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α-Scaling Zeta Functions For Self-Similar Multifractals

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α-Scaling Zeta Functions for Self-Similar Multifractals

A Dissertation submitted in partial satisfaction of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Scott Alan Roby

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Acknowledgments

First, I would like to thank my advisor, Dr. Michel L. Lapidus for all of the inspiration and encouragement he gave me. I came to the University of California, Riverside with the mentality that I would feel around to see which subjects really capture my attention before deciding on a particular topic. Yet, in my first year upon attending the Fractal Research Group and Mathematical Physics and Dynamical Systems seminars, I was already convinced that I had found a home. I truly appreciate your sincerity and passion Dr. Lapidus, and for making me feel welcome as a member of your research group.

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To all my friends and family who supported me in my education.
Visualization of sets in Euclidean space that possess notions of non-integer dimension has lead to a great deal of curiosity and ideas that connect to many traditional fields in and outside of mathematics. Recently, the study of open subsets of $\mathbb{R}$ and their boundaries paved the way for a new definition in the work of Lapidus and van Frankenhuijsen for the term ‘fractal.’ This definition reveals the scaling properties of the relevant sets and serves as continuing motivation for the study of zeta functions that encode a notion of dimension of a corresponding set. There are, in addition, measures called multifractals that possess fractal-like properties. It is the purpose of this dissertation to investigate the properties of self-similar multifractal measures through a very intimate connection to the classically studied hypergeometric functions. One may decompose the support of these measures into a collection of fractal sets each of which possesses the same scaling property. Zeta functions associated to each fractal set in this decomposition encode this generalized notion of dimension, yet they are shown in many cases to be exactly of the form of the generalized hypergeometric functions.

The classical study of hypergeometric functions boasts the attention of mathematicians such as Euler, Gauss, and Riemann, yet as with many special functions the applications continue to be revealed. Some invaluable tools in understanding the generalized hypergeometric functions, as investigated herein, are the associated hypergeometric differential equation and monodromy matrices.
# Contents

<table>
<thead>
<tr>
<th>Table of Contents</th>
<th>vii</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>viii</td>
</tr>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2 Fractal Strings</td>
<td>5</td>
</tr>
<tr>
<td>3.1 Self-Similar Fractal Strings</td>
<td>5</td>
</tr>
<tr>
<td>3.2 Minkowski Dimension</td>
<td>8</td>
</tr>
<tr>
<td>3.3 Geometric Zeta Functions</td>
<td>9</td>
</tr>
<tr>
<td>3.4 Complex Dimensions</td>
<td>10</td>
</tr>
<tr>
<td>3 Multifractal Measures</td>
<td>15</td>
</tr>
<tr>
<td>3.1 Self-Similar Multifractals</td>
<td>15</td>
</tr>
<tr>
<td>3.2 ( \alpha )-Scaling Zeta Functions</td>
<td>17</td>
</tr>
<tr>
<td>3.3 An Example of a Self-Similar Multifractal Which Is Not Uniquely Generated</td>
<td>20</td>
</tr>
<tr>
<td>3.4 A Collection of Uniquely Generated Self-Similar Multifractals</td>
<td>23</td>
</tr>
<tr>
<td>4 Hypergeometric Functions</td>
<td>26</td>
</tr>
<tr>
<td>4.1 The Generalized Hypergeometric Series</td>
<td>27</td>
</tr>
<tr>
<td>4.2 Fuchsian Differential Equations</td>
<td>33</td>
</tr>
<tr>
<td>4.3 The Generalized Hypergeometric Equation</td>
<td>34</td>
</tr>
<tr>
<td>4.3.1 Solutions Around ( z = 0 )</td>
<td>35</td>
</tr>
<tr>
<td>4.3.2 Solutions Around ( z = \infty )</td>
<td>38</td>
</tr>
<tr>
<td>4.4 Monodromy</td>
<td>39</td>
</tr>
<tr>
<td>4.5 Motivating Examples for Monodromy</td>
<td>41</td>
</tr>
<tr>
<td>4.5.1 The Generalized Hypergeometric Equation</td>
<td>42</td>
</tr>
<tr>
<td>4.5.2 Hypergeometric Monodromy: Example 1</td>
<td>44</td>
</tr>
<tr>
<td>4.5.3 Hypergeometric Monodromy: Example 2</td>
<td>46</td>
</tr>
<tr>
<td>4.6 The Case ( N = 2 )</td>
<td>48</td>
</tr>
<tr>
<td>4.6.1 Uniquely Generated Example</td>
<td>49</td>
</tr>
<tr>
<td>5 Future Work</td>
<td>51</td>
</tr>
<tr>
<td>Bibliography</td>
<td>53</td>
</tr>
</tbody>
</table>
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>The first two stages in the construction of a self-similar fractal string with three contraction similarities. Given contraction ratios ( r_1, \ldots, r_N ) one can construct a self-similar fractal string by choosing appropriate gap lengths.</td>
</tr>
<tr>
<td>2.2</td>
<td>The first two stages in the construction of the middle-third Cantor set with removed interval lengths. Here the contraction similarities are ( \Phi_1(x) = \frac{1}{3} x ) and ( \Phi_2(x) = \frac{1}{3} x + \frac{2}{3} ).</td>
</tr>
<tr>
<td>2.3</td>
<td>The construction of the Fibonacci string with removed lengths indicated.</td>
</tr>
<tr>
<td>2.4</td>
<td>The complex dimensions of the Fibonacci string where ( D = \log_2 \phi ), ( p = \frac{2\pi}{\log 2} ), and ( \phi = \frac{1+\sqrt{5}}{2} ).</td>
</tr>
<tr>
<td>2.5</td>
<td>The complex dimensions of the Cantor string where ( D = \log_3 2 ) and ( p = \frac{2\pi}{\log 3} ).</td>
</tr>
<tr>
<td>3.1</td>
<td>The first two stages in the construction of a self-similar multifractal measure on the Cantor set corresponding to some vector ( p ). The area of each rectangle represents the weight on the horizontal interval.</td>
</tr>
<tr>
<td>4.1</td>
<td>The contour of integration for ( I(z) ) with poles of ( \Gamma(-s) ) to the right and poles of the ( \Gamma(s_j + s) ) to the left.</td>
</tr>
<tr>
<td>4.2</td>
<td>The contours ( C_m ) contain the poles 0, 1, \ldots, ( m ) of ( \Gamma(-s) ).</td>
</tr>
<tr>
<td>4.3</td>
<td>The branch cut for the analytic continuation of the generalized hypergeometric function corresponds to a vertical progression of cuts for the ( \alpha )-scaling zeta function.</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

The initiator of the surge of interest in fractals over the last thirty five years is easily attributed to the incredibly influential work of Benoit Mandelbrot. Mandelbrot’s book *The Fractal Geometry of Nature* [Man82] is arguably the work that popularized fractal geometry not only in the mathematical community, but in many other fields such as computer graphics, economics, and meteorology to name a few. One of Mandelbrot’s motivations for investigating fractals is the roughness of nature. In [Man82], he wrote that “Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.” Mandelbrot opened people’s eyes to the fact that fractals are an important tool in understanding the coarse reality of our universe rather than the previous notion that fractals are little more than counterexamples to the traditional smooth theory.

One very interesting fact is that there is currently no universally accepted definition for the term *fractal*. Many definitions have been suggested, but the general rule has been that examples arise that mathematicians immediately label fractal yet do not fall under these definitions. For example, Mandelbrot defined a set as fractal if its Hausdorff dimension is strictly larger than its topological dimension. However, the Devil’s staircase (i.e. the graph of the Cantor function) is an example
of a set which has both Hausdorff and topological dimension equal to 1, and it is widely regarded as a fractal object. An alternative definition one will frequently encounter involves the notion of self-similarity. Roughly speaking, this means that the object is comprised of several scaled-down copies of itself. While this notion provides a great deal of examples which everyone can agree should be classified as fractal, there are, once again, many examples for which this definition is insufficient. The self-similar examples are the main focus of this work, as they provide a rigid framework that allows for success in the computational aspects of the analysis.

As described in chapter 2, this work is done with a foundation in the theory presented in [LvF13]. From this perspective, fractals are analyzed by studying appropriate zeta functions associated to the geometry. The main fractals of interest in this theory are the boundaries of open subsets of \( \mathbb{R} \) called fractal strings, and the associated geometric zeta functions formed by adding the lengths of the intervals in the open set to a complex parameter. Chapter 12 of [LvF13] (further refined in chapter 13) gives a thorough argument for defining fractality as the presence of at least one nonreal pole of the zeta function, called a complex dimension, with positive real part, or if the associated zeta function has a natural boundary of meromorphic continuation. As of yet, all of the examples that one would intuitively call fractal seem to fall under this definition. The zeta functions found in [LvF13] are generally restrictive and best suited for subsets of \( \mathbb{R} \) or, more generally, measures on \( \mathbb{R}^+ \) supported away from zero, but in the more recent theory [LRZ17] the so-called distance zeta functions are used to handle the higher dimensional cases. The poles of the distance zeta function of a fractal string are exactly the same as the poles of the geometric zeta function with an additional pole at zero. The intuition is that the distance zeta function picks up the zero dimension associated to any single point.

A fundamental inquiry of this work concerns measures supported on self-similar sets. Generally, the study of multifractal measures investigates measures which have many different scaling relationships with respect to the geometry. For example, the Lebesgue measure of a ball in \( \mathbb{R}^d \) scales, with respect to the radius of the ball, by a power equal to \( d \) (the measure of a ball in \( \mathbb{R}^2 \) is \( \pi r^2 \) and
the measure of a ball in $\mathbb{R}^3$ is $\frac{4}{3}\pi r^3$). With multifractals, the exponential scaling of the measure with respect to the diameter of a set varies throughout the set. There are two classical approaches to analyzing such measures. The first is called coarse multifractal analysis. The aim here is to decompose the support of the measure into cubes with mesh $r$, count the number of cubes for which the measure is at least scaling by some given exponent, and then take a limit as $r$ tends to 0. The second approach is labeled fine multifractal analysis, and it is the analysis of the collection of points which all have the same local exponent (already taken in a limiting sense). The coarse theory is well suited for real-world applications, while the fine theory provides a very natural mathematical treatment. For more details and discussion of the relationship between the two approaches see [Fal04].

Adaptation of the coarse theory of multifractal analysis to the context of the theory in [LvF13] was done by Lapidus, Lévy-Véhel, Rock in [LLVR09] and further studied in [LR09] by introducing an appropriate class of zeta functions dependent on the exponential scaling relationship. However, determination of the meromorphic continuation of these new zeta functions and subsequent examination of their singularities was not given. This will be presented in chapter 3. Some further work between Essouabri and Lapidus [EL09] showed that the appropriate extension of these zeta functions is to work on a suitable Riemann surface and they demonstrated an intimate connection between a class of these zeta functions and the generalized hypergeometric functions.

In 1655, John Wallis first used the term ‘hypergeometric’ to describe series beyond that of the series $\sum_{n=0}^{\infty} x^n$. During the next one hundred and fifty years, many other mathematicians studied these objects including Euler who gave the first integral representations of the hypergeometric series and Vandermonde who stated an extension of the binomial theorem in terms of the hypergeometric series. A major advance was due to Gauss in 1812 when he introduced the modern notation, proved his famous summation theorem, discussed the convergence, and gave relations between many of these series. In 1836, Kummer showed that the series satisfy what is now known as the hypergeometric differential equation and that the solutions can all be expressed in terms of similar hypergeometric
series. Further analysis of the integral representations for hypergeometric functions was developed by Mellin and Barnes [Sla66].

The first key result, in section 3.4, gives a large collection of examples with this connection to generalized hypergeometric functions to establish the merit in determining the appropriate Riemann surface associated to the generalized hypergeometric functions with the given parameters. The other key effort, in chapter 4, is to determine the monodromy group associated to the solutions of the generalized hypergeometric differential equation which determines the desired Riemann surface, and so monodromy matrices are determined with respect to the local Frobenius bases of the differential equation. These monodromy matrices encode the behavior of a local basis of solutions as they are analytically continued along loops around the singularities of the differential equation. The local Frobenius basis allows one to determine the monodromy associated to a given singularity with relative ease. Once this local basis is given however, the monodromy matrices around any other singularity of the associated differential equation must be given through analytic continuation of the original basis elements in a loop around the singularity. The determination of monodromy matrices with respect to a single local basis in full generality remains an open problem, and so a different local basis around each singularity serves to give a preliminary understanding of the monodromy in general.
Chapter 2

Fractal Strings

We begin our study with the concept of a fractal string. An ordinary fractal string is a bounded open subset $\Omega$ of $\mathbb{R}$. It is well-known that any such set can be written as an at-most countable union of disjoint open connected intervals. In this work, it will be assumed that this collection is countably infinite. It is common to refer to the collection of lengths of each of these intervals $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$ arranged in non-increasing order as the fractal string. A more detailed development and additional examples can be found in [LvF13].

2.1 Self-Similar Fractal Strings

Definition 2.1 For our purposes, a contraction similarity will be defined as a map $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $d(\Phi(x), \Phi(y)) = Cd(x, y)$ for some $0 < C < 1$. The value $C$ is referred to as the scaling ratio of the contraction.

Let $I$ be a closed interval of length $L$ (typically $I = [0, 1]$), let $N \geq 2$ be an integer, and let $\Phi_1, \Phi_2, \ldots, \Phi_N$ be $N$ contraction similarities mapping $I$ to $I$ with scaling ratios $r_1, r_2, \ldots, r_N$ re-
spectively. Assume that
\[ \sum_{j=1}^{N} r_j < 1 \]
and that the images \( \Phi_j(I) \) do not overlap, except possibly at the endpoints. Define the map \( \Psi : K \longrightarrow K \) on the collection of compact subsets of \( I \) by
\[ \Psi(A) = \bigcup_{j=1}^{N} \Phi_j(A). \]
If we define \( \Psi^0(I) = I \), then it follows that for any \( n \in \mathbb{N}_0 \) compositions, \( \Psi \circ \Psi^n(I) \subseteq \Psi^n(I) \). The intersection
\[ F = \bigcap_{j=0}^{\infty} \Psi^j(I) \]
is a compact set which is invariant under the map \( \Psi \), i.e. \( \Psi(K) = K \) [Hut81]. In fact, such a set \( F \) is uniquely defined by this system and is called a self-similar set which is the attractor of the system.

Any fractal string defined by the lengths of the intervals removed at each stage in this construction is called a self-similar fractal string.

Note that given a collection of scaling ratios \( r_1, \ldots, r_N \) such that \( \sum_{j=1}^{N} r_j < 1 \) one can construct a self-similar fractal string with up to \( N + 1 \) gaps as follows: Choose non-negative numbers \( g_1, \ldots, g_N, g_{N+1} \) such that \( \sum_{j=1}^{N} r_j + \sum_{j=1}^{N} g_j = 1 \) and define the contraction similarities by \( \Phi_j(x) = r_j x + \sum_{l=1}^{j-1} g_l + \sum_{l=1}^{j-1} r_l \) (see figure 2.1). The resulting attractor satisfies the strong separation condition, i.e. the images under the contraction similarities are disjoint, exactly if the values \( g_2, g_3, \ldots, g_{N-1}, g_N \) are strictly positive. Most of our examples have \( g_1 = g_{N+1} = 0 \) so that 0 and 1 are fixed points of some contraction similarities. Throughout this work, fractal strings are said to be defined by the scaling ratios in the above sense without specifying the gap lengths due to the fact that our analysis will be independent of choice of the gap lengths.

Take for example the middle-third Cantor set defined by the similarities \( \Phi_1(x) = \frac{1}{3} x \) and \( \Phi_2(x) = \frac{1}{3} x + \frac{2}{3} \) applied to \( I = [0, 1] \). Note that the scaling ratios are \( r_1 = r_2 = \frac{1}{3} \). The Cantor set is the attractor of the system, and the Cantor string is the collection of removed lengths at each stage.
Figure 2.1: The first two stages in the construction of a self-similar fractal string with three contraction similarities. Given contraction ratios $r_1, \ldots, r_N$ one can construct a self-similar fractal string by choosing appropriate gap lengths.

\[ [0, 1] \]

\begin{align*}
g_1 & \quad r_1 & \quad g_2 & \quad r_2 & \quad g_3 & \quad r_3 & \quad g_4 \\
\end{align*}

\[ \vdots \]

Figure 2.2: The first two stages in the construction of the middle-third Cantor set with removed interval lengths. Here the contraction similarities are $\Phi_1(x) = \frac{1}{3}x$ and $\Phi_2(x) = \frac{1}{3}x + \frac{2}{3}$.

\[ [0, 1] \]

\[ \Psi([0, 1]) \quad \Phi_1 \quad \frac{1}{3} \quad \Phi_2 \]

\[ \Psi^2([0, 1]) \quad \Phi_1 \circ \Phi_1 \quad \frac{1}{9} \quad \Phi_1 \circ \Phi_2 \quad \Phi_2 \circ \Phi_1 \quad \frac{1}{9} \quad \Phi_2 \circ \Phi_2 \]

\[ \vdots \]
Thus, the Cantor string \( \mathcal{L}_{CS} = \{\ell_j\}_{j=1}^{\infty} \) is the collection of lengths \( \frac{1}{3^n} \) with multiplicities \( 2^{n-1} \) for all \( n \in \mathbb{N} \).

### 2.2 Minkowski Dimension

Much of the study of fractal strings is an effort to understand the nature of the boundaries of these objects. We denote the boundary of \( \mathcal{L} \) by \( \partial \mathcal{L} \) and define it as the boundary \( \partial \Omega \). It turns out that \( \partial \mathcal{L} \) has very interesting properties such as non-integral dimensionality. Let us first define one notion of dimension. We begin by denoting the volume of the inner tubular neighborhood of \( \partial \Omega \) with radius \( \epsilon \):

\[
V(\epsilon) = \text{vol}_1 \{ x \in \Omega : d(x, \partial \Omega) < \epsilon \}
\]

where \( \text{vol}_1 \) denotes the one-dimensional Lebesgue measure on \( \mathbb{R} \) and \( d(x, A) \) is the distance from the point \( x \) to the set \( A \subseteq \mathbb{R} \).

**Definition 2.2** The dimension of a fractal string \( \mathcal{L} \) is defined as the inner Minkowski dimension of \( \partial \mathcal{L} \),

\[
D_{\mathcal{L}} = \inf \{ \alpha \geq 0 : V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \to 0^+ \}.
\]

Since \( \Omega \) will always consist of a sequence of intervals with lengths tending to zero, the inner \( \epsilon \) neighborhood will cover all but a finite number of intervals and will contribute \( 2\epsilon \) to all the intervals large enough not to be covered. In particular, we have

\[
V(\epsilon) = \sum_{\ell \geq 2\epsilon} 2\epsilon + \sum_{\ell < 2\epsilon} \ell.
\]

For the Cantor string, we have \( 2^{n-1} \) intervals of length \( 3^{-n} \) for all \( n \in \mathbb{N} \). For \( 0 < \epsilon \leq \frac{1}{2} \) we can find \( j \geq 0 \) such that \( 3^{-j} \geq 2\epsilon > 3^{-(j+1)} \). Thus \( j = \lfloor -\log_3(2\epsilon) \rfloor \) where \( \lfloor x \rfloor \) denotes the integer part of \( x \). Adding the multiplicities of all lengths larger than \( 2\epsilon \) gives

\[
\sum_{\ell \geq 2\epsilon} 2\epsilon = 2\epsilon \sum_{n=1}^{j} 2^{n-1} = 2\epsilon (2^j - 1)
\]
which yields the expression
\[
V(\epsilon) = 2\epsilon \left(2^j - 1\right) + \sum_{n=j+1}^{\infty} 2^{n-1} 3^{-n} = 2\epsilon \cdot 2^j + \left(\frac{2}{3}\right)^j - 2\epsilon.
\]

Denoting \( \{x\} = x - [x] \), we have
\[
2^j = 2^{\left[-\log_3(2\epsilon)\right]} = (2\epsilon)^{-\log_3 2} 2^{-\left(-\log_3(2\epsilon)\right)}.
\]

Using a similar calculation for \( \left(\frac{2}{3}\right)^j \) and setting \( D = \log_3 2 \) one can express
\[
V(\epsilon) = (2\epsilon)^{1-D} \left(\frac{1}{2}\right)^{-\log_3(2\epsilon)} + \left(\frac{2}{3}\right)^{-\log_3(2\epsilon)} - 2\epsilon.
\]

This expression shows that the value \( D \) is indeed the inner Minkowski dimension of the boundary of the Cantor string.

Finding this kind of expression for other examples is in general rather difficult, so other techniques are employed to aide the process. In fact, the next section describes arguably the most useful function for calculating the dimension of a fractal string.

### 2.3 Geometric Zeta Functions

**Definition 2.4** The geometric zeta function of a fractal string \( \mathcal{L} = \{\ell_j\}_{j=1}^{\infty} \) is the Dirichlet series
\[
\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n \ell_n^s
\]
defined for complex \( s \) with \( \text{Re}(s) > \sigma \) where the index \( n \) indicates that the \( \ell_n \) are distinct lengths with multiplicity \( m_n \). The abscissa of convergence of the geometric zeta function is defined as
\[
\sigma = \inf\{\alpha \in \mathbb{R} : \zeta_{\mathcal{L}}(\alpha) < \infty\}.
\]

The following theorem demonstrates one of the immediate uses of the geometric zeta functions.

**Theorem 1** ([LvF13]) The abscissa of convergence of the geometric zeta function of \( \mathcal{L} \) coincides with the Minkowski dimension of \( \partial \mathcal{L} \).
Returning to the example of the middle-third Cantor set, we can now calculate the Minkowski dimension via the geometric zeta function. We have

\[ \zeta_{CS}(s) = \sum_{n=1}^{\infty} m_n \ell_n^s \]

\[ = \sum_{n=1}^{\infty} 2^{n-1} \left( \frac{1}{3^n} \right)^s \]

\[ = \sum_{n=1}^{\infty} 2^{-1} (2 \cdot 3^{-s})^n \]

\[ = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}. \]

The abscissa of convergence of the function is the unique real number for which the denominator in the closed form equals 0, namely \( \sigma = \log_3 2 \).

### 2.4 Complex Dimensions

It is clear by the example of the middle-third Cantor set that a geometric zeta function can have analytic continuation to a much larger domain in \( \mathbb{C} \) than the original half-plane of convergence of the Dirichlet series. It is also clear that the geometric zeta function for this example can be analytically continued to all of \( \mathbb{C} \) except for the solutions to \( 1 - 2 \cdot 3^{-s} = 0 \). This is indicative of the behavior of all self-similar strings, and it is suggestive of the value in determining the singularities of the continuations of these zeta functions.

**Definition 2.5** The set of complex dimensions of the fractal string \( \mathcal{L} \) is defined as

\[ \mathcal{D}_\mathcal{L}(W) = \{ \omega \in W : \zeta_\mathcal{L} has a pole at \omega \} \]

where \( W \subseteq \mathbb{C} \) is a domain on which \( \zeta_\mathcal{L} \) has meromorphic extension.

The case of self-similar systems allows for a convenient closed form expression of the geometric zeta function leading to an easily recovered collection of complex dimensions. This can be seen in the following theorem.
**Theorem 2 ([LvF13])** The geometric zeta function of a self-similar string $L$ has a meromorphic continuation to $\mathbb{C}$ given by

$$\zeta_L(s) = \frac{L^s \sum_{k=1}^{K} g_k^s}{1 - \sum_{j=1}^{N} r_j^s}.$$ 

**Proof.**

$$\zeta_L(s) = \sum_{k=1}^{K} \sum_{q=0}^{\infty} \left( \sum_{\nu_1=1}^{N} \cdots \sum_{\nu_q=1}^{N} (r_{\nu_1} r_{\nu_2} \cdots r_{\nu_q} g_k L)^s \right)$$

$$= \sum_{k=1}^{K} (g_k L)^s \sum_{q=0}^{\infty} \left[ \left( \sum_{\nu_1=1}^{N} r_{\nu_1}^s \right) \cdots \left( \sum_{\nu_q=1}^{N} r_{\nu_q}^s \right) \right]$$

$$= \sum_{k=1}^{K} (g_k L)^s \sum_{q=0}^{\infty} \left( \sum_{j=1}^{N} r_j^s \right)^q$$

$$= \frac{L^s \sum_{k=1}^{K} g_k^s}{1 - \sum_{j=1}^{N} r_j^s}$$

Apply the Principle of Analytic Continuation. 

Let us now consider the example of the Fibonacci string. Here the starting interval is $I = [0, 4]$ and the contractions are given by $\Phi_1(x) = \frac{1}{2} x$ and $\Phi_2(x) = \frac{1}{4} x + 3$ (See figure 2.3). Applying the IFS yields the removed lengths $\ell_n = 2^{-n}$ with multiplicities $m_n = F_{n+1}$ where $F_n$ is the $n^{th}$ Fibonacci number and $F_1 = F_2 = 1$. Working with this geometric zeta function is slightly trickier than that of the Cantor string since recognizing it as a geometric series is not as obvious.

We proceed by rearrangement of the terms to give a clearer view of the geometric series. This is justified since the sum is absolutely convergent on a right half-plane. We notice that the sum of terms of the geometric zeta function corresponding to lengths at the $n^{th}$ stage of application of the IFS is $\left( \frac{1}{2^n} + \frac{1}{4^n} \right)^n$. 

\[ 11 \]
Figure 2.3: The construction of the Fibonacci string with removed lengths indicated.

Thus, we have

\[
\zeta_{Fib}(s) = \sum_{n=0}^{\infty} F_{n+1} 2^{-ns} = \sum_{n=0}^{\infty} \left( \frac{1}{2^s} + \frac{1}{4^s} \right)^n = \frac{1}{1 - 2^{-s} - 4^{-s}}.
\]

Hence, the complex dimensions of the Fibonacci string are given by

\[
D_{Fib}(\mathbb{C}) = \{ D + \text{inp} : n \in \mathbb{Z} \} \cup \left\{ -D + i \left( n + \frac{1}{2} \right) p : n \in \mathbb{Z} \right\}
\]
Figure 2.4: The complex dimensions of the Fibonacci string where $D = \log_2 \varphi$, $p = \frac{2\pi}{\log 2}$, and $\varphi = \frac{1 + \sqrt{5}}{2}$.

where $D = \log_2 \varphi$, $p = \frac{2\pi}{\log 2}$, and $\varphi = \frac{1 + \sqrt{5}}{2}$.

The complex dimensions of the Cantor string are given by the complex solutions of

$1 - 2 \cdot 3^{-s} = 0$, which are

$$
\mathcal{D}_{\text{Cantor}} = \{D + np : n \in \mathbb{Z}\}
$$

where $D = \log_3 2$ and $p = \frac{2\pi}{\log 3}$. These complex dimensions can be used to describe the inner tubular neighborhood of the boundary of the Cantor string. Start by fixing $b > 0$ with $b \neq 1$ and considering the pointwise convergent Fourier series of the periodic function $b^{-(u)}$:

$$
b^{-(u)} = \frac{b - 1}{b} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi nu}}{\log b + 2\pi in}.
$$
Figure 2.5: The complex dimensions of the Cantor string where $D = \log_3 2$ and $p = \frac{2\pi}{\log 3}$.

Letting $p = \frac{2\pi}{\log 3}$ and substituting the Fourier series in to (2.3) with $b = \frac{1}{2}$ and $b = \frac{2}{3}$ one gets the expression

$$V(\epsilon) = \frac{1}{2 \log 3} \sum_{n=-\infty}^{\infty} \frac{(2\epsilon)^{1-D-\text{i}np}}{(D + \text{i}np)(1 - D - \text{i}np)} - 2\epsilon,$$

a sum over all of the complex dimensions.
Chapter 3

Multifractal Measures

Now we turn to the study of highly irregular measures called multifractals. In particular, the supports of the measures considered in this section are Cantor-like sets. These measures are introduced in a more general setting in [Fal04], but here we are concerned with applying the theory of fractal strings to particular subsets of the support.

3.1 Self-Similar Multifractals

The measures of interest here are called self-similar due to their construction through the use of iterated function systems. The construction is as follows:

Let \( \{\Phi_j\}_{j=1}^N \) be a collection of contraction similarities with scaling ratios \( r_1, r_2, \ldots, r_N \) such that \( \sum_{j=1}^N r_j \leq 1 \). One can define an iterated function system applied to the unit interval as in chapter 2 where we now allow for the union of the images of the contraction similarities to be the full interval. We are now concerned with the closed intervals in the construction rather than the open intervals defined as the gaps in the construction.

In order to easily examine how the measure will be defined, we need to track the number of applications of each contraction similarity to get to any interval in the construction.
**Definition 3.1** Define the address of an interval in the self-similar construction as a vector $\mathbf{k} = (k_1, k_2, \ldots, k_N) \in \mathbb{N}_0^N$ by identifying each component $k_j$ with the number of times $\Phi_j$ is applied.

This allows us to see that the length of an interval corresponding to a vector $\mathbf{k}$ is given by $\ell(\mathbf{k}) = r_1^{k_1} \cdot r_2^{k_2} \cdots r_N^{k_N}$ and is independent of the order in which the maps are applied. Note that an address does not uniquely define an interval, only the length.

Let $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ be a vector of $N$ real numbers such that $0 < p_j < 1$ for all $j$ and $\sum_{j=1}^N p_j = 1$. Now define the weight on an interval corresponding to a vector $\mathbf{k}$ by $p(\mathbf{k}) = p_1^{k_1} \cdot p_2^{k_2} \cdots p_N^{k_N}$. It is then easily seen that to go from one interval to its image under some $\Phi_j$ one need only multiply the length by $r_j$ and the weight by $p_j$, and that the address uniquely defines weight in addition to length. One sees that the sum of weights of all the intervals in any given stage of this construction is always 1, i.e. for any $n \in \mathbb{N}$ we have that $\sum_{|\mathbf{k}|=n} p(\mathbf{k}) = 1$. In the limit, for each vector $\mathbf{p}$ there exists a unique Borel probability measure $\mu$ supported on the attractor $F$ such that

$$\mu = \sum_{j=1}^N p_j \mu \circ \Phi_j^{-1}.$$

The measure $\mu$ is called a *self-similar multifractal* and it is determined by the vectors $\mathbf{r}$ and $\mathbf{p}$ up to translation of the images of the $\Phi_j$.

A natural thing to consider next is the exponential relationship between the measure of a set and the length of that same set. One can easily see the motivation for this by considering the basic example of $\mathbb{R}^d$ with Lebesgue measure. We have the relation $\text{vol}_d(B(x, r)) = Cr^d$ for any ball $B(x, r)$ centered at $x \in \mathbb{R}^d$ of radius $r$ where $C$ is a constant independent of $x$ and $r$. The importance here is that one can determine the dimension of a ball by examining the exponential relationship between the radius and measure of a ball. This is a particularly nice example due in no small part to translation invariance of Lebesgue measure.

For our purposes, we wish to organize the intervals in the construction of the fractal string into subcollections of intervals corresponding to each exponential relationship. The following definition tracks this behavior.
Figure 3.1: The first two stages in the construction of a self-similar multifractal measure on the Cantor set corresponding to some vector $p$. The area of each rectangle represents the weight on the horizontal interval.

**Definition 3.2** The scaling regularity of an interval determined by $k$ is the value $\alpha(k)$ given by

$$\alpha(k) := \frac{\log p(k)}{\log \ell(k)}$$

For any given self-similar multifractal, the scaling regularities that are attained form an at most countable collection of values.

### 3.2 $\alpha$-Scaling Zeta Functions

**Definition 3.3** For a scaling regularity value $\alpha \in [-\infty, \infty]$, the sequence of $\alpha$-scales, denoted $\mathcal{L}(\alpha)$, is the fractal string given by

$$\mathcal{L}(\alpha) = \{\ell(k) : k \in \mathbb{N}_0^N \text{ and } \alpha(k) = \alpha\}.$$
string case were determined through the use of an appropriate zeta function, and we will use the same approach here.

**Definition 3.4** The \( \alpha \)-scaling zeta function \( \zeta_{\rho, \mathbf{p}}(\alpha, \cdot) \) is the geometric zeta function of the sequence of \( \alpha \)-scales \( \mathcal{L}(\alpha) \). That is

\[
\zeta_{\rho, \mathbf{p}}(\alpha, s) := \zeta_{\mathcal{L}(\alpha)}(s) = \sum_{\alpha(k) = \alpha} f(k)^s.
\]

If the vector \( \mathbf{p} \) is chosen so that all of the scaling regularity values \( \alpha(k) \) are equal, then \( \zeta_{\rho, \mathbf{p}} \) is the geometric zeta function of the fractal string defined by all of the lengths of the closed intervals in the prefractal approximations of the attractor. Such a choice of \( \mathbf{p} \) can be constructed from a vector of contraction ratios \( \mathbf{r} = (r_1, \ldots, r_N) \) by starting with a real number \( 0 < r < 1 \) and positive real numbers \( \gamma_1, \ldots, \gamma_N \) such that \( r_j = r^{\gamma_j} \) for all \( j \). Define the function \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = r^x + \left( r^x \left( \frac{\gamma_1}{\gamma_2} \right) + \cdots + r^x \left( \frac{\gamma_N}{\gamma_1} \right) \right) \). Since \( f(0) > 1 \) and \( \lim_{x \to \infty} f(x) = 0 \) we can use the Intermediate Value theorem to conclude that there exists a unique real number, called \( \eta_1 \), such that \( f(\eta_1) = 1 \). Now define \( \eta_j = \eta_1 \left( \frac{\gamma_1}{\gamma_j} \right) \) for all \( j = 2, \ldots, N \) and let \( p_j = r^{\eta_j} \) for all \( j \). Note that \( \sum_{j=1}^N p_j = 1 \) is the statement \( f(\eta_1) = 1 \), and so \( \mathbf{p} = (p_1, \ldots, p_N) \) is a valid probability vector. Thus, for any \( \mathbf{k} \) we have

\[
\alpha(k) = \frac{\log \left( r^{\eta_1k_1} \cdots r^{\eta_Nk_N} \right)}{\log \left( r^{\gamma_1k_1} \cdots r^{\gamma_Nk_N} \right)}
= \frac{\eta_1k_1 + \cdots + \eta_Nk_N}{\gamma_1k_1 + \cdots + \gamma_Nk_N}
= \frac{\eta_1k_1 + \eta_1 \left( \frac{\gamma_1}{\gamma_2} \right) k_2 + \cdots + \eta_1 \left( \frac{\gamma_N}{\gamma_1} \right) k_N}{\gamma_1k_1 + \cdots + \gamma_Nk_N}
= \frac{\eta_1}{\gamma_1}.
\]

These zeta functions may seem like an entirely different object to that of the classical geometric zeta functions. However, the asymptotic nature of the iterated function system allows one, at least in the self-similar case, to show that the poles of the geometric zeta function associated to the removed interval lengths and the poles of the \( \alpha \)-scaling zeta function associated to the images of the contraction similarities with the unique measure given above coincide [LVM11].
Unlike the above example, there will generally be a countable collection of distinct regularity values for a self-similar multifractal with arbitrary scaling ratios and probabilities.

**Remark 3.5** Notice that for any \( k \in \mathbb{N}_0^N \) and any \( n \in \mathbb{N} \) we have

\[
\alpha(nk) = \frac{\log \left( \prod_{i=1}^{N} p_i^{nk_i} \right)}{\log \left( \prod_{i=1}^{N} r_i^{nk_i} \right)} = n \log \left( \prod_{i=1}^{N} p_i^{k_i} \right) = n \log \left( \prod_{i=1}^{N} r_i^{k_i} \right) = \alpha(k).
\]

This shows that if \( \ell(k) \in \mathcal{L}(\alpha) \) for some \( \alpha \), then \( \ell(nk) = \ell(k)^n \in \mathcal{L}(\alpha) \) for all \( n \in \mathbb{N} \) as well.

Thus, if there exists an interval with a given scaling regularity then we attain a countably infinite collection of lengths with that same scaling regularity. Clearly, an arbitrary address may be the positive integer multiple of a different address, but since the zeta functions require us to determine all addresses with a given regularity it is natural to consider addresses which do not arise as such a multiple for \( n \geq 2 \).

**Definition 3.6** Any address \( k \in \mathbb{N}_0^N \) is said to be a generator if it has the property that \( \gcd(k_1, \ldots, k_N) = 1 \) for the nonzero components. Any self-similar multifractal for which there does not exist generators \( k \) and \( k' \) such that \( k \neq k' \) and \( \alpha(k) = \alpha(k') \) is said to be uniquely generated.

If a self-similar multifractal is uniquely generated then we can easily express the \( \alpha \)-scaling zeta function for a value \( \alpha = \alpha(k) \) corresponding to the unique generator \( k \). We have

\[
\zeta_{r, p}(\alpha, s) = \sum_{n=1}^{\infty} \left( \sum_{nk_1, \ldots, nk_N} \ell(k)^n s \right)
\]

where \( K = k_1 + k_2 + \cdots + k_N \) and

\[
\binom{A}{a_1, \ldots, a_N} = \frac{A!}{(a_1!) \cdots (a_N!)}
\]

is the multinomial coefficient with \( A = a_1 + \cdots + a_N \).
The multinomial coefficient as it appears in the scaling zeta function is the multiplicity of the length \( \ell(nk) \) since it is the number of ways of applying each contraction \( \Phi_j \) exactly \( nk_j \) times and getting the same length each time.

Not every self-similar fractal string is uniquely generated however, but it is still an open problem to express the \( \alpha \)-scaling zeta functions of such fractal strings in a way that allows one to explicitly extract the complex dimensions.

3.3 An Example of a Self-Similar Multifractal Which Is Not Uniquely Generated

Consider the self-similar multifractal determined by the vectors \( r = (\frac{1}{2}, \frac{1}{4}, \frac{1}{10}) \) and \( p = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \). The resulting multifractal is not uniquely generated as demonstrated by the two different generators \( k = (1, 0, 0) \) and \( k' = (0, 1, 0) \) with the same scaling regularity:

\[
\alpha(k) = \frac{\log \left( \left( \frac{1}{2} \right)^{k_1} \left( \frac{1}{4} \right)^{k_2} \left( \frac{1}{10} \right)^{k_3} \right)}{\log \left( \left( \frac{1}{2} \right)^{k_1} \left( \frac{1}{4} \right)^{k_2} \left( \frac{1}{10} \right)^{k_3} \right)} = 1 = \frac{\log \left( \left( \frac{1}{2} \right)^{k_1} \left( \frac{1}{4} \right)^{k_2} \left( \frac{1}{10} \right)^{k_3} \right)}{\log \left( \left( \frac{1}{2} \right)^{k_1} \left( \frac{1}{4} \right)^{k_2} \left( \frac{1}{10} \right)^{k_3} \right)} = \alpha(k').
\]

For any address \( k = (k_1, \ldots, k_N) \), we have the expression

\[
\alpha(k_1, k_2, k_3) = \frac{\log \left( \left( \frac{1}{2} \right)^{k_1} \left( \frac{1}{4} \right)^{k_2} \left( \frac{1}{10} \right)^{k_3} \right)}{\log \left( \left( \frac{1}{2} \right)^{k_1} \left( \frac{1}{4} \right)^{k_2} \left( \frac{1}{10} \right)^{k_3} \right)} = \frac{(k_1 + 2k_2 + 2k_3) \log(\frac{1}{2})}{(k_1 + 2k_2) \log(\frac{1}{2}) + k_3 \log(\frac{1}{10})} = \frac{\log 2 - \log 5}{\log 2 + \log 10}.
\]

From this it is clear that two addresses \( k \) and \( k' \) with \( k_3 \neq 0 \) and \( k'_3 \neq 0 \) yield the same scaling regularity if and only if \( \frac{k_1 + 2k_2}{k_3} = \frac{k'_1 + 2k'_2}{k'_3} \). For \( k_3 = 0 \) the regularity is 1 regardless of the values of \( k_1 \) and \( k_2 \), and so there are a countably infinite collection of distinct generators for regularity 1 [for explicit examples just take any two co-primes for \( k_1 \) and \( k_2 \)], and hence we cannot represent the \( \alpha \)-scaling zeta function for the regularity 1 in the desired form (3.7).
This example was considered in [dSLRR13] where only two addresses were considered due to the complicated nature of the multiplicities of each length with a given scaling regularity. An expression for the $\alpha$-scaling zeta functions can be given explicitly in this example nonetheless. One begins by making the substitution $M = k_1 + 2k_2$, so that regularity values are distinct whenever $\gcd(M, k_3) = 1$. We attain a general formula by counting multiplicities at each stage of the iterations for fixed values of $k_3$, and then increasing $k_3$ until a pattern emerges.

Beginning with $k_3 = 0$, we have

<table>
<thead>
<tr>
<th>Regularity</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\log((\frac{1}{2})^1)}{\log((\frac{1}{2})^1/\pi^0)}$</td>
<td>$\begin{pmatrix} 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\frac{\log((\frac{1}{2})^2)}{\log((\frac{1}{2})^2/\pi^0)}$</td>
<td>$\begin{pmatrix} 1 \end{pmatrix} + \begin{pmatrix} 2 \end{pmatrix} = 2$</td>
</tr>
<tr>
<td>$\frac{\log((\frac{1}{2})^3)}{\log((\frac{1}{2})^3/\pi^0)}$</td>
<td>$\begin{pmatrix} 2 \end{pmatrix} + \begin{pmatrix} 3 \end{pmatrix} = 3$</td>
</tr>
<tr>
<td>$\frac{\log((\frac{1}{2})^4)}{\log((\frac{1}{2})^4/\pi^0)}$</td>
<td>$\begin{pmatrix} 2 \end{pmatrix} + \begin{pmatrix} 3 \end{pmatrix} + \begin{pmatrix} 4 \end{pmatrix} = 5$</td>
</tr>
<tr>
<td>$\frac{\log((\frac{1}{2})^5)}{\log((\frac{1}{2})^5/\pi^0)}$</td>
<td>$\begin{pmatrix} 3 \end{pmatrix} + \begin{pmatrix} 4 \end{pmatrix} + \begin{pmatrix} 5 \end{pmatrix} = 8$</td>
</tr>
<tr>
<td>$\frac{\log((\frac{1}{2})^6)}{\log((\frac{1}{2})^6/\pi^0)}$</td>
<td>$\begin{pmatrix} 3 \end{pmatrix} + \begin{pmatrix} 4 \end{pmatrix} + \begin{pmatrix} 5 \end{pmatrix} + \begin{pmatrix} 6 \end{pmatrix} = 13$</td>
</tr>
<tr>
<td>$\frac{\log((\frac{1}{2})^7)}{\log((\frac{1}{2})^7/\pi^0)}$</td>
<td>$\begin{pmatrix} 4 \end{pmatrix} + \begin{pmatrix} 5 \end{pmatrix} + \begin{pmatrix} 6 \end{pmatrix} + \begin{pmatrix} 7 \end{pmatrix} = 21$</td>
</tr>
</tbody>
</table>

which gives the formula

$$\sum_{j=0}^{\left\lfloor \frac{M}{2} \right\rfloor} \left( M - 2 \left\lceil \frac{M+1}{2} + j \right\rceil - j, 0 \right)$$

for multiplicities.

For $k_3 = 1$ we have
which gives the formula

\[
\sum_{j=0}^{\lfloor M \rfloor} \left( M - 2 \left\lfloor \frac{M}{2} \right\rfloor + j \right) = 22
\]

for multiplicities.

For \( k_3 = 2 \) we have

\[
\begin{align*}
\text{Regularity} & \quad \text{Multiplicity} \\
\log((\frac{1}{2})^3) & \quad \log((\frac{1}{2})(\frac{1}{1})) = 3 \\
\log((\frac{1}{2})^4) & \quad \log((\frac{1}{2})(\frac{1}{2})) = 9 \\
\log((\frac{1}{2})^5) & \quad \log((\frac{1}{2})(\frac{1}{3})) = 22 \\
\log((\frac{1}{2})^6) & \quad \log((\frac{1}{2})(\frac{1}{4})) = 51 \\
\log((\frac{1}{2})^7) & \quad \log((\frac{1}{2})(\frac{1}{5})) = 111 \\
\end{align*}
\]

which gives the formula

\[
\sum_{j=0}^{\lfloor M \rfloor} \left( M - 2 \left\lfloor \frac{M}{2} \right\rfloor + j \right) = 51
\]

for multiplicities.
Thus, the multiplicity for a length \((2^{-M}10^{-k_3})^n\) is given by

\[
\left\lfloor \frac{nM}{2} \right\rfloor \sum_{j=0}^{\left\lfloor \frac{nM+2nk_3+1}{2} \right\rfloor + j} (\left\lfloor \frac{nM}{2} \right\rfloor + 2j, \left\lfloor \frac{nM}{2} \right\rfloor - j, nk_3)
\]

so that the form of the \(\alpha\)-scaling zeta function for given \(M\) and \(k_3\) with \(\gcd(M, k_3) = 1\) is given by

\[
\sum_{n=1}^{\infty} \left[ \left\lfloor \frac{nM}{2} \right\rfloor \sum_{j=0}^{\left\lfloor \frac{nM+2nk_3+1}{2} \right\rfloor + j} (\left\lfloor \frac{nM}{2} \right\rfloor + 2j, \left\lfloor \frac{nM}{2} \right\rfloor - j, nk_3) \right] (2^{-M}10^{-k_3})^ns.
\]

Determining the number of generators \(k'\) satisfying \(k_1 + 2k_2 k_3 = k'_1 + 2k'_2 k_3\) for a fixed generator \(k\) is unsolved. If it can be proven that there are only finitely many such \(k'\) when \(k_3 \neq 0\), then one could express the \(\alpha\)-scaling zeta function as a finite sum of zeta function of the form (3.7) and apply the methods in chapter 4 for determining the appropriate analytic continuation formulas. If, on the other hand, there are always infinitely many generators for every nontrivial scaling regularity then any obvious connections to the methods in chapter 4 are lost.

### 3.4 A Collection of Uniquely Generated Self-Similar Multifractals

Here we develop a large collection of uniquely generated self-similar multifractals. In fact, for every finite collection of rationally independent real numbers under mild conditions, one can construct an example using the following theorem.

**Theorem 3** Let \(p = (p_1, \ldots, p_N)\) be a vector such that \(\sum_{j=1}^{N} p_j = 1\), \(p_j = r^{\eta_j}\) for all \(j\), \(0 < r < 1\), and all the \(\eta_j\) are positive rational numbers. Then for any collection of rationally independent positive real numbers \(\gamma_1, \ldots, \gamma_N\) such that \(\sum_{j=1}^{N} r^{\gamma_j} \leq 1\), the self-similar multifractal defined by the vectors \(p = (r^{\eta_1}, \ldots, r^{\eta_N})\) and \(r = (r^{\gamma_1}, \ldots, r^{\gamma_N})\) is uniquely generated.

**Proof.** Suppose by way of contradiction that there are two nonzero generators \(k = (k_1, \ldots, k_n)\) and \(k' = (k'_1, \ldots, k'_n)\) such that \(k \neq k'\) and \(\alpha(k) = \alpha(k')\). Let \(A = \eta_1 k_1 + \cdots + \eta_N k_N\) and
\[ A' = \eta_1 k'_1 + \cdots + \eta_N k'_N. \] Now consider the expression for \( \alpha \)-scaling regularity:

\[ \alpha(k) = \frac{\eta_1 k_1 + \cdots + \eta_N k_N}{\gamma_1 k_1 + \cdots + \gamma_N k_N}. \]

Thus, we have the equality

\[ \frac{A}{\gamma_1 k_1 + \cdots + \gamma_N k_N} = \frac{A'}{\gamma_1 k'_1 + \cdots + \gamma_N k'_N} \]

which is re-written

\[ \gamma_1 (k_1 A' - k'_1 A) + \gamma_2 (k_2 A' - k'_2 A) + \cdots + \gamma_N (k_N A' - k'_N A) = 0 \]

implying by rational independence that \( k_j A' = k'_j A \) for all \( j = 1, 2, \ldots, N \). Thus, \( k_j = 0 \) if and only if \( k'_j = 0 \). So we need only consider the nonzero terms. Assuming without loss of generality that none of the \( k'_j \) are zero, otherwise re-index all the nonzero integers, we have

\[ \frac{A}{A'} = \frac{k_1}{k'_1} = \frac{k_2}{k'_2} = \cdots = \frac{k_N}{k'_N}. \]

Now, since \( k \neq k' \) we can assume without loss of generality that there is a power of a prime \( p^m \) that divides \( k_{j_0} \) for some \( j_0 \in \{1, 2, \ldots, n\} \) but does not divide \( k'_{j_0} \). Finally, \( k_{j_0} k'_j = k'_j k_j \) implies that \( p \) divides \( k_j \) for all \( j \), which contradicts the fact that \( \gcd(k_1, \ldots, k_N) = 1 \). \( \square \)

Note that the choice of common exponential base of the scaling ratios has no bearing on the rational independence of the exponents since we have that if \( r^{\gamma_j} = q^{\gamma_j} \) for all \( j \) then \( \gamma_j \alpha_j = \gamma_j \log_q(r) \). So if \( a_1 \gamma_1 \alpha_1 + a_2 \gamma_2 \alpha_2 + \cdots + a_N \gamma_N \alpha_N = 0 \) then \( \log_q(r) (a_1 \gamma_1 + \cdots + a_N \gamma_N) = 0 \) and one can simply invoke rational independence of the \( \gamma_j \).

An immediate and obvious collection of examples is found by taking all of the weights in the probability vector \( p \) to be equal. Consider the self-similar multifractal determined by the vectors \( p = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) and \( r = \left( \frac{1}{3}, \sqrt{2}, \sqrt[3]{3} \right) \). In the context of the theorem, \( r = \frac{1}{3}, \eta_1 = \eta_2 = \eta_3 = 1, \gamma_1 = \sqrt{2}, \gamma_2 = \sqrt[3]{3}, \text{and } \gamma_3 = \sqrt[5]{5}. \) By noting that the square roots of distinct primes always yields a collection of rationally independent positive real numbers, one can invoke the theorem to see that the subsequently defined self-similar multifractal is indeed uniquely generated. Thus, the \( \alpha \)-scaling
zeta function for a nontrivial value $\alpha$ is given by

$$\zeta_{r,p}(\alpha, s) = \sum_{n=1}^{\infty} \binom{nK}{nk_1, nk_2, nk_3} \left( \frac{1}{3\sqrt{2k_1} + \sqrt{3k_2} + \sqrt{5k_3}} \right)^n s$$

where $k$ is the unique generator such that $\alpha(k) = \alpha$. 
Chapter 4

Hypergeometric Functions

Consider the $\alpha$-scaling zeta function $\zeta_{r,p}(\alpha,s) = \sum_{n=1}^{\infty} \binom{nK}{nk_1,\ldots,nk_N} \ell(k)^{n\alpha}$ for a uniquely generated self-similar multifractal with the generator $k$. Some basic algebraic manipulations yield the following result for any $n \in \mathbb{N}$:

$$\binom{nK}{nk_1,\ldots,nk_N} = \prod_{j=1}^{K} (a_j)_{n} \frac{\beta(k)^{n}}{n!}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol, $\beta(k) = \frac{k^K}{k_1^1\cdots k_N^N}$, $a_j = \frac{j}{k}$ for $j = 1,2,\ldots,K$, $b_j = \frac{j}{k_1}$ for $j = 1,\ldots,k_1$, and $b_j = \frac{j - \sum_{\ell=1}^{j-1} k_\ell}{k_j}$ whenever $\sum_{\ell=1}^{j-1} k_\ell < j \leq \sum_{\ell=1}^{j} k_\ell$. Thus, the $b_j$ enumerate by row the values

$$\frac{1}{k_1}, \frac{2}{k_1}, \cdots, \frac{k_1}{k_1}, \frac{1}{k_2}, \frac{2}{k_2}, \cdots, \frac{k_2}{k_2}, \cdots, \frac{1}{k_N}, \frac{2}{k_N}, \cdots, \frac{k_N-1}{k_N}.$$ 

Due to the fact that some of values $a_j$ and $b_m$ will coincide, it will be necessary to cancel these terms. For example, consider the generator $k = (1,2,2,3)$. This gives us the values $K = 8,$
\((a_j)_{j=1}^8 = (1, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{8}{8})\) and \((b_m)_{m=1}^7 = (1, \frac{1}{7}, \frac{2}{7}, \frac{1}{2}, \frac{2}{7}, \frac{7}{8}, \frac{3}{8})\). Most notably, regardless of the generator we will have the value 1 in common between the parameters so that there will always be cancellation, but as we see in this example there can be further cancellation than this. The important fact here is that the number of remaining parameters in the numerator will always be one more than the number of parameters in the denominator. By an abuse of notation, we will assume the parameters have been canceled and re-indexed so that we can write \((a_j)_{j=1}^Q\) and \((b_j)_{j=2}^Q\). Further important observations are that all of the \(a_j\) are distinct, all parameters are in the interval \((0,1]\) so that if two differ by an integer then that integer is 0, and that the \(b_j\) may or may not have repeats. The importance of these observations will be apparent in section 4.4.

This allows us to rewrite the zeta function in the following form

\[
\zeta_{r,p}(\alpha, s) = -1 + \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{Q} (a_j)_n (\beta(k)\ell(k))^n}{\prod_{j=2}^{Q} (b_j)_n n!} = -1 + Q F_{Q-1} ((a_j)_{j=1}^Q; (b_j)_{j=2}^Q; \beta(k)\ell(k)^s) \]

and then focus our attention on the generalized hypergeometric series \(Q F_{Q-1}\) in the variable \(z\) (rather than as a function of \(\beta(k)\ell(k)^s\)). To determine the analytic continuation of \(\zeta_{r,p}(\alpha, s)\) it is enough to do so for the generalized hypergeometric series and then pass through the entire function \(s \mapsto \beta(k)\ell(k)^s\).

### 4.1 The Generalized Hypergeometric Series

There is an extensive collection of works done on the hypergeometric functions and their generalizations. Some of the particularly useful references are [Sla66], [BE53], [WW02], and [AAR99]. One formula for the analytic continuation of the hypergeometric series when \(Q = 2\) was due to Barnes [Bar07], and his result readily extends to the general case. The procedure is to define a contour integral which agrees with the hypergeometric series on the unit disk, yet defines an analytic function on the plane with a branch cut.
Throughout this section, $F(z)$ will be used to denote the generalized hypergeometric series

$$qF_{Q-1} \left( \left( a_j \right)_{j=1}^Q; \left( b_j \right)_{j=2}^Q; z \right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{Q} (a_j)_n \ z^n}{\prod_{j=2}^{Q} (b_j)_n \ n!}.$$  

The case where $Q = 2$ with arbitrary complex parameters for the $a_j$ and $b_j$ is generally referred to as Gauss' hypergeometric function and is has been studied extensively. The generalization here is simply to allow an arbitrary number of parameters in the coefficients so long as there is one more in the numerator than there is in the denominator. This restriction guarantees that the series will converge on the unit disk by applying the ratio test:

$$\left| \frac{\prod_{j=1}^{Q} (a_j + n) \ z^n}{\prod_{j=2}^{Q} (b_j + n) \ (n+1)} \right| \to |z|.$$  

Now consider the contour integral

$$I(z) = \int_{C} \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \Gamma(-s)(-z)^s \ ds$$

where the contour of integration $C$ is the imaginary axis with appropriate indentation so that the poles of $\Gamma(-s)$ lie entirely on the region of the plane to the right of the contour and the poles of the $\Gamma(a_j + s)$ all lie to the left (see Figure 4.1).

First we show that $I(z)$ defines an analytic function in $\mathbb{C} \setminus [0, \infty)$. We will need the asymptotic formula

$$\log \Gamma(a + s) = \left( a + s - \frac{1}{2} \right) \log s - s + \frac{1}{2} \log(2\pi) + o(1)$$

as $|s| \to \infty$ which holds for the values of $s$ considered here (see [WW02]), and the identity $\Gamma(1 + s)\Gamma(-s) = \frac{-\pi}{\sin(\pi s)}$. 

28
Thus, for $s = iv$ with $v \in \mathbb{R}$ and $z$ restricted to values in some compact subset $X$ of $\mathbb{C} \setminus [0, \infty)$ we have

$$\prod_{j=1}^{Q} \frac{\Gamma(a_j + s)}{\Gamma(b_j + s)} \Gamma(-s)(-z)^s = \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \Gamma(1 + s) \Gamma(-s)(-z)^s$$

$$\sim \exp \left( \left( \sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1 \right) \log s \right) \frac{-\pi}{\sin(\pi s)} (-z)^s$$

$$= \mathcal{O} \left( |v|^{\sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1} e^{-\text{var}(z) - \pi|v|} \right)$$

as $|s| \to \infty$ along the contour of integration.

The integral $\int_C |v|^{\sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1} e^{-\text{var}(z) - \pi|v|} ds$ converges uniformly for $z$ in $X$, and hence $I(z)$ defines a function analytic in $\mathbb{C} \setminus [0, \infty)$.

Next it will be shown that $I(z)$ is a constant multiple of $F(z)$ in the unit disk, and hence gives the desired analytic continuation. This will be done here by a generalization of the argument found in [Sla66] for the case $K = 2$. 

Figure 4.1: The contour of integration for $I(z)$ with poles of $\Gamma(-s)$ to the right and poles of the $\Gamma(a_j + s)$ to the left.
Figure 4.2: The contours $C_m$ contain the poles $0, 1, \ldots, m$ of $\Gamma(-s)$.

Fix $z$ such that $|z| < 1$, and let $m \in \mathbb{N}$. Define the contour $C_m$ as the rectangle with vertices at $im$, $-im$, $m + \frac{1}{2} - im$, and $m + \frac{1}{2} + im$ with indentation as before to avoid the origin as in figure 4.2.

Note that the integrand of $I$ is analytic in the interior of $C_m$ except for the poles at $0, 1, \ldots, m$ of $\Gamma(-s)$. Since the residue of $\Gamma(-s)$ at $n$ is $\frac{(-1)^{n+1}}{n!}$, we have

$$I_m(z) = \int_{C_m} \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \Gamma(-s)(-z)^s ds$$

$$= 2\pi i \sum_{n=0}^{m} \text{Res} \left[ \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \Gamma(-s)(-z)^s; n \right]$$

$$= 2\pi i \sum_{n=0}^{m} \frac{\prod_{j=1}^{Q} \Gamma(a_j + n)}{\prod_{j=2}^{Q} \Gamma(b_j + n)} \frac{(-1)^{n+1}}{n!} z^n$$

$$= -2\pi i \sum_{n=0}^{m} \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \frac{z^n}{n!}$$

$$\xrightarrow{m \to \infty} -2\pi i F(z).$$
It remains to show that $I_m(z) \xrightarrow{m \to \infty} I(z)$. In particular, we must show that each of the integrals

\begin{align*}
J_1 &= \int_{-im}^{m+\frac{1}{2}-im} \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \Gamma(-s)(-z)^s ds \\
J_2 &= \int_{m+\frac{1}{2}+im}^{m+\frac{1}{2}-im} \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \Gamma(-s)(-z)^s ds \\
J_3 &= \int_{im}^{m+\frac{1}{2}+im} \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \Gamma(-s)(-z)^s ds
\end{align*}

approaches 0 as $m \to \infty$.

For $|s| \to \infty$ with $|\arg(s)| < \pi$ we have

\begin{align*}
\left| \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \frac{1}{\Gamma(1 + s)} \right| &\sim |s|^\sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1 \\
&= O\left( A m^{\sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1} \right).
\end{align*}

In fact, we have

\begin{align*}
\left| \frac{\prod_{j=1}^{Q} \Gamma(a_j + s)}{\prod_{j=2}^{Q} \Gamma(b_j + s)} \frac{1}{\Gamma(1 + s)} \right| &< A m^{\sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1}
\end{align*}

on any of the contours defining $J_1$, $J_2$, or $J_3$ for any $m \in \mathbb{N}$ where $A$ is independent of $m$.

For $J_1$ we have $s = x - im$ so that

\begin{align*}
|\Gamma(-s)\Gamma(1 + s)| &= \left| -\frac{\pi}{\sin(\pi s)} \right| \\
&= \left| -\frac{2\pi i}{e^{\pi(m+ix)} - e^{-\pi(m+ix)}} \right| \\
&= 2\pi e^{-\pi m} \left| \frac{1}{e^{i\pi x} - e^{-2\pi m e^{-i\pi x}}} \right| \\
&\leq 4\pi e^{-\pi m}
\end{align*}

and by picking $\epsilon > 0$ small enough we have

\begin{align*}
|(-z)^s| &= |z|^\epsilon e^{m \arg(-z)} \\
&< |z|^\epsilon e^{(\pi - \epsilon)m}.
\end{align*}
Putting these bounds together, we have

\[ |J_1| \leq \int_{0}^{m+\frac{1}{2}} A m \sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1 4 \pi e^{-\pi m} |z|^x e^{(\pi - \epsilon)m} dx \]
\[ \leq 4 \pi e^{-\epsilon m} A m \sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1 \left( m + \frac{1}{2} \right) \]

since \(|z|^x \leq 1\) for all \(x \in [0, m + \frac{1}{2}]\). This shows that \(J_1 \xrightarrow{m \to \infty} 0\) since \(e^{-\epsilon m}\) is the dominant factor.

The same bounds apply to \(J_3\) and hence we also have \(J_3 \xrightarrow{m \to \infty} 0\).

For \(J_2\) we have \(s = m + \frac{1}{2} + iy\) where \(-m \leq y \leq m\). We obtain the bound

\[ |\Gamma(-s)\Gamma(1 + s)| = \frac{\pi}{|\cosh(\pi y)|} \]
\[ \leq 2\pi e^{-\pi|y|} \]

and note that

\[ |(-z)^s| = |z|^{m+\frac{1}{2}} e^{-y \arg(-z)}. \]

This yields

\[ |J_2| \leq \int_{-m}^{m} A m \sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1 2 \pi e^{-\pi|y|} |z|^{m+\frac{1}{2}} e^{-y \arg(-z)} dy \]
\[ \leq A m \sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1 2 \pi |z|^{m+\frac{1}{2}} \int_{-m}^{m} e^{-\pi|y|} dy \]
\[ \leq A m \sum_{j=1}^{Q} a_j - \sum_{j=2}^{Q} b_j - 1 2 \pi |z|^{m+\frac{1}{2}} 2m \]

since \(e^{-\pi|y|} \leq 1\). This shows that \(J_2 \xrightarrow{m \to \infty} 0\) since \(|z|^{m+\frac{1}{2}}\) is the dominant factor.

In summary, we have

\[ I_m(z) = \int_{-im}^{im} \prod_{j=1}^{Q} \Gamma(a_j + s) \Gamma(-s)(-z)^s ds + J_1 + J_2 + J_3 \xrightarrow{m \to \infty} I(z) \]

so that \(F(z) = -\frac{1}{2\pi i} I(z)\) for all \(z\) in the unit disk minus the branch cut. Hence, \(-\frac{1}{2\pi i} I(z)\) is the analytic continuation of \(F(z)\) to \(\mathbb{C}\) minus the cut \([0, \infty)\), but with the absolutely convergent series representation of \(F(z)\) on the unit disk we have a representation of the analytic continuation for all values in \(\mathbb{C}\) minus the cut \([1, \infty)\).
Figure 4.3: The branch cut for the analytic continuation of the generalized hypergeometric function corresponds to a vertical progression of cuts for the $\alpha$-scaling zeta function.

Returning to the $\alpha$-scaling zeta functions, we have analytic continuation to $\mathbb{C}$ minus the pre-images of the branch cut under the exponential function $s \rightarrow \beta(k)\ell(k)^{-1}$, and this domain can be expressed as $\mathbb{C} \setminus \left\{ \left( \frac{\log \beta(k)}{\log \ell(k)} - r \right) + \frac{2\pi j}{\log \ell(k)} i \mid r \geq 0, j \in \mathbb{Z} \right\}$ (See figure 4.3).

4.2 Fuchsian Differential Equations

The analytic continuation techniques applied to the generalized hypergeometric functions always leave a branch cut, and so any singularities would have to be contained within the cut. To further understand the nature of the function one needs to examine how the function behaves around the branch cut. To this end we adopt Riemann’s point of view to understand the global behavior of the function using Riemann surfaces. To define an appropriate Riemann surface one can study the analytic continuation of these functions around the singularities of the differential equation that they satisfy.

The differential equations of particular importance to this work are homogeneous linear ordinary differential equations in one complex variable $z$, generally written

$$\frac{d^n y}{dz^n} + a_1(z)\frac{d^{n-1} y}{dz^{n-1}} + \cdots + a_{n-1}(z)\frac{dy}{dz} + a_n(z)y = 0 \quad (4.1)$$
where $a_1, \ldots, a_n$ are functions of $z$ in some domain $\Omega \subseteq \mathbb{C}$. A point $z \in \mathbb{C}$ is said to be regular if the functions $a_1, \ldots, a_n$ are holomorphic in a neighborhood of $z$, and we call $z$ a singular point otherwise. We will be concerned more specifically with functions $a_1, \ldots, a_n$ that are meromorphic on $\mathbb{C}$. If a singularity is at most a pole of order $j$ for $a_j(z)$, then we call it a regular singularity. Lazarus Fuchs (1833-1902) extensively studied equations of this type in the spirit of Riemann’s work. In particular, equations such that all singularities are of the regular type are labeled Fuchsian equations as he was the one who discovered the required regularity of the singularities.

Fuchs followed Riemann’s work to the extent of showing that if one has an $n$-tuple of independent solutions $y = (y_1, \ldots, y_n)$ in a neighborhood of a singular point, then one can analytically continue the solutions around the singularity thus yielding a new $n$-tuple of independent solutions $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$. Consequently, there is a matrix $M \in M_n(\mathbb{C})$ such that $\tilde{y} = My$, which we call the monodromy matrix. Fuchs extended this result by examining the eigenvalues of the matrix $M$. If the matrix is diagonalizable, then the eigenvalues $e^{2\pi i \lambda_j}$ are such that $\tilde{y}_j = e^{2\pi i \lambda_j} y_j$ and the solutions are of the form $(z - z_0)^{\lambda_j} \sum_{m=0}^{\infty} c_{j,m}(z - z_0)^m$. However, if the matrix is not diagonalizable, then logarithmic terms are likely present in the solutions.

The method of solving the equation (4.1) by substituting in series of the form $(z - z_0)^{\lambda} \sum_{m=0}^{\infty} c_m(z - z_0)^m$ and performing all the formal calculations to achieve values for $\lambda$ and recurrence relations for $c_m$ is generally referred to as the method of Frobenius. Though Fuchs was the one to make this discovery, Frobenius greatly simplified his method and as a result often is attributed entirely with the result. Frobenius, however, made sure to give credit to Fuchs [Sim15] [Gra84].

### 4.3 The Generalized Hypergeometric Equation

In this section, we treat our parameters $a_j$ and $b_j$ as arbitrary to follow [Nor55] so that the symmetry in the differential equation under transformation of the variable is more apparent. Let $D = \frac{d}{dz}$ and $\theta = zD$. Then the generalized hypergeometric function is a solution to the generalized
hypergeometric differential equation

\[(\theta + b_1 - 1) \cdots (\theta + b_Q - 1) F(z) = z (\theta + a_1) \cdots (\theta + a_Q) F(z) \quad (4.2)\]

where \(b_1 = 1\) for our purposes. Rewriting equation (4.2) in terms of the operator \(D\) yields

\[z^Q (1 - z) D^Q F(z) + \sum_{j=0}^{Q-1} (\beta_j - \alpha_j z) z^j D^j F(z) = 0 \quad (4.3)\]

as can be seen by the following lemma.

**Lemma 4.4** For all \(\xi_1, \ldots, \xi_n \in \mathbb{C}\) there exist \(\mu_0, \ldots, \mu_n \in \mathbb{C}\) such that

\[(\theta + \xi_1) \cdots (\theta + \xi_n) F = \sum_{j=0}^{n} \mu_j z^j D^j F\]

**Proof.** We proceed by induction on \(n\), and note that the base case is clear. Now suppose the claim is true for some value of \(n\), we have

\[(\theta + \xi_1) \cdots (\theta + \xi_n) (\theta + \xi_{n+1}) F = (\theta + \xi_1) \sum_{j=0}^{n} \mu_j z^j D^j F\]

\[= -\xi_1 \mu_0 F + \sum_{j=1}^{n} (j \mu_j - \xi_1 \mu_j + \mu_{j-1}) z^j D^j F + \mu_n z^{n+1} D^{n+1} F\]

where \(\mu_n = 1\).

To investigate the point at infinity, we apply the substitution \(z = \frac{1}{z_1}\) and use the operator \(\theta_1 = z_1 \frac{d}{dz_1}\). One quickly verifies that \((\theta + \xi) F = -(\theta_1 - \xi) F\). Thus, equation (4.2) becomes

\[(\theta_1 - a_1) \cdots (\theta_1 - a_Q) F(z_1) = z_1 (\theta_1 - b_1 + 1) \cdots (\theta_1 - b_Q + 1) F(z_1), \quad (4.5)\]

which, when combined with equation (4.3), shows that equation (4.2) has regular singularities at 0, 1, and \(\infty\).

**4.3.1 Solutions Around \(z = 0\)**

We now employ the method of Frobenius to find a basis of solutions near the singular point \(z = 0\). We consider solutions of the form

\[F(z) = \sum_{n=0}^{\infty} c_n(\lambda) z^{n+\lambda}. \quad (4.6)\]
Let \( P(x) = (x + b_1 - 1) \cdots (x + b_Q - 1) \) and \( R(x) = (x + a_1) \cdots (x + a_Q) \). Thus, equation (4.2) is rewritten \( P(\theta)F - zR(\theta)F = 0 \).

**Lemma 4.7** \( P(\theta)F(z) = \sum_{n=0}^{\infty} P(\lambda + n)c_n(\lambda)z^{n+\lambda} \) and \( R(\theta)F(z) = \sum_{n=0}^{\infty} R(\lambda + n)c_n(\lambda)z^{n+\lambda} \).

**Proof.** It suffices to prove this for \( R \). First note that

\[
(\theta + a_1) \sum_{n=0}^{\infty} c_n(\lambda)z^{n+\lambda} = \sum_{n=0}^{\infty} c_n(\lambda) (\lambda + n + a_1) z^{n+\lambda}.
\]

Let \( R'(x) = (x + a_2) \cdots (x + a_Q) \) and assume that \( R'(\theta)F = \sum_{n=0}^{\infty} R'(\lambda + n)c_n(\lambda)z^{n+\lambda} \). Then

\[
R(\theta) \sum_{n=0}^{\infty} c_n(\lambda)z^{n+\lambda} = (\theta + a_1) \sum_{n=0}^{\infty} R'(\lambda + n)c_n(\lambda)z^{n+\lambda}
\]

\[
= \sum_{n=0}^{\infty} c_n(\lambda)R'(\lambda + n)(\lambda + n + a_1) z^{n+\lambda}
\]

\[
= \sum_{n=0}^{\infty} c_n(\lambda)R(\lambda + n)z^{n+\lambda}.
\]

Thus, the lemma is established by induction. □

Now we can substitute the solution (4.6) into the left side of (4.2) to get

\[
P(\theta)F - zR(\theta)F = \sum_{n=0}^{\infty} P(\lambda + n)c_n(\lambda)z^{n+\lambda} - \sum_{n=1}^{\infty} R(\lambda + n - 1)c_{n-1}(\lambda)z^{n+\lambda}
\]

\[
= P(\lambda)c_0(\lambda)z^{\lambda} + \sum_{n=1}^{\infty} [P(\lambda + n)c_n(\lambda) - R(\lambda + n - 1)c_{n-1}(\lambda)] z^{n+\lambda}
\]

\[
= P(\lambda)c_0(\lambda)z^{\lambda}
\]

whenever

\[
P(\lambda + n)c_n(\lambda) = R(\lambda + n - 1)c_{n-1}(\lambda)
\]

for all \( n \geq 1 \). We can assume that \( c_0(\lambda) \neq 0 \) so that we have a nontrivial solution. Thus, we get solutions whenever \( \lambda \) satisfies the *indicial equation* \( P(\lambda) = 0 \) (we call these roots the *indices*) and the coefficients satisfy the recurrence relation \( c_n(\lambda) = c_{n-1}(\lambda) \frac{R(\lambda + n - 1)}{R(\lambda + n)} \) for all \( n \geq 1 \). Of course, in general this recurrence relation must make sense so that it is necessary for \( P(\lambda + n) \) to not be equal to 0 for all \( n \geq 1 \).

36
To find the solutions corresponding to the higher multiplicity of the repeated indices, we apply the values of the $b_j$ then expression (4.9) for $F$ this condition, one will always have a solution of the form relation will fail to make sense if the differences between any two of the $b_j$ are nonzero integers. Under this condition, one will always have a solution of the form

$$F_1(z) = z^{1-b_1} \sum_{n=0}^{\infty} \prod_{j=1}^{Q} \frac{(1-b_1+a_j)_n}{(1+b_j)_n} z^n.$$  \hfill (4.8)

If the parameters are given as in the case of the $\alpha$-scaling zeta functions with the convention $b_1 = 1$, then expression (4.9) for $F_1(z)$ reduces to the desired generalized hypergeometric function. If the values of the $b_j$ are all distinct, then one finds $Q$ linearly independent solutions to (4.2) around $z = 0$ to be

$$F_m(z) = z^{1-b_m} \sum_{n=0}^{\infty} \prod_{j=1}^{Q} \frac{(1-b_m+a_j)_n}{(1+b_j)_n} z^n.$$  \hfill (4.10)

The collection of parameters that we are interested in will generally have repeats among the $b_j$, and we will henceforth assume that $b_1 = 1$. To keep track of these repeats, assume that we have chosen an ordering for the $b_j$ such that we have integers $(m_j)_{j=0}^{t}$ with $m_0 = 0$, $1 \leq m_1 < m_2 < \cdots < m_t \leq Q$, $b_1 = \cdots = b_{m_1}$, $b_{m_1+1} = \cdots = b_{m_2}$, $\cdots$, $b_{m_{t-1}+1} = \cdots = b_{m_t}$, and the values $b_{m_1}, b_{m_1}, b_{m_1+1}, \ldots, b_Q$ are all distinct.

For each of the distinct values for our parameters we will have a solution of the form (4.11). To find the solutions corresponding to the higher multiplicity of the repeated indices, we apply the
theory of Frobenius to see that the solutions for \( r = 1, \ldots, t \) and \( j = 1, \ldots, m_r - m_{r-1} \) are given by

\[
F_{m_r-j+r}(z) = \left. \left( \frac{d^{j-1}}{d\lambda^{j-1}} z^\lambda \sum_{n=0}^\infty c_n(\lambda) z^n \right) \right|_{\lambda = 1-b_{m_r}}.
\]

\[
= z^{1-b_{m_r}} \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} (\log(z))^\ell \sum_{n=0}^\infty \left( \frac{d^{j-\ell-1}}{d\lambda^{j-\ell-1}} c_n(\lambda) \right|_{\lambda = 1-b_{m_r}} z^n. \tag{4.12}
\]

All solutions presented here involving power series have radius of convergence equal to the minimal distance between the center of the series and the next nearest singularity, namely the convergence is on a unit disc.

### 4.3.2 Solutions Around \( z = \infty \)

To study the differential equation at \( z = \infty \) we need the substitution \( z_1 = \frac{1}{z} \) and the operator \( \theta_1 = z_1 \frac{d}{dz_1} \). Let \( P_1(x) = (x - a_1) \cdots (x - a_Q) \) and \( R_1(x) = (x + 1 - b_1) \cdots (x + 1 - b_Q) \) and assume we have a solution of the form

\[
F^\infty(z_1) = \sum_{n=0}^\infty v_n(\lambda) z_1^{n+\lambda}.
\]

Then

\[
P_1(\theta_1) F(z_1) = \sum_{n=0}^\infty P_1(\lambda + n) v_n(\lambda) z_1^{n+\lambda}
\]

\[
R_1(\theta_1) F(z_1) = \sum_{n=0}^\infty R_1(\lambda + n) v_n(\lambda) z_1^{n+\lambda}.
\]

Plugging in to the left side of (4.5), we get

\[
P_1(\theta_1) F(z_1) - z_1 R_1(\theta_1) F(z_1)
\]

\[
= P_1(\lambda) v_0(\lambda) z_1^{\lambda} + \sum_{n=1}^\infty [P_1(\lambda + n) v_n(\lambda) - R_1(\lambda + n - 1) v_{n-1}(\lambda)] z_1^{n+\lambda}
\]

\[
= 0
\]

whenever \( \lambda \in \{a_1, \ldots, a_Q\} \) and

\[
v_n(\lambda) = v_{n-1}(\lambda) \frac{R_1(\lambda + n - 1)}{P_1(\lambda + n)}
\]

\[
= v_0(\lambda) \prod_{j=1}^Q \frac{(\lambda + 1 - b_j)_n}{(\lambda + 1 - a_j)_n},
\]

38
In the case of the $\alpha$-scaling zeta functions, no two of the parameters $a_1, \ldots, a_Q$ will differ by an integer, so we can write a local basis of solutions around $z = \infty$, or $z_1 = 0$, as

$$F_m^\infty(z) = z^{-a_m} Q F_{Q-1} \left( (a_m + 1 - b_j)_{j=1}^Q : (a_m + 1 - a_j)_{j \neq m} : \frac{1}{z} \right)$$  \hspace{1cm} (4.14)

where the $Q F_{Q-1}$ series define analytic functions for $|z| > 1$.

### 4.4 Monodromy

A well known result called the Monodromy theorem states that if one can analytically continue a function along any path in a simply connected domain from one point to another, then the continuation values will always agree at the terminal point. However, when a function is analytically continued in a loop around a natural singularity of the function then one might return to the initial point with a different value for the continuation.

The basic example is the logarithm $\log(z) = \ln|z| + i \arg(z)$. For any point $z_0 \neq 0$ one can give a Taylor series expansion of the logarithm $f(z)$ centered at $z_0$ with radius of convergence equal to $|z_0|$. When one analytically continues this function along the loop $\gamma(t) = |z_0| e^{it}$ where $t$ starts at $\text{Arg}(z_0)$ and terminates at $\text{Arg}(z_0) + 2\pi$, one arrives at the value $\tilde{f}(z_0) = f(z_0) + 2\pi i$. Due to the fact that the logarithm will change in value by $2\pi i$ whenever analytically continued once around the singularity at $z = 0$, there is no way to express it as a single-valued function on any punctured disc around $z = 0$. However, the concept of Riemann surfaces can be used to give a covering space of $\mathbb{C} \setminus \{0\}$ to serve as a natural domain of definition for the logarithm making it single-valued. The Riemann surface associated with the logarithm is the helicoid, and one can see that analytically continuing along the aforementioned loop results in viewing the function on the next branch of the covering.

We wish to investigate the behavior of the solutions to (4.2) as they are analytically continued around the singularities at 0, 1, and $\infty$. Notice first, that the analytic continuation is entirely dependent on the homotopy equivalence classes of loops around the singularities. We will arbitrarily
choose the base point \( z_0 = \frac{1}{2} \) in \( A := \hat{C} \setminus \{0, 1, \infty\} \). For any element of the fundamental group \( \pi_1(A, z_0) \) and a local solution space \( V \) at \( z_0 \), one analytically continues the elements of \( V \) along a representative loop and arrives back at the base point with new values for the continued functions.

Define a map on the germs of analytic functions at a point \( C_0(f) \) to be the analytic continuation of \( f \) counterclockwise along the path \( \gamma_0(t) = \frac{1}{2} e^{2\pi i t} \) where \( 0 \leq t \leq 1 \). Then by the Monodromy theorem, \( C_0(f) = f \) whenever \( f \) is analytic in a neighborhood of the disc \( \{|z| \leq \frac{1}{2}\} \), and \( C_0(z^\lambda) = e^{2\pi i \lambda} z^\lambda \).

Furthermore, \( C_0(fg) = C_0(f)C_0(g) \) and \( C_0(f + g) = C_0(f) + C_0(g) \).

When a function \( f \) is a solution to some homogeneous linear ordinary differential equation with rational coefficient functions, then \( C_0(f) \) is a solution to that same differential equation. If we have a basis \( (F_j)_{j=1}^n \) of solutions for such an equation of order \( n \), then we can express the \( C_0(F_j) \) as linear combinations of the basis elements as follows:

\[
\begin{align*}
d_{11} F_1 + \cdots + d_{1n} F_n &= C_0 F_1 \\
d_{21} F_1 + \cdots + d_{2n} F_n &= C_0 F_2 \\
& \quad \vdots \quad = \quad \vdots \\
d_{n1} F_1 + \cdots + d_{nn} F_n &= C_0 F_n.
\end{align*}
\]

Define the **monodromy matrix at** \( z = 0 \) of the differential equation, relative to the local basis \( V = (F_j)_{j=1}^n \) of solutions, to be the square matrix of coefficients

\[
M_0 := \begin{bmatrix}
d_{11} & d_{12} & \cdots & d_{1n} \\
d_{21} & d_{22} & \cdots & d_{2n} \\
& & \ddots & \vdots \\
d_{n1} & d_{n2} & \cdots & d_{nn}
\end{bmatrix}.
\]

One can then define the **monodromy representation**

\[
M : \pi_1(\hat{C} \setminus P, z_0) \rightarrow GL(V),
\]
where \( P \) is the collection of singularities of the equation, as the map taking elements of the fundamental group to the corresponding monodromy matrices. The fundamental group is generated by the equivalence classes of loops encircling exactly one of the singularities once, call them \( [\gamma_{p_j}] \) where \( p_1, \ldots, p_m \) are the singularities, with the relation \( [\gamma_{p_1}] \cdots [\gamma_{p_m}] = 1 \). Upon analogously defining the monodromy matrices around \( z = p_j \) for all \( j \) relative to the same base point \( z_0 \), one then has the relation \( M_{p_1}M_{p_2} \cdots M_{p_m} = I_n \). We note in passing that for a fixed base point, the monodromy matrices are unique up to conjugation by elements of \( GL_n(\mathbb{C}) \) as this just corresponds to a change of basis.

### 4.5 Motivating Examples for Monodromy

The differential equation for the square root function is

\[
y' - \frac{1}{2z} y = 0.
\]

This equation has regular singularities at \( z = 0, \infty \) and a basis which is just the function \( y = z^{\frac{1}{2}} \).

Of course, one can employ the method of Frobenius to determine this solution, but any basic course in differential equations provides a much quicker method. The index is nonetheless given by \( \lambda = \frac{1}{2} \) and the recurrence relation is identically zero for all coefficients beyond the lowest power. Thus, we have

\[
C_0 z^{\frac{1}{2}} = e^{2\pi i (\frac{1}{2})} z^{\frac{1}{2}} = -z^{\frac{1}{2}}
\]

so that the associated \( 1 \times 1 \) monodromy matrix is

\[
M_0 = (-1).
\]

Note that since we have the relation \( [\gamma_0] = [\gamma_{\infty}]^{-1} \), we also have that \( M_0 = M_{\infty}^{-1} \) and hence the group generated by the two matrices (also known as the monodromy group) is just the group generated by \( M_0 \). Clearly, this group is isomorphic to \( \langle \mathbb{Z}_2, + \rangle \).
This information determines the appropriate Riemann surface to take as a definition of the domain of $z^{1/2}$. One takes a covering space of $\mathbb{C} \setminus \{0\}$ with two sheets glued together along the branch cut in the typical definition of $z^{1/2}$ in terms of the logarithm to achieve this surface.

The logarithmic differential equation with rational function coefficients

$$y'' + \frac{1}{z} y' = 0$$

has regular singularities at $z = 0, \infty$, and a basis of solutions near $z = 0$ given by $y_1 = 1$ and $y_2 = \log(z)$. We see that

$$C_0 y_1 = y_1$$
$$C_0 y_2 = y_2 + 2\pi i y_1$$

so that the associated monodromy matrix is given by

$$M_0 = \begin{bmatrix} 1 & 0 \\ 2\pi i & 1 \end{bmatrix}.$$  

Note that once again we have the relation $[\gamma_0] = [\gamma_\infty]^{-1}$ so that $M_0 = M_\infty^{-1}$, and hence the monodromy group is generated by $M_0$. Since we have for all $n \in \mathbb{Z}$ that

$$M_0^n = \begin{bmatrix} 1 & 0 \\ n(2\pi i) & 1 \end{bmatrix},$$

it is clear that the monodromy group is isomorphic to $\mathbb{Z}$. This tells us that the appropriate Riemann surface, as a definition of the domain of $\log(z)$, is the helicoid. One takes a covering space of $\mathbb{C} \setminus \{0\}$ with infinitely many sheets glued together along the branch cuts in the typical definition of $\log(z)$ to achieve this surface.

4.5.1 The Generalized Hypergeometric Equation

In the case of the generalized hypergeometric equation, we have two different local bases of solutions for the singularities at $z = 0, \infty$. The basis given by (4.11) and (4.13) is not the same
basis that we will use to determine the monodromy at \( z = \infty \) due to the fact that the solutions are only valid locally, and one would need to give analytic continuation formulas for the full basis to investigate the monodromy representation for a single base point. We will give monodromy matrices for solutions around each singularity at \( z = 0, \infty \) relative to the local Frobenius bases at each point instead. Due to the relation between equivalence classes of loops we have \( M_1 = M_0^{-1}M_{\infty}^{-1} \), so we need only give expressions for the monodromy matrices at \( z = 0, \infty \) (or alternatively we could give monodromy matrices for any two singularities).

If the \( b_j \) parameters are all distinct and no pair differs by an integer, then the local basis is given in (4.11) which allows us to write

\[
C_0 F_m(z) = e^{2\pi i (1-b_m)} F_m(z)
\]

for all \( m = 1, \ldots, Q \). Thus, the monodromy matrix at \( z = 0 \) relative to this basis is given by

\[
M_0 F = \begin{bmatrix}
  e^{2\pi i (1-b_1)} & 0 & 0 & \ldots & 0 & 0 \\
  0 & e^{2\pi i (1-b_2)} & 0 & \ldots & 0 & 0 \\
  0 & 0 & e^{2\pi i (1-b_3)} & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & e^{2\pi i (1-b_{Q-1})} & 0 \\
  0 & 0 & 0 & \ldots & 0 & e^{2\pi i (1-b_Q)}
\end{bmatrix}
\begin{bmatrix}
  F_1 \\
  F_2 \\
  F_3 \\
  \vdots \\
  F_{Q-1} \\
  F_Q
\end{bmatrix}
= \begin{bmatrix}
  C_0 F_1 \\
  C_0 F_2 \\
  C_0 F_3 \\
  \vdots \\
  C_0 F_{Q-1} \\
  C_0 F_Q
\end{bmatrix}
\]

(4.15)

The case for distinct parameters modulo 1 has been studied, and some particularly useful exposition about this case is given in [Beu09]. Unfortunately, the general case for the \( \alpha \)-scaling zeta functions includes repeated indices and will be more complicated. The first three solutions corresponding to a repeated index of order greater than or equal to three in the case of the generalized
hypergeometric equation at $z = 0$ are given by

$$F_{m_r+1}(z) = z^{1-b_m} \sum_{n=0}^{\infty} c_n (1 - b_m) z^n$$

$$F_{m_r+2}(z) = z^{1-b_m} \sum_{n=0}^{\infty} c'_n (1 - b_m) z^n + \log(z) z^{1-b_m} \sum_{n=0}^{\infty} c_n (1 - b_m) z^n$$

$$F_{m_r+3}(z) = z^{1-b_m} \sum_{n=0}^{\infty} c''_n (1 - b_m) z^n + 2 \log(z) z^{1-b_m} \sum_{n=0}^{\infty} c'_n (1 - b_m) z^n + \log(z)^2 z^{1-b_m} \sum_{n=0}^{\infty} c_n (1 - b_m) z^n$$

and their analytic continuations once around $z = 0$ counterclockwise are given by

$$C_0 F_{m_r+1}(z) = e^{2\pi i (1-b_m)} F_{m_r+1}(z)$$

$$C_0 F_{m_r+2}(z) = e^{2\pi i (1-b_m)} \left[ z^{1-b_m} \sum_{n=0}^{\infty} c'_n (1 - b_m) z^n + \log(z) z^{1-b_m} \sum_{n=0}^{\infty} c_n (1 - b_m) z^n \right]$$

$$C_0 F_{m_r+3}(z) = e^{2\pi i (1-b_m)} \left[ z^{1-b_m} \sum_{n=0}^{\infty} c''_n (1 - b_m) z^n + 2 \log(z) z^{1-b_m} \sum_{n=0}^{\infty} c'_n (1 - b_m) z^n + \log(z)^2 z^{1-b_m} \sum_{n=0}^{\infty} c_n (1 - b_m) z^n \right]$$

Thus, in this case there will be a block along the diagonal of the monodromy matrix $M_0$ at $z = 0$ relative to this basis that appears as

$$e^{2\pi i (1-b_m)} \begin{bmatrix} 1 & 0 & 0 \\ 2\pi i & 1 & 0 \\ (2\pi i)^2 & 2(2\pi i) & 1 \end{bmatrix}$$

### 4.5.2 Hypergeometric Monodromy: Example 1

At the beginning of this chapter, we discussed the example $k = (1, 2, 2, 3)$. After cancellation and reordering, this example yields the parameters $a = (a_j)_{j=1}^6 = \left( \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5} \right)$ and
\[ b = (b_j)_{j=2}^6 = (1, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}) \] (recall that \( b_1 \) is always 1 for the zeta functions). Thus the relevant generalized hypergeometric differential equation is

\[
\theta^3 \left( \theta - \frac{1}{2} \right) \left( \theta - \frac{2}{3} \right) \left( \theta - \frac{1}{3} \right) F(z) = z \left( \theta + \frac{1}{8} \right) \left( \theta + \frac{1}{4} \right) \left( \theta + \frac{3}{8} \right) \left( \theta + \frac{5}{8} \right) \left( \theta + \frac{3}{4} \right) \left( \theta + \frac{7}{8} \right) F(z)
\]

and the indices at \( z = 0 \) are 0, 0, 0, \( \frac{1}{2} \), \( \frac{2}{3} \), \( \frac{1}{3} \). The basis given in section 4.3.1 is

\[
F_1(z) = \sum_{n=0}^\infty c_n(0)z^n
\]

\[
F_2(z) = \sum_{n=0}^\infty c_n'(0)z^n + \log(z) \sum_{n=0}^\infty c_n(0)z^n
\]

\[
F_3(z) = \sum_{n=0}^\infty c_n''(0)z^n + 2\log(z) \sum_{n=0}^\infty c_n'(0)z^n + (\log(z))^2 \sum_{n=0}^\infty c_n(0)z^n
\]

\[
F_4(z) = z^{\frac{1}{2}} \sum_{n=0}^\infty c_n \left( \frac{1}{2} \right) z^n
\]

\[
F_5(z) = z^{\frac{2}{3}} \sum_{n=0}^\infty c_n \left( \frac{2}{3} \right) z^n
\]

\[
F_6(z) = z^{\frac{1}{3}} \sum_{n=0}^\infty c_n \left( \frac{1}{3} \right) z^n.
\]

After analytic continuation, we have

\[ C_0 F_1(z) = F_1(z) \]

\[ C_0 F_2(z) = F_2(z) + 2\pi i F_1(z) \]

\[ C_0 F_3(z) = F_3(z) + 2(2\pi i) F_2(z) + (2\pi i)^2 F_1(z) \]

\[ C_0 F_4(z) = e^{2\pi i(\frac{1}{2})} F_4(z) \]

\[ C_0 F_5(z) = e^{2\pi i(\frac{2}{3})} F_5(z) \]

\[ C_0 F_6(z) = e^{2\pi i(\frac{1}{3})} F_6(z). \]
Thus, the monodromy matrix $M_0$ at $z = 0$ relative to this basis of solutions is given by

$$M_0 F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2\pi i & 1 & 0 & 0 & 0 & 0 \\ (2\pi i)^2 & 2(2\pi i) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi i(\frac{1}{2})} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi i(\frac{1}{2})} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2\pi i(\frac{1}{2})} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} \begin{bmatrix} C_0 F_1 \\ C_0 F_2 \\ C_0 F_3 \\ C_0 F_4 \\ C_0 F_5 \\ C_0 F_6 \end{bmatrix} \quad (4.16)$$

The local basis of solutions for $z = \infty$ as given in (4.14) is

$$F^\infty_m(z) = z^{-a_m} \sum_{n=0}^{\infty} v_n(a_m) \left( \frac{1}{z} \right)^n$$

for all $m = 1, \ldots, 6$. If we define $C_\infty$ in the obvious way, then we have for all $m = 1, \ldots, 6$ that

$$C_\infty F^\infty_m(z) = e^{2\pi i(a_m)} F^\infty_m(z)$$

so that the monodromy matrix relative to this basis near $z = \infty$ is given by

$$M_\infty F = \begin{bmatrix} e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2\pi i(\frac{3}{2})} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi i(\frac{3}{2})} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi i(\frac{5}{2})} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2\pi i(\frac{5}{2})} \end{bmatrix} \begin{bmatrix} F_1^\infty \\ F_2^\infty \\ F_3^\infty \\ F_4^\infty \\ F_5^\infty \\ F_6^\infty \end{bmatrix} \begin{bmatrix} C_\infty F_1^\infty \\ C_\infty F_2^\infty \\ C_\infty F_3^\infty \\ C_\infty F_4^\infty \\ C_\infty F_5^\infty \\ C_\infty F_6^\infty \end{bmatrix} \quad (4.17)$$

### 4.5.3 Hypergeometric Monodromy: Example 2

Let $k = (1, 2, 2, 2)$ be a generator. This yields a collection of parameters $a = (\frac{1}{7})_{j=1}^{6}$ and $b = (1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{7}, \frac{1}{7})$ after cancellation and reordering. The indices at $z = 0$ are the values $0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{7}$
and so we have a local solution space near $z = 0$ spanned by the functions

$$F_1(z) = \sum_{n=0}^{\infty} c_n(0) z^n$$

$$F_2(z) = \sum_{n=0}^{\infty} c'_n(0) z^n + \log(z) \sum_{n=0}^{\infty} c_n(0) z^n$$

$$F_3(z) = \sum_{n=0}^{\infty} c''_n(0) z^n + 2 \log(z) \sum_{n=0}^{\infty} c'_n(0) z^n + (\log(z))^2 \sum_{n=0}^{\infty} c_n(0) z^n$$

$$F_4(z) = z^{\frac{1}{2}} \sum_{n=0}^{\infty} c_n \left(\frac{1}{2}\right) z^n$$

$$F_5(z) = z^{\frac{1}{2}} \sum_{n=0}^{\infty} c'_n \left(\frac{1}{2}\right) z^n + z^{\frac{1}{3}} \log(z) \sum_{n=0}^{\infty} c_n \left(\frac{1}{2}\right) z^n$$

$$F_6(z) = z^{\frac{1}{2}} \sum_{n=0}^{\infty} c''_n \left(\frac{1}{2}\right) z^n + 2 z^{\frac{1}{2}} \log(z) \sum_{n=0}^{\infty} c'_n \left(\frac{1}{2}\right) z^n$$

$$+ z^{\frac{1}{2}} (\log(z))^2 \sum_{n=0}^{\infty} c_n \left(\frac{1}{2}\right) z^n.$$

Thus, we have

$$C_0 F_1(z) = F_1(z)$$

$$C_0 F_2(z) = F_2(z) + 2\pi i F_1(z)$$

$$C_0 F_3(z) = F_3(z) + 2(2\pi i) F_2(z) + (2\pi i)^2 F_1(z)$$

$$C_0 F_4(z) = e^{2\pi i \left(\frac{1}{2}\right)} F_4(z)$$

$$C_0 F_5(z) = e^{2\pi i \left(\frac{1}{2}\right)} [F_5(z) + 2\pi i F_4(z)]$$

$$C_0 F_6(z) = e^{2\pi i \left(\frac{1}{2}\right)} [F_6(z) + 2(2\pi i) F_5(z) + (2\pi i)^2 F_4(z)]$$

47
so that the monodromy matrix about $z = 0$ relative to this basis is given by

$$M_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
2\pi i & 1 & 0 & 0 & 0 & 0 \\
(2\pi i)^2 & 2(2\pi i) & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{2\pi i(\frac{1}{2})} & 0 & 0 \\
0 & 0 & 0 & 2\pi i e^{2\pi i(\frac{1}{2})} & e^{2\pi i(\frac{1}{2})} & 0 \\
0 & 0 & 0 & (2\pi i)^2 e^{2\pi i(\frac{1}{2})} & 2(2\pi i) e^{2\pi i(\frac{1}{2})} & e^{2\pi i(\frac{1}{2})}
\end{bmatrix}.$$ 

The local basis of solutions for $z = \infty$ as given in (4.14) is

$$F_\infty^m(z) = z^{-\frac{m}{7}} \sum_{n=0}^{\infty} v_n \left(\frac{m}{7}\right) \left(\frac{1}{z}\right)^n$$

and

$$C_\infty F_\infty^m(z) = e^{2\pi i\left(\frac{m}{7}\right)} F_\infty^m(z)$$

for all $m = 1, \ldots, 6$ so that the monodromy matrix relative to this basis near $z = \infty$ is given by

$$M_\infty F = \begin{bmatrix}
e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 & 0 \\
e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 & 0 \\
e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 & 0 \\
e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 & 0 \\
e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 & 0 \\
e^{2\pi i(\frac{1}{2})} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_1^\infty \\ F_2^\infty \\ F_3^\infty \\ F_4^\infty \\ F_5^\infty \\ F_6^\infty \end{bmatrix} = \begin{bmatrix} C_\infty F_1^\infty \\ C_\infty F_2^\infty \\ C_\infty F_3^\infty \\ C_\infty F_4^\infty \\ C_\infty F_5^\infty \\ C_\infty F_6^\infty \end{bmatrix}.$$ 

(4.18)

4.6 The Case $N = 2$

In the case of the multifractal zeta function arising from an iterated function system containing only two contraction similarities, one will not have to worry about the possibility of repeated parameters. Since we have that $N = 2$, all generators are of the form $k = (k_1, k_2)$. One will still have some cancellation so that $Q < K = k_1 + k_2$. 

48
Theorem 4  The monodromy matrix at \( z = 0 \) with respect to the local Frobenius basis for a uniquely generated self-similar multifractal formed by an iterated function system with exactly two contraction similarities is diagonal.

Proof. Let \( k \) be a generator. The values of \( b_j \) after cancellation are a subset of \( \{ \frac{j}{k_i} \}_{j=1}^{k_i-1} \cup \{ \frac{j}{k_2} \}_{j=1}^{k_2-1} \) since there is exactly one value 1 in both the \( b_j \) and \( a_j \) parameters (if \( k_\ell \) is 0 or 1, then assume \( \{ \frac{j}{k_\ell} \}_{j=1}^{k_\ell-1} = \phi \) so that the statement is trivial). Suppose by way of contradiction that there exist integers \( 1 \leq n < k_1 \) and \( 1 \leq m < k_2 \) such that \( \frac{n}{k_1} = \frac{m}{k_2} \). Then \( nk_2 = mk_1 \), and since \( \gcd(k_1,k_2) = 1 \) we conclude that \( k_1 \) divides \( n \), a contradiction. Therefore there are no repeated indices for the corresponding generalized hypergeometric differential equation. ■

4.6.1 Uniquely Generated Example

The following example was initially treated in [EL09] where the holomorphic continuation formula was given. We begin with a uniquely generated self-similar multifractal with exactly two contraction similarities. For any generator \( k \), the \( \alpha \)-scaling zeta function is given by

\[
\zeta_{r,p}(\alpha,s) = \sum_{n=1}^{\infty} \left( \frac{nK}{nk_1,nk_2} \right) \ell(k)^{ns}.
\]

We have already seen that the Barnes integral gives holomorphic continuation of \( \zeta \) to the region \( \mathbb{C} \setminus \left\{ \left( \frac{\log \beta(k)}{\log \ell(k)} - r \right) + \frac{2\pi i}{\log \ell(k)} j \mid r \geq 0, j \in \mathbb{Z} \right\} \). However, one may find a simpler form to express the continued zeta function in this case. In [EL09] it was shown by applying the theory of generalized hypergeometric series that for any point \( p \) in \( \left\{ \frac{\log \beta(k)}{\log \ell(k)} + \frac{2\pi i}{\log \ell(k)} j \mid j \in \mathbb{Z} \right\} \), one can find functions \( H_p \) and \( G_p \), holomorphic in a neighborhood of \( p \), such that in this neighborhood we have

\[
\zeta_{r,p}(\alpha,s) = \frac{H_p(s)}{\sqrt{1 - \beta(k)\ell(k)^s}} + G_p(s).
\]
In the case where $k = (1, 1)$, $K = 2$, and $\beta(k) = 4$, the functions $H_p$ and $G_p$ have been given explicitly. First note that $\left( -\frac{1}{2} \right) = (2n)(-1)^n 4^{-n}$. Applying the binomial theorem yields

$$\frac{1}{\sqrt{1 - 4\ell(k)^s}} = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) (-4)^n \ell(k)^{ns} = \sum_{n=0}^{\infty} \left( \frac{2n}{n} \right) (-1)^n 4^{-n} (-4)^n \ell(k)^{ns} = \sum_{n=0}^{\infty} \left( \frac{2n}{n} \right) \ell(k)^{ns} = 1 + \zeta_{r,p}(\alpha, s)$$

so one easily sees that $H_p(s) = 1$, $G_p(s) = -1$, and that this formula gives the holomorphic continuation to all of $\mathbb{C}$ except for the branch cuts.

Furthermore, one sees that the generalized hypergeometric equation is first order in this case, and so the above expression corresponds to the one-solution basis near $z = 1$. Thus, we have the monodromy matrices

$$M_1 = (-1), \ M_0 = (1), \text{ and } M_\infty = (-1).$$

The corresponding monodromy group is thus isomorphic to $\mathbb{Z}_2$. Recalling that the differential equation is analyzed through the function $z = \beta(k)\ell(k)^s$, we have the two-sheet covering of $\mathbb{C} \setminus \{1\}$ essentially the same as that of the standard square root function as the desired Riemann surface domain associated to the $\alpha$-scaling zeta function.
Chapter 5

Future Work

We are interested in determining the monodromy group with presentation
\[ \langle M_0, M_1, M_\infty | M_0M_1M_\infty = I \rangle, \]
where the matrices are given relative to the same local basis around some base point. This will likely involve continuation techniques like the use of Barnes integrals.

The monodromy group associated with a differential equation is directly used to determine a suitable Riemann surface on which to define the solution space. This was in fact the motivating idea behind the introduction of differential equations into the current analysis of the \( \alpha \)-scaling zeta functions.

A full classification of scaling ratios and probabilities in the construction of uniquely generated self-similar multifractals is yet to be discovered. An interesting result might be to show that in the case of a self-similar multifractal which is not uniquely generated, a particular scaling regularity might only have finitely many associated generators. If this is the case, then one can express the \( \alpha \)-scaling zeta function associated to such a scaling regularity as a finite sum of series of the form (3.7) corresponding to each generator and subsequently apply the theory of chapter 4 to each of these series.

This theory is restricted to multifractal measures defined on subsets of \( \mathbb{R} \). Another interesting extension would be to define analogous zeta functions for higher dimensional multifractal
measures. The classical theory of fractal strings as discussed at length in [LvF13] was recently shown to extend naturally to higher dimensions as shown in [LRZ17]. If a higher dimensional analogue of the $\alpha$-scaling zeta functions is to be discovered, then it will likely be done in the spirit of the work in [LRZ17].

The poles of the geometric zeta function for a self-similar fractal string have been shown in [LRZ17] and [LvF13] to describe the tube formula for the volume of the inner epsilon neighborhood of the boundary. The known example for the Cantor string was presented at the end of chapter 2. Another open problem is to give a geometric interpretation of the singularities of the $\alpha$-scaling zeta functions in a similar light to that of the geometric and distance zeta functions in [LRZ17] and [LvF13] by describing analogous quantities related to the geometry of the sets and measures being studied.
Bibliography


