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Codazzi Tensors with Two Eigenvalue Functions

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by

Gabriel Merton

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ABSTRACT OF THE DISSERTATION

Codazzi Tensors with Two Eigenvalue Functions

by

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Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Peter Petersen, Chair

This paper addresses a gap in the classification of Codazzi tensors with exactly two eigenfunctions on a Riemannian manifold of dimension three or higher. Derdzinski proved that if the trace of such a tensor is constant and the dimension of one of the eigenspaces is $n - 1$, then the metric is a warped product where the base is an open interval. In [8], Tojeiro generalized this result given the existence of a certain type of function of the eigenvalues of the Codazzi tensor, of which the trace is one example. We will show the equivalence of Tojeiro’s condition to three other conditions. Furthermore, we construct examples of Codazzi tensors having two eigenvalue functions, one of which has eigenspace dimension $n - 1$, where the metric is not a warped product with interval base, refuting a remark in [2] that the warped product conclusion holds without any restriction on the trace.
The dissertation of Gabriel Merton is approved.

Gary Gray
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University of California, Los Angeles
2012
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Publications


CHAPTER 1

Introduction

A symmetric $(0, 2)$ tensor $A$ is Codazzi if it satisfies the symmetry property

$$(\nabla_X A)(Y, Z) = (\nabla_Y A)(X, Z)$$

for any vector fields $X$, $Y$ and $Z$. Alternatively, a $(1, 1)$ tensor $A$ is Codazzi if it is self-adjoint and

$$(\nabla_X A) Y = (\nabla_Y A) X$$

Throughout the paper, $V_\lambda$ denotes the eigendistribution corresponding to the eigenvalue function $\lambda$ of the tensor $A$. That is, we say a vector field $Y$ is in $V_\lambda$ if $AY = \lambda Y$. While it’s conceptually more appropriate to view Codazzi tensors as $(1, 1)$, computations are often easier when the tensor is viewed as $(0, 2)$.

There are many well-known examples of Codazzi tensors: any constant scalar multiple of the metric, and more generally any parallel self-adjoint $(1, 1)$ tensor. One may also ask what it means if certain well known self-adjoint $(1, 1)$ tensors are Codazzi. For example, the second fundamental form of a hypersurface embedded in a space of constant sectional curvature is Codazzi. When the larger space is $\mathbb{R}^3$, this is the content of the famous Codazzi-Mainardi equation. It’s a standard exercise in Riemannian geometry to show that $\text{Ric}$ is Codazzi if and only if the divergence of the full curvature tensor vanishes, i.e. the curvature is harmonic. This is the case, for example, on Einstein manifolds.

There are other more subtle relationships between the behavior of Codazzi tensors and the topology and geometry of the manifold. Berger-Ebin proved in [1] that a constant
trace Codazzi tensor on a compact manifold with non-negative sectional curvature must be parallel. As a nice low-dimension result, Bourguignon showed in [3] that a compact orientable four-manifold admitting a non-trivial Codazzi tensor with constant trace must have signature zero. Derdzinski-Shen proved in [6] that if a Codazzi tensor on $M^n$ has $n$ distinct eigenvalues at all points of $M$, then all the Pontryagin classes of $M$ are zero.

Another result relating the geometry of the manifold to a Codazzi tensor’s spectrum is the departure point of this paper.

**Theorem 1.0.1.** *(Derdzinski)* Suppose $A$ is a Codazzi tensor on $M^n$, $n \geq 3$ having exactly two distinct eigenvalue functions $\lambda, \mu$ in a neighborhood of $p$ with $\dim V\mu \leq \dim V\lambda$. Then, there exists a neighborhood of $p$ such that

1. $M$ is a Riemannian product if and only if $\dim V\mu \geq 2$ or $A$ is parallel in a neighborhood of $p$.

2. $M$ is a warped product with interval base and non-trivial warping function if and only if $\dim V\mu = 1$, $A$ has constant trace, and $A$ is not parallel. In this case, $M = I \times_F N$ where $N$ and $I$ are the integral submanifolds of $V\lambda$ and $V\mu$, respectively.

It’s not immediately obvious that such integral manifolds exists, however, a well known property of Codazzi tensors that we’ll review in Section 2 is the fact that their eigendistributions are integrable.

Theorem 1.0.1 was first proved by Derdzinski in [5] and is reported in Besse’s *Einstein Manifolds*, [2]. Besse precedes the proof with the statement ”A similar argument works without the hypothesis [that trace $A$ is constant].” This statement can be interpreted in two ways. First some terminology. If $A$ is a Codazzi tensor with exactly two distinct eigenfunctions $\mu$ and $\lambda$ and if $M = M_1 \times_F M_2$ is a warped product, we will say that the warped product and eigenspace structures are *consistent* if $M_1$ and $M_2$ are integral submanifolds of the eigendistributions $V\mu$ and $V\lambda$. In Theorem 1.0.1, the structures are consistent. Besse’s remark could be saying that without the constant trace assumption,
i. $M$ is either a product or a warped product with interval base; or

ii. $M$ is either a product or a warped product with interval base and the warping variable
    is the coordinate of the interval.

In Section 4, we prove the following theorems showing neither statement is true.

**Theorem 1.0.2.** There exists a compact Riemannian manifold $(M, g)$ and a Codazzi tensor
$A$ with exactly two distinct eigenfunctions $\mu$ and $\lambda$ with $\dim V_{\mu} = 1$ such that $M$ is
neither a product nor a warped product.

**Theorem 1.0.3.** There exists a Riemannian manifold $(M, g)$ and a Codazzi tensor $A$ with
exactly two distinct eigenfunctions $\mu$ and $\lambda$ with $\dim V_{\mu} = 1$ such that $M$ is a warped
product with interval base but the warped product structure is inconsistent with the eigenspace
structure.

However, the conclusion of Theorem 1.0.1 does hold under weaker conditions than the
trace being constant. In Section 3 we prove,

**Theorem 1.0.4.** Let $A$ be a Codazzi tensor on a manifold $M^n$, $n \geq 3$. Suppose there exists
a neighborhood of a point $p$ where $A$ has two distinct eigenfunctions $\mu$ and $\lambda$, $\dim V_{\mu} = 1$,
and $\lambda$ is not constant. Assume that any one of the following conditions hold:

1. There exists a $C^\infty$ function $F(\lambda(q), \mu(q))$ such that $(D_Y F)(q) = 0$ and $\frac{\partial F}{\partial \mu} |_{q} \neq 0$ for
   all $q \in W$, $Y \in V_{\lambda}$.

2. $D_Y \mu = 0$ for all $Y \in V_{\lambda}$.

3. The integral curves of $V_{\mu}$ are geodesics.

4. If for every unit vector $X \in V_{\mu}$ there exists a function $f \in C^\infty(M)$ such that locally
   $\nabla f = X$
then the metric is a non-trivial warped product with interval base consistent with the eigenspace structure of $A$.

The first condition is a generalization of the constant trace assumption in two ways. First, we can consider functions other than the trace in formulating our hypotheses. Second, that function need not be constant, but merely constant along the foliations of $V_{\lambda}$. Tojeiro proved that condition (1) implies the conclusion in the theorem.

The condition that $\lambda$ be non-constant replaces the condition that $A$ be non-parallel in the constant trace case and guarantees that the warping function is non-trivial.
CHAPTER 2

Background

In this section we assemble the tools needed to prove the proposition and the theorem. All the results in this section can be found in [5] or [2] and seem to have appeared first in [7], though the assumptions are slightly different. Lemma 2.0.5 gives a formula for the image of $\nabla_Y X$ under the Codazzi tensor $A$ given that $X$ and $Y$ are eigenvectors of the same eigenfunction.

**Lemma 2.0.5.** Suppose $A$ is a Codazzi tensor and that $X$ and $Y$ are two sections in $V_\lambda$. Then,

$$A\nabla_Y X = \lambda \nabla_Y X + (D_Y \lambda)X - g(X,Y) \nabla \lambda$$

**Proof.** This follows from the Leibniz rule and the Codazzi condition.

$$g(A(\nabla_Y X), Z) = g(\nabla_Y AX - (\nabla_Y A)X, Z)$$

$$= g((D_Y \lambda)X + \lambda \nabla_Y X, Z) - (\nabla_Z A)(X,Y)$$

Use the Leibniz rule to simplify the second term on the right-hand side.

$$(\nabla_Z A)(X,Y) = \nabla_Z A(X,Y) - A(\nabla_Z X, Y) - A(X, \nabla_Z Y)$$

$$= Z(\lambda g(X,Y)) - \lambda g(\nabla_Z X, Y) - \lambda g(X, \nabla_Z Y)$$

$$= D_Z \lambda g(X,Y)$$

---

1 The author wishes to thank Andrzej Derdziński for bringing this reference to his attention.
This shows that
\[ g(A\nabla_Y X, Z) = g(\lambda \nabla_Y X + (D_Y \lambda) X, Z) - g(X, Y)(D_Z \lambda) \]
for all vector fields \( Z \), proving the lemma.

The next lemma shows that as long as the dimension of the eigendistribution \( V_\lambda \) is at least two, the behavior of the eigenfunction \( \lambda \) is severely restricted in the sense that its directional derivative is zero along any direction belonging to \( V_\lambda \). When we discuss the particular case of interest where there are only two eigenfunctions \( \mu \) and \( \lambda \) with \( \dim V_\mu = 1 \) and \( \dim V_\lambda = n - 1 \), the lemma implies there’s only one linearly independent direction in which \( \lambda \) can vary.

**Lemma 2.0.6.** If \( A \) is a Codazzi tensor and \( V_\lambda \) is the eigendistribution of the eigenfunction \( \lambda \), \( \dim V_\lambda \geq 2 \), then \( D_Y \lambda = 0 \) for all \( Y \in V_\lambda \).

**Proof.** Given any \( Y \in V_\lambda \), choose \( X \in V_\lambda \) such that \( X \) is orthogonal to \( Y \) and \( |X| = 1 \). By the previous lemma,
\[
D_Y \lambda = (D_Y \lambda) g(X, X)
= g(A\nabla_Y X, X) + g(X, Y) g(X, X) - \lambda g(\nabla_Y X, X)
= \lambda g(\nabla_Y X, X) + 0 - \lambda g(\nabla_Y X, X)
= 0
\]

Lemmas 2.0.5 and 2.0.6 are used to prove the eigendistributions of Codazzi tensors are integrable.

**Theorem 2.0.7.** The eigendistributions of a Codazzi tensor \( A \) are integrable.
Proof. Let $V_\lambda$ be an eigendistribution and let $X,Y \in V_\lambda$. The claim is clearly true if $\dim V_\lambda = 1$, so assume $\dim V_\lambda \geq 2$ so that the previous lemma applies. We need to show $[X,Y] \in V_\lambda$.

$$A([X,Y]) = A(\nabla_X Y) - A(\nabla_Y X)$$

$$= (\lambda \nabla_X Y + (D_X \lambda) Y - g(X,Y) \nabla \lambda) - (\lambda \nabla_Y X + (Y \lambda) X - g(X,Y) \nabla \lambda)$$

$$= \lambda (\nabla_X Y - \nabla_Y X)$$

$$= \lambda ([X,Y]) \Rightarrow$$

$[X,Y] \in V_\lambda$

\[ \square \]

Note that by combining Theorem 2.0.7 and Lemma 2.0.6, one can say that if $\dim V_\lambda \geq 2$, then $\lambda$ is constant along the leaves of $V_\lambda$.

The final technical lemma gives a formula for the directional derivative of an eigenfunction $\lambda$ when the direction, $Y$, belongs to a different eigendistribution.

**Lemma 2.0.8.** If $A$ is a Codazzi tensor with $Y \in V_\lambda$ and $X,Z \in V_\mu$, then

$$D_Y \mu \cdot g(X,Z) = (\lambda - \mu) g(\nabla_X Y, Z)$$

**Proof.** Show that both sides of the equation are equal to $(\nabla_Y A)(X,Z)$. On one hand,

$$(\nabla_Y A)(X,Z) = D_Y A(X,Z) - A(\nabla_Y X, Z) - A(X, \nabla_Y Z)$$

$$= D_Y (\mu g(X,Z)) - \mu g(\nabla_Y X, Z) - \mu g(X, \nabla_Y Z)$$

$$= D_Y \mu \cdot g(X,Z)$$
On the other hand, we can use the Codazzi condition.

\[
(\nabla_Y A)(X, Z) = (\nabla_X A)(Y, Z) \\
= \nabla_X A(Y, Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z) \\
= -\mu g(\nabla_X Y, Z) - \lambda g(Y, \nabla_X Z) \\
= (\lambda - \mu)g(\nabla_X Y, Z)
\]
CHAPTER 3

Removing the Constant Trace Assumption

We now turn to the proof of Theorem 1.0.4. The proof consists of two steps. In Proposition 1, four conditions are shown to be equivalent. We then use the proposition to show the existence of a warped product structure.

It’s straightforward to see that the trace of $A$, where $A$ possesses exactly two eigenfunctions $\mu$ and $\lambda$ with $\dim V_\mu = 1$ and $\dim V_\lambda = n - 1$, is

$$\text{trace } A = \mu + (n - 1)\lambda.$$ 

If the trace is constant and $Y \in V_\lambda$, then by Lemma 2.0.6, $Y\mu = 0$. That is, the constant trace assumption implies $Y\mu = 0$ for all $Y \in V_\lambda$. Theorem 1.0.4 shows this conclusion, that $Y\mu = 0$, is sufficient to obtain a warped product structure. Alternatively, using the first characterization given in Proposition 1, it’s sufficient that the trace (or any reasonable function of the eigenvalue functions) be constant in all directions except for one.

Proposition 1. Let $A$ be a Codazzi tensor on a manifold $M^n$, $n \geq 3$, with two distinct eigenfunctions $\mu$ and $\lambda$ on an open domain $W$. Assume $\dim V_\mu = 1$. The following are equivalent.

1. There exists a $C^\infty$ function $F(\lambda(q), \mu(q))$ such that $(D_Y F)(q) = 0$ and $\frac{\partial F}{\partial \mu}|_q \neq 0$ for all $q \in W$, $Y \in V_\lambda$.

2. $D_Y \mu = 0$ for all $Y \in V_\lambda$.

3. The integral curves of $V_\mu$ are geodesics.
4. If $X \in V_\mu$, $|X| = 1$, then there exists a function $f \in C^\infty(M)$ such that locally $\nabla f = X$.

Proof. Throughout the proof assume $X$ is a unit vector in $V_\mu$ and $Y, Z \in V_\lambda$.

(1) $\Leftrightarrow$ (2). By Lemma 2.0.6, for all points $q \in W$, we have

$$D_Y F = \frac{\partial F}{\partial \lambda}(Y \lambda) + \frac{\partial F}{\partial \mu}(Y \mu) = \frac{\partial F}{\partial \mu}(Y \mu)$$

We then have (1) $\Rightarrow$ (2) since $(D_Y F)(q) = 0$ for all $q \in W$ if and only if $(Y \mu)(q) = 0$ for all $q \in W$. On the other hand, if (2) holds, then we can take $F(\lambda, \mu) = \lambda + \mu$, for example.

(2) $\Leftrightarrow$ (3). $g(\nabla_X X, X) = 0$ since $|X| = 1$. By Lemma 2.0.8,

$$D_Y \mu = (\mu - \lambda)g(\nabla_X X, Y)$$

Since $\mu \neq \lambda$, $\nabla_X X = 0$ if and only if $Y \mu = 0$.

(4) $\Leftrightarrow$ (2). To show this implication, recall that a vector field $X$ is gradient if and only if $\nabla X$ is symmetric. We have,

$$(\nabla_Y A)(X, Z) = -A(\nabla_Y X, Z) - A(X, \nabla_Y Z)$$

$$= -\lambda g(\nabla_Y X, Z) - \mu g(X, \nabla_Y Z)$$

$$= (\mu - \lambda)g(\nabla_Y X, Z)$$

Similarly, $(\nabla_Z A)(X, Y) = (\mu - \lambda)g(\nabla_Z X, Y)$. By the Codazzi condition, $g(\nabla_Z X, Y) = g(\nabla_Y X, Z)$. Thus, as a $(0, 2)$ tensor, $\nabla X$ is symmetric on $V_\lambda \times V_\lambda$ regardless of conditions (1) - (3). We also have,
\[
g(\nabla_X X, Y) = (\mu - \lambda)^{-1} D_Y \mu
\]
\[
g(\nabla_Y X, X) = \frac{1}{2} g(Y, X) = 0
\]

This shows $\nabla X$ is symmetric if and only if $Y \mu = 0$.

We now have everything we need to prove the main theorem.

**Proof of Theorem 1.0.4.** In general, the eigenbundles of a Codazzi tensor are integrable and orthogonal, so there exists a chart \( \{ U, r, y_1, \ldots, y_{n-1} \} \) such that \( \partial_r \in V_\mu \) and \( \partial_i = \partial_{y_i} \in V_\lambda \).

Let \( X \in V_\mu \) be a unit vector field. By the fourth criterion in Proposition 1, \( X = \nabla t \) for some local submersion \( t : U \to \mathbb{R} \). Now if \( Y \in V_\lambda \), then \( Y t = g(X, Y) = 0 \) meaning there exists coordinates \( \{ t, x_1, \ldots, x_{n-1} \} \) such that \( \partial_t \in V_\mu \), \( \partial_{x_i} \in V_\lambda \) and \( g(\partial_t, \partial_t) = 1 \). Moreover, \( A(\partial_t, \partial_j) = g(\partial_t, \partial_j) = 0 \) and \( A(\partial_i, \partial_j) = \lambda(t) g_{ij} \).

The next step is to prove that \( \partial_t g_{ij} = f(t) g_{ij} \). Lemma 2.0.8 implies,

\[
\partial_t g(\partial_i, \partial_j) = -2g(\nabla_{\partial_i} \partial_j, \partial_t) = 2(\mu - \lambda)^{-1}(\partial_t \lambda) g_{ij} = 2\eta g_{ij}
\]

where \( \eta = (\mu - \lambda)^{-1}(\partial_t \lambda) \). We can write,

\[
\eta g_{ij} = -g(\nabla_{\partial_i} \partial_j, \partial_t) = \text{Hess } t(\partial_i, \partial_j)
\]

Now show that \( \eta \) depends only on \( t \).

\[
\partial_t \eta = -(\mu - \lambda)^{-2} \cdot (\partial_t \mu - \partial_t \lambda)(\partial_t \lambda) + (\mu - \lambda)^{-1}(\partial_t \partial_t \lambda)
\]
\[
= (\mu - \lambda)^{-1}(\partial_t \partial_t \lambda)
\]
\[
= (\mu - \lambda)^{-1}(\partial_t \partial_t \lambda) = 0
\]
Since $\partial_t g_{ij} = \eta(t) g_{ij}$, integrate $\eta$ to obtain a function $q(t)$ such that $\partial_t (e^{-q} g_{ij}) = 0$. This means $g_{ij} = e^{\eta(t)} h_{ij}$ for some $h_{ij}$. This shows $M$ is a warped product. The warping function is trivial if and only if $\eta = 0$ which happens if and only if $\lambda$ is constant. \qed
CHAPTER 4

Counterexamples

This section presents a class of Codazzi tensors on open sets of $\mathbb{R}^3$ that provide the source of counterexamples for Theorems 1.0.2 and 1.0.3.

Let $\lambda > 0$ be a constant and $\mu(t, x, y)$ a $C^\infty$ function on an open set $V \subset \mathbb{R}^3$ and that there exists a connected open set $U \subset V$ where $\mu \neq \lambda$. Define a metric and tensor on $U$ by

$$
g = (\lambda - \mu(t, x, y))^{-2} \, dt^2 + \lambda \, dx^2 + \lambda \, dy^2$$

$$A(\partial_t) = \mu(t, x, y) \partial_t$$

$$A(\partial_x) = \lambda \partial_x$$

$$A(\partial_y) = \lambda \partial_y$$

As a step toward proving that $A$ is indeed Codazzi, calculate the Christoffel symbols. Throughout this section, the subscripts $i, j$ and $k$ shall refer to the variables $x$ and $y$. For example, $\partial_i$ could mean either $\partial_x$ or $\partial_y$ but not $\partial_t$.

**Lemma 4.0.9.** The non-trivial Christoffel symbols of this metric are,

$$\Gamma^t_{tt} = (\lambda - \mu)^{-1}(\partial_t \mu)$$

$$\Gamma^i_{tt} = -\frac{1}{\lambda}(\lambda - \mu)^{-3}(\partial_i \mu)$$

$$\Gamma^t_{it} = (\lambda - \mu)^{-1}(\partial_i \mu)$$

**Proof.** Recall the formula,

$$\Gamma^m_{pq} = \frac{1}{2} g^{lm} (\partial_q g_{pt} + \partial_p g_{qt} - \partial_t g_{pq})$$
If the coordinates of the metric are orthogonal, then $g^{lm} = 0$ for $l \neq m$ so we have,

$$\Gamma_{pq}^m = \frac{1}{2} g^{mn} (\partial_q g_{pm} + \partial_p g_{qm} - \partial_m g_{pq})$$

Since $g_{ij}$ is constant, the only non-trivial Christoffel symbols are the ones listed above.

$$\Gamma_{tt}^t = \frac{1}{2} g^{tt} (\partial_t g_{tt}) = \frac{1}{2}(\lambda - \mu)^2 (2(\lambda - \mu)^{-3} \partial_t \mu) = (\lambda - \mu)^{-1} (\partial_t \mu)$$

$$\Gamma_{ti}^t = \frac{1}{2} g^{ii} (-\partial_i g_{tt}) = \frac{1}{2\lambda} (-2(\lambda - \mu)^{-3} \partial_i \mu) = -\frac{1}{\lambda} (\lambda - \mu)^{-3} (\partial_i \mu)$$

$$\Gamma_{ti}^t = \frac{1}{2} g^{tt} (\partial_i g_{tt}) = \frac{1}{2}(\lambda - \mu)^2 (2(\lambda - \mu)^{-3}) (\partial_i \mu) = (\lambda - \mu)^{-1} (\partial_i \mu)$$

Proposition 2. The tensor $A$ defined above is Codazzi.

Proof. It suffices to prove $(\nabla_X A)(Y, Z) = (\nabla_Y A)(X, Z)$ where $X$, $Y$ and $Z$ are all coordinate vectors. Again, let $\partial_i$, $\partial_j$ and $\partial_k$ indicate partial derivatives with respect to $x$ or $y$.

Straightforward calculations show,

i. $(\nabla_{\partial_i} A)(\partial_i, \partial_j) = (\nabla_{\partial_i} A)(\partial_i, \partial_j) = 0$

ii. $(\nabla_{\partial_i} A)(\partial_i, \partial_k) = (\nabla_{\partial_i} A)(\partial_i, \partial_k) = (\partial_i \mu) g_{tt}$

iii. $(\nabla_{\partial_i} A)(\partial_j, \partial_k) = 0$
i.

\[(\nabla_{\partial_i} A)(\partial_j, \partial_j) = \partial_i A(\partial_i, \partial_j) - A(\nabla_{\partial_i} \partial_i, \partial_j) - A(\partial_i, \nabla_{\partial_i} \partial_j)\]
\[
= \lambda \partial_i g_{ij} - \lambda g(\nabla_{\partial_i} \partial_i, \partial_j) - \lambda g(\partial_i, \nabla_{\partial_i} \partial_j)
\]
\[
= 0
\]

\[
(\nabla_{\partial_i} A)(\partial_i, \partial_j) = -A(\nabla_{\partial_i} \partial_i, \partial_j) - A(\partial_i, \nabla_{\partial_i} \partial_j)
\]
\[
= -\lambda g(\nabla_{\partial_i} \partial_i, \partial_j) - \mu g(\partial_i, \nabla_{\partial_i} \partial_j)
\]
\[
= (\lambda - \mu)g(\partial_i, \nabla_{\partial_i} \partial_j)
\]
\[
= (\lambda - \mu)g(\partial_i, \Gamma_{ij}^p \partial_p)
\]
\[
= 0
\]

The last step follows from the previous lemma; \(\Gamma_{ij}^p = 0\) for all \(p\) (the only non-trivial Christoffel’s have at least two \(t\) sub/superscripts).

ii.

\[
(\nabla_{\partial_i} A)(\partial_i, \partial_i) = \partial_i(\mu g_{tt}) - 2A(\nabla_{\partial_i} \partial_i, \partial_i)
\]
\[
= (\partial_i \mu) g_{tt} + 2\mu g(\nabla_{\partial_i} \partial_i, \partial_i) - 2\mu g(\nabla_{\partial_i} \partial_i, \partial_i)
\]
\[
= (\partial_i \mu) g_{tt}
\]
\[(\nabla_{\partial_t} A)(\partial_i, \partial_t) = -A(\nabla_{\partial_t} \partial_i, \partial_t) - A(\partial_i, \nabla_{\partial_t} \partial_t)\]
\[= -\mu g(\nabla_{\partial_t} \partial_i, \partial_t) - \lambda g(\partial_i, \nabla_{\partial_t} \partial_t)\]
\[= (\lambda - \mu) g(\nabla_{\partial_t} \partial_i, \partial_t)\]
\[= \frac{\lambda - \mu}{2} \partial_i g_{tt}\]
\[= \frac{\lambda - \mu}{2} \cdot 2(\lambda - \mu)^{-3}(\partial_t \mu)\]
\[= (\lambda - \mu)^{-2}(\partial_t \mu)\]
\[= (\partial_t \mu) g_{tt}\]

iii. \[(\nabla_{\partial_t} A)(\partial_j, \partial_k) = \lambda (\partial_t g(\partial_j, \partial_k) - g(\nabla_{\partial_t} \partial_j, \partial_k) - g(\partial_t, \nabla_{\partial_t} \partial_k)) = 0\]

Our strategy for proving Theorems 1.0.2 and 1.0.3 will be to judiciously select \(\mu\) and \(\lambda\) along with the following well known characterization of warped products with interval bases, first proved by Brinkmann in [4].

Lemma 4.0.10. The following are equivalent

A. There exists a neighborhood \(V\) of \(p\) and a function \(f\) such that \(\text{Hess} f = a \cdot g\) for some function \(a\) and \(\nabla f(p) \neq 0\).

B. There exists a neighborhood \(V\) of \(p\) such that \(V\) is a warped product space \(V = I \times_w F\) with 1-dimensional base \(I\).

Thus, to show a metric is not an interval warped product, it suffices to show if \(f\) and \(a\) satisfy \(\text{Hess} f = a \cdot g\), then \(\nabla f = 0\) in a neighborhood, i.e. \(f\) is locally constant.
Lemma 4.0.11. The components of $\text{Hess} f$ for the metric given above are,

\[
\begin{align*}
\text{Hess} f(\partial_x, \partial_x) &= f_{xx} \\
\text{Hess} f(\partial_y, \partial_y) &= f_{yy} \\
\text{Hess} f(\partial_x, \partial_y) &= f_{xy} \\
\text{Hess} f(\partial_t, \partial_t) &= f_{tt} - (\lambda - \mu)^{-1} \mu f_t + \frac{1}{\lambda} \cdot (\lambda - \mu)^{-3} (f_x \mu_x + f_y \mu_y) \\
\text{Hess} f(\partial_t, \partial_x) &= f_{tx} - (\lambda - \mu)^{-1} \mu_x f_t \\
\text{Hess} f(\partial_t, \partial_y) &= f_{ty} - (\lambda - \mu)^{-1} \mu_y f_t 
\end{align*}
\]

Proof. Use the formula, $\text{Hess} f(\partial_i, \partial_j) = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f$ and the Christoffel symbols calculated above.

From here on will study the particular case where $\lambda = 1$ and $\mu(t, x, y) = \mu(x, y)$. Then by Lemma 4.2, if we are given a particular $\mu(x, y)$ we should look for functions $f(t, x, y)$ and $a(t, x, y)$ that solve the system below.

\[
\begin{align*}
 f_{xx} &= a \\
 f_{yy} &= a \\
 f_{xy} &= 0 \\
 f_{ty}(1 - \mu) &= \mu_y f_t \\
 f_{tx}(1 - \mu) &= \mu_x f_t \\
 f_{tt} + (1 - \mu)^{-3} (f_x \mu_x + f_y \mu_y) &= a(1 - \mu)^{-2}
\end{align*}
\]

Proposition 3. If for a given $\mu(x, y)$, there exists $f$ satisfying the system of PDE’s given above, with $\nabla f(0) \neq 0$, then $\mu$ must be in one of the following forms:

1. $\mu(x, y) = 1 + \frac{c_1}{1 - c_3 x - c_4 y - c_2 (x^2 + y^2)}$
2. \( \mu(x, y) = \frac{ax + G\left(\frac{c+ay}{a(b+ax)}\right)}{ax + b} \)

3. \( \mu(x, y) = \frac{ay + G\left(\frac{x}{c+ay}\right)}{c + ay} \)

4. \( \mu(x, y) = G\left(\frac{by - cx}{b}\right) \)

5. \( \mu(x, y) = \mu(x) \)

6. \( \mu(x, y) = \mu(y) \)

**Proof.** The first three Hessian equations

\[
\begin{align*}
    f_{xx} &= ag(\partial_x, \partial_x) = a \\
    f_{xy} &= ag(\partial_x, \partial_y) = 0 \\
    f_{yy} &= a
\end{align*}
\]

collectively imply \( a(t, x, y) = a(t) \) and

\[
    f(t, x, y) = \frac{a(t)}{2} (x^2 + y^2) + b(t)x + c(t)y + k(t)
\]

If we write \( h(t, x, y) = f_t(t, x, y) \), then we can rewrite and solve the fourth Hessian equation as a first order differential equation.

\[
    h_y(1 - \mu) = \mu_y h
\]

\[
    h_y = \mu_y h + \mu h_y = \frac{\partial}{\partial y}(\mu h) \Rightarrow
\]

\[
    f_t(t, x, y) = h(t, x, y) = C_1(t, x)/(\mu(x, y) - 1)
\]
An analogous argument works for the fifth equation, so

\[ f_t(t, x, y) = \frac{C_1(t, x)}{\mu(x, y) - 1} = \frac{C_2(t, y)}{\mu(x, y) - 1} \Rightarrow f_t(t, x, y) = \frac{C_1(t)}{\mu(x, y) - 1} \]

We can solve for \( C_1(t) \) by using the fact that \( f_t(t, x, y) \) is a polynomial and evaluating at \((t, 0, 0)\).

\[ \frac{C_1(t)}{\mu(x, y) - 1} = \frac{a'(t)}{2} (x^2 + y^2) + b'(t)x + c'(t)y + k'(t) \Rightarrow \]
\[ \frac{C_1(t)}{\mu(0, 0) - 1} = k'(t) \Rightarrow \]
\[ C_1(t) = (\mu(0, 0) - 1)k'(t) \]

Letting \( c_1 = \mu(0, 0) - 1 \), we have

\[ f_t(t, x, y) = \frac{c_1k'(t)}{\mu - 1} \Rightarrow f_t(t, x, y) = \frac{c_1k(t)}{\mu - 1} + K(x, y) \]

Equating to the polynomial expression for \( f \) gives us,

\[ \frac{c_1k(t)}{\mu - 1} + K(x, y) = \frac{a(t)}{2} (x^2 + y^2) + b(t)x + c(t)y + k(t) \Rightarrow \]
\[ K(x, y) = \frac{a(t)}{2} (x^2 + y^2) + b(t)x + c(t)y + H(x, y)k(t) \]

where \( H(x, y) = 1 - \frac{c_1}{\mu - 1} \). Take a \( t \) derivative of each side to get a linear equation of the functions \( x^2 + y^2, x, y \) and \( H(x, y) \).

\[ \frac{a'(t)}{2} (x^2 + y^2) + b'(t)x + c'(t)y + H(x, y)k'(t) = 0 \]

This equation implies either \( a'(t) = b'(t) = c'(t) = k'(t) = 0 \) or

\[ H(x, y) = c_2(x^2 + y^2) + c_3x + c_4y \]
We can now solve for \( \mu \) since,

\[
H = 1 - \frac{c_1}{\mu - 1} \Rightarrow
\]

\[
\mu(x, y) = 1 + \frac{c_1}{1 - H} = 1 + \frac{c_1}{1 - c_3x - c_4y - c_2(x^2 + y^2)}
\]

A continued analysis would investigate the implications of the final Hessian equation. However, the equation for \( \mu \) just derived will be sufficient for our purposes.

We now investigate solutions to the system when \( a, b, c \) and \( k \) are all constant. This simplifies the system considerably; the third and fourth equations automatically hold and the final equation simplifies to

\[
(ax + b)\mu_x + (ay + c)\mu_y = a(1 - \mu)
\]

This PDE is straightforward to solve using the Method of Characteristics.

\[
\mu(x, y) = \frac{ax + G \left( \frac{c + ay}{a(b + ax)} \right)}{ax + b} \text{ if } a \neq 0 \text{ and } b \neq 0
\]

\[
\mu(x, y) = \frac{ay + G \left( \frac{x}{c + ay} \right)}{c + ay} \text{ if } a \neq 0 \text{ and } b = 0
\]

\[
\mu(x, y) = G \left( \frac{by - cx}{b} \right) \text{ if } a = 0 \text{ and } b \neq 0
\]

\[
\mu(x, y) = \mu(x) \text{ if } a = 0, b = 0 \text{ and } c \neq 0
\]

\[
\mu(x, y) = \mu(y) \text{ if } a = 0, b \neq 0 \text{ and } c = 0
\]

By selecting any \( \mu \) not in one of the above forms, we can generate an example of a Codazzi tensor on a compact manifold where the metric is not a warped product at at least one point.
Corollary 4.0.12. Let $\mu(x, y) = \frac{1}{2} \sin x \cos y$. Then the metric $g$ and Codazzi tensor $A$ defined above are periodic and pass to a metric $\bar{g}$ and Codazzi tensor $\bar{A}$ on $S^1 \times S^1 \times S^1$. $\bar{g}$ is not a warped product on a neighborhood of the point $[(0, 0, 0)]$.

This result follows from the fact that $\mu(x, y)$ is clearly not in any of the functional forms listed in the proposition.

For the proof of Theorem 1.0.3 we use the same template but with $\mu = 1 + (y/x^2)$. As before $\lambda = 1$.

Proof of Theorem 1.0.3. Let $M = \{(t, x, y) \in \mathbb{R}^3 : y \neq 0\}$. Define a metric and tensor on $M$ by

$$g = \frac{x^4}{y^2} dt^2 + dx^2 + dy^2.$$ 

$$A\partial_t = \left(1 + \frac{y}{x^2}\right) \partial_t$$

$$A\partial_x = \partial_x$$

$$A\partial_y = \partial_y$$

It’s clear that the metric cannot be written in the form

$$g = dt^2 + F(t) \left(dx^2 + dy^2\right)$$

so either the metric is a warped product with inconsistent warping and eigenspace structures, or the metric is not warped at all. If we let $x = r \cos \theta$ and $y = r \sin \theta$, then

$$g = dr^2 + r^2 \left(\frac{\cos^4 \theta}{\sin^2 \theta} dt^2 + d\theta^2\right)$$

This shows the metric is warped in the $r$ direction. 

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REFERENCES


