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# Rational swept surface constructions based on differential and integral sweep curve properties

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**Abstract**

A *swept surface* is generated from a profile curve and a sweep curve by employing the latter to define a continuous family of transformations of the former. By using polynomial or rational curves, and specifying the homogeneous coordinates of the swept surface as bilinear forms in the profile and sweep curve homogeneous coordinates, the outcome is guaranteed to be a rational surface compatible with the prevailing data types of CAD systems. However, this approach does not accommodate many geometrically intuitive sweep operations based on differential or integral properties of the sweep curve — such as the parametric speed, tangent, normal, curvature, arc length, and offset curves — since they do not ordinarily have a rational dependence on the curve parameter. The use of Pythagorean–hodograph (PH) sweep curves surmounts this limitation, and thus makes possible a much richer spectrum of rational swept surface types. A number of representative examples are used to illustrate the diversity of these novel swept surface forms — including the oriented–translation sweep, offset–translation sweep, generalized conical sweep, and oriented–involute sweep. In many cases of practical interest, these forms also have rational offset surfaces. Considerations related to the automated CNC machining of these surfaces, using only their high–level procedural definitions, are also briefly discussed.

**Keywords**: swept surface; profile curve; sweep curve; homogeneous coordinates; rational surface; Pythagorean–hodograph curve; geometrical transformation.

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# 1 Introduction

*Sweep operations* provide an intuitive approach to constructing surfaces from two parametric curves, a *profile curve* $\mathbf{p}(u)$ and *sweep curve* $\mathbf{s}(v)$. The points of the sweep curve determine a family of transformations acting on the profile curve, such that the continuum of its transformed instances generates a *swept surface* $\mathbf{R}(u, v)$. In addition to the profile and sweep curves, a swept surface construction requires a precise specification of the nature of the geometrical transformation associated with each point of the sweep curve.

Two familiar cases are the *surface of extrusion* and *surface of revolution.* In the former case, the sweep curve is a line segment, whose points specify a family of translations of the profile curve. In the latter case, the sweep curve is a circular arc, whose points specify a family of rotations of the profile curve about an axis orthogonal to the plane of the circle. Ruled surfaces, quadrics, and cyclides are further elementary surfaces that can be defined by intuitive sweep operations. These are just the simplest examples of the diverse surface types that can be constructed using generalized sweep operations.

Although many current commercial CAD systems incorporate some swept surface capability, it is often of limited scope or incurs approximation of the precise surface geometry. The universal reliance of CAD systems on rational parametric surface representations limits the swept surface types that can be exactly generated, and necessitates the use of data–intensive approximations for those that cannot. The intent of the present study is to illustrate how the family of exact rational swept surface constructions can be greatly expanded by employing *Pythagorean–hodograph* (PH) *curves* [10] as sweep curves. The PH curves possess rational differential and integral properties — parametric speed, arc length, tangent and normal, curvature, and offset curves — that facilitate the construction of a rich variety of rational swept surfaces in which the geometrical transformations of the profile curve are explicitly dependent on these properties of the sweep curve, not just its point coordinates.

Perhaps the earliest systematic approach to swept surface constructions was proposed in the late 1970s by Hinds and Kuan in two little–known papers [24, 25] (see also Section 13.8 of [10]). In their methodology, an elegant matrix algebra is employed to directly construct the swept surface $\mathbf{R}(u, v)$ from the profile and sweep curves $\mathbf{p}(u)$ and $\mathbf{s}(v)$. Assuming that they are both rational cubic curves, they can be represented by $4 \times 4$ matrices in which one index is associated with a homogeneous coordinate component, and the other with the cubic basis functions. Similarly, $\mathbf{R}(u, v)$ can be represented as a rational bicubic surface through a $4 \times 4 \times 4$ matrix, in which one index is associated with a homogeneous coordinate component, and the others with cubic basis functions in $u$ and $v$. The $4 \times 4 \times 4$ matrix that defines $\mathbf{R}(u, v)$ is generated by contracting the $4 \times 4$ matrices defining $\mathbf{p}(u)$ and $\mathbf{s}(v)$ with a $4 \times 4 \times 4$ *selector matrix*, that serves to specify the nature of the geometrical transformations of the profile curve $\mathbf{p}(u)$ associated with each point of the sweep curve $\mathbf{s}(v)$.

The essence of this approach lies in judicious choice of the selector matrix elements, so as to implement the desired sweep operation. The coordinates of the sweep curve points may be used to define translations, scalings, rotations, and perspectivities of the profile curve — but only in certain combinations, determined by linear dependence of the homogeneous coordinates of $\mathbf{R}(u, v)$ on those of $\mathbf{p}(u)$ and $\mathbf{s}(v)$. For example, if the profile and sweep curves are planar, it is not possible to define a translational sweep of $\mathbf{p}(u)$ along $\mathbf{s}(v)$, such that $\mathbf{p}(u)$ always lies in the normal plane of $\mathbf{s}(v)$, since the orientation of this plane does not ordinarily have a rational dependence on $v$. However, if $\mathbf{s}(v)$ is a PH curve, it has a rational unit normal vector, so a rational surface $\mathbf{R}(u, v)$ defined by the above sweep operation becomes possible.

The approach proposed herein may be considered a generalization of the rational swept surface constructions introduced by Hinds and Kuan [24, 25] to accommodate geometrical transformations dependent on certain differential and integral properties of a Pythagorean–hodograph sweep curve $\mathbf{s}(v)$. This opens up a much richer class of swept surface constructions amenable to exact representation within the prevailing geometry data types of CAD systems. As in [24, 25] a key motivation is the ability to fabricate such surfaces on CNC machines, using only a high–level procedural definition of the surface. In this context, the ability to efficiently compute certain surface properties, such as the unit surface normal vector, is essential. However, the focus of this paper is on the construction of swept surfaces and computation of their geometrical properties. Since the CNC machining application incurs additional technical considerations, it will be deferred to a subsequent study.

The plan for the remainder of this paper is as follows. Section 2 reviews some preparatory material on the homogeneous–coordinate representation of rational curves and surfaces, the construction and properties of planar PH curves, and some basic principles for the construction and machining of swept surfaces. Section 3 describes the *scaled–rotation sweep*, a representative case of the "traditional" swept surface types defined by bilinear forms in the profile and sweep curve homogeneous coordinates, and some intrinsic limitations of these forms are identified. The generalization to sweep operations dependent on differential or integral properties of a (PH) sweep curve yields a remarkable variety of new possibilities. The goal of this paper is to introduce these novel surface types through illustrative examples of practical interest, rather than to attempt an exhaustive categorization. Consequently, Sections 4–7 present basic definitions, constructions, and examples for several important cases — the oriented–translation sweep, offset–translation sweep, oriented–involute sweep, and generalized conical sweep. For simplicity, these cases all employ planar sweep curves, but Section 8 briefly describes a type of swept surface that addresses the additional orientational freedom associated with a spatial sweep curve: the *rotation–minimizing* sweep. Finally, Section 9 summarizes and assesses the key contributions of this study, and identifies some promising directions for further extending and applying them.

# 2 Rational swept surfaces

The term *swept surface* (or *swept volume*) is often used to refer to the region defined as the union of the instances of a rigid body that executes a general spatial motion — such surfaces are of great importance in modeling material removal in CNC machining, collision avoidance in robotics, and related fields [2, 3, 5, 6, 26, 27, 36, 37]. A closely related concept is the *generalized cylinder* (or *cone*) that has seen widespread application as a basic shape primitive in computer vision and biomedical applications [1, 7, 20, 28, 29, 30, 32].

In the present context, we are interested in the process of using two curves to intuitively *design* a smooth surface, by imposing a continuous family of transformations on an "initial" (profile) curve, parameterized by the points of the other (sweep) curve. The emphasis is on achieving greater flexibility in the types of the allowed transformations, subject to the requirement that the outcome must be a *rational* surface. Before we introduce this generalized family of rational swept surfaces, we must review some basic facts concerning the homogeneous coordinate representation of rational curves and surfaces, the advantages of employing PH curves in sweep transformations, and key principles governing the construction and machining of such surfaces.

## 2.1 Homogeneous coordinates

A point in three–dimensional projective space is specified by its *homogeneous coordinates* $(W, X, Y, Z)$. If $W = 0$, these homogeneous coordinates identify a *point at infinity*. If $W \neq 0$, they identify a finite point, with the Cartesian coordinates $(x, y, z) = (X/W, Y/W, Z/W)$. Note that only the ratios of the homogeneous coordinates matter — i.e., $(W, X, Y, Z)$ and $(cW, cX, cY, cZ)$ identify exactly the same point in projective space for any value $c \neq 0$.

A *projective transformation* is a linear map in projective space, specified by a $4 \times 4$ matrix through the relation

$$\begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} \to \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} . \tag{1}$$

This embodies all *affine transformations*, in which no finite point is mapped to a point at infinity, and vice–versa — including shape–preserving mappings (translation, rotation, reflection, uniform scaling) and non–shape–preserving mappings (shear transformations); and also *perspectivities* in $\mathbb{R}^3$ — in which finite points may be mapped to points at infinity, and vice–versa.

A *rational curve* in $\mathbb{R}^3$ is defined by giving its homogeneous coordinates as polynomial functions $W(\xi), X(\xi), Y(\xi), Z(\xi)$ of a parameter $\xi$. The Cartesian coordinates of such a curve are defined by

$$(x(\xi), y(\xi), z(\xi)) \;=\; \left( \frac{X(\xi)}{W(\xi)}, \frac{Y(\xi)}{W(\xi)}, \frac{Z(\xi)}{W(\xi)} \right) .$$

A *polynomial curve*, corresponding to the special case $W(\xi) =$ constant, has a single point at infinity (namely, $\xi \to \pm\infty$). However, a rational curve may have many points at infinity, identified by the roots of the polynomial $W(\xi)$. Unlike polynomial curves, the set of rational curves of given degree is closed under general projective transformations. Rational forms are essential for an exact specification of basic curves, such as conic sections (the parabola is the only conic that can be exactly specified as a polynomial curve).

Similarly, a *rational surface* in $\mathbb{R}^3$ is defined by specifying its homogeneous coordinates as polynomial functions $W(u, v), X(u, v), Y(u, v), Z(u, v)$ of two surface parameters $u, v$. Such a surface has the Cartesian coordinates

$$(x(u, v), y(u, v), z(u, v)) \;=\; \left( \frac{X(u, v)}{W(u, v)}, \frac{Y(u, v)}{W(u, v)}, \frac{Z(u, v)}{W(u, v)} \right) .$$

A *polynomial surface* corresponds to the special case $W(u, v) =$ constant.

The swept surface approach in [24, 25] takes the homogeneous coordinates $W(u), X(u), Y(u), Z(u)$ of the profile curve $\mathbf{p}(u)$ as elements of the column vector on the right in (1), while the elements $m_{jk}$ of the $4 \times 4$ transformation matrix are chosen to be linearly dependent on the homogeneous coordinates $W(v), X(v), Y(v), Z(v)$ of the sweep curve $\mathbf{s}(v)$. The outcome of the matrix product (1) is then a rational surface specified by homogeneous coordinates $W(u,v), X(u,v), Y(u,v), Z(u,v)$ that are *bilinear* in $W(u), X(u), Y(u), Z(u)$ and $W(v), X(v), Y(v), Z(v)$. The swept surfaces considered herein generalize this approach by allowing the matrix elements $m_{jk}$ to also depend on certain differential/integral properties of the sweep curve $\mathbf{s}(v)$, which is chosen to be a PH curve so as to ensure that the outcome is a rational surface.

## 2.2 Pythagorean–hodograph curves

For a differentiable plane curve $\mathbf{r}(\xi) = (x(\xi), y(\xi))$ the parametric speed

$$\sigma(\xi) \,=\, \|\mathbf{r}'(\xi)\| \,=\, \sqrt{x'^2(\xi) + y'^2(\xi)} \,=\, \frac{\mathrm{d}s}{\mathrm{d}\xi} \tag{2}$$

is the rate of change of the arc length $s$ with respect to the parameter $\xi$. The unit tangent and normal vectors along $\mathbf{r}(\xi)$ are defined by

$$\mathbf{t}(\xi) \,=\, \frac{(x'(\xi), y'(\xi))}{\sigma(\xi)} \qquad \text{and} \qquad \mathbf{n}(\xi) \,=\, \frac{(y'(\xi), -x'(\xi))}{\sigma(\xi)}\,, \tag{3}$$

so that $\mathbf{n}(\xi) = \mathbf{t}(\xi) \times \mathbf{k}$, where $\mathbf{k}$ is a unit vector in the positive $z$ direction — i.e., $(\mathbf{n}(\xi), \mathbf{t}(\xi), \mathbf{k})$ is a right–handed orthonormal frame at each point of $\mathbf{r}(\xi)$. The variation of the vectors (3) along $\mathbf{r}(\xi)$ is characterized by the relations

$$\mathbf{t}'(\xi) \,=\, -\,\sigma(\xi)\,\kappa(\xi)\,\mathbf{n}(\xi) \qquad \text{and} \qquad \mathbf{n}'(\xi) \,=\, \sigma(\xi)\,\kappa(\xi)\,\mathbf{t}(\xi)\,, \tag{4}$$

where the (signed) curvature of $\mathbf{r}(\xi)$ is defined [23] by

$$\kappa(\xi) \,=\, \frac{(\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)) \cdot \mathbf{k}}{\sigma^3(\xi)} \,=\, \frac{x'(\xi)y''(\xi) - x''(\xi)y'(\xi)}{\sigma^3(\xi)}\,. \tag{5}$$

For an "ordinary" polynomial or rational curve $\mathbf{r}(\xi)$, the tangent and normal (3) and the curvature (5) are not rational functions of the parameter $\xi$, since the parametric speed (2) is the square–root of an irreducible polynomial or rational function in $\xi$. Moreover, the arc length integral

$$s(\xi) \,=\, \int_0^\xi \sigma(t)\,\mathrm{d}t \tag{6}$$

has (except in trivial cases) no closed–form analytic reduction, and must be approximated by means of numerical quadrature.

A Pythagorean–hodograph (PH) curve, on the other hand, incorporates a special algebraic structure in its derivative, ensuring that $\sigma(\xi)$ is a *polynomial* in $\xi$. Namely, $\mathbf{r}'(\xi) = (x'(\xi), y'(\xi))$ has components specified [18] in terms of relatively prime polynomials $u(\xi)$, $v(\xi)$ as

$$x'(\xi) \,=\, u^2(\xi) - v^2(\xi)\,, \quad y'(\xi) \,=\, 2\,u(\xi)v(\xi)\,. \tag{7}$$

This is sufficient and necessary[1] for satisfaction of the Pythagorean condition

$$x'^2(\xi) + y'^2(\xi) \,=\, \sigma^2(\xi)\,,$$

where the parametric speed is the polynomial $\sigma(\xi) = u^2(\xi)+v^2(\xi)$. Since $\sigma(\xi)$ is a *polynomial*, the tangent and normal vectors (3) and the curvature (5) of a PH curve are rationally dependent on the curve parameter $\xi$. Furthermore, the arc length (6) is a polynomial in $\xi$, since it is the integral of the polynomial parametric speed $\sigma(\xi)$. Finally, the offset curves

$$\mathbf{r}_d(\xi) \,=\, \mathbf{r}(\xi) \,+\, d\,\mathbf{n}(\xi) \tag{8}$$

at each (signed) distance $d$ from $\mathbf{r}(\xi)$, i.e., the loci obtained by displacing by distance $d$ in the normal direction from each curve point, are *rational* curves. When the sweep curve $\mathbf{s}(v)$ is specified as a PH curve, these properties allow one to prescribe sweep transformations that are explicitly dependent upon its parametric speed, tangent, normal, curvature, arc length, and offset curves, while ensuring that the result is always a *rational* swept surface.

The *complex form* [8], in which the coordinate components $x(\xi)$ and $y(\xi)$ of a plane curve are interpreted as the real and imaginary parts of a complex–valued function $\mathbf{r}(\xi) = x(\xi) + \mathrm{i}\,y(\xi)$, offers a simple means of incorporating the structure (7). Namely, any curve for which the hodograph $\mathbf{r}'(\xi) = \mathbf{w}^2(\xi)$ is the square of a complex polynomial $\mathbf{w}(\xi) = u(\xi)+\mathrm{i}\,v(\xi)$ is a PH curve. This approach facilitates the development of many algorithms for constructing and analyzing planar PH curves. PH quintics, for example, can be constructed as interpolants to first–order Hermite data [17]; as $C^2$ splines interpolating a sequence of points under prescribed end conditions [15]; or through several shape–constrained methods — e.g., see [9, 12, 22, 33, 34, 35].

[1] We focus here on *primitive* Pythagorean hodographs with $\gcd(x'(\xi), y'(\xi)) = \text{constant}$. Non–primitive examples can also be constructed by multiplying the expressions (7) with a third polynomial $w(\xi)$, but the resulting curves are singular at the real roots of $w(\xi)$.

In the present context, the sweep curve $\mathbf{s}(v)$ is usually defined as a planar PH quintic segment, obtained by integrating the hodograph $\mathbf{s}'(v) = \mathbf{w}^2(v)$ defined by a quadratic Bernstein–form polynomial

$$\mathbf{w}(v) \;=\; \mathbf{w}_0(1-v)^2 + \mathbf{w}_1 2(1-v)v + \mathbf{w}_2 v^2 \tag{9}$$

with complex coefficients $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$. Then the parametric speed of $\mathbf{s}(v)$ is the polynomial $\sigma(v) = |\mathbf{w}(v)|^2$, the arc length $s(v)$ is obtained by integrating this polynomial, and the tangent, normal, and curvature can be expressed as

$$\mathbf{t}(v) \;=\; \frac{\mathbf{w}^2(v)}{\sigma(v)}\,, \quad \mathbf{n}(v) \;=\; -\,\mathbf{i}\,\frac{\mathbf{w}^2(v)}{\sigma(v)}\,, \quad \kappa(v) \;=\; 2\,\frac{\mathrm{Im}(\overline{\mathbf{w}}(v)\mathbf{w}'(v))}{\sigma^2(v)}\,.$$

Complete details on the construction and properties of planar PH curves may be found in [10].

## 2.3 Rational swept surface constructions

For simplicity, we typically assume that the profile and sweep curves are both planar. This is not an essential assumption, but it makes the sweep operation easier for the designer to formulate and interpret. To obtain a rational swept surface through transformations of the profile curve $\mathbf{p}(u)$ that depend upon certain differential or integral properties of the sweep curve $\mathbf{s}(v)$, the latter must be a PH curve but the former may be any polynomial or rational curve.

Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be unit vectors in the $(x,y,z)$ directions. The profile curve is typically specified in the $(x,z)$ plane as $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$, and the sweep curve is specified in the $(x,y)$ plane as $\mathbf{s}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}$. For brevity, we use the same symbols $x, y, z, \sigma, s, \mathbf{t}, \mathbf{n}, \kappa$ to denote the coordinate components, parametric speed, arc length, tangent, normal, and curvature of the profile and sweep curves — the independent variable serves to identify which of the two curves they refer to. A surface $\mathbf{R}(u,v)$ generated from $\mathbf{p}(u)$ and $\mathbf{s}(v)$ by a specified sweep operation is defined by homogeneous coordinates $W(u,v)$, $X(u,v)$, $Y(u,v)$, $Z(u,v)$ such that

$$\mathbf{R}(u,v) \;=\; \frac{X(u,v)}{W(u,v)}\,\mathbf{i} + \frac{Y(u,v)}{W(u,v)}\,\mathbf{j} + \frac{Z(u,v)}{W(u,v)}\,\mathbf{k}\,. \tag{10}$$

To ensure that $\mathbf{R}(u,v)$ is a *rational* surface $W(u,v)$, $X(u,v)$, $Y(u,v)$, $Z(u,v)$ must be bivariate polynomials without common factors. The unit normal to the surface (10) is defined in terms of its partial derivatives by

$$\mathbf{N}(u,v) \,=\, \frac{\mathbf{R}_u(u,v)\times\mathbf{R}_v(u,v)}{\|\,\mathbf{R}_u(u,v)\times\mathbf{R}_v(u,v)\,\|} \tag{11}$$

at each point where $\mathbf{R}_u(u,v)$ and $\mathbf{R}_v(u,v)$ are linearly independent, and the *offset surface* at (signed) distance $r$ from $\mathbf{R}(u,v)$ is defined by

$$\mathbf{R}_r(u,v) \,=\, \mathbf{R}(u,v) + r\,\mathbf{N}(u,v)\,. \tag{12}$$

To implement a particular sweep operation, one must specify $W(u,v)$, $X(u,v)$, $Y(u,v)$, $Z(u,v)$ in terms of the quantites $x, y, z, \sigma, s, \mathbf{t}, \mathbf{n}, \kappa$ for the profile and sweep curves. Since the possibility of defining sweep operations in terms of both the point coordinates and differential/integral properties of a PH sweep curve $\mathbf{s}(v)$ opens up a vast array of rational swept surface types, we do not attempt a systematic categorization of them at present. Instead, the versatility of this approach is illustrated by several representative examples that are of the primary interest in practical design problems.

## 2.4 Real–time CNC interpolator

The ultimate goal of this study is to develop a methodology for designing and *machining* a wide variety of rational swept surfaces. The intent is to drive a computer numerical control (CNC) machine from the high–level procedural swept surface definition, rather than voluminous part programs comprising short linear/circular G code segments. We defer details on the machining to a future study, and consider here only some basic geometrical considerations arising in the context of the swept surface constructions.

Spherical (ball–end) tools are typically preferred for machining free–form surfaces, since they can cut any surface whose least concave principal radius of curvature is not less than the tool radius. The machine drives the center of the tool along a path displaced by the surface normal (11) at each point along the contact curve with the surface $\mathbf{R}(u,v)$, i.e., the tool center executes a path on the offset surface (12). The surface normal (11) is explicitly evaluated for each of the rational swept surface forms described below.

For simplicity, we initially choose the isoparametric curves $u =$ constant or $v =$ constant on the surface (10) as toolpaths. However, to ensure a given speed (or *feedrate*) of the tool contact point with the surface, it is necessary to compensate for the non–uniform nature of the parameterization. This is the function of the *real–time interpolator* algorithm [19, 31, 38]. The simplest (first–order) real–time interpolators, which are quite accurate for moderate feedrates and high sampling frequencies, need only the *parametric speed* (i.e., the rate of change of arc length with the curve parameter) along the toolpath. Hence, for each of the swept surfaces described below, the parametric speed along both sets of isoparametric curves will be evaluated.

# 3 Scaled–rotation sweep

To begin, we consider an example of the family of swept surfaces introduced by Hinds and Kuan [24, 25]. For these surfaces $\mathbf{R}(u,v)$ is a bilinear expression in the coordinate components of the profile curve $\mathbf{p}(u)$ and sweep curve $\mathbf{s}(v)$, but does not depend on their differential or integral properties.

For a profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$ in the $(x,z)$ plane and a sweep curve $\mathbf{s}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}$ in the $(x,y)$ plane, consider the surface

$$\mathbf{R}(u,v) \;=\; x(u)\,\mathbf{s}(v) + z(u)\,\mathbf{k}\,. \tag{13}$$

Defining the polar coordinates $\rho(v)$ and $\phi(v)$ of $\mathbf{s}(v)$ by

$$\rho(v) \;=\; \sqrt{x^2(v)+y^2(v)}\,, \quad \cos\phi(v) \;=\; \frac{x(v)}{\rho(v)}\,, \quad \sin\phi(v) \;=\; \frac{y(v)}{\rho(v)}\,, \tag{14}$$

the surface (13) can also be expressed in the form

$$\mathbf{R}(u,v) \;=\; x(u)\,\rho(v)\,[\,\cos\phi(v)\,\mathbf{i} + \sin\phi(v)\,\mathbf{j}\,] + z(u)\,\mathbf{k}\,.$$

Comparing with the profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$, the first term above arises from replacing the vector $\mathbf{i}$ by $\cos\phi(v)\,\mathbf{i} + \sin\phi(v)\,\mathbf{j}$, and multiplying by $\rho(v)$, while the second term is unchanged. Instance $v$ of the transformed profile curve thus lies in the plane spanned by $\cos\phi(v)\,\mathbf{i} + \sin\phi(v)\,\mathbf{j}$ and $\mathbf{k}$, instead of $\mathbf{i}$ and $\mathbf{k}$, and is scaled parallel to the $(x,y)$ plane by the factor $\rho(v)$. The family of these scaled/rotated instances of the profile curve, as $v$ varies, defines a *scaled–rotation* swept surface with homogeneous coordinates

$$\begin{aligned}
W(u,v) &= 1\,, \\
X(u,v) &= x(v)x(u)\,, \\
Y(u,v) &= y(v)x(u)\,, \\
Z(u,v) &= z(u)\,.
\end{aligned}$$

This is clearly a polynomial surface if $\mathbf{p}(u)$ and $\mathbf{s}(v)$ are polynomial curves, and a rational surface if they are rational curves.

The $u =$ constant isoparametric curves on the surface lie in planes parallel to the $(x, y)$ plane, at a height $z(u)$ above it. The $v =$ constant curves lie in planes orthogonal to the $(x, y)$ plane, spanned by the vectors $\mathbf{k}$ and $\mathbf{s}(v)$. To compute the unit normal (11) to the surface defined by (13), we note that its partial derivatives are

$$\mathbf{R}_u(u,v) \;=\; x'(u)\,\mathbf{s}(v) + z'(u)\,\mathbf{k}\,, \quad \mathbf{R}_v(u,v) \;=\; x(u)\,\mathbf{s}'(v)\,.$$

Substituting into (11) and simplifying, the surface normal can be written as

$$\mathbf{N}(u,v) \;=\; \mathrm{sign}(x(u))\,\frac{[\,x'(u)\mathbf{s}(v) + z'(u)\,\mathbf{k}\,] \times \mathbf{s}'(v)}{\|\,[\,x'(u)\mathbf{s}(v) + z'(u)\,\mathbf{k}\,] \times \mathbf{s}'(v)\,\|}\,. \tag{15}$$

Note that the surface normal is singular when $x(u) = 0$, since then $\mathbf{R}_v = \mathbf{0}$. Also, since (15) is not in general a rational expression in $u$ and $v$, the offset (12) to the swept surface (13) is not ordinarily a rational surface. The angle that the surface normal makes with the $(x, y)$ plane

$$\theta(u,v) \;=\; \mathrm{sign}(x(u))\;\sin^{-1}\frac{x'(u)\,(\mathbf{s}(v) \times \mathbf{s}'(v)) \cdot \mathbf{k}}{\|\,[\,x'(u)\mathbf{s}(v) + z'(u)\,\mathbf{k}\,] \times \mathbf{s}'(v)\,\|}\,.$$

In general, this angle is non–constant along both sets of isoparametric curves. The parametric speed along the $u =$ constant and $v =$ constant isoparametric curves is defined by the magnitudes of the derivatives $\mathbf{R}_v(u, v)$ and $\mathbf{R}_u(u, v)$ respectively, namely

$$\|\mathbf{R}_v(u,v)\| \;=\; |x(u)|\,\sigma(v)\,, \quad \|\mathbf{R}_u(u,v)\| \;=\; \sqrt{\rho^2(v)x'^2(u) + z'^2(u)}\,.$$

If the sweep curve $\mathbf{s}(v)$ is an arc of the unit circle centered on the origin, then $\rho(v) \equiv 1$, and the scaling of the profile curve $\mathbf{p}(u)$ parallel to the $(x, y)$ plane is suppressed. In this case we obtain a surface of revolution, generated by rotating $\mathbf{p}(u)$ about the $z$–axis. However, for any other sweep curve, it is not possible to "decouple" the rotation and scaling aspects of this category of sweep operations. It is also not possible to construct a sweep operation that involves orienting the profile curve using *differential* properties of the sweep curve, such as the tangent or normal vector. To achieve this, it is necessary to formulate sweeps in terms of not only the point coordinates of the sweep curve, but also their derivatives, and to do so in a manner that ensures the resulting swept surface has a rational parameterization.

**Example 1.** For the profile curve, we take the 90° circular arc in the $(x,z)$ plane defined by

$$\mathbf{p}(u) \,=\, \frac{(1-u^2)\,\mathbf{i} \,+\, 2u\,\mathbf{k}}{1+u^2}\,, \tag{16}$$

and the sweep curve is a cubic in the $(x,y)$ plane,

$$\mathbf{s}(v) \,=\, \mathbf{p}_0(1-v)^3 + \mathbf{p}_1 3(1-v)^2 v + \mathbf{p}_2 3(1-v)v^2 + \mathbf{p}_3 v^3\,,$$

with the Bézier control points $\mathbf{p}_0 = (1.0, 0.0)$, $\mathbf{p}_1 = (1.0, 0.8)$, $\mathbf{p}_2 = (0.8, 1.0)$, $\mathbf{p}_3 = (0.0, 1.0)$. Figure 1 illustrates the resulting scaled–rotation surface. The initial and final instances of the profile curve are circle arcs, but intermediate instances are ellipse arcs, since the scale factor $\rho(v) = \|\mathbf{s}(v)\|$ parallel to the $(x,y)$ plane is unity when $v=0$ and $v=1$, but exceeds unity at intermediate $v$ values — with a maximum of $\rho(\frac{1}{2}) \approx 1.1844$. The rational scaled–rotation swept surface is specified by the homogeneous coordinates

$$\begin{aligned}
W(u,v) \,&=\, 1+u^2\,, \\
X(u,v) \,&=\, (1-u^2)\,[\,(1-v)^3 + 3(1-v)^2 v + 2.4(1-v)v^2\,]\,, \\
Y(u,v) \,&=\, (1-u^2)\,[\,2.4(1-v)^2 v + 3(1-v)v^2 + v^3\,]\,, \\
Z(u,v) \,&=\, 2\,u\,.
\end{aligned}$$

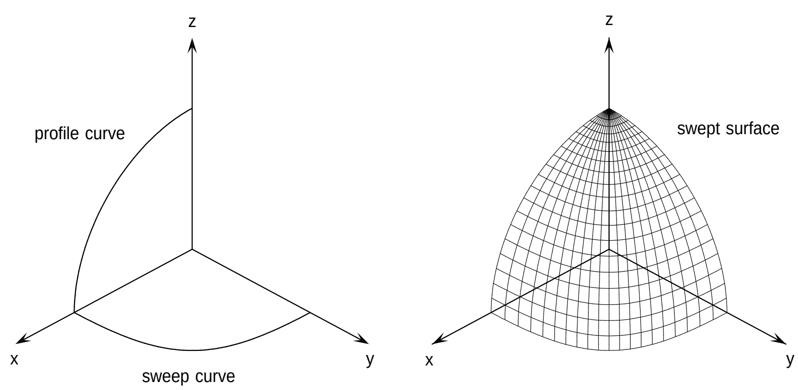


Figure 1: The scaled–rotation surface defined by a circular arc in the $(x,z)$ plane as the profile curve, and a cubic in the $(x,y)$ plane as the sweep curve.

## 4 Oriented–translation sweep

The swept surfaces introduced in [24, 25] may be generalized by considering forms that are bilinear in the coordinates of the profile and sweep curves *and* certain differential/integral properties. For brevity, we focus specifically on the case of most practical interest, in which the dependence of $\mathbf{R}(u,v)$ upon differential/intergal properties is limited to the sweep curve $\mathbf{s}(v)$, assumed to be a PH curve (so as to ensure a rational swept surface).

For a profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$ in the $(x,z)$ plane and a PH sweep curve $\mathbf{s}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}$ in the $(x,y)$ plane, consider the surface

$$\mathbf{R}(u,v) \;=\; \mathbf{s}(v) + x(u)\,\mathbf{n}(v) + z(u)\,\mathbf{k}\,. \tag{17}$$

Comparing this expression with the profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$, we see that the last two terms in (17) arise from replacing the fixed unit vector $\mathbf{i}$ with the varying unit normal $\mathbf{n}(v)$ of the sweep curve $\mathbf{s}(v)$. Instance $v$ of the transformed profile curve thus lies in the plane spanned by $\mathbf{k}$ and $\mathbf{n}(v)$, with normal $\mathbf{t}(v) = \mathbf{k} \times \mathbf{n}(v)$, rather than in the plane spanned by $\mathbf{k}$ and $\mathbf{i}$, with normal $\mathbf{j} = \mathbf{k} \times \mathbf{i}$. The first term in (17) represents a translation of this re–oriented copy of $\mathbf{p}(u)$ by the vector displacement $\mathbf{s}(v)$. As $v$ varies, the continuum of these oriented and translated instances of the profile curve generate an *oriented–translation* swept surface.

Intuitively, the surface $\mathbf{R}(u,v)$ is generated by a translation of the profile curve $\mathbf{p}(u)$ along the sweep curve $\mathbf{s}(v)$, coordinated with a rotation so as to maintain coincidence of the plane containing $\mathbf{p}(u)$ with the normal plane to $\mathbf{s}(v)$. To ensure that $\mathbf{R}(u,0) = \mathbf{p}(u)$, the sweep curve must satisfy

$$\mathbf{s}(0) \;=\; \mathbf{0} \qquad \text{and} \qquad \frac{\mathbf{s}'(0)}{\|\mathbf{s}'(0)\|} \;=\; \mathbf{j}\,.$$

Since $\mathbf{s}(v)$ is a PH curve, it has a rational unit normal $\mathbf{n}(v)$. Then if $\mathbf{p}(u)$ is a polynomial curve, the form (17) defines a *rational* surface, with homogeneous coordinate components

$$\begin{aligned}
W(u,v) &= \sigma(v)\,, \\
X(u,v) &= \sigma(v)x(v) + y'(v)x(u)\,, \\
Y(u,v) &= \sigma(v)y(v) - x'(v)x(u)\,, \\
Z(u,v) &= \sigma(v)\,z(u)\,.
\end{aligned} \tag{18}$$

The $u =$ constant isoparametric curves on this surface lie in planes parallel to the $(x, y)$ plane, at height $z(u)$ above it. The $v =$ constant curves lie in planes orthogonal to the $(x, y)$ plane, spanned by the vectors $\mathbf{k}$ and $\mathbf{n}(v)$.

To determine the unit surface normal, we set $\mathbf{s}'(v) = \sigma(v)\,\mathbf{t}(v)$ and use the relations (4) to obtain

$$\mathbf{R}_u(u,v) \;=\; x'(u)\,\mathbf{n}(v) + z'(u)\,\mathbf{k}\,, \quad \mathbf{R}_v(u,v) \;=\; \sigma(v)\,[\,1 + \kappa(v)x(u)\,]\,\mathbf{t}(v)\,.$$

Hence, the unit surface normal (11) is

$$\mathbf{N}(u,v) \;=\; \mathrm{sign}\,[\,1 + \kappa(v)\,x(u)\,]\,\frac{x'(u)\,\mathbf{k} - z'(u)\,\mathbf{n}(v)}{\sigma(u)}\,. \tag{19}$$

Note that $\mathbf{N}$ is singular when $1 + \kappa(v)\,x(u) = 0$, since we then have $\mathbf{R}_v = \mathbf{0}$. When $\mathbf{p}(u)$ is also a PH curve, the normal (19) has a rational dependence on $u$ and $v$, and hence the offset (12) is a *rational surface*. The angle that the surface normal makes with the $(x, y)$ plane, defined by

$$\theta(u,v) \;=\; \mathrm{sign}\,[\,1 + \kappa(v)\,x(u)\,]\,\sin^{-1}\frac{x'(u)}{\sigma(u)}\,,$$

is constant along the isoparametric curves $u =$ constant if $1 + \kappa(v)\,x(u) \neq 0$. The parametric speed along the $u =$ constant and $v =$ constant isoparametric curves is defined by

$$\|\mathbf{R}_v(u,v)\| \;=\; \sigma(v)\,|\,1 + \kappa(v)x(u)\,|\,, \quad \|\mathbf{R}_u(u,v)\| \;=\; \sigma(u)\,.$$

**Example 2.** As the profile curve, we again take the 90° circle arc (16) in the $(x, z)$ plane. For the sweep curve, we choose the planar PH quintic Hermite interpolant [17] to the $(x, y)$–plane data

$$\mathbf{s}(0) \;=\; (0,0)\,, \quad \mathbf{s}'(0) \;=\; (0,4) \quad \text{and} \quad \mathbf{s}(1) \;=\; (2,2)\,, \quad \mathbf{s}'(1) \;=\; (4,0)\,.$$

This curve is specified by integration of $\mathbf{s}'(v) = \mathbf{w}^2(v)$, where the complex quadratic polynomial (9) must have the Bernstein coefficients

$$\mathbf{w}_0 \;=\; \sqrt{2}(1+\mathrm{i})\,, \quad \mathbf{w}_1 \;=\; \frac{\sqrt{45\sqrt{2}+10}\;\mathrm{e}^{\mathrm{i}\pi/8} - 3(1+\mathrm{e}^{\mathrm{i}\pi/4})}{2}\,, \quad \mathbf{w}_2 \;=\; 2$$

in order to satisfy the prescribed Hermite data. From $\mathbf{w}(v)$, the parametric speed, arc length, tangent, normal, and curvature can all be determined.

Figure 2 illustrates the rational oriented–translation surface generated by the chosen profile and sweep curves. In the present example, $\mathbf{p}(u)$ is a rational quadratic curve, so $x(u)$ and $z(u)$ are both rational quadratic functions of $u$. Noting that $\sigma(v)$ is degree 4, and $x(v)$, $y(v)$ are of degree 5, by clearing denominators one can verify that the homogeneous coordinates (18) defining the rational swept surface are of degree $(2,4)$ in $(u,v)$ for $W(u,v)$ and $Z(u,v)$, and $(2,9)$ for $X(u,v)$ and $Y(u,v)$. The control points and weights defining the rational Bézier representation can be easily constructed.

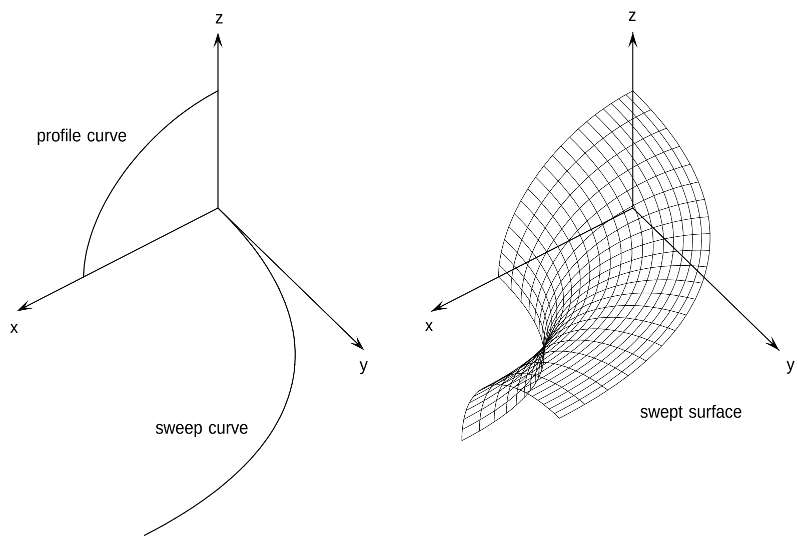


Figure 2: The oriented–translation surface generated by a circular arc profile curve in the $(x,z)$ plane, and a PH quintic sweep curve in the $(x,y)$ plane.

The generation of this swept surface through a continuous translation of the profile curve along the curve, coordinated with a continuous rotation so as to keep it within the sweep curve normal plane, is a geometrically intuitive design procedure with diverse practical applications. The ability to represent the outcome *exactly* as a rational swept surface eliminates the inaccuracy and data volume explosion incurred by prevailing approximation methods.

## 5 Offset–translation sweep

The offset (8) to a given plane curve is the locus defined by displacing each curve point by distance $d$ in the local normal direction. Offseting is a shape transformation that has important applications in specifying tool paths and tolerance zones about a nominal profile. We propose here a sweep operation that combines simultaneous offseting of a planar PH curve with translation orthogonal to its plane, based on a user–specified relation between the offset distance and translational displacement. This provides an intuitive approach to smoothly "rounding" the sharp corners of a solid model.

Consider a planar PH profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + y(u)\,\mathbf{j}$ with unit normal $\mathbf{n}(u)$. Instead of a sweep curve $\mathbf{s}(v)$, the present approach employs a scalar offset distance function $d(v)$. An *offset–translation* swept surface is defined in terms of $\mathbf{p}(u)$ and $d(v)$ by the expression

$$\mathbf{R}(u,v) \,=\, \mathbf{p}(u) + d(v)\,\mathbf{n}(u) + v\,\mathbf{k}\,. \tag{20}$$

For each $v$, the offset curve $\mathbf{p}(u) + d(v)\,\mathbf{n}(u)$ at distance $d(v)$ from the profile curve $\mathbf{p}(u)$ is translated by distance $v$ in the $z$–direction, and the continuum of these offset/translated instances of $\mathbf{p}(u)$ generates the surface (20).

Since $\mathbf{p}(u)$ is a PH curve, its normal $\mathbf{n}(u)$ has a rational dependence on $u$, and if $d(v)$ is a polynomial (or rational) function, the expression (20) defines a *rational* surface, specified by the homogeneous coordinate components

$$\begin{aligned}
W(u,v) &= \sigma(u)\,, \\
X(u,v) &= \sigma(u)x(u) + d(v)y'(u)\,, \\
Y(u,v) &= \sigma(u)y(u) - d(v)x'(u)\,, \\
Z(u,v) &= \sigma(u)\,v\,.
\end{aligned} \tag{21}$$

The $v =$ constant isoparametric curves on the surface (20) are parallel to the $(x, y)$ plane, at a height $v$ above it — they are simply the offsets at distance $d(v)$ to $\mathbf{p}(u)$, displaced vertically by distance $v$. The $u =$ constant curves lie in planes orthogonal to the $(x, y)$ plane, spanned by the vectors $\mathbf{k}$ and $\mathbf{n}(u)$ — these curves have the same shape, determined by the function $d(v)$. The case $d(v) =$ constant defines the linear extrusion of a single fixed–distance offset to $\mathbf{p}(u)$, orthogonal to its plane, while a linear function $d(v)$ defines a simple proportionality between the offset and translation distances.

The surface (20) has the partial derivatives

$$\mathbf{R}_u(u,v) \,=\, \mathbf{p}'(u) + d(v)\,\mathbf{n}'(u)\,, \quad \mathbf{R}_v(u,v) \,=\, d'(v)\,\mathbf{n}(u) + \mathbf{k}\,.$$

Setting $\mathbf{p}'(u) = \sigma(u)\,\mathbf{t}(u)$ and using the relations (4), the surface normal (11) can be expressed as

$$\mathbf{N}(u,v) \;=\; \mathrm{sign}\,[\,1+\kappa(u)\,d(v)\,]\,\frac{\mathbf{n}(u) - d'(v)\,\mathbf{k}}{\sqrt{1+d'^2(v)}}\,. \tag{22}$$

The angle that the surface normal makes with the $(x,y)$ plane is defined by

$$\theta(u,v) \;=\; \mathrm{sign}\,[\,1+\kappa(u)\,d(v)\,]\,\sin^{-1}\frac{-\,d'(v)}{\sqrt{1+d'^2(v)}}\,.$$

Clearly, $\mathbf{N}$ coincides with the profile curve normal $\mathbf{n}$ (i.e., $\theta = 0$) if and only if $d(v) = \text{constant}$, corresponding to a translational sweep of a unique offset to $\mathbf{p}(u)$. Otherwise, $\mathbf{N}$ has a non–zero but constant inclination with the $(x,y)$ plane along each of the isoparametric curves $v = \text{constant}$ on the surface (20). To guarantee a smooth surface patch, with a well–defined normal (22) at each point, the range of the offset distance function $d(v)$ should be restricted[2] so that $1+\kappa(u)\,d(v) \neq 0$ over the parameter domain of the profile curve $\mathbf{p}(u)$.

If $d(v)$ is linear in $v$, the surface normal (22) has a rational dependence on $u$ and $v$, and hence the offset (12) is a *rational surface*. However, if $d(v)$ has a quadratic or higher–order dependence on $v$, the offset (12) is not a rational surface, because of the denominator $\sqrt{1+d'^2(v)}$ in (22). The parametric speed along the $u = \text{constant}$ and $v = \text{constant}$ isoparametric curves is

$$\|\mathbf{R}_v(u,v)\| \;=\; \sqrt{1+d'^2(v)}\,, \quad \|\mathbf{R}_u(u,v)\| \;=\; \sigma(u)\,|\,1+\kappa(u)d(v)\,|\,.$$

**Example 3.** As the profile curve $\mathbf{p}(u)$, we choose the PH quintic Hermite interpolant to the $(x,y)$–plane data

$$\mathbf{p}(0) \;=\; (0,0)\,, \quad \mathbf{p}'(0) \;=\; (0,4) \quad \text{and} \quad \mathbf{p}(1) \;=\; (2,2)\,, \quad \mathbf{p}'(1) \;=\; (4,0)\,.$$

This is identical to the PH quintic used in Section 4 as the sweep curve, but in the present context it is regarded as a profile curve. Figure 3 illustrates the rational offset–translation surface (20) defined by this profile curve and the offset distance function $d(v) = v$. As noted above, in this case the offset surfaces (12) at each distance $r$ are also rational. The surface shape can be altered by choosing alternative offset distance functions $d(v)$. For example, Figure 4 shows the surfaces obtained with the choices $d(v) = v^2$ and $-v^2$.

[2] A point with curvature $\kappa = -1/d$ on a plane curve generates a *cusp* on its offset at distance $d$ — i.e., a sudden tangent reversal [16]. This is obviously undesirable in practice.

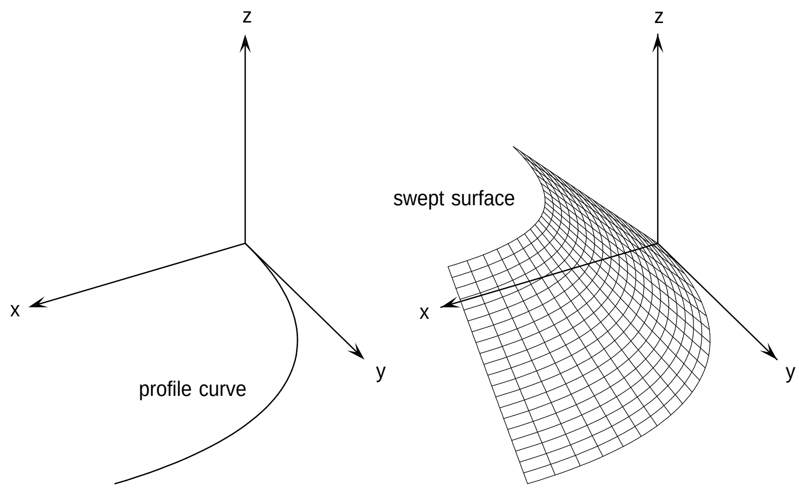


Figure 3: The offset–translation swept surface (20) generated by a PH quintic profile curve $\mathbf{p}(u)$ in the $(x, y)$ plane and offset distance function $d(v) = v$.

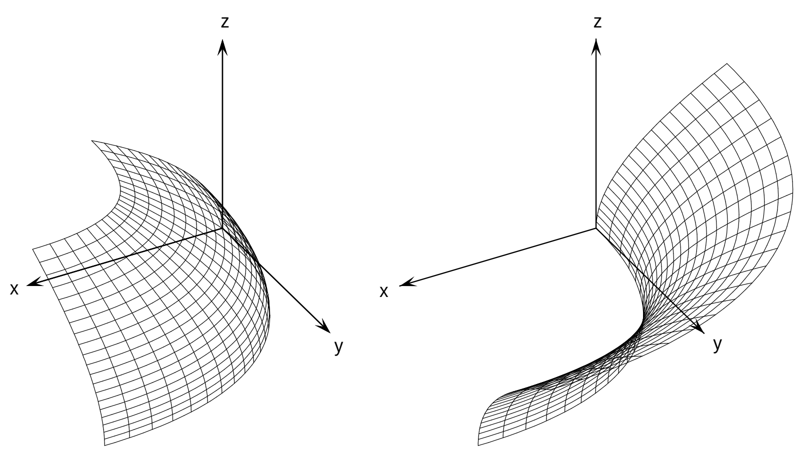


Figure 4: The offset–translation surface generated by the same PH quintic as in Figure 3, using distance functions $d(v) = v^2$ (left) and $d(v) = -\,v^2$ (right).

When $d(v)$ is a polynomial of degree $m$ in $v$, the homogeneous coordinates (21) of the rational offset–translation swept surface are degree $(4,0)$ in $(u,v)$ for $W(u,v)$; $(9,m)$ for $X(u,v)$ and $Y(u,v)$; and $(4,1)$ for $Z(u,v)$. This type of rational swept surface can be further generalized by choosing a distance function $d(u,v)$ dependent on both surface parameters.

**Remark 1.** Although the offset–translation sweep (20) seems conceptually distinct from the oriented–translation sweep (17), it can actually be regarded as a special case of the latter. To see this, consider the modified formulation

$$\mathbf{R}(u,v) \,=\, \mathbf{s}(v) + d(u)\,\mathbf{n}(v) + u\,\mathbf{k}\,, \tag{23}$$

of the former, defining a surface generated by the offsets at distance $d(u)$ to a planar sweep curve $\mathbf{s}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}$, displaced by distance $u$ in the $z$ direction. Comparing (17) and (23), they are seen to coincide with the choice $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k} = d(u)\,\mathbf{i} + u\,\mathbf{k}$ for the profile curve in the former.

# 6 Oriented–involute sweep

A distinctive feature of a PH curve is that the cumulative arc length can be exactly computed by simply evaluating a polynomial. This is a consequence of the fact that the parametric speed is a polynomial, and hence its integral (6) is also a polynomial. The cumulative arc length of a curve is important in constructing the *involutes* of that curve, i.e., loci whose centers of curvature lie on the given curve. Involutes play a key role in defining gear tooth profiles that ensure *conjugate action* (i.e., an exactly constant angular velocity ratio) of meshing gears. Using a PH curve for the sweep curve $\mathbf{s}(v)$ facilitates the construction of rational swept surfaces based on its involute.

Consider a profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$ in the $(x,z)$ plane, and a PH sweep curve $\mathbf{s}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}$ in the $(x,y)$ plane with the polynomial arc length function

$$s(v) \,=\, \int_0^v \|\mathbf{s}'(\xi)\|\,\mathrm{d}\xi\,.$$

The *involute* to $\mathbf{s}(v)$ is the locus defined by

$$\mathbf{c}(v) \,=\, \mathbf{s}(v) - s(v)\,\mathbf{t}(v)\,, \tag{24}$$

i.e., $\mathbf{c}(v)$ is generated by making a displacement, from each point of $\mathbf{s}(v)$, a distance $s(v)$ in a direction opposite to the curve tangent $\mathbf{t}(v)$ at that point.

There are actually infinitely–many involutes to a plane curve, since one may associate any curve point with the parameter value $v = 0$, from which the arc length is measured. Assuming $\mathbf{s}(v)$ is defined on the interval $v \in [\,0,1\,]$ we consider the particular involute defined by (24). Also, to ensure that $\mathbf{c}(v)$ is non–singular, we assume $\mathbf{s}(v)$ has positive curvature for $v \in [\,0,1\,]$. From (24) one can then verify that, at corresponding points, the tangent and normal to $\mathbf{c}(v)$ are the normal and negated tangent to $\mathbf{s}(v)$, respectively.

For the profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$ and involute (24) to the sweep curve $\mathbf{s}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}$, consider the swept surface defined by

$$\mathbf{R}(u,v) \;=\; \mathbf{c}(v) - x(u)\,\mathbf{t}(v) + z(u)\,\mathbf{k}\,. \tag{25}$$

Comparing the last two terms above with $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$, we see that $\mathbf{i}$ has been replaced by $-\,\mathbf{t}(v)$ in the first term, while the second term remains unchanged. Since $-\,\mathbf{t}(v)$ is the *normal* to the involute $\mathbf{c}(v)$, these two terms amount to a re–orientation of the profile curve so it lies in the normal plane to $\mathbf{c}(v)$ for each $v$. The first term in (25) defines a displacement of the profile curve by $\mathbf{c}(v)$ for each $v$. Hence, the surface (25) may be considered as arising from a continuous translation of the profile curve along the involute to the sweep curve, while re–orienting it so it always lies in the normal plane to the involute. This is called an *oriented–involute* swept surface. To ensure that $\mathbf{R}(u,0) = \mathbf{p}(u)$, the sweep curve must satisfy

$$\mathbf{s}(0) \;=\; \mathbf{0} \qquad \text{and} \qquad \frac{\mathbf{s}'(0)}{\|\mathbf{s}'(0)\|} \;=\; -\,\mathbf{i}\,.$$

The homogeneous coordinates for this surface are specified by

$$\begin{aligned}
W(u,v) \;&=\; \sigma(v)\,, \\
X(u,v) \;&=\; \sigma(v)x(v) - x'(v)\,[\,s(v) + x(u)\,]\,, \\
Y(u,v) \;&=\; \sigma(v)y(v) - y'(v)\,[\,s(v) + x(u)\,]\,, \\
Z(u,v) \;&=\; \sigma(v)\,z(u)\,.
\end{aligned} \tag{26}$$

Now $\sigma(v)$ and $s(v)$ are polynomials if $\mathbf{s}(v)$ is a PH curve, and these expressions define a rational surface when $\mathbf{p}(u)$ is a polynomial curve. The $u =$ constant isoparametric curves on this surface lie in planes parallel to the $(x,y)$ plane, at height $z(u)$ above it. The $v =$ constant isoparametric curves lie in planes orthogonal to the $(x,y)$ plane, spanned by the vectors $\mathbf{k}$ and $-\,\mathbf{t}(v)$.

Noting from (24) that $\mathbf{c}'(v) = s(v)\,\sigma(v)\,\kappa(v)\,\mathbf{n}(v)$, and that $s'(v) = \sigma(v)$, the derivatives of the surface (25) can be expressed as

$$\mathbf{R}_u(u,v) \,=\, z'(u)\,\mathbf{k} - x'(u)\,\mathbf{t}(v)\,, \quad \mathbf{R}_v(u,v) \,=\, [\,x(u) + s(v)\,]\,\sigma(v)\,\kappa(v)\,\mathbf{n}(v)\,.$$

Hence, under the assumption that the curvature of $\mathbf{s}(v)$ is positive, the unit normal (11) is given by

$$\mathbf{N}(u,v) \,=\, \mathrm{sign}[\,x(u) + s(v)\,]\,\frac{x'(u)\,\mathbf{k} + z'(u)\,\mathbf{t}(v)}{\sigma(u)}\,. \tag{27}$$

Note that $\mathbf{N}$ is singular when $x(u) + s(v) = 0$, since then $\mathbf{R}_v = \mathbf{0}$. If $\mathbf{p}(u)$ is also a PH curve, the surface normal (27) has a rational dependence on $(u,v)$ and the offsets (12) at each distance $r$ are therefore *rational surfaces*. The parametric speed along the $u$ = constant and $v$ = constant isoparametric curves is

$$\|\mathbf{R}_v(u,v)\| \,=\, |\,(x(u) + s(v))\kappa(v)\,|\,\sigma(v)\,, \quad \|\mathbf{R}_u(u,v)\| \,=\, \sigma(u)\,.$$

**Example 4.** As the profile curve, we again take the 90° circle arc (16) in the $(x,z)$ plane. For the sweep curve, we choose the planar PH quintic Hermite interpolant to the $(x,y)$–plane data

$$\mathbf{s}(0) \,=\, (0,0)\,, \quad \mathbf{s}'(0) \,=\, (-2,0) \quad \text{and} \quad \mathbf{s}(1) \,=\, (0,1)\,, \quad \mathbf{s}'(1) \,=\, (2,0)\,.$$

For this curve, the Bernstein coefficients of the polynomial (9) are

$$\mathbf{w}_0 \,=\, \sqrt{2}\,\mathrm{i}\,, \quad \mathbf{w}_1 \,=\, \frac{\sqrt{35}-3}{2\sqrt{2}}(1+\mathrm{i})\,, \quad \mathbf{w}_2 \,=\, \sqrt{2}\,.$$

Figure 5 shows the profile curve, the sweep curve and its involute, and the resulting oriented–involute swept surface. The homogeneous coordinates (26) of this surface are of degree $(2,4)$ in $(u,v)$ for $W(u,v)$ and $Z(u,v)$, and degree $(2,9)$ for $X(u,v)$ and $Y(u,v)$.

The involute (24) has the sweep curve $\mathbf{s}(v)$ as its *evolute* — i.e., its locus of centers of curvature. One may also construct rational swept surfaces based on the evolutes of PH sweep curves. The evolute $\mathbf{e}(v)$ to a given plane curve $\mathbf{s}(v)$ is defined[3] by

$$\mathbf{e}(v) \,=\, \mathbf{s}(v) \,-\, \rho(v)\,\mathbf{n}(v)\,, \tag{28}$$

[3]The minus sign in (28) arises from the convention adopted in Section 2, namely, that the normal $\mathbf{n}(v)$ points *away* from the center of curvature.

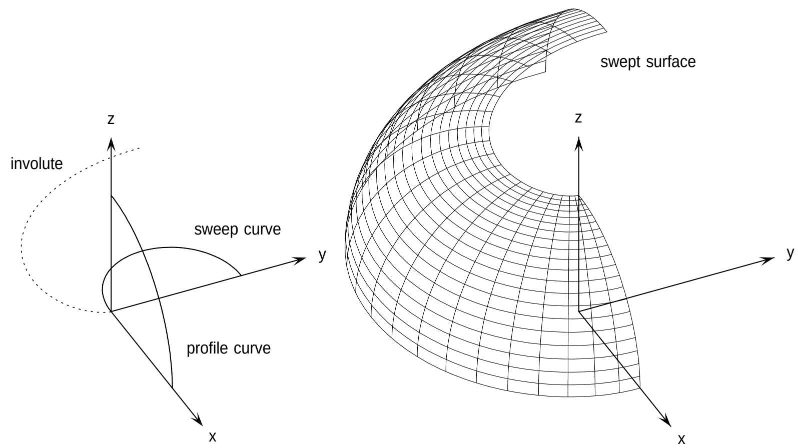


Figure 5: Left: a circle arc in the $(x, z)$ as the profile curve, and a PH quintic sweep curve in the $(x, y)$ plane, together with its involute (the dotted curve). Right: the oriented–involute swept surface generated by these two curves.

where $\mathbf{n}(v)$ is the unit normal to $\mathbf{s}(v)$, and $\rho(v) = 1/\kappa(v)$ is its (signed) radius of curvature. The expression (28) defines a rational curve when $\mathbf{s}(v)$ is a PH curve, since then $\kappa(v)$ and $\mathbf{n}(v)$ depend rationally on $v$. However, the sweep curve must be carefully chosen to ensure a finite smooth swept surface, since an *inflection* on $\mathbf{s}(v)$ (where $\kappa(v) = 0$) incurs a *point at infinity* on $\mathbf{e}(v)$, and a *vertex* on $\mathbf{s}(v)$ (where $\kappa'(v) = 0 \neq \kappa''(v)$) incurs a *cusp* on $\mathbf{e}(v)$.

# 7 Generalized conical sweep

The oriented–translation sweep (see Section 4) was defined by continuously translating the profile curve $\mathbf{p}(u) = x(u)\,\mathbf{i} + z(u)\,\mathbf{k}$ along the PH sweep curve $\mathbf{s}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}$, while orienting it so as to reside in the normal plane of the sweep curve at each point. A *generalized conical sweep* surface is defined by further imposing a uniform scaling of the profile curve, linearly dependent on the sweep curve arc length $s(v)$. Assuming $\mathbf{s}(v)$ is defined on $v \in [\,0, 1\,]$ with total arc length $S = s(1)$, we introduce the scaling function

$$c(v) \,=\, \frac{c_0(S - s(v)) + c_1 s(v)}{S}\,, \tag{29}$$

where $c_0$ and $c_1$ are prescribed initial and final scale factors, chosen such that $c(v) > 0$ for $v \in [\,0,1\,]$. The generalized conical sweep is then defined by

$$\mathbf{R}(u,v) \,=\, \mathbf{s}(v) + c(v)\,[\,x(u)\,\mathbf{n}(v) + z(u)\,\mathbf{k}\,]\,. \tag{30}$$

For each $v$, the translated/oriented instance of the profile curve $\mathbf{p}(u)$ is scaled by the factor $c(v)$. To ensure that $\mathbf{R}(u,0) = \mathbf{p}(u)$, we must have

$$\mathbf{s}(0) \,=\, \mathbf{0}\,, \qquad \frac{\mathbf{s}'(0)}{\|\mathbf{s}'(0)\|} \,=\, \mathbf{j}\,, \qquad c_0 \,=\, 1\,.$$

Since $\mathbf{s}(v)$ is a PH curve, it has a polynomial parametric speed $\sigma(v)$ and arc length $s(v)$, and a rational unit normal $\mathbf{n}(v)$. Then if $\mathbf{p}(u)$ is a polynomial curve, the form (30) defines a *rational* surface, with homogeneous coordinates

$$\begin{aligned}
W(u,v) &= S\sigma(v)\,, \\
X(u,v) &= S\sigma(v)x(v) + [\,c_0 S + (c_1 - c_0)s(v)\,]\,y'(v)x(u)\,, \\
Y(u,v) &= S\sigma(v)y(v) - [\,c_0 S + (c_1 - c_0)s(v)\,]\,x'(v)x(u)\,, \\
Z(u,v) &= [\,c_0 S + (c_1 - c_0)\,s(v)\,]\,\sigma(v)z(u)\,.
\end{aligned} \tag{31}$$

Because of the scale factor $c(v)$, the $u =$ constant isoparametric curves on this surface are, in general, non–planar. The $v =$ constant curves are instances of the profile curve $\mathbf{p}(u)$ displaced by the amount $\mathbf{s}(v)$, oriented in the sweep curve normal plane at that point, and scaled by the factor $c(v)$.

The surface (30) has the partial derivatives

$$\begin{aligned}
\mathbf{R}_u(u,v) &= c(v)\,[\,x'(u)\,\mathbf{n}(v) + z'(u)\,\mathbf{k}\,]\,, \\
\mathbf{R}_v(u,v) &= \sigma(v)\,[\,1 + c(v)\kappa(v)x(u)\,]\,\mathbf{t}(v) + c'(v)\,[\,x(u)\,\mathbf{n}(v) + z(u)\,\mathbf{k}\,]\,,
\end{aligned}$$

with the cross product

$$\begin{aligned}
\mathbf{R}_u(u,v) \times \mathbf{R}_v(u,v) &= \sigma(v)c(v)c'(v)\,[\,x(u)z'(u) - x'(u)z(u)\,]\,\mathbf{t}(v) \\
&\quad + \sigma(v)c(v)\,[\,1 + c(v)\kappa(v)x(u)\,]\,[\,x'(u)\,\mathbf{k} - z'(u)\,\mathbf{n}(v)\,]\,.
\end{aligned}$$

Hence, assuming that $c(v) \neq 0$ and setting

$$f(u,v) \,=\, c'(v)\,[\,x(u)z'(u) - x'(u)z(u)\,]\,, \quad g(u,v) \,=\, 1 + c(v)\kappa(v)x(u)\,,$$

the surface normal can be expressed as

$$\mathbf{N}(u,v) \,=\, \frac{f(u,v)\,\mathbf{t}(v) + g(u,v)\,[\,x'(u)\,\mathbf{k} - z'(u)\,\mathbf{n}(v)\,]}{\sqrt{f^2(u,v) + \sigma^2(u)\,g^2(u,v)}}\,. \tag{32}$$

This is clearly not, in general, a rational expression in $u$ and $v$, so the offset (12) is not a rational surface. Setting $\rho(u) = \sqrt{x^2(u) + z^2(u)}$, the parametric speed along the $u =$ constant and $v =$ constant isoparametric curves is

$$\begin{aligned} \|\mathbf{R}_v(u,v)\| &= \sqrt{\sigma^2(v)\,[\,1 + c(v)\kappa(v)x(u)\,]^2 + c'^2(v)\rho^2(u)}\,, \\ \|\mathbf{R}_u(u,v)\| &= c(v)\,\sigma(u)\,. \end{aligned}$$

**Example 5.** As an example of the generalized conical sweep surface, we use the same profile and sweep curves as in Section 4, together with scale factors $c_0 = 1$ and $c_1 = \frac{1}{2}$ in (29). The resulting rational swept surface is illustrated in Figure 6. The $v =$ constant isoparametric curves on this surface are all circle arcs, displaced to position $\mathbf{s}(v)$ along the sweep curve, oriented in the normal plane at that point, and scaled linearly with respect to arc length $s(v)$ by the factor (29). In the present case $x(u)$, $z(u)$ are quadratic rational functions, while $\sigma(v)$ and $x(v)$, $y(v)$, $s(v)$ are polynomials of degree 4 and 5, respectively. Consequently, the homogeneous coordinates (31) that describe the rational swept surface are of degree $(2,4)$ in $(u,v)$ for $W(u,v)$; $(2,9)$ for $X(u,v)$ and $Y(u,v)$; and $(2,5)$ for $Z(u,v)$. The control points and weights defining the rational Bézier form can be easily computed.

This type of swept surface may be generalized by invoking a quadratic or higher–order dependence of the scaling factor $c(v)$ on the arc length $s(v)$ of the sweep curve, in lieu of the simple linear form (29). For example

$$c(v) \;=\; \frac{c_0(S - s(v))^2 + c_1 2(S - s(v))s(v) + c_2 s^2(v)}{S^2} \tag{33}$$

specifies a quadratic dependence with initial/final values $c(0) = c_0$, $c(1) = c_2$ while the coefficient $c_1$ controls the intermediate variation. Figure 7 shows examples of the surface shapes obtained with the scaling function (33), using the same profile and sweep curves as in Figure 6.

# 8 Rotation–minimizing spatial sweep

To facilitate ease of interpretation, the swept surface constructions described above were restricted to planar profile and sweep curves. As previously noted, this restriction is not essential, but the generalization to space curves requires greater care in specifying the sweep process, and entails difficulties that are absent in the planar case. For example, to specify a spatial PH sweep curve,

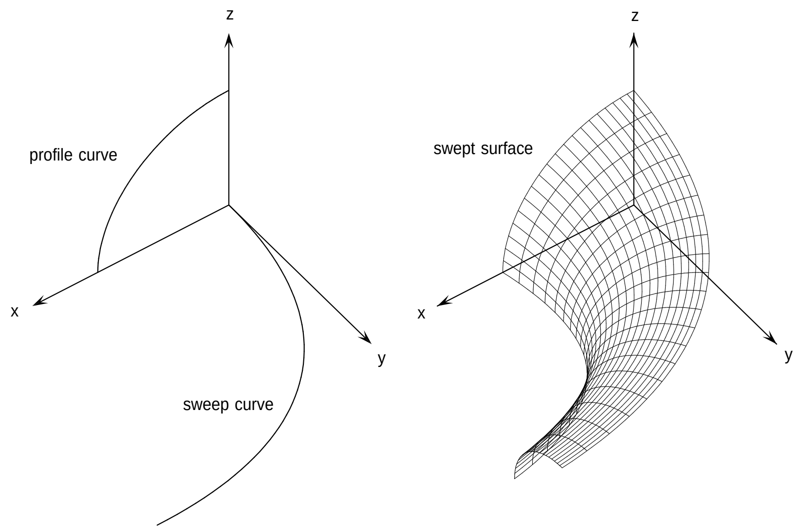


Figure 6: The generalized conical swept surface (30) generated by a circular profile curve $\mathbf{p}(u)$ in the $(x,z)$ plane and a PH quintic sweep curve $\mathbf{s}(v)$ in the $(x,y)$ plane, with factors $c_0 = 1$ and $c_1 = \frac{1}{2}$ in the scaling function (29).

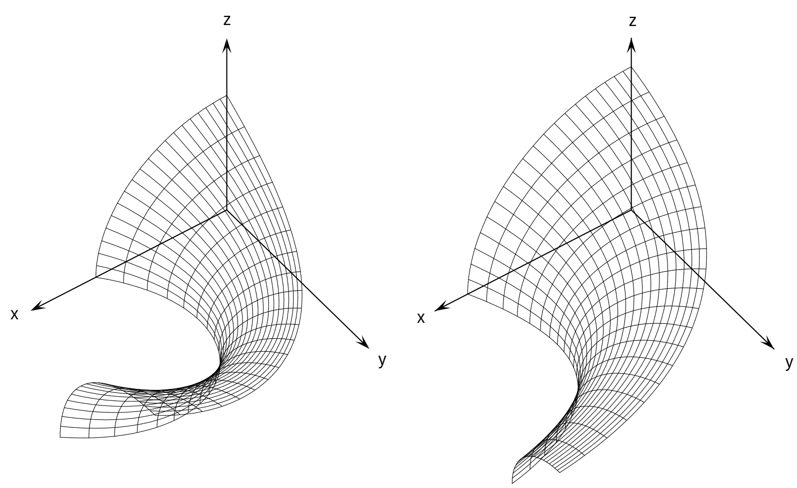


Figure 7: Generalized conical swept surfaces generated by a circular profile curve $\mathbf{p}(u)$ in the $(x,z)$ plane and a PH quintic sweep curve $\mathbf{s}(v)$ in the $(x,y)$ plane, using the quadratic scaling function (33) with coefficients $(c_0, c_1, c_2) = (1.0, 0.0, 1.0)$ on the left, and $(c_0, c_1, c_2) = (1.25, 0.75, 0.50)$ on the right.

the quaternion representation [4, 14] must be invoked in place of the complex form for planar PH curves discussed in Section 2.

A thorough treatment of swept surfaces based on spatial PH sweep curves is beyond the scope of the present paper. Instead, the noteworthy case of a rational rotation–minimizing motion of a planar profile curve along a spatial sweep curve is used to briefly illustrate some of the issues encountered in this context. In Section 4, the oriented–translation swept surface was defined by translating a planar profile curve $\mathbf{p}(u)$ along a planar sweep curve $\mathbf{s}(v)$, while orienting it so as to always lie in the sweep curve normal plane, spanned by $\mathbf{n}(v)$ and $\mathbf{k}$. During this motion, the profile curve exhibits no instantaneous rotation within the normal plane, i.e., about the sweep curve tangent $\mathbf{t}(v)$.

If a spatial sweep curve $\mathbf{s}(v)$ is employed, the normal plane is spanned by the *binormal* and *principal normal* vectors, defined [23] by

$$\mathbf{b}(v) \,=\, \frac{\mathbf{s}'(v)\times\mathbf{s}''(v)}{\|\,\mathbf{s}'(v)\times\mathbf{s}''(v)\,\|} \qquad \text{and} \qquad \mathbf{n}(v) \,=\, \mathbf{b}(v)\times\mathbf{t}(v)\,,$$

$\mathbf{t}(v)=\mathbf{s}'(v)/\|\mathbf{s}'(v)\|$ being the tangent. The vectors $(\mathbf{t}(v),\mathbf{n}(v),\mathbf{b}(v))$ define the orthonormal *Frenet frame* along $\mathbf{s}(v)$, but this frame is not suitable for defining the orientation of a profile curve in the normal plane in attempting to generalize the oriented–translation swept surface, since $\mathbf{n}(v)$ and $\mathbf{b}(v)$ are not rational in $v$ and have, in general, a non–zero angular velocity about $\mathbf{t}(v)$. To remedy these problems, we need to identify curves that admit orthonormal frames $(\mathbf{t}(v),\mathbf{f}(v),\mathbf{g}(v))$ in which the normal–plane vectors $\mathbf{f}(v)$ and $\mathbf{g}(v)$ are rational in $v$ and have no instantaneous rotation about $\mathbf{t}(v)$.

The polynomial curves that have this property are a subset of the spatial PH curves, called *rational rotation–minimizing frame* (RRMF) *curves*. The simplest non–trivial instances are the RRMF quintics, and are identified by a simple algebraic constraint on the coefficients of the quadratic quaternion polynomials that generate them [11]. The RRMF quintics have been used in the design of rational rotation–minimizing spatial motions, interpolating prescribed initial/final locations and orientations of a rigid body [13].

Figure 8 shows how the normal–plane orientation of a profile curve that is swept along a space curve can strongly influence the resulting surface shape. In this case, the profile curve is an ellipse and the sweep curve is an RRMF quintic. If the Frenet frame is used to specify the normal–plane orientation of the sweep curve, the result has an undesirable "twisted" shape and is not a rational surface. However, using the rational rotation–minimizing frame of the RRMF quintic yields a rational swept surface with a very natural shape (the variation of the profile curve orientation is the least possible, consistent with the constraint that it must always lie in the sweep curve normal plane).

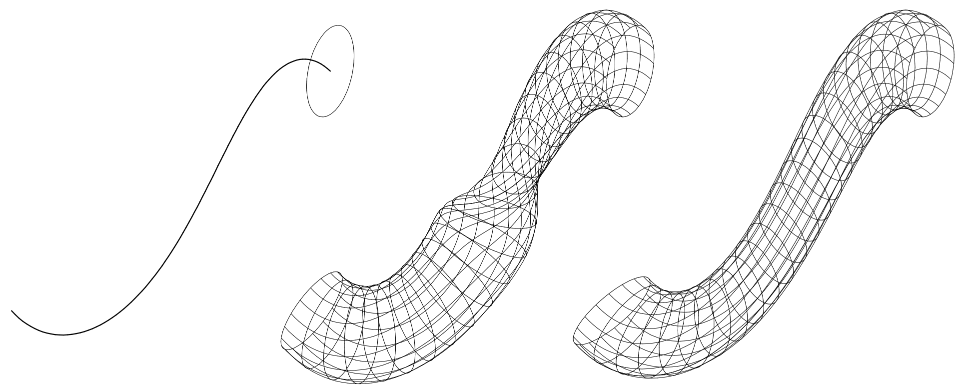

Figure 8: The tubular surface generated by sweeping an ellipse profile curve along an RRMF quintic sweep curve (left). The profile curve lies in the sweep curve normal plane at each point, and within this plane it is oriented using either the Frenet frame (center) or the rotation–minimizing frame (right).

# 9 Closure

The use of Pythagorean–hodograph sweep curves opens up a rich spectrum of rational swept surface constructions, arising from families of transformations of a profile curve that depend explicitly on differential and integral properties of the sweep curve. A number of representative examples have been presented herein, to illustrate the diversity of surface geometries that can be intuitively generated in a manner exactly compatible with the prevailing representation schemes of modern CAD systems. Preliminary considerations regarding the automated CNC machining of such surfaces, based only on their high–level procedural definitions, have also been presented (the technical aspects of this topic will be more fully addressed in a forthcoming study). The versatility of these generalized rational swept surfaces makes their complete categorization a non–trivial task, that is worthy of further investigation.

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