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SMOOTHNESS OF HOROCYCLE FOLIATIONS

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1. Introduction

Let SM denote the unit tangent bundle of a compact $C^\infty$ Riemannian manifold $M$. Suppose that $M$ has everywhere negative sectional curvature. In [1] Anosov proved that the geodesic flow $\varphi$ on SM is of a certain type, called "Anosov" by later writers, and defined below. Associated with any Anosov flow $\varphi$ is a foliation by "strong stable manifolds"; this is called the horocycle foliation in the special case where $\varphi$ is the geodesic flow on SM and $M$ has negative curvature. The strong unstable manifold provide another isomorphic horocycle foliation.

The leaves of these foliations are as smooth as the Anosov flow $\varphi$, but Anosov showed that the foliations are not in general of class $C^1$, even when $\varphi$ is real analytic. However, when $M$ has dimension two or the curvature is $C^2$, we shall prove that the horocycle foliations (and even their tangent plane fields) are of class $C^2$. In the course of the proof, the fact that "negative curvature $\Rightarrow$ Anosov geodesic flow" fails out naturally. Our methods in §§ 5, 6 resemble those of Anosov and Sinai [2].

This smoothness result was suggested to us by an analogous situation we encountered in [5]; there, we showed that the strong stable manifold foliation of an Anosov diffeomorphism $f$ of class $C^3$ provided that either the strong stable manifolds have codimension one in $M$ or the spectrum of $Df$ is "bunched". These cases are analogous to (i), (ii) below.

Thanks are due to Pat Eberlein, Rob Gardiner, Leon Green, and Joe Wolf for helpful conversations.

2. The smoothness theorem

Let $M$ be a $C^\infty$ compact boundaryless manifold with a $C^\infty$ Riemann structure $\mathcal{F}$. The geodesics of $\mathcal{F}$ give rise to the geodesic flow $\varphi$ on the tangent bundle $TM$ of $M$.

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It is amusing that, to mean "geometric", Russian mathematicians, such as Anosov, use a word translated from Russian to English as "rough". Here is an example where roughness is likely to be generic.
if \( \nu \in TM \) and \( t \to \bar{r}(t) \) is the unique \( \mathbb{R} \)-geodesic with 
\[
\bar{r}(0) = \nu, \quad \text{then} \quad \varphi(t) = \bar{r}(t) \in T_{\varphi(t)}M.
\]
\( \varphi \) is tangent to a vector field \( X \), called the geodesic spray. Geodesics have constant speed, so \( \varphi \) preserves the unit sphere bundle \( SM \) of \( TM \).

The geodesic flow \( \varphi \) on \( SM \) is Anosov if there is a continuous splitting \( T(SM) = E' \oplus E' \ominus E' \), invariant under the tangent flow \( T \varphi \) on \( T(SM) \), such that \( E' \) is the subbundle spanned by the geodesic spray \( X \), \( T \varphi \) exponentially expands \( E' \), and \( T \varphi \) exponentially contracts \( E' \). This means that for some (hence any) Riemann structure or Finsler on \( T(SM) \), there are constants \( C, c > 0, \lambda > 1 \) such that
\[
|T \varphi(x)| \geq c \lambda^t |x| \quad \text{if} \quad x \in E' \quad \text{and} \quad t \geq 0, \\
|T \varphi(x)| \leq C \lambda^{-t} |x| \quad \text{if} \quad x \in E' \quad \text{and} \quad t \geq 0.
\]
The subbundle \( E' \), \( E'' \) are known to be uniquely integrable. They are tangent to the horocycle foliations. Thus, to prove the horocycle foliations are of class \( C^2 \), it suffices to prove \( E' \), \( E'' \) are of class \( C^2 \).

The sectional curvature of \( \mathbb{R} \) at a 2-plane \( P \subset T_pM \) is \( K_s(P) = \frac{K_s((P))}{|p|} \) the Gaussian curvature of \( \exp_p(P) \) relative to the inclusion-induced Riemannian structure. If \( K_s(P) < 0 \) for all \( p \in M \) and all 2-planes \( P \subset T_pM \), then \( \mathbb{R} \) is said to have negative curvature.

**Definition.** The curvature of \( \mathbb{R} \) is absolutely \( \psi \)-pinched if
\[
\alpha < \inf (K_s((P)) / K_s((P))').
\]
The inf is taken over all \( P, P' \in M \) and all 2-planes \( P, P' \) in \( T_pM, T_{p'}M \). The curvature of \( \mathbb{R} \) is relatively \( \psi \)-pinched if
\[
\alpha < \inf (K_s((M)) / K_s((M))).
\]
The inf is taken over all \( p, p' \in M \) and all 2-planes \( P, P' \) in \( T_pM \).

**Smoothness Theorem.** Let \( \mathbb{R} \) be a Riemannian structure on \( M \). If either

(i) the curvature of \( \mathbb{R} \) is negative and \( M \) has dimension two or

(ii) the curvature of \( \mathbb{R} \) is negative and absolutely \( \psi \)-pinched, then the Anosov splitting \( T(SM) = E' \oplus E' \ominus E' \) for the geodesic flow is of class \( C^2 \).

In particular, the horocycle foliations are of class \( C^2 \). Under natural uniformly bounded curvature assumptions on the curvature, compactness of \( M \) can be relaxed to completeness.


**Question.** Is this theorem true for relative \( \psi \)-pinching? If it is, then it includes (i) and (ii) as special cases. For negative curvature on a 2-manifold is
always relatively 0-pinch for all \( \alpha < 1 \). Originally we were sure this world "follow easily" from the \( C^1 \) section theorem (see below), but now we doubt it. Also we conjecture that there are many cases when the horocycle foliation is not of class \( C^1 \). Even if the curvature is 1-pinch, we expect the horocycle foliations are hardy ever of class \( C^1 \). Such results might follow from methods of R. Metz who proved a converse to the \( C^1 \) section theorem [13]. Anosov said the horocycle foliation is "obviously not smooth in general" [1, p. 12].

### 3. Background

In [9] we proved, with Mike Shub, a general theorem giving sufficient conditions for an invariant section of a bundle to be smooth. Let \( E \) be a \( C^r \) finite dimensional vector bundle over the compact \( C^r \) manifold \( M \). Assume \( E \) has a Finsler (= continuous family of norms on fibres). Let \( D \) be a disc subbundle of \( E \).

**Definition.** The minimum norm (also called the contour) of an operator \( A \) is \( m(A) = \inf_{\|x\| = 1} \|Ax\| = \{A^{-1}\}^+ \).

**Definition.** An \( r \)-fiber contraction is a \( C^r \) fibre map \( F: D \to D \) covering a \( C^1 \) diffeomorphism \( f: M \to M \) such that for some Finslers on \( E \) and \( TM \)

\[
\sup_{\gamma \in \mathbb{H}} k_p \sigma^j \gamma < 1, \quad 0 \leq j \leq r,
\]

where \( k_p \) is the Lipschitz constant of \( F(D_p)D_p \) is the \( D \)-fiber at \( p \in M \), and \( \sigma_p = m(\gamma_f) \).

\( k_p \) is the fiber contraction rate; \( \sigma_p \) is the base contraction rate. The assumption \( \sup_{\gamma \in \mathbb{H}} k_p \sigma^j \gamma < 1 \) implies \( F \) uniformly contracts the \( D \)-fibers (let \( j = 0 \)) and contracts \( D_p \) more sharply than it contracts the base at \( p \) (let \( j = 1 \)).

**\( C^r \) section theorem.** If \( F \) is an \( r \)-fiber contraction of \( D, r \geq 0 \) then there is a unique \( F \)-invariant section \( \sigma : M \to D \). Besides, \( \sigma \) is of class \( C^r \).

This is a central result of [9].

A second concept we use from [8] is that of the "graph-transform" \( F_g \).

If \( F: D \to D \) is a fiber map as above, then \( F \) induces a natural map \( F_g \), sec \((D) \subseteq \) on the sections of \( D \) defined by \( F_g(s) = F \circ s \circ f^{-1}\). This can be re-expressed as

\[
\text{image}(F_\sigma) = F(\text{image} \sigma).
\]

Finally, we use the uniqueness of the hyperbolic splitting of a hyperbolic bundle automorphism. This result is part of [9, 2.9].

### 4. Proof of (i)

Let \( X \) be the geodesic spray generating the geodesic flow \( \sigma \). Then \( \sigma(t) \) preserves the subbundle \( T_0(M) \) orthogonal to \( X \). And, since the Anosov splitting is unique,
Since $E$ is a smooth bundle, we can approximate $E^1, E^2$ by smooth subbundles $E^1, E^2$ of $E$. Let $\mathcal{G}$ be the smooth bundle over $SM$ whose fiber at $\psi$ is

$$\mathcal{G}_\psi = \{ (G \in L(\hat{E}^1, \hat{E}^2) : |G| \leq 1) \}.$$

Put the "max Finser" on $T(SM)$ so that

$$|z| = \max \{ \max_{\mu \neq 0} |a_{\mu}|, |\gamma_{\mu}| \},$$

where $z = x \oplus w \oplus y \in E^1 \oplus \text{span} X(\nu) \oplus E^2$, and $|\cdot|_\lambda$ is length respecting $\mathcal{G}$. This is a Finser on the base-space of $\mathcal{G}$.

Since $\mathcal{F}_\psi$ preserves $E^1 \oplus E^2 = \hat{E}^1 \oplus \hat{E}^2$, the $T \mathcal{F}_\psi$-graph transform $(T \mathcal{F}_\psi)_\psi$ is a fiber map $\mathcal{G} \rightarrow \mathcal{G}$ covering $\psi$, the time-one map of the geodesic flow. $(T \mathcal{F}_\psi)_\psi$ is defined by

$$(T \mathcal{F}_\psi)(\text{graph } G) = \text{graph}((T \mathcal{F}_\psi)_\psi G), \quad G \in \mathcal{G},$$

where $G = \{ x + G(\nu) \in \hat{E}^1 \oplus \hat{E}^2 \}$. Let $T \mathcal{F}^1 = T \mathcal{F}_\psi | E^1, T \mathcal{F}^2 = T \mathcal{F}_\psi | E^2$. The fiber $\mathcal{G}_\psi$ is contracted at a rate $\approx \| T \mathcal{F}_\psi \| m(T \mathcal{F}_\psi)^{-1}$, and the base is contracted at the rate $\approx m(T \mathcal{F}_\psi)$. (To say this about the base-map we need the max Finser.) The hypothesis of the $C^1$ section theorem ($s = 1$) is that (fiber contraction) $\times$ (base contraction)$^{-1} < 1$, and we have shown this product to be

$$\| T \mathcal{F}_\psi \| m(T \mathcal{F}_\psi)^{-1} \cdot (m(T \mathcal{F}_\psi))^{-1} = m(T \mathcal{F}_\psi)^{-1} < 1,$$

since $E^2$ is one-dimensional. Hence the unique $(T \mathcal{F}_\psi)_\psi$-invariant section of $\mathcal{G}$ is of class $C^1$. The section whose graphs give $E^2$ is clearly invariant, since $E^2$ is $T \mathcal{F}_\psi$-invariant. Hence $E^2 \in C^1$. Symmetrically, $E^1 \in C^1$.

Remarks. If for any other reason $\text{bol}(T \mathcal{F}_\psi)m(T \mathcal{F}_\psi)^{-1} < 1$, then we get $E^2 \in C^1$. By bol( ) we mean the "bollicity" which measures how nonconformal an isomorphism is:

$$\text{bol}(T) = \frac{\| T \|}{\| T \|} = \sup_{x, y \neq 0} \left\{ \frac{|T x|}{|T y|} \right\} = \frac{\| T \|}{\| T \|} = 1.$$

5. Second order linear differential equations

To prove (ii) we need good norm-estimates on $T \mathcal{F}_\psi, T \mathcal{F}_\psi^2$: the next lemma will provide them. By $\mathcal{S}(\mathbb{R}^n)$ we mean symmetric linear endomorphisms of $\mathbb{R}^n$, i.e., self adjoint operators. By $\mathcal{S}^*(\mathbb{R}^n)$ we mean the convex cone of positive or negative definite operators.

Lemma 1. Suppose $i \rightarrow F_i$ is a continuous map $R \rightarrow \mathcal{S}(\mathbb{R}^n)$, and $\alpha, \beta$ are positive constants with
Let $\Phi$ be the flow on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ generated by the artificially autonomous differential equation

$$\dot{\tau} = 1, \quad \dot{x} = y, \quad \dot{y} = P_\tau x; \quad \tau \in \mathbb{R}, \quad x, y \in \mathbb{R}^n.$$

Then there exists a unique $\Phi$-invariant splitting $E_1 \oplus E_t = \tau \times \mathbb{R}^n$ such that $E_1, E_t$ are graphs of uniformly bounded linear maps $\mathbb{R}^n \to \mathbb{R}^n$. Besides

$$E_1 = \text{graph} \ G_1, \ G_1 \in \mathcal{L}^\omega_\tau (\mathbb{R}^n), \quad a^{\omega_\tau} < \|G_1(x, \cdot)\| < b^{\omega_\tau} ;$$

$$E_t = \text{graph} \ G_t, \ G_t \in \mathcal{L}^\omega_\tau (\mathbb{R}^n), \quad a^{\omega_\tau} < \|G_t(x, \cdot)\| < b^{\omega_\tau}$$

for all $x \in \mathbb{R}^n$ with $|x| = 1$. This splitting $E^\omega \oplus E^\tau$ of the product bundle $\mathbb{R} \times \mathbb{R}^n$ exhibits the hyperbolicity of $\Phi$. Norms on $E^\omega, E^\tau$ can be chosen, which are uniformly equivalent to the induced norms and make

$$e^{\omega_\tau} < \|\Phi_t \| < e^{b^{\omega_\tau}}, \quad e^{a^{\omega_\tau}} < m(\Phi_t) \| \Phi_t \| < e^{b^{\omega_\tau}}$$

for all $t > 0$. If $P$ has period $\omega$, then so do $E^\omega$ and $E^\tau$.

**Remark.** A special case of this lemma is enlightening. Consider the autonomous constant coefficient linear differential equation:

$$\dot{x} = y, \quad \dot{y} = px, \quad p > 0$$

arising from the second order equation $\ddot{x} = px$. This vector field on $\mathbb{R}^2$ generates the linear flow

$$t \mapsto \Phi_t = \begin{bmatrix} \cosh (pt) & \sinh (pt) \\ p \sinh (pt) & \cosh (pt) \end{bmatrix},$$

which has the constant invariant splitting

$$E^\omega = \{(x, px) : x \in \mathbb{R}\}, \quad E^\tau = \{(x, -px) : x \in \mathbb{R}\}.$$

It is a delightful coincidence that the hyperbolic trigonometric functions occur in a hyperbolic flow, and that this flow represents the tangent flow on the standard Poincaré hyperbolic plane (when $p = 1$).

**Proof of Lemma 1.** The flow $\Phi$ on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ naturally induces a (local) flow $\Phi_t$ on $\mathbb{R} \times \mathcal{G}_L (\mathbb{R})$ as follows. Fix $\tau \in \mathbb{R}$. For each $S \in \mathcal{G}_L (\mathbb{R})$ put $\Phi_t (\tau, S) = (\tau + t, S)$. Here $S_t$ is the unique linear map $\mathbb{R}^n \to \mathbb{R}^n$ such that

$$(\tau + t) \times \text{graph} (S_t) = \Phi_t (\tau \times \text{graph} S).$$
When $S = S_i$ is fixed and $i$ is small, $S_i$ is well defined.

Fix $\tau$ and consider the solution $w_{\tau} \in C_{[C_i, D_i]}$ of

$$W = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} w_{\tau}, \quad w_{\tau} = I.$$ 

Thus $\Phi_i : \tau \times R^p \times R^q = W_{\tau}$. If $t > 0$ is small, then

$$S_t = (C_i + D_i)t(A_i + B_iS_i)^{-1}.$$ 

The tangent to the curve $S_t$ is

$$\frac{dS_t}{dt} = (C + D_iS_i(A + BS_i))^{-1} - (C + D_iS_i(A + BS_i))^{-1}(A + BS_i)(A + BS_i)^{-1}.$$ 

At $t = 0$ this reduces to $F_P - S'$ since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} C & D \\ P_A & P_B \end{bmatrix}, \quad \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Thus the flow $\Phi_i$ is tangential to the vector field (on $R \times GL(n)$) given by $(r, S) \mapsto (0, P - S')$. (Note that its integral curves are solutions to the Riccati equation $S = P - S'$.) Since this vector field is tangent to $R \times \mathcal{G}(R^n)$ by inspection, the flow $\Phi_i$ leaves $R \times \mathcal{G}(R^n)$ invariant.

We claim that all points of the boundary $\partial(R \times \mathcal{G}(s))$ are strict ingress points for $\Phi_i$, where

$$\mathcal{G}(s) = \{ s \in \mathcal{G} : e^{\alpha s} \leq s(x, \xi) \leq e^{\beta s} \text{ for all } x \in R^n, |x| = 1 \}.$$ 

A boundary point $p$ of a region $U$ is a strict ingress point for a local flow $\varphi$ if $\varphi_{0} p \in \text{Int}(U)$ for all small $t > 0$. This is an idea due to Ważewski.

For $x \in R^n$ and $S \in \mathcal{G}$ we have

$$S_t = y_t, \quad x_t = x \in R^n,$$

$$y_t = P_x, \quad y_t = S_t x_t,$$

and compute

$$\frac{d}{dt} \langle S_t x_t, x_t \rangle = \langle (S_t x_t + S_t^* x_t) + (S_t x_t, x_t) \rangle \langle x_t, x_t \rangle - \langle (S_t x_t, x_t) + \langle (S_t x_t, x_t) \rangle \langle x_t, x_t \rangle \rangle = \langle (P_t x_t + S_t x_t) + (S_t x_t, x_t) = 2\langle S_t x_t, x_t \rangle \langle x_t, x_t \rangle \rangle.
For small $t$, $x \mapsto x_t$ defines an embedding of the unit sphere $S^{n-1}$ of $R^n$ into $R^n$ which is near the identity; therefore it is surjective. This implies that

\[ \inf_{i \leq i+1} \langle \tilde{x}_i, x_t \rangle = \inf_{i \leq i+1} \frac{\langle \tilde{x}_i, x_t \rangle}{\langle x_i, x_t \rangle} \]

for small $t$.

Choose $\alpha_t, \alpha_i, \beta_i, \beta_t$ such that

\[ \alpha < \alpha_t < \alpha_i \leq \inf \{ m(P_z) \}, \quad \sup \{ P_z \} \leq \beta_i < \beta_t < \beta, \]

\[ \alpha_t = \alpha < \alpha_i - \alpha, \quad \beta - \beta_i < \beta_t - \beta. \]

Since $P_z$ is symmetric, $\langle P_z, x \rangle \geq \alpha_i |x|^2$.

Suppose $S \in \tilde{B}_n$ and consider the sets

\[ X_S(S) = \{ x \in S^{n-1} : \langle x, x \rangle < \alpha_i |x|^2 \}, \]

\[ X_S(S) = \{ x \in S^{n-1} : \beta - \langle x, x \rangle \leq \beta - \beta_i \}. \]

For each $x \in X_S(S)$ we have from (1)

\[ \frac{d}{dt} \bigg|_{t=\tau} \frac{\langle \tilde{x}_i, x_t \rangle}{\langle x_i, x_t \rangle} = \langle P_x, x \rangle + \langle S_x, x \rangle - 2 \langle S_x, x \rangle^2. \]

It follows from (2) that if $x \in X_S(S)$, then

\[ \langle S_x, x \rangle > \alpha_i^{1/2} \]

for all small $i > 0$.

But if $x \in S^{n-1} - X_S(S)$ and $t$ is small, then

\[ \langle S_t x, x \rangle \approx \langle S_x, x \rangle > \alpha_i^{1/2} \]

by continuity. Thus (3) holds for all $x \in S^{n-1}$, that is,

\[ \inf_{i \leq i+1} \langle S_t x, x \rangle > \alpha_i^{1/2} \]

for all small $t > 0$.

The same reasoning proves that also

\[ \sup_{i \leq i+1} \langle S_t x, x \rangle < \beta_t^{1/2} \]

for all small $t > 0$.

This shows that $x \times S$ is a strict ingress point of $\tilde{R}(\mathcal{U} \times S_t)$ for the local flow $\tilde{\Phi}$. 

The set $\mathcal{S}_\tau$ is a compact convex subset of the (finite dimensional) linear space $\mathcal{S}$. All the points of its boundary were shown to be strict ingress points. Since $\mathcal{S}(R \times \mathcal{S})$ is not a retract of $\mathcal{S}_\tau$, Ważewski's Principle [6, p. 279] says there must be a trajectory of $\Phi_t$ remaining in $R \times \mathcal{S}$ for all time. Let $\tau \to \tau \times G' \subset \mathcal{S}$ be such a trajectory, and set $E_\tau = \text{graph } G'$. $\tau \in \mathcal{R}$. Clearly $G'$ is interior to $\mathcal{S}_\tau$, and $\Phi_t(E_\tau) = E_t$. Let $\mathcal{S}_{\tau \tau} = \{ \xi \in \mathcal{S}; \alpha^{\xi \tau} \leq \langle -\xi, x \rangle \leq \beta^{\xi \tau} \text{ for all } x \in \mathbb{R}^n, |x| = 1 \}$. Then all points of $\partial \mathcal{S}(R \times \mathcal{S})$ are strict egress points. This can be seen by some reasoning similar to the above. Again by Ważewski's Principle, there is a $\Phi_t$-trajectory remaining in $\mathcal{S}_{\tau \tau}$ for all time. This gives $G_0; E_\tau$ as claimed and completes the existence part of Lemma 1.

Uniqueness of $E_t; E_\tau$ follows from hyperbolicity of $\Phi$ and Hirsch-Pugh-Shub [9, 2.9]. To prove hyperbolicity and the asserted estimates on its strength, we introduce the new inner product in $\mathbb{R}^n \times \mathbb{R}^n$ by setting

$$\langle z^i, z'^i \rangle = \langle x^i, x'^i \rangle, z^i = (x^i, y^i) \in \mathbb{R}^n \times \mathbb{R}^n; \quad i = 1, 2.$$ 

By restriction we get new inner products on each $E_t, E_\tau (\xi \in \mathcal{R})$. This makes $x \rightarrow (x, G' x), x \rightarrow (x, G \xi)$ isometries of $\mathbb{R}^n$ onto $E_t, E_\tau$.

Denote $\Phi_t(x, \xi)$ by $(x + \xi, \xi)$ and put $z_t = (x_t, y_t) \in \mathbb{R}^n \times \mathbb{R}^n$. Then

$$x_t = y_t, \quad y_t = P_t x_t,$$

and so

$$\begin{align*}
\frac{d}{dt} \langle z_t, z_t \rangle &= \frac{d}{dt} \langle x_t, x_t \rangle = 2 \langle x_t, y_t \rangle \\
&= 2 \langle x_t, x_t \rangle = \langle x_t, G_t x_t \rangle
\end{align*}$$

by invariance of $E_t$. Since $G_t \in \mathcal{S}_{\tau \tau}$, this last quantity lies between $2\alpha^{\tau \alpha}$ and $2\beta^{\tau \beta}$. Hence $\langle z_t, z_t \rangle$ satisfies the differential inequality

$$2\alpha^{\tau \alpha} \frac{d}{dt} \langle z_t, z_t \rangle < 2\beta^{\tau \beta}, \quad t > 0,$$

while

$$\langle z_t, z_t \rangle = |z_t|^2, \quad 0 \neq z \in E_\tau.$$

From Hartman [6, p. 24] we conclude that

$$e^{2\alpha^{\tau \alpha} t} |z_t|^2 < \langle z_t, z_t \rangle < e^{2\beta^{\tau \beta} t} |z_t|^2$$

for all $t > 0$. Taking square roots gives the growth estimate on $\Phi_t$ in Lemma 1. Similarly, if $z \in E_\tau$ then
\[
\frac{d}{dt}\langle x, z_t \rangle_t = 2\langle x, G_1(x) \rangle,
\]
which lies between \(-2\omega^{1/3}\) and \(-2\omega^{1/3}\) since \(G_1 \in \mathcal{X}_2\). This gives the growth estimate on \(\Phi_t\) in Lemma 1.

As remarked before, hyperbolicity of \(\Phi\) implies the uniqueness of \(E^+, E^-.\) Suppose \(F\) has period \(\omega\). Set \(F^+ = S\omega\), \(F^- = E_{\omega}\). Then \(F^+ \circ F^+\) is a \(\Phi\)-invariant splitting of \(R \times R^3\) since \(\Phi F^+ = \omega, F^+ \circ F^+ = \Phi\). Clearly \(F^+ \circ F^+\) also exhibits the hyperbolicity of \(\Phi\) so by \([9, 2, 9]\) \(E^+ \cong F^+, E^- \cong F^-\), and \(\omega\)-periodicity of \(E^+, E^-\) is proved. This completes the proof of Lemma 1.

Remark. An alternative proof that \(E^+, E^-\) exist can be devised by showing that the flow \(\Phi\) contracts \(\mathcal{X}_2\), instead of using Wazewski's principle. Contractiveness of \(\Phi\) on \(\mathcal{X}_2\) follows from considering the first variation equation of \(S = F - S\), along a \(\Phi\)-trajectory \(S_t\), namely, \(F = \dot{S} = (FS_t + SF_t)\). While \(S_t\) is in \(\mathcal{X}_2\), it is a positive operator so the above \(F\) is "negative", showing that \(\Phi\) contracts infinitesimally. \(t \geq 0\). Contractiveness of \(\Phi\) in the large follows by the mean value theorem since \(\mathcal{X}_2\) is convex. The details of this argument involve use of the inner product

\[
\langle A, B \rangle = \text{trace}(A^*B)
\]
on \(L(R^+, R^\nu)\) and its corresponding norm. This is not the operator norm on \(L(R^+, R^\nu)\), and it does not have an analogue for an infinite dimensional real Hilbert space \(E\). The estimates in the proof of Lemma 1 remain valid for \(E\), but Wazewski's Principle fails because \(\mathcal{X}_2\) probably is a retract of \(\mathcal{X}_2\). Compare Klee [11]. Thus the generalization of Lemma 1 to Hilbert space remains unproved by us.

6. Fermi Coordinates

The next lemma concerns a special coordinate system along a geodesic, called a "Fermi chart." For the geodesic flow, the bundle-chart over a Fermi chart serves the same purpose as a flowbox does for a flow. Let \(\mathcal{X}\) be a smooth Riemannian structure on \(TM\), and let \(\nu \in S_p M\) be given. \(\nu \in S_M\). Let \(X\) be the geodesic spray of \(S\). Let \(e_1(\cdot), \ldots, e_n(\cdot)\) be an orthonormal basis for \(T_x M\) with \(\nu = e_\nu\), and let \(\gamma\) be the geodesic initially tangent to \(\nu\). Parallel translation along \(\gamma\) gives smooth orthonormal vector fields \(e_\nu(\cdot), \ldots, e_n(\cdot)\) on \(\gamma\) such that \(e_\nu(\cdot) \equiv \nu(\cdot)\). Since \(e_\nu\) is tangent to the identity,

\[
f_* (\sum e_i(\cdot)) = \exp_{x_\nu} \left( \sum e_i(\cdot) \right)
\]
defines an immersion \(f_*\), called the "Fermi chart" associated with \(\nu \in S_p M\). The domain of \(f_*\) includes
\[ \mathcal{D}_r = \{ v \in \mathcal{E} \mid \diamond v \in \mathcal{E} \cap \mathcal{V}, |v| \leq c, r \in \mathcal{R} \}, \] 
where \( c \) is a positive constant. \( \mathcal{D}_r \) sends span \( (v) \) isometrically onto \( r \). Since \( f_r \) is an immersion, \( \mathcal{D}_r \) pulls back \( w \) to a Riemann structure \( f_\mathcal{D}_r \) on \( T \mathcal{D}_r \times T \mathcal{D}_r \). Thus \( f_\mathcal{D}_r \) is \( \mathcal{D}_r \) expressed in the \( f_r \)-chart. Let \( g_{ab}, \Gamma^c_{ab} \) and \( R^a_{bcd} \) be the components of \( f_\mathcal{D}_r \), its Christoffel symbols and its Riemannian curvature tensor in the \( f_r \)-chart.

**Lemma 2.** The Fermi chart \( f_r \) has the following properties at all points of span \( (v) \):

(0th order) \[ g_{ab} = \delta_{ab} , \]
(1st order) \[ \Gamma^a_{ab} = 0 , \]
(2nd order) \[ R^a_{bcd} = -\frac{1}{2} \frac{\partial g_{bi}}{\partial x^i} \frac{\partial g_{ai}}{\partial x^j} - \frac{\partial g_{aj}}{\partial x^i} \frac{\partial g_{ai}}{\partial x^j} . \]

**Proof.** The 0th and 1st order assertions are proved in Gromov, Klingenberg-Mauer [5]. In any chart \[ \Gamma^a_{ab} = \frac{1}{2} \sum g^{ac} (\partial_x \beta_{bc} + \partial_x \beta_{ca} - \partial_x \beta_{ac}) , \] where \( (g^{ac}) \) is the matrix inverse to \( (g_{ab}) \). By \( \partial_x \) etc. we mean \( \partial / \partial x^a \) where \( x^1, \cdots, x^n \) are the coordinates in the chart. Juggling indices and summing as in Weatherburns [15] we get \[ \partial_x \beta_{ab} = 0 , \] \[ 1 \leq a, b, c \leq m \] at any point of a chart where \( \Gamma^a = 0 \) and \( (g_{ab}) = (\delta_{ab}) \). This means the map \[ x \rightarrow (g_{ab}(x)) \in \text{real} \times m \times m \text{ matrices} \] has zero derivative at all points of span \( (v) \) in the Fermi chart. By the chain rule the same is true of \[ x \rightarrow (g^{ac}(x))^{-1} = (g^{ac}(x)) . \] Thus all first partials of \( g_{ab} \) and \( g^{ac} \) vanish along span \( (v) \). From this constancy we conclude \( \partial_x \beta_{ab} = 0, \partial_x g_{ab} = 0 \) along span \( (v) \). \( (v) = \omega \times \mathcal{D}_r \).

In any chart the components \( R^a_{bcd} \) are related to the \( \Gamma^a_{ab} \) by \[ R^a_{bcd} = \partial_x \Gamma^a_{bc} - \partial_x \Gamma^a_{bd} + \sum (\Gamma^a_{bc} \Gamma^b_{ad} - \Gamma^a_{bd} \Gamma^b_{cd}) \] (see Hicks [7]), so in the Fermi chart along span \( (v) \)
\[ R^\mu_\mu = \partial_j \Gamma^\mu_j - \partial_j \Gamma^\mu_j = \frac{1}{2} \sum_i \partial^a \partial^b \partial_j \Gamma^\mu_{ab} + \partial_j \partial_k \Gamma^\mu_{ab} - \partial_k \partial_j \Gamma^\mu_{ab} \]
\[ + \frac{1}{2} \sum_i \partial^a \partial^b \partial_j \partial_k \Gamma^\mu_{ab} + \partial_j \partial_k \partial_l \Gamma^\mu_{ab} - \partial_k \partial_j \partial_l \Gamma^\mu_{ab} \]
\[ - \frac{1}{2} \sum_i \partial^a \partial^b \partial_j \partial_k \partial_l \Gamma^\mu_{ab} + \partial_j \partial_k \partial_l \partial_m \Gamma^\mu_{ab} - \partial_k \partial_j \partial_l \partial_m \Gamma^\mu_{ab} \]
\[ - \frac{1}{2} \sum_i \partial^a \partial^b \partial_j \partial_k \partial_l \partial_m \Gamma^\mu_{ab} + \partial_j \partial_k \partial_l \partial_m \partial_n \Gamma^\mu_{ab} - \partial_k \partial_j \partial_l \partial_m \partial_n \Gamma^\mu_{ab} \]
\[ = -\frac{1}{2} (\partial_\alpha \partial_\beta \partial_\gamma + \partial_\beta \partial_\gamma \partial_\alpha - \partial_\gamma \partial_\alpha \partial_\beta) \cdot \frac{\partial \theta^\alpha}{\partial x^\beta} \cdot \frac{\partial \theta^\beta}{\partial x^\gamma} \cdot \frac{\partial \theta^\gamma}{\partial x^\alpha}. \]

For along span (v): $\partial_4 (\theta^\nu)$ vanishes, $\partial_4 \partial_\nu \theta_k$ etc. vanishes, $\partial_4 (\theta^\nu)$ vanishes, and $\theta^\nu = 0$. For the same reasons
\[ \frac{\partial^2 \theta^\nu}{\partial x^4 \partial x^i} = \frac{1}{2} \sum \partial^a \partial^b \partial_4 (\theta^a \partial_i + \theta^b \partial_i - \theta^i \partial_4) \]
\[ + \frac{1}{2} \sum \partial^a \partial^b \partial_4 (\theta^a \partial_\nu + \theta^b \partial_\nu - \theta^\nu \partial_4) \]
\[ = -\frac{1}{2} \partial_4 \partial_\nu \theta_k = -\frac{1}{2} \frac{\partial \theta^\nu}{\partial x^4} \frac{\partial \theta^\nu}{\partial x^i} \]
along span (v). This completes the proof of Lemma 2.

7. Proof of (ii)

Let $\mathcal{S}$ be the given Riemann structure on $TM$. Let $v \in \mathcal{S}_p$, $p \in M$, and choose an orthonormal basis of $T_pM$, $e_1, \ldots, e_n$ with $e_1 = v$. Let $f_4$ be the Frenet chart determined by $e_1, \ldots, e_n$, and let $\Gamma_4$ be the bundle chart of $TM$ tangent to $f_4$:
\[ \mathcal{S}_p \times T_pM \xrightarrow{f_4^*} TM \]
\[ (x, \xi) \mapsto T_{f_4(x)}(\xi) \in T_\mathcal{S}_{f_4(x)}M. \]
$\mathcal{S}_p$ is the domain of $f_4$. The geodesic spray $X$ is represented in any $TM$-bundlechart for $TM$ as the first order ordinary differential equation
\[ (1) \quad \left[ \begin{array}{c} \dot{x} \\ \dot{\xi} \end{array} \right] = \left[ \begin{array}{c} 0 \\ \Gamma(\xi) \frac{\partial f_4}{\partial \xi} \end{array} \right], \]
where $\Gamma(\xi): T_pM \times T_pM \to T_pM$ is the symmetric bilinear map such that
\[ \Gamma(\xi)(e_i, e_j) = \sum_{a,b} \frac{\partial \Gamma^a}{\partial x^b} (x) e_a, \quad x \in \mathcal{S}_p. \]
The $\Gamma_{ij}$ are the Christoffel symbols of $\mathcal{S}_p$ expressed in the $f_4$-chart.

The geodesic flow $\varphi$ of $\mathcal{S}_p$, represented in the $F_4$-chart, is the solution of (1). The assertion of the smoothness theorem concerns the tangent flow $T\varphi$ on
T(TM). When represented in the $F_{\bar{\alpha}}$-chart, $T_{\bar{\alpha}}$ is the solution of the first variation equation of (1):

$$\mathbf{W} = D(F_{\bar{\alpha}}X)_{\bar{\alpha}}\mathbf{W}, \quad \mathbf{W}(0) = \mathbf{I}$$

for $w_1 = F_{\bar{\alpha} \bar{\beta}}g_{\bar{\beta}}F_{\bar{\alpha}}(w)$, $w \in \partial \mathcal{S} \times T_p\mathcal{M}$. By $F_{\bar{\alpha}}X$ we mean the vector field $X \times F_{\bar{\alpha}}$ on $\partial \mathcal{S} \times T_p\mathcal{M}$. At $F_{\bar{\alpha}}(p, v)$ we calculate

$$D(F_{\bar{\alpha}}X)_{\bar{\alpha} \bar{\beta}} = D \begin{pmatrix} \zeta \\ \Gamma(x, \xi, \xi) \\ \frac{\partial \Gamma(x, \xi, \xi)}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

by Lemma 2, since

$$\begin{pmatrix} \frac{\partial \Gamma(x, \xi, \xi)}{\partial x} \\ \frac{\partial \Gamma(x, \xi, \xi)}{\partial \xi} \\ \frac{\partial \Gamma(x, \xi, \xi)}{\partial \xi} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ \delta_{\bar{\alpha} \bar{\beta}} \\ \delta_{\bar{\alpha} \bar{\beta}} \end{pmatrix}$$

(3) $\mathbf{W} = \begin{pmatrix} 0 \\ -R^i_{\bar{\alpha} \bar{\beta}}(v) \\ 0 \end{pmatrix} \mathbf{I}, \quad \mathbf{W}(0) = \mathbf{I}.$

In general, $R^i_{\bar{\alpha} \bar{\beta}}$ is skew-symmetric in $\bar{\beta}$ and $R^i_{\bar{\alpha} \bar{\beta}} = 0$, so we see that

$$\begin{pmatrix} R^i_{\bar{\alpha} \bar{\beta}} \\ \vdots \\ R^i_{\bar{\alpha} \bar{\beta}} \end{pmatrix}, \quad 2 \leq k, l \leq m.$$
In any chart at a point where the coordinates are orthonormal, the sectional curvature of a pair of vectors $Y, Z$ at $T_p M$ is
\[ K_{p}(Y, Z) = \langle R(Y, Z)Z, Y \rangle, \quad Y = \sum y_i e_i, \quad Z = \sum z_i e_i, \]
and thus finally using the negative curvature hypothesis we have
\[ \langle P, Z \rangle = -\sum_{i,j} \langle P_{i,j} e_i e_j, Z \rangle = -K e_i, Z \rangle > 0, \]
where $P_{i,j} = R_{i,j}(e_i) e_j$.

Choose constants $K > k > 0$ such that every sectional curvature lies strictly between $-K^2$ and $-K$. By (4), in applying Lemma 1 we can take $k = \beta = K$.

By Lemma 1, $\Phi$ is hyperbolic and the strength of its hyperbolicity can be estimated. Using the $F_r$-chart we get a well-defined $T_0$-invariant splitting $E^c \oplus E^c$ of $E$ over the $\Phi$-orbit of $v$. If $t \mapsto \Phi_t v$ is periodic in $t$, then $F_t$ is periodic and, by Lemma 1, so is the $\Phi$-invariant-splitting. Hence $E^c \oplus E^c$ is well defined. Choose one $v$ on each $\Phi$-orbit and make the preceding construction. This gives a well-defined $T_0$-invariant splitting of $E$ over all $SM$.

Since the Finsler on span $(e)$ in $\mathcal{H} \times V_u$ adapted to $\Phi$ is uniformly equivalent to the standard Finsler, and since $f$ is a Fermi-chart, we use that the estimates
\[ e^{i\tau} < m(\Phi_t) \leq |\Phi_t| \leq e^{i\tau}, \quad e^{-i\tau} < m(\phi_t) \leq |\Phi_t| \leq e^{-i\tau}, \]
which are valid for all $t > 0$ when the adapted Finsler is used—imply
\[ e^{i\tau} < m(T_{\Phi_t} o) \leq |T_{\Phi_t} o| \leq e^{i\tau}, \quad e^{-i\tau} < m(T_{\Phi_t} o) \leq |T_{\Phi_t} o| \leq e^{-i\tau} \]
respecting the $\delta^c$-norms for all large $t$. By $T_{\Phi_t} o_T \Phi_t$ we mean $T_{\Phi_t} o_T E_c$. Thus, respecting the fixed $\delta^c$-norms, $T_{\Phi_t} o_T E_c$ is a linear uniformly hyperbolic flow and so, by [9, (2.9)], $E^c$ and $E^c$ are automatically continuous and independent of which $v$ was chosen on each $\Phi$-orbit. Hence $\phi$ is Anosov.

By (5) we get
\[ \text{bol}(T_{\Phi_t} o) < e^{i\tau}, \quad m(T_{\Phi_t} o) > e^k, \]
\[ \text{bol}(T_{\Phi_t} o) < e^{-i\tau}, \quad |T_{\Phi_t} o| > e^{-k} \]
for all large $t$. Now return to the proof of (ii). Since $E$ is a smooth bundle we can approximate $E^c$, $E$ by smooth subbundles $E^c$, $E$ of $E$. Then we can consider, for a large fixed $t$, the $\Phi$-map $T_{\Phi_t} o : \mathcal{H} \times \mathcal{H}$ where $\mathcal{H} = \{ G \in L(\mathcal{H}, E), \|G\| \leq 1 \}$. As in the proof of (i), $T_{\Phi_t} o$ is a fiber contraction with (fiber contraction)\-(base contraction)$-^{1}$
\[ = (T_{\Phi_t} o)(m(T_{\Phi_t} o)^{-1})(m(T_{\Phi_t} o)^{-1})^{-1} \]
Since the curvature is $\frac{1}{2}$-pinched, we have $K - 2k < 0$ and the hypothesis of the $C^1$ section theorem is satisfied; therefore the unique $(T^3)$-invariant section of $\varphi$ is of class $C^3$. Since $E^2$ gives such a section, $E^3$ is of class $C^3$. Working with the reverse flow and $x^c = \{E \in L(E); E^3; |E^3| \leq 1\}$, (5) gives the same result for $E^3$. This completes the proof of (ii).

Remarks on the smoothness of $\varphi$. For simplicity, we assumed the Riemannian structure $\varphi$ was $C^3$. However, the above constructions work equally naturally when $\varphi$ is $C^4$, the smoothness theorem holds when $\varphi$ is $C^4$, and $\varphi$ is Anosov when $\varphi$ is $C^5$ with negative curvature. This can be seen by $C^4$-approximating $\varphi$ by a $C^4$ Riemann structure $\tilde{\varphi}$ and using the uniformities in the hyperbolicity estimate. Alternatively, the Fermi chart could be smooth as were flow boxes in Pugh-Robinson [14].

Standard question. If the geodesic flow $\varphi$ of $\varphi$ is Anosov, then does $M$ admit a Riemann structure $\varphi$ with negative curvature? Wilhelm Klingenberg showed in [12, [16] that all known topological properties of $M$ which are implied by negative curvature are equally implied by $\varphi$ being Anosov.

References


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