Lawrence Berkeley National Laboratory

Recent Work

Title
NUMERICAL STUDY OF THE DISCRETE MINIMAL SURFACE EQUATION IN A NONCONVEX DOMAIN

Permalink
https://escholarship.org/uc/item/7zc3f1s3

Author
Concus, Paul.

Publication Date
1973-06-01
NUMERICAL STUDY OF THE DISCRETE MINIMAL SURFACE EQUATION IN A NONCONVEX DOMAIN

Paul Concus

June 1973

Prepared for the U.S. Atomic Energy Commission
under Contract W-7405-ENG-48

For Reference
Not to be taken from this room
DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.
Numerical study of the discrete minimal surface equation in a nonconvex domain

Paul Concus

Lawrence Berkeley Laboratory
University of California
Berkeley, California

ABSTRACT

A discrete approximation to the minimal surface equation is solved numerically on an L-shaped domain with Dirichlet boundary conditions. The nature of the solution discontinuity at the vertex of the reentrant corner is depicted graphically.

1. In this study, a discrete approximation to the minimal surface equation

\[
\frac{\partial}{\partial x} \left( \frac{u_x}{W} \right) + \frac{\partial}{\partial y} \left( \frac{u_y}{W} \right) = 0 , \quad W = (1+u_x^2+u_y^2)^{\frac{1}{2}},
\]

is solved numerically on an L-shaped domain D with Dirichlet boundary conditions (see Fig. 1a). The domain D is the unit square \((0,1) \times (0,1)\) with the smaller square \([\frac{1}{2}, 1] \times [\frac{1}{2}, 1]\) deleted from it, and the boundary conditions are that \(u = 0\) on the outside border of D and that \(u\) increase linearly to the value \(c\) at \((\frac{1}{2}, \frac{1}{2})\) along the reentrant legs. That is,

\[
u = 0 \quad \text{on} \quad \begin{cases}
x = 0 \\
y = 0 \\
x = 1, \quad 0 < y < \frac{1}{2} \\
y = 1, \quad 0 < x < \frac{1}{2} \\
\end{cases}
\]

\[
u = 2c(1-x) \quad \text{on} \quad y = \frac{1}{2}, \quad \frac{1}{2} \leq x \leq 1,
\]

and \(u = 2c(1-y) \quad \text{on} \quad x = \frac{1}{2}, \quad \frac{1}{2} \leq y \leq 1.
\]

The analogous Dirichlet problem on the nonconvex quadrilateral considered by Radó [5, 6] is discussed by Nitsche [3, 4], who proved that a solution \(u \in C^2(D) \cap C^0(D)\) does not exist for any value of \(c > 0\). A solution does exist, however, if it is permitted to have a discontinuity at the vertex of
the reentrant corner. It is the purpose of the present study to indicate the behavior of such a solution surface in the neighborhood of a reentrant corner. An L-shaped domain, rather than the quadrilateral one, is chosen here because of its practical convenience: it can be subdivided by means of a uniform square mesh with no special treatment required at the boundary.

2. The discrete approximation used is that given in [1, 2]. Let \( U_{ij} \) denote the approximating value to \( u(x,y) \) at the node point \( x_i = ih, y_j = jh \) on the uniform square mesh of width \( h = \frac{1}{N} (N \text{ an integer}) \); then in place of (1) one has the system of nonlinear algebraic equations

\[
F_{ij} = \frac{1}{W_{ij}} \left( 2U_{ij} - U_{i-1,j} - U_{i,j-1} + U_{i-1,j-1} \right) + \frac{1}{W_{i+1,j}} \left( 2U_{ij} - U_{i+1,j} - U_{i,j} + U_{i-1,j+1} \right) \\
+ \frac{1}{W_{i,j+1}} \left( 2U_{ij} - U_{i-1,j} - U_{i,j+1} + U_{i+1,j} \right) + \frac{1}{W_{i+1,j+1}} \left( 2U_{ij} - U_{i+1,j} - U_{i,j+1} \right) = 0.
\]

In (3)

\[
W_{ij} = (1 + \left[ \frac{u_x^2 + u_y^2}{i,j} \right])^{\frac{1}{2}}
\]

denotes \( W \) for the mesh cell with center \((1, \frac{1}{2}, j, \frac{1}{2})\), evaluated by use of

\[
\left[ u_x^2 + u_y^2 \right]_{i,j} = \frac{1}{2h^2} \left[ (U_{ij} - U_{i-1,j})^2 + (U_{ij} - U_{i,j-1})^2 + (U_{i,j-1} - U_{i-1,j})^2 \\
+ (U_{i-1,j} - U_{i-1,j-1})^2 \right].
\]

The system of equations (3) can be derived directly from the variational integral \( A = \iint_D W \, dx \, dy \) by using (4) to obtain the corresponding discrete sum and then by setting equal to zero the partial derivatives with respect to the unknown nodal values of \( U \). Equation (3) is to be solved for the interior nodal values of \( U \) subject to (2) at the boundary nodes.
3. The numerical solution of (3) was carried out using the technique of block nonlinear successive overrelaxation described in [2]. Approximate solutions, accurate to within $10^{-5}c$, were obtained for $c = 0.1(0.1)1.0$ and for mesh spacings $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$. The results are displayed graphically in Figs. 2-9. (The automatic plotter that prepared the figures interpolated linearly between the data points.)

In Fig. 2 are plotted the values of $U_{ij}$ for $c = 1.0$ along the line segment $l_1$ ($x-y = 0$ - see Fig. 1b) for the four mesh spacings used. These curves indicate the behavior of the numerical solution as $h$ is reduced. Figs. 3 and 4 depict the corresponding values of $U_{ij}$ for $c = 1.0$ along the line segments $l_2$ and $l_3$ ($y = \frac{1}{2}$ and $x + y = 1$, resp. - see Fig. 1b). Note that, as one would expect for the original problem, the solutions of the discrete problems have the greatest jump in approaching $(\frac{1}{2}, \frac{1}{2})$ along the direction of $l_1$.

Figs. 5-7 depict the analogous graphs for $c = 0.1$.

In Fig. 8 are depicted the values of $U_{ij}$ for $h = \frac{1}{40}$ along the line $l_1$ for $c = 0.1(0.1)1.0$. Note that, as in the previous graphs, the vertical scale is normalized to $c$. These curves illustrate the behavior of the jump discontinuity at $(\frac{1}{2}, \frac{1}{2})$ as a function of $c$.

Finally, in Fig. 9 is shown the extrapolated estimate as a function of $c$ for the limiting value $u(\frac{1}{2}, \frac{1}{2})$ along the line $l_1$. The limiting values $u(\frac{1}{2}, \frac{1}{2})$ in this figure were obtained for each $c$ by passing a parabola through the computed approximation to $u(\frac{1}{2}-h, \frac{1}{2}-h)$ for the cases with $h = \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$ and extrapolating to $h = 0$. In addition to the values for $c = 0.1(0.1)1.0$, those for $c = 0.05$ and $c = 1.5$ were calculated as well.
4. A rigorous estimate for the accuracy with which Fig. 9 represents the solution of the original non-discrete problem (1, 2) would be very difficult to obtain, because of the complications introduced by the discontinuity at \((\frac{1}{2}, \frac{1}{2})\). However simple heuristic checks give evidence that the extrapolation of the numerical results to \(h = 0\) can legitimately be carried out. One such check that was used was to include the value of \(u(\frac{1}{2}-h, \frac{1}{2}-h)\) for \(h = \frac{1}{10}\) in the extrapolation to \(h = 0\), using a cubic polynomial to estimate the limit. These limits differed from those plotted in Fig. 9 by less than one percent.

The author expresses his appreciation to J.C.C. Nitsche, who, in conversations, suggested that these computations be carried out and to D. Johnson, who prepared the computer plots from the tabular data. This work was performed under the auspices of the U.S. Atomic Energy Commission.
[1] P. Concus, Numerical solution of the minimal surface equation, 

[2] P. Concus, Numerical solution of the minimal surface equation by block 
nonlinear successive overrelaxation, Information Processing 1968, 


Springer-Verlag, Berlin.

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720
Figure 1a

The L-shaped domain
Figure 1b

Line segments $l_1$, $l_2$, $l_3$ along which tabular data are plotted
Figure 2.  \( U \) as a function of distance along \( h \), for \( c = 1.0 \). Curves from top to bottom are for \( h = 10, 20, 40, \) and \( 80 \).
Figure 3. $U$ as a function of distance along $L_d$ for $c = 1.0$. Curves from top to bottom are for $h = 10, 20, 40, \text{ and } 80$. 
Figure 4. $U$ as a function of distance along $l_3$ for $c = 1.0$. Curves from top to bottom are for $h = 10, 20, 40$, and $80$. 
Figure 5. Same as Fig. 2 except that \( c = 0.1 \)
Figure 6. Same as Fig. 3 except that $c = 0.1$
Figure 7. Same as Fig. 4 except that $c = 0.1$
Figure 8. $U$ as a function of distance along $l_1$ for $h = \frac{1}{40}$. Curves from top to bottom are for $c = 0.1, 0.2, \ldots, 1.0$. 
Figure 9. Estimate of $\lim_{h \to 0} u(1/2-h, 1/2-h)$ as a function of $c$. 
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.