Complex Boundary Integral Equation Formulation and Stability Analysis of a Maxwell Model and of an Elastic Model of Solid-Solid Phase Transformations

by

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A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Mathematics in the Graduate Division of the University of California, Berkeley

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Abstract

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We study a viscoelastic model of the solid-solid phase change of olivine to its denser $\beta$-spinel state at high pressures and temperatures reachable in laboratory experiments matching conditions typical of Earth’s mantle. Using a previously unknown technique, the equations are transformed to the problem of finding two complex analytic functions in the sample satisfying certain conditions on the outer boundary. The Sherman-Lauricella boundary integral equation is used in a numerical algorithm that eliminates the bottleneck of having to solve a large matrix equation at every timestep. The method is implemented and used to compute the solution of a number of non-axisymmetric test problems, some static and some dynamic in time. Next we develop an alternative formulation in which the Lamé equations of linearized elasticity are used to model the deformation of the two phases, and we allow for compressibility. The formulation is novel in that separate reference configurations are maintained for the core and shell regions of the sample that grow or shrink in time by accretion or removal at the boundary, one at the expense of the other. We then compare the behavior of the evolution of this system to the incompressible viscoelastic case and to an alternative elastic model. Finally, we study the stability of circular interfaces with axisymmetric initial data under the evolution equations. For various parameter values of the circular interface evolution, we find families of small perturbations of the circular interface and radial interface velocity jump that either grow or decay exponentially in time. In unstable cases, the growth rate increases without bound as the wave number of the perturbation increases. In stable cases, the evolution equations are well-posed until the interface leaves the stability regime, at which point the numerical solutions blow up in an oscillatory manner. Examples of stable and unstable behavior are presented.
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Chapter 1

Introduction

In Earth’s mantle, the material olivine, which is one of the planet’s most abundant minerals [8, 69], undergoes a solid-solid phase transformation to its denser $\beta$-spinel state. This transformation has been used to explain the sudden increase in the propagation velocity of seismic waves at depths of 410 kilometers below Earth’s surface [2, 17, 61, 69] and is important to geophysics as it affects mantle circulation and the cooling rate of Earth [45]. It has also been used to attempt to explain the occurrence of deep earthquakes [6, 14, 25, 43, 45].

Laboratory experiments by Kubo et al. [31, 32] on small samples of olivine put under pressures and temperatures found at depths of hundreds of kilometers under the surface of the earth show that a thin ring of $\beta$-spinel appears near the edge of the sample and begins to nucleate further inward. Many models of this transformation have been proposed, focused mainly on spherically symmetric geometries.

This dissertation studies the equations first presented by Morris in [43] to model the three-dimensional olivine to $\beta$-spinel transformation. We study the equations in a cylinder, using a two-dimensional geometry with the problem of plane strain, and allow for asymmetric samples and interfaces. The governing equations are difficult to solve because the stress tensor and velocity vectors are related by an equation that has a time derivative on the stress but not on velocity. We resolve this issue by employing a previously unknown technique to reformulate the system so that the evolution at any point in time is written in terms of variables that exist purely at the interface at only that point in time. This important variable change allows us to reduce the system to a complex analysis problem that we show how to solve using robust, fast, and accurate boundary integral equation methods that we combine with a high-order Runge-Kutta method to evolve the system.

We then propose a new model of the olivine to $\beta$-spinel transformation that uses the Lamé equations of linear elasticity and show that it is in fact equivalent to the Morris equations with certain parameters. We compare these models to a recent model that Morris [44] has proposed that uses linear elasticity rather than a Maxwell creeping flow. We discuss the different physical assumptions in the models that lead to different systems of equations.

Finally, a detailed stability analysis is presented in which we describe a family of perturbations that under certain conditions grows exponentially. Stability regions are found and
plotted, and for the unstable regions we give examples of how the perturbations grow both in time and with wave number. We explain why in numerical simulations, some interfaces evolve in a stable way for some time before suddenly blowing up.

This dissertation is organized as follows. In Chapter 2, we develop a new technique to transform the governing equations into a mathematically equivalent system that is easier to work with both theoretically and numerically. In Chapter 3, solving that system is reduced to the problem of finding two complex analytic functions satisfying certain boundary conditions. In Chapter 4, we describe robust, fast, and accurate numerical algorithms that we have developed to solve the fixed time equations and evolve the interface in time. In Chapter 5, we propose an alternative to the Maxwell model that uses the Lamé equations of linearized elasticity, and the resulting system of equations is shown to be equivalent to the Maxwell model with appropriately chosen constants. We and compare and contrast this formulation with an alternative elastic model recently proposed by Morris [44]. Chapter 6 presents a technical and detailed stability analysis that has not been done before.

1.1 Governing equations

Given an initial sample of olivine, we assume initial conditions of zero initial velocity and deviatoric stress. A pressure \( P_a \) is applied to the outer boundary from the outside, which causes the material near the boundary to change to the denser phase. Figure 1.1 depicts the geometry of the problem. The initial sample is mapped to the inside of a simple closed curve \( \Gamma_O \). The interface between lower and higher density phases, called \( \Gamma_I \), moves in with time. We call the region inside \( \Gamma_I \) the core, denoted \( \Omega_c \), and the region between \( \Gamma_I \) and \( \Gamma_O \) the shell, denoted \( \Omega_s \). We’ll use subscripts “c” and “s” to describe core and shell variables, respectively.

At each point in \( \Omega \), the Cauchy stress tensor \( \sigma \) is a symmetric, second rank tensor in three dimensions (i.e., \( \sigma(x) \) is a self-adjoint linear map \( \sigma(x) : \mathbb{R}^3 \to \mathbb{R}^3 \)) that describes the state of stress of the material at that point. For a point \( x \) in \( \Omega \) and a unit length vector \( v \in \mathbb{R}^3 \), \( \sigma(x)v \) is the internal force per unit area across a surface orthogonal to \( v \) at \( x \). The pressure is defined as \( p = -\frac{1}{3} \text{tr}(\sigma) \) and deviatoric stress is \( \sigma' = \sigma + pI \). In plane strain we are not concerned with the entries of \( \sigma \) that describe the stress in the direction of the axis of the cylinder, so from here we will consider \( \sigma \) as a \( 2 \times 2 \) matrix. We write \( \tilde{\mathbf{u}} \) rather than \( \mathbf{u} \) to denote the stress tensor at the point \( x \) at time \( t \), and \( \tilde{\mathbf{u}}(x,t) \) is the velocity at \( (x,t) \). The initial conditions of the system are \( \Gamma_I = \Gamma_O \), \( \sigma'(\cdot,0) = 0 \) and \( \tilde{\mathbf{u}}(\cdot,0) = \mathbf{0} \). (The reason we write \( \tilde{\mathbf{u}} \) rather than \( \mathbf{u} \) will become clear in Section 2.3 below, where we solve an auxiliary Stokes problem to connect \( \sigma \) to \( \tilde{\mathbf{u}} \).)

In each of the core and shell, at all times, the equations describing the stress tensor \( \sigma \) and velocity \( \tilde{\mathbf{u}} \) of the material are

\[
\nabla \cdot \sigma = 0 \quad ; \quad \nabla \cdot \tilde{\mathbf{u}} = 0 \quad ; \quad \mu(\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^T) = \frac{\partial}{\partial t}\sigma' + \frac{\mu}{\eta}\sigma'.
\]
CHAPTER 1. INTRODUCTION

Figure 1.1: Geometry of the problem. The core $\Omega_c$ consists of olivine, and the shell $\Omega_s$ is the $\beta$-spinel.

These are, respectively, the momentum equation for slow flow, the continuity equation for incompressible flow, and the flow rule for a Maxwell solid. The divergence of a second rank tensor is the vector of the divergences of the rows, so $\nabla \cdot \sigma = 0$ means that for each $i = 1, 2$, \[
\frac{\partial}{\partial x_1} \sigma_{i1} + \frac{\partial}{\partial x_2} \sigma_{i2} = 0.
\]
On the outer boundary a hydrostatic stress is applied with pressure $P_a$, so that at each point on $\Gamma_O$ with outward facing normal $n$ the traction $\sigma n$ satisfies
\[
\sigma n = -P_a n. \tag{1.2}
\]

The interface conditions will involve the higher density $\rho_s$ of the $\beta$-spinel in the shell and the lower density $\rho_c$ of the olivine in the core, the (positive) difference $\Delta \rho = \rho_s - \rho_c$, and the outward normal velocity of the interface $V$. For a function $f$ defined near the interface and a point $\zeta \in \Gamma_I$, the jump across the interface of $f$ is defined as
\[
[f](\zeta) = \lim_{z \to \zeta} f(z) - \lim_{z \to \zeta} f(z).
\]
Interfacial traction and mass balance require, respectively,
\[
[\sigma n] = 0 \quad \text{and} \quad [\rho(\tilde{u} \cdot n - V)] = 0. \tag{1.3}
\]
The mass balance equation reduces to $\rho_s(\tilde{u} \cdot n) - \rho_c(\tilde{u} \cdot n) = \Delta \rho V$. In real-world examples the density difference is small, and here we treat $\frac{\Delta \rho}{\rho}$ as a small parameter, where $\rho = \rho_c \approx \rho_s$. The above equation becomes $[\tilde{u} \cdot n] = \frac{\Delta \rho}{\rho} V$, which is the form we will use. Finally, we impose a no-slip condition $[\tilde{u} \cdot s] = 0$, where $s$ is the tangent to the interface. This together
with the mass balance equation gives us both normal and tangential components of $\tilde{\mathbf{u}}$ and we get

$$[\tilde{\mathbf{u}}] = \frac{\Delta \rho}{\rho} \mathbf{Vn}. \quad \text{(1.4)}$$

Finally, Morris [43] uses a thermodynamic argument to model the relationship between the interface velocity and traction as

$$V = c(-P - \mathbf{n} \cdot \mathbf{\sigma}), \quad \text{(1.5)}$$

where $c$ is a constant depending on the material and $P$ is the coexistence pressure. Similar models have been developed by Herring [22, 3] and Mullins [48, 47] in the context of surface and grain-boundary diffusion, where the growth rate is proportional to $\partial^2 s \sigma_{nn}$. Here $s$ is arclength along the interface and $\sigma_{nn} = \mathbf{n} \cdot \mathbf{\sigma}$. These stress-driven grain boundary diffusion models have been employed by many researchers to model electromigration and microchip failure [4, 72, 64, 73, 74].

In Chapter 5, we present a new elastic model of solid-solid phase transformations as an alternative to the above Maxwell model. We formulate the problem in an Eulerian framework on a domain with a moving interface. This formulation is novel in that we maintain separate reference configurations for the core and the shell whose boundaries evolve in time as mass is transported from one to the other. The displacement field (defined on the current configuration, which has no gaps or overlaps) is discontinuous across the interface due to the separate reference configurations of the two regions. We also compare this model to a recent elastic model of Morris [44], in which the initial hydrostatic state is chosen as the reference configuration, and the displacement field is continuous. This elastic model suggests that a term is missing from the deviatoric strain rate equation in the earlier paper [43] in which the Maxwell model was first proposed. For certain choices of parameters, our elastic model leads to the same equations as the Maxwell model, still missing the term. We identify the modeling assumptions in the present analysis and implicit in the analysis of [44] which lead to the discrepancy, and discuss the merits of the two approaches.
Chapter 2

Mathematical formulation

In this chapter we convert equations (1.1)–(1.5) to a system that lends itself nicely to solving numerically. As the system currently stands, equation (1.4) causes problems with advancing the interface because $\tilde{u}$ and $\sigma$ are linked only by equation (1.1), which has a time derivative on $\sigma$. This issue is resolved in Section 2.3, where the function $u$ is introduced and the evolution equations are transformed so that the system advances through time in terms of quantities that evolve purely on the interface.

2.1 Nondimensionalizing the variables

We start with nondimensionalizing equations (1.1)–(1.5). Let $\Delta P = P_a - P$, and for length scale $L$ and time scale $\tau$, let $C = \frac{L}{\tau} = c \Delta P$. To nondimensionalize, using star subscripts for the dimensionalized variables, we make the transformations

\[
\begin{align*}
x &= \frac{x}{L} \\
t &= \frac{t}{\tau} \\
p &= \frac{p - P}{\Delta P} \\
\sigma' &= \frac{\sigma}{\Delta P} \\
\tilde{u} &= \left(\frac{\Delta \rho}{\rho}\right)^{-1} \tilde{u}_s. \\
\end{align*}
\]

After this scaling, the system defined by equations (1.1)–(1.5) becomes

\[
\begin{align*}
\nabla \cdot \sigma &= 0 \\
\nabla \cdot \tilde{u} &= 0 \\
\nabla \tilde{u} + \nabla \tilde{u}^T &= \left(c_1 \frac{\partial}{\partial t} + c_2\right) \sigma' \\
\end{align*}
\]

in $\Omega_c$ and $\Omega_s$. 

\[ \begin{cases} \left[ \sigma n \right] = 0 & \text{on } \Gamma_I, \\ \hat{\mathbf{u}} = V n & \text{on } \Gamma_I, \\ \sigma n = -n & \text{on } \Gamma_O, \end{cases} \]

where

\[ c_1 = \frac{\Delta P}{\rho \mu} \quad \text{and} \quad c_2 = \frac{1}{c \frac{\Delta \rho}{\rho \eta}}. \quad (2.1) \]

The equation for the movement of the interface becomes

\[ V = n \cdot \sigma n. \]

This quantity arises frequently and we will often write \( \sigma_{nn} \) to mean \( n \cdot \sigma n \). 

### 2.2 Exact solution in a circle

In the case where \( \Gamma_I \) and \( \Gamma_O \) are circles, following [43], which solves the problem in the spherically symmetric case, we can give a closed form for the right-hand side of the differential equation for \( R \) and \( f \).

If \( \hat{\mathbf{u}} \) is written as \( \hat{\mathbf{u}}(r, \theta) = \hat{w}(r, \theta) \hat{r} \), where \( \hat{r} \) is the outward unit radial vector, then

\[ \nabla \cdot \hat{\mathbf{u}} = 0 \implies \frac{1}{r} \partial_r (r \hat{w}(r)) = 0, \]

so that \( \hat{w}(r, t) = \frac{\tilde{F}(t)}{r} \) for some \( \tilde{F} \) that depends on time. This implies that \( \tilde{w}^c \equiv 0 \) and \( \Delta \hat{w} = \Delta \hat{w} - \hat{w}_{r} = \frac{1}{r} \partial_r (r \partial_r \hat{w}) - \hat{w}_r = 0 \). Thus \( p \) is constant in the core and the shell. Let \( R(t) \) the radius of the interface at time \( t \). The equation \( \left[ \hat{\mathbf{u}} \right] = -V n \) gives us \( \tilde{F} \):

\[ \left[ \hat{\mathbf{u}} \right] = -V n \implies \hat{w}(R) = -V \implies \frac{\tilde{F}(t)}{R} = -V \implies \tilde{F}(t) = -RV. \]

The radial part of \( \nabla \hat{\mathbf{u}} + \nabla \hat{\mathbf{u}}^T = (c_1 \frac{\partial}{\partial r} + c_2) \sigma' \) is \( (c_1 \frac{\partial}{\partial r} + c_2) \sigma'_{rr} = 2 \partial_r \hat{w}(r) = -2 \frac{\tilde{F}}{r^2} = 2RV \frac{2}{r^2}. \)

So \( \sigma' = \frac{S(t)}{r^2} \) for some \( S(t) \) and we get the equation

\[ \left( c_1 \frac{d}{dt} + c_2 \right) S = 2RV. \]

The interfacial traction balance

\[ \sigma_c n = \sigma_s n \implies -p_c = -p_s + \sigma'_{rr} \implies -p_c = -p_s + \frac{S}{R^2} \]

and zero traction condition on the outer edge

\[ \sigma_s n = 0 \implies -p_s + \sigma'_{rr} = 0 \implies -p_s + S = 0. \]
Combining these, we get $p_c = S(1 - R^{-2})$. Since $V = 1 - \sigma_{rr} = 1 + p_c$, we get a closed system with the two equations
\[
\begin{align*}
\dot{R} &= -1 - S(1 - R^{-2}) \quad (2.2) \\
c_1 \dot{S} + c_2 S &= -2 R \dot{R}. \quad (2.3)
\end{align*}
\]

The evolution equations (2.2)–(2.3) agree in spirit with the three dimensional solution given in [43], with differences that are inherent when moving to the two dimensional problem.

### 2.3 Conversion to Stokes

This system is difficult to deal with because the velocity $\tilde{u}$ and stress $\sigma$ are linked only through the Maxwell flow rule. We’ll find an equivalent system in variables that aren’t related by a time derivative. At each point $x \in \Omega$, let $u$ (without a tilde) be the solution of the ordinary differential equation
\[
\left( c_1 \frac{\partial}{\partial t} + c_2 \right) u = \tilde{u} \quad (2.4)
\]
with initial condition $u(x, 0) = 0$. Since $\nabla \cdot \tilde{u} = 0$, $u$ must also be divergence free because $0 = \nabla \cdot \left( \left( c_1 \frac{\partial}{\partial t} + c_2 \right) u \right) = \left( c_1 \frac{\partial}{\partial t} + c_2 \right) (\nabla \cdot u)$ and $\nabla \cdot u(x, 0) \equiv 0$.

We will sometimes add a subscript $s$ or $c$ to $u$ or $\tilde{u}$ to indicate its restriction to the shell or core. We note that care must be taken when defining $u_s$ by (2.4) above. At time 0 the shell is negligible, so there is no initial condition at $t = 0$ from which to evolve (2.4). For $x \in \Gamma_O$, we set $u_s(x, 0) = 0$, as in the core. For a given $x_0 \in \Omega$, at the time $t_0$ that the interface crosses $x_0$ we define $u_s(x_0, t_0)$ to be $\lim_{x \to x_0} u_s(x, t_0)$, $x \in \Omega_s$, so that the function is continuous up to the boundary from the shell.

Just as in the above arguments, the initial conditions $\sigma' \equiv 0$, $u_s = 0$, and $u_c = 0$ ensure that $\nabla u + (\nabla u)^T = \sigma'$. The divergence of this equation is
\[
\begin{pmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy} \end{pmatrix} + \begin{pmatrix} u_{xx} + v_{xy} \\ u_{yx} + v_{yy} \end{pmatrix} = \nabla \cdot \sigma'.
\]
Because $u_x + v_y = 0$ the second term drops out, and we are left with $\Delta u = \nabla \cdot \sigma'$. This together with
\[
0 = \nabla \cdot \sigma = \nabla \cdot (p I + \sigma') = \nabla p + \nabla \cdot \sigma'
\]
gives us $-\nabla p + \Delta u = 0$, so $p$ and $u$ solve the Stokes equations.

### 2.4 Evolving the interface

We would like to solve this system to find the interface velocity and advance in time, but in order to do that we need to know the interface condition for $[u]$ and how to evolve it.
The initial condition $J_u = 0$ tells us how to start this process. If we track a fixed interface point $x(t)$ through time we can find an ordinary differential equation for $[u](x(t), t) = (u_s - u_c)(x(t), t)$. There is a choice here for $x(t)$ in how to move the point with the interface, and there is no natural choice because the interface doesn’t consist of physical atoms moving. First it will be shown how to evolve $J_u$ in the case that $x(t)$ moves normally to the interface, as that’s easier because of our definition of $V$ and the interface velocity in the normal direction. We will then consider a more general case. If $x$ moves in the outward unit normal direction $n$, then for both the shell and core the time derivative of $u(x(t), t)$ is

$$\frac{d}{dt} \left( [u](x(t), t) \right) = (\nabla u(x(t), t)) \dot{x}(t) + \frac{\partial}{\partial t} u(x(t), t) = V (\nabla u) n + \frac{\partial}{\partial t} u$$

because $\dot{x}(t) = V n$. So

$$\frac{d}{dt} \left( [u](x(t), t) \right) = \left( \frac{d}{dt} u(x(t), t) \right) = V [\nabla u] n + \left[ \frac{\partial}{\partial t} u \right].$$

Note that we could not have skipped the step of writing $\frac{d}{dt} u_s$ and $\frac{d}{dt} u_c$ separately because $\nabla [u]$ is nonsensical as $[u]$ exists only on the interface. On the other hand, we don’t know $\frac{\partial}{\partial t} u_s$ or $\frac{\partial}{\partial t} u_c$, but $\left[ \frac{\partial}{\partial t} u \right]$ can be derived from $[\tilde{u}]$. We have $\left[ (c_1 \frac{\partial}{\partial t} + c_2) u \right] = [\tilde{u}] = V n$, so

$$\frac{\partial}{\partial t} [u] = \frac{1}{c_1} V n - \frac{c_2}{c_1} [u].$$

We are left with

$$\frac{d}{dt} [u] = V [\nabla u] n + \frac{1}{c_1} V n - \frac{c_2}{c_1} [u].$$

This is a formula for $\frac{d}{dt} [u]$ that involves the variables only at the current time, so there is no potential problem with having to use finite differences or another such method to compute time derivatives of interface evolution.

For the case where $x(t)$ doesn’t move normally to the interface, $\left[ \frac{\partial}{\partial t} u \right]$ will be the same but the advection term $V [\nabla u] n$ will change. If $x$ moves in the $w$ direction with velocity $\dot{x} = |\dot{x}| w$, we decompose $\dot{x}$ as

$$|\dot{x}| w = V n + Ts,$$

where $n$ is the outward unit normal and $s$ is the unit tangent vector in the positive (counterclockwise) direction. We will write $|\dot{x}|$ and $T$ in terms of $V$. Taking the dot product of both sides with $n$ determines $|\dot{x}|$ as

$$|\dot{x}| = \frac{V}{w \cdot n}.$$

Taking the dot product with $s$ gives us

$$T = \frac{w \cdot s}{w \cdot n} V. \quad (2.5)$$

So

$$\frac{d}{dt} [u] = \frac{V}{w \cdot n} [\nabla u] w + \frac{1}{c_1} V n - \frac{c_2}{c_1} [u].$$
In our numerical method we’ll move each interface point in the radial direction \( r \), so for a fixed initial interface point \((R_0 \cos \alpha, R_0 \sin \alpha)\), its movement is described by \( x(\alpha, t) = (R(\alpha, t) \cos \alpha, R(\alpha, t) \sin \alpha)\), \( R(\alpha, 0) = R_0 \), and \( \dot{R} = \frac{V}{r \cdot n} \). The initial condition for

\[
 f(R(\alpha, t), t) := [u](R(\alpha, t), t)
\]

is \( f(R(\alpha, 0), 0) = 0 \), and

\[
 \dot{f} = \dot{R} \nabla u \cdot r + \frac{1}{c_1} V n - \frac{c_2}{c_1} f.
\]

Another variant that is often useful is derived from

\[
 \frac{d}{dt} [u] = [\nabla u] \dot{x} + \left[ \frac{\partial}{\partial t} u \right]
 = T [\nabla u] s + V [\nabla u] n + \left[ \frac{\partial}{\partial t} u \right]
 = T [\nabla u] s + V [\nabla u] n + \frac{1}{c_1} V n - \frac{c_2}{c_1} [u],
\]

which yields

\[
 \dot{f} = T \frac{f_\alpha}{s_\alpha} + V [\nabla u] n + \frac{1}{c_1} V n - \frac{c_2}{c_1} f,
\]

where \( s_\alpha = |x_\alpha(\alpha, t)| \) and \( T = \frac{r \cdot n}{r n} V \).

We can now give a mathematical description of the problem. We are given an initial interface, which is also the fixed outer edge, described by \( R_0 \), and initially \( f \equiv 0 \). We then evolve \( R \) and \( f \) at each interface point by

\[
 \begin{pmatrix}
 \dot{R} \\
 \dot{f}
\end{pmatrix} = \begin{pmatrix}
 \frac{V}{r \cdot n} \\
 \dot{R} \nabla u \cdot r + \frac{1}{c_1} V n - \frac{c_2}{c_1} f
\end{pmatrix}.
\]

Given \( R \) and \( f \), the right-hand side of the ordinary differential equation is determined by the solution to the system

\[
\begin{cases}
 -\Delta u + \nabla p = 0 & \text{in } \Omega_c \text{ and } \Omega_s, \\
 \nabla \cdot u = 0 & \\
 \begin{cases}
 [\sigma n] = 0 \\
 [u] = f
\end{cases} & \text{on } \Gamma_I, \\
 \sigma n = -n & \text{on } \Gamma_O.
\end{cases}
\]
Chapter 3

Complex variable formulation

Complex analysis is a very useful tool widely used in the study of elasticity. For the Maxwell-Stokes model of Morris, which in the last chapter we transformed to equations (2.8)–(2.11), it will allow us to reduce the fixed time equations (2.9)–(2.11) to the problem of finding two complex functions \( \phi \) and \( \psi \) on the inside of \( \Gamma \) that satisfy a boundary condition on \( \Gamma \), and we can get evolution equations for \( \phi \) and \( \psi \) from equation (2.8).

3.1 Complex differential operators

For \( f = u + iv \), with \( u \) and \( v \) real and differentiable on some subset of \( \mathbb{R}^2 \), we define

\[
\partial_z f := \frac{1}{2} ( \partial_x - i \partial_y )(u + iv) = \frac{1}{2} (u_x + v_y) + i \frac{1}{2} (-u_y + v_x) \\
\partial_{\bar{z}} f := \frac{1}{2} ( \partial_x + i \partial_y )(u + iv) = \frac{1}{2} (u_x - v_y) + i \frac{1}{2} (u_y + v_x).
\]

For analytic \( f \), we have

\[
\partial_z f = f' ; \quad \partial_{\bar{z}} \overline{f} = 0 ; \quad \partial_{\bar{z}} f = 0 ; \quad \partial_z \overline{f} = \overline{f}'.
\]

Conversely, if \( u \) and \( v \) have continuous partial derivatives and \( \partial_z(u + iv) = 0 \), then \( u + iv \) is analytic, and the same conclusion can be made for \( \partial_{\bar{z}}(u - iv) = 0 \).

It will be useful to use these equalities later to compute directional derivatives. For a vector \( \mathbf{t} = (t_1, t_2) \) and a complex function \( u + iv \) we will use the notation

\[
\partial_{\mathbf{t}}(u + iv) := (\nabla u)\mathbf{t} + i(\nabla v)\mathbf{t}.
\] (3.1)

If \( u \) and \( v \) are smooth, then \( \partial_x(u + iv) = \partial_z(u + iv) + \partial_{\bar{z}}(u + iv) \) and \( \partial_y(u + iv) = i(\partial_z(u + iv) - \partial_{\bar{z}}(u + iv)) \).
$iv) - \partial \bar{z}(u + iv))$. We can decompose the operator $\partial_4$ into

$$\partial_4 = t_1 \partial_z + t_2 \partial_y$$
$$= t_1(\partial_z + \partial_{\bar{z}}) + t_2 i(\partial_z - \partial_{\bar{z}})$$
$$= (t_1 + it_2)\partial_z + (t_1 - it_2)\partial_{\bar{z}}$$
$$= t\partial_z + i\partial_{\bar{z}},$$

where $t := t_1 + it_2$, the complex version of the vector $t$.

### 3.2 Complex formulation of Stokes equations in a two-dimensional simply connected domain

Let $\Omega \in \mathbb{R}^2$ be open and simply connected with smooth boundary $\partial \Omega$. We will also write $\Omega$ to mean the associated subset of $\mathbb{C}$. Consider the problem of finding the stresses in $\Omega$ given the traction on $\partial \Omega$. We’ve already rewritten the system as a Stokes system in pressure $p$ and velocity $u$, and now we define variables

$$(u, v) = u \ ; \ \gamma = -u_x + v_y \ ; \ \tau = u_y + v_x \ ; \ q = -u_y + v_x.$$ 

The deviatoric stress can be written in terms of $\gamma$ and $\tau$ as $\sigma' = \begin{pmatrix} -\gamma & \tau \\ \tau & -p + \gamma \end{pmatrix}$ and therefore

$$\sigma = \begin{pmatrix} -p - \gamma & \tau \\ \tau & -p + \gamma \end{pmatrix}.$$ 

If $(p, u)$ is a solution to the system (2.9), then

$$p_x = u_{xx} + u_{yy} = -v_{yx} + u_{yy} = -q_y$$
$$p_y = v_{xx} + v_{yy} = v_{xx} - u_{xy} = q_x.$$ 

Thus $-p$ and $q$ are harmonic conjugates and $-p + iq$ is analytic in $\Omega$. Since complex analytic functions in simply connected domains have single-valued antiderivatives, there exists a complex analytic function $\phi$ such that

$$-p + iq = 2\phi'.$$  \hspace{1cm} (3.2)
We also have
\[ \partial \overline{z} (\gamma + i \tau) = \frac{1}{2} (\partial_x + i \partial_y) (\gamma + i \tau) \]
\[ = \frac{1}{2} (\gamma_x - \tau_y + i (\gamma_y + \tau_x)) \]
\[ = \frac{1}{2} (-u_{xx} + v_{xy} - u_{yy} - v_{xy} + i (-u_{xy} + v_{yy} + u_{xy} + v_{xx})) \]
\[ = \frac{1}{2} (-\Delta u + i \Delta v) \]
\[ = \frac{1}{2} (-p_x + ip_y) \]
\[ = \frac{1}{2} (-p_x + iq_x) \]
\[ = \phi'' \]
\[ = \partial \overline{z} (\bar{\phi}'') \]

Since \( \partial \overline{z} (\gamma + i \tau - \bar{\phi}'') = 0 \), \( \gamma + i \tau - \bar{\phi}'' \) is analytic. Because \( \Omega \) is simply connected, there exists \( \chi \) such that \( \chi'' = \gamma + i \tau - \bar{\phi}'' \). Define \( \psi := \chi' \), so that \( \gamma + i \tau = \bar{\phi}'' + \psi' \). This \( \phi \) and \( \psi \) are called Kolosov functions \([38, 49, 50, 72]\).

The real biharmonic function \( W := \Re (\bar{\phi} + \chi) \) is called the Airy stress function, and the stress can be written in terms of partial derivatives of \( W \). We will make extensive use of the identity
\[ W_x + i W_y = 2 \partial \overline{z} W \]
\[ = \partial \overline{z} (\bar{\phi}' + \chi' + \bar{\phi}' + \bar{\chi}) \]
\[ = \phi' + z \bar{\phi}' + \bar{\psi}. \] (3.3)

To write the stress tensor in terms of \( W \), start with
\[ \gamma - i \tau = z \bar{\phi}'' + \bar{\psi}' \]
\[ = \partial \overline{z} (\phi + z \bar{\phi}' + \bar{\psi}) \]
\[ = \partial \overline{z} (W_x + i W_y) \]
\[ = \frac{1}{2} (\partial_x + i \partial_y) (W_x + i W_y) \]
\[ = \frac{1}{2} (W_{xx} - W_{yy}) + i W_{xy}. \]

Thus
\[ \gamma = \frac{1}{2} (W_{xx} - W_{yy}) \]
\[ \tau = -W_{xy}. \]
Similarly,
\[-p = \phi' + \overline{\phi'} = \partial_z(W_x + iW_y) = \frac{1}{2}(W_{xx} + W_{yy}).\]

We conclude that
\[
\sigma = \begin{pmatrix} -p - \gamma & \tau \\ \tau & -p + \gamma \end{pmatrix} = \begin{pmatrix} W_{yy} & -W_{xy} \\ -W_{xy} & W_{xx} \end{pmatrix}.
\]

### 3.3 Velocity in terms of $\phi$ and $\psi$

First we observe that
\[
\partial_z(u + iv) = \frac{1}{2}(\gamma - i\tau)
\]
\[
\partial_z(u + iv) = \frac{1}{2}(0 + iq)
\]
\[
\partial_z(\phi - z\overline{\phi} - \overline{\psi}) = -z\overline{\phi'} - \overline{\psi'} = -(\gamma - i\tau)
\]
\[
\partial_z(\phi - z\overline{\phi} - \overline{\psi}) = \phi' - \overline{\phi'} = 2i\Im(\phi') = iq.
\]

So $u + iv$ and $a + ib := \frac{1}{2}(\phi - z\overline{\phi} - \overline{\psi})$ satisfy
\[
u_x + v_y = a_x + b_y
\]
\[
u_x - v_y = a_x - b_y
\]
\[
u_y + v_x = a_y + b_x
\]
\[
u_y - v_x = a_y - b_x
\]

and thus $\nabla(u - a) = \nabla(b - v) = 0$. So there is some complex constant $A$ such that
\[
u + iv = \frac{1}{2}(\phi - z\overline{\phi} - \overline{\psi}) + A.
\]

Without loss of generality, we may set $A = 0$, absorbing it into $\phi$ or $\psi$.

### 3.4 Boundary and interface conditions in terms of $\phi$ and $\psi$

In general, if on some smooth, simple closed curve $\Gamma$ we are given that $\sigma\mathbf{n} = \mathbf{g}$, where $\mathbf{n}$ is pointing outward, then the tangential vector in the counterclockwise direction is $\mathbf{s} =$
(-n_2, n_1), and we have that
\[ g = \sigma \mathbf{n} \]
\[ = \begin{pmatrix} W_{yy} & -W_{xy} \\ -W_{xy} & W_{xx} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \]
\[ = \begin{pmatrix} n_1 W_{yy} - n_2 W_{xy} \\ -n_1 W_{xy} + n_2 W_{xx} \end{pmatrix} \]
\[ = \begin{pmatrix} \partial_s W_y \\ \partial_s (-W_x) \end{pmatrix}. \]

We can then integrate along \( \Gamma \) to find \( W_x \) and \( W_y \) up to an additive constant. From (3.3), we have
\[ \int_A^B \sigma \mathbf{n} \, ds = -i \left( \phi + z \overline{\phi'} + \psi \right) \bigg|_A^B \] (3.6)
where \( A \) and \( B \) are the endpoints of \( \Gamma \), and the result depends only on the endpoints.

Given \( g \) on \( \Gamma_O \), fix a point \( \xi_0 \in \Gamma_O \) and define \( G(\xi) = i \int_{\xi_0}^{\xi} (g_1(t) + ig_2(t)) \, dt \). Then if the outer boundary condition
\[ \phi(\xi) + \xi \overline{\phi'(\xi)} + \overline{\psi(\xi)} = G(\xi) + C \] (3.7)
is satisfied for some constant \( C \), we have
\[ \partial_s (W_x + iW_y) = \partial_s (\phi + \xi \overline{\phi'} + \overline{\psi}) = \partial_s G = -g_2 + ig_1 = i \sigma \mathbf{n} \] (3.8)
and therefore the boundary traction obtained from \( \phi \) and \( \psi \) is correct. Like \( A \) in (3.5), \( C \) may be set to 0 without loss of generality by adjusting \( \phi \) or \( \psi \).

In our case, the outer boundary traction is \( g = \sigma \mathbf{n} = -\mathbf{n} \), which means that for a smooth parametrization \( \gamma \) of \( \Gamma_O \) with \( \gamma(0) = \xi_0 \), we have
\[ G(\gamma(\beta)) = i \int_{\xi_0}^{\gamma(\beta)} g(t) \, dt \]
\[ = i \int_0^\beta g(\gamma(\alpha))|\gamma'(\alpha)| \, d\alpha \]
\[ = i \int_0^\beta \frac{\gamma'(\alpha)}{|\gamma'(\alpha)|} |\gamma'(\alpha)| \, d\alpha \]
\[ = -\int_0^\beta \gamma'(\alpha) \, d\alpha \]
\[ = -\gamma(\beta) + \gamma(0). \]
Thus \( G(\xi) = -\xi \) satisfies \( \partial_s G = ig \) and therefore
\[ \phi(\xi) + \xi \overline{\phi'(\xi)} + \overline{\psi(\xi)} = -\xi + C \]
for some constant \( C \). We will generally use \( \xi \) to denote points on \( \Gamma_O \), \( \zeta \) for points on \( \Gamma_I \), and \( z \) for points in \( \Omega_c \) or \( \Omega_s \).
3.5 The complex problem

Given the development so far, we now consider the problem of finding $\phi$ and $\psi$, analytic in $\Omega$, such that for some constant $C$,

$$\lim_{z \to \xi, z \in \Omega} \left( \phi(z) + z\phi'(z) + \psi(z) \right) = -\xi + C$$

(3.9)

for all $\xi \in \partial\Omega$. If $(\phi_0, \psi_0)$ is a solution, then $(\phi_1, \psi_1)$ is a solution if and only if there is a real constant $A$ and complex constants $B$ and $B'$ such that $\phi_1(z) = \phi_0(z) + Aiz + B$ and $\psi_1(z) = \psi_0(z) + B'$. Not coincidentally, these are precisely the transformations that preserve the stress variables $p = -2\Re(\phi')$ and $\gamma + i\tau = \phi'' + \psi'$. They do change $u + iv = \phi - z\phi' - \psi$, but account for a rigid body deformation. We haven’t shown that this pressure and velocity solve the Stokes equations, and we’ll do that now.

Writing $p$ and $u$ in terms of $\phi$ and $\psi$ will force $(p, u)$ to be a Stokes system independently of any boundary condition. Let $\phi$ and $\psi$ be any analytic functions in a (possibly multiply connected) domain, $p = -2\Re(\phi')$ and $u = \frac{1}{2} \left( \phi - z\phi' - \psi \right)$. Let $q = 2\Im(\phi')$. Then $2\phi' = -p + iq$ and $2\phi'' = -p_x + iq_x = -p_x + ip_y$, so

$$p_x + ip_y = -2\phi''.$$

To find partial derivatives of $u$ we first compute

$$\partial_z(u + iv) = \frac{1}{2} \partial_z(\phi(z) - z\phi'(z) - \psi(z)) = \frac{1}{2} \left( \phi'(z) - \phi'(z) \right)$$

and

$$\partial_z(u + iv) = \frac{1}{2} \left( -z\phi''(z) - \psi'(z) \right).$$

Then

$$u_x + iv_x = (\partial_z + \partial_z)(u + iv) = \frac{1}{2} \left( \phi'(z) - \phi'(z) - z\phi''(z) - \psi'(z) \right)$$

and

$$u_y + iv_y = i(\partial_z - \partial_z)(u + iv) = \frac{1}{2} i \left( \phi'(z) - \phi'(z) + z\phi''(z) + \psi'(z) \right).$$

Multiplying the last equation by $-i$, we get

$$v_y - iu_y = \frac{1}{2} \left( \phi'(z) - \phi'(z) + z\phi''(z) + \psi'(z) \right),$$

and so

$$\nabla \cdot u = u_x + v_y = \Re(u_x + iv_x + v_y - iu_y) = \Re(\phi' - \phi') = 0.$$

To compute the Laplacian of $u$, we start with

$$u_{xx} + iv_{xx} = (\partial_z + \partial_z)(u_x + iv_x) = \frac{1}{2} \left( \phi'' - \phi'' + (-\phi'' - z\phi''' - \psi''') \right)$$

and

$$u_{yy} + iv_{yy} = i(\partial_z - \partial_z)(u_y + iv_y) = -\frac{1}{2} \left( \phi'' + \phi'' - (\phi'' + z\phi''' + \psi''') \right).$$
Adding these two, we get
\[ \Delta u + i \Delta v = -2 \bar{\phi}'' = p_x + ip_y. \]
Thus, \( \nabla \cdot \mathbf{u} = 0 \) and \( -\Delta \mathbf{u} + \nabla p = 0 \), as required.

We must address two issues that make our problem more complicated. The first is that in the multiply connected shell the development of the complex variable problem doesn’t necessarily hold. Secondly, the velocity appears in the right-hand side of the ordinary differential equation (2.8) for the evolution of \( R \) and \( f \) but isn’t completely determined by the problem of finding \( \phi \) and \( \psi \) that satisfy equation (3.9).

In a multiply connected domain like the shell, where we can’t necessarily take antiderivatives (e.g., in our problem in the case where \( \Gamma_O \) and \( \Gamma_I \) are concentric circles, \( \psi \) is a multiple of \( 1/z \) and doesn’t have a single-valued antiderivative), the standard method (see [50]) is to write \( \phi \) and \( \psi \) in the form
\[
\begin{align*}
\phi(z) &= A \log(z) + \phi^*(z) \\
\psi(z) &= B \log(z) + \psi^*(z),
\end{align*}
\]
where \( A \) and \( B \) are complex constants and \( \phi^* \) and \( \psi^* \) are analytic in the whole shell. Instead of doing this and coupling the shell and core equations at the interface, we will transform the core and shell problems into one problem in all of \( \Gamma^+_O \), the region inside \( \Gamma_O \). We want single-valued \( \phi_s \) and \( \psi_s \) written in terms of physical variables, just as they are in the core, and if we are successful, then the interface condition \( [\sigma \mathbf{n}] = 0 \) will become
\[
\left[ \phi(\zeta) + \zeta \bar{\phi}'(\zeta) + \bar{\psi}(\zeta) \right] = 0.
\]
The velocity jump \( [\mathbf{u}] = f \) becomes
\[
\frac{1}{2} \left[ \phi(\zeta) - \zeta \phi'(\zeta) - \bar{\psi}(\zeta) \right] = f.
\]
These two conditions can be reduced to jumps in \( \phi \) and \( \psi \):
\[
\begin{align*}
[\phi] &= f \\
[\psi] &= \left[ -\bar{\phi} - \zeta \phi' \right] = -\bar{f} - \zeta f'.
\end{align*}
\]
Here, \( f' \) means \( [\phi'] \), which we know from just \( f \) because it is the same as the derivative of \( f \) as a limit along the interface, i.e.
\[
[\phi'](\zeta_0) = f'(\zeta_0) := \lim_{\zeta \to \zeta_0, \zeta \in \Gamma_I} \frac{f(\zeta) - f(\zeta_0)}{\zeta - \zeta_0}. \quad (3.12)
\]
If we have a smooth parametrization \( \gamma \) of \( \Gamma_I \), this is simply
\[
f' = \frac{(f \circ \gamma)'}{\gamma'}.
\]
Now define
\[
\phi_0(z) = \begin{cases} 
\phi_s(z) + \frac{1}{2\pi i} \int_{\Gamma_s} \frac{[\phi](\zeta)}{\zeta - z} d\zeta, & z \in \Omega_s \\
\phi_c(z) + \frac{1}{2\pi i} \int_{\Gamma_c} \frac{[\phi](\zeta)}{\zeta - z} d\zeta, & z \in \Omega_c
\end{cases}
\] (3.13)
and
\[
\psi_0(z) = \begin{cases} 
\psi_s(z) + \frac{1}{2\pi i} \int_{\Gamma_s} \frac{[\psi](\zeta)}{\zeta - z} d\zeta, & z \in \Omega_s \\
\psi_c(z) + \frac{1}{2\pi i} \int_{\Gamma_c} \frac{[\psi](\zeta)}{\zeta - z} d\zeta, & z \in \Omega_c.
\end{cases}
\] (3.14)
The idea is that if \(\phi_s\) and \(\psi_s\) exist, which will be shown in the course of the analysis below, then \(\phi_0\) and \(\psi_0\) are not only continuous across \(\Gamma\), but are in fact analytic in the whole region \(\Gamma^+_O\). We just need a boundary condition for \(\phi_0\) and \(\psi_0\), which we’ll get from the boundary condition \(\phi_s(\xi) + \xi \overline{\phi_s'}(\xi) + \psi(\xi) = -\xi\). To get the \((\phi_0, \psi_0)\) boundary condition it will be useful to integrate by parts
\[
\frac{d}{dz} \left( \frac{1}{2\pi i} \int_{\Gamma_t} \frac{[\phi](\zeta)}{\zeta - z} d\zeta \right) = \frac{1}{2\pi i} \int_{\Gamma_t} \frac{[\phi](\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma_t} \frac{[\phi]'}(\zeta) d\zeta.
\]
As we approach a point \(\xi \in \Gamma_O\) from the inside, we get
\[
\phi_0(\xi) + \xi \overline{\phi_0'}(\xi) + \psi_0(\xi) = \phi_s(\xi) + \xi \overline{\phi_s'}(\xi) + \psi_s(\xi) + 
\]
\[
+ \frac{1}{2\pi i} \int_{\Gamma_t} \left( \frac{[\phi]'}(\zeta) d\zeta - \xi \frac{[\phi]'}(\zeta) d\zeta - \frac{[\psi]'}(\zeta) d\zeta \right)
\]
\[
= -\xi + \frac{1}{2\pi i} \int_{\Gamma_t} \left( \frac{f(\zeta)}{\zeta - \xi} d\zeta + \frac{f(\zeta)}{\zeta - \xi} d\zeta + \frac{f(\zeta)}{\zeta - \xi} d\zeta \right). \tag{3.15}
\]
Once we solve this equation, which is in the simply connected \(\Gamma^+_O\), we can compute the physical variables in the core and shell as functions of \(\phi_0\) and \(\psi_0\).

We have gone through the development of how to arrive at the complex variable formulation of the problem, but haven’t yet given a logical description of the process. In order to do that we will make extensive use of the Sokhotski-Plemelj theorem.

### 3.6 The Sokhotski-Plemelj theorem

Let \(\Gamma\) be a smooth closed curve in \(\mathbb{C}\). Define \(\Gamma^+\) to be the region inside \(\Gamma\) and \(\Gamma^-\) the region outside \(\Gamma\). If \(h\) is a complex analytic function in \(\Gamma^+\), then Cauchy’s integral formula says that \(\frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - z} dt = h(z)\) for \(z \in \Gamma^+\) and Cauchy’s theorem says \(\frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - z} dt = 0\) for \(z \in \Gamma^-\). Thus for a point \(t_0 \in \Gamma\),
\[
\lim_{z \to t_0} \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - z} dt = \lim_{z \to t_0} \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - z} dt = h(t_0).
\]
The Sokhotski-Plemelj theorem [49] shows that this holds for a more general class of functions \(h\) and gives formulas for the above limits.
Now let \( h \) be a function defined only on \( \Gamma \) that is \( \mu \)-Hölder continuous for positive \( \mu \). Here \( h \) is not necessarily the boundary value of an analytic function. Define \( H(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t-z} dt \) for all \( z \in \mathbb{C} \). For \( z \in \Gamma \), \( H(z) \) must be interpreted as a Cauchy principal value integral.

From here on, we use the dashed integral notation \( \oint \) to denote Cauchy principal value integrals.

In order to analyze the behavior of \( H \) near \( \Gamma \), it will be useful to compute the integral \( \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{t-t_0} dt \). For \( \varepsilon \) small enough that \( \Gamma \) intersects the circle of radius \( \varepsilon \) around \( t_0 \) exactly twice, once we cut out the section of \( \Gamma \) inside that circle we are left with a portion of the curve near which \( \log(t-t_0) \) is a single-valued antiderivative of \( \frac{1}{t-t_0} \). We can thus compute \( \int_{\Gamma \setminus B_z(t_0)} \frac{1}{t-t_0} dt \) directly, and taking the limit as \( \varepsilon \to 0 \) we get

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{t-t_0} dt = \frac{1}{2}.
\]

Now we rewrite \( H(t_0) \) as

\[
H(t_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t-t_0} dt = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t) - h(t_0)}{t-t_0} dt + \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t_0)}{t-t_0} dt = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t) - h(t_0)}{t-t_0} dt + \frac{1}{2} h(t_0),
\]

and \( \frac{h(t) - h(t_0)}{t-t_0} \) is integrable in the usual Riemann sense because

\[
\left| \frac{h(t) - h(t_0)}{t-t_0} \right| = \frac{\left| h(t) - h(t_0) \right|}{\left| t-t_0 \right|^{1-\mu}} \leq \frac{C}{\left| t-t_0 \right|^{1-\mu}}
\]

for some constant \( C \), and \( \mu > 0 \) implies that \( |t-t_0|^{\mu-1} \) is integrable.

Let \( J(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)-h(t_0)}{t-z} dt \). A key lemma which we will not prove here (see [49]) is that

\[
\lim_{z \to t_0} \frac{J(z)}{z \in \Gamma^+} = \lim_{z \to t_0} \frac{J(z)}{z \in \Gamma^-}.
\]

Our above calculation of \( H(t_0) \) shows that \( J(t_0) = H(t_0) - \frac{1}{2} h(t_0) \).

We know exactly what \( H(z) - J(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t-z} dt \) is both inside and outside \( \Gamma \) by Cauchy’s theorems since the integrand is analytic in both regions. Cauchy’s theorem implies that \( H(z) - J(z) = 0 \) for \( z \in \Gamma^- \) and by Cauchy’s integral formula \( H(z) = J(z) + h(t_0) \) for \( z \in \Gamma^+ \). Thus

\[
H(z) = \begin{cases} 
J(z) + h(t_0), & z \in \Gamma^+ \\
J(z), & z \in \Gamma^-.
\end{cases}
\]
Putting this all together, we get that
\[
H^+(t_0) := \lim_{z \to t_0, z \in \Gamma^+} H(z) = J(t_0) + h(t_0) = H(t_0) + \frac{1}{2} h(t_0) \quad (3.16)
\]
and
\[
H^-(t_0) := \lim_{z \to t_0, z \in \Gamma^-} H(z) = J(t_0) = H(t_0) - \frac{1}{2} h(t_0).
\]
These are often written in the equivalent form
\[
H^+(t_0) + H^-(t_0) = 2H(t_0) \\
H^+(t_0) - H^-(t_0) = h(t_0),
\]
called the Plemelj formulas.

It will be necessary to figure out the limit of the related quantity \(\frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - z} d\bar{t}\). This can be done by taking complex conjugates. We have
\[
\lim_{z \to t_0, z \in \Gamma^+} \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - z} d\bar{t} = \lim_{z \to t_0, z \in \Gamma^+} \left( -\frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{h(t)}}{t - z} dt \right)
\]
\[
= -\left( \frac{1}{2} h(t_0) + \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - t_0} dt \right)
\]
\[
= -\frac{1}{2} h(t_0) + \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - t_0} d\bar{t}. \quad (3.17)
\]
Similarly,
\[
\lim_{z \to t_0, z \in \Gamma^-} \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - z} d\bar{t} = \frac{1}{2} h(t_0) + \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t)}{t - t_0} d\bar{t}.
\]
Thus, integrating against \(\frac{d\bar{t}}{t - t_0}\) instead of \(\frac{dt}{t - t_0}\) has the effect of changing the sign of the jump across the path of integration in the Plemelj formulas.

### 3.7 Resultant moment of external forces

If for some function \(H\) defined on \(\Gamma_O\), \(\phi(\xi) + \xi \phi'(\xi) + \overline{\psi(\xi)} = H(\xi)\) for all \(\xi \in \Gamma_O\), then
\[
\int_{\Gamma_O} H(\xi) d\xi = \int_{\Gamma_O} \left( \overline{\phi(\xi)} + \overline{\xi \phi'(\xi) + \psi(\xi)} \right) d\xi
\]
\[
= \int_{\Gamma_O} \overline{\phi(\xi)} d\xi + \int_{\Gamma_O} \overline{\xi \phi'(\xi)} d\xi
\]
\[
= \int_{\Gamma_O} \overline{\phi(\xi)} d\xi - \int_{\Gamma_O} \phi(\xi) d\xi
\]
\[
= 2i\Im \int_{\Gamma_O} \overline{\phi(\xi)} d\xi.
\]
The last equality uses the useful integration by parts \( \int_L f(z) \overline{g(z)} \, dz = -\int_L f'(z) \overline{g(z)} \, dz \) when \( L \) is a smooth closed curve and \( f \) and \( g \) are differentiable on \( L \). Thus

\[
\Re \int_{\Gamma_0} H(\xi) \, d\xi = 0 \tag{3.18}
\]

must hold for a solution to exist, and, using (3.8), we note that this is equivalent to

\[
\Re \int_{\Gamma_0} H(\xi) \, d\xi = -\Re \int_{\Gamma_0} \overline{\xi} \, d\xi = \int_{\Gamma_0} \Re (i \overline{\xi}) \, d\xi = \int_{\Gamma_0} (\xi \times \overline{\sigma}) \, d\xi = 0.
\]

This expresses the condition that the resultant moment of the external forces must be zero ([38, 50, 51]).

We’ll now show that this condition is automatically satisfied by the right-hand side of equation (3.15), which we rewrite

\[
\phi_0(\xi) + \xi \phi'(\xi) + \psi_0(\xi) = -\xi + \frac{1}{2\pi i} \int_{\Gamma_1} \left( \frac{\phi(\zeta)}{\xi - \zeta} \right) \, d\zeta - \xi \frac{\phi'(\zeta)}{\xi - \overline{\zeta}} \, d\overline{\zeta} + \frac{\psi(\zeta)}{\xi - \overline{\zeta}} \, d\overline{\zeta} = 0 \tag{3.19}
\]

\[
+ \frac{1}{2\pi i} \int_{\Gamma_1} \left( \frac{f(\zeta)}{\xi - \zeta} - 1 \frac{f'(\zeta)}{\zeta - \overline{\zeta}} \, d\zeta \right) \tag{3.20}
\]

\[
+ \frac{1}{2\pi i} \int_{\Gamma_1} \left( \frac{f(\zeta) + \zeta f'(\zeta)}{\xi - \overline{\zeta}} \, d\zeta \right). \tag{3.21}
\]

Let \((a + ib) : [0, L] \to \Gamma_O\) be a smooth parametrization of the outer boundary for real functions \(a\) and \(b\). For the first line, (3.19), we have

\[
\Re \int_{\Gamma_0} -\xi \, d\overline{\xi} = \Re \int_{0}^{L} (a + ib)(\alpha)(a' + ib')(\alpha) \, d\alpha
\]

\[
= \int_{0}^{L} (aa' + bb')(\alpha) \, d\alpha
\]

\[
= \frac{1}{2} \left( a^2 + b^2 \right) \bigg|_{0}^{L}
\]

\[
= 0,
\]

since \( \Gamma_O \) is closed and therefore \((a + ib)(0) = (a + ib)(L)\).
Consider the second term in the second line (3.20), \(-\frac{1}{2\pi i} \int_{\Gamma_I} \frac{f'(\zeta)}{\bar{\zeta} - \xi} d\zeta\). Taking the complex conjugate of this term, we get
\[
\frac{1}{2\pi i} \bar{\xi} \int_{\Gamma_I} \frac{f'(\zeta)}{\zeta - \xi} d\zeta = -\frac{1}{2\pi i} \bar{\xi} \int_{\Gamma_I} f(\zeta) \frac{-1}{(\zeta - \xi)^2} d\zeta.
\]
Integrating this function of \(\xi\) around \(\Gamma_O\), we get
\[
\frac{1}{2\pi i} \int_{\Gamma_O} \bar{\xi} \int_{\Gamma_I} f(\zeta) \frac{1}{(\zeta - \xi)^2} d\zeta d\xi = \frac{1}{2\pi i} \int_{\Gamma_I} f(\zeta) \left[ \int_{\Gamma_O} \frac{\bar{\xi}}{(\zeta - \xi)^2} d\xi \right] d\zeta
= \frac{1}{2\pi i} \int_{\Gamma_I} f(\zeta) \left[ -\int_{\Gamma_O} \frac{d\bar{\xi}}{\zeta - \xi} \right] d\zeta
= -\frac{1}{2\pi i} \int_{\Gamma_I} \int_{\Gamma_O} \frac{f(\zeta)}{\bar{\xi} - \xi} d\xi d\zeta.
\]
Thus
\[
\Re \left[ \int_{\Gamma_O} \frac{1}{2\pi i} \int_{\Gamma_I} \left( \frac{f(\zeta)}{\zeta - \xi} d\zeta - \xi \frac{f'(\zeta)}{\zeta - \xi} d\zeta \right) d\xi \right] = 0.
\]
For the third line, (3.21), we have
\[
\Re \left[ \int_{\Gamma_O} \frac{1}{2\pi i} \int_{\Gamma_I} \frac{f(\zeta)}{\zeta - \xi} d\zeta d\xi \right] = \Re \left[ \int_{\Gamma_I} \left( \frac{f(\zeta)}{\bar{\zeta} + \xi f'(\zeta)} \right) \left( \int_{\Gamma_O} \frac{1}{\zeta - \xi} d\zeta \right) d\zeta \right]
= \Re \left[ -\int_{\Gamma_I} \left( \frac{f(\zeta)}{\bar{\zeta} + \xi f'(\zeta)} \right) d\zeta \right]
= \Re \left[ -\int_{\Gamma_I} \left( \frac{f(\zeta)}{\bar{\zeta} + \xi f'(\zeta)} - f(\zeta) d\zeta \right) \right]
= 0.
\]
Thus the condition (3.18) on the boundary values of \(\phi(\xi) + \xi \phi'(\xi) + \psi(\xi)\) necessarily holds for all \(H\) that we may consider.

### 3.8 Complex variable formulation

In this section we show the process for using complex variable methods to solve (2.9)–(2.11). Given an interface \(\Gamma_I\), outer boundary \(\Gamma_O\), and interface velocity jump \(f\), find a Stokes solution \((p, u)\) in the core and shell that satisfy the interface and outer boundary conditions
\[
\begin{align*}
\mathbf{[u]} &= \mathbf{f} \quad ; \quad [\sigma \mathbf{n}] = 0 \quad ; \quad \sigma \mathbf{n}|_{\Gamma_O} = -\mathbf{n}.
\end{align*}
\]
(3.22)

To solve the problem, we first find \(\phi_0\) and \(\psi_0\) analytic in \(\zeta_1\) such that
\[
\phi_0(\xi) + \xi \phi'_0(\xi) + \psi_0(\xi) = -\xi + \frac{1}{2\pi i} \int_{\Gamma_I} \left( \frac{f(\zeta)}{\zeta - \xi} d\zeta + \frac{f(\zeta)}{\bar{\zeta} - \xi} d\bar{\zeta} + \frac{f'(\zeta)}{\zeta - \xi} d\zeta \right). \tag{3.23}
\]
Existence of such \( \phi_0 \) and \( \psi_0 \) will be shown in section 4.2. Define

\[
\phi_s(z) = \phi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \Omega_s
\]

\[
\phi_c(z) = \phi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \Omega_c
\]

\[
\psi_s(z) = \psi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{-\overline{f(\zeta)} - \overline{\zeta f'(\zeta)}}{\zeta - z} d\zeta, \quad z \in \Omega_s
\]

\[
\psi_c(z) = \psi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{-\overline{f(\zeta)} - \overline{\zeta f'(\zeta)}}{\zeta - z} d\zeta, \quad z \in \Omega_c.
\]

Then, as shown above, since \( \phi_s, \psi_s, \phi_c, \) and \( \psi_c \) are analytic, \((p_s, u_s)\) and \((p_c, u_c)\) solve the Stokes equations. Here \( p \) and \( u \) are defined by \( p = -2\Re \phi' \) and \((u + iv)(z) = \phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\). We now show that these \((p, u)\) do indeed satisfy the interface and boundary conditions (3.22) and are therefore a solution to (2.9)–(2.11).

The Sokhotski-Plemelj theorem will tell us that the interface conditions are met. We have for a point \( \zeta \in \Gamma_I \) that

\[
\left[ \phi(\zeta) \right] = \left( \lim_{z \to \zeta, z \in \Omega_s} \phi_s(z) \right) - \left( \lim_{z \to \zeta, z \in \Omega_c} \phi_c(z) \right) = f(\zeta)
\]

and

\[
\left[ \psi(\zeta) \right] = \left( \lim_{z \to \zeta, z \in \Omega_s} \psi_s(z) \right) - \left( \lim_{z \to \zeta, z \in \Omega_c} \psi_c(z) \right) = -\overline{f(\zeta)} - \overline{\zeta f'(\zeta)},
\]

as they should be. Now the interface equation \( [u + iv] = \frac{1}{2} [\phi(\zeta) - \zeta \overline{\phi'(\zeta)} - \overline{\psi(\zeta)}] = f \) is satisfied, and also \( [\phi(\zeta) + z \overline{\phi'}/(\zeta) + \overline{\psi(\zeta)}] = 0 \).

The boundary and interface traction conditions can be computed from taking the tangential derivative of \( \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \) along the interface and boundary. Let \( n_1 + in_2 \) be the unit vector in the normal (outward facing) direction at a point, and let \( s = in \), the unit tangential vector in the positive direction. Then

\[
\partial_s (\phi + z\overline{\phi'} + \overline{\psi}) = (s\partial_z + \overline{s\partial_\bar{z}}) (\phi + z\overline{\phi'} + \overline{\psi})
\]

\[
= s (\phi' + \overline{\phi'}) + \overline{s} (\zeta \overline{\phi''} + \overline{\psi'})
\]

\[
= in (-2p) - i\overline{m} (\gamma - i\tau)
\]

\[
= i(-2pn - (\gamma - i\tau)n),
\]
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so

$$\sigma \mathbf{n} = -p \mathbf{n} + \left( \begin{array}{cc} -\gamma & \tau \\ \tau & -\gamma \end{array} \right) \mathbf{n}$$

$$= -p \mathbf{n} + \left( \begin{array}{c} \Re \left((-\gamma + i\tau)\mathbf{n}\right) \\ \Re \left((\tau + i\gamma)\mathbf{n}\right) \end{array} \right)$$

$$= -p \mathbf{n} + \left(\begin{array}{c} -\Re \left((-\gamma - i\tau)\mathbf{n}\right) \\ -\Im \left((\gamma - i\tau)\mathbf{n}\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \Im \left(\partial_s \left(\phi + z\bar{\phi'} + \bar{\psi}\right)\right) \\ -\Re \left(\partial_s \left(\phi + z\bar{\phi'} + \bar{\psi}\right)\right) \end{array}\right).$$

On the interface, since \([\phi + \zeta\bar{\phi'} + \bar{\psi}] = 0\), this says that \([\sigma \mathbf{n}] = 0\). The outer boundary condition (3.23) implies that \(\phi_s(\xi) + \xi\phi'_s(\xi) + \psi_s(\xi) = -\xi\) for \(\xi \in \Gamma_O\) (as in (3.15)), so on \(\Gamma_O\),

$$\sigma \mathbf{n} = \left(\begin{array}{c} \Im \left(\partial_s \left(\phi + \xi\bar{\phi'} + \bar{\psi}\right)\right) \\ -\Re \left(\partial_s \left(\phi + \xi\bar{\phi'} + \bar{\psi}\right)\right) \end{array}\right) = -\mathbf{n}.$$ 

We have shown that \(p\) and \(u\) solve the Stokes equations and the interface and boundary conditions (3.22).

In this method of solving (2.9)–(2.11), the bulk of the problem is in finding \(\phi_0\) and \(\psi_0\) that satisfy (3.23). Equivalently, given \(f\) on the interface, we are looking for complex analytic functions \(\phi_s, \phi_c, \psi_s,\) and \(\psi_c\) that satisfy

$$\begin{align*}
\left\{ \phi(\xi) + \xi\phi'(\xi) + \psi(\xi) = -\xi, \quad \xi \in \Gamma_O \\
\left[ \phi(\zeta) + \zeta\phi'(\zeta) + \psi(\zeta) \right] = 0, \quad \zeta \in \Gamma_I \\
\frac{1}{2} \left[ \phi(\zeta) - \zeta\phi'(\zeta) - \psi(\zeta) \right] = f(\zeta), \quad \zeta \in \Gamma_I
\end{align*}$$

This section has given a complex formulation of the Stokes equations, and in the next section we discuss how to transform the interface evolution into complex variables.

### 3.9 The right-hand side of the differential equation for \(R\) and \(f\)

The evolution of the interface point \(R(\theta,t)\) and velocity jump \(f(\theta,t) = [u](\theta,t)\) is given by

$$\frac{d}{dt} \left( \begin{array}{c} R \\ [u] \end{array} \right) = \left( \begin{array}{c} \frac{V}{c_2} n \right) r + \frac{V}{c_1} V n - c_2 \left( \begin{array}{c} r \end{array} \right).$$

(3.24)

We haven’t written the right-hand side in terms of \(f\) because \([\nabla u] r\) depends on \(f\) through solving the Stokes equations. However, with a complex solution to the fixed-time problem,
we can write (3.24) in terms of \( R, f, \phi, \psi, \) and the unit normal to the interface, \( n(\zeta) \). As before, the unit tangent to the curve is \( s(\zeta) = in(\zeta) \). Similarly to the way we computed directional derivatives of \( \phi \) and \( \psi \) above, we have

\[
(\nabla u)r = \frac{1}{2} \frac{\partial r}{\partial r} (\phi - \zeta \bar{\phi} - \bar{\psi})
\]

\[
= \frac{1}{2} \left( r \frac{\partial z}{\partial r} + \frac{\zeta R}{R} (\phi' - \bar{\phi}') + \frac{\bar{\zeta} R}{R} (-\zeta \bar{\psi}' - \bar{\psi}') \right)
\]

\[
= \frac{1}{2} R^{-1} \left( \zeta \phi' - \zeta \bar{\phi}' - R^2 \bar{\phi}' - \zeta \bar{\psi}' \right) .
\]

The normal interface velocity is

\[
V = n \cdot \sigma n = \Re \left( \bar{n}(-i \partial_s (\phi + \zeta \bar{\phi} + \bar{\psi})) \right) = \Re \left( \bar{s} \partial_s (\phi + \zeta \bar{\phi} + \bar{\psi}) \right),
\]

where we used \( \bar{n} = is \).

It is now simple to put everything together in terms of the interface position and interface values of \( \phi_s, \phi_c, \psi_s, \) and \( \psi_c \). The result is

\[
\frac{d}{dt} \left( R \frac{\partial r}{\partial r} \right) = \left( \frac{1}{n \cdot r} \Re \left( \bar{s} \partial_s (\phi + \zeta \bar{\phi} + \bar{\psi}) \right) \right) \left( \frac{1}{n \cdot r} \left[ \frac{1}{2} R^{-1} \left( \zeta \phi' - \zeta \bar{\phi}' - R^2 \bar{\phi}' - \zeta \bar{\psi}' \right) \right] + \frac{1}{c_1} n \right) - \frac{\alpha}{c_1} f .
\]

(3.26)

Alternatively, using (2.7), we obtain

\[
\frac{d}{dt} \left( R \frac{\partial r}{\partial r} \right) = \left( T \frac{L_s}{s_s} + \left( \frac{1}{c_1} - \frac{L_{\zeta s}}{\zeta_s} \right) V n - \frac{\alpha}{c_1} f \right) .
\]

(3.27)

where \( V \) is given by (3.25), \( T = \frac{r \cdot s}{r \cdot n} V \) (recall (2.5)), and we used \( [\nabla u] n = -\frac{L_s}{\zeta_s} n \). To see why \( [\nabla u] n = -\frac{L_s}{\zeta_s} n \), first note that

\[
[\sigma n] = 0 \iff \begin{bmatrix} -p - \gamma & \tau \\ \tau & -p + \gamma \end{bmatrix} n = 0
\]

\[
\iff [p] n = \begin{bmatrix} -\gamma & \tau \\ \tau & \gamma \end{bmatrix} n .
\]

(3.28)

Now, recalling that \( \gamma = v_y - u_x, \tau = u_y + v_x, q = v_x - u_y \) and \( 0 = u_x + v_y \), we have

\[
[\nabla u] n = \frac{1}{2} \begin{bmatrix} -\gamma & \tau - q \\ \tau + q & \gamma \end{bmatrix} n
\]

\[
= \frac{1}{2} \begin{bmatrix} -\gamma & \tau \\ \tau & \gamma \end{bmatrix} n + \frac{1}{2} \begin{bmatrix} 0 & -q \\ q & 0 \end{bmatrix} n
\]

\[
= \frac{1}{2} \left( [p] n_1 - [q] n_2 \right) + \frac{1}{2} \left( [p] n_2 + [q] n_1 \right) .
\]

(3.29)
In the complex plane, this is
\[
\frac{1}{2} \left[p + iq\right] (n_1 + in_2) = -\frac{1}{2} \left[-p + iq\right] n = -\frac{f}{\zeta} n,
\]
where we used (3.2) and (3.12).

### 3.10 Alterations to $\phi$ and $\psi$ that don’t affect the stress state

Suppose some $\phi$ and $\psi$ solve the zero boundary traction problem
\[
\phi(\xi) + \xi \phi'(\xi) + \psi(\xi) = 0, \quad \xi \in \Gamma_O.
\]
It is well known that this is only possible with a zero stress state in the body (see [50, 51]). Since the pressure must be zero throughout, we have that $\Re \phi' = 0$ everywhere and conclude that $\phi'$ is a complex constant. Let $Aiz + B = \phi'(z)$, where $A$ is a real constant and $B$ is complex. From $\gamma + i\tau = 0$ we get that $z\phi'' + \psi' = 0$. Since $\phi'' = 0$, we have that $\psi$ is constant. Let $C = \psi(z)$, $C \in \mathbb{C}$. From $\phi(\xi) + \xi \phi'(\xi) + \psi(\xi) = 0$ we have $Aiz + B + \xi(-Ai) + C = 0$, and therefore $C = -\overline{B}$. Thus $\phi$ and $\psi$ must be in the form
\[
\begin{align*}
\phi(z) &= Aiz + B \\
\psi(z) &= -\overline{B}
\end{align*}
\]
for some real $A$ and complex $B$.

If $\phi_1, \psi_1$ and $\phi_2, \psi_2$ both solve
\[
\phi(\xi) + \xi \phi'(\xi) + \psi(\xi) = H(\xi)
\]
for an arbitrary $H$, then $(\phi_1 - \phi_2)(\xi) + \xi \left( (\phi_1 - \phi_2)'(\xi) \right) + (\psi_1 - \psi_2)(\xi) = 0$, and therefore $\phi_1$ and $\psi_1$ differ from $\phi_2$ and $\psi_2$ by $Aiz + B$ and $-\overline{B}$.

### 3.11 Exact solution in the circle

We can write down the exact solution to this PDE in the case that $\Gamma_O$ is the unit circle, $\Gamma_I$ is the circle of radius $R$ with center 0, and $f(\theta, t) = f_0(t)e^{i\theta}$ for real $f_0$. By symmetry and the initial conditions of the problem, at each point in time $f$ must have this form. The system
\[
\begin{cases}
\phi(\xi) + \xi \phi'(\xi) + \psi(\xi) = -\xi, \quad \xi \in C_1 \\
\phi(\zeta) + \zeta \phi'(\zeta) + \psi(\zeta) = 0, \quad \zeta \in C_R \\
\frac{1}{2} \left[ \phi(\zeta) - \zeta \phi'(\zeta) - \psi(\zeta) \right] = f_0 \zeta R^{-1}, \quad \zeta \in C_R
\end{cases}
\]
is solved by

\[ \phi_s(z) = f_0 R z - \frac{1}{2} z \]
\[ \psi_s(z) = -2f_0 Rz^{-1} \]
\[ \phi_c(z) = f_0 R z - \frac{1}{2} z - f_0 z R^{-1} \]
\[ \psi_c(z) = 0. \]

The physical variables can be retrieved from \( \phi \) and \( \psi \). In this section, we write \( \hat{s} \) to mean the unit tangent vector to the interface in the counterclockwise direction, while a subscript \( s \) refers to a shell as opposed to a core variable, e.g. \( \partial_{\hat{s}} \phi_s \) means the tangential derivative of the shell \( \phi \) along the interface. In the shell,

\[ p_s(z) = -2 \Re \phi'_s(z) = -2f_0 R + 1 \]
\[ (\gamma_s + i\tau_s)(z) = z \phi''_s(z) + \psi'_s(z) = 2f_0 R \frac{1}{z^2}. \]
\[ (u_s + iv_s)(z) = \frac{1}{2} \left( \phi_s(z) - z \phi'_s(z) - \psi_s(z) \right) \]
\[ = \frac{1}{2} \left( 2f_0 R \bar{z}^{-1} \right) \]
\[ = f_0 R \bar{z}^{-1}, \]

or \((u - iv)(z) = f_0 Rz^{-1}\). From

\[ \phi_s(z) + z \phi'_s(z) + \psi_s(z) = 2f_0 R \left( 1 - \frac{1}{|z|^2} \right) z - z \]

we can compute the interface traction

\[ \sigma_s n(\zeta) = -i \partial_{\hat{s}} \left( \phi_s(\zeta) + \zeta \phi'_s(\zeta) + \psi_s(\zeta) \right) \]
\[ = -i \partial_{\hat{s}} \left( 2f_0 \left( R - \frac{1}{R} \right) \zeta - \zeta \right) \]
\[ = -i \left( 2f_0 \left( 1 - \frac{1}{R^2} \right) i \zeta - i \frac{\zeta}{R} \right) \]
\[ = \left( 2f_0 - 2f_0 R^{-2} - R^{-1} \right) \zeta. \]

In the core,

\[ p_c(z) = -2 \Re \phi'_c(z) = -2f_0 R + 1 + 2f_0 R^{-1} \]
\[ (\gamma_c + i\tau_c)(z) = z \phi''_c(z) + \psi'_c(z) = 0 \]
\[ (u_c + iv_c)(z) = \frac{1}{2} \left( \phi_c(z) - z \phi'_c(z) - \psi_c(z) \right) = 0 \]

The interfacial traction balance is satisfied, so

\[ \phi_c(z) + z \phi'_c(z) + \psi_c(z) = \phi_s(z) + z \phi'_s(z) + \psi_s(z) = 2f_0 Rz - z - 2f_0 R^{-1} z \]
and
\[ \sigma_s n(\zeta) = \sigma_s n(\zeta) = (2f_0 - 2f_0R^{-2} - R^{-1}) \zeta. \]

And now we can figure out the ODE for \( R \) and \( f \). We start with
\[ V(\zeta) = n(\zeta) \cdot \sigma_s n(\zeta) = \Re((R^{-1}\zeta) \sigma_s n(\zeta)) = 2f_0R - 2f_0R^{-1} - 1, \]
and
\[ \dot{R} = \frac{V}{\tau n} = V = 2f_0R - 2f_0R^{-1} - 1. \]

Next,
\[ (\nabla u_s) r(\zeta) = \frac{1}{2} R^{-1} (\zeta \phi' - \zeta \overline{\phi'} - R^2 \phi'' - \overline{\zeta \phi''}) = -\frac{1}{2} R^{-1} \overline{\zeta} (2f_0R \overline{\zeta}^{-2}) = -f_0 \overline{\zeta}^{-1} = -f_0 R^{-2} \zeta, \]
and since \( u_c = 0 \) this quantity is equal to \( [\nabla u] r \).

Now
\[ \frac{d}{dt} [u] = \dot{R} [\nabla u] r + \frac{1}{c_1} V n - \frac{c_2}{c_1} [u] = -(2f_0R - 2f_0R^{-1} - 1)f_0 R^{-2} \zeta + \frac{1}{c_1}(2f_0R - 2f_0R^{-1} - 1)R^{-1} \zeta - \frac{c_2}{c_1} f_0 R^{-1} \zeta. \]

Since \( [u] \) and \( \frac{d}{dt} [u] \) are multiples of \( \zeta \), at any point in time \( f_0 = [u] R \zeta^{-1} \) is constant along the interface. If we think of \( f_0 \) as a quantity moving in time, then \( f_0 \) and \( R \) solve the real ordinary differential equation
\[ \frac{d}{dt} \left( R f_0 \right) = \left( -(2f_0R - 2f_0R^{-1} - 1)f_0 R^{-1} - \frac{1}{c_1}(2f_0R - 2f_0R^{-1} - 1) - \frac{c_2}{c_1} f_0 \right). \quad (3.30) \]

Given initial conditions \( R(0) = 1, f_0(0) = 0 \), we can solve the system of ordinary and partial differential equations by solving the two real variable ODE above and writing the physical variables in terms of \( f_0 \) and \( R \). Note that equation (3.30) uses different variables but agrees with the real variable solution found in Section 2.2. The results are below, and agree, \textit{mutatis mutandis}, with those reported in [43], where the three-dimensional spherically symmetric problem was solved using a different approach. In Chapter 4, we develop a numerical method to evolve general (noncircular) interface geometries, which has not been previously investigated in the literature.
CHAPTER 3. COMPLEX VARIABLE FORMULATION

Figure 3.1: Interface radius evolution for various $c_1$, $c_2$, where $c_1$ and $c_2$ are defined as in equation (2.1)

Figure 3.2: Interface velocity and velocity jump versus radius for various $c_1$, $c_2$, where $c_1$ and $c_2$ are defined as in equation (2.1)
Chapter 4

Numerical methods

In the previous chapter, we boiled down the system

\[
\begin{cases}
-\Delta u + \nabla p = 0 & \text{in } \Omega_c \text{ and } \Omega_s, \\
\nabla \cdot u = 0 & \text{in } \Omega_c \text{ and } \Omega_s,
\end{cases}
\]

\[
\begin{cases}
[\sigma n] = 0 & \text{on } \Gamma_I, \\
[u] = f & \text{on } \Gamma_I, \\
\sigma n = -n & \text{on } \Gamma_O
\end{cases}
\]

to the problem of finding analytic functions \( \phi_s, \phi_c, \psi_s, \) and \( \psi_c \) such that

\[
\begin{cases}
\phi(\xi) + \xi \phi'(\xi) + \overline{\psi(\xi)} = -\xi, & \xi \in \Gamma_O \\
\left[ \phi(\zeta) + \zeta \phi'(\zeta) + \overline{\psi(\zeta)} \right] = 0, & \zeta \in \Gamma_I, \\
\phi(\zeta) = f(\zeta), & \zeta \in \Gamma_I
\end{cases}
\]

or, more simply, finding two analytic functions \( \phi_0 \) and \( \psi_0 \) in \( \Gamma_O^+ \) such that for all \( \xi \in \Gamma_O \),

\[
\phi_0(\xi) + \xi \phi_0'(\xi) + \overline{\psi_0(\xi)} = -\xi + \frac{1}{2\pi i} \int_{\Gamma_I} \left( \frac{f(\zeta)}{\zeta - \xi} d\zeta + \frac{f(\zeta)}{\zeta - \overline{\xi}} d\overline{\zeta} + \frac{f'(\zeta)}{\zeta - \xi} d\zeta + \frac{f'(\zeta)}{\zeta - \overline{\xi}} d\overline{\zeta} \right). \tag{4.1}
\]

The Sherman-Lauricella integral equation \([15, 50, 51]\) will lead to a fast numerical algorithm for solving (4.1).
4.1 The Sherman-Lauricella integral equation

Define $H(\xi)$ to be the right-hand side of equation (4.1). To find $\phi_0$ and $\psi_0$, we will look for functions of the form

$$
\phi_0(z) = \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi)}{\xi - z} \, d\xi
$$

$$
\psi_0(z) = \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi) d\xi + \omega(\xi) d\bar{\xi}}{\xi - z} - \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\xi \omega(\xi)}{(\xi - z)^2} \, d\xi
$$

(4.2)

(4.3)

for some density $\omega : \Gamma_o \to \mathbb{C}$.

When $\omega$ is differentiable along the outer boundary, an integration by parts shows that

$$
\int_{\Gamma_o} \frac{\xi \omega(\xi)}{(\xi - z)^2} \, d\xi = -\int_{\Gamma_o} \left( -\frac{1}{\xi - z} \right) \left( \xi \omega'(\xi) + \omega(\xi) \frac{d\bar{\xi}}{d\xi} \right) \, d\xi = \int_{\Gamma_o} \frac{\omega(\xi) d\bar{\xi}}{\xi - z} + \int_{\Gamma_o} \frac{\xi \omega'(\xi) d\xi}{\xi - z},
$$

and substituting this into the formula for $\psi_0$ we get a simpler way to write $\psi_0$ in terms of $\omega$,

$$
\psi_0(z) = \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi) - \bar{\xi} \omega'(\xi)}{\xi - z} \, d\xi.
$$

We still need the original formula (4.3) to derive an integral equation for $\omega$, but this will help when we need to compute $\psi_0$ later.

The boundary condition on $\phi_0(\xi) + \xi \phi_0'(\xi) + \psi_0(\xi)$ turns into an integral equation that $\omega$ must satisfy. For $z \in \Omega_o$,

$$
\phi_0(z) + z \phi_0'(z) + \psi_0(z) = \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\xi \omega(\xi)}{(\xi - z)^2} \, d\xi
$$

$$
- \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi) d\xi + \omega(\xi) d\bar{\xi}}{\xi - \bar{\xi}} + \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\xi \omega(\xi)}{(\xi - z)^2} \, d\xi
$$

$$
= \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi)}{\xi - \bar{\xi}} \, d\xi
$$

$$
+ \frac{1}{2\pi i} \int_{\Gamma_o} \frac{(\xi - z) \omega(\xi)}{(\xi - z)^2} \, d\xi - \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi)}{\xi - \bar{\xi}} \, d\xi.
$$

(4.4)

(4.5)

Taking the limit as $z$ goes to a point $z_o \in \Gamma_o$, the limit of the first line (4.4) is

$$
\frac{1}{2} \omega(z_o) + \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi)}{\xi - z_o} \, d\xi - \left( -\frac{1}{2} \omega(z_o) + \frac{1}{2\pi i} \int_{\Gamma_o} \frac{\omega(\xi)}{\xi - \bar{z}_o} \, d\xi \right).
$$

Here we have used the Plemelj formula (3.16) and the related equation (3.17).

For the second line (4.5), we compute

$$
\frac{d\xi - \bar{\xi}}{\xi - \bar{\xi}} = \frac{1}{\xi - \bar{\xi}} - \frac{\xi - z}{(\xi - z)^2} \, d\xi.
$$
CHAPTER 4. NUMERICAL METHODS

So
\[
\frac{1}{2\pi i} \int_{\Gamma_O} \frac{(\xi - z)\omega(\xi)}{(\xi - z)^2} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - \bar{z}} d\xi = -\frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - \bar{z}} \xi - z,
\]
which is continuous across \( \Gamma_O \). We end up with the integral equation
\[
\omega(z_O) + \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - z_O} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - \bar{z}_O} d\xi
+ \frac{1}{2\pi i} \int_{\Gamma_O} \left( \frac{(\xi - z_O)\omega(\xi)}{(\xi - z_O)^2} d\xi - \frac{\omega(\xi)}{\xi - \bar{z}_O} \xi - z_O \right)
= H(z_O).
\tag{4.6}
\]

This integral equation is not quite invertible. To see that, write out \( \phi_0(z) + z\phi'_0(z) + \psi_0(z) \) using the formula for \( \psi_0 \) in terms of \( \omega' \). For \( z \in \Gamma_O^+ \),
\[
\phi_0(z) + z\phi'_0(z) + \psi_0(z) = \frac{1}{2\pi i} \int_{\Gamma_O} \left[ \frac{\omega(\xi)}{\xi - z} d\xi - \frac{z\omega'(\xi)}{\xi - \bar{z}} d\xi - \frac{\omega(\xi) - \xi\omega'(\xi)}{\xi - \bar{z}} \xi - z \right] d\xi.
\]

For \( \omega(\xi) = \xi \),
\[
\phi_0(z) + z\phi'_0(z) + \psi_0(z) = \frac{1}{2\pi i} \int_{\Gamma_O} \left[ \frac{\omega(\xi)}{\xi - z} d\xi - \frac{z\omega'(\xi)}{\xi - \bar{z}} d\xi - \frac{\omega(\xi) - \xi\omega'(\xi)}{\xi - \bar{z}} \xi - z \right] d\xi
= \frac{1}{2\pi i} \int_{\Gamma_O} \left[ \frac{\xi}{\xi - z} d\xi - \frac{z}{\xi - \bar{z}} d\xi - \frac{\xi - \xi}{\xi - \bar{z}} d\xi \right]
= z - z \frac{1}{2\pi i} \int_{\Gamma_O} \frac{1}{\xi - \bar{z}} d\xi
= 0.
\]

Thus a term must be added to make the integral equation invertible. Let
\[
b_0 = \frac{1}{2\pi i} \int_{\Gamma_O} \left( \frac{\omega(\xi)}{\xi^2} d\xi + \frac{\omega(\xi)}{\xi^2} d\xi \right) \tag{4.7}
\]
As will be shown in section 4.2, the integral equation
\[
\omega(z_O) + \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - z_O} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - \bar{z}_O} d\xi
+ \frac{1}{2\pi i} \int_{\Gamma_O} \left( \frac{(\xi - z_O)\omega(\xi)}{(\xi - z_O)^2} d\xi - \frac{\omega(\xi)}{\xi - \bar{z}_O} \xi - z_O \right)
+ \frac{b_0}{z_O} = H(z_O) \tag{4.8}
\]
is invertible.
Furthermore, as we now show, if
\[ \Re \int_{\Gamma_O} H(\xi) d\xi = 0, \tag{4.9} \]
the solution \( \omega \) has \( b_0 = 0 \). Let \( \omega \) be a solution of (4.8) with an \( H \) that satisfies (4.9). Writing \( \phi_0 \) and \( \psi_0 \) in terms of \( \omega \), as above, the integral equation implies \( \phi_0(\xi) + \xi \phi'_0(\xi) + \psi_0(\xi) + \frac{\psi_0(\xi)}{\xi} = H(\xi) \) on the outer boundary. Integrating both sides against \( d\xi \),
\[
\int_{\Gamma_O} H(\xi)d\xi = \int_{\Gamma_O} \left( \phi_0(\xi) + \xi \phi'_0(\xi) + \psi_0(\xi) \right) d\xi + \int_{\Gamma_O} \frac{b_0}{\xi} d\xi
\]
\[
= \int_{\Gamma_O} \left( \phi_0(\xi)d\xi + \xi \phi'_0(\xi)d\xi \right) - 2\pi ib_0
\]
\[
= \int_{\Gamma_O} \left( \phi_0(\xi)d\xi - \phi'_0(\xi)d\xi \right) - 2\pi ib_0.
\]
Taking the real part of both sides, we get \( \Re(2\pi ib_0) = 0 \). Since \( b_0 \) is necessarily purely imaginary, as can be seen from its definition (4.7) in terms of \( \omega \), this implies that \( b_0 = 0 \).

As we showed in section 3.7, in our problem \( H \) always satisfies equation (4.9). Thus the solution of the amended integral equation (4.8) is also a solution of the original equation (4.6). Therefore to find the density \( \omega \), the problem that we solve is to find \( \omega \) defined on \( \Gamma_O \) such that for all \( z_O \in \Gamma_O \),
\[
\omega(z_O) + \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - z_O} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - z_O} d\xi
\]
\[
+ \frac{1}{2\pi i} \int_{\Gamma_O} \frac{(\xi - z_O)\omega(\xi)}{(\xi - z_O)^2} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - z_O} d\xi
\]
\[
+ \frac{-1}{2\pi i z_O} \int_{\Gamma_O} \left( \frac{\omega(\xi)}{\xi^2} d\xi + \frac{\omega(\xi)}{\xi^2} d\xi \right)
\]
\[
= -z_O + \frac{1}{2\pi i} \int_{\Gamma_i} \left( \frac{f(\zeta)}{\zeta - z_O} d\zeta + \frac{f(\zeta)}{\zeta - z_O} d\zeta + \frac{\xi - z_O}{\xi - z_O} d\zeta \right). \tag{4.10}
\]
Once we have solved for \( \omega \), we have formulas (4.2) and (4.3) for \( \phi_0 \) and \( \psi_0 \) from their
definitions in terms of $\omega$. Then, as we did above, defining

\[ \phi_s(z) = \phi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in \Omega_s \]

\[ \phi_c(z) = \phi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in \Omega_c \]

\[ \psi_s(z) = \psi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{-\overline{f(\zeta)} - \overline{\zeta f'(\zeta)}}{\zeta - z} \, d\zeta, \quad z \in \Omega_s \]

\[ \psi_c(z) = \psi_0(z) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{-\overline{f(\zeta)} - \overline{\zeta f'(\zeta)}}{\zeta - z} \, d\zeta, \quad z \in \Omega_c, \]

we can compute for any point $\zeta_0 \in \Gamma_I$

\[ \phi_s(\zeta_0) := \lim_{z \to \zeta_0} \phi_s(z) = \phi_0(\zeta_0) + \frac{1}{2\pi i} \int_{\Gamma_I} \frac{f(\zeta)}{\zeta - \zeta_0} \, d\zeta + \frac{1}{2} f(\zeta_0) \quad (4.11) \]

\[ \phi_c(\zeta_0) := \lim_{z \to \zeta_0} \phi_c(z) = \phi_0(\zeta_0) + \frac{1}{2\pi i} \int_{\Gamma_I} \frac{f(\zeta)}{\zeta - \zeta_0} \, d\zeta - \frac{1}{2} f(\zeta_0) \quad (4.12) \]

\[ \psi_s(\zeta_0) := \lim_{z \to \zeta_0} \psi_s(z) = \psi_0(\zeta_0) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\overline{f(\zeta)} + \overline{\zeta f'(\zeta)}}{\zeta - \zeta_0} \, d\zeta - \frac{1}{2} \left( \overline{f(\zeta_0)} + \overline{\zeta_0 f'(\zeta_0)} \right) \quad (4.13) \]

\[ \psi_c(\zeta_0) := \lim_{z \to \zeta_0} \psi_c(z) = \psi_0(\zeta_0) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\overline{f(\zeta)} + \overline{\zeta f'(\zeta)}}{\zeta - \zeta_0} \, d\zeta + \frac{1}{2} \left( \overline{f(\zeta_0)} + \overline{\zeta_0 f'(\zeta_0)} \right). \quad (4.14) \]

This is everything that we need in order to calculate the evolution of the interface and $\mathbb{R}[u]$. Moving the interface points along rays originating at 0, the position and interface jumps satisfy the evolution

\[
\frac{d}{dt} \left( R [u] \right) = \left( \Re \left( s \partial_s (\phi + \zeta \overline{\phi}' + \overline{\psi}) \right) \left( \frac{1}{n_r} \Re \left( \overline{s \partial_s (\phi + \zeta \overline{\phi}' + \overline{\psi})} \right) \left( \frac{1}{n_r} \left[ \frac{1}{2} R^{-1} \left( \zeta \phi' - \zeta \overline{\phi}' - R^2 \phi'' - \overline{\psi} \right) \right] \right) + \frac{1}{c_1} n \right) - \frac{c_2}{c_1} f \right),
\]

as shown in section 3.9.
4.2 Uniqueness of solutions to the Sherman-Lauricella integral equation

With a parametrization $\gamma : [0, 1] \to \Gamma_O$ of $\Gamma_O$, the left-hand side of the Sherman-Lauricella equation (4.10) at a point $\gamma(\beta)$ can be written

\[
(\omega \circ \gamma)(\beta) + \frac{1}{2\pi i} \int_0^1 \frac{(\omega \circ \gamma)(\alpha)}{\gamma(\alpha) - \gamma(\beta)} \frac{\gamma'(\alpha) d\alpha}{\gamma'(\alpha)} - \frac{1}{2\pi i} \int_0^1 \frac{(\omega \circ \gamma)(\alpha)}{\gamma(\alpha) - \gamma(\beta)} \frac{\gamma'(\alpha) d\alpha}{\gamma(\alpha) - \gamma(\beta)} \gamma(\alpha) d\alpha
\]

\[
+ \frac{1}{2\pi i} \int_0^1 \frac{(\omega \circ \gamma)(\alpha)}{\gamma(\alpha) - \gamma(\beta)} \frac{\gamma'(\alpha) d\alpha}{(\gamma(\alpha) - \gamma(\beta))^2} - \frac{1}{2\pi i} \int_0^1 \frac{(\omega \circ \gamma)(\alpha)}{\gamma(\alpha) - \gamma(\beta)} \frac{\gamma'(\alpha) d\alpha}{\gamma(\alpha) - \gamma(\beta)} \gamma(\alpha) d\alpha
\]

\[
= (\omega \circ \gamma)(\beta) + \int_0^1 K_1(\alpha, \beta)(\omega \circ \gamma)(\alpha) d\alpha + \int_0^1 K_2(\alpha, \beta)(\omega \circ \gamma)(\alpha) d\alpha
\]

for integral kernels

\[
K_1(\alpha, \beta) = \frac{1}{2\pi i} \left( \frac{\gamma'(\alpha)}{\gamma(\alpha) - \gamma(\beta)} - \frac{\gamma'(\alpha)}{\gamma(\alpha) - \gamma(\beta)} - \frac{\gamma'(\alpha)}{\gamma(\alpha)^2} \right)
\]

and

\[
K_2(\alpha, \beta) = \frac{1}{2\pi i} \left( \frac{(\gamma(\alpha) - \gamma(\beta)) \gamma'(\alpha)}{(\gamma(\alpha) - \gamma(\beta))^2} - \frac{\gamma'(\alpha)}{\gamma(\alpha) - \gamma(\beta)} - \frac{\gamma'(\alpha)}{\gamma(\alpha)^2} \right).
\]

Let $\omega_1$ and $\omega_2$ be the real and imaginary parts, respectively, of $\omega \circ \gamma$. These integrals can be further decomposed into real and imaginary parts as

\[
\omega_1(\beta) + \int (\Re (K_1 + K_2))(\alpha, \beta) \omega_1(\alpha) d\alpha + \int (\Im (-K_1 + K_2))(\alpha, \beta) \omega_2(\alpha) d\alpha
\]

\[
i \left( \int (\Im (K_1 + K_2))(\alpha, \beta) \omega_1(\alpha) d\alpha \right) + i \left( \omega_2(\beta) + \int (\Re (K_1 - K_2))(\alpha, \beta) \omega_2(\alpha) d\alpha \right).
\]

(4.16)

This is a coupled system of real second kind Fredholm equations to which the Fredholm alternative applies. Thus to show that the Sherman-Lauricella equations have unique solutions, we show that the integral operator described by (4.16) has a trivial kernel. This is equivalent to showing that in (4.10), $\omega \equiv 0$ whenever the right-hand side is zero. The proof below is modeled on Parton and Perlin [51].


Suppose

\[ 0 = \omega(z_O) + \frac{1}{2\pi i} \int_{\Gamma O} \frac{\omega(\xi)}{\xi - z_O} d\xi - \frac{1}{2\pi i} \int_{\Gamma O} \frac{\omega(\xi)}{\xi - \overline{z}_O} d\xi \]

\[ + \frac{1}{2\pi i} \int_{\Gamma O} \frac{(\xi - z_O)\omega(\xi)}{(\xi - \overline{z}_O)^2} d\xi - \frac{1}{2\pi i} \int_{\Gamma O} \frac{\omega(\xi)}{\xi - \overline{z}_O} d\xi \]

\[ + \frac{-1}{2\pi iz_O} \int_{\Gamma O} \left( \frac{\omega(\xi)}{\xi^2} d\xi + \frac{\overline{\omega}(\xi)}{\xi^2} d\xi \right) \]

for all \( z_O \in \Gamma_O \). As shown in section 4.1, since 0 satisfies equation (4.9), we have

\[ 0 = b_0 \]

\[ = \frac{1}{2\pi i} \int_{\Gamma O} \left( \frac{\omega(\xi)}{\xi^2} d\xi + \frac{\overline{\omega}(\xi)}{\xi^2} d\xi \right) \]

\[ = \frac{1}{\pi i} \Re \int \frac{\omega(\xi)}{\xi^2} d\xi \]

\[ = i\Im \left( \frac{1}{\pi i} \int \frac{\omega(\xi)}{\xi^2} d\xi \right). \]  

(4.17)

Define

\[ \phi(z) = \frac{1}{2\pi i} \int_{\Gamma O} \frac{\omega(\xi)}{\omega - z} d\xi \]

\[ \psi(z) = \frac{1}{2\pi i} \int_{\Gamma O} \frac{\omega(\xi) - \overline{\xi} \omega' (\xi)}{\xi - z} d\xi \]

for \( z \in \Gamma_O^+ \), the region inside \( \Gamma_O \). Then \( \lim_{z \to \xi \atop z \in \Gamma_O^+} \left( \phi(z) + z \overline{\phi'(z)} + \overline{\psi(z)} \right) = 0 \) for \( \xi \in \Gamma_O \). So \( \phi \) and \( \psi \) solve the stress problem with vanishing specified boundary traction. From the results of section 3.10, we have that \( \phi(z) = Aiz + B \) and \( \psi(z) = -\overline{B} \) for some \( A \in \mathbb{R} \) and \( B \in \mathbb{C} \). From \( \phi'(0) = \frac{1}{2\pi i} \int_{\Gamma O} \frac{\omega(\xi)}{(\xi - 0)^2} d\xi \) and equation (4.17) we get that \( 0 = \Im \phi'(0) = A \). Put all together, \( \phi \) and \( \psi \) are in the form

\[ \phi = B \quad \text{and} \quad \psi = -\overline{B} \]

for some complex constant \( B \).

Define

\[ F_\phi(z) = \frac{1}{2\pi i} \int_{\Gamma O} \frac{-i(\omega(\xi) - B)}{\xi - z} d\xi \]  

(4.18)

\[ F_\psi(z) = \frac{1}{2\pi i} \int_{\Gamma O} \frac{-i(\overline{\omega(\xi)} - \overline{\xi} \omega'(\xi) + \overline{B})}{\xi - z} d\xi. \]  

(4.19)
For \( z \in \Gamma_O^+ \),

\[
F_\phi(z) = -i \left( \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{B}{\xi - z} d\xi \right) = -i(\phi(z) - B) = 0,
\]

and similarly \( F_\psi(z) = 0 \) for \( z \in \Gamma_O^+ \). We also have that, by an integration by parts, for \( z \in \Gamma_O^+ \),

\[
0 = \phi'(z) = \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega'()(\xi)}{\xi - z} d\xi = iF_\phi'(z).
\]

By the Plemelj formulas, for \( \xi \in \Gamma_O \),

\[
- \lim_{z \to \xi^+} \frac{F_\phi(z)}{z} = \lim_{z \to \xi^-} F_\phi(z) - \lim_{z \to \xi^-} F_\phi(z) = -i(\omega(\xi) - B) \quad (4.20)
\]

\[
- \lim_{z \to \xi^+} \frac{F_\psi(z)}{z} = \lim_{z \to \xi^-} F_\psi(z) - \lim_{z \to \xi^-} F_\psi(z) = -i(\bar{\omega}(\xi) - \bar{\xi}\omega'(\xi) + B) \quad (4.21)
\]

Therefore

\[
\lim_{z \to \xi^+} \left( F_\phi(z) + zF_\phi'(z) + F_\psi(z) \right) = i(\omega(\xi) - B) + \xi(i\omega'(\xi)) + \left( i\left( \bar{\omega}(\xi) - \bar{\xi}\omega'(\xi) + B \right) \right)
\]

\[= -2iB.\]

This means that \( F_\phi \) and \( F_\psi \) solve the zero traction condition problem with constant \(-2iB\), and, just as in the simply connected case, this corresponds to zero stress state in \( \Gamma_O^- \), the infinite plane with a hole. Thus \( F_\phi \) and \( F_\psi \) must take the form \( F_\phi(z) = A'iz + B' \), \( F_\psi(z) = C' \) for real constant \( A' \) and complex constants \( B' \) and \( C' \). From the formulas for \( F_\phi \) and \( F_\psi \), it is immediate that \( F_\phi(z) \to 0 \) as \( z \to \infty \) and \( F_\psi(z) \to 0 \) as \( z \to \infty \). This implies that \( A' = B' = C' = 0 \).

Equation (4.20) then implies that \( \omega \equiv B \), and plugging this into (4.21) we conclude that \( 0 = B = \omega \).

We have shown that equation (4.10) has unique solutions. Note that this means that \( \omega \) exists and is unique, but does not mean that \( \phi \) and \( \psi \) are uniquely determined. What we can say is that if \( (\phi_1, \psi_1) \) and \( (\phi_2, \psi_2) \) are solutions, then \( \phi_1 \) and \( \phi_2 \) are equal up to a term of the form \( Aiz + B \) and \( \psi_1 \) and \( \psi_2 \) are the same up to a constant \( C \), and that the physical variables \( p = -2\Re \phi' \) and \( \gamma + i\tau = \bar{z}\phi'' + \psi' \) are unaffected by these changes in the Kolosov functions. Therefore, the stress state \( \sigma = \begin{pmatrix} -p - \gamma & \tau \\ \tau & -p + \gamma \end{pmatrix} \) is uniquely determined.
4.3 Numerical solution of the Sherman-Lauricella integral equation

We wish to numerically find a density \( \omega \) on the outer boundary that satisfies equation (4.10),

\[
\omega(z_O) + \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - z_O} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\omega(\xi)}{\xi - \bar{z}_O} d\bar{\xi} \\
+ \frac{1}{2\pi i} \int_{\Gamma_O} \frac{(\xi - z_O)\bar{\omega}(\xi)}{(\xi - z_O)^2} d\xi - \frac{1}{2\pi i} \int_{\Gamma_O} \frac{\bar{\omega}(\xi)}{\xi - z_O} d\xi \\
+ \frac{-1}{2\pi iz_O} \int_{\Gamma_O} \left( \frac{\omega(\xi)}{\xi^2} d\xi + \frac{\bar{\omega}(\bar{\xi})}{\bar{\xi}^2} d\bar{\xi} \right) \\
= -z_O + \frac{1}{2\pi i} \int_{\Gamma_I} \left( \frac{f(\zeta)}{\zeta - z_O} d\zeta + \frac{f(\zeta)}{\zeta - z_O} d\zeta + \frac{f'(\zeta)}{\zeta - z_O} d\zeta \right),
\]

for all \( z_O \in \Gamma_O \). Recall that we are solving a nonstandard problem in which the velocity \( u \) has a specified jump \( f \) across the interface, and this quantity enters into the integral equation (4.10) on the right-hand side. We did this so that instead of solving a coupled interface problem involving separate domains, in equations (4.11)–(4.14) we introduced a Cauchy integral for which the Plemelj formulae enforce the desired jump conditions and we can therefore reduce the problem to solving just one integral equation in the whole sample.

We’ll find an \( \omega \) that satisfies the integral equation by discretizing the integrals. Computing the right-hand side is simply a matter of numerically integrating a complex function around a closed contour. We approximate the integrals involving the unknown \( \omega \) on the left-hand side by a linear operator on \( \omega \) evaluated at quadrature nodes on the outer boundary and solve the matrix equation that results.

Let \( \gamma : [0, 1] \rightarrow \Gamma_O \) be a parametrization of the outer boundary. We’ll use a trapezoidal rule quadrature scheme for both sides of the integral equation, which is spectrally accurate for integrating periodic analytic functions over a period (see [71]). For \( N \) boundary points, let \( h = 1/N, \xi_j = \gamma(hj) \) for \( j = 0, \ldots, N - 1 \) and \( d_j = \gamma'(hj) \). Let \( \kappa_j \) be the curvature of \( \Gamma_O \) at \( \xi_j \).

Let \( \gamma^I : [0, 1] \rightarrow \Gamma_I \) be a parametrization of the interface, \( N^I \) the number of interface points in the quadrature (which can be different from \( N \)), \( h^I = 1/N^I, \zeta_j = \gamma^I(h^Ij) \) for \( j = 0, \ldots, N^I - 1 \), \( d^I_j = \gamma^I'(\zeta_j) \), \( f^I_j = f(\zeta_j) \), and \( f^I'_j = f'(\zeta_j) \).

Define \( H(z_O) \) to be the right-hand side of the integral equation. For each \( \xi_j \), we compute the right hand side of the integral equation \( H_j \) using the trapezoidal rule for the integrals around \( \Gamma_I \). The result is

\[
H_j = -\xi_j + \frac{1}{2\pi i} \sum_{k=0}^{N^I-1} \left( \frac{f_k}{\zeta_k - \xi_j} d_k + \frac{f_k}{\zeta_k - \bar{\xi}_j} \bar{d}_k + \frac{f_k}{\zeta_k - \xi_j} \bar{d}_k \right).
\]
Discretizing the left-hand integral operator on $\omega$, we get, for each $\omega_j := \omega(\xi_j)$, the equation

\[
\frac{1}{2 \pi i} \sum_{k \neq j} \left( \frac{\omega_k d_k}{\xi_k - \xi_j} - \frac{\omega_k \bar{d}_k}{\bar{\xi}_k - \bar{\xi}_j} + \frac{(\xi_k - \xi_j) \omega_k \bar{d}_k}{(\xi_k - \xi_j)^2} - \frac{\bar{\omega}_k d_k}{\bar{\xi}_k - \bar{\xi}_j} \right) + \frac{h}{2 \pi} \left( \kappa_j |d_j| - \kappa_j \frac{d^2_j |d_j|}{|d_j|^2} \omega_j \right) - \frac{1}{2 \pi i \xi_j} \sum_k \left( \frac{\omega_k d_k}{\xi_k^2} + \frac{\bar{\omega}_k \bar{d}_k}{\bar{\xi}_k^2} \right) = H_j.
\]

The second line represents the limit of the first line as $\xi_k \to \xi_j$. The last line corresponds to the $b_0$ term (see [15]). These are $N$ complex equations in $N$ complex unknowns $\omega_j$, but they are not quite linear in $\omega$ as they involve terms of the form $\omega \bar{\omega}$. Breaking these $N$ complex equations into $2N$ real equations in the real and imaginary parts of $\omega_j$, we get a real $2N \times 2N$ linear system that is expected to be invertible since the integral equation we started with is invertible.

In more detail, write the left-hand sides of the $N$ equations as $L_1 \omega + L_2 \bar{\omega}$, where $L_1$ and $L_2$ are the matrices

\[
L_1(j, k) = \begin{cases} 
\frac{1}{2 \pi i} h \left( \frac{d_k}{\xi_k - \xi_j} - \frac{\bar{d}_k}{\bar{\xi}_k - \bar{\xi}_j} - \frac{d_k}{\xi_k^2} \right), & j \neq k \\
1 + \frac{h}{2 \pi} \kappa_j |d_j| - \frac{1}{2 \pi i} h \frac{d_j}{\xi_j |\xi_j|^2}, & j = k
\end{cases}
\]

and

\[
L_2(j, k) = \begin{cases} 
\frac{1}{2 \pi i} h \left( \frac{(\xi_k - \xi_j) \bar{d}_k}{(\xi_k - \xi_j)^2} - \frac{d_k}{\xi_k^2} \bar{\xi}_k \bar{\xi}_j \right), & j \neq k \\
-\frac{h}{2 \pi} \kappa_j \frac{d^2_j |d_j|}{|d_j|^2} - \frac{1}{2 \pi i} h \frac{\bar{d}_j}{\bar{\xi}_j}, & j = k
\end{cases}
\]

Solving the discretized integral operator equations is thus equivalent to solving the real $2N \times 2N$ system

\[
\begin{pmatrix} \Re L_1 + \Re L_2 & -\Im L_1 + \Im L_2 \\ \Im L_1 + \Im L_2 & \Re L_1 - \Re L_2 \end{pmatrix} \begin{pmatrix} \Re \omega \\ \Im \omega \end{pmatrix} = \begin{pmatrix} \Re H \\ \Im H \end{pmatrix}.
\]

(4.22)

Moving forward in time, the right-hand side, which depends on $f$, will change, but the left-hand side depends only on the geometry of the outer edge, which never changes as the boundary is fixed. Thus before evolving, the matrix can be factored or fully inverted and stored in order to not have to do a full matrix solve every time the right-hand side of the ordinary differential equations needs to be evaluated.

Below are contour plots of $p$ and $\|u\|$ at fixed times, for various outer boundary geometries and interface data. Also plotted are checks to make sure that the computed $[u]$ is in fact equal to $f$. 

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Figure 4.1: Contour plots of pressure and magnitude of velocity for interface defined by \( R(\alpha) = 0.5 + 0.1 \cos(5\alpha) \) with \( f(\alpha) = -e^{i\alpha} \). The constants are \( c_1 = c_2 = \kappa = 1 \), where \( c_1 \) and \( c_2 \) are defined as in equation (2.1).

Figure 4.2: Contour plots of pressure and magnitude of velocity for interface defined by \( R(\alpha) = 0.5 \) with \( f(\alpha) = (-1 + 0.1 \cos(10\alpha))e^{i\alpha} \). The constants are \( c_1 = c_2 = \kappa = 1 \), where \( c_1 \) and \( c_2 \) are defined as in equation (2.1).
Figure 4.3: Contour plots of pressure and magnitude of velocity for interface defined by $R(\alpha) = 0.5 + 0.03 \cos(4\alpha + \pi)$ with $f(\alpha) = -e^{i\alpha}$. The constants are $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).
Figure 4.4: Contour plots of pressure and magnitude of velocity. In this case the outer boundary is an ellipse that is twice as tall as it is wide. The interface is defined by $R(\alpha) = 0.5$ with $f(\alpha) = -e^{i\alpha}$. The constants are $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).
Figure 4.5: Contour plots of pressure and magnitude of velocity. In this case the outer boundary is an ellipse that is twice as tall as it is wide. The interface is defined by $R(\alpha) = 0.5 + 0.06 \cos(3(\alpha + \pi/6))$ with $f(\alpha) = -e^{i\alpha}$. The constants are $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).
Figure 4.6: As a check on our method, we compute $u_s$ and $u_c$ on the interface and check whether it is equal to $f$. In this example the outer boundary is the unit circle and the interface is defined by $R(\alpha) = 0.5 + 0.1 \cos(5\alpha) + 0.1 \sin(10\alpha)$ with $f(\alpha) = (-1 + 0.1 \cos(10\alpha))e^{i\alpha}$. The constants are $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).
Figure 4.7: As a check on our method, we compute $\mathbf{u}_s$ and $\mathbf{u}_c$ on the interface and check whether it is equal to $f$. In this example the outer boundary is an ellipse twice as tall as it is wide, and the interface is defined by $R(\alpha) = 0.5 + 0.1\cos(4\alpha)$ with $f(\alpha) = (-1 + 0.1\sin(15\alpha))e^{i\alpha}$. The constants are $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).
4.4 Evaluation of Kolosov functions on the interface

Once we have $\omega$, we need to compute $\dot{R}$ and $\dot{f}$. As shown in section 3.9, these quantities are determined by the interface values of $\phi_c$, $\phi_s$, $\psi_c$, and $\psi_s$.

These quantities all depend on $\phi_0$ and $\psi_0$. At an interface node $\zeta_k$, we compute $\phi_0(\zeta_k) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\xi)}{\xi - \zeta_k} d\xi$ by a trapezoidal rule on the nodes where we have already computed $\omega$, i.e. our approximation of $\phi_0(\zeta_k)$ is

$$\phi_{0k} = \frac{1}{2\pi i} h \sum_{j=0}^{N} \frac{\omega_j d_j}{\xi_j - \zeta_k}.$$  

Similarly, $\psi_0(\zeta_k)$ is approximated as

$$\psi_{0k} = \frac{1}{2\pi i} h \sum_{j=0}^{N} \left( \frac{\omega_j d_j + \omega_j d_j}{\xi_j - \zeta_k} - \frac{\xi_j \omega_j}{(\xi_j - \zeta_k)^2} \right).$$

As

$$\phi_s(\zeta_k) = \phi_0(\zeta_k) + \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta)}{\zeta - \zeta_k} d\zeta + \frac{1}{2} f(\zeta_k)$$

and

$$\psi_s(\zeta_k) = \psi_0(\zeta_k) - \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta)}{\zeta - \zeta_k} d\zeta - \frac{1}{2} \left( f(\zeta_k) + \zeta_k f'(\zeta_k) \right),$$

we need to compute the Cauchy principal value of the singular integrals around $\Gamma_I$.

Calculating Cauchy principal value integrals

We need to compute integrals of the form $\frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta)}{\zeta - \zeta_0} d\zeta$ for some density $\mu$ on $\Gamma_I$, and we’ll use two different methods. Let $\gamma : [0,2\pi] \to \Gamma_I$ be a smooth parametrization of $\Gamma_I$. Fix the point $\zeta_0 = \gamma(\beta)$.

The first method decomposes the integral as

$$\frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta)}{\zeta - \zeta_0} = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{\gamma'(\alpha)}{\gamma(\alpha) - \gamma(\beta)} \mu(\gamma(\alpha)) d\alpha$$

$$= \frac{1}{2\pi i} \int_{0}^{2\pi} \left( \frac{\gamma'(\alpha)}{\gamma(\alpha) - \gamma(\beta)} - \frac{1}{2} \cot \left( \frac{\alpha - \beta}{2} \right) \right) \mu(\gamma(\alpha)) d\alpha$$

$$+ \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{1}{2} \cot \left( \frac{\alpha - \beta}{2} \right) \mu(\gamma(\alpha)) d\alpha.$$  

In the first integrand, at $\alpha = \beta$, the leading terms cancel and the limit is $\frac{\gamma'(\beta)}{2\gamma(\beta)}$. So the first integrand is actually Riemann integrable, and can be computed by a trapezoidal scheme.
The second integral, while still a principal value integral, is equal to \( \frac{i}{2} H[\mu \circ \gamma](\beta) \), where \( H \) is the periodic Hilbert transform. This can be computed by a Fourier transform. Using the notation \( \hat{f}_k \) to mean the \( k \)th Fourier coefficient of \( f \), we have

\[
\hat{H}f_k = \begin{cases} 
-\hat{f}_k, & k > 0 \\
0, & k = 0 \\
\hat{f}_k, & k < 0.
\end{cases}
\]

Thus an FFT of \( \mu \circ \gamma \) at the nodes \( 2\pi j/N, j = 0, \ldots, N - 1 \) and an inverse FFT on the transformed coefficients gives us \( H[\mu \circ \gamma] \).

We also implement another method if \( \mu \) is differentiable along \( \Gamma_I \) and we know its derivative. Similarly to the derivation of the Plemelj formulas, we write

\[
\frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta)}{\zeta - \zeta_0} d\zeta = \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta) - \mu(\zeta_0)}{\zeta - \zeta_0} d\zeta + \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta_0)}{\zeta - \zeta_0} d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mu(\zeta) - \mu(\zeta_0)}{\zeta - \zeta_0} d\zeta + \frac{1}{2} \mu(\zeta_0).
\]

The first term is integrable as the integrand’s limit at \( \zeta_0 \) is our definition of \( \mu'(\zeta_0) \). Thus in the trapezoidal rule computation of \( \frac{1}{2\pi i} \int_0^{2\pi} \frac{\mu(\gamma(\alpha)) - \mu(\gamma(\beta))}{\gamma(\alpha) - \gamma(\beta)} \gamma'(\alpha) d\alpha \), the node at \( \beta \) contributes \( h\mu'(\gamma(\beta)) \) to the sum. The two methods give identical results up to roundoff error (~13 digits in double precision). We use them to compute the core and shell values of \( \phi \) and \( \psi \) on the interface from the formulas (4.11)–(4.14).

### 4.5 Summary

To advance the interface, we need to know \( (\dot{R}, \dot{f}) \) from \( (R, f) \). In the previous sections we described how to numerically compute \( (\dot{R}, \dot{f}) \) from \( (R, f) \), and that process is summarized here. Given \( R \) and \( f \), we get the right-hand side of the integral equation (4.10) by a trapezoidal rule quadrature. We factor the linear operator that maps the density \( \omega \) to the right-hand side, described in (4.22), so we can rapidly apply it to get \( \omega \) at each timestep. From knowing \( \omega \) we have shown how to retrieve the Kolosov functions \( \phi \) and \( \psi \) on the interface. This is in fact the only place that we need to evaluate them, since our goal is to get \( \dot{R} \) and \( \dot{f} \). After differentiating \( \phi \) twice and \( \psi \) once along the interface, we have everything we need to get \( \dot{R} \) and \( \dot{f} \) as in equation (4.15). Armed with the ability to compute \( \dot{R} \) and \( \dot{f} \) from knowing \( R \) and \( f \), any explicit ODE scheme can be used to evolve the system in time. We used 4th, 5th, and 8th order Runge-Kutta methods for this purpose [16]. Some results of the numerical methods are plotted below.
Figure 4.8: Evolution of a rounded square interface with two choices of initial velocity jump \( f \). The initial velocity jump has a large effect on the evolution of the interfaces. In this plot, we evolve two initial interfaces defined by \( R(\alpha) = 0.8 + 0.05 \cos(4\alpha + \pi) \), one with radial velocity jump \( f(\alpha) = 0.3e^{i\alpha} \) and one with normal velocity jump \( f(\alpha) = 0.3n(\alpha) \), where \( n \) is the unit normal vector. In this evolution the outer boundary is the unit circle and \( c_1 = c_2 = \kappa = 1 \), where \( c_1 \) and \( c_2 \) are defined as in equation (2.1).
Figure 4.9: Evolution of a rounded triangle interface with two choices of initial velocity jump $f$. The initial velocity jump has a large effect on the evolution of the interfaces. In this plot, we evolve two initial interfaces defined by $R(\alpha) = 0.8 + 0.09 \cos(3\alpha + \pi/2))$, one with radial velocity jump $f(\alpha) = 0.3e^{i\alpha}$ and one with normal velocity jump $f(\alpha) = 0.3n(\alpha)$, where $n$ is the unit normal vector. In this evolution the outer boundary is the unit circle and $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).
Figure 4.10: Spatial Fourier modes at several times $t$ of the solution $R(\alpha, t)$ from Figure 4.9 with initial conditions $R(\alpha) = 0.8 + 0.09 \cos(3\alpha + \pi/2)$ and $f(\alpha) = 0.3e^{i\alpha}$. The real and imaginary parts of $\hat{R}_k$ are plotted at position $2k$ and $2k+1$. The solution remains stable up to $t = 0.15$, after which roundoff errors begin to grow rapidly, with higher frequency modes growing at a faster rate. This is even true of many circular interface evolutions, which will be analyzed in Chapter 6 to identify stable and unstable parameter regimes. In the unstable case, these instabilities will be shown to be a property of the PDE solution rather than of the numerical method used to compute it.
Chapter 5

Elastic model

In this chapter, instead of modeling the $\alpha$- and $\beta$-spinel phases of olivine as incompressible Maxwell solids, which we will refer to as the Maxwell-Stokes model, we now consider the Lamé equations of linearized elasticity.

Our model differs significantly from a recent elastic model of Morris, proposed in [44] for the special case of incompressible materials in a spherically symmetric geometry. Morris adopts an Eulerian description and defines displacement $u(x, t)$ on the current configuration at time $t$ as the vector from the initial position (at $t = 0$) of a material point to its current position, $x$. In other words, the reference configuration for the elastic problem is taken as the initial (hydrostatically stressed) state of the system, when the (spherical) body consists entirely of $\alpha$-olivine and the shell has zero thickness. This choice of reference configuration causes the displacement field to be continuous across the interface for all $t > 0$, but also causes the divergence of the displacement field in the shell to be nonzero due to the density difference between the reference configuration and the deformed configuration containing the transformed material. Morris accounts for this by defining the deviatoric stress using a Hooke law measuring the difference between the actual strain and the strain of isotropic compression matching the density difference. This is common practice in solid mechanics when thermal expansion is accounted for, as well as in materials science. Lee and Tromp [35] refer to this uniform strain as the ‘free’ or ‘stress-free’ strain.

Our approach is similar in that we also adopt an Eulerian description and use the current configuration as the domain on which displacement and stress are defined; see also [72]. Linearization of finite-deformation elasticity models [7] leads to the same (Lamé) equations whether these fields are defined in the reference configuration or the deformed configuration, and by choosing the latter, there will be no gaps or overlaps in the union of the core and shell regions on which we define the displacement and stress fields. However, in our model, the shell and core maintain separate reference configurations as the core shrinks and the shell grows. Motion of the interface causes mass transport from one reference configuration (e.g. that of the core) to the other, and this causes gaps and/or overlaps in the union of the reference configurations of the two regions. Thus, the displacement field will generally be discontinuous across the interface (as coincident points on the interface would like to relax to separated
points on the boundaries of their respective reference configurations). In our model, the chemical reactions involved in the phase change are assumed to eliminate any memory of the relative positions of the material points prior to the time at which the interface passed a given material point, so we do not measure strain with respect to displacement from the initial configuration, but rather from two time-varying reference configurations with possibly separate elastic constants matching the material properties of \( \alpha \)-olivine and \( \beta \)-spinel. As we will see, our elastic model is in many ways closer in spirit to Morris’s original Maxwell-Stokes model [43] studied in Chapters 1–4 than it is to his recent elastic model [44].

After summarizing the changes required in representing stress and displacement for possibly compressible materials, we derive the equations of motion for our model in two ways, one in which the boundaries of the reference configurations are dealt with explicitly, and one using an Eulerian framework to show the connection with [43]. We then study the cylindrically and spherically symmetric cases to compare and contrast the modeling assumptions made in Morris’s recent incompressible elastic model [44] versus this new model, and highlight their consequences.

5.1 Complex variable formulation for compressible elastic materials

In our elastic model, we have the force balance equation \( \nabla \cdot \sigma = 0 \) and Hooke’s Law \( \sigma = \lambda \text{tr}(\varepsilon) I + 2\mu \varepsilon \), where \( \varepsilon = \frac{1}{2}(\nabla u + \nabla u^T) \). Here \( u \) is interpreted as a displacement rather than a velocity, and we are solving the Lamé equations of linear elasticity rather than an auxiliary Stokes system to connect the physical stress to the physical creeping flow \( \tilde{u} \). (There is no \( \tilde{u} \) in the model considered here — the quantity \( u \) itself is the nondimensionalized physical displacement.) We note that the incompressible limit of the Lamé equations are the Stokes equations, aside from the change in meaning of \( u \) from displacement to velocity. We define \( \gamma \), \( \tau \) and the pressure \( p \) by

\[
\sigma = \mu \begin{pmatrix} -p - \gamma & \tau \\ \tau & -p + \gamma \end{pmatrix}.
\]

So \( \gamma = -u_x + v_y \) and \( \tau = u_y + v_x \) as in the Maxwell-Stokes model, but here \( \nabla \cdot u \neq 0 \), and instead

\[
-p = \frac{1}{2\mu} \text{tr}(\sigma) = \frac{\lambda + \mu}{\mu} \nabla \cdot u.
\]

Define \( \kappa = \frac{\lambda + 3\mu}{\lambda + \mu} \) and \( q = \frac{2}{\kappa + 1}(-u_y + v_x) \). The parameter \( \kappa \) falls in the range \([1, 3]\) and the limit \( \kappa = 1 \) is the incompressible case. The pressure is then \( -p = q \). We now show that \( -p \) and \( q \) solve the Cauchy-Riemann equations. The force balance equation gives us

\[
\begin{align*}
-p_x - \gamma_x + \tau_y &= 0 \\
\tau_x - p_y + \gamma_y &= 0,
\end{align*}
\]
so
\[ p_x = -\gamma_x + \tau_y = u_{xx} - v_{xy} + u_{yy} + v_{xy} = \Delta u \]
and
\[ p_y = \tau_x + \gamma_y = u_{xy} + v_{xx} - u_{xy} + v_{yy} = \Delta v. \]
Thus
\[ \frac{\kappa + 1}{2} (q_x - p_y) = \frac{\kappa + 1}{2} q_x - \frac{\kappa - 1}{2} p_y - p_y \\
= -u_{xy} + v_{xx} + u_{xy} + v_{yy} - \Delta v \\
= 0 \]
and, similarly, \[ \frac{\kappa + 1}{2} (p_x + q_y) = p_x - (u_{xx} + v_{xy}) + (-u_{yy} + v_{xy}) = 0. \]
We have shown that there is an analytic function \( \phi'(z) \) such that
\[ -p + iq = 2\phi'. \tag{5.1} \]
In general one would worry that \( -p + iq \) might not have a single-valued antiderivative \( \phi \) due to logarithm terms of the form (3.10) in the multiply-connected shell region. However, proceeding as in the Maxwell-Stokes model, by using Cauchy integrals to represent the jump conditions explicitly, the problem will reduce to solving for analytic functions \( \phi_0(z) \) and \( \psi_0(z) \) defined in (3.13) and (3.14) on the whole domain \( \Omega_c \cup \Omega_s \), which is not multiply-connected.

Continuing from (5.1),
\[ \partial_{\bar{z}} (\gamma + i\tau) = \frac{1}{2} (-\Delta u + i\Delta v) = \frac{1}{2} (-p_x + iq_x) = \phi'' = \partial_{\bar{z}} (\bar{z}\phi'') \]
So \( \chi'' = \psi' = \gamma + i\tau - \bar{z}\phi'' \) is analytic.

Let \( W = \Re(\bar{z}\phi + \chi) \). Just as in the Maxwell-Stokes model, \( W_x + iW_y = \phi + z\phi' + \psi, \gamma = \frac{1}{2}(W_{xx} - W_{yy}), \tau = -W_{xy}, \) and \( -p = \frac{1}{2}(W_{xx} + W_{yy}) \). So
\[ \sigma = \mu \begin{pmatrix} W_{yy} & -W_{xy} \\ -W_{xy} & W_{xx} \end{pmatrix}, \]
and the traction \( \sigma n \) can be computed in terms of \( W \) as \( \sigma n = \begin{pmatrix} \partial_{\bar{z}}W_y \\ -\partial_{\bar{z}}W_x \end{pmatrix} \).

The displacement satisfies
\[ \partial_{\bar{z}}(u + iv) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x) = \frac{1}{2}(\gamma - i\tau) = \frac{1}{2}(\gamma - i\tau) \]
\[ \partial_{\bar{z}}(u + iv) = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(-u_y + v_x) = -\frac{\kappa - 1}{4} p + \frac{\kappa + 1}{4} q = \frac{1}{2} \left(-\frac{\kappa - 1}{2} p + i\frac{\kappa + 1}{2} q \right), \]
\[ \partial_z \left( \frac{1}{2} (\kappa \phi - z \bar{\phi} - \bar{\psi}) \right) = \frac{1}{2} \left( -z \bar{\phi}' - \bar{\psi}' \right) = -\frac{1}{2} (\gamma - i\tau) \]

\[ \partial_z \left( \frac{1}{2} (\kappa \phi - z \bar{\phi} - \bar{\psi}) \right) = \frac{1}{2} (\kappa \phi' - \bar{\phi}') \]

\[ = \frac{1}{2} \left( \frac{\kappa}{2} (-p + iq) - \frac{1}{2} (-p - iq) \right) \]

\[ = \frac{1}{2} \left( -\frac{\kappa - 1}{2} p + i \frac{\kappa + 1}{2} q \right). \]

Thus the real and imaginary parts of \( u + iv \) and \( \frac{1}{2} (\kappa \phi - z \bar{\phi} - \bar{\psi}) \) have the same partial derivatives and there exist \( \phi \) and \( \psi \) such that \( u + iv = \frac{1}{2} (\kappa \phi - z \bar{\phi} - \bar{\psi}) \). From here, following the nondimensionalization and development of the Maxwell-Stokes model and noting that the formula for traction in terms of \( \phi \) and \( \psi \) has not changed, the boundary condition \( \sigma n \) remains \( \phi(\xi) + \xi \phi'(\xi) + \bar{\psi}(\xi) = -\xi \) and the interfacial traction balance is still \( \left[ \phi(\zeta) + \zeta \phi'(\zeta) + \bar{\psi}(\zeta) \right] = 0 \). The displacement jump \( \left[ u \right] \), however, becomes \( \left[ u \right] = \frac{1}{2} \left[ (\kappa \phi - \zeta \bar{\phi}' - \bar{\psi}) \right] \). Adding the interfacial traction balance to this equation gives us \( \frac{2}{\kappa + 1} \left[ \phi \right] = \left[ u \right] \). For a specified interfacial displacement jump \( g = \left[ u \right] \), the equations at a fixed time become

\[
\begin{cases}
\phi(\xi) + \xi \phi'(\xi) + \bar{\psi}(\xi) = -\xi, & \xi \in \Gamma_O \\
\left[ \phi(\zeta) + \zeta \phi'(\zeta) + \bar{\psi}(\zeta) \right] = 0, & \zeta \in \Gamma_I \\
\left[ \phi(\zeta) \right] = \frac{2}{\kappa + 1} g(\zeta), & \zeta \in \Gamma_I
\end{cases}
\]

which is equivalent to solving the Stokes problem with a scaled \( \left[ \phi \right] \). This would not be true if the traction boundary conditions were replaced by Dirichlet (displacement) boundary conditions.

### 5.2 Evolution equations in a framework with time-varying reference configurations

Next we derive the equations governing the time evolution of the interface. As mentioned at the beginning of this chapter, instead of using the initial hydrostatic state (in which the body consists entirely of \( \alpha \)-olivine) as a static reference configuration for our elastic formulation, as was done in [44] in a spherically symmetric setting, we give the core and shell regions separate reference configurations that evolve in time as mass is transported from one to the other due to the phase transformation; see Figure 5.1.

Because the reference configurations are not stationary in time, we adopt an Eulerian framework in which \( u \) is defined in the actual configuration rather than on the reference
Figure 5.1: The boundaries of the core (−) and shell (+) reference configurations are determined by the interface location in the actual configuration, which has no gaps or overlaps, together with the displacement field, which jumps discontinuously across the interface in this Eulerian framework.

Figure 5.2: Given a surface flux \( J \) along the interface and a mass flux \( F \) from the core to the shell, we can determine the normal velocity \( V^\pm \) of each reference frame from mass conservation:

\[
\rho^\pm V^\pm = \left(-F \pm \frac{1}{2} \frac{dJ}{ds}\right) \frac{ds}{ds^\pm}, \quad \frac{ds}{ds^\pm} = \frac{s_\alpha}{s^\pm_\alpha}.
\]

Here \( s_\alpha = |\zeta_\alpha| \) is the rate at which arclenth is swept out along the curve \( \zeta(\alpha, t) \) as \( \alpha \) varies, with \( t \) fixed, and \( s^\pm_\alpha \) are defined similarly. It is also useful to decompose the velocities of the curves, e.g. \( \dot{\zeta}(\alpha, t) = \frac{d}{dt}\zeta(\alpha, t) \), into tangential and normal components:

\[
\dot{\zeta} = Vn + Ts, \quad \dot{\zeta}^\pm = V^\pm n^\pm + T^\pm s^\pm.
\]

Note that \( \zeta(\alpha, t) \) can be any parametrization of the interface — its normal velocity \( V \) is determined by the physics of the problem, but its tangential velocity \( T \) is arbitrary; changing \( T \) corresponds to evolving the interface with a different family of parametrizations. We
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Figure 5.2: The flux $F$ across the interface and flux $J$ along the interface in the actual configuration determine the rates of growth and recession in the reference configurations of the shell and core by mass conservation.

will later specialize to the case that $\zeta(\alpha, t) = R(\alpha, t)e^{i\alpha}$, but for now it is an arbitrary parametrization.

Next we consider reparametrizations $a^\pm(\beta, t)$ for which the corresponding points on the boundaries of the reference configurations move normal to these curves:

$$\frac{d}{dt} \zeta^\pm(a^\pm(\beta, t), t) = V^\pm n^\pm. \quad (5.4)$$

Using (5.3), this gives $\zeta^\pm a^\pm + [V^\pm n^\pm + T^\pm s^\pm] = V^\pm n^\pm$, which determines how each $a^\pm$ evolves in time:

$$\dot{a}^\pm = -\frac{T^\pm}{s_\alpha^\pm}. \quad (5.5)$$

In our model, we assume a no sliding condition, in which a point moving normal to the core reference boundary remains in contact with the corresponding point moving normal to the shell boundary. By definition, $\zeta^+(\alpha, t)$ is in contact with $\zeta^-(\alpha, t)$, so we assume

$$\dot{a}^+ = \dot{a}^- \quad \text{(no sliding condition)}. \quad (5.6)$$

This yields the evolution equations

$$\dot{\zeta}^\pm \cdot n^\pm = V^\pm, \quad \frac{\dot{\zeta}^+ \cdot s^+}{s_\alpha^+} = \frac{\dot{\zeta}^- \cdot s^-}{s_\alpha^-}. \quad (5.7)$$

Now let $\varepsilon = \frac{\Delta \rho}{\rho}$, where $\rho = \rho^-$, $\Delta \rho = \rho^+ - \rho^-$, and thus $\rho^+ = (1 + \varepsilon)\rho$. We rescale $u$ and $g$ by $\varepsilon$ and recall that $s = \zeta_\alpha/s_\alpha$, so that

$$\zeta^\pm(\alpha, t) = \zeta(\alpha, t) - \varepsilon g^\pm(\alpha, t),$$

$$s^\pm_\alpha = |\zeta^\pm_\alpha| = |\zeta_\alpha - \varepsilon g^\pm_\alpha| = |s_\alpha s - \varepsilon g^\pm_\alpha|.$$
Decomposing \( g^\pm_\alpha = (g_\alpha \cdot s)s + (g_\alpha \cdot n)n \) gives
\[
s^\pm_\alpha = s_\alpha \left( 1 - \varepsilon \frac{g^\pm_\alpha \cdot s}{s_\alpha} \right) + O(\varepsilon^2).
\]
Neglecting \( O(\varepsilon^2) \) terms and identifying vectors in \( \mathbb{R}^2 \) with the corresponding points in \( \mathbb{C} \), we compute
\[
s^\pm = \zeta^\pm = \frac{\zeta_\alpha - \varepsilon g^\pm_\alpha}{s_\alpha} = \frac{s_\alpha s - \varepsilon (g^\pm_\alpha \cdot s) s - \varepsilon (g^\pm_\alpha \cdot n)n}{s_\alpha - \varepsilon g^\pm_\alpha \cdot s} = s - \varepsilon \frac{g^\pm_\alpha}{s_\alpha} n,
\]
\[
n^\pm = -is^\pm = n + \varepsilon \frac{g^\pm_\alpha \cdot n}{s_\alpha} s.
\]
Substitution of (5.8) and (5.9) in (5.7), and using (5.2), yields
\[
\dot{\zeta}^\pm = \dot{\zeta} - \varepsilon \dot{g}^\pm
\]
\[
\dot{\zeta}^\pm \cdot n^\pm = \dot{\zeta} \cdot n - \varepsilon \dot{g}^\pm \cdot n + \varepsilon \frac{g^\pm_\alpha \cdot n}{s_\alpha} \dot{\zeta} \cdot s
\]
\[
= V^\pm = \begin{cases} -\rho^{-1}(1 + \varepsilon)^{-1}(F - \frac{\varepsilon}{2} J_s)s_\alpha/s_\alpha^\pm & \text{on shell side,} \\ -\rho^{-1}(F + \frac{\varepsilon}{2} J_s)s_\alpha/s_\alpha^- & \text{on core side,} \end{cases}
\]
\[
= -\rho^{-1}\left(F \mp \varepsilon J_s\right) \left(1 + \varepsilon \frac{g^\pm_\alpha \cdot s}{s_\alpha} - \varepsilon \delta^\pm\right),
\]
where \( \delta^+ = 1 \) and \( \delta^- = 0 \). The \( O(1) \) terms are
\[
\dot{\zeta} \cdot n = -\rho^{-1} F + O(\varepsilon).
\]
Differencing (5.11) across the interface and using \( g = g^+ - g^- \) gives
\[
-\dot{g} \cdot n + \frac{g_\alpha \cdot n}{s_\alpha} \dot{\zeta} \cdot s = \rho^{-1}\left[J_s - F\left(\frac{g_\alpha \cdot s}{s_\alpha} - 1\right)\right] = \frac{\dot{\zeta} \cdot n}{V} \left(\frac{g_\alpha \cdot s}{s_\alpha} - 1\right) + \rho^{-1} J_s.
\]
Next we compute
\[
\frac{\dot{\zeta}^\pm \cdot s^\pm}{s^\pm_\alpha} = \frac{(\dot{\zeta} - \varepsilon \dot{g}^\pm) \cdot \left(s - \varepsilon \frac{g^\pm_\alpha \cdot n}{s_\alpha} n\right)}{s_\alpha \left(1 - \varepsilon \frac{g^\pm_\alpha \cdot s}{s_\alpha}\right)} = \frac{\dot{\zeta} \cdot s}{s_\alpha} \left(1 + \varepsilon \frac{g^\pm_\alpha \cdot s}{s_\alpha}\right) - \varepsilon \frac{g_\alpha \cdot n}{s_\alpha} - \varepsilon \frac{g^\pm_\alpha \cdot n}{s_\alpha} \dot{\zeta} \cdot n.
\]
Thus
\[
0 = \frac{\dot{\zeta}^+ \cdot s^+}{s^+_\alpha} - \frac{\dot{\zeta}^- \cdot s^-}{s^-_\alpha} = \varepsilon \left[\frac{(\dot{\zeta} \cdot s)(g_\alpha \cdot s)}{s^2_\alpha} - \frac{\dot{g} \cdot s}{s_\alpha} - \frac{g_\alpha \cdot n}{s^2_\alpha} \dot{\zeta} \cdot n\right],
\]
which means \( \dot{g} \cdot s = (\dot{\zeta} \cdot s) \left( \frac{g_\alpha}{s_\alpha} \cdot s \right) - (\dot{\zeta} \cdot n) \left( \frac{g_\alpha}{s_\alpha} \cdot n \right) \). Combined with (5.12) and (5.13), we get

\[
\dot{\zeta} \cdot n = V = -\rho^{-1} F, \quad T = \dot{\zeta} \cdot s, \tag{5.14}
\]

\[
\dot{g} \cdot n = T \left( \frac{g_\alpha}{s_\alpha} \cdot n \right) - V \left( \frac{g_\alpha}{s_\alpha} \cdot s \right) + V - \rho^{-1} J_s, \tag{5.15}
\]

\[
\dot{g} \cdot s = T \left( \frac{g_\alpha}{s_\alpha} \cdot s \right) - V \left( \frac{g_\alpha}{s_\alpha} \cdot n \right). \tag{5.16}
\]

Combining the last two equations gives

\[
\dot{g} = (\dot{g} \cdot n) n + (\dot{g} \cdot s) s = T \frac{g_\alpha}{s_\alpha} - V \left[ \left( \frac{g_\alpha}{s_\alpha} \cdot s \right) n + \left( \frac{g_\alpha}{s_\alpha} \cdot n \right) s \right] + (V - \rho^{-1} J_s) n. \tag{5.17}
\]

Let \( a = \left( \frac{g_\alpha}{s_\alpha} \cdot n \right) \) and \( b = \left( \frac{g_\alpha}{s_\alpha} \cdot s \right) \) so that \( \frac{g_\alpha}{s_\alpha} = an + bs \). Identifying vectors with complex numbers, we have

\[
\frac{g_\alpha}{\zeta_\alpha} = a n + b = -ia + b.
\]

Thus, we find that

\[
\left( \frac{g_\alpha}{\zeta_\alpha} \right) n = (b + ia) n = bn + as,
\]

which is the bracketed expression in (5.17). This finally gives

\[
\dot{g} = T \frac{g_\alpha}{s_\alpha} + \left( 1 - \frac{g_\alpha}{\zeta_\alpha} \right) V n - \rho^{-1} J_s n.
\]

Recalling that \( f = [\phi] = \frac{2}{\kappa + 1} [u + iv] = \frac{2}{\kappa + 1} q \), and specializing to the polar coordinate parametrization of the curve (see Figure 5.3), we obtain

\[
\dot{f} = T \frac{f_\alpha}{s_\alpha} + \left( \frac{2}{\kappa + 1} - \frac{f_\alpha}{\zeta_\alpha} \right) V n - \frac{2}{\kappa + 1} \rho^{-1} J_s n, V = -\rho^{-1} F, \quad \dot{R} = \frac{V}{r \cdot n}, \quad T = \frac{r \cdot s}{r \cdot n}, \tag{5.18}
\]

which, if we set \( J = 0 \) and \( F = -\rho \sigma_{nn} \), is the same as (3.27) with \( c_1 = \frac{\kappa + 1}{2} \) and \( c_2 = 0 \). Thus, mathematically, this elastic model with these flux laws is equivalent to the Maxwell-Stokes model if we use these values for \( c_1 \) and \( c_2 \).

### 5.3 Alternative derivation of the evolution equations in an Eulerian framework

As the derivation of the previous section is quite technical and led to a surprising connection with the Maxwell-Stokes model, we present a second derivation of equations (5.18) in
an Eulerian framework, staying as close as possible to Morris’s original derivation of the Maxwell-Stokes model [43], which was posed in terms of creeping flow and did not refer to reference configurations at all.

Let \( \rho = \rho_c^{(0)} \), \( \Delta \rho = \rho_s^{(0)} - \rho_c^{(0)} \), and \( \varepsilon = \Delta \rho / \rho \), assumed small. The superscript \( (0) \) notation indicates the density is in the reference configuration rather than the deformed configuration. On non-dimensionalizing the problem, we include a factor of \( \varepsilon \) in the displacement so that it remains \( O(1) \) in the \( \varepsilon \to 0 \) limit. Thus, a material point at \( X \) in the reference configuration corresponds to a material point in the deformed configuration through

\[
X(x, t) = x - \varepsilon u(x, t).
\]

Here we recall that we solve the Lamé equations in the current (i.e. deformed) configuration instead of in the reference configuration, since the former has a simple interface running through it (so jumps in displacement can be represented easily with Cauchy integrals) while the latter has gaps and overlaps that make the problem awkward to work with. As time evolves, we want material points to remain stationary in the reference configuration, which grows or shrinks by accretion or removal at the boundary rather than by deformation. Thus,

\[
\dot{X} = \dot{x} - \varepsilon (\nabla u) \dot{x} - \varepsilon u_t = 0,
\]

and therefore

\[
\dot{x} = (I - \varepsilon \nabla u)^{-1} \varepsilon u_t \approx \varepsilon u_t
\]

represents the motion of material points in the deformed configuration. (Note that \( \dot{x} \) in the present model has the same physical meaning as the creeping flow velocity \( \tilde{u} \) in the Maxwell-Stokes model described in Chapter 1. Also note that setting \( \dot{X} = 0 \) in (5.20) holds material points fixed in the reference configuration even though we have avoided parametrizing their boundaries.) Parametrizing the interface by \( \zeta(\alpha, t) \), we define

\[
\dot{u} = \frac{d}{dt} u(\zeta(\alpha, t), t) = (\nabla u) \dot{\zeta} + u_t.
\]
Combined with (5.21), this gives
\[ \dot{x} = \varepsilon [\dot{u} - (\nabla u)\dot{\zeta}]. \] (5.23)

Here the dot on \( x \) is a derivative in time for a fixed material point while a dot on \( u \) is a derivative holding \( \alpha \) constant, i.e. tracking the value of \( u \) on the interface as it moves (rather than tracking \( u \) at a fixed material point). We note that different parametrizations \( \zeta(\alpha, t) \) will yield different values of \( \dot{u} \) since holding \( \alpha \) fixed in one parametrization will generally lead to tangential motion along the curve in another. This will lead to different values of \( T \) in the decomposition \( \dot{\zeta} = Ts + Vn \), but will yield identical physical interpretations of the solutions. Differencing across the interface and defining \( g = [u] \) gives
\[ \varepsilon \dot{g} = \varepsilon g_t, \] (5.24)

where \( g_t := [u_t] \) is the jump in \( \frac{\partial u}{\partial t}(x, t) \) across the interface. If we draw a small control volume with sides perpendicular to the interface and top and bottom parallel to it, as shown in Figure 5.4, we see that mass conservation requires
\[ (\rho_c - \rho_s)V = (\rho_c\dot{x}_c - \rho_s\dot{x}_s) \cdot n, \] (5.25)

which is the analogue of (1.3) in the Maxwell-Stokes model, which was derived using a similar control volume argument in [43]. We conclude that
\[ V = \frac{\rho_s\dot{x}_s - \rho_c\dot{x}_c}{\rho_s - \rho_c} \cdot n. \] (5.26)

These densities are measured in the deformed configuration. If \( \Phi(X) \) is the deformation, \( F = \nabla \Phi \) is the deformation gradient, and \( J = \det F \) is the Jacobian, then \( \rho_c = \rho_c^{(0)}/J_c \) and \( \rho_s = \rho_s^{(0)}/J_s \), where \( \rho_c^{(0)} = \rho \) and \( \rho_s^{(0)} = \rho + \Delta \rho \) are the reference densities. Note that \( F^{-1} = \partial X/\partial x = I - \varepsilon \nabla u \), so
\[ 1/J = \det(F^{-1}) = \det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \varepsilon \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \right] = 1 - \varepsilon(u_x + v_y) + O(\varepsilon^2). \] (5.27)
As a result, neglecting $O(\varepsilon^2)$ terms,\[ \frac{\rho_s - \rho_c}{\rho} = \frac{1 + \varepsilon}{J_s} - \frac{1}{J_c} = (1 + \varepsilon)(1 - \varepsilon \text{div} u_s) - (1 - \varepsilon \text{div} u_c) \] (5.28)

Moreover, using \( \dot{x}_s = O(\varepsilon) \), \( \dot{x}_c = O(\varepsilon) \), and \( \dot{x} = \varepsilon g_t \), we obtain
\[ \frac{\rho_s \dot{x}_s - \rho_c \dot{x}_c}{\rho} = (1 + \varepsilon)(1 - \varepsilon \text{div} u_s) \dot{x}_s - (1 - \varepsilon \text{div} u_c) \dot{x}_c = \dot{x} = \varepsilon g_t. \] (5.29)

Thus \( V = \frac{g_t \cdot n}{1 + \frac{\kappa - 1}{2} [p]} + O(\varepsilon) \). We make the additional assumption that \( \dot{x} \cdot s = 0 \), i.e. the tangential velocity of material points is continuous across the interface so that the two phases do not slide tangentially relative to one another as the interface propagates. This implies
\[ g_t = (g_t \cdot n) n = (1 + \frac{\kappa - 1}{2} [p]) V n. \] (5.30)

Next we define
\[ \dot{g} = \frac{d}{dt} \left( \langle u \rangle (\zeta(\alpha, t), t) \right) = \frac{d}{dt} [u_s(\zeta(\alpha, t), t) - u_c(\zeta(\alpha, t), t)] = [u_t] + [\nabla u] \dot{\zeta} = g_t + [\nabla u] (T s + V n) \] (5.31)

and note that \( [\nabla u] s = \partial_s [u] = g_\alpha/s_\alpha \), where \( s_\alpha = |\zeta_\alpha| \). Using (5.28) and (5.1) together with \( u_x - v_y = -\gamma, u_y + v_x = \tau, v_x - u_y = \frac{\kappa - 1}{2} q \), and \( u_x + v_y = -\frac{\kappa - 1}{2} p \) gives
\[ [\nabla u] n = \frac{1}{2} \left[ \begin{pmatrix} -\gamma & \tau \end{pmatrix} - \frac{\kappa - 1}{2} \begin{pmatrix} p & p \end{pmatrix} + \frac{\kappa + 1}{2} \begin{pmatrix} q & -q \end{pmatrix} \right] n \]
\[ = \frac{1}{2} \left( [p] - \frac{\kappa - 1}{2} [p] + i \frac{\kappa + 1}{2} [q] \right) n \]
\[ = \frac{1}{2} \left( [p] + \kappa [\phi] - [\phi'] \right) n \]
\[ = \frac{1}{2} \left( [p] + \kappa [\phi] - [\phi'] \right) n. \] (5.32)

Combining with (5.30) and (5.31) yields
\[ \dot{g} = T \frac{g_\alpha}{s_\alpha} + \left( 1 + \frac{\kappa - 1}{2} [p] + \frac{\kappa - 1}{2} [\phi'] - [\phi'] \right) V n \]
\[ = T \frac{g_\alpha}{s_\alpha} + \left( 1 - \frac{\kappa + 1}{2} [\phi'] \right) V n. \] (5.33)

Finally, matching the notation of Chapters 2–4, we define \( f = [\phi] \), which is related to \( g = [u] \) via \( f = \frac{2}{\kappa + 1} g \). From (5.33), we have
\[ \dot{f} = T \frac{f_\alpha}{s_\alpha} + \left( \frac{2}{\kappa + 1} - [\phi'] \right) V n. \] (5.34)
Parametrizing $\zeta(\alpha, t) = R(\alpha, t)e^{i\alpha}$ and writing $\mathbf{r} = e^{i\alpha}$, the decomposition $\dot{\zeta} = T\mathbf{s} + V\mathbf{n}$ yields $\dot{R} = \frac{1}{\mathbf{n} \cdot \mathbf{r}}V$ and $T = \frac{\mathbf{r} \times \mathbf{s}}{\mathbf{r} \cdot \mathbf{n}}V$, as before. Since $[\varphi'] = f_0/\zeta_0$, we finally arrive at the evolution equation

$$
\frac{d}{dt} \left( R f_0 \right) = \left( T \frac{f_0}{s_0} + \frac{1}{\mathbf{n} \cdot \mathbf{r}}V \left( \frac{2}{\kappa + 1} - \frac{f_0}{\zeta_0} \right) V \mathbf{n} \right),
$$

which is the same as (5.18).

### 5.4 Behavior of the evolution in the case of a circle

We now compare the behavior of the Maxwell-Stokes and elastic cases in the circle, where we know the exact solution of the elasticity equations and solve the ODE numerically.

In the circular case $f$ is always in the form $f(\zeta) = f_0\zeta R^{-1}$, and we now write down the differential equations for $R$ and $f_0$. The problem with interface radius $R$ has

$$
\dot{R} = \sigma_{nn} = 2f_0R - 2f_0R^{-1} - 1,
$$

exactly as in the Maxwell-Stokes model. In the circle, $T = \frac{\mathbf{r} \times \mathbf{s}}{\mathbf{n} \times \mathbf{s}} = 0$ and $\dot{f} = \left( \frac{2}{\kappa + 1} - \frac{f_0}{\zeta_0} \right) V \mathbf{n}$ means that $f_0$ evolves as

$$
\dot{f}_0 = \left( \frac{2}{\kappa + 1} - f_0 R^{-1} \right) \left( 2f_0R - 2f_0R^{-1} - 1 \right) = \left( \frac{2}{\kappa + 1} - f_0 R^{-1} \right) \dot{R}.
$$

Thus if $2f_0R - 2f_0R^{-1} - 1$ ever becomes zero, both $\dot{R}$ and $\dot{f}_0$ are zero and the interface stops moving. Unlike the viscoelastic Maxwell-Stokes model (with $c_2 \neq 0$), this does indeed happen before the interface reaches the origin, as can be seen in the plots below.
Figure 5.5: Interface radius evolution for various $\kappa$. The viscoelastic case uses $c_1 = c_2 = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).

Figure 5.6: Interface velocity and displacement jump versus radius for various $\kappa$. The viscoelastic case uses $c_1 = c_2 = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1).
5.5 Comparison

In this section we briefly compare our elastic model of pressure-induced phase transformations to the recent model of Morris [44]. Since that model was described in the incompressible, spherically symmetric setting, we briefly consider that setup for our model. Adopting the notation of [44], let the radius of the interface (assumed spherical) be denoted \( R(t) \), let \( r(t) \) denote the (radial) trajectory of a material point in an Eulerian frame, and let \( \rho_1 \) be the density of the core material and \( \rho_2 \) be the (higher) density of the shell material. Let \( R_0 \) and \( r_0 \) be the initial values of \( R \) and \( r \). Because the material is incompressible and the geometry is assumed to be spherically symmetric, mass conservation alone determines \( r \) as a function of \( r_0 \), \( R \) and \( R_0 \). Our model will do exactly the same as Morris’s model here, so we have

\[
\hat{r}_3 = \begin{cases} 
0, & 0 < r_0 < R, \\
\frac{\rho_1}{\rho_2} r_3^3 + \left(1 - \frac{\rho_1}{\rho_2}\right) R^3, & R < r_0 < R_0.
\end{cases}
\] (5.36)

If we define \( \varepsilon \) by \( \rho_2 = (1 + \varepsilon)\rho_1 \) and assume \( 0 < \varepsilon \ll 1 \), we can Taylor expand \( u = r - r_0 \) to obtain

\[
u = r - r_0 = \left((1 - \varepsilon)r_3^3 + \varepsilon R^3\right)^{1/3} - r_0 = \begin{cases} 
0, & 0 < r < R, \\
-\varepsilon \left(r - \frac{R^3}{r}\right), & R < r < r_1.
\end{cases}
\] (5.37)

where \( r_3^3 = R^3 + \frac{\rho_1}{\rho_2}(R_0^3 - R^3) \) is the position of a material point that started on the interface at \( t = 0 \), i.e. \( (r_1)_0 = R_0 \). Because the reference configuration is taken to be the initial, hydrostatically stressed state consisting entirely of olivine, the mapping \( \phi \) from the reference state to the current configuration of the shell is not incompressible, i.e. \( J = \det F \neq 0 \), where \( F = \nabla \phi \) is the deformation gradient. In the small-strain approximation \( \varepsilon \ll 1 \), this manifests itself as the divergence of displacement being non-zero in the shell:

\[
\text{div } \mathbf{u} = \begin{cases} 
0, & 0 < r < R; \\
-\varepsilon, & R < r < r_1.
\end{cases}
\] (5.38)

Thus, if we were to leave the incompressible, spherically symmetric setting where the kinematics are determined completely by mass conservation, the force balance equations would not be the incompressible Lamé equations, and so a different PDE would have to be solved. This would, among other things, break the solution strategy of using Kolosov functions and complex analysis tools to solve the two dimensional problem using the approach presented in this dissertation.

By contrast, in our model, the shell and core maintain separate reference configurations, which are also determined uniquely by mass conservation in the incompressible, spherically symmetric setting. The core reference configuration remains unchanged in this symmetric setting, but in the shell’s reference frame, the position \( \tilde{r}_0 \) of a particle initially located at \( r_0 \)
satisfies \( \rho_2(R_0^3 - \tilde{r}_0^3) = \rho_1(R_0^3 - r_0^3) \). Defining \( \tilde{r}_0 = r_0 \) in the core as well, we find

\[
\tilde{r}_0^3 = \begin{cases} 
  r_0^3, & r < R; \\
  R_0^3 - \frac{\rho_1}{\rho_2} (R_0^3 - r_0^3), & r > R.
\end{cases}
\]

(5.39)

The displacement in our Eulerian model is then

\[
u = r - \tilde{r}_0 = \left( (1 - \varepsilon)r_0^3 + \varepsilon R^3 \right)^{1/3} - \left( R_0^3 - (1 - \varepsilon)(R_0^3 - r_0^3) \right)^{1/3},
\]

(5.40)

which is divergence-free in both regions. The strain tensor in both formulations is diagonal in spherical polar coordinates. In the core it is zero, and in the shell it is

Morris:

\[
\begin{align*}
  e_{rr} &= -\frac{\varepsilon}{3} \left( 1 + 2 \frac{R^3}{r^3} \right) \\
  e_{\theta\theta} = e_{\varphi\varphi} &= -\frac{\varepsilon}{3} \left( 1 - \frac{R^3}{r^3} \right)
\end{align*}
\]

Our model:

\[
\begin{align*}
  e_{rr} &= \frac{2}{3} \varepsilon \frac{R_0^3 - R^3}{r^3} \\
  e_{\theta\theta} = e_{\varphi\varphi} &= -\frac{1}{3} \varepsilon \frac{R_0^3 - R^3}{r^3}
\end{align*}
\]

Morris removes the 'stress-free' strain [35] of uniform compression and defines the stress in terms of the deviatoric strain

\[
\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}', \tag{5.41}
\]

where \( e_{rr}' = -\frac{2}{3} \varepsilon \frac{R^3}{r^3} \) and \( e_{\theta\theta}' = e_{\varphi\varphi}' = \frac{1}{3} \varepsilon \frac{R^3}{r^3} \). In our model, the strain is already deviatoric, so \( \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \).

We see that a significant difference in the predictions of the two models is that the radial deviatoric strain \( e_{rr}' \) is compressive in the Morris model [44], and tensile in our model (and in Morris’s earlier model [43]). In [44], Morris discredits his earlier model for having failed to account for the jump in uniform strain that occurs when the interface passes a point. The strain rates are correct, but there is a missing \( \delta \)-function in the strain-rate equation

\[
\frac{\partial e_{rr}'}{\partial t} = -2\varepsilon \frac{dR}{dt} \left\{ \frac{R^2}{r^3} H(r - R) - \frac{1}{3} \delta(r - R) \right\}, \tag{5.42}
\]

where \( H \) is the Heaviside function and \( \delta \) is the Dirac delta function. However, we have provided an alternative viewpoint in which the core and shell maintain separate reference configurations that evolve in time, and which agree with the earlier work [44] for certain choices of constants. It is therefore instructive to identify precisely where the difference in modeling assumptions yields this qualitative difference in the sign of the radial deviatoric strain.

This is demonstrated in Figure 5.7. In the Morris model, following e.g. Lee and Tromp [35], the 'stress free' strain associated with the phase changed is assumed to be a uniform compression. In our model, when mass is transported from the core reference configuration
Figure 5.7: The key modeling assumption that yields different predictions for the radial deviatoric strain in a spherically symmetric geometry is whether the ‘stress free’ strain associated with the phase change is uniform or depends on the curvatures of the reference configuration boundaries.

to the shell reference configuration, there will be a deviatoric strain component associated with the phase change before the strain associated with the deformation from the reference configuration to the actual configuration is computed. In the spherically symmetric case, the radial deviatoric strain of the phase transformation is so compressive that the radial strain of deformation (which is automatically deviatoric) is tensile. The assumptions of the Morris model are widely used in thermal expansion problems as well as in materials science. One argument in favor of our model is that if a monolayer of atoms joins a laterally compressed surface in the actual configuration, the spacing will match the lattice and attach in a similarly compressed state. We also see no strong argument that the atoms should remember their relative positions before the chemical reactions occur that are responsible for the phase transformation. The truth may be somewhere in between, and can only be answered experimentally.
Chapter 6

Stability Analysis

In this chapter we explore an instability that we have discovered in the evolution equations (2.8)–(2.11). Instabilities are common in fluid-interface problems. The Kelvin-Helmholtz instability describes an instability in the evolution of the interface between two fluids when there is a jump in tangential velocity across the interface [1, 42]. Water being pushed into oil is the classic example of the Rayleigh-Taylor instability, where the evolution of an initial planar interface is unstable to perturbations in the interface [33, 59, 65]. In Hele-Shaw flow, when a fluid is injected into a Hele-Shaw cell containing a more viscous fluid, fingers form in the interface between the two fluids, and this is called the Saffman-Taylor instability [62, 70]. The supercooled Stefan problem, where a liquid is cooled below its freezing temperature before solidifying, has an unstable evolution of the interface between the solid and liquid phases, and is similar to the problem of dendritic crystal growth [5, 58]. Like the olivine-spinel transformation, the Stefan problem is a free boundary problem with a moving interface separating two phases of a material. In the Stefan problem, the interface speed is governed by heat flux, whereas in the Maxwell-Stokes and elastic models that we study here, the interface evolution is determined by non-equilibrium chemical potential considerations [43].

We find that a circular interface evolving according to (2.8)–(2.11) can be either stable or unstable to certain perturbations in the interface position and data. If the circular interface begins in a stable regime, it often evolves into an unstable regime later in time. In this chapter we carry out a stability analysis to study this behavior. In unstable cases, we will show that for a particular family of perturbations of the interface and interface jump conditions, the perturbations grow exponentially in time, and the growth rate increases without bound as the wave number of the perturbation increases.

The Kelvin-Helmholtz and Rayleigh-Taylor instabilities can be stabilized by taking into account surface tension, but in our case more work is needed to determine how to stabilize the interface evolution in the unstable regimes.

The analysis of this chapter is done in the context of the incompressible Maxwell-Stokes problem, but it also applies to the compressible elastic case as explained in the comments after equation (5.35).
6.1 Family of perturbations

In this chapter we always assume the outer boundary \( \Gamma_O \) is the unit circle, \( S^1 \). Suppose at some point in time the state of the evolution is such that the interface \( \Gamma_I \) is parametrized by

\[
\zeta(\alpha) = (R_0 + \varepsilon a_1 \cos(n\alpha))e^{i\alpha}
\]  

and the interface displacement jump by

\[
f(\zeta(\alpha)) = (f_0 + \varepsilon a_2 \cos(n\alpha) + \varepsilon a_3 i \sin(n\alpha))e^{i\alpha},
\]

where \( a = (a_1, a_2, a_3) \in \mathbb{R}^3 \) and \( \varepsilon \) is small. When \( \varepsilon = 0 \), this is the unperturbed problem in the circle, which, as shown in section 3.11, evolves as

\[
\dot{R}_0 = V^{(0)} = \sigma^{(0)}_{nn} = 2f_0R_0 - 2f_0R_0^{-1} - 1
\]

\[
\dot{f}^{(0)} = \left[-(2f_0R_0 - 2f_0R_0^{-1} - 1)f_0R_0^{-1} + \frac{1}{c_1}(2f_0R_0 - 2f_0R_0^{-1} - 1) - \frac{c_2}{c_1}f_0 \right]e^{i\alpha}.
\]

The superscript \((0)\) is used to denote quantities in the unperturbed problem.

Let us define

\[
Q(z) = \begin{cases} 
\phi_c(z) + z\phi'_c(z) + \psi_c(z), & z \in \Omega_c, \\
\phi_s(z) + z\phi'_s(z) + \psi_s(z), & z \in \Omega_s.
\end{cases}
\]

To find \( \dot{f} \) and \( \dot{R} \) for \( \varepsilon \neq 0 \), we solve

\[
\begin{cases}
Q(\xi) = -\xi, & \xi \in \Gamma_O = S^1 \\
[Q(\zeta)] = 0, & \xi \in \Gamma_I \\
[\phi(\zeta)] = f, & \zeta \in \Gamma_I
\end{cases}
\]

for \( \phi \) and \( \psi \) and compute \( \dot{f} \) and \( \dot{R} = \frac{d}{dt} |\zeta| = \frac{V}{r_n} = \frac{\sigma_{nn}}{r_n} \) according to the evolution equations (3.26). It is shown in this chapter that for small \( \varepsilon \),

\[
\sigma_{nn} = \sigma^{(0)}_{nn} + \varepsilon b_1 \cos(n\alpha) + O(\varepsilon^2)
\]

\[
\dot{f} = \dot{f}^{(0)} + \varepsilon b_2 \cos(n\alpha)e^{i\alpha} + \varepsilon b_3 i \sin(n\alpha)e^{i\alpha} + O(\varepsilon^2)
\]

for some constants \( b = (b_1, b_2, b_3) \) that depend linearly on \( a \). We often write \( b(a) \) when we explicitly think of \( b \) as a (linear) function of the perturbation \( a \).

This means that there exists a \( 3 \times 3 \) matrix \( J(R_0(t), f_0(t)) \) for each \( t \) such that if at some point in time a perturbation to the circular problem of the form described in equations (6.1) and (6.2) is introduced, then \( \zeta \) and \( f \) evolve as

\[
\zeta(\alpha, t) = (R_0(t) + \varepsilon a_1(t) \cos(n\alpha))e^{i\alpha} + O(\varepsilon^2)
\]

\[
f(\zeta(\alpha, t)) = (f_0(t) + \varepsilon a_2(t) \cos(n\alpha) + \varepsilon a_3(t)i \sin(n\alpha))e^{i\alpha} + O(\varepsilon^2)
\]
with
\[ \dot{\mathbf{a}}(t) = J(R_0(t), f_0(t))\mathbf{a}(t) \] (6.10)
and \( f_0 \) and \( R_0 \) satisfying the exact circular solution ODE given in (3.30).

As will be shown in section (6.6), for the perturbed problem we can find formulas for the Kolosov functions \( \phi \) and \( \psi \) that are correct to \( O(\varepsilon^2) \). They have the form

\[
\phi_s(z) = \left( f_0 R_0 - \frac{1}{2} \right) z + \varepsilon \left( A_s z^{1+n} + B z^{1-n} \right) + O(\varepsilon^2) \quad (6.11)
\]
\[
\phi_c(z) = \left( f_0 R_0 - \frac{1}{2} - f_0 R_0^{-1} \right) z + \varepsilon A_c z^{1+n} + O(\varepsilon^2) \quad (6.12)
\]
\[
\psi_s(z) = -\frac{2 f_0 R_0}{z} z^{-1} + \varepsilon \left( C_s z^{-1+n} + D z^{-1-n} \right) + O(\varepsilon^2) \quad (6.13)
\]
\[
\psi_c(z) = 0 + \varepsilon C_c z^{-1+n} + O(\varepsilon^2) \quad (6.14)
\]

with all coefficients real. The \((0)\) superscript here is for the exact solution, when \( \varepsilon = 0 \), and the \((1)\) superscript is for the perturbation. We write the exact and perturbed parts of \( Q \) as \( Q(z) = Q^{(0)}(z) + \varepsilon Q^{(1)}(z) + O(\varepsilon^2) \). Equations (6.11)–(6.14) give explicit formulas for \( Q^{(0)}(z) \) and \( Q^{(1)}(z) \). Note that \( Q(z) \) is continuous across \( \Gamma_I \) by (6.5).

Since in this chapter we are assuming that \( \phi \) and \( \psi \) are rational functions in the form given in equations (6.11)–(6.14), we may extend the core functions into the shell and the shell functions into \( \Omega \setminus \{0\} \) in the obvious way while maintaining analyticity. Whereas until now the functions \([\phi] := \phi_s - \phi_c\) and \([\psi] := \psi_s - \psi_c\) were defined only at the interface, for the purposes of this chapter we extend their definitions to all points in \( \Omega \setminus \{0\} \), and moreover \([\phi]\) and \([\psi]\) are analytic throughout \( \Omega \setminus \{0\} \). This will be useful in the case of a perturbed interface when we discuss the extension of \([\phi]\) and \([\psi]\) from the perturbed interface to the circle.

Let \( \{e_i\}, i = 1, 2, 3 \) be the standard basis for \( \mathbb{R}^3 \). We write \( b_i(e_j) \) to mean the \( b_i \) in equations (6.6)–(6.7) obtained by solving (6.5) when \( a = e_j \). In this way, the entries of \( J \) are simply \( J_{ij} = b_i(e_j) \). To find these entries, we will first compute the velocity in the cases of perturbed \( f \) (\( J_{12} \) and \( J_{13} \)), then \( \dot{f} \) in the cases of perturbed \( f \) (\( J_{22}, J_{23}, J_{32}, \) and \( J_{33} \)), and finally the more difficult case of \( V \) and then \( \dot{f} \) when the interface itself is perturbed (\( J_{11} \) and then \( J_{21} \) and \( J_{31} \)).
6.2 Interface velocity in the case of a perturbed displacement jump \((J_{12} \text{ and } J_{13})\)

The entry \(J_{12}\) is \(b_1(0, 1, 0)\) and \(J_{13}\) is \(b_1(0, 0, 1)\). In both cases we assume that we already have the coefficients for \(\phi\) and \(\psi\) from (6.11)–(6.14), and the interface is a circle, so we will be able to compute \(\sigma_{nn}\) for both problems at the same time.

To compute \(V = \sigma_{nn}\), we start with \(\mathbf{n} \cdot \sigma = \Re\{\mathbf{n}(-i\partial_s Q)}\}, where \(\mathbf{s}\) is the normalized tangent to \(\Gamma_I\). Since \(Q\) is continuous across \(\Gamma_I\), we can compute

\[
\partial_s Q(\zeta) = \partial_s (\phi + \zeta \phi' + \psi) = s\phi' + \zeta s\phi'' + \zeta s\psi' + \zeta s\psi''
\]

from either side of the interface. The same is true of \(\partial_s Q^{(0)}(\zeta)\) and \(\partial_s Q^{(1)}(\zeta)\). Let’s start with \(Q^{(1)}\) in terms of \(\phi_c\) and \(\psi_c\) as in (6.11)–(6.14):

\[
Q^{(1)}(\zeta) = A_c \zeta^{1+n} + \zeta(1+n)A_c R_0^{2+n} \zeta^{-n} + C_c R_0^{-2+2n} \zeta^{1-n} = A_c \zeta^{1+n} + ((1+n)A_c R_0^{2+n} + C_c R_0^{-2+2n}) \zeta^{1-n}.
\]

Then

\[
\partial_s Q^{(1)}(\zeta(\alpha)) = [(1+n)A_c \zeta(\alpha)^n + ((1-n)(1+n)A_c R_0^{2+n} + (1-n)C_c R_0^{-2+2n}) \zeta(\alpha)^{-n}] s
= [(1+n)A_c R_0^{\alpha} e^{-in\alpha} + ((1-n)(1+n)A_c R_0^{\alpha} + (1-n)C_c R_0^{-2+2n}) e^{-in\alpha}] i e^{i\alpha}
\]

and \(\sigma_{nn}^{(1)} = \Re\{ -i \mathbf{n} \partial_s Q^{(1)}(\zeta(\alpha))\}\) is given by

\[
\sigma_{nn}^{(1)} = (1+n)A_c R_0^{\alpha} \cos(n\alpha) + ((1-n)(1+n)A_c R_0^{\alpha} + (1-n)C_c R_0^{-2+2n}) \cos(n\alpha)
= ((2-n)(1+n)A_c R_0^{\alpha} + (1-n)C_c R_0^{-2+2n}) \cos(n\alpha).
\]

As a sanity check, we note that

\[
Q^{(0)}(\zeta) = \left(f_0 R_0 - \frac{1}{2} - f_0 R_0^{-1}\right) \zeta + \zeta \left(f_0 R_0 - \frac{1}{2} - f_0 R_0^{-1}\right) + 0
\]

and \(\Re\{ -i \mathbf{n} \partial_s Q^{(0)}(\zeta)\}\) = \(-i \mathbf{n} \Re\{ 2f_0 R_0 - 1 - 2f_0 R_0^{-1}\}\), which agrees with \(\sigma_{nn}^{(0)}\) in (6.3). So \(J_{12}\) and \(J_{13}\) are

\[
(2-n)(1+n)A_c R_0^{\alpha} + (1-n)C_c R_0^{-2+2n}.
\]

Note that \(J_{12}\) and \(J_{13}\) will generally not be equal since \(A_c\) and \(C_c\) depend on \(\mathbf{a}\), which equals \(\mathbf{e}_2\) for \(J_{12}\) and \(\mathbf{e}_3\) for \(J_{13}\). Explicit formulas for \(A_c\) and \(C_c\) for \(\mathbf{a} \in \{\mathbf{e}_2, \mathbf{e}_3\}\) will be derived in Section 6.6 below.
6.3 Time derivative of displacement jump in the case of a perturbed displacement jump ($J_{22}$, $J_{23}$, $J_{32}$, and $J_{33}$)

Now we compute $\dot{f}$ in these cases. We have $\frac{d}{dt}[u + iv] = \dot{R}[\partial_r(u + iv)] + \frac{1}{c_1}Vn - \frac{\alpha}{c_1}[u + iv]$. Only the first term here is a bit complicated.

Instead of writing $u + iv = \frac{1}{2}(\phi - z\phi' - \psi)$ and expanding derivatives to compute $u_r + iv_r$ in terms of partial derivatives of the real and imaginary parts of $\phi$, $\phi'$, and $\psi$, we use the simpler form

$$\partial_r(u + iv) = r_1u_x + r_2u_y + i(r_1v_x + r_2v_y)$$

$$= r_1(u_x + iv_x) + ir_2(u_y + iv_y)$$

$$= r_1(\partial_x + \partial_z)(u + iv) + ir_2(\partial_y - \partial_z)(u + iv)$$

$$= (r\partial_x + r\partial_z)(u + iv).$$

This is easy to compute because

$$\partial_z(u + iv) = \frac{1}{2}(\phi' - \bar{\phi})$$

and

$$\partial_{\bar{z}}(u + iv) = \frac{1}{2}(-z\phi'' - \bar{\psi}).$$

So

$$\left[\partial_z(u + iv)\right](\zeta(\alpha)) = \frac{1}{2} \left[ \varepsilon \left( (1 + n)(A_s - A_c) - (1 - n)BR_0^{-2n} \right) (\zeta(\alpha))^n 
+ \varepsilon \left( (1 - n)B - (1 + n)(A_s - A_c)R_0^{2n} \right) (\zeta(\alpha))^{-n} \right]$$

$$= \frac{1}{2} \left[ \varepsilon \left( (1 + n)(A_s - A_c)R_0^n - (1 - n)BR_0^{-n} \right) e^{in\alpha} 
+ \varepsilon \left( (1 - n)BR_0^{-n} - (1 + n)(A_s - A_c)R_0^n \right) e^{-in\alpha} \right]$$

$$= \varepsilon \left( (1 + n)(A_s - A_c)R_0^n - (1 - n)BR_0^{-n} \right) i\sin(n\alpha).$$
and

\[
\begin{align*}
\left[ \partial_r (u + iv) \right] (\zeta(\alpha)) \\
= & \frac{1}{2} \left( 2 f_0 R_0 \zeta(\alpha)^{-2} + \varepsilon (n(1+n)(A_s - A_c) R_0^{n-2} + (n-1)(C_s - C_c) R_0^{n-2}) (\zeta(\alpha))^{2-n} \\
& + \varepsilon (n(n-1)BR_0^{-2} + (n+1)DR_0^{-2}) (\zeta(\alpha))^{2+n} \right) \\
= & -\frac{1}{2} e^{2i\alpha} \left[ 2 f_0 R_0^{-1} \\
& + \varepsilon \left[ n(1+n)(A_s - A_c) R_0^n + (n-1)(C_s - C_c) R_0^{-2} \\
& + n(n-1)BR_0^{-n} - (1+n)DR_0^{-2} \right] \cos(n\alpha) \\
& + \varepsilon \left[ -n(1+n)(A_s - A_c) R_0^n - (n-1)(C_s - C_c) R_0^{-2} \\
& + n(n-1)BR_0^{-n} - (1+n)DR_0^{-2} \right] i \sin(n\alpha) \right].
\end{align*}
\]

Thus

\[
\begin{align*}
\left[ \partial_r (u + iv) \right] (\zeta(\alpha)) & = r(\zeta(\alpha)) \left[ \partial_r (u + iv) \right] (\zeta(\alpha)) + \frac{1}{2} \varepsilon \left[ (n(1+n)(A_s - A_c) R_0^n + (n-1)BR_0^{-n} \\
& - (n-1)(C_s - C_c) R_0^{-2} + (1+n)DR_0^{-2} \right] \cos(n\alpha) e^{i\alpha} \\
& + \frac{1}{2} \varepsilon \left[ (2+n)(1+n)(A_s - A_c) R_0^n - (2-n)(1-n)BR_0^{-n} \\
& + (n-1)(C_s - C_c) R_0^{-2} + (1+n)DR_0^{-2} \right] i \sin(n\alpha) e^{i\alpha} \\
& =: \partial_r^{ex} e^{i\alpha} + \varepsilon u_c \cos(n\alpha) e^{i\alpha} + \varepsilon u_s i \sin(n\alpha) e^{i\alpha}.
\end{align*}
\] (6.15)

In the above equation we have defined \( \partial_r^{ex}, u_c, \) and \( u_s, \) and we also define

\[
\sigma_{mn}^{(1)} := (2-n)(1+n)A_c R_0^n + (1-n)C_c R_0^{-2+n}
\]

so that \( \sigma_{nn} = \sigma_{nn}^{(0)} + \varepsilon \sigma_{nn}^{(1)} \cos(n\alpha), \) as shown in the previous section. We can now write out

\[
\begin{align*}
\hat{R}(\zeta(\alpha)) \left[ \partial_r (u + iv) \right] (\zeta(\alpha)) & = (\sigma_{nn}^{(0)} + \varepsilon \sigma_{nn}^{(1)} \cos(n\alpha))(\partial_r^{ex} e^{i\alpha} + \varepsilon u_c \cos(n\alpha) + \varepsilon u_s i \sin(n\alpha)) \\
& = \sigma_{nn}^{(0)} \partial_r^{ex} e^{i\alpha} + \varepsilon \left( \sigma_{nn}^{(1)} \partial_r^{ex} + \sigma_{nn}^{(0)} u_c \right) \cos(n\alpha) e^{i\alpha} + \varepsilon \left( \sigma_{nn}^{(0)} u_s \right) i \sin(n\alpha) e^{i\alpha} + O(\varepsilon^2).
\end{align*}
\] (6.16)
Now we have
\[
\frac{d}{dt} f(\zeta(\alpha)) = \frac{d}{dt} \left[ u + iv \right] (\zeta(\alpha))
= \hat{R} \left[ \partial_r (u + iv) \right] + \frac{1}{c_1} \hat{R} \hat{n} - \frac{c_2}{c_1} \left[ u + iv \right]
= \left( \sigma_{nn}^0 \sigma_{ex} \right) e^{i\alpha}
+ \varepsilon \left( \sigma_{nn}^0 \sigma_{ex} \sigma_{mn}^0 u_c + \frac{1}{c_1} \sigma_{nn}^1 \right) \cos(n\alpha) e^{i\alpha}
+ \varepsilon (\sigma_{nn}^0 u_s) i \sin(n\alpha) e^{i\alpha}
- \frac{c_2}{c_1} \hat{f}(\zeta(\alpha)) + O(\varepsilon^2).
\]

This gives us 4 more entries of $J$:
\[
\begin{align*}
J_{22} &= \sigma_{nn}(1) \sigma_{ex} + \sigma_{nn}^0 u_c + \frac{1}{c_1} \sigma_{nn}^1 - \frac{c_2}{c_1} \\
J_{23} &= \sigma_{nn}(1) \sigma_{ex} + \sigma_{nn}^0 u_c + \frac{1}{c_1} \sigma_{nn}^1 \\
J_{32} &= \sigma_{nn}^0 u_s \\
J_{33} &= \sigma_{nn}^0 u_s - \frac{c_2}{c_1}.
\end{align*}
\]

### 6.4 Velocity of a perturbed interface ($J_{11}$)

We now tackle the first column, where $\zeta(\alpha) = R_0 e^{i\alpha} + \varepsilon \cos(n\alpha) e^{i\alpha}$ and $f(\zeta(\alpha)) = e^{i\alpha}$. Since we have already computed $\hat{f}$ and $\hat{R}$ on the circle in terms of $\phi$ and $\psi$ in the form given in equations (6.11)–(6.14), instead of writing out all of the formulas again, we will just find the $O(\varepsilon)$ changes when we evaluate on the perturbed interface instead of the circle.

Let $\zeta(0)(\alpha) = R_0 e^{i\alpha}$. We'll start with the $\sigma_{nn}$ term, and first show that
\[
\partial_s Q^{(1)}(\zeta(\alpha)) = \partial_s Q^{(1)}(\zeta(0)(\alpha)) + O(\varepsilon).
\]

Since $\zeta(\alpha)^k = \zeta(0)(\alpha)^k + O(\varepsilon)$ for any $k$, we get
\[
Q^{(1)}(\zeta(\alpha)) = A_c \zeta(\alpha)^{1+n} + \zeta(\alpha)(1+n)A_c \zeta(\alpha)^n + C_c \zeta(\alpha)^{-1+n} = Q^{(1)}(\zeta(0)(\alpha)) + O(\varepsilon).
\]

Thus
\[
\partial_s Q^{(1)}(\zeta(\alpha)) = (\partial_s (Q^{(1)} \circ \zeta)(\alpha)) \zeta'(\alpha)^{-1}
= \left[ \partial_s (Q^{(1)} \circ \zeta(0)(\alpha) + O(\varepsilon)) \right] \left[ \zeta'(0)(\alpha)^{-1} + O(\varepsilon) \right]
= \partial_s Q^{(1)}(\zeta(0)(\alpha)) + O(\varepsilon).
\]
As before, \[ \Re\{ -i \mathbf{n} \partial_{\mathbf{v}} Q^{(1)}(\zeta(\alpha)) \} = ((2 - n)(1 + n)A_e R_0^n + (1 - n) C_e R_0^{2+n}) \cos(n\alpha) + O(\varepsilon). \]

We also confirm that \( Q^{(0)}(z) = (2f_0 R_0^{-1} - 1 - 2f_0 R_0) z \) everywhere, so \( \Re\{ -i \mathbf{n} \partial_{\mathbf{v}} Q^{(0)}(\zeta) \} = \sigma_{nn}^{(0)}. \)

We conclude
\[ J_{11} = (2 - n)(1 + n)A_e R_0^n + (1 - n) C_e R_0^{2+n}. \quad (6.17) \]

### 6.5 Time derivative of displacement jump in the case of a perturbed interface (\( J_{21} \) and \( J_{31} \))

Figuring out \( \frac{d}{dt} [u + iv] = \dot{R} \left[ \partial_t (u + iv) \right] + \frac{1}{\Omega} V \mathbf{n} - \frac{\partial}{\partial z} [u + iv] \) in the case of a perturbed interface gets complicated, but we will be able to use the calculations from Section 6.3.

Following the same reasoning as in showing \( Q^{(1)}(\zeta) = Q^{(1)}(\zeta^{(0)}) + O(\varepsilon) \), we have
\[ \frac{d}{dt} [u + iv]^{(1)}(\zeta(\alpha)) = \dot{\zeta}(\alpha) \overline{Q^{(1)}(\zeta(\alpha))} - \overline{\dot{\psi}(\zeta(\alpha))} = (u + iv)^{(1)}(\zeta^{(0)}(\alpha)) + O(\varepsilon). \]

Similarly,
\[ \partial_z (u + iv)^{(1)}(\zeta(\alpha)) = \partial_z (u + iv)^{(1)}(\zeta^{(0)}(\alpha)) + O(\varepsilon). \]

Thus the formulas for computing \( \frac{d}{dt} [u + iv]^{(1)}(\zeta) \) up to \( O(\varepsilon) \) in terms of \( \phi \) and \( \psi \) in the form given in equations (6.11)–(6.14) are the same as those that were written out in Section 6.3.

So we turn our attention to finding the \( O(\varepsilon) \) changes in \( (u + iv)^{(0)} \) and \( \partial_z (u + iv)^{(0)} \) from the circular to the perturbed interface. Since \( \phi^{(0)'} \) and \( \phi^{(0)''} \) are constant, \( \left[ \phi^{(0)'} \right] (\zeta) = \left[ \phi^{(0)''} \right] (\zeta^{(0)}) \) and the same holds for \( \left[ \phi^{(0)'} \right] \) and \( \left[ \phi^{(0)''} \right] \). This means
\[ \left[ \partial_z (u + iv)^{(0)}(\zeta) \right] = \left[ \frac{1}{2} (\phi' - \overline{\phi'}) (\zeta) \right] = \left[ \partial_z (u + iv)^{(0)}(\zeta^{(0)}) \right] + O(\varepsilon^2). \]

The \( \overline{\psi} \) case is a bit trickier:
\begin{align*}
- \frac{1}{2} \left( \left[ \psi^{(0)'}(\zeta(\alpha)) \right] - \left[ \psi^{(0)'}(\zeta^{(0)}(\alpha)) \right] \right) &= - \frac{1}{2} \left( 2f_0 R_0 \Re \zeta^{(0)}(\alpha) - 2f_0 R_0 \Re \zeta^{(0)}(\alpha) \right) \\
&= - f_0 R_0 \left( R_0 e^{-i\alpha} + \frac{\varepsilon}{2} e^{i(-1-n)\alpha} + \frac{\varepsilon}{2} e^{i(-1+n)\alpha} \right) \\
&= - f_0 R_0 \left( R_0 e^{2i\alpha} \right) \\
&= - 2R_0^{-3} e^{3i\alpha} \left( \frac{\varepsilon}{2} e^{i(-1-n)\alpha} + \frac{\varepsilon}{2} e^{i(-1+n)\alpha} \right) \\
&= \varepsilon f_0 R_0^{-2} e^{i(2-n)\alpha} + \varepsilon f_0 R_0^{-2} e^{i(2+n)\alpha} + O(\varepsilon^2) \\
&= 2\varepsilon f_0 R_0^{-2} \cos(n\alpha) e^{2i\alpha} + O(\varepsilon^2). 
\end{align*}
This in addition to $[\phi^{(0)''}] = 0$ says

$$
[\partial_z(u + iv)(\zeta(\alpha))] = \left[ \frac{1}{2} ( - z \overline{\phi'} - \overline{\psi'} ) (\zeta(\alpha)) \right]
= [\partial_z(u + iv)(\zeta(0)(\alpha))] + 2 \varepsilon f_0 R_0^{-2} \cos(n\alpha)e^{2i\alpha} + O(\varepsilon^2).
$$

Now

$$
[\partial_r(u + iv)](\zeta(\alpha)) = r(\zeta(\alpha)) [\partial_z(u + iv)](\zeta(\alpha)) + \overline{r(\zeta(\alpha))} [\overline{\partial_z(u + iv)}](\zeta(\alpha))
= [\partial_r(u + iv)](\zeta(0)(\alpha)) + |r(\zeta(\alpha))| 2 \varepsilon f_0 R_0^{-2} \cos(n\alpha)e^{2i\alpha} + O(\varepsilon^2)
= [\partial_r(u + iv)](\zeta(0)(\alpha)) + 2 \varepsilon f_0 R_0^{-2} \cos(n\alpha)e^{i\alpha} + O(\varepsilon^2).
$$

So the new term that gets added to $[\partial_r(u + iv)]$ is

$$
r(\zeta(\alpha)) 2 \varepsilon f_0 R_0^{-2} \cos(n\alpha)e^{2i\alpha} = 2 \varepsilon f_0 R_0^{-2} \cos(n\alpha)e^{i\alpha}.
$$

Using

$$
\sigma_{nn}^{(1)} = (2 - n)(1 + n) A_c R_0^n + (1 - n) C_c R_0^{-2+n}
$$

that we computed above in equation (6.17) and

$$
u_s = \frac{1}{2} \left[ (2 + n)(1 + n)(A_s - A_c) R_0^n - (2 - n)(1 - n) BR_0^{-n}
+ (n - 1)(C_s - C_c) R_0^{-2+n} + (1 + n) DR_0^{-2-n} \right]
$$

and the new equation

$$
u_c = \frac{1}{2} \left[ -(2 + n)(A_s - A_c) R_0^n + n(1 - n) BR_0^{-n}
- (n - 1)(C_s - C_c) R_0^{-2+n} + (1 + n) DR_0^{-2-n} \right] + 2 f_0 R_0^{-2},
$$

both adapted from (6.15), the formula (6.16) for $\hat{R} [\partial_r(u + iv)]$ holds just as before but with the new $u_c$. The only possible issue would be that $\hat{R} = \frac{\sigma_{nn}}{\bar{\gamma}_n}$ could differ from $\sigma_{nn}$ by more than $O(\varepsilon^2)$, but it does not. To figure out $r \cdot n$, we start with computing $|\zeta'(\alpha)|$:

$$
|\zeta'(\alpha)| = |i R_0 e^{i\alpha} + \varepsilon \cos(n\alpha) e^{i\alpha} + \varepsilon (-n \sin(n\alpha)) e^{i\alpha}|
= |R_0 + \varepsilon \cos(n\alpha) + i \varepsilon n \sin(n\alpha)|
= ((R_0 + \varepsilon \cos(n\alpha))^2 + \varepsilon^2 n^2 \sin(n\alpha)^2)^{\frac{1}{2}}
= (R_0^2 + 2 \varepsilon R_0 \cos(n\alpha) + O(\varepsilon^2))^{\frac{1}{2}}
= R_0 + \frac{1}{2} (R_0^2)^{-\frac{1}{2}} (2 \varepsilon R_0 \cos(n\alpha) + O(\varepsilon^2))
= R_0 + \varepsilon \cos(n\alpha) + O(\varepsilon^2),
$$
and \(|\zeta'(\alpha)|^{-1} = R_0^{-1} - R_0^{-2}\varepsilon \cos(n\alpha) + O(\varepsilon^2)\). So

\[
\hat{n} = -i\zeta'(\alpha)|\zeta'(\alpha)|^{-1}
= \left[ R_0 \varepsilon^{i\alpha} + \varepsilon \cos(n\alpha)e^{i\alpha} + i\varepsilon n \sin(n\alpha)e^{i\alpha} \right] \left[ R_0^{-1} - R_0^{-2}\varepsilon \cos(n\alpha) + O(\varepsilon^2) \right]
= e^{i\alpha} + iR_0^{-1}\varepsilon n \sin(n\alpha)e^{i\alpha} + O(\varepsilon^2)
\]

and

\[
r \cdot n = \Re\{\hat{r}\hat{n}\}
= \Re\{e^{-i\alpha}\hat{n}\}
= \Re\{1 + iR_0^{-1}\varepsilon n \sin(n\alpha) + O(\varepsilon^2)\}
= 1 + O(\varepsilon^2).
\]

The term \(\frac{1}{c_1}V\hat{n}\) will be different from the perturbed displacement jump case. We get

\[
\frac{1}{c_1}V\hat{n} = \frac{1}{c_1} \left( \sigma_{nn}^{(0)} + \varepsilon \sigma_{nn}^{(1)} \cos(n\alpha) \right) \left( e^{i\alpha} + iR_0^{-1}\varepsilon n \sin(n\alpha)e^{i\alpha} + O(\varepsilon^2) \right)
= \frac{1}{c_1} \left( \sigma_{nn}^{(0)}e^{i\alpha} + \varepsilon\sigma_{nn}^{(1)} \cos(n\alpha)e^{i\alpha} + \varepsilon\sigma_{nn}^{(0)}R_0^{-1}n \sin(n\alpha)e^{i\alpha} + O(\varepsilon^2) \right).
\]

We can put all of this together now to get

\[
\frac{d}{dt}f(\zeta(\alpha)) = \frac{d}{dt} \left[ u + iv \right] (\zeta(\alpha))
= \hat{R} \left[ \partial_r(u + iv) \right] + \frac{1}{c_1} \hat{R}\hat{n} - \frac{c_2}{c_1} \left[ u + iv \right]
= \left( \sigma_{nn}^{(0)} \partial_r^{ex} + \frac{1}{c_1} \sigma_{nn}^{(0)} \right) e^{i\alpha}
+ \varepsilon \left( \sigma_{nn}^{(1)} \partial_r^{ex} + \sigma_{nn}^{(0)} u_c + \frac{1}{c_1} \sigma_{nn}^{(1)} \right) \cos(n\alpha)e^{i\alpha}
+ \varepsilon \left( \sigma_{nn}^{(0)} u_s + \frac{1}{c_1} \sigma_{nn}^{ex} R_0^{-1} n \right) i \sin(n\alpha)e^{i\alpha}
- \frac{c_2}{c_1} f(\zeta(\alpha)) + O(\varepsilon^2).
\]

This gives us the last two entries of \(J\)

\[
J_{21} = \sigma_{nn}^{(1)} \partial_r^{ex} + \sigma_{nn}^{(0)} u_c + \frac{1}{c_1} \sigma_{nn}^{(1)}
J_{31} = \sigma_{nn}^{(0)} u_s + \frac{1}{c_1} \sigma_{nn}^{(0)} R_0^{-1} n.
\]
6.6 The coefficients of the solution to the perturbed system

As promised, we now show that $\phi$ and $\psi$ do indeed take the form (6.11)–(6.14) and we find the coefficients $A_c$, $A_s$, $B$, $C_c$, $C_s$, and $D$ in terms of $a$ for each of the three problems $a \in \{e_1, e_2, e_3\}$.

Coefficients for column 2

We'll start with $\zeta(\alpha) = R_0 e^{i\alpha}$ and $f(\zeta(\alpha)) = f_0 e^{i\alpha} + \varepsilon \cos(n\alpha)e^{i\alpha}$ and find six linear equations for the six coefficients, two equations for each of the boundary/interface conditions. First we have for $\xi \in \Gamma_O$

$$Q(\xi) = -\xi \implies Q^{(0)}(\xi) + \varepsilon Q^{(1)}(\xi) = -\xi$$
$$\implies Q^{(1)}(\xi) = 0$$
$$\implies A_s \xi^{1+n} + B \xi^{1-n} + \xi \left( (1 + n) A_s \xi^{-n} + (1 - n) B \xi^n \right) + C_s \xi^{1-n} + D \xi^{1+n} = 0$$
$$\implies \begin{cases} A_s + (1-n)B + D = 0 \\ (1+n)A_s + C_s + B = 0. \end{cases}$$ (6.18)

The outer boundary condition is not affected by the perturbations, so equation 6.18 will hold for all three perturbed problems.

Next,

$$\left[ Q(\zeta) \right] = 0 \implies \left[ Q^{(0)}(\zeta) \right] + \varepsilon \left[ Q^{(1)}(\zeta) \right] = 0$$
$$\implies \left[ Q^{(1)}(\zeta) \right] = 0$$
$$\implies A_s \xi^{1+n} + B \xi^{1-n} + \xi \left( (1 + n) A_s R_0^{2n} \xi^{-n} + (1 - n) B R_0^{-2n} \xi^n \right)$$
$$+ C_s R_0^{-2+2n} \xi^{-1-n} + D R_0^{-2-2n} \xi^{1+n}$$
$$- A_c \xi^{1+n} - \xi \left( (1 + n) A_c R_0^{2n} \xi^{-n} \right) - C_c R_0^{-2+2n} \xi^{1-n}$$
$$= 0$$
$$\implies \begin{cases} A_s + (1-n)B R_0^{-2n} + D R_0^{-2-2n} - A_c = 0 \\ B + (1+n)A_s R_0^{2n} + C_s R_0^{-2+2n} - (1+n)A_c R_0^{2n} - C_c R_0^{-2+2n} = 0. \end{cases}$$ (6.19)
Finally,

\[
\phi(\zeta) = f_0 e^{i\alpha} + \varepsilon \cos(n\alpha) e^{i\alpha}
\]

\[
\Rightarrow \phi^{(1)}(\zeta) = \frac{1}{2} e^{i(1+n)\alpha} + \frac{1}{2} e^{i(1-n)\alpha}
\]

\[
A_s \zeta^{1+n} + B \zeta^{1-n} - A_c \zeta^{1+n} = \frac{1}{2} R_{0}^{-1-n} \zeta^{-1+n} + \frac{1}{2} R_{0}^{-1+n} \zeta^{1-n}
\]

\[
\Rightarrow \begin{cases} 
A_s - A_c = \frac{1}{2} R_{0}^{-1-n} \\
B = \frac{1}{2} R_{0}^{1+n}
\end{cases}
\]  \hspace{1cm} (6.20)

It turns out that there is a real-valued solution to the six equations in (6.18)–(6.20), so it is not necessary to break them up into a 12 \times 12 system of the coefficients’ real and imaginary parts. The solution to these equations is

\[
A_s = \frac{1}{2} \left[(n - 1) R_{0}^{-1+n} + (2 - n) R_{0}^{1+n}\right]
\]

\[
B = \frac{1}{2} R_{0}^{-1+n}
\]

\[
C_s = -\frac{1}{2} (1 + n) \left((n - 1) R_{0}^{-1+n} + (2 - n) R_{0}^{1+n}\right) - \frac{1}{2} R_{0}^{-1+n}
\]

\[
D = \frac{1}{2} (n - 2) R_{0}^{1+n}
\]

\[
A_c = \frac{1}{2} \left[(n - 1) R_{0}^{-1+n} + (2 - n) R_{0}^{1+n} - R_{0}^{-1-n}\right]
\]

\[
C_c = \frac{1}{2} \left[(2 + n) R_{0}^{1-n} - (1 + n) \left((n - 1) R_{0}^{-1+n} + (2 - n) R_{0}^{1+n}\right) - R_{0}^{-1+n}\right].
\]

**Coefficients for column 3**

For $\zeta(\alpha) = R_0 e^{i\alpha}$ and $f(\zeta(\alpha)) = f_0 e^{i\alpha} + \varepsilon i \sin(n\alpha) e^{i\alpha}$, equations (6.18) and (6.19) are the same and the third just has a different right-hand side:

\[
\phi(\zeta) = f_0 e^{i\alpha} + \varepsilon i \sin(n\alpha) e^{i\alpha}
\]

\[
\Rightarrow \phi^{(1)}(\zeta) = i \frac{1}{2i} e^{i(1+n)\alpha} - i \frac{1}{2i} e^{i(1-n)\alpha}
\]

\[
A_s \zeta^{1+n} + B \zeta^{1-n} - A_c \zeta^{1+n} = \frac{1}{2} R_{0}^{-1-n} \zeta^{-1+n} - \frac{1}{2} R_{0}^{-1+n} \zeta^{1-n}
\]

\[
\Rightarrow \begin{cases} 
A_s - A_c = \frac{1}{2} R_{0}^{-1-n} \\
B = -\frac{1}{2} R_{0}^{-1+n}
\end{cases}
\]
The solution is

\[ A_s = \frac{1}{2} \left[ (1-n)R_0^{-1+n} + nR_0^{1+n} \right] \]
\[ B = -\frac{1}{2} R_0^{-1+n} \]
\[ C_s = \frac{1}{2} \left[ n^2 R_0^{-1+n} - n(1+n)R_0^{1+n} \right] \]
\[ D = -\frac{1}{2} n R_0^{1+n} \]
\[ A_c = \frac{1}{2} \left[ (1-n)R_0^{-1+n} + n R_0^{1+n} - R_0^{-1-n} \right] \]
\[ C_c = \frac{1}{2} \left[ n R_0^{1-n} + n^2 R_0^{-1+n} - n(1+n)R_0^{1+n} \right]. \]

**Coefficients for column 1**

For \( \zeta(\alpha) = R_0 e^{i\alpha} + \varepsilon \cos(n\alpha) e^{i\alpha} \) and \( f(\zeta(\alpha)) = f(0) e^{i\alpha} \), the outer boundary equation (6.18) is the same, but we have to do something more complicated for the interface equations because of the perturbed interface.

What we will do is find what \([ \phi ]\) and \([ Q ]\) have to be on the circle so that when moved to the perturbed interface, they are equal to \([ \phi(\zeta) ]\) and \([ Q(\zeta) ]\) to within \( O(\varepsilon^2) \). Let \( \zeta(0)(\alpha) = R_0 e^{i\alpha} \).

We guess that \( f \) must have the form

\[ f(\zeta(0)) = f(0) e^{i\alpha} \]

\[ = f_0 e^{i\alpha} + \varepsilon \delta_1 e^{i(1+n)\alpha} + \varepsilon \delta_2 e^{i(1-n)\alpha} \]

\[ = f_0 R_0^{-1} \zeta(0) + \varepsilon \delta_1 R_0^{-1-n} \zeta(0)^{1+n} + \varepsilon \delta_2 R_0^{-1+n} \zeta(0)^{1-n} \]

for some \( \delta_1 \) and \( \delta_2 \). Then

\[ f(\zeta(\alpha)) = f(0) e^{i\alpha} + \varepsilon \left( \frac{1}{2} f_0 R_0^{-1} + \delta_1 \right) e^{i(1+n)\alpha} + \varepsilon \left( \frac{1}{2} f_0 R_0^{-1} + \delta_2 \right) e^{i(1-n)\alpha} + O(\varepsilon^2). \]

Thus when \( \delta_1 = \delta_2 = -\frac{1}{2} f_0 R_0^{-1} \), i.e. when

\[ f(\zeta) = f_0 R_0^{-1} \zeta - \varepsilon \frac{1}{2} f_0 R_0^{-2-n} \zeta^{1+n} - \varepsilon \frac{1}{2} f_0 R_0^{-2+n} \zeta^{1-n}, \]
we get \( f(\zeta(\alpha)) = f_0 e^{i\alpha} + O(\varepsilon^2) \) as desired.

Now we have to figure out what \( [\psi] \) is on the circle. We will first get \( [\psi(\zeta)] \) from the equation \( [Q(\zeta)] = 0 \) and then figure out what \( [\psi] \) must be on the circle so that when moved to the interface it becomes this \( [\psi(\zeta)] \). Since \( [Q(\zeta)] = 0 \),

\[
[\psi(\zeta(\alpha))] = -f(\zeta(\alpha)) - \zeta(\alpha)f'(\zeta(\alpha)).
\]

To write down what this is in terms of \( f_0 \) and \( R_0 \), we will need

\[
\zeta'(\alpha)^{-1} = \left( iR_0 e^{i\alpha} + \frac{\varepsilon}{2} i(1+n) e^{i(1+n)\alpha} + \frac{\varepsilon}{2} i(1-n) e^{i(1-n)\alpha} \right)^{-1}
\]

\[
= (iR_0 e^{i\alpha})^{-1} - (iR_0 e^{i\alpha})^{-2} \left( \frac{\varepsilon}{2} i(1+n) e^{i(1+n)\alpha} + \frac{\varepsilon}{2} i(1-n) e^{i(1-n)\alpha} \right) + O(\varepsilon^2)
\]

\[
= -iR_0^{-1} e^{-i\alpha} + \frac{\varepsilon}{2} iR_0^{-2} ((1+n) e^{i(1+n)\alpha} + (1-n) e^{i(1-n)\alpha}) + O(\varepsilon^2)
\]

and

\[
\overline{\zeta(\alpha)}\zeta'(\alpha)^{-1}
\]

\[
= \left( R_0 e^{-i\alpha} + \frac{\varepsilon}{2} e^{i(-1-n)\alpha} + \frac{\varepsilon}{2} e^{i(-1+n)\alpha} \right)
\]

\[
\times \left[ -iR_0^{-1} e^{-i\alpha} + \frac{\varepsilon}{2} iR_0^{-2} ((1+n) e^{i(-1+n)\alpha} + (1-n) e^{i(-1-n)\alpha}) \right]
\]

\[
= -i e^{-2i\alpha} + \frac{\varepsilon}{2} R_0^{-1} i n e^{i(-2+n)\alpha} - \frac{\varepsilon}{2} R_0^{-1} i n e^{i(-2-n)\alpha} + O(\varepsilon^2).
\]

Now

\[
[\psi(\zeta(\alpha))] = -f_0 e^{-i\alpha} - \frac{f_0 i e^{i\alpha}}{-f(\zeta(\alpha))} \left[ -i e^{-2i\alpha} + \frac{\varepsilon}{2} R_0^{-1} i n e^{i(-2+n)\alpha} - \frac{\varepsilon}{2} R_0^{-1} i n e^{i(-2-n)\alpha} \right] + O(\varepsilon^2)
\]

\[
= -2f_0 e^{-i\alpha} + \frac{\varepsilon}{2} n f_0 R_0^{-1} e^{i(-1+n)\alpha} - \frac{\varepsilon}{2} n f_0 R_0^{-1} e^{i(-1-n)\alpha} + O(\varepsilon^2). \tag{6.21}
\]

We guess \( [\psi(\zeta)] \) has the form \(-2f_0 R_0 \zeta^{-1} + \varepsilon \delta_1 \zeta^{-1+n} + \varepsilon \delta_2 \zeta^{-1-n} \), and we find what \( \delta_1 \) and \( \delta_2 \) have to be to satisfy equation (6.21). Using

\[
\zeta(\alpha)^{-1} = \left( R_0 e^{i\alpha} + \frac{\varepsilon}{2} e^{i(1+n)\alpha} + \frac{\varepsilon}{2} e^{i(1-n)\alpha} \right)^{-1}
\]

\[
= R_0^{-1} e^{-i\alpha} - R_0^{-2} e^{-2i\alpha} \left( \frac{\varepsilon}{2} e^{i(1+n)\alpha} + \frac{\varepsilon}{2} e^{i(1-n)\alpha} \right) + O(\varepsilon^2)
\]

\[
= R_0^{-1} e^{-i\alpha} - \frac{\varepsilon}{2} R_0^{-2} e^{i(-1+n)\alpha} - \frac{\varepsilon}{2} R_0^{-2} e^{i(-1-n)\alpha} + O(\varepsilon^2),
\]

\[
\delta_1 \zeta(\alpha)^{-1+n} = \delta_1 R_0^{-1+n} e^{i(-1+n)\alpha} + O(\varepsilon),
\]

and

\[
\delta_2 \zeta(\alpha)^{-1-n} = \delta_2 R_0^{-1-n} e^{i(-1-n)\alpha} + O(\varepsilon),
\]
we get

\[
[\psi]\left(\zeta(\alpha)\right) = -2f_0R_0\zeta(\alpha)^{-1} + \varepsilon\delta_1\zeta(\alpha)^{-1+n} + \varepsilon\delta_2\zeta(\alpha)^{-1-n}
\]

\[
= -2f_0R_0\left(\frac{R_0^{-1}e^{-i\alpha}}{2} - \frac{\varepsilon}{2}R_0^{-2}e^{i(-1+n)\alpha} - \frac{\varepsilon}{2}R_0^{-2}e^{i(-1-n)\alpha}\right)
\]

\[
+ \varepsilon\delta_1R_0^{-1+n}e^{i(-1+n)\alpha} + \varepsilon\delta_2R_0^{-1-n}e^{i(-1-n)\alpha} + O(\varepsilon^2)
\]

\[
= -2f_0e^{-i\alpha} + \varepsilon\left(\delta_1R_0^{-1+n} + f_0R_0^{-1}\right)e^{i(-1+n)\alpha} + \varepsilon\left(\delta_2R_0^{-1-n} + f_0R_0^{-1}\right)e^{i(-1-n)\alpha}
\]

\[+ O(\varepsilon^2).\]

So

\[
\delta_1R_0^{-1+n} + f_0R_0^{-1} = -\frac{1}{2}nf_0R_0^{-1} \implies \delta_1 = \frac{1}{2}(-2 + n)f_0R_0^{-n}
\]

and

\[
\delta_2R_0^{-1-n} + f_0R_0^{-1} = -\frac{1}{2}nf_0R_0^{-1} \implies \delta_2 = \frac{1}{2}(-2 - n)f_0R_0^{n},
\]

and we end up with

\[
[\psi]\left(\zeta\right) = -2f_0R_0\zeta^{-1} + \frac{\varepsilon}{2}(-2 + n)f_0R_0^{-n}\zeta^{-1+n} + \frac{\varepsilon}{2}(-2 - n)f_0R_0^{n}\zeta^{-1-n}.
\]

It is easy now to write down the equations for the coefficients. The outer boundary equation is the same as always,

\[
A_s + (1 - n)\mathcal{B} + \mathcal{D} = 0
\]

\[
(1 + n)\overline{A_s} + \overline{C_s} + B = 0,
\]

knowing \(f\) directly gives us

\[
A_s - A_c = -\frac{1}{2}f_0R_0^{-2-n}
\]

\[
B = -\frac{1}{2}f_0R_0^{-2+n},
\]

and we don’t have to reconstruct \([Q]\) from \(f\) and \([\psi]\) because knowing \([\psi]\) tells us

\[
C_s - C_c = \frac{1}{2}(-2 + n)f_0R_0^{-n}
\]

\[
D = \frac{1}{2}(-2 - n)f_0R_0^{n}.
\]
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The solution is

\[ A_s = \frac{1}{2} f_0 \left[ (1 - n)R_0^{-2+n} + (2 + n)R_0^n \right] \]

\[ B = -\frac{1}{2} f_0 R_0^{-2+n} \]

\[ C_s = \frac{1}{2} f_0 \left[ n^2 R_0^{-2+n} - (1 + n)(2 + n)R_0^n \right] \]

\[ D = \frac{1}{2} f_0 (-2 - n)R_0^n \]

\[ A_c = \frac{1}{2} f_0 \left[ (1 - n)R_0^{-2+n} + (2 + n)R_0^n + R_0^{-2-n} \right] \]

\[ C_c = \frac{1}{2} f_0 \left[ n^2 R_0^{-2+n} - (1 + n)(2 + n)R_0^n + (2 - n)R_0^{-n} \right]. \]

6.7 The full matrix J

Now we can put it all together to get the matrix for \( J(f_0, R_0) \). In the elastic case, the formulas hold with \( c_1 = \frac{\kappa + 1}{2} \) and \( c_2 = 0 \).
\[ J_{11} = (2 - n)(1 + n)A_c R_0^n + (1 - n)C_c R_0^{-2+n} \]
\[ = - \frac{f_0 (-2n (n^2 - 1) R_0^2 + (n - 1)n^2 + (n - 2)(n + 1)(n + 2)R_0^4) R_0^{2n} + 2f_0(n - 2)R_0^2}{2R_0^4} \]
\[ J_{12} = (2 - n)(1 + n)A_c R_0^n + (1 - n)C_c R_0^{-2+n} \]
\[ = \frac{R_0^{2n} ((n - 1)n^2 + (n - 2)^2(n + 1)R_0^4 - 2(n - 2)(n - 1)(n + 1)R_0^2) - 2nR_0^2}{2R_0^4} \]
\[ J_{13} = (2 - n)(1 + n)A_c R_0^n + (1 - n)C_c R_0^{-2+n} \]
\[ = \frac{R_0^{2n} (2R_0^2 - n (R_0^2 - 1) ((n - 1)n (R_0^2 - 1) - 2R_0^2)) - 2R_0^2}{2R_0^4} \]
\[ J_{21} = \sigma_{nm}^{(1)} \sigma_{rr}^{ex} + \sigma_{mn}^{(0)} u_c + \frac{1}{c_1} \sigma_{mn}^{(1)} \]
\[ = - \frac{f_0}{4c_1 R_0^4} \]
\[ \times \left[ (-2n (n^2 - 1) R_0^2 + (n - 1)n^2 + (n - 2)(n + 1)(n + 2)R_0^4) R_0^{2n}(-2c_1 f_0 + 2R_0) \right. \]
\[ + 2R_0^2 (-2c_1 (f_0 (n + 2R_0^2 - 4) - R_0) + 2(n - 2)R_0) \right] \]
\[ J_{22} = \sigma_{nn}^{(1)} \sigma_{rr}^{ex} + \sigma_{nn}^{(0)} u_c + \frac{1}{c_1} \sigma_{nn}^{(1)} - \frac{c_2}{c_1} \]
\[ = \frac{1}{4c_1 R_0^4} \left[ 2R_0^2 (2c_1 (f_0 (n - 2R_0^2 + 2) + R_0) - R_0(2c_2 R_0 + 2n)) \right. \]
\[ + ((n - 1)n^2 + (n - 2)^2(n + 1)R_0^4 - 2(n - 2)(n - 1)(n + 1)R_0^2) R_0^{2n}(-2c_1 f_0 + 2R_0) \]
\[ J_{23} = \sigma_{nn}^{(1)} \sigma_{rr}^{ex} + \sigma_{nn}^{(0)} u_c + \frac{1}{c_1} \sigma_{nn}^{(1)} \]
\[ = - \frac{1}{4c_1 R_0^4} \left[ 2R_0^2 (2R_0 + 2c_1 (f_0 (2n (R_0^2 - 1) - 1) - nR_0)) \right. \]
\[ + (n (R_0^2 - 1) ((n - 1)n (R_0^2 - 1) - 2R_0^2) - 2R_0^2) R_0^{2n}(-2c_1 f_0 + 2R_0) \]
\[ J_{31} = \sigma_{nn}^{(0)} u_s + \frac{1}{c_1} \epsilon \sigma_{mn}^{(0)} R_0^{-1} n \]
\[ = - \frac{n (2f_0 (R_0^2 - 1) - R_0) (6c_1 f_0 - 2R_0)}{2c_1 R_0^3} \]
\[ J_{32} = \sigma_{nn}^{(0)} u_s \]
\[ = n (2f_0 (R_0^2 - 1) - R_0) \]
\[ = \frac{c_2}{c_1} \]
\[ J_{33} = \sigma_{nn}^{(0)} u_s - \frac{c_2}{c_1} \]
\[ = - \frac{c_2}{c_1} - \frac{2f_0}{R_0^2} + 2f_0 - \frac{1}{R_0} \].
We have confirmed these formulas by estimating them using finite difference methods. To numerically compute $J_{ij} = b_i(e_j)$, we set $\varepsilon = \pm (2 \times 10^{-16})^{1/3}$, and solve equations (6.5) with $a = e_j$ in (6.1) and (6.2) using the numerical methods described in Chapter 4 using 256 collocation points on $\Gamma_I$ and $\Gamma_O$. This gives us the centered difference approximations

$$b_1 \approx \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sigma_{nn}(\varepsilon) - \sigma_{nn}(-\varepsilon)}{2\varepsilon} \cos n\alpha \, d\alpha,$$

$$b_2 \approx \frac{1}{\pi} \int_{-\pi}^{\pi} \Re \{ \hat{f}(\varepsilon)e^{-i\alpha} - \hat{f}(-\varepsilon)e^{-i\alpha} \} \cos n\alpha \, d\alpha,$$

$$b_3 \approx \frac{1}{\pi} \int_{-\pi}^{\pi} \Im \{ \hat{f}(\varepsilon)e^{-i\alpha} - \hat{f}(-\varepsilon)e^{-i\alpha} \} \sin n\alpha \, d\alpha,$$

where the integrals are computed with the trapezoidal rule. We denote these estimates of $b_i(e_j)$ by $J_{est}$. (Actually, we read off the coefficients $b_i$ from an FFT of the difference quotients in the integrands above. Except for the $b_i$ above, the other Fourier modes are small, of the same order as the errors in $J_{est}$ reported below.)

We now give two examples of how closely $J_{est}$ matches the formula for $J$ given at the beginning of this section. First consider the case $c_1 = c_2 = \kappa = 1$. In the evolution of the unperturbed circle, at the point that the radius of the interface is $R = \frac{1}{2}$, $f_0$ is approximately $-0.3$. For $n = 10$, the matrix $J$ is

$$J(0.5, -0.3) = \begin{pmatrix} 9.60107803 & -19.99790955 & -2.00194359 \\ 15.48172485 & -32.79665527 & -1.20310974 \\ -5.6 & -2 & -1.2 \end{pmatrix}$$

and the numerical estimate is off by

$$J_{est}(0.5, -0.3) - J(0.5, -0.3) = 10^{-7} \begin{pmatrix} -1.10193975 & -0.00169308 & -0.00227155 \\ -2.65386602 & -0.0081687 & -0.00364052 \\ -6.68932508 & 0.0006782 & -0.0000042 \end{pmatrix}.$$ 

We don’t have to use values of $R$ and $f_0$ that are achieved in our simulations to estimate $J$. For a compressible case, when $\kappa = 2, n = 15, R = 0.8$, and $f_0 = -0.6$, we have

$$J = \begin{pmatrix} 12.45728292 & -18.10902946 & -1.75034055 \\ 18.07906747 & -25.07945841 & 6.14535089 \\ -25.15625 & -8.625 & -0.575 \end{pmatrix}$$

and

$$J_{est} - J = 10^{-7} \begin{pmatrix} -0.962784892 & 0.0179896986 & 0.0118338317 \\ -1.36167419 & 0.0122005162 & 0.00344014150 \\ -4.18201388 & 0.014677319 & 0.00108812737 \end{pmatrix}.$$
6.8 Results on growth of perturbations

Assume now a fixed \(n, c_1, c_2, \text{and } \kappa\). What has been done so far in this chapter is to find a family of perturbations such that a member of the family of size \(\varepsilon\), when advanced in time, remains in the family up to a negligible difference of size \(O(\varepsilon^2)\). We assume \(\varepsilon \ll 1\) and ignore the \(O(\varepsilon^2)\) terms. This gives a differential equation that describes the evolution of a perturbation, namely

\[
\dot{a}(t) = J(R_0(t), f_0(t))a(t),
\]

where \(J(R_0(t), f_0(t))\) is given in the previous section. If such a perturbation \(a_0\) is introduced at time \(t_0\), since we have the differential equation (3.30) for \((R_0, f_0)\), we can solve

\[
\begin{cases}
\dot{a}(t) = J(R_0(t), f_0(t))a(t) \\
a(t_0) = a_0
\end{cases}
\]

to model the growth of \(a\) in time. Since \(J(R_0(t), f_0(t))\) varies in time, we can’t simply write down the solution of the differential equation as the matrix exponential of \(J\). We solve this linear ordinary differential equation numerically, and plot several results in Figures 6.1–6.4.

We have found some conditions \((R_0, f_0)\) under which any perturbation introduced at that time initially begins to decrease in size. However, in all of these situations that we
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Figure 6.2: The perturbations grow tremendously before the interface reaches even a tenth of its initial radius in the unperturbed problem. The viscoelastic case uses $c_1 = c_2 = 1$ the elastic cases use $c_1 = \frac{\kappa + 1}{2}$ and $c_2 = 0$. Here $c_1$ and $c_2$ are defined as in equation (2.1).

Figure 6.3: At a fixed time, in this case $t_1 = 5$, as the frequency of the initial perturbation increases, the size of the perturbation at $t_1$ increases, and the relationship is exponential.
Figure 6.4: Not all perturbations grow exponentially. Here $c_1 = c_2 = \kappa = 1$ and $n = 15$, where $c_1$ and $c_2$ are defined as in equation (2.1). For most choices of constants, there is one initial perturbation that grows exponentially but the initial $a_0$ that are orthogonal to it do not grow and in fact decrease exponentially, so that the evolution equations have a stabilizing effect for some perturbations.

have looked at, as the evolution continues, the perturbations at some point begin to increase again at an exponential rate.

For example, let $n = 15$, $c_1 = c_2 = \kappa = 1$. If at $t = 0$ the state of the interface is $R_0 = 0.8$ and $f_0 = 0.3$, then any perturbation of frequency $n = 15$ introduced at that time has its size decrease at $t = 0.15$. We have

$$\max_{\|a_0\|=1} \|a(.15)\| = 0.95.$$  

However, by $t = 1$ there are perturbations that have grown by a factor of over 10000. In the case that $n = 100$, we have

$$\max_{\|a_0\|=1} \|a(.15)\| = 0.009$$

while by $t = 0.5$ there are perturbations that have grown by a factor of over $10^{13}$.

An explanation for these phenomena can be seen in Figures 6.6 and 6.7, which show the evolution of this interface, along with the initial interface with $R(0) = 1$, $f_0(0) = 0$, through contour plots of the largest real part of an eigenvalue of $J(R_0, f_0)$.

The way this manifests itself when running the numerical simulation described in Chapter 4 is illustrated in Figures 6.8 and 6.9. For $c_1 = c_2 = \kappa = 1$, an evolution was run with 192 grid points on $\Gamma_O$ and $\Gamma_I$, starting with circular interface at $R = 0.8$ with $f_0 = 0.3$, evolving
Figure 6.5: This plot shows an example where we can see the instabilities growing right away. Here $R_0 = 0.5$, $f_0 = -1$, and the perturbation is $\varepsilon a = 0.01(-0.92, -0.19, 0.33)$. The constants are $c_1 = c_2 = \kappa = 1$ and $n = 15$, where $c_1$ and $c_2$ are defined as in equation (2.1). The velocity in the unperturbed case is 2, and the arrow at each interface point is $(\dot{R} - 2)\hat{r}$.

to $t = 0.5$ using 100 timesteps of the DOPRI8 8th order Runge-Kutta method. We see that the problem is well-posed while the trajectory is inside the stable region, and once the trajectory leaves the stable region it becomes ill-posed and the Fourier modes grow rapidly. When the simulation is run with more grid points, we find that roundoff error is amplified to $O(1)$ faster once the trajectory leaves the stable region. This is expected as high frequency modes lead to eigenvalues of $J$ with larger real part in the unstable region.
Figure 6.6: Evolution of interfaces through a contour plot of the largest eigenvalue of $J$ for $n = 15$, $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1). The dashed black line is the 0 level set. Perturbations of an interface at state $R = 0.8$, $f_0 = 0.3$ decrease with time and then begin to increase.
Figure 6.7: Evolution of interfaces through a contour plot of the largest eigenvalue of $J$ for $n = 100$, $c_1 = c_2 = \kappa = 1$, where $c_1$ and $c_2$ are defined as in equation (2.1). The dashed black line is the 0 level set. Perturbations of an interface at state $R = 0.8$, $f_0 = 0.3$ decrease with time and then begin to increase.
Figure 6.8: Snapshots of the interface at different points in time
Figure 6.9: The Fourier modes of $R$ at different times ($t = 0.05m$, $0 \leq m \leq 10$) during the simulation. The real and imaginary parts of $\hat{R}_k$ are plotted at position $2k$ and $2k + 1$. The unperturbed circular solution involves only the constant mode $\hat{R}_0$. The corresponding interface curves were given in Figure 6.8, and the trajectory in $(R, f_0)$ space was given in Figures 6.6 and 6.7. Once the solution leaves the stable regime, roundoff errors are amplified by the evolution equations, eventually causing the solution to blow up. Note that high-frequency modes grow faster than low-frequency modes after $t = 0.2$, whereas all non-constant modes stay small up $t = 0.15$. 
Chapter 7

Conclusion

We have presented a new mathematical framework for modeling the olivine to $\beta$-spinel transformation in a domain that is not spherically symmetric. The Maxwell flow rule

$$\mu(\nabla \tilde{u} + \nabla \tilde{u}^T) = \frac{\partial}{\partial t} \sigma' + \frac{\mu}{\eta} \sigma'$$

(7.1)

that appears in the original model proposed in [43] (without the tildes) suggests that to be able to accurately evolve the system, it is necessary to compute temporal derivatives throughout the bulk. In asymmetric domains, where in general it is not possible to write down the solution to the fixed time equations, this makes evolving the interface computationally expensive, both in terms of number of operations and the memory required to track variables in the bulk through time. It also likely limits the accuracy of any method to the order of the time differencing formula used for $\frac{\partial}{\partial t} \sigma'$. Reformulating the equations by writing $\tilde{u}$ for the physical velocity and introducing an auxiliary Stokes system $(p, u)$ to couple the physical velocity to the physical stress was the key step that allows us to describe the evolution purely in terms of quantities on the interface. This means we do not have to discretize the bulk in spite of the Maxwell constitutive equation. This enables us to reduce the system to a complex analysis boundary value problem at each point in time. To avoid solving a coupled, complex variable interface problem involving separate domains, we started with two unknown complex analytic functions defined in the whole sample and subtracted Cauchy integrals for which the Plemelj formulae enforce the desired jump conditions across the interface. Furthermore, this method eliminates the bottleneck of having to solve large matrix equations many times as we can factor one large matrix at the beginning and use the factorization at each subsequent timestep. We implemented the method and presented several stationary and time-dynamic test cases to showcase the new Maxwell-Stokes solver.

We then presented a new elastic model of the olivine to $\beta$-spinel phase transformation that allows for compressibility, using the Lamé equations of linear elasticity as a constitutive model. We showed that this model in fact agrees with [43] with certain parameters even though it is derived from different physical assumptions. We presented two derivations of the new model from different perspectives, one emphasizing the motion of the boundaries of
the respective reference configurations of the core and shell regions, and one in an Eulerian framework that attempts to make a connection with the derivation in [43]. We then compared this new elastic model to a recently proposed elastic model of Morris [44] using the same geometry studied in [44], and identified the difference in predicting the radial deviatoric strain as a modeling assumption about whether the phase change necessarily should involve only uniform compression, or should have a deviatoric component to it.

The stability analysis shown in Chapter 6 is new, and demonstrates properties of the solutions to the equations that are mathematically interesting. We identified regions in which solutions to the problem are unstable to perturbations in a circular interface and used this to explain why numerical simulations of the evolution of the circle often begin to oscillate wildly for certain initial conditions in numerical simulations. This is also used to explain why some initial conditions lead to stable evolutions for a long period of time while others almost immediately blow up.

It is surprising that the original Maxwell-Stokes model studied in the first 4 chapters of this dissertation would exhibit instabilities as the term \((\mu/\eta)\sigma\)' in (7.1) is supposed to have a smoothing effect on the solution. A similar instability was discovered in the context of finite-deformation elasticity in rapidly spinning viscoelastic cylinders by Govindjee, Potter and Wilkening [13]. They found that a popular model of viscoelastic solids due to Simo [67] can violate the second law of thermodynamics, whereas a Sidoroff-style law [66], which has been proven to satisfy the second law, does not lead to instabilities. The Simo model is similar in structure to the Maxwell flow rule, but written in terms of the second Piola-Kirchhoff stress tensor in a finite deformation framework.

Experiments performed in [37, 46, 11, 12] evolve circular interfaces that do not produce large oscillations in the interface over time. Thus, as our equations have instabilities for many choices of constants and initial conditions, more work needs to be done to find a physically correct model. A proper accounting of the chemical potential driving force in the elastic case would also involve the jump in energy density in the elastic field across the interface, and would lead to a more complicated formula for \(V\). These considerations, along with other effects such as surface tension, tangential sliding, diffusion along the interface, or heat released in the phase transformation may have a stabilizing effect on the interface evolution, but are beyond the scope of the current work.

It is unknown whether the alternative elastic model [44] will also exhibit instabilities as it has so far only been considered in the incompressible, spherically symmetric setting. This is a mathematical question that cannot be answered on physical grounds. Mathematically, the elastic model we presented involving separately evolving reference configurations for the core and shell has the advantage that it leads to the Lamé equations of linearized elasticity in both the core and shell regions, which makes it possible to use complex function theory and boundary integral methods to efficiently solve the problem in non-axisymmetric geometries. This is not true of Morris's model [44] due to the non-zero divergence constraint \(\nabla \cdot \mathbf{u} = -\varepsilon\) in the shell. Therefore a different numerical approach would have to be developed to generalize that model to a non-axisymmetric setting, which is beyond the scope of the current work. We hope that by proposing a new viewpoint, it may inspire more sophisticated models.
that explicitly address whether the phase change should really be modeled with a uniform compressive strain. The presence of an interface offers a preferred direction that breaks local isotropic symmetry.

The numerical framework that we have created is flexible and will be useful in future work on this problem, and the techniques developed to perform the stability analysis will be widely applicable to new models of solid-solid phase transformations.
Bibliography


