Title
Joint first-passage probability, and reliability of systems under Stochastic excitation

Permalink
https://escholarship.org/uc/item/7zf2f3pv

Journal

ISSN
0733-9399

Authors
Song, J
Kiureghian, A D

Publication Date
2006

Peer reviewed
Joint First-Passage Probability and Reliability of Systems under Stochastic Excitation

Junho Song¹ and Armen Der Kiureghian²

Abstract: The first-passage probability, describing the probability that a scalar process exceeds a prescribed threshold during an interval of time, is of great engineering interest. This probability is essential for estimating the reliability of a structural component whose response is a stochastic process. When considering the reliability of an engineering system composed of several interdependent components, the probability that two or more response processes exceed their respective safe thresholds during the operation time of the system is an equally essential quantity. This paper proposes simple and accurate formulas for approximating this joint first-passage probability of a vector process. The nth order joint first-passage probability is obtained from a recursive formula involving lower order joint first-passage probabilities and the out-crossing probability of the vector process over a safe domain. Interdependence between the crossings is approximately accounted for by considering the clumping of these events. The accuracy of the proposed formulas is examined by comparing analytical estimates with those obtained from Monte Carlo simulations for stationary Gaussian processes. As an example application, the reliability of a system of interconnected equipment items subjected to a stochastic earthquake excitation is estimated by linear programming bounds employing marginal and joint component fragilities obtained by the proposed formulas.

DOI: 10.1061/(ASCE)0733-9399(2006)132:1(65)

CE Database subject headings: Earthquake loads; Envelope; Probability; Random vibration; Stochastic processes; System reliability; Vector analysis.

Introduction

Engineering systems consisting of multiple structural components, e.g., electrical substations with interconnected equipment items, highway transportation networks consisting of bridges, roadways, and tunnels, and harbor and port facilities, are often subject to stochastic loads, such as earthquakes, wind, or sea waves. Under these conditions, the response of each structural component is a stochastic process and its reliability can be estimated in terms of the first-passage probability, i.e., the probability that the stochastic response process exceeds a prescribed safe threshold during a given interval of time.

The reliability of a system, however, cannot be directly deduced from the marginal first-passage probabilities of its components when the component failure events are statistically dependent. Such dependence is usually present in systems composed of structural components, particularly when the components are subjected to a common source of excitation. For such systems, bounds on the system failure probability can be obtained from analytical bounding formulas (Ditlevsen 1979; Hohenbichler and Rackwitz 1983; Zhang 1993) or linear programming (LP) (Song and Der Kiureghian 2003). However, in order to achieve narrow bounds, it is necessary to have information on the joint failure probabilities of pairs, triplets, or, in general, subsets of the components. For systems under stochastic loading, the joint probability of interest is the probability that the response of each component in the subset exceeds its respective safe threshold during the given period of time. We denote this as a joint first-passage probability. It is noted that this probability is different from the out-crossing probability of a vector process from a safe domain.

After a brief review of the marginal first-passage probability and its approximate solutions, this paper proposes approximate formulas for estimating the joint first-passage probability of vector processes. By applying the addition rule of probability, the joint first-passage probability is computed in a recursive manner in terms of all lower order joint first-passage probabilities, including the marginal first-passage probabilities, and the out-crossing probability of the vector process. The latter probability is approximated by use of the mean crossing rate of the vector process out of a safe domain, which is the sum of the crossing rates of the component processes over their respective double-barriers with finite dimensions. Dependence between the crossings is approximately accounted for by considering the clumping of their occurrences. Detailed formulas are provided for the case of two- and three-dimensional stationary, zero-mean Gaussian vector processes. These include new results for the cross-correlation and joint probability density function (PDF) of two or more envelope processes. The proposed formulas for the joint first-passage probability are verified by comparing the analytical estimates with Monte Carlo simulation results for stationary Gaussian processes.
The comparison includes an investigation of the effects of the correlation between the component processes and their bandwidths on the accuracy of the approximations.

As an application of the proposed joint first-passage probability concept and formulas, the reliability of a system of interconnected equipment items subjected to a stationary stochastic excitation is considered. The connected equipment items are modeled as single-degree-of-freedom (SDOF) linear oscillators and the connecting elements are modeled as linear springs. The second moments of the responses of the connected system are computed by linear random vibration analysis and the marginal and joint fragilities of the equipment items are obtained by use of the approximate first-passage probability functions introduced in the paper. Bounds on the reliability of the entire system are then obtained by use of LP (Song and Der Kiureghian 2003), employing the marginal and joint component fragilities.

Marginal First-Passage Probability

Let \( p(a;\tau)=P[a=\max_{0=t<r=t}[X(t)] \) denote the first-passage probability of a zero-mean stochastic process \( X(t) \) over a prescribed double-sided threshold \( |x|=a \) during an interval of time \( t \in (0,\tau) \). No exact solution of this probability exists for general cases. Attempts have been made to derive approximate solutions in the form (Lutes and Sarkani 2004)

\[
p(a;\tau) \equiv 1 - A \exp \left[ - \int_0^\tau v(a;t) \, dt \right]
\]

(1)

where \( A \) is probability that the process is in the safe domain at \( t=0 \) and \( v(a;t) \) is conditional mean crossing rate at time \( t \), given no crossings prior to that time. In most cases, this conditional crossing rate is impossible to obtain because the necessary conditional joint density function of the process and its rate is unknown (Lutes and Sarkani 2004). A well-known approximation is to replace \( v(a;t) \) in Eq. (1) with the unconditional mean crossing rate of \( |X(t)| \) over \( a \), which we denote by \( v(a) \). Since this approximation neglects the statistical dependence between the crossing events, it is often called the “Poisson approximation” (Rice 1944, 1945). With this approximation

\[
p(a;\tau) \equiv 1 - A \exp \left[ - \int_0^\tau v(a) \, dt \right]
\]

(2a)

\[A = P[|X(0)| < a] = \int_0^a f_X(x;0) \, dx \]

(2b)

\[v(a) = v(-a;\tau) + v(a;\tau) \]

(2c)

where \( f_X(x;0) \) is marginal PDF of \( X(t) \), and \( v(-a;\tau) \) and \( v(a;\tau) \) denote the unconditional mean rates of down- and up-crossings of the process \( X(t) \) over the levels \(-a\) and \( a \), respectively. These rates are computed by Rice’s formula (Rice 1944, 1945) in Eq. (2c), in which \( f_X(x,-;t) \) denotes the joint PDF of the process \( X(t) \) and its time derivative \( X(t) \) at the same time instant.

For a stationary, zero-mean Gaussian process, it is well-known that \( A=1-2\Phi(-r) \), where \( \Phi(\cdot) \) is standard normal cumulative probability function; \( r=a/\sigma_x \) is normalized threshold; and \( v(a)=\sigma_X/(\pi\sigma_x)\exp(-r^2/2) \) (Rice 1944, 1945), in which \( \sigma_X \) and \( \sigma_x \) are standard deviations of \( X(t) \) and \( \dot{X}(t) \), respectively. Note that in this case the mean crossing rate is constant in time.

Though convenient, the Poisson approximation can result in significant errors, which depend on the bandwidth of the process and the time it spends in the unsafe domain. VanMarcke (1975) proposed an improved formula, accounting for the dependence between the crossing events and the time that the process spends above the threshold, during which obviously no up-crossings can occur. The dependence between the crossings was approximately accounted for by considering the crossings of the envelope process and the clumping of the process crossings associated with each crossing of the envelope. In essence, the conditional crossing rate in Eq. (1) is replaced by \( \eta(a;t) \), the unconditional mean crossing rate of the envelope process, discounted by the probability that crossings by the process will indeed occur during a single envelope excursion. For consistency, \( A \) in Eq. (1) is replaced by \( B \), the probability that the envelope process is in the safe domain at \( t=0 \). The resulting approximation is

\[
p(a;\tau) \equiv 1 - B \exp \left[ - \int_0^\tau \eta(a;t) \, dt \right]
\]

(3a)

\[B = P[E(0) < a] = \int_0^a f_E(e;0) \, de \]

(3b)

\[\eta(a;t) = \frac{P[E(t) > a]v(0;t)}{P[E(t) < a]} \left[ 1 - \exp \left( - \frac{\nu_E^2(a;t)}{\nu_E(0;t)} \right) \right] \]

(3c)

where \( E(t) \) denotes the envelope process of \( X(t) \); \( f_E(e;0) \) is marginal PDF of \( E(t) \); and \( \nu_E^2(a;t) \) is unconditional mean up-crossing rate of \( E(t) \).

When \( X(t)=\text{stationary} \), zero-mean Gaussian process and the envelope defined by Cramer and Leadbetter (1967) is employed, the corresponding formulas are

\[
B = \frac{1 - \exp(-\delta)}{1 - \exp(-r^2/2)} \quad \text{and} \quad \eta(a) = v(a) \left( \frac{1 - \exp(-\sqrt{2}\delta)}{1 - \exp(-r^2/2)} \right)
\]

(3m)

where \( \delta=(1-\lambda_0/\lambda_0.2^{1/2})=\text{shape factor that characterizes the bandwidth of the process, in which } \lambda_m, m=0,1,2,\ldots, \text{=spectral moments defined by}

\[
\lambda_m = \int_0^\infty \omega^m G_{XX}(\omega) \, d\omega, \quad m=0,1,2,\ldots
\]

(4)

where \( G_{XX}(\omega) \) is one-sided power spectral density (PSD) function of the process. For nonstationary Gaussian processes, the formulas will involve time-dependent spectral moments (Lutes and Sarkani 2004).

Various analytical estimates of the first-passage probability have been compared with simulated stationary responses of SDOF oscillators under zero-mean, Gaussian white noise processes (Lutes and Sarkani 2004). VanMarcke’s formula in Eq. (3) performs better than other available approximations, including the Poisson approximation in Eq. (2). VanMarcke’s approximation provides accurate estimates, especially when the damping ratio of the oscillator is not less than 5%.
Joint First-Passage Probability

Consider a zero-mean stochastic vector process \( X(t) \) = \([X_1(t), X_2(t), \ldots, X_n(t)]^T\) and define \( Y_i = \max_{t \in [0, \tau]} |X_i(t)| \) as the maximum of the absolute value of the component process \( X_i(t) \), \( i = 1, \ldots, n \), over the time interval \( t \in (0, \tau) \). We define the joint first-passage probability of the vector process \( X(t) \) over the \( n \)-dimensional double-sided barrier \( |x| = a_i, \ i = 1, \ldots, n \) as

\[
p_{1\ldots n}(a_1, \ldots, a_n; \tau) = P(a_1 \leq Y_1, \ldots, a_n \leq Y_n, \tau) \tag{5}
\]

As can be seen, the joint first-passage probability is the probability that each of the component processes of \( X(t) \) exceeds its respective double-sided threshold during the specified interval. This probability is different from the so-called out-crossing probability of the vector process, which for the specified domain can be defined as

\[
P_{1\ldots n}(a_1, \ldots, a_n; \tau) = P(a_1 \leq \max_{t \in (0, \tau)} \{X_1(t) \ldots X_n(t)\} \leq a_n, \tau) \tag{6}
\]

While Eq. (5) requires that each component of the process exceed its respective threshold, Eq. (6) only requires that at least one of the component processes exceed its respective threshold. Fig. 1 illustrates the difference between the two probabilities for a case with \( n = 2 \).

The probabilities in Eqs. (5) and (6) are related through the addition rule of probability for the union of events. Specifically, for \( n = 2 \)

\[
p_{12}(a_1, a_2; \tau) = p_1(a_1; \tau) + p_2(a_2; \tau) - p_{12}(a_1, a_2; \tau) \tag{7}
\]

where \( p_i(a_i; \tau) \) = marginal first-passage probability of the component process \( X_i(t) \), \( i = 1, 2 \). The above result is easily obtained by solving for \( P(E_1 \cup E_2) \) in the addition rule \( P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \), where the notation \( E_i = \{a_i \leq Y_i, \tau\} \) and \( E_1 \cap E_2 = E_1 \cap E_2 \) is used. In a similar manner, the joint first-passage probability for \( n = 3 \) is obtained as

\[
p_{123}(a_1, a_2, a_3; \tau) = -p_1(a_1; \tau) - p_2(a_2; \tau) - p_3(a_3; \tau) + p_{12}(a_1, a_2; \tau) + p_{13}(a_1, a_3; \tau) + p_{23}(a_2, a_3; \tau) + p_{123}(a_1, a_2, a_3; \tau) \tag{8}
\]

Generalizing this result, the joint first-passage probability for an \( n \)-dimensional vector process is given by

\[
p_{1\ldots n}(a_1, \ldots, a_n; \tau) = (-1)^n \left[ \sum_{i=1}^{n} p_i(a_i; \tau) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{ij}(a_i, a_j; \tau) + \cdots \right. \\
+ \left. (-1)^n \sum_{i=1}^{n} p_{1\ldots i-1;i+1\ldots n}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n; \tau) \right] - p_{1\ldots n}(a_1, \ldots, a_n; \tau) \tag{9}
\]

It is seen that the \( n \)-dimensional joint first-passage probability can be computed in a recursive manner in terms of the \( n \)-dimensional out-crossing probability and all lower order joint first-passage probabilities, including the marginal first-passage probabilities of the component processes.

In the following two sections, we derive the necessary expressions to compute the joint first-passage probabilities for two- and three-dimensional vector processes. These expressions employ approximations similar to the “Poisson” and “VanMarcke” approximations for the marginal first-passage probabilities, as described in the preceding section. The proposed formulas can be generalized to vector processes with higher dimensions.

Joint First-Passage Probability of Two Processes

First, analogous to the Poisson approximation in Eq. (2), the out-crossing probability \( p_{12}(a_1, a_2; \tau) \) in Eq. (7) is approximated by an exponential function employing the unconditional mean out-crossing rate over the rectangular barrier \( |x| = a_i, \ i = 1, 2 \)

\[
p_{12}(a_1, a_2; \tau) \approx 1 - A_{12} \exp \left[ - \int_{0}^{\tau} r_{12}(a_1, a_2; t) dt \right] \tag{10}
\]

Here, \( A_{12} \) = probability that \( X(t) = [X_1(t), X_2(t)]^T \) is inside the rectangular domain \( \{x_1, x_2: |x_1| < a_1, |x_2| < a_2\} \) at \( t = 0 \) and is obtained from integration of the joint PDF of \( X(0); \) and \( r_{12}(a_1, a_2; t) \) = unconditional mean out-crossing rate of \( X(t) \) over the rectangular domain shown in Fig. 2(a). This rate is written as the sum of the two mean crossing rates of the scalar processes over their respective double-barriers with finite dimensions, as shown in Figs. 2(b and c). That is

![Fig. 1. Trajectories of a vector process and relation to the joint failure event](image-url)
where \( v_{12}(a_1; a_2; t) = v_{12}(a_1; t | a_2) = v_{21}(a_2; t | a_1) \) (11)

\[
v_{12}(a_1; a_2; t) = \int_{-a_2}^{a_2} \left[ \int_{-a_1}^{a_1} \hat{x}_1 f_{X_1, X_2}(x_1, x_2; t) dx_1 \right] dx_2
\]

where \( f_{X_1, X_2}(x_1, x_2; t) \) denotes the joint PDF of \( X_1(t) \) and \( X_2(t) \) at the same time instant. Using symmetry, \( v_{12}(a_1; a_2; t | a_1) \) is obtained by interchanging the indices 1 and 2 in Eq. (12). Substituting the first-passage probabilities from Eqs. (2) and (10) into Eq. (7), one obtains the approximate joint first-passage probability. Hereafter we call this the extended VanMarcke approximation, since it neglects the dependence between the crossing events. An expression for \( v_{12}(a_1; a_2) \) (independent of time) is derived in Appendix I for the case of a two-dimensional, stationary, zero-mean Gaussian vector process. Similar expressions can be derived for nonstationary processes.

An improved approximation is obtained by using Eq. (3) for \( p_1(a_1; \tau) \) and \( p_2(a_2; \tau) \) and a similar approximation developed herein for \( p_{12}(a_1; a_2; \tau) \). The latter approximation employs an exponential form analogous to Eq. (3),

\[
p_{12}(a_1; a_2; \tau) \approx 1 - B_{12} \exp \left( - \int_0^\tau \eta_{12}(a_1, a_2; t) dt \right) \quad (13)
\]

where \( B_{12} \) is the probability that the vector of envelope processes is inside the rectangular domain \( \{ (x_1, x_2) : |x_1| < a_1, |x_2| < a_2 \} \) at \( t = 0 \) and \( \eta_{12}(a_1, a_2; t) \) is the crossing rate over the rectangle barrier accounting for the clumping of the crossings. For the latter term, we employ the approximation

\[
\eta_{12}(a_1, a_2; t) = v_{12}(a_1; t | a_2) \frac{\eta_1(a_1; t)}{v_1(a_1; t)} + v_{21}(a_2; t | a_1) \frac{\eta_2(a_2; t)}{v_2(a_2; t)} \quad (14)
\]

where the bracketed quotients are intended to account for the types of corrections that are inherent in the VanMarcke approximation of the marginal first-passage probabilities. They approximately account for the clumping of the crossings and the time spent by the process above the respective thresholds. The form adopted in Eq. (14) is motivated by the assumption that the correction for these effects for the crossings over a finite boundary is essentially the same as that for the infinite boundary (each term inside square brackets). Substituting the marginal first-passage probabilities from Eq. (3) and the out-crossing probability from Eq. (13) into Eq. (7), one obtains the joint first-passage probability. Hereafter we call this the extended VanMarcke approximation.

Joint First-Passage Probability of Three Processes

To obtain an extended Poisson approximation of \( p_{123}(a_1, a_2, a_3; \tau) \), \( p_1, p_2, p_3 \) by Eq. (2) and \( p_{12}, p_{13}, \) and \( p_{23} \) by an extended Poisson approximation are substituted into Eq. (8). \( p_{123}(a_1, a_2, a_3; \tau) \) is approximated by use of an unconditional mean out-crossing rate over a cuboid-barrier:

\[
p_{123}(a_1, a_2, a_3; \tau) \approx 1 - A_{123} \exp \left( - \int_0^\tau v_{123}(a_1, a_2, a_3; t) dt \right) \quad (15)
\]

where \( A_{123} \) is the probability that \( X(t) = [X_1(t), X_2(t), X_3(t)]^T \) inside the cuboidal domain \( \{ (x_1, x_2, x_3) : |x_1| < a_1, |x_2| < a_2, |x_3| < a_3 \} \) at \( t = 0 \), obtained from integration of the joint PDF of \( X(0) \); and \( v_{123}(a_1, a_2, a_3; t) \) is the unconditional mean out-crossing rate of \( X(t) \) over the cuboidal domain, shown in Fig. 3(a). This rate is written as the sum of three mean out-crossing rates of the individual processes over their respective double-barriers with finite dimensions, shown in Figs. 3(b-d). Specifically

\[
v_{123}(a_1, a_2, a_3; t) = v_{123}(a_1; a_2, a_3) + v_{213}(a_2; a_3, a_1) + v_{312}(a_3; a_1, a_2) \quad (16)
\]

where \( v_{123}(a_1; a_2, a_3) \) denotes the unconditional mean out-crossing rate of \( X_1(t) \) over the finite edges defined by \( \{ (x_1, x_2, x_3) : |x_1| = a_1, |x_2| < a_2, |x_3| < a_3 \} \). This is computed by the generalized Rice formula (Belyaev 1968)

\[
v_{123}(a_1; a_2, a_3) = \int_{-a_3}^{a_3} \int_{-a_2}^{a_2} \left[ \int_{-\infty}^{0} \hat{x}_1 f_{X_1, X_2, X_3}(\hat{x}_1, x_2, x_3; t) d\hat{x}_1 \right] dx_2 dx_3 \quad (17)
\]
where \( f_{X_1 X_2 X_3}(t, \cdots ; t) \) denotes the joint PDF of \( X_1(t), X_2(t), X_3(t) \), and \( X_3(t) \), all taken at the same time instant. Using symmetry, \( \nu_{2|3}(a_2; t|a_1, a_3) \) and \( \nu_{3|2}(a_3; t|a_1, a_2) \) are obtained by interchanging the indices in Eq. (17). The expression for \( \nu_{1|23}(a_1|a_2, a_3) \) (independent of time) for the case of a threedimensional, stationary, zero-mean Gaussian vector process is derived in Appendix I. Similar expressions can be derived for nonstationary processes.

To obtain an extended VanMarcke approximation of \( p_{123}(a_1, a_2, a_3; \tau) \), \( p_1 \), \( p_2 \), and \( p_3 \) by Eq. (3) and \( p_{12} \), \( p_{13} \), and \( p_{23} \) by an extended VanMarcke approximation are substituted into Eq. (8). A similar approximation of \( p_{1+2+3}(a_1, a_2, a_3; \tau) \) employs the exponential form

\[
p_{1+2+3}(a_1, a_2, a_3; \tau) \equiv 1 - B_{123} \exp \left[ - \int_0^\tau \eta_{123}(a_1, a_2, a_3; t) \, dt \right]
\]

where \( B_{123} \) = probability that the vector of envelope processes is inside the cuboidal domain \( \{(x_1, x_2, x_3): |x_1| < a_1, |x_2| < a_2, |x_3| < a_3\} \) at \( t = 0 \); and \( \eta_{123}(a_1, a_2, a_3; t) \) = out-crossing rate over the cuboidal barrier accounting for the clumping of the crossings. The latter is approximated as

\[
\eta_{123}(a_1, a_2, a_3; t) = \nu_{1|23}(a_1; t|a_2, a_3) \nu_{1|23}(a_1; t|a_2, a_3) + \nu_{2|3}(a_2; t|a_1, a_3) \nu_{2|3}(a_2; t|a_1, a_3) + \nu_{3|1}(a_3; t|a_1, a_2) \nu_{3|1}(a_3; t|a_1, a_2)
\]

where the bracketed quotients account for the types of corrections inherent in VanMarcke approximation. In Appendix III, an approximate expression is derived for the joint PDF of Cramer and Leadbetter (1967) envelopes of three- and higher-dimensional stationary, zero-mean Gaussian vector processes, which can be used to compute \( B_{123} \) and similar terms for higher-dimensional vector processes.

**Verification by Monte Carlo Simulation for Stationary Processes**

In this section, the proposed approximate formulas for the joint first-passage probability of two- and three-dimensional vector processes are verified through comparisons with simulated stationary responses of SDOF oscillators subjected to white noise excitation.

**Verification of Joint First-Passage Probability of Two Processes**

Consider the displacement responses \( X_1(t) \) and \( X_2(t) \) of two SDOF oscillators subjected to a white-noise excitation having a one-sided PSD, \( G_0 = 1 \). The oscillators have natural frequencies \( f_1 \) and \( f_2 \), respectively, and equal damping ratios \( \zeta_1 = \zeta_2 = \zeta \). Nine combinations of the oscillator frequencies and damping ratios are selected to investigate the effect of the bandwidth and the correlation between the processes on the accuracy of the proposed formulas. For each set of these parameters, the statistical mo-
ments of the displacement and velocity responses given by Der Kiureghian (1980) and Igusa et al. (1984) are used to compute the approximate first-passage probability formulas. Table 1 lists the parameter values and the statistical moments for the selected cases (neglect entries with $X_i$). As can be seen, three categories of bandwidth (narrow, medium, wide) and three categories of correlation between the processes (low, medium, high) are selected. The cases are named by their bandwidth and correlation categories. For example, “narrow-medium” denotes the case with the narrow bandwidth $\delta=0.158$ and the medium correlation coefficient $\rho_{X_1X_2}=0.50$. A total of 2,000 discretized sample realizations of the white-noise process with time step $\Delta t=0.02$ s are generated, each having a duration of 60 s. For each sample, the displacements $X_i(t)$ and $X_j(t)$ of the two oscillators are numerically computed. The last 30 s of each displacement time history, where the response has effectively achieved full stationarity, is used to observe the crossing events.

Fig. 4 compares the results based on the proposed approximations of the first-passage probability with the simulation results for the “medium-medium” case. The shape factors of the displacement processes are $\delta=0.246$ and the correlation coefficient of the displacement processes is $\rho_{X_1X_2}=0.5$. All first-passage probabilities are computed with respect to three normalized thresholds: $r_i=a_i/\sigma_{X_i}=1, 2,$ and 3 for $i=1, 2, 3$. Figs. 4(a and b) show the marginal first-passage probabilities $p_{i}(a_i;\tau)$ and $p_{2}(a_2;\tau)$. It can be seen that the estimates based on VanMarcke formula are significantly more accurate than those based on the Poisson approximation. Fig. 4(c) shows the out-crossing probability $p_{12}(a_1,a_2;\tau)$, and $p_{2}(a_2;\tau)$. The accuracy of the extended VanMarcke formula in estimating this probability is similar to that of VanMarcke formula for the marginal probability estimates. Fig. 4(d) demonstrates the joint first-passage probabilities $p_{12}(a_1,a_2;\tau)$ for the three thresholds. The extended Poisson formula leads to significant errors, whereas the extended VanMarcke formula provides excellent agreement with the simulation results.

Fig. 5 shows the joint first-passage probability estimates for the remaining eight cases. Figs. 5(a and b) show the results for the “medium-low” and “medium-high” cases, which are for correlation coefficient values $\rho_{X_1X_2}=0.1$ and 0.9, respectively, with the medium shape factor $\delta=0.246$. Comparing these results with the results in Fig. 4(d), one concludes that the performance of the extended VanMarcke formula is not affected by the correlation coefficient between the component processes. Figs. 5(c and d) show the joint first-passage probabilities for the “narrow-medium” and “wide-medium” cases, which are for the shape factors $\delta=0.158$ and 0.339, respectively, and the medium correlation coefficient $\rho_{X_1X_2}=0.5$. Comparing these results with those in Fig. 4(d), one can see that the extended VanMarcke formula does not perform as well in the case of strongly narrow-band processes, but still leads to reasonably accurate estimates of the joint first-passage probability. The error in this case is inherited from the inaccuracy of VanMarcke formula for strongly narrow-band processes. Similar comparisons of the results in Figs. 5(e–h) show that the extended VanMarcke formula also provides good or reasonable accuracy for the “narrow-low,” “narrow-high,” “wide-low,” and “wide-high” cases.

**Verification of Joint First-Passage Probability of Three Processes**

We now consider the responses $X_1(t)$, $X_2(t)$, and $X_3(t)$ of three SDOF oscillators in order to examine the accuracy of the proposed formulas for the joint first-passage probability of three-dimensional vector processes. The damping ratios for all three oscillators are equal, $\zeta_1=\zeta_2=\zeta_3=\zeta$, and the frequency $f_3$ of the third oscillator is selected such that $\rho_{X_1X_2}^2=\rho_{X_1X_3}=0.5$. The marginal first-passage probabilities $p_{i}(a_i;\tau)$ and $p_{2}(a_2;\tau)$ are shown in Figs. 4(a and b). Although not shown here, the same

---

**Table 1. Parameters of Three Single-Degree-of-Freedom Oscillators and Statistical Moments of the Responses under White Noise Excitation ($f_1=2$ Hz)**

<table>
<thead>
<tr>
<th>Bandwidth ((\delta/\zeta))</th>
<th>Narrow (0.158/0.02)</th>
<th>Medium (0.246/0.05)</th>
<th>Wide (0.339/0.10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correlation (\rho_{X_1X_2}) (\rho_{X_1X_3}) (\rho_{X_2X_3})</td>
<td>Low (0.10)</td>
<td>Medium (0.50)</td>
<td>High (0.90)</td>
</tr>
<tr>
<td>(f_1) (Hz)</td>
<td>2.25</td>
<td>2.08</td>
<td>2.03</td>
</tr>
<tr>
<td>(f_2) (Hz)</td>
<td>2.54</td>
<td>2.17</td>
<td>2.05</td>
</tr>
<tr>
<td>(\sigma_{x_1})</td>
<td>0.141</td>
<td>0.141</td>
<td>0.141</td>
</tr>
<tr>
<td>(\sigma_{x_2})</td>
<td>0.118</td>
<td>0.133</td>
<td>0.138</td>
</tr>
<tr>
<td>(\sigma_{x_3})</td>
<td>0.0982</td>
<td>0.125</td>
<td>0.135</td>
</tr>
<tr>
<td>(\sigma_{x_1})</td>
<td>1.77</td>
<td>1.77</td>
<td>1.77</td>
</tr>
<tr>
<td>(\sigma_{x_2})</td>
<td>1.67</td>
<td>1.73</td>
<td>1.76</td>
</tr>
<tr>
<td>(\sigma_{x_3})</td>
<td>1.57</td>
<td>1.70</td>
<td>1.74</td>
</tr>
<tr>
<td>(p_{x_1x_1})</td>
<td>0.0268</td>
<td>0.200</td>
<td>0.693</td>
</tr>
<tr>
<td>(p_{x_1x_2})</td>
<td>0.319</td>
<td>0.510</td>
<td>0.302</td>
</tr>
<tr>
<td>(p_{x_1x_3})</td>
<td>0.182</td>
<td>0.416</td>
<td>0.468</td>
</tr>
<tr>
<td>(p_{x_2x_2})</td>
<td>-0.283</td>
<td>-0.490</td>
<td>-0.298</td>
</tr>
<tr>
<td>(p_{x_2x_3})</td>
<td>0.318</td>
<td>0.510</td>
<td>0.302</td>
</tr>
<tr>
<td>(p_{x_3x_3})</td>
<td>-0.143</td>
<td>-0.384</td>
<td>-0.455</td>
</tr>
<tr>
<td>(p_{x_3x_3})</td>
<td>-0.282</td>
<td>-0.490</td>
<td>-0.298</td>
</tr>
</tbody>
</table>
level of accuracy is achieved for $p_3/a_3; H_20849$.

Figs. 6(a and b) show the comparisons for $p_{1+3}(a_1; a_3; \tau)$ and $p_{2+3}(a_2; a_3; \tau)$, while $p_{1+2}(a_1; a_2; \tau)$ can be seen in Fig. 4(c). Fig. 6(c) shows $p_{1+2+3}(a_1; a_2; a_3; \tau)$, the out-crossing probability for the cuboidal domain. The accuracy of the extended VanMarcke formula in estimating this probability is similar to that of VanMarcke formula for estimating the marginal first-passage probabilities and the extended VanMarcke formula for estimating the two-dimensional out-crossing probabilities. Fig. 6(d) compares the joint first-passage probabilities $p_{123}(a_1; a_2; a_3; \tau)$ for the three thresholds. It is seen that the extended VanMarcke formula provides excellent agreement with the simulation results.

Fig. 7 shows the results for the joint first-passage probabilities for the remaining eight cases, demonstrating the effects of the bandwidth and the correlation between the component processes on the accuracy of the approximate formulas. Examination of the results for these cases leads to the same conclusions as made for two-dimensional vector processes.

**Application to System of Interconnected Equipment Items**

Consider the system of five interconnected equipment items in Fig. 8, which may represent a small electrical substation. Equipment items 1 and 2 and equipment items 3 and 4 are connected to each other through a set of identical springs having linear elastic behavior. Other connections are assumed to be sufficiently flexible so as not to cause dynamic interaction.

With the equipment items modeled as linear SDOF oscillators, the equation of motion of the connected equipment items 1 and 2 is written in a matrix form as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = -\mathbf{L}\ddot{x}_g$$  \hfill (20a)

where

$$\mathbf{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 + c_0 & -c_0 \\ -c_0 & c_2 + c_0 \end{bmatrix}$$  \hfill (20b)

$$\mathbf{K} = \begin{bmatrix} k_1 + k_0 & -k_0 \\ -k_0 & k_2 + k_0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$  \hfill (20c)

where $x_g(t)$=ground acceleration; $u_i(t)$=displacement of the $i$th equipment item at its attachment point relative to its base; $m_i$, $c_i$, $k_i$, and $l_i$, $i=1,2$,=effective mass, damping, stiffness, and external inertia force values of the equipment items, respectively; and $c_0$ and $k_0$, respectively, denote the viscous damping coefficient and the stiffness of the connector. The equation of motion for equipment items 3 and 4 is obtained by replacing indices 1 and 2 of Eq. (20) with 3 and 4, respectively. Equipment item 5, which has no significant dynamic interaction with other equipment items, is modeled as a stand-alone SDOF oscillator having the equation of motion.
Fig. 5. Joint first-passage probability \( p_{12}(a_1, a_2; \tau) \) for (a) “medium-low,” (b) “medium-high,” (c) “narrow-medium,” (d) “wide-medium,” (e) “narrow-low,” (f) “narrow-high,” (g) “wide-low,” and (h) “wide-high” categories.
Note, however, that the response $u_3(t)$ of this equipment item is correlated with the responses of the other equipment items because of the common excitation. See Song (2004) for details on the characterization of electrical substation equipment as SDOF oscillators.

The following values of the parameters are used for this example: $m_1=438$ kg, $m_2=210$ kg, $m_3=403$ kg, $m_4=193$ kg, $m_5=200$ kg, $k_1=k_3=158$ kN/m, $k_2=k_4=k_5=198$ kN/m, $l_1=m_1=1.0$, and $\zeta_1=c_1/(2\sqrt{m_1k_1})=0.02$ for $i=1,\ldots,5$, $c_0=0$, and $k_0=25.7$ kN/m. The ground acceleration $\ddot{x}_g(t)$ is assumed to be a stationary, zero-mean Gaussian, filtered white-noise process defined by the Kanai-Tajimi PSD (Clough and Penzien 1993)

$$m_3\ddot{x}_g + c_3\dot{x}_g + k_3x_g = -l_3\ddot{x}_g$$

The connectivity failure event of the system, denoted $E_{\text{system}}$, is described by

$$E_{\text{system}} = E_1E_3 \cup E_2E_4 \cup E_1E_4E_5 \cup E_2E_3E_5$$

where $E_i$, $i=1,\ldots,5$, denotes the event that the displacement of the $i$th equipment item exceeds its prescribed safe threshold. Since the system is neither series nor parallel, no theoretical bounding formulas are available, while Monte Carlo simulation is costly and would require complete information, i.e., the probability of joint failure of all five equipment items. Song and Der Kiureghian (2003) have shown that linear programming (LP) can be used to obtain the narrowest possible bounds on the failure probability of a system for any given partial information on marginal and joint component failure probabilities. The lower (upper) bound on the system failure probability is obtained by solving the LP problem

$$\text{minimize} (\text{maximize}) \; c^T p$$

subject to \quad $a_1 p = b_1$ \quad (24b)

$$a_2 p \geq b_2$$ \quad (24c)

where $p=[p_1, p_2, \ldots, p_{24}]$ = vector containing the probabilities of all mutually exclusive and collectively exhaustive events in the system as a function of the root-mean-square (rms) of the ground acceleration. This is commonly known as the fragility function. The duration of the stationary response is assumed to be 20 s. For each intensity level, the spectral moments $\lambda_0, \lambda_1$, and $\lambda_2$ for each equipment item in the system are computed by linear random vibration analysis (Lutes and Sarkani 2004). Using the spectral moments, the marginal and joint first-passage probabilities, $P_1, \ldots, P_5, P_{12}, P_{13}, \ldots, P_{55}$, are computed by the original and extended Poisson and VanMareke formulas. The prescribed safe displacement thresholds are ±7.62 cm for equipment 1 and 3, and ±3.81 cm for equipment 2, 4, and 5.
Fig. 7. Joint first-passage probability $p_{123}(a_1, a_2, a_3; \tau)$ for (a) “medium-low,” (b) “medium-high,” (c) “narrow-medium,” (d) “wide-medium,” (e) “narrow-low,” (f) “narrow-high,” (g) “wide-low,” and (h) “wide-high” categories.
the sample space of the $n$ components; $e^2 p$ the probability of the system failure event; and Eqs. (24b) and (24c) are equality and inequality constraints expressing known values and bounds, respectively, on marginal or joint component probabilities. Employing this approach, the proposed joint first-passage probability estimates can be used to compute narrow bounds on the failure probability of multicomponent systems under stochastic demands. In this example, only marginal and bicomponent failure probabilities are employed. Fig. 9 shows the fragility of each equipment item and the lower and upper bounds on the system fragility. For this example, the system probability bounds are practically coinciding.

Summary and Conclusions

The notion of joint first-passage probability and its essential role in estimating the reliability of systems subjected to stochastic excitation is introduced. It is shown that the joint first-passage probability of a vector process can be approximated by a recursive formula involving lower order joint first-passage probabilities and the out-crossing probability of the vector process. Approximate formulas are derived for the out-crossing probability by extending the Poisson and VanMarcke approximations for scalar processes. The extended VanMarcke formula approximately accounts for the dependence between the crossing events. Detailed formulas are derived for the cases of two- and three-dimensional stationary, zero-mean Gaussian vector processes. These include new results for the cross-correlation and joint PDF of Cramer-Leadbetter envelopes of two or more processes.

The accuracy of the proposed formulas is examined through comparisons with simulated stationary responses of SDOF oscillators subjected to white noise excitation. It is found that the extended VanMarcke formula provides excellent agreement with the simulation results, while the extended Poisson formula shows significant errors. The extended VanMarcke formula is less accurate in the case of strongly narrow-band processes—an error that is inherited from the original formula for marginal first-passage probability—but it still leads to reasonably accurate estimates. The performance of extended VanMarcke formula is not affected by the correlation coefficient between the processes.

The usefulness of the proposed formulas is demonstrated by their application to a small system consisting of five interconnected equipment items subjected to a stochastic ground motion. Each equipment item is modeled by an SDOF oscillator, while the connecting elements are modeled as linear springs. Random vibration analysis is used to compute the spectral moments of the response vector process, which are used to compute the marginal and joint first-passage probabilities. These probabilities are used to compute bounds on the probability of failure of the system by use of linear programming.

Acknowledgments

This paper is based on research supported by the Lifelines Program of the Pacific Earthquake Engineering Research Center funded by the Pacific Gas & Electric Co. and the California Energy Commission. Partial support was also provided by the Earthquake Engineering Research Centers Program of the National Science Foundation under Award No. EEC-9701568 and by the Taisei Chair in Civil Engineering. This support is gratefully acknowledged.

Appendix I. Mean Crossing Rate of Vector Process over Finite Edges

For the case of a two-dimensional, stationary, zero-mean Gaussian vector process, an expression for $v_{12}(a_1|a_2)$ (independent of time) is derived by repeated conditioning of the joint PDF’s in Eq. (12) and analytically evaluating the integral over $x_i$. The result is

$$v_{12}(a_1|a_2) = \frac{2\phi(a_2/x_1)}{\sigma_{x_1}^2} \int_{a_2}^{\infty} \phi \left( \frac{x_2 - \mu_x^{(1)}}{\sigma_{x_2}^{(1)}} \right) \left[ \phi \left( \frac{\mu_x^{(12)}}{\sigma_{x_1}^{(12)}} \right) + \frac{\mu_x^{(12)}}{\sigma_{x_1}^{(12)}} \Phi \left( \frac{\mu_x^{(12)}}{\sigma_{x_1}^{(12)}} \right) \right] dx_2$$

(25)

where $\phi$ denotes the PDF of the standard normal distribution; and $\mu_{x_1}^{(1)} = E[X_1|X_1 = a_1]$; $\sigma_{x_1}^{(1)} = (\text{Var}[X_1|X_1 = a_1])^{1/2}$; $\mu_{x_1}^{(12)} = E[X_1|X_1 = a_1, X_2 = x_2]$; and $\sigma_{x_1}^{(12)} = (\text{Var}[X_1|X_1 = a_1, X_2 = x_2])^{1/2}$. There exist well-known expressions for these conditional means and standard deviations of the normal distribution (Stone 1996).
Table 2. Relative Error $\varepsilon_{r}$ (%) in Estimate of $F_{12}(a_1,a_2;\tau)$ Based on the Nataf Approximation of the Bivariate Probability Density Function of Envelopes

<table>
<thead>
<tr>
<th>$\varepsilon_{r}$</th>
<th>$\eta_{12}\tau=0$</th>
<th>0.01</th>
<th>0.1</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>$-0.933 - 0.924$</td>
<td>$-0.919 - 0.914$</td>
<td>$-0.806 - 0.833$</td>
<td>$-0.262 - 0.332$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.551 - 1.52$</td>
<td>$-0.539 - 1.50$</td>
<td>$-0.451 - 1.37$</td>
<td>$-0.122 - 0.543$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.0933 - 2.31$</td>
<td>$-4.11 - 2.29$</td>
<td>$-2.86 - 2.07$</td>
<td>$-0.485 - 0.802$</td>
</tr>
</tbody>
</table>

In the same manner, $\nu_{1|2}(a_1|a_2,a_3)$ is written as

$$\nu_{1|2}(a_1|a_2,a_3) = \frac{2\phi(a_i/\sigma_{X_i})}{\sigma_{X_i}^{(12)}(\sigma_{X_2}^{(12)})} \int_{-\infty}^{\infty} \phi \left( \frac{x_2 - \mu_{X_2}^{(12)}}{\sigma_{X_2}^{(12)}} \right) \phi \left( \frac{x_3 - \mu_{X_3}^{(12)}}{\sigma_{X_3}^{(12)}} \right) \sigma_{X_1}^{(12)} \sigma_{X_2}^{(12)} \sigma_{X_3}^{(12)} dx_2 dx_3$$

where $\mu_{X_i}^{(12)} = \mu_{X_i}(X_1|X_1=a_1, X_2=x_2); \sigma_{X_i}^{(12)} = \sigma_{X_i}(X_1|X_1=a_1, X_2=x_2)$. This result is derived in a manner similar to the derivation of the moments of an envelope process at two different time points as reported by Middleton (1960).

The Nataf approximation of the joint PDF of $n$ envelopes $E_{i,n}, i=1, \ldots, n$, is then given by (Liu and Der Kiureghian 1986)

$$f_{E_{1,n}}(e_1, e_2, \ldots, e_n) = f_{E_{1,2}}(e_1, e_2) \cdots f_{E_{1,n}}(e_n)$$

where $f_{E_{1,i}}(e_i)$, $i=1, \ldots, n$, is marginal PDF of $E_{i}(t)$. $u$ vector with elements $u_i = \Phi^{-1}[F_{E_{1,i}}(e_i)]$, $i=1, \ldots, n$, in which $F_{E_{1,i}}(e_i)=$ Rayleigh CDF of $E_{i}$; $\mathbf{R}_n=$ correlation matrix of $u$; and $\varphi_n(u; \mathbf{R}_n)$ is the corresponding $n$-variate normal PDF. The element $\rho_{0,ij}$ of $\mathbf{R}_n$ is then given by the correlation coefficient $\rho_{E_{1,i},E_{1,j}}$ through the double integral formula

$$\rho_{E_{1,i},E_{1,j}} = \frac{\pi}{4 - \pi} \left[ 2F_1 \left( -\frac{1}{2}; -\frac{1}{2}; 1; k_{0,ij}^2 \right) - 1 \right]$$

(28)

where $2F_1$ denotes the Gauss hypergeometric function and $k_{0,ij}^2=(R_{0,ij}^2+R_{ij}^2)\left(\sigma_{X_i}^2/\sigma_{X_j}^2\right)$. This result is derived in a manner similar to the derivation of the moments of an envelope process at two different time points as reported by Middleton (1960).

For a given $\rho_{E_{1,i},E_{1,j}}$ of Eq. (28), one can find the corresponding $\rho_{0,ij}$ by iteratively solving Eq. (30) or using approximate formulas developed by Liu and Der Kiureghian (1986). For the Rayleigh random variables $E_{1,i}$ and $E_{1,j}$, the formula is

$$\rho_{0,ij} = \rho_{E_{1,i},E_{1,j}}(1.028 - 0.029 \rho_{E_{1,i},E_{1,j}})$$

(31)

In order to examine the accuracy of the Nataf joint distribution for the envelopes, the probability $p_{1,2}(a_1,a_2;\tau)$ that a two-dimensional, stationary, zero-mean Gaussian vector process crosses a rectangular domain during an interval of time $T=0, \tau$ is computed by use of Eq. (13) employing the exact bivariate PDF in Eq. (27) and the approximation bivariate PDF obtained from Eq. (29). The relative error $\varepsilon_{r}$ is defined as

$$\varepsilon_{r} = \left[ \frac{1 - B_{12}^{exact} \exp(-\eta_{12}\tau)}{1 - B_{12}^{Nataf} \exp(-\eta_{12}\tau)} \right] \times 100(\%)$$

(32)

where $B_{12}^{exact}$ and $B_{12}^{Nataf}$ denote the probability that the vector of envelope processes is inside the rectangular domain at $t=0$, computed by use of the exact and approximate bivariate PDFs, respectively. The relative errors are computed for a total of 12 cases defined by the values of the mean number of out-crossings $\eta_{12}\tau$ and the aspect ratio $r_x = (a_1/\sigma_{X_1})(a_2/\sigma_{X_2})$ of the rectangular domain. The specific values $\eta_{12}\tau=0, 0.01, 0.1, 1$, and $r_x = 1, 2, 3$ are considered, and for each case the range of errors for the complete range of envelope correlation values $0 \leq \rho_{E_{1,i},E_{1,j}} \leq 1$ is determined. Table 2 lists the computed percent error values for each
case. As expected, the errors are larger when the mean number of out-crossings is small, since in those cases the probability $p_{1+2}(a_1,a_2;\tau)$ is dominated by the outcome at $t=0$. The error is also larger when the aspect ratio is close to 1, since in that case the probability is not dominated by one process. However, error values are all small, with a maximum of slightly higher than 4% for $\eta_{12}\tau=0$ and $r_e=1$ and values much smaller than 1% for $\eta_{12}\tau=1$. These results confirm that the Nataf model provides a good approximation of the bivariate PDF of the envelopes for the purpose of computing the out-crossing probability of two processes. Although this examination is limited to the case of a two-dimensional vector process, for which an exact solution of the bivariate envelope distribution is available, we can conjecture that similar accuracy exists for cases with higher dimensions.

References


