r} \right)^2 - (1 - \alpha) \left( \sqrt{P_o \gamma \cos \phi + N_r} \right)^2 \]

\[ Z_i = \alpha \Im \left( \sqrt{P_o + N} \right)^2 - (1 - \alpha) \Im \left( \sqrt{P_o \gamma (\cos \phi + j \sin \phi) + N} \right)^2 \]

\[ = \alpha N_i^2 - (1 - \alpha) \left( \sqrt{P_o \gamma \sin \phi + N_i} \right)^2 \]

to numerical results from a Monte Carlo simulotion. The Monte Carlo results are histograms computed from one million trials of the zero-mean, unit variance, Gaussian noise variables \( N_r \) and \( N_i \).

Using the parameters \( \rho = 1, \alpha = 0.51, \gamma = 0.9, \phi = \pi/4 \) and \( OSIR = 0 \text{ dB} \), we note that the distributions are very nearly Gaussian because \( \alpha \) is nearly equal to \( \frac{1}{2} \), and the means and variances are both large because \( \gamma \) is nearly equal to 1.
Figure D.3: Analytic vs. Monte Carlo Distributions of $Z$: $\rho = 1$, $\alpha = 0.51$, $\gamma = 0.9$, $\phi = \pi/4$, $OSIR = 0$ dB.
Appendix E

Finite Limit Representation of the Marcum-Q Function

Lemma E.0.1. We wish to prove that for all \( 0 \leq z \leq \infty \)

\[
Q_1 \left( \sqrt{2A}, \sqrt{2Bz} \right) = \begin{cases} 
\phi_1 (A, B, z) & \text{if } z > \xi, \\
1 + \phi_2 (A, B, z) & \text{if } z < \xi,
\end{cases}
\]  
(E.1)

with \( \xi \equiv \frac{A}{B} \) and

\[
\phi_1 (A, B, z) \equiv \frac{1}{\pi} \int_0^\pi F(\xi, \theta, z) \exp \left[ -H(A, B, \theta, z) \right] d\theta,
\]  
(E.2)

\[
\phi_2 (A, B, z) \equiv \frac{1}{\pi} \int_0^\pi G(\xi, \theta, z) \exp \left[ -H(A, B, \theta, z) \right] d\theta,
\]  
(E.3)

where

\[
F(\xi, \theta, z) = \frac{1 \pm \sqrt{\frac{z}{\xi}} \cos \theta}{1 \pm 2 \sqrt{\frac{z}{\xi}} \cos \theta + \frac{\xi}{2}} \quad \text{and} \quad G(\xi, \theta, z) = \frac{\xi \pm \sqrt{z} \cos \theta}{1 \pm 2 \sqrt{\frac{z}{\xi}} \cos \theta + \frac{\xi}{2}}
\]

\[
H(A, B, \theta, z) \equiv A + Bz \pm 2\sqrt{ABz} \cos \theta.
\]  
(E.4)

Proof. From [24, Eq. 4.20 and 4.21], a finite limit representation for the first order Marcum-Q
function is
\[ Q_1(\alpha, \beta) = \begin{cases} 
Q_1(\xi_\beta, \beta) = \frac{1}{\pi} \int_0^\pi \frac{1 + \xi \cos \theta}{1 + 2 \xi \cos \theta + \xi^2} \exp \left( -\frac{\beta^2}{2} \left( 1 \pm 2 \xi \cos \theta + \xi^2 \right) \right) d\theta & \text{if } 0 \leq \alpha < \beta, \\
Q_1(\alpha, \xi_\alpha) = 1 + \frac{1}{\pi} \int_0^\pi \frac{\xi^2 \pm \xi \cos \theta}{1 + 2 \xi \cos \theta + \xi^2} \exp \left( -\frac{\alpha^2}{2} \left( 1 \pm 2 \xi \cos \theta + \xi^2 \right) \right) d\theta & \text{if } 0 \leq \beta < \alpha, 
\end{cases} \]  
(E.5)

where, in the first case, \( \xi \equiv \frac{\alpha}{\beta} \), and in the second case, \( \xi \equiv \frac{\beta}{\alpha} \).

For our function-of-interest, \( Q_1(\sqrt{2A}, \sqrt{2Bz}) \), the ratio of the arguments \( \sqrt{\frac{A}{Bz}} \) will be greater than unity for \( z < \frac{\sqrt{A}}{\sqrt{Bz}} \) and less than unity for \( z > \frac{\sqrt{A}}{\sqrt{Bz}} \). The inconsistency of the argument ratios in the above definition (E.5) is problematic since \( z \) is an integration variable over the region \( 0 \leq z \leq \infty \). We begin by defining ratios \( \xi \equiv \frac{\sqrt{A}}{\sqrt{Bz}} \) and \( \chi = \frac{\sqrt{A}}{\sqrt{Bz}} = \sqrt{\frac{\xi}{z}} \) that are consistent for all \( 0 \leq z \leq \infty \), in order to make the piecewise definition

\[ Q_1(\sqrt{2A}, \sqrt{2Bz}) = \begin{cases} 
Q_1(\chi \sqrt{2Bz}, \sqrt{2Bz}) & \text{if } 0 \leq \sqrt{2A} < \sqrt{2Bz}, \\
Q_1(\sqrt{2A}, \chi^{-1} \sqrt{2A}) & \text{if } 0 \leq \sqrt{2Bz} < \sqrt{2A}, 
\end{cases} \]
(E.6)

= \left\{ \begin{array}{ll}
\frac{1}{\pi} \int_0^\pi \frac{1 + \chi \cos \theta}{1 + 2 \chi \cos \theta + \chi^2} \exp \left( -Bz \left( 1 \pm 2 \chi \cos \theta + \chi^2 \right) \right) d\theta & \text{if } z > \xi, \\
1 + \frac{1}{\pi} \int_0^\pi \frac{\chi^2 \pm \chi \cos \theta}{1 + 2 \chi \cos \theta + \chi^2} \exp \left( -A \left( 1 \pm 2 \chi^{-1} \cos \theta + \chi^{-2} \right) \right) d\theta & \text{if } z < \xi.
\end{array} \right.

The exponential terms are

\[ \exp \left( -Bz \left( 1 \pm 2 \chi \cos \theta + \chi^2 \right) \right) = \exp \left( -Bz \mp 2Bz \sqrt{\frac{A}{Bz}} \cos \theta - Bz \frac{A}{Bz} \right) \]

= \exp \left( -A - Bz \mp 2\sqrt{ABz} \cos \theta \right) \]

= \exp \left[ -H(A, B, \theta, z) \right], \quad (E.7)

\[ \exp \left( -A \left( 1 \pm 2 \chi^{-1} \cos \theta + \chi^{-2} \right) \right) = \exp \left( -A \mp 2A \sqrt{\frac{Bz}{A}} \cos \theta - \frac{Bz}{A} \right) \]

= \exp \left( -A - Bz \mp 2\sqrt{ABz} \cos \theta \right) \]

= \exp \left[ -H(A, B, \theta, z) \right]. \quad (E.8)
Using the parameter definitions (E.4) to simplify (E.6), we obtain the desired result:

\[
Q_1 \left( \sqrt{2A}, \sqrt{2Bz} \right) = \begin{cases} 
\phi_1 (A, B, z) & \text{if } z > \xi, \\
1 + \phi_2 (A, B, z) & \text{if } z < \xi,
\end{cases}
\] (E.9)

with \( \xi \equiv \frac{A}{B} \) and

\[
\phi_1 (A, B, z) \equiv \frac{1}{\pi} \int_{0}^{\pi} F(\xi, \theta, z) \exp \left[ -H(A, B, \theta, z) \right] d\theta,
\] (E.10)

\[
\phi_2 (A, B, z) \equiv \frac{1}{\pi} \int_{0}^{\pi} G(\xi, \theta, z) \exp \left[ -H(A, B, \theta, z) \right] d\theta.
\] (E.11)
Appendix F

Autocorrelation of a Time-spread Pulse

We wish to show that the nonstationary autocorrelation of a single time-spread pulse is approximately the product of the nonstationary variance and the short pulse autocorrelation function, \( R_p(x_o) \). The proof uses steps similar to the proof of Corrolary 1 in [15].

For large \( N_{\text{eff}} \) we can combine Proposition 1 and Lemma 1 from [15] to obtain

\[
R_y(x, x_o) \approx \frac{P_o}{N_{\text{eff}}} \text{sinc} \left( \frac{x}{N_{\text{eff}}} \right) \text{sinc} \left( \frac{x + x_o}{N_{\text{eff}}} \right) \sum_{n=-\infty}^{\infty} R_p(x_o - nN_{\text{eff}}) \tag{F.1}
\]

\[
\approx \frac{P_o}{N_{\text{eff}}} \text{sinc} \left( \frac{x}{N_{\text{eff}}} \right) \text{sinc} \left( \frac{x + x_o}{N_{\text{eff}}} \right) R_p(x_o) \tag{F.2}
\]

\[
\approx \frac{P_o}{N_{\text{eff}}} \text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) R_p(x_o) \tag{F.3}
\]

The second approximation follows because \( R_p(nN_{\text{eff}}) \approx 0 \) unless \( n = 0 \). By the same rationale, if \( x_o \gg 1 \), then \( R_y(x, x_o) \approx 0 \), independent of \( x \), since the effective width of \( R_p(x_o) \) is on the order of 1 and \( R_p(x_o) \approx 0 \) unless \( |x_o| \) is on the order of 1. The last approximation follows because when \( R_y(x, x_o) \neq 0 \), then \( x_o = O(1) \), and thus \( \text{sinc} \left( \frac{x}{N_{\text{eff}}} + \frac{x_o}{N_{\text{eff}}} \right) = \text{sinc} \left( \frac{x}{N_{\text{eff}}} + O \left( \frac{1}{N_{\text{eff}}} \right) \right) \).

With \( N_{\text{eff}} \gg 1 \), then \( \text{sinc} \left( \frac{x}{N_{\text{eff}}} \right) \approx \text{sinc} \left( \frac{x}{N_{\text{eff}}} + O \left( \frac{1}{N_{\text{eff}}} \right) \right) \).

For example, let \( N_{\text{eff}} = 100 \) and consider the same Gaussian pulse used earlier. For
\( R_p(x_o) = \exp(-y), x_o = \frac{3}{\sqrt{\pi}} \sqrt{y} \) and \( \frac{x_o}{N_{\text{eff}}} = \frac{\sqrt{y}}{50\sqrt{\pi}} \). Therefore,

- For \( R_p(x_o) = \exp(-1) \approx 0.37 \), \( \frac{x_o}{N_{\text{eff}}} = \frac{1}{50\sqrt{\pi}} \approx 0.011 \).

- For \( R_p(x_o) = \exp(-4) \approx 0.02 \), \( \frac{x_o}{N_{\text{eff}}} = \frac{1}{25\sqrt{\pi}} \approx 0.023 \).

- For \( R_p(x_o) = \exp(-16) \approx 10^{-7} \), \( \frac{x_o}{N_{\text{eff}}} = \frac{2}{25\sqrt{\pi}} \approx 0.045 \).

Thus, even for moderate processing gain, it is clear that when \( R_p(x_o) \) is non-zero, \( \frac{x_o}{N_{\text{eff}}} = O(1/N_{\text{eff}}) \) is sufficiently small to ensure that \( \text{sinc}\left(\frac{x}{N_{\text{eff}}}\right) \approx \text{sinc}\left(\frac{x+x_o}{N_{\text{eff}}}\right) \).
Appendix G

Asymptotic Stationarity

Theorem G.0.2. \( \forall x \in (0, N_s) \),

\[
E \left\{ \frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \sin^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right) \right\} = \frac{P_o J}{2N_s} \quad \text{(G.1)}
\]

\[
\text{Var} \left\{ \frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \sin^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right) \right\} = \frac{2}{3} \frac{P_o}{N_{\text{eff}}} \left( \frac{P_o J}{2N_s} \right) \quad \text{(G.2)}
\]

as \( J \) and \( N_s \) grow large.

Proof. Let \( \nu \) be the point process formed by rearranging the delays \( \lambda \) in increasing order. With \( J \) points uniformly distributed in the interval \( (0, N_s) \), where \( J \gg 1 \), the resulting point process \( \nu \) is nearly Poisson with an average density \( \Upsilon = \frac{J}{T_s} \). This is exact in the limit as \( J \) and \( N_s \) tend to infinity ([17], pg. 357).

When \( \nu \) is a Poisson point process with average density \( \Upsilon \), then

\[
s(x) = \sum_i h(x - \nu_i) \quad \text{(G.3)}
\]

is a shot process that is strict sense stationary.
By Campbell’s theorem [17], the mean and variance of \( s(x) \) are:

\[
E \{ s(x) \} = \Upsilon \int_{-\infty}^{\infty} h(x) \, dx \tag{G.4}
\]
\[
\text{Var} \{ s(x) \} = \Upsilon \int_{-\infty}^{\infty} h^2(x) \, dx \tag{G.5}
\]

Thus as \( J \) and \( T_s \) tend to infinity,

\[
s(x) = \frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \text{sinc}^2 \left( \frac{x - \lambda_i}{N_{\text{eff}}} \right) \tag{G.6}
\]
\[
= \frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \text{sinc}^2 \left( \frac{x - \nu_i}{N_{\text{eff}}} \right) \tag{G.7}
\]
\[
= h \left( \frac{x - \nu_i}{N_{\text{eff}}} \right) \tag{G.8}
\]

is a shot process with average density

\[
\Upsilon = \frac{J}{N_s} \tag{G.10}
\]

and transfer function

\[
h(x) = \frac{P_o}{2N_{\text{eff}}} \sum_{i=1}^{J} \text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) \tag{G.11}
\]

Therefore, the mean and variance of \( s(t) \) are

\[
E \{ s(x) \} = \Upsilon \int_{-\infty}^{\infty} h(x) \, dx \tag{G.12}
\]
\[
= \frac{J}{N_s} \frac{P_o}{2N_{\text{eff}}} \int_{-\infty}^{\infty} \text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) \, dx \tag{G.13}
\]

and

\[
\text{Var} \{ s(x) \} = \Upsilon \int_{-\infty}^{\infty} h^2(x) \, dx \tag{G.14}
\]
\[
= \frac{J}{N_s} \left( \frac{P_o}{2N_{\text{eff}}} \right)^2 \int_{-\infty}^{\infty} \text{sinc}^4 \left( \frac{x}{N_{\text{eff}}} \right) \, dx \tag{G.15}
\]

respectively.
To readily find \(\int_{-\infty}^{\infty} \text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) \, dx\) and \(\int_{-\infty}^{\infty} \text{sinc}^4 \left( \frac{x}{N_{\text{eff}}} \right) \, dx\), we note the following Fourier transform pairs:

\[
\begin{align*}
\text{sinc} \left( \frac{x}{N_{\text{eff}}} \right) & \quad \leftrightarrow \quad \text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} \right) \, N_{\text{eff}} \quad (G.16) \\
\text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) & \quad \leftrightarrow \quad \text{tri} \left( \frac{\chi N_{\text{eff}}}{2\pi} \right) \, N_{\text{eff}} \quad (G.17)
\end{align*}
\]

where

\[
\begin{align*}
\text{rect} \ (x) & \equiv \begin{cases} 
1 & |x| \leq 1/2 \\
0 & \text{otherwise}
\end{cases} \quad (G.19) \\
\text{tri} \ (x) & \equiv \begin{cases} 
1 - |x| & |x| \leq 1 \\
0 & \text{otherwise}
\end{cases} \quad (G.20)
\end{align*}
\]

By Parseval’s Theorem,

\[
\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\chi)|^2 \, d\chi, \quad (G.21)
\]

we obtain

\[
\begin{align*}
\int_{-\infty}^{\infty} \text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) \, dx & = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} \right) \, N_{\text{eff}} \right|^2 \, d\chi \\
& = \frac{N_{\text{eff}}^2}{2\pi} \int_{-\pi/N_{\text{eff}}}^{\pi/N_{\text{eff}}} 1 \, d\chi \\
& = N_{\text{eff}} \quad (G.24)
\end{align*}
\]
and

\[
\int_{-\infty}^{\infty} \left( \text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) \right)^2 \, dx
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \text{tri} \left( \frac{xN_{\text{eff}}}{2\pi} \right) N_{\text{eff}} \right|^2 \, d\chi
\]

\[
= \frac{N_{\text{eff}}^2}{2\pi} \int_{-2\pi/N_{\text{eff}}}^{2\pi/N_{\text{eff}}} \left( 1 - \left| \frac{xN_{\text{eff}}}{2\pi} \right| \right)^2 \, d\chi
\]

\[
= \frac{N_{\text{eff}}^2}{\pi} \int_{0}^{2\pi/N_{\text{eff}}} \left( 1 - \frac{\chi N_{\text{eff}}}{2\pi} \right)^2 \, d\chi
\]

\[
= \frac{N_{\text{eff}}^2}{\pi} \left[ \int_{0}^{2\pi/N_{\text{eff}}} 1 \, d\chi - 2 \int_{0}^{2\pi/N_{\text{eff}}} \frac{\chi N_{\text{eff}}}{2\pi} \, d\chi + \int_{0}^{2\pi/N_{\text{eff}}} \left( \frac{\chi N_{\text{eff}}}{2\pi} \right)^2 \, d\chi \right]
\]

\[
= \frac{N_{\text{eff}}^2}{\pi} \left( \frac{2\pi}{N_{\text{eff}}} - 2 \frac{2\pi}{N_{\text{eff}}} + 1 \frac{2\pi}{3N_{\text{eff}}} \right)
\]

\[
= \frac{2}{3} N_{\text{eff}}.
\]

Therefore, the mean and variance of the shot process \( s(x) \) are:

\[
E \{ s(x) \} = \frac{J}{N_s} \frac{P_o}{2N_{\text{eff}}} \int_{-\infty}^{\infty} \text{sinc}^2 \left( \frac{x}{N_{\text{eff}}} \right) \, dx
\]

\[
= \frac{P_o J}{2N_s}
\]

\[
\text{Var} \{ s(x) \} = \frac{J}{N_s} \left( \frac{P_o}{2N_{\text{eff}}} \right)^2 \int_{-\infty}^{\infty} \text{sinc}^4 \left( \frac{x}{N_{\text{eff}}} \right) \, dx
\]

\[
= \frac{2}{3} \frac{P_o}{4N_{\text{eff}}} \left( \frac{P_o J}{N_s} \right)
\]

\[
= \frac{1}{3} \frac{P_o}{N_{\text{eff}}} E \{ s(x) \}
\]

In order to prevent a time-spread pulse from overlapping in adjacent symbols, as a practical limit the processing gain cannot exceed \( N_s \). With a large processing gain approaching this limit (e.g. \( N_{\text{eff}} \rightarrow N_s \)), the variance \( \text{Var} \{ s(x) \} \rightarrow 0 \) as \( N_{\text{eff}} \rightarrow N_s \rightarrow \infty \), while \( N_s \rightarrow \infty \) and \( J \rightarrow \infty \) ensures the point process \( \nu \) is Poisson.
Appendix H

Matched Filter Output - MAI statistics

The interference component at the output of the matched filter is

\[
\hat{Z}_i(\chi) = \sum_{i=1}^{J} P(\chi) C_i(\chi) C_o^*(\chi) \hat{H}(\chi) \exp(-j\chi \lambda_i)
\]  

(H.1)

The spectral code cross-correlation, \(C_i(\chi) C_o^*(\chi)\), is

\[
C_i(\chi) C_o^*(\chi) = \left[ \sum_{n=-\infty}^{\infty} c_n^{(i)} \text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} - n \right) \right] \left[ \sum_{m=-\infty}^{\infty} c_m^{(o)*} \text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} - m \right) \right]
\]  

(H.2)

And

\[
\text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} - n \right) \text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} - m \right) = \begin{cases} 
\text{rect}^2 \left( \frac{\chi N_{\text{eff}}}{2\pi} - n \right) = \text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} - n \right) & \text{if } m = n \\
0 & \text{if } m \neq n
\end{cases}
\]  

(H.3)

Since the spectral chips can only overlap if the indices are equal. Thus

\[
C_i(\chi) C_o^*(\chi) = \sum_{n=-\infty}^{\infty} c_n^{(i)} c_n^{(o)*} \text{rect} \left( \frac{\chi N_{\text{eff}}}{2\pi} - n \right)
\]
For the desired user, $i = 0$:

$$C_o(\chi)C_o^*(\chi) = \sum_{n=-\infty}^{\infty} c_n^{(o)}c_n^{(o)*}\text{rect}\left(\frac{\chi N_{\text{eff}}}{2\pi} - n\right)$$

$$= \sum_{n=-\infty}^{\infty} |c_n^{(o)}|^2\text{rect}\left(\frac{\chi N_{\text{eff}}}{2\pi} - n\right)$$

$$= 1 \forall \chi$$

And for any other user $i \neq 0$, we can define a new spectral filter $\widetilde{C}_i(\chi)$:

$$\widetilde{C}_i(\chi) \equiv C_i(\chi)C_o^*(\chi)$$

$$= \sum_{n=-\infty}^{\infty} c_n^{(i)}c_n^{(o)*}\text{rect}\left(\frac{\chi N_{\text{eff}}}{2\pi} - n\right)$$

that is statistically identical to $C_i(\chi)$ since $c_n^{(o)}$ is deterministic. Furthermore, since all of the other sequences $c_n^{(i)}$ are random and uncorrelated, all of the $\widetilde{C}_i(\chi)$ are also uncorrelated. Therefore, $\widetilde{Y}_i(\chi) = \widetilde{C}_iP(\chi)$ is a spread pulse that is statistically identical to the original spread pulse $Y_i(\chi) = C_iP(\chi)$.

The interference component at the output of the decoding spectral filter can thus be expressed as:

$$\hat{Z}_i(\chi) = \hat{H}(\chi)\sum_{i=1}^{J} \hat{Y}_i(\chi)\exp(-j\chi\lambda_i) \quad \text{(H.4)}$$

$$\hat{z}_i(x) = \hat{h}(x) \star \sum_{i=1}^{J} \hat{g}_i(x - \lambda_i) \quad \text{(H.5)}$$

$$\hat{z}_i(x) = \hat{h}(x) \star \hat{n}(x) \quad \text{(H.6)}$$

since $\hat{g}_i(x)$ and $\hat{y}_i(x)$ are statistically identical, the interference term $\hat{n}(x)$ is statistically identical to the original definition of $n(x)$ in Equation 13.16.
Appendix I

Recursive Bisection Algorithm

I.0.1 Bilinear Interpolation Function

The bilinear interpolation function [18, Section 3.6] is

\[
2P_{\text{interp}}(\{P_{00}, P_{10}, P_{01}, P_{11}\}; (x_0, y_0); (\Delta x, \Delta y); (x, y)) =
\]

\[
P_{00} + (P_{10} - P_{00}) \frac{x - x_0}{\Delta x} + (P_{01} - P_{00}) \frac{y - y_0}{\Delta y} + (P_{11} - P_{10} - P_{01} + P_{00}) \frac{(x - x_0)(y - y_0)}{\Delta x \Delta y},
\]

(I.1)

and the integral (i.e., the volume) is

\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} P_{\text{interp}}(\{P_{00}, P_{10}, P_{01}, P_{11}\}; (x_0, y_0); (\Delta x, \Delta y); (x, y)) \, dx \, dy =
\]

\[
= P_{00} \Delta x \Delta y
\]

\[
+ (P_{10} - P_{00}) \frac{\Delta y}{\Delta x} \int_{x_0}^{x_1} (x - x_0) \, dx
\]

\[
+ (P_{01} - P_{00}) \frac{\Delta x}{\Delta y} \int_{y_0}^{y_1} (y - y_0) \, dy
\]

\[
+ (P_{11} - P_{10} - P_{01} + P_{00}) \frac{1}{\Delta x \Delta y} \int_{x_0}^{x_1} (x - x_0) \, dx \int_{y_0}^{y_1} (y - y_0) \, dy.
\]

(I.2)
The integral in $x$ simplifies to

\[
\int_{x_0}^{x_1} (x - x_0) \, dx = \frac{1}{\Delta x} \left[ \frac{1}{2} x^2 \right]_{x_0}^{x_1} - x_0
\]

\[
= \frac{1}{2 \Delta x} \left[ x_1^2 - x_0^2 - 2x_0(x_1 - x_0) \right]
\]

\[
= \frac{1}{2 \Delta x} [x_1 - x_0]^2
\]

\[
= \frac{\Delta x}{2},
\]

where the definition $\Delta x = x_1 - x_0$ has been used. Similarly, the integral in $y$ simplifies to $\frac{\Delta y}{2}$.

Therefore, the integral of the bilinear interpolation function is

\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} P_{\text{interp}} \{ P_{00}, P_{10}, P_{01}, P_{11} \}; (x_0, y_0); (\Delta x, \Delta y); (x, y) \, dx \, dy = 
\]

\[
= P_{00} \Delta x \Delta y + (P_{10} - P_{00}) \frac{\Delta x \Delta y}{2} + (P_{01} - P_{00}) \frac{\Delta x \Delta y}{2}
\]

\[
+ (P_{11} - P_{10} - P_{01} + P_{00}) \frac{\Delta x \Delta y}{4}
\]

\[
= \Delta x \Delta y \left[ P_{00} \left( 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{4} \right) + P_{10} \left( \frac{1}{2} - \frac{1}{4} \right) + P_{01} \left( \frac{1}{2} - \frac{1}{4} \right) + P_{11} \frac{1}{4} \right]
\]

\[
= \frac{\Delta x \Delta y}{4} [P_{00} + P_{10} + P_{01} + P_{11}],
\]

which is the area $\Delta x \Delta y$ times the average height, $[P_{00} + P_{10} + P_{01} + P_{11}] / 4$. 

Algorithm 1: One-Dimensional Recursive Bisection Algorithm

Function \([U, T, L] = \text{RecursiveBisection} (\text{SIR}, \{P_0, P_1\}, \{x_0, x_1\}, \epsilon')\)

Given: Conditional Error Probability \(P(\text{SIR} \mid x)\), Parameter Distribution \(f_x(x)\)

\[
\Delta x = x_1 - x_0
\]

\[
U = \max\{P_0, P_1\} \Delta x \quad // \text{upper bound}
\]

\[
L = \min\{P_0, P_1\} \Delta x \quad // \text{lower bound}
\]

\[
T = \frac{1}{2} (P_U + P_L) \Delta x \quad // \text{average}
\]

if \(E = (U - L) < \epsilon'\) then  //if the error is within the convergence tolerance
    return \([U, T, L]\)
else  //perform recursive bisection
    \[
x_{\text{mid}} = \frac{x_0 + x_1}{2}
\]
    \[
P_{\text{mid}} = P(\text{SIR} \mid x_{\text{mid}}) f_x(x_{\text{mid}})
\]
    \[
[U_{\text{left}}, T_{\text{left}}, L_{\text{left}}] = \text{RecursiveBisection} (\text{SIR}, \{P_0, P_{\text{mid}}\}, \{x_0, x_{\text{mid}}\}, \epsilon')
\]
    \[
[U_{\text{right}}, T_{\text{right}}, L_{\text{right}}] = \text{RecursiveBisection} (\text{SIR}, \{P_{\text{mid}}, P_1\}, \{x_{\text{mid}}, x_1\}, \epsilon')
\]
    return \([(U_{\text{left}} + U_{\text{right}}), (T_{\text{left}} + T_{\text{right}}), (L_{\text{left}} + L_{\text{right}})]\)
end if

Function \([U, T, L] = \text{AverageErrorRates} (N, \Delta \text{dB}, \text{RelativeError})\)

Given: Conditional Error Probability \(P(\text{SIR} \mid x)\), Parameter Distribution \(f_x(x)\)

\((x_{\text{min}}, x_{\text{max}}) = \text{Range}[f_x(x)]\)

PreviousError = \(\frac{1}{2}\)  //initialize convergence tolerance for 1dB SIR relative to

\(\text{SIRdB} = 1\)  //an average error rate of \(\frac{1}{2}\)

for \(n = 1\) to \(N\) do
    \(\epsilon' = (\text{PreviousError} \times \text{RelativeError})\)  //determine convergence tolerance for SIRdB relative to
    \(\text{SIRdB} = \text{SIRdB} + \Delta \text{dB}\)  //the average error rate computed for (SIRdB-\Delta dB)
    \(\text{SIR} = 10^{\frac{\text{SIRdB}}{10}}\)
    \[
P_0 = P(\text{SIR} \mid x_{\text{min}}) f_x(x_{\text{min}})
\]
    \[
P_1 = P(\text{SIR} \mid x_{\text{max}}) f_x(x_{\text{max}})
\]
    \([U_{n}, T_{n}, L_{n}] = \text{RecursiveBisection} (\text{SIR}, \{P_0, P_1\}, \{x_{\text{min}}, x_{\text{max}}\}, \epsilon')\)
    PreviousError = \(T_{n}\)
end for

return \([U, T, L]\)
Algorithm 2: Two-Dimensional Recursive Bisection Algorithm

Function \([U, A, L] = \text{RecursiveBisection}(\text{SIR}, \{P_{00}, P_{01}, P_{10}, P_{11}\}, \{x_0, x_1\}, \{y_0, y_1\}, \epsilon')\)

Given: Conditional Error Probability \(P(\text{SIR}|x,y)\), Parameter Distributions \(f_x(x), f_y(y)\)

\[
\Delta x = x_1 - x_0 \\
\Delta y = y_1 - y_0 \\
U = \max\{P_{00}, P_{01}, P_{10}, P_{11}\} \Delta x \Delta y \quad \text{//upper bound} \\
L = \min\{P_{00}, P_{01}, P_{10}, P_{11}\} \Delta x \Delta y \quad \text{//lower bound} \\
A = \frac{1}{2} (P_{00} + P_{01} + P_{10} + P_{11}) \Delta x \Delta y \quad \text{//average} \\
\text{if } E = (U - L) < \epsilon' \text{ then} \quad \text{//if the error is within the convergence tolerance} \\
\text{return } [U, A, L] \\
\text{else} \\
\text{//perform recursive bisection} \\
x_{\text{mid}} = \frac{1}{2} (x_0 + x_1) \\
y_{\text{mid}} = \frac{1}{2} (y_0 + y_1) \\
P_{\text{left}} = P(\text{SIR}|x_0, y_{\text{mid}}) f_x(x_0) f_y(y_{\text{mid}}) \\
P_{\text{right}} = P(\text{SIR}|x_1, y_{\text{mid}}) f_x(x_1) f_y(y_{\text{mid}}) \\
P_{\text{up}} = P(\text{SIR}|x_{\text{mid}}, y_0) f_x(x_{\text{mid}}) f_y(y_0) \\
P_{\text{down}} = P(\text{SIR}|x_{\text{mid}}, y_1) f_x(x_{\text{mid}}) f_y(y_1) \\
P_{\text{center}} = P(\text{SIR}|x_{\text{mid}}, y_{\text{mid}}) f_x(x_{\text{mid}}) f_y(y_{\text{mid}}) \\
[U_{00}, A_{00}, L_{00}] = \text{RecursiveBisection}(\text{SIR}, \{P_{00}, P_{01}, P_{10}, P_{11}\}, \{x_0, x_{\text{mid}}\}, \{y_0, y_{\text{mid}}\}, \epsilon') \\
[U_{01}, A_{01}, L_{01}] = \text{RecursiveBisection}(\text{SIR}, \{P_{00}, P_{01}, P_{10}, P_{11}\}, \{x_{\text{mid}}, x_1\}, \{y_0, y_{\text{mid}}\}, \epsilon') \\
[U_{10}, A_{10}, L_{10}] = \text{RecursiveBisection}(\text{SIR}, \{P_{00}, P_{01}, P_{10}, P_{11}\}, \{x_0, x_{\text{mid}}\}, \{y_{\text{mid}}, y_1\}, \epsilon') \\
[U_{11}, A_{11}, L_{11}] = \text{RecursiveBisection}(\text{SIR}, \{P_{00}, P_{01}, P_{10}, P_{11}\}, \{x_{\text{mid}}, x_1\}, \{y_{\text{mid}}, y_1\}, \epsilon') \\
\text{return } [(U_{00} + U_{01} + U_{10} + U_{11}), (A_{00} + A_{01} + A_{10} + A_{11}), (L_{00} + L_{01} + L_{10} + L_{11})] \\
\text{end if} \\
\]

Function \([U, A, L] = \text{AverageErrorRates}(N, \Delta \text{dB}, \text{RelativeError})\)

Given: Conditional Error Probability \(P(\text{SIR}|x,y)\), Parameter Distributions \(f_x(x), f_y(y)\)

\[
(x_{\text{min}}, x_{\text{max}}) = \text{Range}[f_x(x)] \\
(y_{\text{min}}, y_{\text{max}}) = \text{Range}[f_y(y)] \\
\text{PreviousError} = \frac{1}{2} \\
\text{//initialize convergence tolerance for 1dB SIR relative to} \\
\text{SIRdB = 1} \\
\text{//an average error rate of } \frac{1}{2} \\
\text{for } n = 1 \text{ to } N \text{ do} \\
\epsilon' = (\text{PreviousError} \times \text{RelativeError}) \\
\text{//determine convergence tolerance for SIRdB relative to} \\
\text{SIRdB = SIRdB + } \Delta \text{dB} \\
\text{//the average error rate computed for (SIRdB-}\Delta \text{dB)} \\
\text{SIR} = 10^{\frac{\text{SIRdB}}{10}} \\
P_{00} = P(\text{SIR}|x_{\text{min}}, y_{\text{min}}) f_x(x_{\text{min}}) f_y(y_{\text{min}}) \\
P_{01} = P(\text{SIR}|x_{\text{max}}, y_{\text{min}}) f_x(x_{\text{max}}) f_y(y_{\text{min}}) \\
P_{10} = P(\text{SIR}|x_{\text{min}}, y_{\text{max}}) f_x(x_{\text{min}}) f_y(y_{\text{max}}) \\
P_{11} = P(\text{SIR}|x_{\text{max}}, y_{\text{max}}) f_x(x_{\text{max}}) f_y(y_{\text{max}}) \\
[U_n, A_n, L_n] = \text{RecursiveBisection}(\text{SIR}, \{P_{00}, P_{01}, P_{10}, P_{11}\}, \{x_{\text{min}}, x_{\text{max}}\}, \{y_{\text{min}}, y_{\text{max}}\}, \epsilon') \\
\text{PreviousError} = A_n \\
\text{end for} \\
\text{return } [U, A, L]
Bibliography


