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PRODUCTION CROSS SECTIONS IN THE MULTIPERIPHERAL MODEL

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ABSTRACT

The total cross section for the multiperipheral model is obtained by solving a Bethe-Salpeter equation. We show that the N-particle production cross sections are given directly in terms of the eigenvalues and eigenfunctions of this Bethe-Salpeter equation, and hence are directly related to the Regge pole parameters. In simple models of Regge poles based on the multiperipheral model the production cross sections are completely determined. Results are given for several simple cases including the Chew and Snider model of the Pomeranchuk.

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I. INTRODUCTION

It has been proposed by Fubini and co-workers\(^1\) that a multiperipheral model of high-energy scattering will provide a dynamical description of Regge poles and hence a dynamical model of the low-mass particles that lie on these trajectories. The basic form of this model is a ladder structure of the type shown in Fig. 1, where the multiparticle production contribution to the total cross section is constructed by the iteration of an elastic cross section. In our previous work\(^2\) and in the work of Chew, Rogers, and Snider\(^3\) the ladder graphs were summed by means of the Bethe-Salpeter equation. While the values of the various Regge pole parameters were obtained in the neighborhood of \( t = 0 \) little was said about the \( N \)-particle production cross sections which are the quantities most directly predicted by the model and are also directly measurable experimentally.

In this paper we would like to approach the same problem but beginning with a different point of view. We will assume that both the total cross section and the elastic cross section are known from experiment or theory and use the multiperipheral model to predict the \( N \)-particle production cross sections. It is hoped that this will allow the direct comparison of experimental data to check the validity of this model without depending on a specific calculation, and at the same time clarify the effect of certain assumptions made in previous calculations in regard to the parameterization of the elastic cross section and the assumed off-mass-shell dependence of the scattering amplitude.
Our starting point will be the assumption that the total cross section has a ladder structure of the type shown in Fig. 1 and that it can be calculated from a Bethe-Salpeter-type equation in which the inhomogeneous term and the kernel are related to the elastic cross section. It is further assumed that this equation is partially diagonalized when one invokes either \( O(4) \) or \( O(3) \) symmetry.

In the following section we will obtain a formal expression for the \( N \)-particle production cross sections in terms of the elastic and total cross sections. In Sec. 3 we investigate several applications to simple models, and we obtain the production cross sections as a function of energy and the number of pairs produced.
II. FORMULATION OF THE MULTIPERIPHERAL MODEL

The basic equation of the multiperipheral model can be written as the following Bethe-Salpeter equation:

\[ T = T_0 + T_0 G T, \quad (2.1) \]

which corresponds formally to the diagram in Fig. 2. The kernel \( T_0 \), when evaluated on the mass shell, is related to the elastic cross section by

\[ \sigma_{\ell}(s) = \frac{\partial}{\partial t} \left| \frac{1}{s(s - 4\mu^2)} \right| \im T_0(s, t = 0). \quad (2.2) \]

The solution \( T(s, t) \) bares the same relation to the total cross section, and \( G \) is the two-particle propagator.

If we now expand both \( T \) and \( T_0 \) in partial waves appropriate to whichever symmetry is assumed, i.e., \( O(4) \) at \( t = 0 \) and \( O(3) \) elsewhere, the Eq. (2.1) is diagonal with respect to this quantum number \( \lambda \). The continuation of these quantities into the complex \( \lambda \) plane is accomplished in the standard manner. For simplicity we will assume that \( T_0 \) has only a branch cut for positive \( s \) and hence no signature need be introduced. The resulting continuation is

\[ T_\lambda(t) = \int_{4\mu^2}^{\infty} ds \, s^{-\lambda-1} \im T(s, t), \quad (2.3) \]

where \( \lambda \) stands for either \( \ell \) or \( n \) and we have used the asymptotic form of \( Q_{\ell} \) or \( f_n \) to simplify the discussion. The inverse of this relation is:
where \( c \) is the usual contour to the right of all singularities of \( T^\lambda \). The solution of the partial-wave Bethe-Salpeter equation can be expressed in terms of the eigenfunctions and eigenvalue of the homogeneous form of the equation:

\[
\psi^i_\lambda = E^i_\lambda T^0_{\lambda\alpha} G \psi^i_\lambda. \tag{2.5}
\]

Then \( T^\lambda \) and \( T^0_{\lambda\alpha} \) have the following expansions:

\[
T^\lambda(t) = \sum_i \frac{[\psi^i_\lambda(t)]^2}{E^i_\lambda(t) - 1} \tag{2.6}
\]

and

\[
T^0_{\lambda\alpha}(t) = \sum_i \frac{[\psi^i_\lambda(t)]^2}{E^i_\lambda(t)}. \tag{2.7}
\]

Note here that the off-mass shell dependence is contained in the functions \( \psi \). It is clear from Eq. \( 2.6 \) that \( \lambda \)-plane poles of the amplitude \( T^\lambda(t) \) arise from the denominator vanishing and hence the Regge trajectories \( \alpha_i(t) \) and residues \( \beta_i(t) \) are determined by the following equations:

\[
E^{i}_\alpha(t) = 1, \tag{2.8}
\]

\[
\beta_i(t) = \psi^{i}_\alpha(t)^2 \left[ \frac{\partial E^{i}_\lambda(t)}{\partial \lambda} \right]_{\lambda = \alpha_i}. \tag{2.8}
\]
In order to determine the cross section for production of $N$-particle pairs we define the amplitude $T_N$ such that the imaginary part of $T_N$ when evaluated on the mass shell is proportional to $\sigma_N$. It is clear that the $T_N$'s satisfy the following recursion relation:

$$T_{N+1} = T_0 G T_N .$$

(2.9)

We can now define a generating function for the $T_N$'s as follows:

$$T_h = \sum_{N=0}^{\infty} h^N T_N ,$$

(2.10)

which will be the solution of the Bethe-Salpeter equation

$$T_h = T_0 + hT_0 G T_h ,$$

(2.11)

and the production cross section can be extracted from $T_h$ by projecting out the coefficient of $h^N$. The projection $T_{h,\lambda}$ has the same type of continuation into the $\lambda$ plane as $T_\lambda$ and $T_{0,\lambda}$ and hence satisfy the same partial-wave Bethe-Salpeter equation. The solution to Eq. (2.11) can again be expressed in terms of the eigenfunction solutions of the homogeneous Bethe-Salpeter equation:

$$T_{h,\lambda} = \sum_i \frac{(\Psi_\lambda^i)^2}{E_\lambda^i - h} .$$

(2.12)

Since this expression is readily expandable in powers of $h$ we obtain the following result for $T_N$: 
\[ T_{N,\lambda} = \sum_i \left( \frac{1}{E_\lambda} \right)^{N+1} (\psi_i^\dagger)^2, \]  

which, when evaluated on the mass shell and substituted into Eq. (2.4), yields the N-particle production cross section

\[ \sigma_N(s) = \frac{1}{[s(s - 4\mu^2)]^{\frac{1}{2}}} \frac{1}{2\pi i} \int_c d\lambda \, s^\lambda T_{N,\lambda}. \]  

The average multiplicity for the production process can easily be expressed directly in terms of \( T_h \):

\[ \langle N \rangle = \sum_N (N + 1) \frac{\sigma_N/\sigma_T}{\sigma_T} = 1 + \frac{3}{\alpha_h} \frac{\text{Im} \, T_{h=1}(s,0)/\text{Im} \, T_{h=1}(s,0)}. \]  

Finally, we note that if the sum in Eq. (2.12) is well represented by the first term, i.e., the eigenvalues are widely separated, \( \sigma_N \) can be expressed directly in terms of integrals over \( \sigma_{el} \) and \( \sigma_T \) as follows:

\[ \sigma_N(s) = \frac{1}{2\pi i} \int_c d\lambda \, s^{-\lambda-1} T_{\alpha\lambda} \left( \frac{T_\lambda - T_{\alpha\lambda}}{T_\lambda} \right)^N, \]  

where

\[ T_\lambda = \int_{s_0}^{\infty} ds \, s^{-\lambda} \sigma_T(s), \]

and

\[ T_{\alpha\lambda} = \int_{s_0}^{\infty} ds \, s^{-\lambda} \sigma_{el}(s). \]
III. SPECIFIC MODELS

To illustrate the method presented in the previous sections, we will consider several specific examples. Our general procedure will be to assume certain types of λ-plane singularities and determine the resulting relation between the elastic, total, and production cross sections together with the multiplicity implied by these singularities. Since these relations are determined by the imaginary part of the amplitude at \( t = 0 \) we will simply parameterize the amplitude at this point, but the reader should keep in mind that in general all of the parameters introduced in the following discussion are functions of \( t \).

We study examples where a single term in the expansion given in Eq. (2.12) dominates the amplitude.

A. Single Pole

The simplest case we can consider is that \( T_\lambda \) has only a simple pole at \( \lambda = \alpha_0 \). Then \( E_\lambda \) is a linear function of \( \lambda \):

\[
E_\lambda = 1 + (\lambda - \alpha_0)/f
\]

and the asymptotic form of \( \sigma_T \) is

\[
\sigma_T(s) = \beta s^{\alpha_0 - 1},
\]

where \( \beta \) is the residue of the pole. The \( \sigma_N \)'s have the following Poisson distribution:

\[
\sigma_N = \beta \frac{(f \ln s)^N}{N!} s^{\alpha_0 - f - 1}.
\]
The average multiplicity for pairs is then the following:

\[ \langle N \rangle = 1 + f \ln s . \]  

(3.4)

**B. Two Poles**

The simplest generalization of the single-pole model is to consider an eigenvalue which is a quadratic function of \( \lambda \) with the resulting amplitude containing two poles at \( \alpha_+ \) and \( \alpha_- \) with residues \( r_+ \) and \( r_- \), respectively. If we assume

\[ E_\lambda = (\lambda - \alpha_+)(\lambda - \alpha_-)/f + 1 , \]  

(3.5)

the amplitude \( T_{h,\lambda} \) is

\[ T_{h,\lambda} = f \left( r_+ (\lambda - \alpha_-) + r_- (\lambda - \alpha_+) \right) / \left( (\lambda - \alpha_+)(\lambda - \alpha_-) + f(1 - h) \right) . \]  

(3.6)

Here \( f \) is a parameter which will be related to the average multiplicity. The Mellin transform of Eq. (3.6) is:

\[ \text{Im} T_h(s) = fs^{\alpha_R} \left\{ (r_+ + r_-) \cosh(R \ln s) + \alpha_D (r_+ - r_-) \frac{\sinh(R \ln s)}{R} \right\}, \]  

(3.7)

where \( \alpha_R = (\alpha_+ + \alpha_-)/2, \alpha_D = (\alpha_+ - \alpha_-)/2, \) and \( R = [\alpha_D^2 + f(h - 1)]^{1/2}. \)

By differentiation with respect to \( h \) we obtain the cross section for producing \( N \) pairs of particles \( \sigma_N \):
\[ \sigma_N(s) = \int s^{\alpha_N - 1} \left\{ R(r_+ + r_-) i_{N-1}(R \ln s) + \alpha_D(r_+ - r_-) i_N(R \ln s) \right\} \times \frac{\ln s}{N!} \left( \frac{r}{2R} \ln s \right)^N , \]  

where \[ i_N(z) = \left( \frac{z}{2z} \right)^{\frac{1}{2}} I_{N+\frac{1}{2}}(z) \] and \( R \) is evaluated at \( h = 0 \). Note that this is not a Poisson distribution except for large \( R \ln s \) where the asymptotic behavior of \( i_N \) is independent of \( N \). The average multiplicity of pairs in this limit is \( \langle N \rangle = 1 + \frac{f}{2R(h = 1)} \ln s \).  

If the poles are a complex conjugate pair the \( i_N \) Bessel functions go into \( j_N \)'s which are oscillating functions and \( \sigma_N \) is not positive definite. Therefore the complex poles must either be the nonleading singularities or must be an approximation to a cut of the type suggested by Ball, Marchesini, and Zachariasen which is valid only over a limited energy region.

C. Approximate Solutions to the Multiperipheral Bootstrap

In our previous work on a realistic treatment of the multiperipheral model the Bethe-Salpeter equation was solved numerically. From these calculations we obtain a reasonable Pomeranchuk and \( \rho \) Regge poles. What this calculation predicts for \( \sigma_N \) is directly relevant to experiment, however, the fact that the eigenvalues were obtained numerically makes the required integration in Eq. (2.14) difficult. Recently, Chew and Snider, investigating the same type of model, have employed a factorable kernel and have obtained results quite similar to
our earlier result. Because of their approximation they have only one eigenvalue with the explicit \(\lambda\) dependence known. It is of interest to obtain the \(a_N\) predicted by their model, and one would expect these results to be similar to those predicted by a realistic model. In their model the eigenvalue and eigenfunction are the following:

\[
E_\lambda = \frac{(\lambda - \beta_\pi)(\lambda - \beta_p)}{g_p^4 + g_R^4 (\lambda - \beta_p)}
\]  

(3.10)

and \(\psi_\lambda^2 = (\lambda - \beta_\pi)\), where in the interest of simplicity we have set the internal and external couplings of the CS model equal. Here \(\beta_p\) is the position of the effective singularity due to the high-energy component of the kernel and \(\beta_\pi\) is the position of the singularity associated with the pion propagator. The parameters \(g_p\) and \(g_R\) control the strength of the high-energy and resonance components of the kernel, respectively.

Using Eq. (2.14), we obtain

\[
a_N(s) = \frac{1}{2\pi i} \int_c d\lambda \ s^{\lambda-1} \left( \frac{1}{\lambda - \beta_\pi} \right)^N \left( g_R^4 + \frac{g_p^4}{\lambda - \beta_p} \right)^{N+1}.
\]

(3.11)

The evaluation of these integrals is straightforward but lengthy.

In Fig. 3 are shown the total cross section and some typical production cross sections, for the value of the parameters determined by Chew and Snider\(^5\) which are the following: \(g_R^4 = 1.0, \ g_p^4 = 0.03, \ \beta_\pi = -0.3, \) and \(\beta_p = 0.9.\) In Fig. 4 we compare the \(N\) dependence of production cross sections with the Poisson distribution for several values of the
energy. The average multiplicity predicted by this model is shown in Fig. 5 together with two straight lines which agree with this curve in the accelerator energy region and in the asymptotic region. Note that the effective coefficient of \( \ln s \) changes from 2 to 0.65 at ultrahigh energy. This effect is similar to that we suggested in our previous paper, in which the low-energy multiplicity is controlled by an "effective" Regge-pole that is actually the contribution of a cut, while at sufficiently high energy the "true" Pomeranchuk pole dominates and produces a much smaller average multiplicity.
CONCLUSION

The multiperipheral model provides a mathematical framework in which to study dynamical Regge trajectories and their associated cuts. In application of this model, previous work has generally concentrated on the elastic cross section which is directly related to the kernel of the integral equation and the total cross section which allows one to identify various Regge poles. These calculations also contain implicitly all the production cross sections which are also experimentally measured quantities. We have computed the production cross sections for a few simple models including the approximate multiperipheral model of Chew and Snider, where the assumption of factorizability of the kernel leads to very simple predictions. While the production cross sections in this model deviate from a Poisson distribution, this deviation is not in disagreement with current experimental data.
FOOTNOTES AND REFERENCES


5. G. F. Chew and D. R. Snider, Multiperipheral Mechanism for a Schizophrenic Pomeron, Lawrence Radiation Laboratory Report UCRL-19455, Jan. 1970; hereafter referred to as CS.
FIGURE CAPTIONS

Fig. 1. Diagram representing the multiperipheral contribution to the total cross section.

Fig. 2. Diagram representing the multiperipheral Bethe-Salpeter equation.

Fig. 3. Production cross section as function of energy from CS model.
   Energy units: 1 GeV$^2$.

Fig. 4. Production cross section from CS model as function of number of pair (solid curves) and corresponding Poisson distribution (dashed curves) for several energies.

Fig. 5. Multiplicity of particles as function of energy from CS model.
Fig. 2
$\sigma_N(s)$ (arbitrary units)

$\sigma_T$

$\sigma_2$

$\sigma_3$

$\sigma_6$

$\sigma_9$

$\sigma_{el}$

$\sigma_1$

$\Ins$

Fig. 3
Fig. 4
Fig. 5
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