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6. A stable analytic foliation with only exceptional minimal sets. M. Hirsch.

Let $M$ be a compact manifold admitting a foliation $\mathcal{F}$. A subset $\Lambda \subset M$ is said to be minimal if it is (1) compact and nonempty and (2) a union of leaves and if (3) no proper subset of $\Lambda$ satisfies (1) and (2). Examples of minimal sets are a compact leaf and, if every leaf is dense, $M$ itself. All other minimal sets are called exceptional. An exceptional minimal set must be nowhere dense.

Denjoy found a $C^1$ codimension 1 foliation of $T^2$ with an exceptional minimal set and showed that this cannot happen in class $C^2$. A corollary to the Poincaré–Bendixson theorem says that if $M^2$ is an open subset of $S^2$ then a foliation of $M$ cannot have exceptional minimal sets. Raymond has an example of a $C^\infty$ foliation on $S^3$ with an exceptional minimal set.

Theorem. For $n \geq 3$ there is a compact n-manifold $M$ with a codimension 1 foliation $\mathcal{F}$ s.t.

(a) $\mathcal{F}$ is analytic.

(b) $\mathcal{F}$ is stable under $C^1$-small perturbations of the tangent field $T\mathcal{F}$.

(c) $\mathcal{F}$ has a unique minimal set $\Lambda$ and $\Lambda$ is exceptional.

(d) for $n > 3$ $\pi_1(M)$ is solvable but not polycyclic.

Proof. We shall only describe this foliation in the case $n = 3$. For any immersion $f:S^1 \to S^1$ choose an embedding $g$ of the solid torus $S^1 \times D^2$ into its interior so that $pg = fp$ where $p:S^1 \times D^2 \to S^1$ is the projection. Let $V = S^1 \times D^2 - \text{int } g(S^1 \times D^2)$. Then the boundary of $V$ is two copies of $S^1 \times S^1$ and we define a closed 3-manifold $M$ from $V$ by identifying $x$ with $g(x)$ for each $x$ on the boundary of $S^1 \times D^2$. Let $\pi:V \to M$ and $q = p|V$ be the projections. There is a unique foliation $\mathcal{F}$ of $M$ that induces on $V$ the fibration by $q^{-1}$ of points.
For \( x \in S^1 \) we define the extended \( f \)-orbit of \( x \), \( E(x) \), as \( \{ y \in S^1 ; n \in \mathbb{N}, f^n x = f^m y \} \). For \((x, z) \in V \subset S^1 \times D^2 \), the leaf of \( \mathcal{F} \) through \( \pi(x, z) \) is \( M_{E(x)} \) where, for \( A \subset S^1 \), we define \( M_A = \pi^{-1} A \). We say \( A \subset S^1 \) is minimal for \( f \) if it is compact, non-empty, invariant by \( f \) and \( f^{-1} \) and minimal with respect to these properties. The \( A \subset S^1 \) is minimal for \( f \) if and only if \( M_A \subset M \) is minimal for \( \mathcal{F} \).

Now we choose a particular structurally stable \( C^\omega \) immersion of degree 2, \( f:S^1 \to S^1 \). \( f \) has one attracting fixed point \( \delta \) and no other attracting periodic orbits. There are just two other fixed points and they divide \( S^1 \) into two semicircles \( I \) and \( J \). \( fI = I \) but \( fJ \) wraps around \( 1\frac{1}{2} \) times with \( f|J \) expanding. Such an \( f \) has a unique minimal set which is \( \Gamma \), the complement of \( W^S(\delta) = \bigcup f^{-n}(\text{int} I) \). \( \Gamma \) is a Cantor set and \( f|\Gamma \) is conjugate to the one-sided 2-shift. So \( \mathcal{F} \) has a unique minimal set \( M_\Gamma \) and \( M_\Gamma \) is clearly exceptional.

A foliation \( \mathcal{G} \) of \( M \) that is \( C^1 \) close to \( \mathcal{F} \) gives rise to a foliation \( \mathcal{G}' \) of \( V \) meeting \( \partial V \) in circles and \( \mathcal{G}' \) can be shown to fibre \( V \) over a map \( S^1 \to S^1 \) that is close to \( f \). Thus the stability of \( f \) leads to the stability of \( \mathcal{F} \).

A presentation of \( \pi_1(M) \) can be obtained by considering the identification of the boundary of \( V \) and from this presentation it is not hard to see (d).

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7. Accessiblility and foliations. Peter Stefan.

Let \( S \) be a set of \( C^q \) vector fields, \( q \geq 1 \), defined on open subsets of a finite dimensional paracompact manifold \( M \). The \( S \)-orbit of a point \( x \in M \) is \( \{ y \in M ; x \) can be joined to \( y \) by a finite sequence of segments of integral curves of vector fields in \( S \} \). What can be said about the partition \( \mathcal{P}(S) \) of \( M \) into \( S \)-orbits?