Title
LAMB SHIFT IN A STRONG COULOMB POTENTIAL

Permalink
https://escholarship.org/uc/item/8f42m928

Author
Mohr, Peter J.

Publication Date
1975-01-24
LAMB SHIFT IN A STRONG COULOMB POTENTIAL

Peter J. Mohr

January 24, 1975

Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48

For Reference
Not to be taken from this room
DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.
Results of a calculation of the self-energy radiative level shift of order $\alpha$ of the $2S_{1/2}$ and $2P_{3/2}$ states in a strong Coulomb potential are given. The shift is evaluated numerically to all orders in $Z\alpha$ for $Z$ in the range 10-110. An estimate is obtained for the effect of terms of high order in $Z\alpha$ on the Lamb shift in hydrogen. With this estimate taken into account, the theoretical value is $S = 1057.864(14)$ MHz.
Recent experiments with a variety of hydrogen-like ions have determined the values of the Lamb shift in these systems.\textsuperscript{1} Comparison of these values to the values of the Lamb shift predicted by quantum electrodynamics is one of the fundamental tests of the theory. Furthermore, carrying out this comparison over a wide range of values of the nuclear charge $Z$ is important in order to test whether the theory correctly predicts the $Z$ dependence of the Lamb shift. For atomic hydrogen, new experimental techniques in measuring the Lamb shift give promise of increasing the precision of the measurements by an order of magnitude.\textsuperscript{2} Similar precision in the theory requires knowledge of the contribution of terms of high order in $Z\alpha$. For high-$Z$ atoms, a comparison can be made between the experimental and theoretical binding energies of the innermost electrons.\textsuperscript{3} In this case the theoretical value of the radiative level shift in a Coulomb potential with nuclear charge $Z$ provides a first approximation to the shift of the corresponding level in the neutral atom with the same $Z$. For the applications listed above, it is necessary to have accurate values predicted by quantum electrodynamics for the radiative shift of levels in a Coulomb potential for a wide range of $Z$.

In this letter we report the results of a calculation of the self-energy contribution to the Lamb shift of electron levels in a strong Coulomb potential. The self-energy radiative level shift of order $\alpha$ of the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states, corresponding to the Feynman
diagram in Fig. 1(a), is considered to all orders in $Z\alpha$. We have evaluated it numerically with no approximations by a slightly modified version of a method used previously to evaluate the $1S_2$ state self-energy.\(^4\) The evaluation has been done for values of $Z$ given by $Z = 10, 20, 30, \ldots, 110$. We have estimated the small $Z$ Lamb shift by extrapolating the calculated values with a procedure which takes into account the known behavior of the Lamb shift at small $Z$. Our method of evaluating the energy shifts is based on the expansion of the bound electron propagator in terms of the known Coulomb radial Green's functions. This expansion was used by Wichmann and Kroll in their study of vacuum polarization.\(^5\) We employ the covariant regulator scheme to carry out the mass renormalization. Divergent terms and terms of order lower than $(Z\alpha)^4$ are isolated and treated analytically. Detailed results and modifications of the method necessary for the $n = 2$ states are described in a forthcoming paper.

Because of the approximate $Z^4/n^3$ scaling of the self-energy level shift $\Delta E$, it is convenient to express the shift of each state in terms of a slowly varying function $F(Z\alpha)$ defined by

$$\Delta E = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{n^3} F(Z\alpha) mc^2$$

(1)

where $n$ is the principal quantum number of the state. Values we have obtained for $F(Z\alpha)$ are listed in Table I. The numbers in parentheses
are estimates of the uncertainty associated with the numerical integration in the evaluation of $F(Z\alpha)$. Only the self-energy corresponding to Fig. 1(a) is included in $F(Z\alpha)$. Values of $F(Z\alpha)$ for intermediate values of $Z$ can easily be obtained by ordinary polynomial interpolation. Erickson has obtained an approximation for the $Z$ dependence of the Lamb shift which agrees qualitatively with our results.\textsuperscript{6}

Table I. Values for the function $F(Z\alpha)$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$2S_{\frac{1}{2}}(Z\alpha)$</th>
<th>$2P_{\frac{1}{2}}(Z\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.893(2)</td>
<td>-0.1145(4)</td>
</tr>
<tr>
<td>20</td>
<td>3.5065(4)</td>
<td>-0.0922(4)</td>
</tr>
<tr>
<td>30</td>
<td>2.8391(3)</td>
<td>-0.0641(4)</td>
</tr>
<tr>
<td>40</td>
<td>2.4550(3)</td>
<td>-0.0308(4)</td>
</tr>
<tr>
<td>50</td>
<td>2.2244(2)</td>
<td>0.0082(3)</td>
</tr>
<tr>
<td>60</td>
<td>2.0946(4)</td>
<td>0.0549(3)</td>
</tr>
<tr>
<td>70</td>
<td>2.0435(8)</td>
<td>0.1129(3)</td>
</tr>
<tr>
<td>80</td>
<td>2.065(2)</td>
<td>0.1884(3)</td>
</tr>
<tr>
<td>90</td>
<td>2.169(3)</td>
<td>0.2934(3)</td>
</tr>
<tr>
<td>100</td>
<td>2.387(3)</td>
<td>0.453(1)</td>
</tr>
<tr>
<td>110</td>
<td>2.798(3)</td>
<td>0.725(2)</td>
</tr>
</tbody>
</table>

We obtain an estimate for the contribution of the terms of high order in $Z\alpha$ to the small $Z$ Lamb shift as follows. We isolate the known low order terms of the self-energy contribution to the
Lamb shift $S_{SE} = \Delta E(2S_\frac{1}{2}) - \Delta E(2P_\frac{1}{2})$ by writing

$$S_{SE} = \frac{\alpha}{\pi} \frac{(2\alpha)^4}{6} mc^2 \left[ \ln(2\alpha)^2 - \ln \frac{K_0(2,0)}{K_0(2,1)} + \frac{11}{24} + \frac{1}{2} \right]$$

$$+ 3\pi \left( 1 + \frac{11}{128} - \frac{1}{2} \ln 2 \right) (2\alpha) - \frac{3}{4} (2\alpha)^2 \ln^2(2\alpha)^2$$

$$+ \left( \frac{299}{240} + 4\ln 2 \right) (2\alpha)^2 \ln(2\alpha)^2 + (2\alpha)^2 G_{SE}(2\alpha).$$ (2)

As a consequence of this definition, the function $G_{SE}(2\alpha)$ approaches a constant as $2\alpha \to 0$. We expect that the small $2\alpha$ behavior of $G_{SE}(2\alpha)$ has the form

$$G_{SE}(2\alpha) = a + b(2\alpha)\ln(2\alpha)^2 + c(2\alpha) + \ldots$$ (3)

where the omitted terms are higher order in $2\alpha$. This behavior is suggested by the form of the high order terms of the vacuum polarization. Fitting the function on the right-hand side of (3) to the values of $G_{SE}(2\alpha)$ corresponding to our calculated values of the self-energy at $Z = 10, 20,$ and $30$ yields a value of $G_{SE}(\alpha) = -23.4 \pm 1.2$ for hydrogen. The upper and lower limits for $G_{SE}(\alpha)$ are obtained by similar extrapolations with the value of $G_{SE}(2\alpha)$ at $2\alpha = 10$ replaced by its upper and lower limits corresponding to the uncertainty listed in Table I. Although this procedure does not give rigorous limits to the error in $G_{SE}(\alpha)$, we feel that they are valid limits to the uncertainty.
This view is supported by the fact that extrapolations from the calculated points at \( Z = 20, 30, \) and 40 and from the calculated points at \( Z = 30, 40, \) and 50 both yield a value for \( G_{SE}(\alpha) \) within the stated limits. In addition, extrapolation from the calculated points at \( Z = 10, 20, \) and 30 with an ordinary 2nd degree polynomial in \( Z\alpha \) yields a value within the limits. Fig. 2 shows the calculated values of \( G_{SE}(Z\alpha) \) from \( Z = 10 \) to 50. The error bar at \( Z = 10 \) corresponds to the uncertainty in \( F(Z\alpha) \). The point at \( Z = 1 \) is the extrapolated value \( G_{SE}(\alpha) \).

To order \( \alpha \), the remaining radiative correction to be included is the vacuum polarization corresponding to the Feynman diagram in Fig. 1(b). Wichmann and Kroll have considered this correction in detail.\(^5\) They have shown that for small \( Z \) the dominant contribution is given by the Uehling potential which is the part of the vacuum polarization linear in the external potential. In particular, they found that the part which is third order in the external potential contributes only 308 Hz to the \( 2S_{1/2} \) state level shift in hydrogen. (Even powers in the external potential give no contribution as a consequence of Furry's theorem.) We express the total vacuum polarization contribution of order \( \alpha \) to the Lamb shift as

\[
S_{VP} = \frac{\alpha}{\pi} \left( \frac{Z\alpha}{6} \right)^4 \frac{mc^2}{\hbar} \left[ -\frac{1}{5} + \frac{5}{64} \pi (Z\alpha) - \frac{1}{10} (Z\alpha)^2 \ln(Z\alpha)^{-2} + (Z\alpha)^2 G_{VP}(Z\alpha) \right].
\]
The known low order terms displayed in (4) are obtained from the Uehling potential.\textsuperscript{7} In view of the Wichmann-Kroll result, the function $G_{\text{VP}}(Z\alpha)$ is well approximated for small $Z$ by the part $G_{U}(Z\alpha)$ which arises from the Uehling potential. We have calculated the small $Z\alpha$ behavior of $G_{U}(Z\alpha)$ and obtain

$$G_{\text{VP}}(Z\alpha) \approx G_{U}(Z\alpha) = -\frac{1199}{2100} + \frac{5}{128} n(Z\alpha) \ln(Z\alpha)^{-2}$$

\[+ 0.5 \ (Z\alpha) + \ldots\]  

(5)

The sum of the self-energy and vacuum polarization contributions $G(Z\alpha) = G_{SE}(Z\alpha) + G_{\text{VP}}(Z\alpha)$ for hydrogen is $G(\alpha) = -24.0 \pm 1.2$ and gives a shift of $-0.173(9)$ MHz. The value corresponding to this correction according to the compilation of Lautrup, Peterman, and de Rafael is $-0.126(5)$ MHz.\textsuperscript{7} That value is based on a calculation of the high order binding correction by Erickson.\textsuperscript{6} We do not know the source of the discrepancy between that value and our value. Our result is consistent with the earlier estimate of Erickson and Yennie:\textsuperscript{8} $G(0) = -19.08 \pm 5$. Combining our value for the high order binding correction with the values for the other contribution listed by Lautrup, Peterman, and de Rafael, we obtain the theoretical Lamb shift value of $S = 1057.864(14)$ MHz in hydrogen. The effect of the high order binding terms is quite important in comparison to the accuracy of the recent preliminary experimental result:\textsuperscript{9} $S = 1057.893(20)$ MHz.
I am grateful to Professor Eyvind H. Wichmann for many helpful discussions. I wish to thank Professor Glen Erickson for communicating unpublished results on the Lamb shift.
REFERENCES

† Work supported by the U.S. Atomic Energy Commission.


7. B. E. Lautrup, A. Peterman, and E. de Rafael, Phys. Reports 3, 193 (1972). We note two misprints in Table 1.1 of this reference: on line 5, (Zα^2) should read (Zα)^2 and the quantity b should be -55/48 -4log2; see their Ref. 127.


FIGURE CAPTIONS

Fig. 1 Feynman diagrams for the radiative corrections of order $\alpha$:
(a) self-energy and (b) vacuum polarization.

Fig. 2 Calculated values of $G_{SE}(Z\alpha)$ for $Z = 10$ to 50 and the extrapolated value at $Z = 1$. 
Fig. 1
Fig. 2
LEGAL NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.