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HA-spaces and Commutative Homology Rings

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Nicholas D. Nguyen

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2013
The dissertation of Nicholas D. Nguyen is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

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ABSTRACT OF THE DISSERTATION

HA-spaces and Commutative Homology Rings

by

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Doctor of Philosophy in Mathematics

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In the study of Lie groups, topological groups, and H-spaces, one common problem involved finding characterizations of spaces whose mod $p$ homology is commutative, and in the case that it is not commutative, one could ask what kinds of commutators occurred in the mod $p$ homology.

A common theme in H-space theory is trying to extend results for Lie groups to H-spaces by creating new techniques and finding new proofs. As such, our dissertation stems from recent work in characterizing Lie groups whose mod $p$ homology is commutative using a tool called the adjoint action of the Lie group on its loop space.
Our main contribution starts with defining and studying the adjoint action, along with similar maps like the commutator, for a generalization of compact simply-connected Lie groups called finite simply-connected HA-spaces, and the homomorphism it induces on mod $p$ cohomology ($p$ odd). After finding a formula for the induced homomorphism in terms of the coproduct, we proceed to several applications.

Given a finite simply-connected HA-space $X$, we use the adjoint action, along with similar maps, and their induced homomorphisms to characterize HA-spaces whose mod $p$ homology is commutative. This generalizes earlier work by showing that $X$ does not need the full structure of a Lie group; it can be weakened to an HA-space structure.

For our second application, given a finite simply-connected HA-space $X$, we study the free loop space of $X$ and use the adjoint action and related maps to compute products in the mod $p$ homology of the free loop space.

Finally, given a finite simply-connected HA-space $X$, we demonstrate how to use the commutator on $X$ in order to construct a new multiplication map on $X$ which induces a commutative algebra structure in mod $p$ homology.
1 Introduction

A major milestone in the study of topological spaces is the usage of functors from the category of (pointed) topological spaces to some category of algebraic objects like groups and modules. These algebraic objects can be used to distinguish spaces: if a functor takes two topological spaces to different algebraic objects, then the spaces are fundamentally different. We will focus on the singular cohomology over a ring $R$, a contravariant functor that takes a space $X$ to the graded $R$-module $H^*(X; R)$, and the singular homology over a ring $R$, a covariant functor which takes $X$ to the graded $R$-module $H_*(X; R)$. The cohomology $H^*(X; R)$ is also an algebra over $R$, in that objects inside it can be added, multiplied, and scaled by an element of $R$. Due to the many structures available in $H^*(X; R)$, it has received much focus in mathematical research. In particular, given a space, one can compute its cohomology and specify its algebra structure (how objects are added and multiplied). Conversely, given an algebra, one can ask if it arises as the cohomology of some space.

Now suppose our space has a continuous map $\mu : X \times X \to X$ such that if $x_0$ is the basepoint of $X$, then $\mu(x, x_0) = \mu(x_0, x) = x$. Examples of spaces with such a map are (from specific to general) Lie groups, topological spaces, and H-spaces; the
map $\mu$ is called a multiplication map on $X$, and the Lie group, topological space, or H-space is denoted $(X, \mu)$. In this case, the existence of $\mu$ imposes more structure to the cohomology and homology of $X$ over a field $\mathbb{F}_p$ (in this dissertation, $p$ will always denote an odd prime integer). The cohomology $H^*(X; \mathbb{F}_p) = H^*(X, \mu; \mathbb{F}_p)$ is now a Hopf algebra, with a unary operation $\mu^*$ (induced by $\mu$) called the coproduct.

Furthermore, suppose that $X$ is a space of finite type as well: $X$ is homotopy equivalent to a CW complex with a finite number of cells in each dimension. Then the homology $H_*(X; \mathbb{F}_p) = H_*(X, \mu; \mathbb{F}_p)$ can be given an algebra structure induced by $\mu$.

The presence of the coproduct in cohomology actually limits the possibilities for the algebra structure of $H^*(X, \mu; \mathbb{F}_p)$, and consequently, there has been much effort on computing the algebra and Hopf algebra structures of the cohomology of Lie groups, topological groups, and H-spaces.

Mathematicians have made computations of the coproduct operations in the cohomology over $\mathbb{F}_p$ of a compact connected Lie group. It started in the 1950’s with Borel’s work, where he found that if $(G, \mu)$ is a classical Lie group, $H^*(G, \mu; \mathbb{F}_p)$ is a primitively generated exterior algebra, the simplest kind of Hopf algebra possible [4]. From there, mathematicians like Borel, Toda, Mimura, Shimada, Araki, and Kono worked on the exceptional groups; the problem is that for certain combinations of exceptional groups $(G, \mu)$ and primes $p$, $H^*(G, \mu; \mathbb{F}_p)$ is not primitively generated. It was not until 1977 that Kono finished these computations with his calculation of coproducts in $H^*(E_8, \mu; \mathbb{F}_p)$ [20].

Borel also showed that if $G$ is a compact connected Lie group, $H^*(G, \mu; \mathbb{F}_p)$
is primitive if and only if the homology \( H_*(G, \mu; \mathbb{F}_p) \) is a (graded) commutative algebra [3]. From here, one can ask if there is a relationship between commutativity of the homology \( H_*(G, \mu; \mathbb{F}_p) \) and commutativity of the group \( G \) itself. If \( G \) is abelian, then \( H_*(G, \mu; \mathbb{F}_p) \) would be commutative. The converse is false: \( E_8 \) is a nonabelian Lie group, but \( H_*(E_8, \mu; \mathbb{F}_7) \) is commutative. Instead, we can turn to Kono and Kozima’s work on the adjoint action of \( G \) on its loop space, a map \( \text{Ad} : G \times \Omega G \to \Omega G \) given pointwise by \( \text{Ad}(g, l)(t) = gl(t)g^{-1} \). Notice that if \( G \) is an abelian compact connected Lie group, then \( \text{Ad}(g, l) = l \) for any element \( g \) and loop \( l \). In other words, the map \( \text{Ad} \) is equal to the projection on the second factor \( p^\Omega_2 : G \times \Omega G \to \Omega G \) given by \( p^\Omega_2(g, l) = l \). Kono and Kozima prove that \( H_*(G, \mu; \mathbb{F}_p) \) is commutative if and only if the algebra homomorphisms \( \text{Ad}^* \) and \( p^\Omega_2^* \) (from \( H^*(G; \mathbb{F}_p) \) to \( H^*(G \times G; \mathbb{F}_p) \)) are equal [21].

As H-space theorists, we would like to know if this result holds for a larger class of spaces than the Lie groups. This question has been asked for many results about the cohomology of Lie groups in the history of H-spaces, and Kane, in his textbook on the subject, even calls Lie groups the “experimental data” of H-space theory [19]. One of the goals of this dissertation is to introduce a generalization of Lie groups that we will call HA-spaces, and show how we can generalize \( \text{Ad} \) and other maps from Kono and Kozima’s work to finite simply-connected HA-spaces in a way that allows us to prove some of their results for this larger class of spaces. From there, we will use our maps for other applications: the computation of products in the homology of a type of H-space called the free loop space of an HA-space, and the construction of an
H-space whose homology is a commutative (if possibly nonassociative) algebra which is isomorphic as a vector space to the homology of a given finite simply-connected HA-space.

The remainder of the introduction will proceed as follows. We will start with some concepts and definitions from the theory of H-spaces that will be used in this dissertation; assumptions and notation will also be stated as well. We divide this background material into two sections: the algebraic material, and the topological material. Afterward, we will proceed by topic, giving a brief background of each problem and stating our results. First, we will discuss Kono and Kozima’s work in more detail, along with some subsequent developments by other authors, and introduce some obstacles to generalizing their results to HA-spaces. Our results will include details on how we overcame these challenges. From there, we will discuss the problem of computing products in the homology of a free loop space, and find formulas for products in the homology over $\mathbb{F}_p$ (for any odd prime $p$) of a free loop space of a finite simply-connected HA-space. Afterward, we will look at the question of finding multiplication maps on a finite simply-connected topological space: given such a space $X$, we want to know if there is a map $\nu : X \times X \to X$ such that $(X, \nu)$ is an H-space whose homology over $\mathbb{F}_p$ ($p$ is a fixed odd prime) is a commutative and associative algebra. In the case that our topological space is a finite simply-connected HA-space $(X, \mu)$, we find a formula for a map $\nu : X \times X \to X$ in terms of $\mu$ such that the multiplication induced by $\nu$ on the homology of $X$ over $\mathbb{F}_p$ is commutative (but possibly nonassociative), and determine conditions on $H_*(X, \mu; \mathbb{F}_p)$ which make
$H_{\nu}(X, \nu; \mathbb{F}_p)$ a commutative and associative algebra.

**1.1 Algebraic Assumptions and Definitions**

Let us begin with the algebraic assumptions and definitions that will be needed. After stating our assumptions, we will introduce Hopf algebras and discuss some of their characteristics and properties. For background material on Hopf algebras, we recommend Milnor and Moore’s paper, [29], and Kane’s book [19].

Throughout this entire dissertation, the symbol $p$ will always denote an odd prime. All algebras will have $\mathbb{F}_p$ as their base field. They will be assumed to be associative unless otherwise stated. Given an algebra $A$, we will assume $A$ is graded, and denote the subspace of homogenous elements of degree $n$ by $A^n$. For any positive integer $n$, we will require $A^n$ to be a vector space (over $\mathbb{F}_p$) of finite dimension (it has a spanning set with a finite number of elements), and $A^0$ must equal $\mathbb{F}_p$. If we say an algebra is commutative, we mean in the graded sense. We will assume that all homomorphisms between algebras are algebra homomorphisms which preserve addition, multiplication, and scalar multiplication by the base field $\mathbb{F}_p$, and we will just write homomorphism for algebra homomorphism. The reader must be aware that we will deal with functions between algebras which might only be linear transformations which preserve addition and scalar multiplication. We will indicate when we are dealing with such a function. If a homomorphism $\phi$ is said to equal zero or be trivial, this means that $\phi(x) = 0$ if $x$ has positive degree, and $\phi(x) = x$ if $x$ has degree zero.
If \( A \) is a graded algebra, then \( \bar{A} \) will denote positive degree elements of \( A \), \( A^{\text{odd}} \) will denote the odd degree elements, and \( A^{\text{even}} \) will denote the even degree elements.

Before we continue, let us review the algebra structure on the tensor product of graded algebras.

**Definition 1.1.1.** Let \( A \) and \( B \) be algebras. The vector space \( A \otimes B \) can be given an algebra structure as follows: let \( a, c \in A \) and \( b, d \in B \). Then in \( A \otimes B \), we multiply the elements \( a \otimes b \) and \( c \otimes d \) as follows (where \(|x|\) denotes the degree of an element \( x \)):

\[
(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd.
\]

(1.1.1)

We will also abbreviate \( 0 \otimes 0 \) as 0.

First, let us recall that the cohomology over \( \mathbb{F}_p \) of a Lie group or topological group \((G, \mu)\) has a special structure:

**Definition 1.1.2.** An algebra \( A \) together with an algebra homomorphism \( \Delta : A \to A \otimes A \) is a Hopf algebra \( A = (A, \Delta) \) if the diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{q_1} & A \otimes A \\
\downarrow{q_2} & & \downarrow{id_A} \\
A \otimes A & \xrightarrow{\Delta} & A
\end{array}
\]

where

\[
q_1(a \otimes b) = \begin{cases} 
0, & |b| > 0 \\
a, & b = 1 
\end{cases}
\]
The homomorphism $\Delta$ is called the \textit{comultiplication or coproduct} on $A$. According to the diagram, for any $a \in A$,

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a' \otimes a''$$  \hspace{1cm} (1.1.2) 

where $|a'|, |a''| > 0$ and $|a'| + |a''| = |a|$.

The term $\sum a' \otimes a'' = \Delta(a) - a \otimes 1 - 1 \otimes a$ is called the \textit{reduced coproduct} and is denoted $\overline{\Delta}(a)$.

An element $a \in A$ is \textit{primitive} if $\overline{\Delta}(a) = 0$ (that is, $\Delta(a) = a \otimes 1 + 1 \otimes a$). The \textit{submodule of primitive elements} of $A$ is denoted $PA$. Elements of degree $n$ in $PA$ are denoted $P^n A$, elements of odd degree are denoted $P^{odd} A$, and elements of even degree are denoted $P^{even} A$.

A nonzero element $a \in A$ has \textit{height} $n$, $1 < n < \infty$, if $a^n = 0$, and $a^{n-1} \neq 0$. If no such integer exists, we say $a$ has \textit{infinite height}.

The \textit{module of indecomposables} is $QA = A/\overline{A} \overline{A}$, the quotient module of $A$ by the image of the multiplication on positive degree elements. Elements of $QA$ are cosets of $\overline{A} \overline{A}$. Representatives of positive degree elements of $QA$ in $A$ are called \textit{generators}; given a basis of $QA$, a set consisting of a representative for each and every element of this basis of $QA$ is called a \textit{generating set} for $A$. That is, any element of $A$ can
be written as a sum, scalar multiple, and product of elements in the generating set. Elements of degree \( n \) in \( QA \) will be denoted \( Q^n A \), elements of odd degree are denoted \( Q^{\text{odd}} A \), and elements of even degree are denoted \( Q^{\text{even}} A \). Elements of \( \bar{A} \bar{A} \) are called *decomposables.*

If there is a choice of generating set for \( A \) which consists of primitive elements, then we say \( A \) is *primitively generated.*

If we denote the dual of \( A \) by \( A^* \), then we say \( A \) is *coassociative* if \( A^* \) is associative, and \( A \) is *cocommutative* if \( A^* \) is commutative.

An algebra homomorphism \( \varphi : (A, \Delta_A) \to (B, \Delta_B) \) between two Hopf algebras is called a *Hopf algebra homomorphism* if

\[
(\varphi \otimes \varphi) \Delta_A = \Delta_B \varphi.
\]

If \( \varphi \) is an algebra isomorphism which is a Hopf algebra homomorphism as well, we say that \( \varphi \) is a *Hopf algebra isomorphism* and that \( (A, \Delta_A) \) and \( (B, \Delta_B) \) are isomorphic as Hopf algebras.

*Remark.* We do not assume coassociativity or cocommutativity in our definition of a Hopf algebra.

We have the following elementary properties of \( PA \) and \( QA \) from [19]:

**Theorem 1.1.3.** The vector spaces \( (QA)^* \) and \( P(A^*) \) are equal, and \( (PA)^* = Q(A^*) \).

Moreover, if \( A \) is a commutative (and associative) Hopf algebra (over \( \mathbb{F}_p \)), denote the \( p \)th power map on \( A \) by \( \xi : A \to A \) (so \( \xi(a) = a^p \)). Then the following sequence is
exact for each positive integer \( n \):

\[
0 \to P^n(\xi(A)) \to P^nA \to Q^nA.
\]

Furthermore, any primitive element whose degree is not a multiple of \( 2p \) is a generator of \( A \). That is, when \( n \) is not a multiple of \( 2p \), the linear transformation

\[
P^nA \to Q^nA
\]

is injective, and if \( A \) is also cocommutative and coassociative, the linear transformation is an isomorphism.

Remark. As we will see in Example 1.1.5, odd degree elements cannot have nontrivial \( p \)th powers, so although the exact sequence tells us that primitive elements whose degrees are not a multiple of \( p \) are generators, since odd degree elements are never nontrivial \( p \)th powers, we can be more specific and state that any primitive element whose degree is not a multiple of \( 2p \) is a generator.

We also have the following characterization of primitively generated Hopf algebras as well from [19]:

**Theorem 1.1.4.** A Hopf algebra \( A \) is primitively generated if and only if \( A^* \) is commutative, associative, and has no \( p \)th powers (that is, if \( \bar{a} \in A^* \), then \( \bar{a}^p = 0 \)).

**Example 1.1.5.** Let us list all the possible commutative Hopf algebras over \( \mathbb{F}_p \) with a single algebra generator. These examples are called monogenic Hopf algebras. In
each of these cases, the generator is primitive.

1) An exterior algebra on an odd degree generator: \( \wedge(x_{2n+1}) \)

2) A polynomial algebra on an even degree generator: \( \mathbb{F}_p[x_{2n}] \)

3) A truncated polynomial algebra on an even degree generator: \( \mathbb{F}_p[x_{2n}]/(x_{2n}^p) \)

An important fact about Hopf algebras is that once these monogenic Hopf algebras are known, we can describe the algebra structure of any other Hopf algebra using the Borel Decomposition Theorem (quoted from [19]):

**Theorem 1.1.6.** Let \( A \) be a commutative (and associative) Hopf algebra. Then there are monogenic Hopf algebras \( A_i \) such that \( A \) and \( \bigotimes_i A_i \) are isomorphic as algebras.

**Definition 1.1.7.** We will call \( \bigotimes_i A_i \) in Theorem 1.1.6 the *Borel decomposition* of \( A \).

Note that the Hopf algebra \( \bigotimes_i A_i \) is primitively generated, but since the isomorphism might not preserve coproducts, \( A \) itself could have nontrivial reduced coproducts. Nevertheless, the Borel Decomposition Theorem tells us that Hopf algebras are much easier to visualize than generic algebras, so spaces whose cohomology algebras are Hopf algebras are easier to study. Thus, in the next section, we will introduce topological spaces whose cohomology algebras over \( \mathbb{F}_p \) are Hopf algebras.

### 1.2 Topological Assumptions and Definitions

In this section, we will state any assumptions on the topological spaces we will be studying. After that, we will introduce the main objects we will be studying:
the H-spaces, and the HA-spaces. Our exposition will compare these objects to the more familiar topological groups as well. Next, we will quote an important property of the cohomology of finite simply-connected HA-spaces from Lin’s paper [25] that will be referenced in our results. After that, we will introduce some examples of HA-spaces that we will focus on: loop spaces and free loop spaces. Finally, we will end this section with a discussion of the differences between topological groups and HA-spaces. For further background material on topological spaces and cohomology, we recommend the books [8], [12], [26], [33], and [38]. For further background on H-spaces, we recommend the books [19], [35], and [40].

For the rest of this document, unless specified, all spaces will have a basepoint (typically denoted with a zero subscript or superscript), and are homotopy equivalent to a CW complex which has a finite number of cells in each dimension and whose basepoint is a subcomplex. All maps and homotopies are continuous and respect the basepoint. If we say a space is finite, we mean it has the homotopy type of a CW complex with a finite number of cells. If we say a diagram of topological spaces and maps commutes (with no adjectives), we will mean it commutes up to homotopy. The symbol “≃” will mean “homotopic relative basepoints.”

Let us define some important maps which will occur frequently in our work:

**Definition 1.2.1.** Given a space $X$ with basepoint $x_0$, we define the *diagonal map* $\Delta_X : X \to X \times X$ and *constant (trivial) map* $k : X \to X$ by

$$\Delta_X(x) = (x, x), \text{ and } k(x) = x_0.$$
The diagonal map induces the (cup) product in cohomology,

\[ H^*(X \times X; \mathbb{F}_p) \xrightarrow{\Delta^*} H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \xrightarrow{\Delta^*} H^*(X; \mathbb{F}_p) \]

which we will also denote as \( \Delta^*_X \). That is, if \( a, b \in H^*(X; \mathbb{F}_p) \),

\[ \Delta^*_X(a \otimes b) = ab. \]

The identity map is denoted by \( id_X : X \to X \), where \( id_X(x) = x \).

The \( (j) \)th projection map \( p_j : X_1 \times X_2 \times \ldots \times X_n \to X_j \) is given by

\[ p_j(x_1, x_2, \ldots, x_j, \ldots, x_n) = x_j, \]

and the \( (j) \)th inclusion map \( i_j : X_j \to X_1 \times X_2 \times \ldots \times X_n \) is given by

\[ i_j(x_j) = (x^0_1, x^0_2, \ldots, x^0_j, \ldots, x^0_n), \]

where \( x^0_k \) is the basepoint of \( X_k \).

For any two spaces \( Y \) and \( Z \), let \( T_{Y,Z} : Y \times Z \to Z \times Y \) be defined by \( T_{Y,Z}(y, z) = (z, y) \). If \( Y = Z \), we will abbreviate the map as \( T_Y \), or even just \( T \).

Remark. Whenever we have a map \( f : X \times Y \to Z \) where \( X, Y, Z \) are topological spaces, we will abuse notation in cohomology over \( \mathbb{F}_p \) and call \( f^* \) to be the composit-
tion:

\[ H^*(Z; \mathbb{F}_p) \xrightarrow{f^*} H^*(X \times Y; \mathbb{F}_p) \xrightarrow{\cong} H^*(X; \mathbb{F}_p) \otimes H^*(Y; \mathbb{F}_p). \]

We do the same for a map \( g : X \to Y \times Z \) and denote the following composition by \( g^* \):

\[ H^*(Y \times Z; \mathbb{F}_p) \xrightarrow{\cong} H^*(Y; \mathbb{F}_p) \otimes H^*(Z; \mathbb{F}_p) \xrightarrow{g^*} H^*(X; \mathbb{F}_p). \]

Let us introduce the most general type of space whose cohomology over \( \mathbb{F}_p \) is a Hopf algebra:

**Definition 1.2.2.** Let \( X \) be a topological space with basepoint \( x_0 \). Suppose there exists a continuous map \( \mu : X \times X \to X \) such that

\[ \mu(x, x_0) = \mu(x_0, x) = x. \]

That is, the following diagram commutes strictly:

\[
\begin{array}{ccc}
X & \xrightarrow{i_1} & X \times X \\
\downarrow{i_2} & & \downarrow{id_X} \\
X \times X & \xrightarrow{\mu} & X \\
\end{array}
\]

(1.2.1)

Then \( X = (X, \mu) \) is called an \( H \)-space, \( \mu \) is called a multiplication map (or an \( H \)-structure) on \( X \), and \( x_0 \) is called a (strict) unit or (strict) identity. We may abbreviate \( \mu(x_1, x_2) \) using concatenation:

\[ \mu(x_1, x_2) = x_1x_2. \]

(1.2.2)
We can weaken the condition that $X$ has an identity element $x_0$ by requiring the diagram to commute (up to homotopy). In this case, we would say that $X = (X, \mu)$ has a homotopy identity or homotopy unit. However, if $X$ has the homotopy type of a CW complex, then if $(X, \mu)$ has a homotopy identity, there exists a map $\mu_0 : X \times X \to X$ such that $\mu_0(x, x_0) = \mu_0(x_0, x) = x$ and $\mu \simeq \mu_0$ [40]. Therefore, unless specified, we will assume that our H-spaces have strict identities.

Given an H-space $X$ with multiplication map $\mu$, we can define the induced homomorphism on its cohomology over the group $\mathbb{F}_p$ and use the induced homomorphism to define a coproduct on $H^*(X; \mathbb{F}_p)$. By definition, $\mu^*$ will be a homomorphism from $H^*(X; \mathbb{F}_p)$ to $H^*(X \times X; \mathbb{F}_p)$, and since the algebras $H^*(X \times X; \mathbb{F}_p)$ and $H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ are isomorphic, we have the following composition, which we will show is a coproduct on $H^*(X; \mathbb{F}_p)$:

\[
H^*(X; \mathbb{F}_p) \xrightarrow{\mu^*} H^*(X \times X; \mathbb{F}_p) \xrightarrow{\cong} H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p).
\]

We will abuse notation and call this composition $\mu^*$ as well, since we may work with more than one multiplication map on the same space $X$, and we will need a way to keep track of which multiplication map induces which coproduct. Now consider the diagram in the definition of an H-space. If we apply cohomology with coefficients in
\[ \mathbb{F}_p, \text{ we get} \]

\[
\begin{array}{c}
\xymatrix{ 
H^*(X; \mathbb{F}_p) \ar[r]^{i^*_1} & H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \ar[r]^{\mu^*} & H^*(X; \mathbb{F}_p) \ar[l]_{i^*_2} & \mu^* \ar[l] \ar[u]_{id} \ar[u]_{\mu^*} \ar[r] & H^*(X; \mathbb{F}_p) \ar[l]_{\mu^*} \ar[u]_{\mu^*} \ar[u]_{\mu^*} 
} 
\end{array}
\]

where

\[
i^*_1(x \otimes y) = \begin{cases} 
0, & |y| > 0 \\
x, & y = 1 
\end{cases}
\]

and

\[
i^*_2(x \otimes y) = \begin{cases} 
0, & |x| > 0 \\
y, & x = 1 
\end{cases}
\]

Hence the cohomology of an H-space \( H^*(X; \mathbb{F}_p) = H^*(X, \mu; \mathbb{F}_p) \) is a Hopf algebra with coproduct \( \mu^* \) induced by the H-space multiplication map (the reduced coproduct is denoted by \( \bar{\mu}^* \)).

Since \( X = (X, \mu) \) has finite type, then in homology, we have a linear transformation

\[
H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \xrightarrow{\cong} H_*(X \times X; \mathbb{F}_p) \xrightarrow{\mu_*} H_*(X; \mathbb{F}_p).
\]

We will abuse notation and call this composition \( \mu_* \) as well. This composition defines a product on \( H_*(X; \mathbb{F}_p) = H_*(X, \mu; \mathbb{F}_p) \), making it an algebra:

\[
\mu_*(\bar{x} \otimes \bar{y}) = \bar{x}\bar{y}.
\]
We can obtain a Hopf algebra structure on $H_*(X;\mathbb{F}_p)$ by using the diagonal map $\Delta_X$:

$$H_*(X;\mathbb{F}_p) \xrightarrow{\Delta_X^*} H_*(X \times X;\mathbb{F}_p) \xrightarrow{\cong} H_*(X;\mathbb{F}_p) \otimes H_*(X;\mathbb{F}_p).$$

We will call this composition $\Delta_{X^*}$ as well. It is a coproduct on $H_*(X,\mu;\mathbb{F}_p)$ because its dual map in cohomology, $\Delta_X^*$, is a product. Note that since the following diagram commutes (strictly):

\[
\begin{array}{ccc}
X \times X & \xrightarrow{\mu} & X \\
\downarrow{\Delta_{X \times X}} & & \downarrow{\Delta_X} \\
X \times X \times X \times X & \xrightarrow{\mu \times \mu} & X \times X
\end{array}
\]

This implies that in homology, the induced linear transformation $\Delta_{X^*}$ is an algebra homomorphism with respect to the product $\mu_*$. 

*Remark.* The preceding arguments also work for the case when $(X,\mu)$ only has a homotopy unit.

Notice that H-spaces require only the existence of an identity (the basepoint); nothing about associativity or existence of multiplicative inverses is mentioned. Let us introduce generalizations of these two conditions, along with a space that satisfies both conditions:

**Definition 1.2.3.** If the compositions $\mu(\mu \times id_X)$ and $\mu(id_X \times \mu)$ are (basepoint preserving) homotopic maps, then $(X,\mu)$ is said to be *homotopy associative*. That is,
the following diagram commutes:

\[
\begin{array}{ccc}
X \times X \times X & \xrightarrow{id \times X \times \mu} & X \times X \\
\mu \times id_X & & \mu \\
X \times X & \xrightarrow{\mu} & X 
\end{array}
\]  \quad (1.2.4)

Elementwise, \( \mu (\mu \times id_X) \) is given by

\[
\mu (\mu \times id_X) (x, y, z) = (xy)z,
\]  \quad (1.2.5)

and \( \mu (id_X \times \mu) \) is given by

\[
\mu (id_X \times \mu) (x, y, z) = x(yz).
\]  \quad (1.2.6)

If the compositions are actually equal, then we call \( (X, \mu) \) an *associative H-space*.

A map \( i : X \to X \) is called a *(two-sided) homotopy inverse operation* for \( (X, \mu) \) if the compositions \( \mu(i \times id_X) \Delta_X \) and \( \mu(id_X \times i) \Delta_X \) are homotopic to \( k \). In other
words, the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \times X \\
\downarrow{i \times id_X} & & \downarrow{i \times id_X} \\
X \times X & \xrightarrow{id_X \times i} & X \times X
\end{array}
\]

(1.2.7)

Elementwise, \(\mu(i \times id_X)\Delta_X\) is given by

\[
\mu(i \times id_X)\Delta_X(x) = i(x)x, \quad (1.2.8)
\]

and \(\mu(id_X \times i)\Delta_X\) is given by

\[
\mu(id_X \times i)\Delta_X(x) = xi(x). \quad (1.2.9)
\]

An H-space which is homotopy associative and has a two-sided homotopy inverse operation will be called an \textit{HA-space}.

\textit{Remark.} Work of James [14], Sibson [32], and Sugawara [37] show that if \((X, \mu)\) is a homotopy associative H-space, one can construct a two-sided homotopy inverse operation \(i : X \to X\) which satisfies \(\mu(i \times id_X)\Delta_X \simeq \mu(id_X \times i)\Delta_X \simeq k\). They also show that the map \(i\) is unique up to homotopy in that any other map \(j : X \to X\) which satisfies \(\mu(j \times id_X)\Delta_X \simeq \mu(id_X \times j)\Delta_X \simeq k\) is homotopic to \(i\) [14, 32, 37]. The reader must be aware that many authors like James and Sibson refer to the map \(i\) as
a “homotopy inverse,” but that term is also used by modern textbooks like Hatcher’s [12] in a different context with homotopy equivalences. Hence, we will refer to the map $i$ as a “homotopy inverse operation” (in an analogy with a strict inverse operation in a group) and will often drop “two-sided.”

Notice that if Diagrams 1.2.1, 1.2.4, and 1.2.7 actually commute strictly, then $(X, \mu)$ would be a topological group. Hence one can think of an HA-space as a generalization of topological groups. However, unlike groups, we cannot state the definitions for H-spaces and HA-spaces in terms of elements due to the ability to use homotopy.

If $(X, \mu)$ is an HA-space, then its homology $H_*(X, \mu; \mathbb{F}_p)$ will be an (associative) algebra, and its cohomology $H^*(X, \mu; \mathbb{F}_p)$ is a coassociative Hopf algebra. In the case that $(X, \mu)$ is a finite simply-connected HA-space, we have found that by choosing an appropriate generating set for $H^*(X, \mu; \mathbb{F}_p)$, we could simplify our calculations and find patterns that led to our results. We will use Theorem 2.1 from [25] to choose generators whose coproducts have a specified form. We will quote it here to introduce notation as well:

**Theorem 1.2.4 (Lin).** Let $(X, \mu)$ be a finite simply-connected HA-space. Then every even degree nonzero element of $QH^*(X, \mu; \mathbb{F}_p)$ has a primitive representative in $H^*(X, \mu; \mathbb{F}_p)$ (a primitive generator). Let $B$ be the sub Hopf algebra generated by the even degree primitive generators of $H^*(X; \mathbb{F}_p)$. Let

$$R = \{ r \in H^*(X, \mu; \mathbb{F}_p) : \tilde{\mu}^*(r) \in B \otimes H^*(X, \mu; \mathbb{F}_p) \}.$$
The set $R$ is a sub-coalgebra of $H^*(X, \mu; \mathbb{F}_p)$ with $R^{\text{even}} = B$, and the vector space $R^{\text{odd}}$ is isomorphic to the vector space $QH^{\text{odd}}(X, \mu; \mathbb{F}_p)$. All odd degree nonzero elements of $QH^*(X, \mu; \mathbb{F}_p)$ have a representative in $H^*(X, \mu; \mathbb{F}_p)$ whose reduced coproduct lies in $B \otimes R^{\text{odd}}$. Hence there is a generating set for $H^*(X, \mu; \mathbb{F}_p)$ whose even degree elements are primitive and whose odd degree elements have reduced coproduct in $B \otimes R^{\text{odd}}$.

**Definition 1.2.5.** We will refer to the generating set of $H^*(X, \mu; \mathbb{F}_p)$ described in the previous theorem as $S$.

We can also generalize the idea of abelian groups with the next definition:

**Definition 1.2.6.** Given an H-space $(X, \mu)$, if the maps $\mu$ and $\mu T_{X,X}$ are homotopic maps, then $(X, \mu)$ is a homotopy commutative H-space.

If $(X, \mu)$ is an HA-space which is also homotopy commutative, then its homology $H_*(X, \mu; \mathbb{F}_p)$ will be a commutative algebra. However, it turns out that when $p$ is odd, many compact simply-connected Lie groups which are not homotopy commutative (the groups $A_n$, $B_n$, $C_n$, $D_n$, and $G_2$) have homology algebras $H_*(G, \mu; \mathbb{F}_p)$ which are commutative. One central problem we will address in our results is how we can characterize such spaces and their cohomology over $\mathbb{F}_p$. Before we do so, let us give some examples of H-spaces. First, let us introduce an example that actually motivated Serre to consider a definition for H-spaces in his work on fiber spaces in [31]:

**Definition 1.2.7.** Let $X$ be a simply-connected topological space with basepoint $x_0$. Its loop space $\Omega X$ is the space of functions $l : [0, 1] \to X$ with the property that
\( l(0) = l(1) = x_0 \). The space will have the compact-open topology, and its basepoint is the constant loop \( l_0 \), where \( l_0(t) = x_0 \) for all \( t \in [0, 1] \). We can define a multiplication map \( \mu_c : \Omega X \times \Omega X \to \Omega X \) given by concatenation of loops: if \( a \) and \( b \) are loops in \( \Omega X \) given by \( a(t) \) and \( b(t) \), then for any \( t \in [0, 1] \), \( \mu_c(a, b) \) is a loop given by

\[
\mu_c(a, b)(t) = \begin{cases} 
  a(2t) & 0 \leq t < 1/2 \\
  b(2t - 1) & 1/2 \leq t \leq 1
\end{cases}.
\]

(1.2.10)

We also define a homotopy inverse operation \( i_c : \Omega X \to \Omega X \), where for any \( t \in [0, 1] \),

\[
i_c(a)(t) = a(1 - t).
\]

(1.2.11)

Then \( (\Omega X, \mu_c) \) is an HA-space. Note that it has a homotopy unit rather than a strict unit.

If \( X = (X, \mu) \) is an HA-space with homotopy inverse operation \( i \), then we can define a second multiplication map \( \mu_\Omega : \Omega X \times \Omega X \to \Omega X \) given by pointwise multiplication of loops. For any \( t \in [0, 1] \),

\[
\mu_\Omega(a, b)(t) = \mu(a(t), b(t)) = a(t)b(t),
\]

(1.2.12)

along with another homotopy inverse operation \( i_\Omega : \Omega X \to \Omega X \) given by pointwise
application of $i$. For any $t \in [0,1],$

\[ i_\Omega(a)(t) = i(a(t)). \]  \hspace{1cm} (1.2.13)

Then $(\Omega X, \mu_\Omega)$ is an HA-space. Note that the constant loop $l_0$ is a strict unit.

Furthermore, if $(X, \mu)$ is a topological group, then so is $(\Omega X, \mu_\Omega)$.

When $X = (X, \mu)$ is an HA-space, the maps $\mu_c$ and $\mu_\Omega$ are homotopic (see Corollary 1.2.5 of [40]), so the induced coproducts $\mu_c^*$ and $\mu_\Omega^*$ on $H^*(\Omega X; \mathbb{F}_p)$ are the same. Furthermore $(\Omega X, \mu_c)$ is homotopy commutative (see Corollary 1.2.5 of [40]), so $H^*(\Omega X, \mu_c; \mathbb{F}_p) = H^*(\Omega X, \mu_\Omega; \mathbb{F}_p)$ is a cocommutative Hopf algebra, and $H_*(\Omega X, \mu_c; \mathbb{F}_p) = H_*(\Omega X, \mu_\Omega; \mathbb{F}_p)$ is a commutative algebra.

The following definition introduces an important map that allows us to work with pointwise definitions of loops during calculations.

**Definition 1.2.8.** Let us identify $S^1$ as the quotient space $[0,1]/\{0,1\}$. Given a simply-connected topological space $X$ and its loop space $\Omega X$, we can define the *evaluation map* $\varepsilon : S^1 \wedge \Omega X \to X$ as follows:

\[ \varepsilon(t, l) = l(t). \]  \hspace{1cm} (1.2.14)

The evaluation map will induce a homomorphism $\varepsilon^* : H^*(X; \mathbb{F}_p) \to H^*(S^1 \wedge \Omega X; \mathbb{F}_p)$, and since for each $n > 1$, the vector spaces $H^n(S^1 \wedge \Omega X; \mathbb{F}_p)$ and $H^{n-1}(\Omega X; \mathbb{F}_p)$ are
isomorphic, we have a composition of linear transformations in each degree $n > 1$:

$$H^n(X; \mathbb{F}_p) \xrightarrow{\varepsilon^*} H^n(S^1 \wedge \Omega X; \mathbb{F}_p) \xrightarrow{\cong} H^{n-1}(\Omega X; \mathbb{F}_p).$$

(Warning: these linear transformations will lower degree by one) We will collectively denote all of these linear transformations as the cohomology suspension map $\sigma^*$. Elements in the image of $\sigma^*$ will be called suspension elements.

We also have the related map $\hat{\varepsilon}: S^1 \times \Omega X \to X$ given by $\hat{\varepsilon}(t, l) = l(t)$. It also induces the cohomology suspension map, since the following diagram commutes:

$$\begin{array}{ccc}
S^1 \times \Omega X & \xrightarrow{\hat{\varepsilon}} & X \\
\downarrow & & \downarrow \varepsilon \\
S^1 \wedge \Omega X & \xrightarrow{\varepsilon} & \end{array}$$

In general, if $x \in H^n(X; \mathbb{F}_p)$, then there is a generator $s \in H^1(S^1; \mathbb{F}_p)$ such that

$$\varepsilon^*(x) = s \otimes \sigma^*(x). \tag{1.2.15}$$

As a word of warning, the reader must know that there is another linear transformation called the cohomology suspension, $\sigma: H^n(X; \mathbb{F}_p) \to H^{n-1}(\Omega X; \mathbb{F}_p)$. For example, this map appears in Proposition 2.13 of Kono and Kozima’s paper [21].

The following diagram (based on material from Switzer’s textbook [38]) relates the
two cohomology suspension maps for \( n > 1 \):

\[
\begin{array}{ccc}
H^{n-1}(\Omega X; \mathbb{F}_p) & \xleftarrow{\text{id}_{\Omega X}} & H^{n-1}(\Omega X; \mathbb{F}_p) \\
\cong & \sigma^* & \sigma \\
H^n(S^1 \wedge \Omega X; \mathbb{F}_p) & \xrightarrow{\epsilon_*} & H^n(X; \mathbb{F}_p)
\end{array}
\]

Hence \( \sigma^* = \sigma \) for any degree \( n > 1 \). We refer the reader to Switzer’s textbook for further information about \( \sigma \).

Related to loop spaces are the free loop spaces, which can receive an H-space structure from its base space as follows:

**Definition 1.2.9.** Let \( X \) be a simply-connected topological space. Its free loop space \( \Lambda X \) is the space of functions \( l : [0, 1] \to X \) with the property that \( l(0) = l(1) \). The space will have the compact-open topology, and its basepoint is the constant loop \( l_0 \), where \( l_0(t) = x_0 \) for all \( t \in [0, 1] \). Notice that \( \Omega X \subset \Lambda X \).

If \( X = (X, \mu) \) is an HA-space, then we can define a multiplication map \( \mu_\Lambda : \Lambda X \times \Lambda X \to \Lambda X \) given by pointwise multiplication of loops: for any \( t \in [0, 1] \),

\[
\mu_\Lambda(a, b)(t) = \mu(a(t), b(t)),
\]

\[
= a(t)b(t), \tag{1.2.16}
\]

and a homotopy inverse operation \( i_\Lambda : \Lambda X \to \Lambda X \) defined so that for any \( t \in [0, 1] \),

\[
i_\Lambda(a)(t) = i(a(t)). \tag{1.2.17}
\]
Then $(\Lambda X, \mu_A)$ is an HA-space with a strict unit. Furthermore, if $(X, \mu)$ is a topological group, then so is $(\Lambda X, \mu_A)$.

Given a simply-connected topological space $X$ and its free loop space $\Lambda X$, we can define the free evaluation map $\hat{\varepsilon}_f : S^1 \times \Lambda X \to X$ as follows:

$$\hat{\varepsilon}_f(t, l) = l(t).$$  \hfill (1.2.18)

The loop space and free loop space are part of an important fibration sequence which we will need for many of our definitions and results:

**Definition 1.2.10.** Let $X$ be any simply-connected topological space. Let $j : \Omega X \to \Lambda X$ be the inclusion

$$j(l) = l,$$

and let $\varepsilon_0 : \Lambda X \to X$ be evaluation at $t = 0$:

$$\varepsilon_0(\varphi) = \varphi(0).$$

Then we have a fibration sequence which we will call the free loop fibration:

$$\begin{array}{c}
\Omega X \\
\downarrow j \\
\Lambda X \\
\downarrow \varepsilon_0 \\
X
\end{array}$$
Now let us discuss an important property of topological groups, loop spaces, and (strictly) associative H-spaces that distinguishes them from spaces which are merely HA-spaces.

**Definition 1.2.11.** Let $Y$ be a space. If $X$ is a space which satisfies $Y \simeq \Omega X$, then $X$ is called a classifying space of $Y$.

For example, any loop space has a classifying space. Milnor has shown that given a topological group $G$, one can construct a space $B_G$ such that $G \simeq \Omega B_G$, so $B_G$ is a classifying space of $G$ [28]. He also shows a reverse procedure: given a loop space $Y = \Omega X$, one can find a topological group $G$ such that $Y \simeq G$. Later, Dold, Lashof, and Milgram find ways to construct a classifying space for any associative H-space: given an associative H-space $X$, one can find a space $B_X$ such that $X \simeq \Omega B_X$ [7, 27]. Overall, any associative H-space is homotopy equivalent to a loop space, which in turn is homotopy equivalent to a topological group.

However, the same construction cannot be done for HA-spaces in general. As we will see in Section 2.5, HA-spaces may not be associative enough to possess classifying spaces. Thus, a challenge that occurs when generalizing a result from H-spaces with classifying spaces to H-spaces which do not is that we can no longer assume that the space is homotopy equivalent to a topological group with an associative multiplication and a strict inverse operation. Let us illustrate with some concepts from Lie group theory which have found applications in characterizing finite simply-connected topological groups whose homology over $\mathbb{F}_p$ is commutative. The following definitions are due to Kono and Kozima from [21]:
**Definition 1.2.12.** Let \((G, \mu)\) be a finite simply-connected topological group (for example a compact simply-connected Lie group). The *adjoint action of \(G\) on itself* \(ad : G \times G \to G\) is given by

\[
\text{ad}(g, h) = g h g^{-1},
\]

and the *commutator (map) com : \(G \times G \to G\) is given by*

\[
\text{com}(g, h) = g h g^{-1} h^{-1} = (g h)(h g)^{-1}.
\]

The *adjoint action of \(G\) on its loop space \(Ad : G \times \Omega G \to \Omega G\) is defined so that given \(g \in G\) and \(l \in \Omega G\), for any \(t \in [0, 1]\),

\[
\text{Ad}(g, l)(t) = g l(t) g^{-1}.
\]

Notice that at \(t = 0\),

\[
\text{Ad}(g, l)(0) = g l(0) g^{-1} = gg_{0} g^{-1} = gg^{-1} = g_{0}.
\]
so $Ad(g,l)$ is indeed a loop in $\Omega G$. There is also a map we will call $Com : G \times \Omega G \to \Omega G$. It is defined so that given $g \in G$ and $l \in \Omega G$, for any $t \in [0,1]$,

$$Com(g,l)(t) = gl(t)g^{-1}l(t)^{-1}$$

$$= (gl(t))(l(t)g)^{-1}$$

$$= com(g,l(t)). \quad (1.2.22)$$

Notice that at $t = 0$,

$$Com(g,l)(0) = gl(0)g^{-1}l(0)^{-1}$$

$$= gg^{-1}$$

$$= g0.$$  

so $Com(g,l)$ is indeed a loop in $\Omega G$.

Notice how these definitions rely on strict associativity in order to unambiguously multiply more than two elements. Furthermore, notice how strict inverses were needed when we verified that the maps $Ad$ and $Com$ give outputs in $\Omega G$.

Previously, Kono and Kozima have used these maps on Lie groups to find a characterization of compact simply-connected Lie groups whose homology over $\mathbb{F}_p$ is commutative. Let $p^\Omega_2 : G \times \Omega G \to \Omega G$ be projection onto the second factor. Kono and Kozima prove that the induced homomorphism $Ad^* : H^*(\Omega G; \mathbb{F}_p) \to$
$H^*(G; \mathbb{F}_p) \otimes H^*(\Omega G; \mathbb{F}_p)$ is equal to the induced homomorphism of the projection onto the second factor $p^{\Omega^*}_2 : H^*(\Omega G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p) \otimes H^*(\Omega G; \mathbb{F}_p)$ if and only if the algebra $H_*(G, \mu; \mathbb{F}_p)$ is commutative [21]. Iwase proves this result for finite simply-connected loop spaces $(\Omega X, \mu_{\Omega})$ by using the fact that loop spaces, despite being homotopy associative and possessing only a homotopy inverse operation, are homotopy equivalent to topological groups [13].

Remark. Kono, Kozima, and Iwase actually use their maps to characterize $p$-torsion in $H_*(G; \mathbb{Z})$. However, the absence of $p$-torsion in $H_*(G; \mathbb{Z})$ is equivalent to commutativity of $H_*(G, \mu; \mathbb{F}_p)$ [17].

1.3 Commutator Maps for HA-spaces

We will commence stating the main results of this dissertation. For our first set of results, we will construct a commutator map for HA-spaces (that generalizes the definition of a commutator map for topological groups) and use it to find a new characterization of finite simply-connected HA-spaces whose homology over $\mathbb{F}_p$ is a commutative algebra.

First, let us define our commutator for a simply-connected HA-space.

**Definition 1.3.1.** Let $(X, \mu)$ be a simply-connected HA-space. We define the com-
mutator $\com : X \times X \to X$ as

$$
\com(x, y) = \mu(\mu(x, y), i(\mu(y, x)))
= (xy)i(yx).
$$

(1.3.1)

Now let us look at generalizing Kono and Kozima’s definition of $\Com : G \times \Omega G \to \Omega G$, where $G = (G, \mu)$ is a simply-connected topological group with identity $g_0$. Again, notice that at $t = 0$,

$$
\Com(g, l)(0) = gl(0)g^{-1}l(0)^{-1}
= gg_0g^{-1}g_0
= gg^{-1}
= g_0,
$$

so $\Com(g, l)$ is indeed a loop in $\Omega G$. This calculation works because of the strict inverses.

Now consider a simply-connected HA-space $(X, \mu)$ with basepoint $x_0$ and homotopy inverse operation $i$. Given any $x \in X$ and $l \in \Omega X$, suppose we have a loop $\varphi \in \Lambda X$ given pointwise by

$$
\varphi(t) = (xl(t))i(l(t)x).
= \com(x, l(t)).
$$

(1.3.2)
Is $\varphi \in \Omega X$? Let us plug in $t = 0$:

\[
\varphi(0) = (xl(0))i(l(0)x) = (xx_0)i(x_0x) = xi(x).
\]

We must be careful: $i$ is merely a homotopy inverse operation, so it is not necessarily true that $xi(x) = x_0$ for all $x \in X$. Thus, a function which takes a pair $(x, l) \in X \times \Omega X$ to a loop given by $(xl(t))i(l(t)x)$ may give values (loops) in $\Lambda X$ that might not be in $\Omega X$.

Nevertheless, we can develop a map whose domain is $X \times \Omega X$ and whose codomain is $\Omega X$ which generalizes Kono and Kozima’s definition of $Com$. We can use the fibration in Definition 1.2.10:

**Theorem 1.3.2.** Let $(X, \mu)$ be a simply-connected HA-space. Define a map $\widehat{Com} : X \times \Omega X \to \Lambda X$ so that given $x \in X$ and $l \in \Omega X$, for any $t \in [0, 1]$,

\[
\widehat{Com}(x, l)(t) = (xl(t))i(l(t)x) = com(x, l(t)).
\] (1.3.3)

Then there exists a map $Com : X \times \Omega X \to \Omega X$ which is unique up to homotopy such that

\[
\widehat{Com} \simeq j \circ Com,
\]
and if \((X,\mu)\) is also a topological group, then this definition of \(\text{Com}\) will agree with Kono and Kozima’s definition.

Our first result using \(\text{com}\) and \(\text{Com}\) is the computation of the induced homomorphism \(\text{com}^* : H^*(X,\mu;\mathbb{F}_p) \rightarrow H^*(X,\mu;\mathbb{F}_p) \otimes H^*(X,\mu;\mathbb{F}_p)\) for a finite simply-connected HA-space \((X,\mu)\).

**Lemma 1.3.3.** Let \((X,\mu)\) be a finite simply-connected HA-space and \(p\) be an odd prime. If \(x\) is an element of \(S\) (see Theorem 1.2.4), then

\[
\text{com}^*(x) = \overline{\mu}^*(x) - T^*_{X,X} \overline{\mu}^*(x). \tag{1.3.4}
\]

This lemma serves as a starting point for the main results of this dissertation. From here, we can use this formula for \(\text{com}^*\) to find a formula for \(\text{Com}^* : H^*(\Omega X;\mathbb{F}_p) \rightarrow H^*(X;\mathbb{F}_p) \otimes H^*(\Omega X;\mathbb{F}_p)\) on suspension elements.

**Lemma 1.3.4.** Let \((X,\mu)\) be a finite simply-connected HA-space and \(p\) be an odd prime. If \(x\) is an element of \(S\), then

\[
\text{Com}^*(\sigma^*(x)) = (1 \otimes \sigma^*)\overline{\mu}^*(x). \tag{1.3.5}
\]

Using these formulas, we can find additional characterizations of finite simply-connected HA-spaces whose homology over \(\mathbb{F}_p\) is a commutative algebra. This allows us to present a connection between the commutator \(\text{com}\) of an HA-space and commutativity of \(H_*(X,\mu;\mathbb{F}_p)\):
Theorem 1.3.5. Let \((X, \mu)\) be a finite simply-connected HA-space and \(p\) be an odd prime. Then

a) the algebra \(H_*(X, \mu; \mathbb{F}_p)\) is commutative iff \(\text{com}^*\) is a trivial homomorphism.

b) the algebra \(H_*(X, \mu; \mathbb{F}_p)\) is commutative iff \(\text{Com}^*\) is a trivial homomorphism.

These results improve on previous ones by Kono, Kozima, and Iwase, who prove Theorem 1.3.5 for the special cases when \((X, \mu)\) is a finite simply-connected loop space, a topological group, or a Lie group. Their results rely on the fact that these kinds of H-spaces either have an associative multiplication and strict inverse, or are homotopy equivalent to a space with those properties (and hence possess a classifying space). We show that only homotopy associativity and existence of a homotopy inverse operation are needed to prove these results.

1.4 The Homology of Free Loop Spaces

We will use Lemmas 1.3.3 and 1.3.4 to compute coproducts in \(H^*(\Lambda X; \mathbb{F}_p)\) and determine when \(H^*(\Lambda X; \mathbb{F}_p)\) and \(H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)\) are not only isomorphic as algebras, but also as Hopf algebras.

Definition 1.4.1. Given a simply-connected HA-space \((X, \mu)\) with two-sided homotopy inverse operation \(\iota\), we give the product space \(X \times \Omega X\) a multiplication map \(\mu_{X \times \Omega X}\), where

\[
\mu_{X \times \Omega X} = (\mu \times \mu_\Omega)(id_X \times T_{\Omega X,X} \times id_{\Omega X}).
\]
Pointwise, given \((x_1, l_1), (x_2, l_2) \in X \times \Omega X\), at \(t \in [0, 1]\),

\[
\mu_{X \times \Omega X}((x_1, l_1), (x_2, l_2)) = (x_1 x_2, l_1(t)l_2(t)). \tag{1.4.2}
\]

We also define a homotopy inverse operation

\[
i_{X \times \Omega X} = i \times i_{\Omega}
\]

This makes \((X \times \Omega X, \mu_{X \times \Omega X})\) an HA-space.

We focus on a homotopy equivalence \(h_1 : \Lambda X \to X \times \Omega X\). First, we will look at the definition of \(h_1\) for topological groups, and then discuss what happens for HA-spaces.

**Definition 1.4.2.** If \((X, \mu)\) is a simply-connected topological group, then we can define \(h_1(\varphi) = (\varphi(0), l)\), where \(l \in \Omega X\) is the loop given by

\[
l(t) = \varphi(0)^{-1}\varphi(t). \tag{1.4.3}
\]

Notice that at \(t = 0\),

\[
l(0) = \varphi(0)^{-1}\varphi(0)
\]

\[
= x_0,
\]

so \(l\) is indeed an element of \(\Omega X\).
In the case that \((X, \mu)\) is a simply-connected topological group, we will show that the map \(h_1\) is a homeomorphism.

In the case that \((X, \mu)\) is a simply-connected HA-space that is not known to be (homotopy equivalent to) a topological group, then like \(Com\), we cannot write down \(h_1\) pointwise. This is because the loop \(l\) given by

\[
l(t) = \mu (i(\varphi(0)), \varphi(t))
\]

may not be an element of \(\Omega X\), because at \(t = 0\),

\[
l(0) = i(\varphi(0))\varphi(0),
\]

which might not equal \(x_0\) in general. Instead, we can use a map \(h'_1 : \Lambda X \to X \times \Lambda X\) to help us define \(h_1\):

**Theorem 1.4.3.** Let \((X, \mu)\) be a simply-connected HA-space. Define a map \(h'_1 : \Lambda X \to X \times \Lambda X\) so that given a loop \(\varphi \in \Lambda X\), for any \(t \in [0, 1]\), \(h'_1\) takes a loop \(\varphi \in \Lambda X\) to a pair \((\varphi(0), l)\) where \(l \in \Lambda X\) is the loop given by

\[
l(t) = \mu (i(\varphi(0)), \varphi(t)). \tag{1.4.4}
\]

Then there exists a map \(h_1 : \Lambda X \to X \times \Omega X\) which is unique up to homotopy such that

\[
h'_1 \simeq (id_X \times j)h_1, \tag{1.4.5}
\]
and \( h_1 \) is a homotopy equivalence. In particular, if we define \( h_2 : X \times \Omega X \to \Lambda X \) by

\[
h_2(x, l)(t) = xl(t),
\]

(1.4.6)

then \( h_2 h_1 \simeq \text{id}_{\Lambda X} \) and \( h_1 h_2 \simeq \text{id}_{X \times \Omega X} \).

Hence \( H^*(\Lambda X; \F_p) \) and \( H^*(X; \F_p) \otimes H^*(\Omega X; \F_p) \) are isomorphic as algebras via the isomorphisms

\[
h_1^* : H^*(X; \F_p) \otimes H^*(\Omega X; \F_p) \to H^*(\Lambda X; \F_p),
\]

\[
h_2^* : H^*(\Lambda X; \F_p) \to H^*(X; \F_p) \otimes H^*(\Omega X; \F_p),
\]

while \( H_*(\Lambda X; \F_p) \) and \( H_*(X; \F_p) \otimes H_*(\Omega X; \F_p) \) are isomorphic as vector spaces via the linear transformations

\[
h_{1*} : H_*(\Lambda X; \F_p) \to H_*(X; \F_p) \otimes H_*(\Omega X; \F_p),
\]

\[
h_{2*} : H_*(X; \F_p) \otimes H_*(\Omega X; \F_p) \to H_*(\Lambda X; \F_p).
\]

We can ask if \( h_1 \) is an H-map; that is, we ask whether the following diagram commutes or not:

\[
\begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{h_1 \times h_1} & X \times \Omega X \times X \times \Omega X \\
\mu_X \downarrow & & \mu_X \times \Omega X \\
\Lambda X & \xrightarrow{h_1} & X \times \Omega X
\end{array}
\]
If \( h_1 \) is an H-map, then it would induce a Hopf algebra isomorphism

\[
h_1^*: H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p),
\]

and an algebra isomorphism

\[
h_1*: H_*(\Lambda X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p).
\]

Consequently, we can multiply elements of \( H_*(\Lambda X, \mu; \mathbb{F}_p) \) by using \( h_1* \) and \( h_2* \) to carry out the multiplication in \( H_*(X, \mu; \mathbb{F}_p) \otimes H_*(\Omega X, \mu; \mathbb{F}_p) \).

However, it is not clear if \( h_1 \) is an H-map in general. In this case, we can use a basic tool from H-space theory, the H-deviation, to study how \( h_1 \) preserves the multiplication. The H-deviation of \( h_1 \) is a map \( D_{h_1} : \Lambda X \times \Lambda X \to X \times \Omega X \), and it is defined (up to homotopy) so that \( D_{h_1} \) is nullhomotopic if and only if \( h_1 \) is an H-map.

We will find a formula for \( D_{h_1} \) in terms of \( \text{Com} \) which will allow us to compute products in \( H_*(\Lambda X, \mu; \mathbb{F}_p) \) using three pieces of information: the product structures of \( H_*(X, \mu; \mathbb{F}_p) \) and \( H_*(\Omega X, \mu; \mathbb{F}_p) \), and knowledge of

\[
\text{Com}_*: H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p).
\]

**Lemma 1.4.4.** Let \((X, \mu)\) be a simply-connected HA-space. Given any \( x \otimes y \in \)
\[ H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p), \text{ we have:} \]

\[ \mu^*_\Lambda(h^*_1(x \otimes y)) = \Delta^*_\Lambda \left( D^*_{h_1} \otimes [(h^*_1 \otimes h^*_1)(\mu_{X \times \Omega X})^*] \right)(\mu^*_{X \times \Omega X}(x \otimes y)). \tag{1.4.7} \]

Let \( p_2^\Omega : X \times \Omega X \to \Omega X \) be projection onto the second factor, \( \varepsilon_0 : \Lambda X \to X \) be defined by \( \varepsilon_0(\varphi) = \varphi(0) \) and \( j_2 : \Omega X \to X \times \Omega X \) be the inclusion \( j_2(l) = (x_0, l) \). Then we can write \( D_{h_1} \) as the composition

\[ D_{h_1} \simeq j_2 \circ \text{Com} \circ (i \times \text{id}_{\Omega X}) \circ (\varepsilon_0 \times (p_2^\Omega \circ h_1)) \circ T_{\Lambda X}. \tag{1.4.8} \]

From here, we can use \( D_{h_1} \) to find a multiplication map

\[ \tilde{\mu} : X \times \Omega X \times X \times \Omega X \to X \times \Omega X \]

such that the following diagram commutes:

\[ \begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{h_1 \times h_1} & X \times \Omega X \times X \times \Omega X \\
\mu_{\Lambda} \downarrow & & \downarrow \tilde{\mu} \\
X & \xrightarrow{h_1} & X \times \Omega X
\end{array} \tag{1.4.9} \]

**Theorem 1.4.5.** Let \( \omega : X \times \Omega X \to \Omega X \) be given by

\[ \omega(x, l) = \text{Com} \circ (i \times \text{id}_{\Omega X})(x, l) \]

\[ = \text{Com}(i(x), l). \]
Then the following choice of \( \tilde{\mu} : X \times \Omega X \times X \times \Omega X \to X \times \Omega X \) will make diagram \( 1.4.9 \) commute:

\[
\tilde{\mu} = (id_X \times \mu_\Omega)(\mu \times \omega \times \mu_\Omega)(id_X \times \Delta_X \times \Delta_{\Omega X} \times id_{\Omega X})(id_X \times T_{\Omega X,X} \times id_{\Omega X}),
\]

\[
\tilde{\mu}(x_1, l_1, x_2, l_2) = (\mu(x_1, x_2), \mu_\Omega(Com(i(x_2), l_1), \mu_\Omega(l_1, l_2))).
\]

With these results, we can use \( Com^* \) to determine when \( H^*(\Lambda X, \mu_\Lambda; \mathbb{F}_p) \) and \( H^*(X, \mu; \mathbb{F}_p) \otimes H^*(\Omega X, \mu_\Omega; \mathbb{F}_p) \) are isomorphic as Hopf algebras in the case that \( (X, \mu) \) is a finite simply-connected HA-space.

**Theorem 1.4.6.** Let \( (X, \mu) \) be a finite simply-connected HA-space. Then

\begin{enumerate}
  \item the homomorphism \( D_{h_1}^* \) is trivial iff \( Com^* \) is a trivial homomorphism.
  \item the algebra \( H_*(X, \mu; \mathbb{F}_p) \) is commutative iff

\[
h_1^* : H^*(X \times \Omega X, \mu_{X \times \Omega X}; \mathbb{F}_p) \to H^*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)
\]

is a Hopf algebra isomorphism.
\end{enumerate}

We can use Theorem 1.4.5 and \( Com_* \) to compute products in \( H_*(\Lambda X, \mu_\Lambda; \mathbb{F}_p) \).

We will study the multiplication map

\[
\mu_{\Lambda_*} : H_*(\Lambda X; \mathbb{F}_p) \otimes H_*(\Lambda X; \mathbb{F}_p) \to H_*(\Lambda X; \mathbb{F}_p)
\]
in the homology of \((\Lambda X, \mu_\Lambda)\). Given

\[
\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u} \in H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p),
\]

we have

\[
\mu_{X \times \Omega X}((\bar{x} \otimes \bar{t}), (\bar{y} \otimes \bar{u})) = \bar{x}\bar{y} \otimes \bar{t}\bar{u},
\]

where \(\mu_*(\bar{x} \otimes \bar{y}) = \bar{x}\bar{y}\) and \(\mu_{\Omega_*}(\bar{t} \otimes \bar{u}) = \bar{t}\bar{u}\). In contrast, the product of \((\bar{x} \otimes \bar{t}) \in H_*(\Lambda X; \mathbb{F}_p)\) and \((\bar{y} \otimes \bar{u}) \in H_*(\Lambda X; \mathbb{F}_p)\) can be more complicated. The following theorem gives an idea of what can happen; we will restate and prove the theorem in more generality in Chapter 4:

**Theorem 1.4.7.** Let \(\bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p)\), \(\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)\) all have positive degree and be primitive. Then

\[
\mu_{\Lambda X_*}((h_{2*} \otimes h_{2*})(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})) = h_{2*}[\bar{x} \otimes \text{Com}_*(i_*(\bar{y}) \otimes \bar{t})\bar{u} + \bar{x}\bar{y} \otimes \bar{t}\bar{u}].
\]

Dually, in cohomology, we can use Lemma 1.4.4, Theorem 1.4.5, and Lemma 1.3.4 to compute coproducts of elements in the image of

\[
h_1^*p_2^*\sigma^* : H^*(X; \mathbb{F}_p) \rightarrow H^*(\Lambda X; \mathbb{F}_p).
\]
Theorem 1.4.8. Let $x \in S$. Then

$$
\mu^\Lambda_h(h^*_1(1 \otimes \sigma^*(x))) = (h^*_1 \otimes h^*_1)[1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \sigma^*(x) + 1 \otimes T^\Lambda_{\Omega, X}[(i^* \otimes \sigma^*)(\mu^\Lambda(x))] \otimes 1].
$$

Kono and Kozima briefly mention an application for their work describing the difference between the group structure of $(\Lambda G, \mu^\Lambda)$ and that of $(G \times \Omega G, \mu_{G \times \Omega G})$. They define a homeomorphism of topological spaces $\Phi : \Omega G \times G \rightarrow \Lambda G$, where $\Phi(l, g)$ is the loop given by $l(t)g$. If $\Phi$ is a group homomorphism, then the following diagram would strictly commute:

$$
\begin{array}{ccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{(\mu_{\Omega \times G} \times \mu)(id_{\Omega G} \times T \times id_G)} & \Omega G \times G \\
\Phi \times \Phi & & \Phi \\
\Lambda G \times \Lambda G & \xrightarrow{\mu^\Lambda} & \Lambda G
\end{array}
$$

In general, however, $\Phi$ is not known to be a group homomorphism unless $(G, \mu)$ is abelian [13]. Instead, Kono and Kozima present a modified diagram which will always commute. First, they define a map

$$
\omega : \Omega G \times G \times \Omega G \times G \rightarrow \Omega G \times \Omega G \times G \times G
$$

given by the composition involving the adjoint action $Ad$:

$$
\omega = (id_{\Omega G} \times T_{G, \Omega G} \times id_G)(id_{\Omega G \times G} \times Ad \times id_G)(id_{\Omega G} \times \Delta_G \times id_{\Omega G \times G});
$$
for any $t \in [0, 1]$,

$$\omega(l_1, g_1, l_2, g_2)(t) = (l_1, Ad(g_1, l_2), g_1, g_2)(t)$$

$$= (l_1(t), g_1l_2(t)g_1^{-1}, g_1, g_2).$$

Then the following diagram strictly commutes whether or not $\Phi$ is a group homomorphism:

$$\begin{array}{ccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{\omega} & \Omega G \times \Omega G \times G \times G \\
\downarrow \Phi \times \Phi & & \downarrow \mu \times \mu \\
\Lambda G \times \Lambda G & \xrightarrow{\mu_{\Lambda}} & \Lambda G
\end{array}$$

According to this diagram, if the coproduct structure of $H^*(G, \mu; \mathbb{F}_p)$, the coproduct structure of $H^*(\Omega G, \mu_{\Omega}; \mathbb{F}_p)$, and values of the homomorphism $Ad^*$ are known, then we can use those pieces of information to compute coproducts in $H^*(\Lambda G, \mu_{\Lambda}; \mathbb{F}_p)$. However, Kono, Kozima, and Iwase do not offer any results in this direction beyond mentioning the diagram. Hamanaka applies homology to this diagram to compute products in $H_*(\Lambda G_2; \mathbb{F}_2)$ using his calculations of $Ad_*$, but remarks that for the other exceptional Lie groups, “it is complicated to write them out exactly” [9].

### 1.5 Constructing New H-spaces

Given a finite simply-connected HA-space $(X, \mu)$ and an odd prime $p$, we will find an explicit form for a map $\nu : X \times X \to X$ so that $\nu$ induces a new multiplication
map on $H_*(X; \mathbb{F}_p) = H_*(X, \nu; \mathbb{F}_p)$ which makes it a commutative (but possibly nonassociative) algebra. We also find conditions on the original multiplication map $\mu$ and its induced homomorphism which make $H_*(X, \nu; \mathbb{F}_p)$ commutative and associative as well.

One interesting aspect of H-space theory is that we do not restrict our spaces to just one multiplication map. The idea of finding alternate multiplication maps on H-spaces which induced given conditions on cohomology or homology comes from Harper and Zabrodsky’s work [11]. Their process of constructing new multiplication maps from old ones is called altering the H-structure. Their technique can prove that the underlying topological spaces of the Lie groups $(F_4, \mu)$ and $(E_6, \mu)$ have a multiplication map $\nu$ which induces a commutative multiplication map on their homology algebras over $\mathbb{F}_3$. We adapted their methods to finite simply-connected HA-spaces $(X, \mu)$ with homotopy inverse operation $i$ to find an explicit form for a multiplication map $\nu$ which makes $H_*(X, \nu; \mathbb{F}_p)$ a commutative (if not associative) algebra:

**Theorem 1.5.1.** Let $(X, \mu)$ be a finite simply-connected HA-space and $p$ be a fixed odd prime. Define an H-structure $\nu : X \times X \to X$ as a product

$$\nu(x, y) = \underbrace{(\text{com}(x, y) \ldots \text{com}(x, y))}_\text{$p-1$ times} \mu(x, y)$$

(1.5.1)

where concatenation means multiply using $\mu$. Then $(X, \nu)$ is an H-space for which
$H_*(X, \nu; \mathbb{F}_p)$ is a commutative algebra. Furthermore, if $x$ is an element of $S$, then

$$\nu^*(x) = \frac{1}{2} (\mu^*(x) + T^*_{X,X} \mu^*(x)).$$

(1.5.2)

We can also ask when $H_*(X, \nu; \mathbb{F}_p)$ is associative. In fact, we look at certain cases when $H_*(X, \nu; \mathbb{F}_p)$ turns out to be commutative and associative, and determine a way to measure lack of associativity in terms of the original coproduct $\mu^*$:

**Theorem 1.5.2.** Let $(X, \mu)$ be a finite simply-connected HA-space. Let $\nu$ be defined as in equation 1.5.1. The Hopf algebra $H_*(X, \nu; \mathbb{F}_p)$ is associative if and only if in $H^*(X, \mu; \mathbb{F}_p)$,

$$\frac{1}{4} ((\mu^* \otimes 1)T^* \mu^* + (T^* \mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^* \mu^* - (1 \otimes T^* \mu^*)\mu^*) = 0.$$ (1.5.3)

Finally, if $\nu^*$ is coassociative, then $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated as a Hopf algebra with coproduct $\nu^*$.

### 1.6 Outline of the Dissertation

Our first main goal is to generalize $Com$ and $Ad$ beyond Lie groups and topological groups. In Chapter 2, we will review and present properties of H-spaces, HA-spaces, and their cohomology. We start off with a description of some common techniques we will use in our calculations that involve (free) loop spaces, products of spaces, homotopy associativity, and homotopy inverse operations. Next, we will col-
lect important results about the cohomology of finite H-spaces in Section 2.2. After that, we will look at HA-spaces and find properties of the homotopy inverse operation which parallel properties of a group inverse operation. From there, we discuss maps between H-spaces and introduce the H-deviation of a map, a way of measuring how maps between H-spaces preserve the multiplication on the spaces. Once we have presented this background material, then in Chapter 3, we discuss Kono, Kozima, and Iwase’s results and generalize them to finite simply-connected HA-spaces. We start with some properties of the free loop fibration, and then use it to give definitions of the maps $\text{com}$, $\text{ad}$, $\text{Com}$, and $\text{Ad}$ that agree with Kono and Kozima’s definitions when dealing with Lie groups and topological groups. From there, we prove Lemma 1.3.3, Lemma 1.3.4, and Theorem 1.3.5. Then, in Chapter 4, we will introduce and study the map $h_1$ and its H-deviation $D_{h_1}$, along with their induced homomorphisms in cohomology. This analysis allows us to prove Lemma 1.4.4, Theorem 1.4.5 and Theorem 1.4.6. We also use Theorem 1.4.5 to compute products in $H_*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$ and obtain a generalization of Theorem 1.4.7. Finally, in Chapter 5, we give a brief introduction to alterations of H-structures: creating new multiplication maps on a space using an old one. Then we use Lemma 1.3.3 to prove Theorem 1.5.1, and afterward, we look at some examples and discuss associativity of our new multiplication map (and the product structure it induces in homology), proving Theorem 1.5.2 along the way.
2 Properties of H-spaces and HA-spaces

This chapter will cover some additional background material that we will need in this dissertation. First, we will describe the techniques we will be using in many of our calculations, especially those which involve maps whose codomain is a (free) loop space or a product of spaces. In the next section, we will go over some important properties of the cohomology of finite H-spaces and their loop spaces. We review some results from Lin’s work on generators of the cohomology of finite H-spaces in [24], and collect some important properties of the cohomology suspension map defined in Section 1.2. Afterward, we will study HA-spaces and focus on maps whose codomain is an HA-space. This will allow us to give some additional properties of the homotopy inverse operation on an HA-space. Then, we will discuss maps between H-spaces and how we will deal with maps that do not preserve the multiplication. After that, we give a brief discussion of measuring associativity on an H-space using $A_n$-forms.

Remark. From now on, we will abbreviate the H-space $(X, \mu)$ as just $X$, unless we are considering multiple multiplication maps on the same space.
2.1 Loop Spaces, Products, and Homotopy

This first section presents elementary material, which we collect here for convenience. These techniques will be used in many of our more involved calculations, and as such, we feel it is important to describe them.

Throughout Chapters 3 and 4, we will be working with maps whose codomain is a loop space or a free loop space. Oftentimes, it will be convenient to work with pointwise definitions of the map (where we evaluate the loops at \( t \in S^1 \)) during calculations. The following lemmas will justify this practice.

**Lemma 2.1.1.** Let \( X \) and \( Y \) be simply-connected topological spaces and \( g_1, g_2 : X \to \Lambda Y \). Let \( \hat{\epsilon}_f : S^1 \times \Lambda Y \to Y \) be the free evaluation map given by

\[
\hat{\epsilon}_f(t, \varphi) = \varphi(t).
\]

Consider the compositions \( \hat{\epsilon}_f(id_{S^1} \times g_1), \hat{\epsilon}_f(id_{S^1} \times g_2) : S^1 \times X \to Y \). Elementwise, these are given by

\[
\begin{align*}
\hat{\epsilon}_f(id_{S^1} \times g_1)(t, x) &= g_1(x)(t), \\
\hat{\epsilon}_f(id_{S^1} \times g_2)(t, x) &= g_2(x)(t).
\end{align*}
\]

Suppose that

\[
\hat{\epsilon}_f(id_{S^1} \times g_1) \simeq \hat{\epsilon}_f(id_{S^1} \times g_2).
\]
Then
\[ g_1 \simeq g_2. \]

Proof. Since
\[ \hat{\epsilon}_f(id_{S^1} \times g_1) \simeq \hat{\epsilon}_f(id_{S^1} \times g_2), \]
there exists a map \( F : S^1 \times X \times I \to Y \) such that
\begin{align*}
F(t, x, 0) &= \hat{\epsilon}_f(id_{S^1} \times g_1)(t, x) = g_1(x)(t), \\
F(t, x, 1) &= \hat{\epsilon}_f(id_{S^1} \times g_2)(t, x) = g_2(x)(t).
\end{align*}
(2.1.1)

Define a map \( G : X \times I \to \Lambda Y \) so that \( G(x, u) \) is a loop given by
\[ G(x, u)(t) = F(t, x, u). \]

Then according to equation 2.1.1,
\begin{align*}
G(x, 0) &= g_1(x), \\
G(x, 1) &= g_2(x),
\end{align*}
so \( G \) is a homotopy from \( g_1 \) to \( g_2 \). \( \square \)

A similar proof can be used to show a corresponding result for maps into a loop space:
Lemma 2.1.2. Let $X$ and $Y$ be simply-connected topological spaces and $g_1, g_2 : X \to \Omega Y$. Let $\varepsilon : S^1 \land \Omega Y \to Y$ be the evaluation map given by

$$\varepsilon(t, \varphi) = \varphi(t).$$

Consider the compositions $\varepsilon(id_{S^1} \land g_1), \varepsilon(id_{S^1} \land g_2) : S^1 \land X \to Y$. Elementwise, these are given by

$$\varepsilon(id_{S^1} \land g_1)(t, x) = g_1(x)(t),$$
$$\varepsilon(id_{S^1} \land g_2)(t, x) = g_2(x)(t).$$

Suppose that

$$\varepsilon(id_{S^1} \land g_1) \simeq \varepsilon(id_{S^1} \land g_2).$$

Then

$$g_1 \simeq g_2.$$

Next, let us explain how we work with maps into a product of spaces. Given topological spaces $X$, $Y$, and $Z$, and a map $f : X \to Y \times Z$, we will use projections $p_1 : Y \times Z \to Y$ and $p_2 : Y \times Z \to Z$ to break down the map $f$ into its components:

$$f(x) = (p_1 f(x), p_2 f(x)).$$

We can then perform calculations on each component separately in order to determine
whether the map $f$ is equal to or homotopic to some other map.

**Lemma 2.1.3.** Let $X, Y, Z$ be topological spaces and $f, g : X \to Y \times Z$ be maps. Let $p_1 : Y \times Z \to Y$ and $p_2 : Y \times Z \to Z$ be the projections. If

$$p_1 f \simeq p_1 g$$

and

$$p_2 f \simeq p_2 g,$$

then

$$f \simeq g.$$

**Proof.** Let $H_1 : I \times X \to Y$ be a homotopy from $p_1 f$ to $p_1 g$, and let $H_2 : I \times X \to Z$ be a homotopy from $p_2 f$ to $p_2 g$. That is,

$$H_1(0, x) = p_1 f(x),$$

$$H_1(1, x) = p_1 g(x),$$

$$H_2(0, x) = p_2 f(x),$$

$$H_2(1, x) = p_2 g(x).$$

Define $H : I \times X \to Y \times Z$ by

$$H(t, x) = (H_1(t, x), H_2(t, x)).$$

(2.1.2)
Then

\[ H(0, x) = (H_1(0, x), H_2(0, x)) \]
\[ = (p_1 f(x), p_2 f(x)) \]
\[ = f(x), \]

and

\[ H(1, x) = (H_1(1, x), H_2(1, x)) \]
\[ = (p_1 g(x), p_2 g(x)) \]
\[ = g(x), \]

so \( H \) is a homotopy from \( f \) to \( g \).

Remark. In the language of category theory, if we work in the category of pointed topological spaces with morphisms being (based) homotopy classes of maps, then the previous lemma states that the Cartesian product of two pointed topological spaces is a product in the category.

Let us end this section with an example on working with homotopy associativity and homotopy inverse operations for an HA-space \((X, \mu)\) with homotopy inverse operation \(i : X \to X\).
Example 2.1.4. Let $X$ be an HA-space and $f: X \times X \to X$ be defined by

$$f(x, y) = (xy)i(y),$$

and let $p_1: X \times X \to X$ be projection onto the first factor:

$$p_1(x, y) = x.$$

We want to show that

$$f \simeq p_1.$$

We can expand $f(x, y)$ as

$$f(x, y) = \mu_X(\mu_X \times id_X)(id_X \times id_X \times i)(id_X \times \Delta_X)(x, y). \quad (2.1.3)$$

We can use homotopy associativity: $\mu_X(\mu_X \times id_X) \simeq \mu_X(id_X \times \mu_X)$, to see that $f$ is homotopic to a map which takes $(x, y)$ to

$$\mu_X(id_X \times \mu_X)(id_X \times id_X \times i)(id_X \times \Delta_X)(x, y)$$

$$= \mu_X (id_X \times (\mu_X (id_X \times i)\Delta_X)) (x, y)$$

$$= x (i(y)y) \quad (2.1.4)$$

By Definition 1.2.3,

$$\mu_X(id_X \times i)\Delta_X \simeq k,$$
so $f$ is homotopic to a map which takes $(x, y)$ to

$$
\mu_X (id_X \times k)(x, y) = \mu_X(x, x_0) = x = p_1(x, y).
$$

Therefore, $f \simeq p_1$.

As we have seen, we can use homotopy associativity to move the parentheses and obtain equation 2.1.4, and apply the homotopy inverse operation to “cancel” the product $i(y)y$ that appeared in equation 2.1.4. In full detail, we had to expand $f(x, y)$ in equation 2.1.3 in order to identify $\mu_X(\mu_X \times id_X)$ in the composition, and use homotopy associativity to make $\mu_X(id_X \times i)\Delta_X$ appear in the composition so we could replace it with $k$.

In practice, the process of expanding compositions with $\mu$ and $i$ can get cumbersome, especially when more elements are being multiplied (we might have to use homotopy associativity multiple times and include numerous $id_X$ factors). Therefore, in Chapters 3 and 4, we will refrain from writing expansions like the one in equation 2.1.3, so that calculations will look more like the next example:

**Example 2.1.5.** Let $X$ be an HA-space and $f : X \times X \to X$ be defined by

$$
f(x, y) = (xy)i(y),
$$
and let $p_1 : X \times X \to X$ be projection onto the first factor:

$$p_1(x, y) = x.$$ 

We want to show that

$$f \simeq p_1.$$ 

We can use homotopy associativity to see that $f$ is homotopic to a map which takes $(x, y)$ to $x(i(y)y)$. Hence $f$ is homotopic to a map which takes $(x, y)$ to

$$x = p_1(x, y).$$

Therefore, $f \simeq p_1$.

### 2.2 H-spaces and their Cohomology

In Section 1.1, we saw that there are only three kinds of monogenic Hopf algebras (recall that all algebras have $\mathbb{F}_p$, $p$ odd, as their base field), and any other Hopf algebra is isomorphic (as an algebra) to a tensor product of monogenic Hopf algebras. One can ask which Hopf algebras occur as the cohomology of some H-space.

**Example 2.2.1.** Let us look at exterior algebras first. Given any odd natural number $n$, Adams shows that there is an H-space $X$ (which might not be a topological group), called an odd sphere localized at the odd primes, such that the cohomology of $X$ over
$\mathbb{F}_p$ is

$$H^*(X; \mathbb{F}_p) = \wedge(x_n),$$

and $X$ itself is a CW complex of finite type (finite number of cells in each dimension) that has the odd sphere $S^n$ as a subcomplex [1]. Hence given an odd prime $p$, any exterior algebra (with any number of generators in any odd degree) over $\mathbb{F}_p$ can be realized as the cohomology of an H-space, namely the product of some odd spheres localized at the odd primes.

In contrast, when dealing with a finite H-space $X$, the possible degrees of even degree generators in $H^*(X; \mathbb{F}_p)$ are much more limited. Lin’s Binary Theorem from [24] describes the degrees in which even degree generators can occur. Let us quote part of it:

**Theorem 2.2.2 (Lin).** Let $X$ be a finite simply-connected H-space, and let $m$ be a positive integer. Then $Q^{2m}H^*(X; \mathbb{F}_p) = 0$ unless $m = \sum_{s=1}^{j} m_s p^s + 1$, where the $m_s$ coefficients are either zero or one, and at most one of the $m_s$ coefficients is zero (unless $j = 1$, in which case $m_1 = 1$).

Lin uses the Binary Theorem to prove the following result about the cohomology of $\Omega X$ and the cohomology suspension map $\sigma^*$:

**Lemma 2.2.3 (Lin).** Let $X$ be a finite simply-connected H-space. Then

$$H^{\text{odd}}(\Omega X; \mathbb{F}_p) = 0$$
and $\sigma^*$ vanishes on even degree elements of $H^*(X; \mathbb{F}_p)$.

In general, we have the following facts about the cohomology suspension map $\sigma^*$ from [39] and [6] respectively:

**Theorem 2.2.4.** Let $X$ be a simply-connected $H$-space.

1) *(Whitehead)* Decomposables in $H^*(X; \mathbb{F}_p)$ vanish on the cohomology suspension map, and the image of $\sigma^*$ is contained in $PH^*(X; \mathbb{F}_p)$.

2) *(Clark)* Moreover, for $n > 1$, if we look at the linear transformations

$$Q^n H^*(X; \mathbb{F}_p) \rightarrow P^{n-1} H^*(\Omega X; \mathbb{F}_p),$$

induced by $\sigma^*$, then if $n$ is odd, then this induced linear transformation is injective on $Q^n H^*(X; \mathbb{F}_p)$, and unless $n - 1 \equiv -2 \mod 2p$, it is surjective onto $P^{n-1} H^*(X; \mathbb{F}_p)$.

In the case that $X$ is a finite simply-connected HA-space and $S$ is the generating set from Theorem 1.2.4, $\sigma^*$ is injective on the set of odd degree elements of $S$.

### 2.3 HA-spaces and their Cohomology

Let $(X, \mu)$ be an HA-space with homotopy inverse operation $i$. We will introduce some properties of $i$, along with its induced homomorphism on cohomology. To do this, we need to look at homotopy classes of maps whose codomain is an HA-space. While it has been known that the set of (basepoint-preserving) homotopy classes of
maps whose codomain is an HA-space has a group structure (see James’ paper [14] and Zabrodsky’s book [40]), we would like to discuss this group structure in more detail and provide our own presentation.

For any space $Y$ and any HA-space $X$, given a map $f : Y \to X$, let us denote the homotopy class of $f$ by $[f]$. That is,

$$[f] = [g] \text{ if and only if } f \simeq g.$$ 

Let $\langle Y, X \rangle$ be the set of (basepoint-preserving) homotopy classes of maps from $Y$ to $X$ (our notation is due to Hatcher from [12]). Let us show that the HA-space structure of $X$ can be used to define a group structure on $\langle Y, X \rangle$. First, we will work with the functions (from $Y$ to $X$) themselves, and define a binary operation on functions from $Y$ to $X$:

**Definition 2.3.1.** Let $Y$ be any space and $X$ be an H-space. Given $f : Y \to X$ and $g : Y \to X$, we define a function $f * g : Y \to X$ by

$$f * g = \mu (f \times g) \Delta_Y.$$ 

Elementwise, this means that for any $y \in Y$,

$$(f * g) (y) = f(y)g(y).$$ 

Let us show how the homotopy associativity of $X$ will impose an associativity
condition on the binary operation $*$:

**Proposition 2.3.2.** Given $f : Y \to X$, $g : Y \to X$, and $h : Y \to X$, the binary operation $*$ satisfies

$$(f * g) * h \simeq f * (g * h).$$

**Proof.** By definition,

$$(f * g) * h = \mu ((f * g) \times h) \Delta_Y$$

$$= \mu ((\mu (f \times g) \Delta_Y) \times h) \Delta_Y. \quad (2.3.1)$$

Elementwise, this map takes an element $y \in Y$ to $(f(y)g(y))h(y) \in X$, so equation 2.3.1 becomes

$$(f * g) * h = \mu (\mu \times \text{id}_X) (f \times g \times h) (\text{id}_Y \times \Delta_Y) \Delta_Y. \quad (2.3.2)$$

Since $X$ is homotopy associative, $\mu (\mu \times \text{id}_X) \simeq \mu (\text{id}_X \times \mu)$. Thus, we have

$$(f * g) * h \simeq \mu (\text{id}_X \times \mu) (f \times g \times h) (\text{id}_Y \times \Delta_Y) \Delta_Y. \quad (2.3.3)$$

Elementwise, this map takes $y$ to $f(y)(g(y)h(y))$, so the right hand side of equation
2.3.3 becomes

\[(f * g) * h \simeq \mu (f \times (\mu (g \times h) \Delta_Y)) \Delta_Y \]

\[= \mu (f \times (g \times h)) \Delta_Y \]

\[= f \times (g \times h). \quad (2.3.4)\]

The binary operation passes to homotopy classes in \( \langle Y, X \rangle \) as follows:

\[[f] * [g] = [f \times g]. \quad (2.3.5)\]

In particular, by the previous proposition,

\[(([f] * [g]) * [h]) = [f \times g] * [h] \]

\[= [(f \times g) * h] \]

\[= [f \times (g \times h)] \]

\[= [f] * [g \times h] \]

\[= [f] * ([g] * [h]), \quad (2.3.6)\]

so \(*\) is an associative binary operation on \( \langle Y, X \rangle \). In fact, it gives a group structure on \( \langle Y, X \rangle \):

**Proposition 2.3.3.** Given an HA-space \( X \) and any space \( Y \), \( \langle Y, X \rangle \) is a group with
binary operation $\ast$.

Proof. Let us show that the identity element of $\langle Y, X \rangle$ is the homotopy class of the constant map $k : Y \to X$. To do this, we look at the map $f \ast k$. In $X$,

$$(f \ast k)(y) = (\mu(f \times k) \Delta_Y)(y) = \mu(f(y), x_0) = f(y),$$

so $f \ast k = f$. Similarly, $k \ast f = f$, so in $\langle Y, X \rangle$,

$$[f] \ast [k] = [k] \ast [f] = [f]. \quad (2.3.7)$$

Inverse elements are defined using the homotopy inverse operation $i : X \to X$.

Notice that

$$f \ast (i \circ f) = \mu(f \times (i \circ f)) \Delta_Y$$

$$= \mu(id_X \times i)(f \times f) \Delta_Y$$

$$= \mu(id_X \times i) \Delta_X f,$$

and the last map is nullhomotopic since $\mu(id_X \times i) \Delta_X$ is nullhomotopic. Similarly, $$(i \circ f) \ast f \simeq k,$$ so in $\langle Y, X \rangle$,

$$[f] \ast [i \circ f] = [i \circ f] \ast [f] = [k]. \quad (2.3.8)$$

Hence we will denote $[i \circ f]$ by $[f]^{-1}$. 

Equations 2.3.7 and 2.3.8, along with Proposition 2.3.2, show that $\langle Y, X \rangle$ is a group under the operation $*$ with identity $[k]$. 

Here are some properties of the homotopy inverse operation $i$ that are analogous to properties of inversion in a group, namely, that in a group $G$, $(g^{-1})^{-1} = g$ and $(g_1g_2)^{-1} = g_2^{-1}g_1^{-1}$. They have been used throughout the literature, but we were unable to find a reference, so for convenience, we have provided a proof using homotopy classes of maps into an HA-space. Our proof exploits the fact that $\langle Y, X \rangle$ is a group, so the properties of the group inverse operation in $\langle Y, X \rangle$ can be used to prove statements about maps being homotopic.

**Lemma 2.3.4.** In $X$, $i \circ i \simeq id_X$ and $i \circ \mu \simeq \mu(i \times i)T_{X,X}$.

*Proof.* Recall that in any group $G$, $(g^{-1})^{-1} = g$. Since $\langle X, X \rangle$ is a group, for any class $[f]$ (where $f$ is a map $f : X \to X$), we have

$$( [f]^{-1} )^{-1} = [f].$$

In particular,

$$[i \circ i] = [i]^{-1} = ([id_X]^{-1})^{-1} = [id_X],$$

so $i \circ i \simeq id_X$.

Recall that in any group $G$, $(g_1g_2)^{-1} = g_2^{-1}g_1^{-1}$. Since $\langle X \times X, X \rangle$ is a group,
for any classes \([f]\) and \([g]\), we have

\[
([f] * [g])^{-1} = [g]^{-1} * [f]^{-1}.
\]

Let \(f = p_1\) and \(g = p_2\), where \(p_1, p_2 : X \times X \to X\) are projection maps. Then we have

\[
(p_1 * p_2) (x, y) = \mu(p_1(x, y), p_2(x, y)) = \mu(x, y), \text{ and}
\]

\[
((i \circ p_2) * (i \circ p_1)) (x, y) = \mu(i(p_2(x, y)), i(p_1(x, y)))
\]

\[
= \mu(i(y), i(x))
\]

\[
= \mu(i \times i)(y, x),
\]

\[
= (\mu(i \times i) T_{X,X}) (x, y),
\]

so in \(\langle X \times X, X \rangle\),

\[
[i \circ \mu] = [\mu]^{-1} = (\mu(p_1 \times p_2))^{-1} = [p_2]^{-1} * [p_1]^{-1} = [\mu(i \times i) T_{X,X}].
\]

Therefore, \(i \circ \mu \simeq \mu(i \times i) T_{X,X} \). \(\Box\)

Next, we examine \(i^*\), the induced algebra homomorphism on \(H^*(X; \mathbb{F}_p)\). In Definition 1.2.3, we see that

\[
\mu(i \times id_X) \Delta_X \simeq \mu(id_X \times i) \Delta_X \simeq k.
\] (2.3.9)
Thus, if $x \in H^*(X; \mathbb{F}_p)$ has positive degree and $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$, then we can apply cohomology to equation 2.3.9 to determine a formula for $i^*(x)$:

**Proposition 2.3.5.** Suppose $x \in H^*(X; \mathbb{F}_p)$ has positive degree and $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$. Then

$$i^*(x) = -x - \sum i^*(c)d = -x - \sum c[i^*(d)]. \tag{2.3.10}$$

**Proof.** When we apply cohomology to equation 2.3.9, we see that

$$\Delta_X^*(i^* \otimes 1)\mu^*(x) = \Delta_X^*(i^* \otimes 1) \left( x \otimes 1 + 1 \otimes x + \sum c \otimes d \right)$$

$$= \Delta_X^* \left( i^*(x) \otimes 1 + 1 \otimes x + \sum i^*(c) \otimes d \right)$$

$$= i^*(x) + x + \sum i^*(c)d,$$

$$\Delta_X^*(1 \otimes i^*)\mu^*(x) = \Delta_X^* \left( 1 \otimes i^* \right) \left( x \otimes 1 + 1 \otimes x + \sum c \otimes d \right)$$

$$= \Delta_X^* \left( x \otimes 1 + 1 \otimes i^*(x) + \sum c \otimes i^*(d) \right)$$

$$= x + i^*(x) + \sum c[i^*(d)],$$

so

$$i^*(x) + x + \sum i^*(c)d = x + i^*(x) + \sum c[i^*(d)] = 0,$$

and therefore,

$$i^*(x) = -x - \sum i^*(c)d = -x - \sum c[i^*(d)]. \tag{2.3.11}$$
In particular,

\[ \sum i^*(c)d = \sum c[i^*(d)]. \tag{2.3.12} \]

By dualizing to homology, we obtain the following result:

**Corollary 2.3.6.** Let \( \bar{x} \in H_*(X; \mathbb{F}_p) \) have positive degree and \( \Delta_{X_*}(\bar{x}) = \bar{x} \otimes 1 + 1 \otimes \bar{x} + \sum \bar{c} \otimes \bar{d} \). Then

\[ i_*(\bar{x}) = -\bar{x} - \sum i_*(\bar{c})\bar{d} = -\bar{x} - \sum \bar{c}[i_*(\bar{d})]. \tag{2.3.13} \]

In addition, equation 2.3.11 tells us that the map \( i \) induces multiplication by \(-1\) on \( QH^*(X; \mathbb{F}_p) \).

**Example 2.3.7.** If \( x \in H^*(X; \mathbb{F}_p) \) is primitive, then \( i^*(x) = -x \). In general, if \( x \) is a product of primitive classes \( x_1, \ldots, x_n \),

\[ i^*(x) = i^*(x_1, \ldots, x_n) = i^*(x_1) \cdots i^*(x_n) = (-1)^n x. \]

In general, \( i^* \) may not be multiplication by \(-1\) on generators.

**Example 2.3.8.** Let \( X = E_8 \) and \( p = 3 \). For reference,

\[ H^*(E_8; \mathbb{F}_3) \cong \wedge (x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathbb{F}_p [x_8, x_{20}] / (x_8^3, x_{20}^3). \]
In particular, we have the following coproducts [10]:

\[ \mu^*(x_j) = x_j \otimes 1 + 1 \otimes x_j, \text{ where } j = 3, 7, 8, 19, 20, \]

\[ \mu^*(x_{15}) = x_{15} \otimes 1 + 1 \otimes x_{15} + x_8 \otimes x_7, \]

\[ \mu^*(x_{35}) = x_{35} \otimes 1 + 1 \otimes x_{35} + x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8x_{20} \otimes x_7. \]

Since \( x_i \) is primitive for \( j = 3, 7, 8, 19, 20 \), we have \( i^*(x_j) = -x_j \) in those cases. For \( x_{15} \) and \( x_{35} \), we have

\[
\begin{align*}
i^*(x_{15}) &= -x_{15} - i^*(x_8)x_7 \\
&= -x_{15} + x_8x_7,
\end{align*}
\]

\[
\begin{align*}
i^*(x_{35}) &= -x_{35} - i^*(x_8)x_{27} - i^*(-x_8^2)x_{19} - i^*(x_8^2)x_7 - i^*(x_8x_{20})x_7 \\
&= -x_{35} + x_8x_{27} + x_8^2x_{19} + x_{20}x_{15} - x_8x_{20}x_7.
\end{align*}
\]

Given a generator \( x \), it is possible to find another generator \( \tilde{x} \) that represents the same class as \( x \) in \( QH^*(X; \mathbb{F}_p) \) (again, \( p \) is odd), with \( i^*(\tilde{x}) = -\tilde{x} \).

**Definition 2.3.9.** If \( \tilde{x} \) is an algebra generator for which \( i^*(\tilde{x}) = -\tilde{x} \), then \( \tilde{x} \) is a \(-1\) characteristic generator of \( H^*(X; \mathbb{F}_p) \).

**Proposition 2.3.10.** Suppose \( x \) is a generator of \( H^*(X; \mathbb{F}_p) \) and \( \mu^*(x) = x \otimes 1 + 1 \otimes \)
\[ x + \sum c \otimes d. \text{ Let } \tilde{x} = x - \frac{1}{p-2} \sum i^*(c)d. \text{ Then } i^*(\tilde{x}) = -\tilde{x}. \]

**Proof.** Since \( \mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d, \) \( i^*(x) = -x - \sum i^*(c)d. \) Let \( \tilde{x} = x - \frac{1}{p-2} \sum i^*(c)d. \) Then

\[
i^*(\tilde{x}) = \left( \frac{p-2}{p-2} \right) i^*(\tilde{x}) = \frac{1}{p-2} \left[ -(p-2)x - (p-2) \sum i^*(c)d - \sum i^*(i^*(c))i^*(d) \right] = \frac{1}{p-2} \left[ -(p-2)x - (p-2) \sum i^*(c)d - \sum c[i^*(d)] \right].
\]

By equation 2.3.12, this equals

\[
\frac{1}{p-2} \left[ -(p-2)x - (p-2) \sum i^*(c)d - \sum i^*(c)d \right] = \frac{1}{p-2} \left[ -(p-2)x + \sum i^*(c)d \right] = -\tilde{x}.
\]

\[ \square \]

**Remark.** Kane’s book (specifically Chapter 48) contains a similar proposition [19]. However, his proof only works on maps \( \Psi : X \to X \) which induce multiplication by an integer \( \lambda \) which multiplicatively generates \( \mathbb{F}_p \setminus \{0\} \).

Although \( x \) and \( \tilde{x} \) represent the same element in \( QH^*(X; \mathbb{F}_p) \), their coproducts may not be equal.

**Proposition 2.3.11.** \( \mu^*(\tilde{x}) = \frac{1}{p-2} (\mu^*(i^*(x)) - \mu^*(x)). \)
Proof. By definition,

\[ \tilde{x} = x - \frac{1}{p-2} \sum i^*(c)d. \]

Since \( i^*(x) = -x - \sum i^*(c)d \), we can rewrite this as

\[ \tilde{x} = x + \frac{1}{p-2} x + \frac{1}{p-2} i^*(x) \]

and hence

\[ \tilde{x} = -\frac{1}{p-2} x + \frac{1}{p-2} i^*(x). \]

We obtain the result by taking the coproduct of both sides.

These \( -1 \) characteristic generators will be useful in Chapter 5 when we look for different coproducts on \( H^*(X; \mathbb{F}_p) \) and study whether they give primitively generated Hopf algebras.

2.4 Maps Between HA-spaces

Both Lie groups and topological groups have their own versions of homomorphisms - maps that preserve the structures on the spaces. We can define a similar notion for H-spaces as well. One interesting feature of H-space theory is that unlike group theory, we do not assume every map between H-spaces preserves the multiplication. Consequently, H-space theory has a tool for examining maps which do not preserve the multiplication, and we will need this tool in our definitions and results.
Material in this section can be found in [40].

Let us start with a generalization of group homomorphisms:

**Definition 2.4.1.** Let \((X, \mu_X)\) and \((Y, \mu_Y)\) be H-spaces, and \(f : X \to Y\) be a map. Then \(f\) is a \(\mu_X - \mu_Y\) \(H\)-map if \(f \circ \mu_X \simeq \mu_Y \circ (f \times f)\). That is, the following diagram commutes:

\[
\begin{array}{ccc}
X \times X & \xrightarrow{f \times f} & Y \times Y \\
\mu_X \downarrow & & \downarrow \mu_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

Frequently, we will drop “\(\mu_X - \mu_Y\)” and simply write “\(H\)-map.”

A map that is an \(H\)-map induces a special kind of homomorphism on cohomology:

**Definition 2.4.2.** Let \((A, \mu_A^*)\) and \((B, \mu_B^*)\) be Hopf algebras and \(\phi : A \to B\) be an algebra homomorphism. Suppose that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\mu_A \downarrow & & \downarrow \mu_B \\
A \otimes A & \xrightarrow{\phi \otimes \phi} & B \otimes B
\end{array}
\]

Then we call \(\phi\) a *Hopf algebra homomorphism*. If \(\phi\) is also an algebra isomorphism, then we will call it a *Hopf algebra isomorphism*, and \(A\) and \(B\) are said to be *isomorphic as Hopf algebras*.

In particular, if \((X, \mu_X)\) and \((Y, \mu_Y)\) are H-spaces and \(f : X \to Y\) is an H-map,
then $f^* : H^*(Y; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$ is a Hopf algebra homomorphism:

\[
\begin{array}{ccc}
H^*(Y; \mathbb{F}_p) & \xrightarrow{f^*} & H^*(X; \mathbb{F}_p) \\
\mu_Y^* & & \mu_X^* \\
\downarrow & & \downarrow \\
H^*(Y; \mathbb{F}_p) \otimes H^*(Y; \mathbb{F}_p) & \xrightarrow{f^* \otimes f^*} & H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)
\end{array}
\]

and $f_* : H_*(X; \mathbb{F}_p) \to H_*(Y; \mathbb{F}_p)$ is an algebra homomorphism:

\[
\begin{array}{ccc}
H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) & \xrightarrow{f_* \otimes f_*} & H_*(Y; \mathbb{F}_p) \otimes H_*(Y; \mathbb{F}_p) \\
\mu_X^* & & \mu_Y^* \\
\downarrow & & \downarrow \\
H_*(X; \mathbb{F}_p) & \xrightarrow{f_*} & H_*(Y; \mathbb{F}_p)
\end{array}
\]

**Example 2.4.3.** Let $(X, \mu)$ be an H-space and consider the diagonal map $\Delta_X : X \to X \times X$. If we give $X \times X$ the multiplication map $\mu_{X \times X} : X^4 \to X \times X$ given by

$$
\mu_{X \times X}(x, y, z, w) = (\mu(x, z), \mu(y, w)),
$$

then

$$
\mu_{X \times X} (\Delta_X \times \Delta_X) (x, y) = (\mu(x, y), \mu(x, y)),
$$

and

$$
\Delta_X \mu(x, y) = (\mu(x, y), \mu(x, y)),
$$

so the diagonal map is an H-map. Therefore, the cup product $\Delta_X^* : H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$ is a Hopf algebra homomorphism. Indeed, this must occur because the coproduct map, by definition, is an algebra homomorphism.
Let us find a relationship between H-maps and homotopy inverse operations that generalizes the fact that group homomorphisms preserve multiplicative inverses.

**Lemma 2.4.4.** Let \((X, \mu_X)\) and \((Y, \mu_Y)\) be HA-spaces with homotopy inverse operations \(i_X\) and \(i_Y\) respectively. Let \(f : X \rightarrow Y\) be an H-map. Define a map \(f^* : \langle X, X \rangle \rightarrow \langle X, Y \rangle\) as follows: given \([g] \in \langle X, X \rangle\),

\[
    f^*[g] = [fg]. \tag{2.4.1}
\]

Then \(f^*\) is a group homomorphism, and

\[
    i_Y f \simeq f i_X. \tag{2.4.2}
\]

**Proof.** Let us show that \(f^*\) is a group homomorphism. Before we proceed, let us indicate that the group operation in \(\langle X, X \rangle\) will be denoted \(\ast_X\), and the group operation in \(\langle X, Y \rangle\) will be denoted by \(\ast_Y\). Since \(f\) is an H-map, \(f \circ \mu_X \simeq \mu_Y \circ (f \times f)\), so for any \(g_1, g_2 : X \rightarrow X\), we have

\[
    f \mu_X (g_1 \times g_2) \Delta_X \simeq \mu_Y (f \times f) (g_1 \times g_2) \Delta_X \\
    = \mu_Y ((fg_1) \times (fg_2)) \Delta_X \tag{2.4.3}
\]

(elementwise, the right hand side of equation 2.4.3 takes \(x\) to \((fg_1(x)) (fg_2(x))\)). This
means that in \( \langle X, Y \rangle \),

\[
f^\sharp([g_1]_X [g_2]) = [fg_1]_Y [fg_2] = f^\sharp([g_1])_Y f^\sharp([g_2]),
\]

so \( f^\sharp \) is a group homomorphism. In particular, it preserves group inverses, so for any \([g] \in \langle X, X \rangle\),

\[
[f i_X g] = f^\sharp([i_X g]) = f^\sharp([g]^{-1}) = (f^\sharp([g]))^{-1} = [fg]^{-1} = [i_Y fg],
\]

so if we pick \( g = id_X \), we see

\[
[f i_X] = [i_Y f],
\]

so \( f i_X \simeq i_Y f \). \( \square \)

One major difference between group theory and H-space theory is that we do not assume that every map between H-spaces is an H-map. When dealing with maps which are not H-maps, we need a way to express how they fail to preserve the multiplication.
**Definition 2.4.5.** Let \((X, \mu_X)\) and \((Y, \mu_Y)\) be \(H\)-spaces, and \(f : X \to Y\) be a map. Then the \(H\)-deviation of \(f\) (from a \(\mu_X - \mu_Y\) \(H\)-map) is the map \(D_f : X \times X \to Y\) which satisfies
\[
\mu_Y(D_f \times [\mu_Y \circ (f \times f)]) \Delta_{X \times X} \simeq f \circ \mu_X.
\] (2.4.4)

In \(\langle X \times X, Y \rangle\), this means
\[
[D_f] * [\mu_Y \circ (f \times f)] = [f \circ \mu_X].
\]

If \(Y\) is an HA-space, let \(i_Y\) denote the homotopy inverse of \(Y\). Then \(D_f\) can be chosen as
\[
D_f = \mu_Y(((f \circ \mu_X) \times (i_Y \circ \mu_Y \circ (f \times f))) \Delta_{X \times X}).
\] (2.4.5)

Elementwise,
\[
D_f(x_1, x_2) = f(x_1 x_2) i_Y (f(x_1) f(x_2)).
\] (2.4.6)

In particular, \(f\) is an \(H\)-map iff \(D_f\) is null-homotopic.

**Example 2.4.6.** Suppose we have an HA-space \(X\), with homotopy inverse operation \(i\). Let us examine when \(i\) is a \(\mu - \mu\) \(H\)-map by studying \(D_i\). Elementwise, we have
\[
D_i(x_1, x_2) = i(x_1 x_2) i(i(x_1) i(x_2)).
\] (2.4.7)
Hence we can rewrite $D_i$ as the composition

$$D_i = (i \circ \mu) \ast (i \circ ((i \circ p_1) \ast (i \circ p_2))) \quad (2.4.8)$$

By Lemma 2.3.4, $D_i$ is homotopic to a map which takes $(x_1, x_2)$ to

$$i(x_1x_2)(x_2x_1).$$

Let us find a relationship between the map $D_i$ and the commutativity of $X$ as an HA-space. The following result generalizes the fact that a group is abelian if and only if its inverse operation is a homomorphism.

**Theorem 2.4.7.** The homotopy inverse operation $i$ is a $\mu - \mu$ H-map if and only if $X$ is a homotopy commutative HA-space.

**Proof.** If $X$ is a homotopy commutative HA-space, then $\mu \simeq \mu T$, so $D_i$ is homotopic to a map which takes $(x_1, x_2)$ to $i(x_1x_2)(x_1x_2)$, and hence is null-homotopic. Thus, $i$ is a $\mu - \mu$ H-map.

If $D_i$ is null-homotopic, then in $\langle X \times X, X \rangle$,

$$[D_i] = [k]. \quad (2.4.9)$$

Let $p_1, p_2 : X \times X \to X$ be projection maps. By equation 2.4.8, the left hand side of
equation 2.4.11 becomes

\[
[\mu]^{-1} \ast ([p_1]^{-1} \ast [p_2]^{-1})^{-1} = [k],
\]  

(2.4.10)

so if we (left) multiply both sides of equation 2.4.10 by \([\mu]\), we obtain

\[
([p_1]^{-1} \ast [p_2]^{-1})^{-1} = [\mu],
\]

so

\[
[p_2] \ast [p_1] = [\mu].
\]  

(2.4.11)

Since \([p_2] \ast [p_1] = [\mu T]\) (elementwise, the maps \(p_2 \ast p_1\) and \(\mu T\) take \((x, y)\) to \(yx\)), equation 2.4.11 becomes

\[
[\mu T] = [\mu],
\]

so \(X\) is a homotopy commutative HA-space.

\[\square\]

H-maps can play a role similar to group homomorphisms: if we have a familiar H-space \((Y, \mu_Y)\) and a less familiar H-space \((X, \mu_X)\), we could look for an H-map \(f : X \to Y\) to help us study \(X\) and its cohomology. For example, if \(f\) is a homotopy equivalence, then \(f^* : H^*(Y, \mu_Y; \mathbb{F}_p) \to H^*(X, \mu_X; \mathbb{F}_p)\) would be a Hopf algebra isomorphism. This would allow computations of coproducts in \(H^*(X, \mu_X; \mathbb{F}_p)\) using the coproducts of \(H^*(Y, \mu_Y; \mathbb{F}_p)\).
If $f$ is not (known to be) an H-map, we can actually use its H-deviation to examine the difference in the coproduct structures of $H^*(X, \mu_X; \mathbb{F}_p)$ and $H^*(Y, \mu_Y; \mathbb{F}_p)$:

**Lemma 2.4.8.** Let $(X, \mu_X)$ and $(Y, \mu_Y)$ be H-spaces, and $f : X \to Y$ be a homotopy-equivalence. Let $y \in H^*(Y, \mu_Y; \mathbb{F}_p)$. Then

$$\mu_X^*(f^*(y)) = \Delta_X^* \left( D_f^* \otimes ((f^* \otimes f^*) \mu_Y^*) \right) \mu_Y^*(y).$$

(2.4.12)

In particular, the induced homomorphism $D_f^*$ is trivial if and only if $f^*$ is a Hopf algebra isomorphism:

$$\mu_X^*(f^*(y)) = (f^* \otimes f^*) \mu_Y^*(y).$$

(2.4.13)

**Proof.** Let us verify equation 2.4.12 first. By definition, $D_f$ must satisfy

$$\mu_Y(D_f \times [\mu_Y \circ (f \times f)]) \simeq f \circ \mu_X,$$

so when we apply cohomology, we see that for any $y \in H^*(Y, \mu_Y; \mathbb{F}_p)$,

$$\mu_X^*(f^*(y)) = \Delta_X^* \left( D_f^* \otimes ((f^* \otimes f^*) \mu_Y^*) \right) \mu_Y^*(y).$$

(2.4.14)

Now let us check equation 2.4.13. Write $\mu_Y^*(y) = y \otimes 1 + 1 \otimes y + \sum y' \otimes y''$. 


Then if $D_f^* = 0$ on positive degree elements,

$$
\mu_X^*(f^*(y)) = \Delta^*_{XX} \left( D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) \mu_Y^*(y) \\
= \Delta^*_{XX} \left( D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) (y \otimes 1 + 1 \otimes y \\
+ \sum y' \otimes y'').
$$

(2.4.15)

Since $D_f^*(y) = D_f^*(y') = 0$ and $D_f^*(1) = 1$, we can simplify equation 2.4.15 to

$$
\mu_X^*(f^*(y)) = \Delta^*_{XX} \left( D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) (1 \otimes y) \\
= (f^* \otimes f^*)\mu_Y^*(y).
$$

Hence if $D_f^*$ is trivial, $f^*$ is a Hopf algebra isomorphism.

On the other hand, suppose $D_f^*$ is not a trivial homomorphism. Choose a positive degree class $y \in H^*(Y, \mu_Y; \mathbb{F}_p)$ so that $D_f^*(y) \neq 0$, and any element of smaller positive degree than $y$ is in the kernel of $D_f^*$. Write $\mu_Y^*(y) = y \otimes 1 + 1 \otimes y + \sum y' \otimes y''$. Then

$$
\mu_X^*(f^*(y)) = \Delta^*_{XX} \left( D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) \mu_Y^*(y) \\
= \Delta^*_{XX} \left( D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) (y \otimes 1 + 1 \otimes y \\
= + \sum y' \otimes y'').
$$

(2.4.16)

By our hypotheses, since each $y'$ has degree less than that of $y$, $D_f^*(y') = 0$, so equation
2.4.16 gives us

\[ \mu_X^*(f^*(y)) = D_f^*(y) + (f^* \otimes f^*)\mu_Y^*(y) \]

\[ \neq (f^* \otimes f^*)\mu_Y^*(y). \]

Hence if \( D_f^* \) is not trivial, \( f^* \) is not a Hopf algebra isomorphism. \( \square \)

Another application of H-maps and H-deviations concerns H-spaces with the homotopy type of a product. Given H-spaces \((X, \mu_X), (Y, \mu_Y), \) and \((Z, \mu_Z), \) and a homotopy equivalence \( f : X \to Y \times Z \) with homotopy inverse \( g : Y \times Z \to X \) (so \( gf \simeq id_X \) and \( fg \simeq id_{Y \times Z}, \) let

\[ \mu_{Y \times Z} : Y \times Z \times Y \times Z \to Y \times Z \]

be defined by

\[ \mu_{Y \times Z} = (\mu_Y \times \mu_Z)(id_Y \times T_{Z,Y} \times id_Z). \]

We can ask if the map \( f \) is an H-map; that is, we can ask if the following diagram commutes:

\[
\begin{array}{ccc}
X \times X & \xrightarrow{f \times f} & Y \times Z \times Y \times Z \\
\mu_X & \downarrow & \mu_{Y \times Z} \\
X & \xrightarrow{f} & Y \times Z
\end{array}
\] (2.4.17)

As an analogy, in group theory, we may ask if a group \( G \) is isomorphic to a direct sum or direct product of groups. In our situation, if \( f \) and \( g \) are H-maps, then diagram
2.4.17 commutes and thus the following diagram commutes:

\[
\begin{array}{ccc}
X \times X & \xrightarrow{g \times g} & Y \times Z \times Y \times Z \\
\downarrow \mu_X & & \downarrow \mu_Y \times Z \\
X & \xrightarrow{f} & Y \times Z \\
\end{array}
\]

However, if \(f\) and \(g\) are not H-maps, we can instead ask if the homotopy equivalences \(f\) and \(g\) induce what is called a twisted H-structure on \(Y \times Z\). This twisted H-structure, if it exists, would be a map

\[
\tilde{\mu} : Y \times Z \times Y \times Z \to Y \times Z
\]

which makes the diagram commute:

\[
\begin{array}{ccc}
X \times X & \xrightarrow{g \times g} & Y \times Z \times Y \times Z \\
\downarrow \mu_X & & \downarrow \tilde{\mu} \\
X & \xrightarrow{f} & Y \times Z \\
\end{array}
\]

Let us explain the meaning of the term twisted H-structure. The following definition is based on one from Zabrodsky’s paper [41]:

**Definition 2.4.9.** Let \((Y, \mu_Y)\), and \((Z, \mu_Z)\) be H-spaces, and \(i_1, i_2 : Y \to Y \times Y\) be inclusions. Suppose we have a map \(\omega : Y \times Y \to Z\) for which \(\omega i_1\) and \(\omega i_2\) are both nullhomotopic. Then the \(\omega\text{-twisted } H\text{-structure on } Y \times Z\) is a map \(\tilde{\mu} : Y \times Z \times Y \times Z \to \)
$Y \times Z$ given by the following composition:

$$
\tilde{\mu} = (id_Y \times \mu_Z)(\mu_Y \times \omega \times \mu_Z)(\Delta_{Y \times Y} \times id_{Z \times Z})(id_Y \times T_{Z,Y} \times id_Z).
$$

Elementwise, given $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$, we have

$$
\tilde{\mu}(y_1, z_1, y_2, z_2) = (\mu_Y(y_1, y_2), \mu_Z(\omega(y_1, y_2), \mu_Z(z_1, z_2)).
$$

If we abbreviate $\mu_Y(y_1, y_2) = y_1y_2$ and $\mu_Z(z_1, z_2) = z_1 \cdot z_2$, we have

$$
\tilde{\mu}(y_1, z_1, y_2, z_2) = (y_1y_2, \omega(y_1, y_2) \cdot (z_1 \cdot z_2)).
$$

The map $\omega$ is called the **twisting factor**.

There are similar definitions where the domain of $\omega$ is $Y \times Z, Z \times Y, \text{or } Z \times Z$.

**Example 2.4.10.** In this example, let $Y = Z = F_4$ with identity element $g_0$. Let us find a twisted $H$-structure for $F_4 \times F_4$. Let $com : F_4 \times F_4 \to F_4$ be the commutator:

$$
com(g, h) = ghg^{-1}h^{-1}.
$$
Note that if \( i_1, i_2 : F_4 \to F_4 \times F_4 \) are inclusions, then

\[
com \circ i_1(g) = \com(g, g_0)
\]

\[
= gg_0 g^{-1} g_0^{-1}
\]

\[
= g_0.
\]

\[
com \circ i_2(g) = \com(g_0, g)
\]

\[
= g_0 g g_0^{-1} g^{-1}
\]

\[
= g_0.
\]

The \textit{com}-twisted H-structure on \( F_4 \times F_4 \) is the map

\[
\tilde{\mu} : F_4 \times F_4 \times F_4 \times F_4 \to F_4 \times F_4,
\]

\[
\tilde{\mu}(g_1, h_1, g_2, h_2) = (g_1 g_2, (g_1 g_2 g_1^{-1} g_2^{-1}) h_1 h_2).
\]

In the context of diagram 2.4.18, we would like to know if there is a map \( \omega \) (in finding \( \omega \), we must also determine its domain among the choices in Definition 2.4.9) and an \( \omega \)-twisted H-structure

\[
\tilde{\mu} : Y \times Z \times Y \times Z \to Y \times Z
\]
that makes diagram 2.4.18 commute. We will encounter this problem in Chapter 4. As we will also see in Chapter 4, functions which are not H-maps, along with their H-deviations, allow us to study new H-spaces using information from old ones, just like how one studies unfamiliar groups using homomorphisms and isomorphisms involving familiar ones.

2.5 $A_n$-spaces and $A_n$-forms

At this point, we have seen two “degrees” of associativity for an H-space $(X, \mu)$:

- (Strict) associativity: $\mu(\mu \times \text{id}_X) = \mu(\text{id}_X \times \mu)$
- Homotopy associativity: $\mu(\mu \times \text{id}_X) \simeq \mu(\text{id}_X \times \mu)$

In this section, we will introduce $A_n$-spaces and $A_n$-forms, which describe and measure how associative a multiplication map is. Although this section is optional (it will only come up in parenthetical remarks), it will allow the reader to appreciate the wide gulf between HA-spaces and topological groups. References for this material can be found in Stasheff’s paper [34] (which introduced the ideas of $A_n$-forms) and Kane’s book [19].

First, suppose we have three elements $x$, $y$, and $z$ in the H-space $(X, \mu)$. There are two ways to multiply them using $\mu$:

\[ \mu(\mu \times \text{id}_X)(x, y, z) = (xy)z \text{ and } \mu(\text{id}_X \times \mu)(x, y, z) = x(yz). \]
If the maps $\mu(\mu \times \text{id}_X)$ and $\mu(\text{id}_X \times \mu)$ are homotopic, we can represent this using a 1-cell called $K_3$:

$$K_3 = \begin{array}{ccc}
(xy)z & x(yz) \\
0 & 0 & 0
\end{array}$$

The left vertex (0-cell) represents the map $\mu(\mu \times \text{id}_X)$, and the right vertex represents $\mu(\text{id}_X \times \mu)$. The edge (1-cell) between them represents a homotopy between the maps.

Now let us step up to multiplying four elements $x, y, z,$ and $w$. There are five ways to multiply them:

$$(xy)(zw), ((xy)z)w, (x(yz))w, x((yz)w), \text{ and } x(y(zw)).$$

We can first consider homotopies between pairs like $(xy)(zw)$ and $((xy)z)w$, giving a union of five 0-cells and five 1-cells, but we can impose a stronger condition and assume all possible homotopies among all five maps exist. We can represent them using a 2-cell called $K_4$:

$$K_4 = \begin{array}{ccc}
(xy)(zw) & ((xy)z)w \\
0 & x(y(zw)) & x((yz)w)
\end{array}$$
Notice that each edge is a copy of $K_3$, whose vertices are maps applied to three elements formed from products of $x$, $y$, $z$, and $w$. For example, the top edge is a copy of $K_3$, and the vertices represent the maps $\mu(id_X \times \mu)$ and $\mu(\mu \times id_X)$ applied to the elements $xy$, $z$, and $w$. The 2-cell itself represents a homotopy between the homotopies (the edges).

We can continue this process of forming $(n-2)$-cells $K_n$ to represent homotopies between maps that multiply $n$ elements in $(X, \mu)$. For reference, here is $K_2$, a single vertex that represents the only way of multiplying two elements using $\mu$:

$$K_2 = \frac{xy}{\circ}$$

Given $n$ elements of $X$, $x_1, x_2, \ldots, x_n$, each map, homotopy, homotopy between homotopies, and so on represents placing pairs of parentheses in the expression “$x_1x_2 \ldots x_n$” in order to obtain a product of elements. In particular, each map from $X^n$ to $X$ corresponds to placing $n-2$ pairs of parentheses, obtaining an unambiguous order of multiplying elements in $X$ using $\mu$. For example, in our picture of $K_4$, the bottom vertex corresponds to placing two pairs of parentheses in “$xyzw$”, one pair around “$yz$” and another pair around “$yzw$.” A homotopy between two of these maps corresponds to placing only the $n-3$ pairs of parentheses that the two maps have in common. For example, the top edge in our picture of $K_4$ corresponds to placing a pair of parentheses around “$xy$,” and the edge in the upper right corresponds to a pair of parentheses around “$xyz$.” Homotopies between homotopies correspond to placing
Given $K_n$, a face ($n-1$ dimensional cell) of $K_n$ will correspond to placing only one pair of parentheses. We can formalize the idea of copies of $K_j$ being subsets of the boundary of $K_n$ for $j < n$ by introducing notation for faces of $K_n$. The face $(K_{n-s+1} \times K_s)_k$ corresponds to placing a parenthesis to the left of $x_k$ and to the right of $x_{k+s-1}$: $x_1 \ldots x_{k-1} (x_k \ldots x_{k+s-1}) x_{k+s} \ldots x_n$.

Hence we can construct $K_n$ from $K_j$, $j < n$, as follows: $K_n$ will be the cone on $\partial K_n$, where

$$\partial K_n = \bigcup_{2 \leq s \leq n-1, 1 \leq k \leq n-s+1} (K_{n-s+1} \times K_s)_k.$$ 

Now let us formalize the idea of homotopies (and homotopies of homotopies, and so on) and introduce $A_n$-forms using the cells $K_n$:

**Definition 2.5.1.** Let $(X, \mu)$ be an H-space. The space $X$ is said to admit an $A_n$-form (where $2 \leq n < \infty$) if there exist maps $M_i : K_i \times X^i \to X$ for $2 \leq i \leq n$ where

$$M_2 : K_2 \times X \times X \to X$$ is given by $M_2(k, x, y) = \mu(x, y),$

and if $i > 2$, we have compatibility conditions that allow $M_i$ restricted to $\partial K_i \times X^i$ to be written in terms of $M_j$, $j < i$: the composition

$$\xymatrix{(K_{i-s+1} \times K_s)_k \times X^i \ar[r]^{M_i} & \partial K_i \times X^i \ar[r]^{M_i} & X}$$
can be written as

\[(\alpha, \beta, x_1, \ldots, x_i) \mapsto M_{n-s+1}(\alpha, x_1, \ldots, x_{k-1}, M_s(\beta, x_k, \ldots, x_{k-s+1}), x_{k+s}, \ldots, x_i)).\]

A space with an $A_n$-form will be called an $A_n$-space. If the maps $M_i$ exist for all $i \geq 2$, then $X$ is an $A_\infty$-space.

For example, any H-space is automatically an $A_2$-space by definition.

If $(X, \mu)$ is an HA-space, then let $H : I \times X^3 \to X$ be a homotopy between $\mu(id_X \times \mu)$ and $\mu(\mu \times id_X)$ (where $I$ is the unit interval $[0, 1]$) such that

\[H(0, x, y, z) = \mu(\mu \times id_X)(x, y, z) = (xy)z, \text{ and}\]
\[H(1, x, y, z) = \mu(id_X \times \mu)(x, y, z) = x(yz)\]

Denote the point in $K_2$ by $k_2$. Since $I$ and $K_3$ are homeomorphic, let $f : K_3 \to I$ be a homeomorphism taking the left vertex to 0 and the right vertex to 1. Then define

\[M_2(k_2, x, y) = \mu(x, y) \text{ and}\]
\[M_3(k, x, y, z) = H(f(k), x, y, z).\]

Then at the vertices, we have

\[M_3(f^{-1}(0), x, y, z) = H(0, x, y, z) = \mu(\mu \times id_X)(x, y, z) = (xy)z\]
and

\[ M_3(f^{-1}(1), x, y, z) = H(1, x, y, z) = \mu (id_X \times \mu)(x, y, z) = x(yz). \]

Let us verify the compatibility conditions. The face \((K_2 \times K_2)_1\) corresponds to the left vertex \(f^{-1}(0)\). Indeed, we have

\[ M_2(k_2, M_2(k_2, x, y), z) = \mu (\mu \times id_X)(x, y, z) = (xy)z. \]

Furthermore, the face \((K_2 \times K_2)_2\) corresponds to the right vertex \(f^{-1}(1)\). This time, we have

\[ M_2(k_2, x, M_2(k_2, y, z)) = \mu (id_X \times \mu)(x, y, z) = x(yz). \]

Thus, an HA-space is also an \(A_3\)-space. Conversely, if \((X, \mu)\) is an H-space which is an \(A_3\)-space, we can use \(M_3\) to define a homotopy \(H : I \times X^3 \to X\) between \(\mu (id_X \times \mu)\) and \(\mu (\mu \times id_X)\):

\[ H(r, x, y, z) = M_3(f^{-1}(r), x, y, z). \]

Then

\[ H(0, x, y, z) = M_3(f^{-1}(0), x, y, z) = \mu (\mu \times id_X)(x, y, z) = (xy)z, \quad \text{and} \]

\[ H(1, x, y, z) = M_3(f^{-1}(1), x, y, z) = \mu (id_X \times \mu)(x, y, z) = x(yz). \]

Therefore, an \(A_3\)-space is a homotopy associative H-space. From here, we can appeal to Sibson’s theorem: given a homotopy associative H-space, a homotopy inverse oper-
ation $i : X \to X$ can be constructed that satisfies $\mu(i \times id_X)\Delta_X \simeq \mu(id_X \times i)\Delta_X \simeq k$ [32]. Hence any $A_3$-space is an HA-space.

If $(X, \mu)$ is a topological group or associative H-space, we can take advantage of strict associativity and define all the maps to simply take the product of the elements of $X$:

$$M_i(k, x_1, \ldots, x_i) = x_1 \ldots x_i.$$ 

Hence any topological group and any associative H-space is an $A_\infty$-space.

More generally, $A_n$-forms are designed to be homotopy invariants in the sense that if $X$ has an $A_n$-form and $f : X \to Y$ is a homotopy equivalence, then we can use $f$ to induce an $A_n$-form on $Y$. For example, any loop space $\Omega X$ is homotopy equivalent to an associative H-space (with a homotopy unit)

$$\Omega'X = \{l : [0, r] \to X : r \geq 0, \text{ and } l(0) = l(r) = x_0\}$$

whose multiplication map $\mu_{\epsilon'}$ is given as follows: if $a : [0, r] \to X$ and $b : [0, s] \to X$ are two loops in $\Omega'X$, then $\mu_{\epsilon'}(a, b)$ is given by the loop

$$\mu_{\epsilon'}(a, b) = \begin{cases} 
    a(t) & 0 \leq t \leq r \\
    b(t - r) & r \leq t \leq r + s
\end{cases}.$$ 

Hence any loop space is an $A_\infty$-space. Conversely, any $A_\infty$-space possesses a classifying space (see Stasheff’s paper [34] and Kane’s book [19] for a construction). In
conclusion, we can say that any $A_\infty$-space is homotopy equivalent to a loop space, which is homotopy-equivalent to an associative H-space, which is homotopy equivalent to a topological group.

Overall, there is a countably infinite difference in degrees of associativity between HA-spaces and topological groups: HA-spaces are only guaranteed to have $A_3$-forms, but topological groups have $A_n$-forms for any integer $n \geq 2$.

To summarize, we have:

- $A_\infty$-spaces: topological groups, loop spaces, associative H-spaces
- $A_3$-spaces: HA-spaces
- $A_2$-spaces: H-spaces

Again, we note that $A_\infty$-spaces possess classifying spaces, while $A_3$-spaces might not.

Chapter 2, in part, has been submitted for publication of the material as it may appear in Journal of Topology and its Applications, 2013. Nguyen, Nicholas, Elsevier, 2013. The dissertation author was the primary investigator and author of this paper.
3 The Adjoint Action and Related Maps

One outcome of Kono, Kozima, and Iwase’s work is that given a finite simply-connected topological group $G$, the adjoint action $ad : G \times G \to G$ given by

$$ad(g, h) = ghg^{-1}$$

may not equal $p^G_2 : G \times G \to G$, the projection onto the second factor, but the homomorphisms they induce in cohomology and homology over $\mathbb{F}_p$ are equal if and only if $H_*(G, \mu; \mathbb{F}_p)$ is a commutative algebra [13]. In total, they introduce the following maps:

- the commutator map $com : G \times G \to G$ given by

$$com(g, h) = ghg^{-1}h^{-1}.$$  

- the adjoint action $Ad : G \times \Omega G \to \Omega G$, where $Ad(g, l)$ is the loop given pointwise
by

$$Ad(g, l)(t) = gl(t)g^{-1}.$$  

- a map that we will denote as $Com : G \times \Omega G \to \Omega G$, where $Com(g, l)$ is the loop given pointwise by

$$Com(g, l)(t) = gl(t)g^{-1}l(t)^{-1}.$$  

Overall, they use these maps to characterize commutativity of $H_*(G; \mathbb{F}_p)$. Let us quote (and combine) Theorems 2.2 and 2.3 from Iwase’s paper [13]:

**Theorem 3.0.2 (Iwase).** Let $G$ be a finite simply-connected topological group and $p$ be any odd prime. Let $BG$ and $B\Lambda G$ be the classifying spaces of $G$ and $\Lambda G$ respectively. Let $j : G \to B\Lambda G$ be the inclusion, and $p^G_2 : G \times \Omega G \to \Omega G$ and $p^G_2 : G \times G \to G$ be projections from the second factor. Then the following conditions are equivalent:

i) The induced homomorphism $j^* : H^*(B\Lambda G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p)$ is surjective.

ii) The Pontryagin ring $H_*(G; \mathbb{F}_p)$ is a commutative Hopf algebra. In other words, the adjoint action of $G$ on itself induces the trivial action

$$ad_* = p^G_2 : H_*(G; \mathbb{F}_p) \otimes H_*(G; \mathbb{F}_p) \to H_*(G; \mathbb{F}_p).$$

Note: Iwase means “trivial” in that $ad_* = p^G_2$, not that these maps equal zero on positive degree elements.

iii) The Hopf algebra $H^*(G; \mathbb{F}_p)$ is primitively generated.
iv) There is a $H^*(BG; \mathbb{F}_p)$-module isomorphism

$$H^*(B\Lambda G; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p).$$

v) The integral homology $H_*(G; \mathbb{Z})$ has no $p$-torsion.

vi) The adjoint action (of $G$ on $\Omega G$) induces the trivial action

$$\text{Ad}_* = p_2^\Omega : H_*(G; \mathbb{F}_p) \otimes H_*(\Omega G; \mathbb{F}_p) \to H_*(\Omega G; \mathbb{F}_p).$$

vii) There is an isomorphism of algebras $H^*(B\Lambda G; \mathbb{F}_p) \cong H^*(BG \times G; \mathbb{F}_p)$.

It was not known if these results require $G$ to be a topological group, or if the statements can be proven to be equivalent for a larger class of spaces. If we want to generalize these results beyond topological groups, we must be careful: if we weaken the associativity condition on $G$ (to be more precise, if we no longer require $G$ to be an $A_\infty$-space), we are no longer guaranteed to have classifying spaces for $G$ and its free loop space. Consequently, we cannot generalize statements (i), (iv), and (vii) from Iwase’s theorems, and we will have to find other methods of proving the equivalence of the remaining statements. A more immediate challenge is defining generalizations of the four maps: without a strict inverse, it is not immediately clear how we should define the maps and find formulas for them. Even if we allow a homotopy inverse operation, there is the question of how to define $\text{Com}$ and $\text{Ad}$ so that their codomain is a loop space, not a free loop space (see the discussion at the beginning of Section
1.3).

We overcome these challenges as follows. We weaken the requirements for associativity and existence of inverses to homotopy associativity and existence of a two-sided homotopy inverse operation, resulting in an HA-space. Given a simply-connected HA-space $X$, we will use the free loop fibration to develop our own versions of the maps $ad$, $com$, $Ad$, and $Com$ that generalize the maps introduced by Kono and Kozima:

**Theorem 3.0.3.** Let $X$ be a simply-connected HA-space and $p$ be any odd prime. There exist maps $ad : X \times X \to X$, $com : X \times X \to X$, $Ad : X \times \Omega X \to \Omega X$, and $Com : X \times \Omega X \to \Omega X$ such that if $X$ is a topological group, these maps agree with the maps defined by Kono and Kozima.

We will proceed to prove the following theorem, generalizing Kono, Kozima, and Iwase's results and removing the need for classifying spaces. For convenience, we will number each statement to match with the corresponding statement from Iwase’s paper:

**Theorem 3.0.4.** Let $X$ be a finite simply-connected HA-space and $p$ be any odd prime. Then the following conditions are equivalent:

1. The homomorphism $com^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ is trivial
2. The Hopf algebra $H^*(X; \mathbb{F}_p)$ is primitively generated.
3. The integral homology $H_*(X; \mathbb{Z})$ has no $p$-torsion.
4. The homomorphism $Com^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ is
Moreover, in statement (ii) of Iwase’s theorem, note that he assumes the equivalence of the commutativity of $H_*(G; \mathbb{F}_p)$ and the equality of $ad_*$ and $pG_2$. We will go into more detail and prove:

**Theorem 3.0.5.** The homomorphism $com^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ is trivial iff $H_*(X; \mathbb{F}_p)$ is commutative.

The key to the proof of our theorems will be the calculation of formulas for $com^*$ and $Com^*$. Once we define the four maps, we will deduce these formulas by taking advantage of a property of the cohomology of finite simply-connected HA-spaces, Theorem 1.2.4.

Before we go further, let us point out why we work with $com$ and $Com$ instead of $ad$ and $Ad$. A classical theorem from the topology of Lie groups is that a compact connected Lie group is abelian if and only if it is a torus [30]. According to a theorem by Araki, James, and Thomas, if a compact connected Lie group is not abelian, then it is not homotopy commutative [2]. Thus, the commutator map $com$ is not nullhomotopic for these groups. Nevertheless, Borel observes that when $p$ is odd and $G$ is one of the classical groups $A_n, B_n, C_n,$ or $D_n$, $H_*(G; \mathbb{F}_p)$ is a commutative exterior algebra [3]. Our results stem from a desire to find a relationship between the commutator $com$ on a compact simply-connected Lie group $G$, and the commutativity of $H_*(G; \mathbb{F}_p)$. 
This chapter will start with the free loop fibration from Definition 1.2.10. We will prove some important properties about the free loop fibration that will be needed for our definitions and results. Next, we will define our maps $\text{com}$, $\text{ad}$, $\text{Com}$, and $\text{Ad}$ for simply-connected HA-spaces and study their induced homomorphisms in cohomology. Finally, we will use $\text{com}$ and $\text{Com}$ to characterize finite simply-connected HA-spaces whose homology over $\mathbb{F}_p$ is a commutative algebra.

3.1 The Free Loop Fibration

We will need to use the free loop fibration to not only prove our results, but also to define our maps in the first place. Frequently, we will face the following situation: for some space $Y$ and a simply-connected HA-space $X$, we have a map $\hat{f} : Y \to \Lambda X$, and we would like to know whether:

1. If there exists a map $f : Y \to \Omega X$ such that $jf \simeq \hat{f}$:

\[
\begin{array}{c}
\Lambda X \\
\downarrow j \\
Y \rightarrow \Lambda X
\end{array}
\]

2. If the map $f$ is unique up to homotopy: if there is another map $f_2 : Y \to \Omega X$ such that $jf_2 \simeq jf \simeq \hat{f}$, then does this imply that $f \simeq f_2$?

In this section, we will discuss these questions using a fibration sequence called the free loop fibration. We will explain why the free loop fibration is indeed a fibration,
and then prove some properties of the fibration and the maps in it.

Let us address existence of a lift by verifying that $\varepsilon_0$ is a fibration:

**Proposition 3.1.1.** Let $X$ be any simply-connected topological space, $j : \Omega X \to \Lambda X$ be the inclusion

$$j(l) = l,$$

and $\varepsilon_0 : \Lambda X \to X$ be evaluation at $t = 0$:

$$\varepsilon_0(\varphi) = \varphi(0).$$

Then $\varepsilon_0$ is a fibration, and we have a fibration sequence which we call the free loop fibration:

$$
\Omega X \quad \xrightarrow{j} \quad \Lambda X \quad \xrightarrow{\varepsilon_0} \quad X
\$$

**Proof.** Let $Y$ be any space and $f : Y \to \Lambda X$ be any continuous map, and suppose that we have a homotopy $G : Y \times I \to X$ such that for any $y \in Y$,

$$G(y, 0) = \varepsilon_0(f(y)) = f(y)(0). \quad (3.1.1)$$

Furthermore, since $f(y) \in \Lambda X$, pointwise, we have

$$f(y)(0) = f(y)(1). \quad (3.1.2)$$
We will prove that \( \varepsilon_0 \) is a fibration by constructing a homotopy \( F : Y \times I \to \Lambda X \) such that

\[
F(y, 0) = f(y),
\]

\[
(3.1.3)
\]

\[
G(y, t) = \varepsilon_0(F(y, t)).
\]

\[
(3.1.4)
\]

Notice that at \( t = 0 \), we must have

\[
\varepsilon_0(F(y, 0)) = G(y, 0) = f(y)(0) = f(y)(1).
\]

To find \( F \), we can instead look for a homotopy

\[
F''' : (S^1 \times Y) \times I \to X
\]

such that

\[
F''' = \varepsilon_f(id_{S^1} \times F).
\]

In other words, we would use \( F''' \) to define \( F \) pointwise:

\[
F(y, t)(s) = F'''(s, y, t).
\]

Finding \( F''' \) is equivalent to finding a map

\[
F'' : Y \times I \times S^1 \to X
\]
where

\[ F''' = (T_{Y,S^1} \times id_I)(id_Y \times T_{I,S^1})F'', \]

which in turn is equivalent to finding a map

\[ F' : Y \times I \times I \to X \]

such that

\[
F'(y, t, s) = F''(y, t, s) \\
= F'''(s, y, t) \\
= F(y, t)(s)
\]

(recall that we defined \( S^1 \) to be the quotient space of \( I = [0, 1] \) by \( \{0, 1\} \)). Hence \( F' \) must satisfy the following properties: since \( F(y, t) \in \Lambda X, F(y, t)(0) = F(y, t)(1) \), and thus

\[ F'(y, t, 0) = F'(y, t, 1), \]

while by equation 3.1.3, when \( t = 0, \)

\[ F'(y, 0, s) = f(y)(s), \]
and by equation 3.1.4, when $s = 0$,

$$F'(y, t, 0) = G(y, t).$$

We can take

$$F'(y, t, s) = \begin{cases} 
G(y, t - 3s) & 0 \leq s \leq \frac{t}{3} \\
 f(y) \left( \frac{3s-t}{3-2t} \right) & \frac{t}{3} \leq s \leq 1 - \frac{t}{3} \\
G(y, t + 3s - 3) & 1 - \frac{t}{3} \leq s \leq 1 
\end{cases}$$

Let us show that this map is well-defined and continuous. In particular, when $s = \frac{t}{3}$, we have

$$F'(y, t, \frac{t}{3}) = G(y, t - 3\left(\frac{t}{3}\right))$$

$$= G(y, 0)$$

$$= f(y)(0)$$

$$= f(y) \left( \frac{3\left(\frac{t}{3}\right) - t}{3 - 2t} \right),$$

and when $t = 0$ as well, we get

$$f(y) \left( \frac{3(0) - 0}{3 - 2(0)} \right) = f(y)(0)$$

$$= G(y, 0)$$

$$= G(y, 0 - 3(0)).$$
and when \( t = 1 \), we get

\[
\begin{align*}
    f(y) \left( \frac{3(\frac{1}{3}) - 1}{3 - 2} \right) & = f(y)(0) \\
    & = f(y)(1) \\
    & = G(y, 0) \\
    & = G(y, 1 - 3(\frac{1}{3})).
\end{align*}
\]

When \( s = 1 - \frac{t}{3} \), we have

\[
\begin{align*}
    F'(y, t, 1 - \frac{t}{3}) & = G(y, t + 3(1 - \frac{t}{3}) - 3) \\
    & = G(y, 0) \\
    & = f(y)(0) \\
    & = f(y)(1) \\
    & = f(y) \left( \frac{3 - 2t}{3 - 2t} \right) \\
    & = f(y) \left( \frac{3(1 - \frac{t}{3}) - t}{3 - 2t} \right).
\end{align*}
\]

so at \( t = 0 \), we get

\[
\begin{align*}
    f(y) \left( \frac{3(1) - 0}{3 - 0} \right) & = f(y)(1) \\
    & = f(y)(0) \\
    & = G(y, 0) \\
    & = G(y, 0 + 3(1 - 0) - 3),
\end{align*}
\]
and at $t = 1$, we get

\[
f(y) \left( \frac{3(\frac{2}{3}) - 1}{3 - 2} \right) = f(y)(1) = f(y)(0) = G(y, 0)
\]

\[
= G(y, 1 + 3(1 - \frac{1}{3}) - 3).
\]

We see that $F'$ is well-defined and continuous. Let us now check that $F'$ preserves the basepoint $y_0 \in Y$. Depending on what $s$ and $t$ are, we either have

\[
F'(y_0, t, s) = G(y_0, t - 3s),
\]

\[
F'(y_0, t, s) = f(y_0) \left( \frac{3s - t}{3 - 2t} \right),
\]

or

\[
F'(y_0, t, s) = G(y_0, t + 3s - 3).
\]

Since $G$ is a basepoint-preserving homomorphism, we must have

\[
G(y_0, t - 3s) = G(y_0, t + 3s - 3) = x_0,
\]

and since $f$ preserves basepoints, $f(y_0) = l_0$, the constant loop at $x_0$, so

\[
f(y_0) \left( \frac{3s - t}{3 - 2t} \right) = x_0.
\]
Hence no matter what $t$ and $s$ are,

$$F'(y_0, t, s) = x_0.$$ 

We now verify the other requirements of $F'$. If $s = 0$, we have

$$F'(y, t, 0) = G(y, t - 3(0))$$
$$= G(y, t),$$

and when $s = 1$, we have

$$F'(y, t, 1) = G(y, t + 3(1) - 3)$$
$$= G(y, t).$$

When $t = 0$, we have

$$F'(y, 0, s) = f(y) \left( \frac{3s - 0}{3 - 2(0)} \right)$$
$$= f(y)(s).$$

Therefore, when we use $F'$ to define $F : Y \times I \to \Lambda X$ using equation 3.1.5, we see that $F$ is a homotopy that lifts $G$. Hence $\varepsilon_0 : \Lambda X \to X$ is a fibration.
Let us determine the fiber. By definition,

\[ \varepsilon_0^{-1}(\{x_0\}) = \{ \varphi \in \Lambda X : \varepsilon_0(\varphi) = \varphi(0) = \varphi(1) = x_0 \} \]

\[ = \Omega X, \]

so the fiber is indeed the loop space of \( X \).

In particular, given \( \hat{f} : Y \to \Lambda X \) such that \( \varepsilon_0 \hat{f} : Y \to X \) is nullhomotopic, there exists a map \( f \) such that \( jf \simeq \hat{f} \):

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & \Lambda X \\
\downarrow{\hat{f}} & & \downarrow{\varepsilon_0} \\
X & & X
\end{array}
\]

In addition to the maps in the fibration sequence, we also have the following map from \( X \) to \( \Lambda X \):

**Definition 3.1.2.** We define \( \chi : X \to \Lambda X \) as follows: given \( x \in X \) and \( t \in [0,1] \),

\[ \chi(x)(t) = x. \]

Let us prove two properties of the map \( j \). The first property will allow us to see how \( j \) interacts with the multiplication maps on \( \Omega X \) and \( \Lambda X \):

**Lemma 3.1.3.** The map \( j : \Omega X \to \Lambda X \) is an H-map.
Proof. By definition, if \( l_1, l_2 \in \Omega X \), at any \( t \in [0, 1] \),

\[
\mu_\Lambda(l_1, l_2)(t) = l_1(t)l_2(t) = \mu_\Omega(l_1, l_2)(t),
\]

so

\[
\mu_\Lambda(l_1, l_2) = \mu_\Omega(l_1, l_2).
\]

Hence for any \( l_1, l_2 \in \Omega X \),

\[
\mu_\Lambda(j(l_1), j(l_2)) = \mu_\Lambda(l_1, l_2) = \mu_\Omega(l_1, l_2) = j(\mu_\Omega(l_1, l_2)),
\]

so \( \mu_\Lambda(j \times j) = j \mu_\Omega \). Thus, \( j \) is an H-map. \( \square \)

Let us determine a uniqueness condition on lifts of a map \( \hat{f} : Y \to \Lambda X \). That is, suppose there exists two maps \( f_1, f_2 : Y \to \Omega X \) such that \( jf_1 \simeq jf_2 \simeq \hat{f} \). What conditions will guarantee that \( f_1 \simeq f_2 \)? The following lemma will show that if \( X \) is a simply-connected HA-space, then given a homotopy \( G : Y \times I \to \Lambda X \) from \( jf_1 \) to
$jf_2$, we can find a homotopy $F : Y \times I \to \Omega X$ from $f_1$ to $f_2$.

\[ jf_1 \simeq jf_2. \]

**Lemma 3.1.4.** Let $Y$ be any topological space, $X$ be any simply-connected HA-space with homotopy inverse operation $i : X \to X$, and suppose we are given maps $f_1, f_2 : Y \to \Omega X$, such that

$jf_1 \simeq jf_2$.

Then

$f_1 \simeq f_2$.

**Proof.** Let $G : Y \times I \to \Lambda X$ be a homotopy from $jf_1$ to $jf_2$. That is,

$G(y, 0) = jf_1(y), \text{ and } G(y, 1) = jf_2(y).$

Let us recall a crucial property of $X$, and specifically its homotopy inverse operation $i$: the composition

$\mu(id_X \times i)\Delta_X(x) = xi(x)$

is nullhomotopic. In other words, there is a homotopy $K : X \times I \to X$ such that

$K(x, 0) = x_0, \text{ and } K(x, 1) = xi(x). \quad (3.1.6)$
For each fixed \( x \in X \), the map \( K \) defines paths in \( PX \) from \( x_0 \) to \( xi(x) \) via \( K(x, s) \), \( s \in [0, 1] \). Since we require homotopies to preserve basepoints, for any \( s \in [0, 1] \), we must have

\[
K(x_0, s) = x_0. 
\]

We will use these paths to modify \( G \) to obtain a homotopy \( F' : Y \times I \to \Lambda X \) from a map homotopic to \( f_1 \) to another map homotopic to \( f_2 \) with the property that

\[
F'(x, t)(0) = F'(x, t)(1) = x_0,
\]

so \( F' \) defines a map \( F : Y \times I \to \Omega X \) which is a homotopy from a map homotopic to \( f_1 \) to another map homotopic to \( f_2 \). Hence \( f_1 \simeq f_2 \).

Let us commence modifying the homotopy \( G \). Currently, we have

\[
G(y, 0)(0) = jf_1(y)(0) = x_0,
\]

\[
G(y, 1)(0) = jf_2(y)(0) = x_0,
\]

but if \( 0 < t < 1 \), it may not be true that \( G(y, t)(0) \) is equal to \( x_0 \). Our first modification will be to replace the loops defined by \( G(y, t) \) with loops in \( \Lambda X \) whose starting points in \( X \) are of the form \( xi(x) \). This will allow us to use the paths defined by the homotopy \( K \). Let us define \( G' : Y \times I \to \Lambda X \) as follows:

\[
G' = \mu_{\Lambda}(id_{\Lambda X} \times [\chi \circ i \circ \varepsilon_0])\Delta_{\Lambda X}G \tag{3.1.7}
\]
\[ G'(y, t) = \mu_{\Lambda}(G(y, t), \chi(i(G(y, t)(0)))) \]  
(3.1.8)

\[ G'(y, t)(s) = \mu(G(y, t)(s), i(G(y, t)(0))). \]  
(3.1.9)

Then at \( s = 0 \),

\[ G'(y, t)(0) = \mu(G(y, t)(0), i(G(y, t)(0))). \]

Notice that at \( t = 0 \), since \( G(y, 0) = jf_1(y) \) is a loop that starts at \( x_0 \), \( G(y, 0)(0) = x_0 \), we have

\[ G'(y, 0)(s) = \mu(G(y, 0)(s), i(G(y, 0)(0))) \]
\[ = \mu(G(y, 0)(s), x_0) \]
\[ = G(y, 0)(s) \]
\[ = jf_1(y)(s) \]
\[ = f_1(y)(s), \]

and similarly, at \( t = 1 \), we get

\[ G'(y, 1)(s) = G(y, 1)(s) \]
\[ = jf_2(y)(s) \]
\[ = f_2(y)(s). \]

Therefore, \( G' \) is actually a homotopy from \( jf_1 \) to \( jf_2 \). When \( 0 < t < 1 \), it is still possible that \( G'(y, t)(0) \) might not equal \( x_0 \) since \( i \) is only known to be a homotopy
inverse operation. To overcome this obstacle, we use the paths from the homotopy $K$. Define $K' : Y \times I \times I \to X$ as the composition

$$K' = K((\varepsilon_0 G) \times id_I), \quad (3.1.10)$$

so for each $y \in Y$, $t \in I$, we have

$$K'(y,t,s) = K(G(y,t)(0),s), \quad (3.1.11)$$

and at $s = 0$,

$$K'(y,t,0) = K(G(y,t)(0),0) = x_0,$$

while at $s = 1$,

$$K'(y,t,1) = K(G(y,t)(0),1) = \mu(G(y,t)(0),i(G(y,t)(0))) = G'(y,t)(0) = G'(y,t)(1),$$

so for each $y \in Y$, $t \in I$, $K'$ defines a path from $x_0$ to $\mu(G(y,t)(0),i(G(y,t)(0)))$. 
Now we will define \( F' : Y \times I \to \Lambda X \) pointwise:

\[
F'(y, t)(s) = \begin{cases} 
K'(y, t, 4s) & 0 \leq s \leq \frac{1}{4} \\
G'(y, t)(4s - 1) & \frac{1}{4} \leq s \leq \frac{1}{2} \\
K'(y, t, 2 - 2s) & \frac{1}{2} \leq s \leq 1 
\end{cases}
\]

Let us check that \( F' \) is well defined. At \( s = \frac{1}{4} \), we have

\[
K'(y, t, 1) = G'(y, t)(0),
\]

and at \( s = \frac{1}{2} \), we get

\[
G'(y, t)(1) = G'(y, t)(0) = K'(y, t, 1).
\]

Now we will verify that \( F' \) is a basepoint-preserving homotopy. Let \( y_0 \in Y \) be the basepoint, and \( l_0 \) be the constant loop at \( x_0 \). Depending on what \( t \) and \( s \) are, we either have

\[
F'(y_0, t)(s) = G'(y_0, t)(4s - 1)
\]

\[
F'(y_0, t)(s) = K'(y_0, t, 4s),
\]

or

\[
F'(y_0, t)(s) = K'(y_0, t, 2 - 2s).
\]
For any \( t \) and \( s \), we have

\[ G(y_0, t) = l_0, \]

so

\[
G'(y_0, t) = \mu_\Lambda(G(y_0, t), \chi(i(G(y_0, t)(0))))
\]

\[ = \mu_\Lambda(l_0, \chi(x_0)) \]

\[ = \mu_\Lambda(l_0, l_0) \]

\[ = l_0, \]

which implies

\[ G'(y_0, t)(s) = x_0. \]

Meanwhile,

\[
K'(y_0, t, s) = K(G(y_0, t)(0), s)
\]

\[ = K(x_0, s) \]

\[ = x_0. \]

Therefore, no matter what \( t \) and \( s \) are, we must have

\[ F'(y_0, t)(s) = x_0, \]
so for any $t$,

$$F'(y_0, t) = l_0,$$

and hence $F'$ is a basepoint-preserving homotopy. Let us check that $F'(y, t)(0) = F'(y, t)(1) = x_0$:

$$F'(y, t)(0) = K'(y, t, 0)$$

$$= x_0,$$

$$F'(y, t)(1) = K'(y, t, 2 - 2)$$

$$= x_0.$$

Thus, we can define a homotopy $F : Y \times I \to \Omega X$ pointwise by

$$F(y, t)(s) = F'(y, t)(s). \quad (3.1.12)$$

Let us continue working with $F$ and examine what is happening at $t = 0$. We have

$$G(y, 0)(s) = jf_1(s)$$

$$= f_1(s),$$
and for any \( u \in [0, 1] \),

\[
K'(y, 0, u) = K(G(y, 0)(0), u) = K(jf_1(y)(0), u) = K(x_0, u) = x_0
\]

so \( l_0(u) \),

so the loop defined by \( F \) at \( t = 0 \) is given pointwise by

\[
F(y, 0)(s) = \begin{cases} 
  l_0(4s) = x_0 & 0 \leq s \leq \frac{1}{4} \\
  f_1(4s - 1) & \frac{1}{4} \leq s \leq \frac{1}{2} \\
  l_0(2 - 2s) = l_0(2s - 2) = x_0 & \frac{1}{2} \leq s \leq 1
\end{cases}
\]

We can write down a formula for it using loop concatenation \( \mu_c \) and inclusion maps \( i_1, i_2 : \Omega X \to \Omega X \times \Omega X \):

\[
F(y, 0) = \mu_c(\mu_c(l_0, f_1), l_0) = \mu_c((\mu_c i_2) \times id_{\Omega X})i_1 f_1(y),
\]

so \( F(y, 0) \) defines a function \( f'_1 : Y \to \Omega X \) with

\[
f'_1 = \mu_c((\mu_c i_2) \times id_{\Omega X})i_1 f_1,
\]
which is homotopic to $f_1$ since $(\Omega X, \mu_c)$ is an H-space with a homotopy identity:

\[
\begin{align*}
\mu_c((\mu_c i_2) \times id_{\Omega X}) i_1 f_1 &= \mu_c(id_{\Omega X} \times id_{\Omega X}) i_1 f_1 \\
&= \mu_c i_1 f_1 \\
&\simeq id_{\Omega X} f_1 \\
&= f_1.
\end{align*}
\]

Similarly, $F(y, 1)$ defines a function $f'_2 : Y \to \Omega X$ with

\[
\begin{align*}
f'_2 &= \mu_c((\mu_c i_2) \times id_{\Omega X}) i_1 f_2
\end{align*}
\]

which is homotopic to $f_2$. Therefore $F$ represents a homotopy from $f'_1$ to $f'_2$, so we have

\[
\begin{align*}
f_1 &\simeq f'_1 \\
&\simeq f'_2 \text{ (via the homotopy $F$)} \\
&\simeq f_2
\end{align*}
\]

as functions from $Y$ to $\Omega X$.

In addition to the fibration and the spaces in it, we will often need to work with the products $X \times \Omega X$, $X \times \Lambda X$, and $X \times X$, along with projections from these spaces.
Let $p^O_1 : X \times \Omega X \to X$, $p^O_2 : X \times \Omega X \to \Omega X$, $p^A_1 : X \times \Lambda X \to X$, $p^A_2 : X \times \Lambda X \to \Lambda X$, $p^X_1 : X \times X \to X$, and $p^X_2 : X \times X \to X$ be projections. We can relate these projection maps in the following commutative diagrams:

\[ (3.1.13) \]

\[ (3.1.14) \]

### 3.2 Defining The Maps for HA-spaces

In this section, we will use the free loop fibration to define the adjoint actions, commutator, and related maps for HA-spaces. We will state and prove properties of our maps that generalize some of the results from Section 2 of Kono and Kozima’s paper [21]; additional properties will be stated and proven in Section 3.5 at the end of this chapter.

Our first task is to define these maps for simply-connected HA-spaces. Let us start with the maps $ad : X \times X \to X$ and $com : X \times X \to X$. 
**Definition 3.2.1.** We define \( ad : X \times X \to X \) and \( com : X \times X \to X \) as follows:

\[ com = \mu (id_X \times i) (\mu \times (\mu T)) \Delta_{X\times X}, \]  
(3.2.1)

\[ com(x, y) = (xy)i(yx), \]  
(3.2.2)

and

\[ ad = \mu (com \times p_2^X) \Delta_{X\times X}, \]  
(3.2.3)

\[ ad(x, y) = ((xy)i(yx)) y. \]  
(3.2.4)

Notice that if \( X = G \) is also topological group with \( i(g) = g^{-1} \), then these definitions simplify to those of Kono and Kozima:

\[ com(g, h) = (gh)(hg)^{-1} \]
\[ = ghg^{-1}h^{-1}, \]

and

\[ ad(g, h) = ((gh)(hg)^{-1}) h \]
\[ = ghg^{-1}h^{-1}h \]
\[ = ghg^{-1}. \]

To define \( Com \) for simply-connected HA-spaces, we must start by defining
another function whose codomain is $\Lambda X$ and mimic Kono and Kozima’s definition of $Com$.

**Definition 3.2.2.** Given $x \in X$ and $l \in \Omega X$, we define a function $\widehat{Com} : X \times \Omega X \to \Lambda X$ to be the loop given by

\[
\widehat{Com}(x,l)(t) = com(x,l(t)) = (xl(t))i(l(t)x).
\]

In other words, the composition $\hat{\varepsilon}_f(id_{S^1} \times \widehat{Com}) : S^1 \times X \times \Omega X \to X$ is

\[
\hat{\varepsilon}_f(id_{S^1} \times \widehat{Com})(t,x,l) = com(x,l(t)) = (xl(t))i(l(t)x).
\]

**Theorem 3.2.3.** Let $X$ be a simply-connected $HA$-space. There exists a map $Com : X \times \Omega X \to \Omega X$ which is unique up to homotopy such that

\[
\widehat{Com} \simeq j \circ Com,
\]

and if $X = G$ is also a topological group, then this definition of $Com$ will agree with Kono and Kozima’s definition.
Proof. We need to show that $\widehat{\text{Com}}$ lifts to $\Omega X$ in the free loop fibration. To verify this, we compute $\varepsilon_0 \widehat{\text{Com}}$ and determine if it is nullhomotopic:

$$\varepsilon_0 \widehat{\text{Com}}(x,l) = \widehat{\text{Com}}(x,l)(0)$$
$$= \text{com}(x,l(0))$$
$$= (x_0x)i(x_0x)$$
$$= xi(x),$$

so $\varepsilon_0 \widehat{\text{Com}}$ is nullhomotopic. By the fibration sequence in Definition 1.2.10, there is a map we will call $\text{Com} : X \times \Omega X \to \Omega X$, such that

$$j \circ \text{Com} \simeq \widehat{\text{Com}}, \quad (3.2.5)$$

and by Lemma 3.1.4, $\text{Com}$ is unique up to homotopy in the following sense: if $\text{Com}_2 : X \times \Omega X \to \Omega X$ is another map which satisfies

$$j \circ \text{Com}_2 \simeq \widehat{\text{Com}},$$

then

$$\text{Com}_2 \simeq \text{Com}.$$
If $X$ is a topological group, then we may take

$$Com(x, l)(t) = xl(t)x^{-1}l(t)^{-1},$$

because for any $t \in [0, 1]$,

$$j \circ Com(x, l)(t) = xl(t)x^{-1}l(t)^{-1}$$

$$= \widehat{Com}(x, l)(t).$$

Hence our definition agrees with Kono and Kozima’s definition in the case that $X$ is a topological group. \hfill \Box

Now that we have defined $Com$, we can perform a similar process for $Ad : X \times \Omega X \to \Omega X$.

**Theorem 3.2.4.** Let $X$ be a simply-connected HA-space. Define a map $\widehat{Ad} : X \times \Omega X \to \Lambda X$ so that given $x \in X$ and $l \in \Omega X$, for any $t \in [0, 1]$,

$$\widehat{Ad}(x, l)(t) = (xl(t))i(x). \quad (3.2.6)$$

Then there exists a map $Ad : X \times \Omega X \to \Omega X$ which is unique up to homotopy such that

$$\widehat{Ad} \simeq j \circ Ad,$$
and if $X = G$ is also a topological group, then this definition of $Ad$ will agree with Kono and Kozima’s definition.

Example 3.2.5. Given a simply-connected HA-space $X$ with homotopy inverse operation $i$, let $K : X \times I \to X$ be a homotopy from $k$ (where $k(x) = x_0$) to $\mu(id_X \times i)\Delta_X$:

$$K(x, 0) = x_0, \text{ and } K(x, 1) = xi(x). \quad (3.2.7)$$

We can use $K$ to define a map $Com : X \times \Omega X \to \Omega X$ pointwise:

$$Com(x, l)(s) = \begin{cases} K(x, 4s) & 0 \leq s \leq \frac{1}{4} \\ \widehat{Com}(x, l)(4s - 1) & \frac{1}{4} \leq s \leq \frac{1}{2} \\ K(x, 2 - 2s) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

For each $x$ and $l$, $Com(x, l)$ is a concatenation of paths and loops. First we use the homotopy $K$ to find a path from $x_0$ to $xi(x)$, the starting point of $\widehat{Com}(x, l)$. Next, we follow around the loop $\widehat{Com}(x, l)$. Finally, we return to $x_0$ by moving backwards along the path from $x_0$ to $xi(x)$ given by $K$.

A homotopy $F : X \times \Omega X \times I \to \Lambda X$ from $\widehat{Com}$ to $j \circ Com$ (which equals $Com$
pointwise) is given by

\[ F(x, l, t)(s) = \begin{cases} 
K(x, 4s + (1 - t)) & 0 \leq s \leq \frac{t}{4} \\
\widehat{\text{Com}}(x, l)(\frac{4s-t}{4-3t}) & \frac{t}{4} \leq s \leq 1 - \frac{t}{2} \\
K(x, 2 - 2s + (1 - t)) & 1 - \frac{t}{2} \leq s \leq 1 
\end{cases} \]

We can perform a similar procedure for finding an explicit choice of \( Ad : X \times \Omega X \to \Omega X \).

The proof of the following lemma will demonstrate our main technique for proving statements involving \( \text{Com} \) and \( Ad \); it will let us demonstrate the utility of defining the maps \( \widehat{\text{Com}} \) and \( \widehat{Ad} \).

**Lemma 3.2.6.** Let \( X \) be a simply-connected HA-space, and let \( i_2 : \Omega X \to X \times \Omega X \) be inclusion in the second factor. Then

\[ Ad \circ i_2 \simeq id_{\Omega X}. \]  \hspace{1cm} (3.2.8)

**Proof.** According to our definition of \( Ad \) and \( \widehat{Ad} \), the following diagram commutes:

\[ \begin{array}{ccc}
\Omega X & \xrightarrow{i_2} & X \times \Omega X \\
& & \downarrow j \\
& & \Lambda X \\
\end{array} \xrightarrow{Ad} \Omega X \xrightarrow{j} \Lambda X \xrightarrow{\widehat{Ad}} \Lambda X \]

Let us prove that

\[ \widehat{Ad} \circ i_2 \simeq j \circ id_{\Omega X}. \]
Then we can obtain equation 3.2.8 by using Theorem 3.2.4 (the definitions of \( Ad \) and \( \hat{Ad} \)) and Lemma 3.1.4:

\[
j \circ Ad \circ i_2 \simeq \hat{Ad} \circ i_2
\]
\[
\simeq j \circ id_{\Omega X}
\]

will imply

\[
Ad \circ i_2 \simeq id_{\Omega X}.
\]

For any \( t \in [0, 1] \) and \( l \in \Omega X \),

\[
\hat{Ad} \circ i_2(l)(t) = \hat{Ad}(x_0, l)(t)
\]
\[
= (x_0(t))i(x_0)
\]
\[
= l(t)x_0
\]
\[
= l(t)
\]
\[
= [j(l)](t)
\]
\[
= (j \circ id_{\Omega X}(l))(t),
\]

so

\[
\hat{Ad} \circ i_2 = j \circ id_{\Omega X}.
\]
Therefore, equation 3.2.9 holds, so by Lemma 3.1.4,

\[ Ad \circ i_2 \simeq id_{\Omega X}. \]

Notice that working with \( \hat{Ad} \) and \( \hat{Com} \) allowed us to perform calculations, since \( \hat{Ad} \) and \( \hat{Com} \) have simple pointwise formulas while \( Ad \) and \( Com \) do not (the pointwise formula for \( Com \) in Example 3.2.5 requires us to work with the homotopy \( K \), which does not have an explicit definition, and requires us to work with a concatenation of paths and loops on \( X \)). Future applications of this technique will be more complicated when homotopy associativity and homotopy inverse operations become involved.

Unlike Kono, Kozima, Hara, and Hamanaka, we will focus on \( com \) and \( Com \) instead of \( ad \) and \( Ad \), but it will help to look at relationships among all four maps. Let us start with some properties involving inclusion maps:

**Lemma 3.2.7.** Let \( X \) be a simply-connected HA-space and \( i_1 : X \to X \times \Omega X \) and \( i_2 : \Omega X \to X \times \Omega X \) be inclusions. Then the compositions \( Ad \circ i_1 : X \to \Omega X \), \( Com \circ i_1 : X \to \Omega X \), and \( Com \circ i_2 : \Omega X \to \Omega X \) are nullhomotopic.

Before we prove the theorem, let us discuss our presentation for the proofs of these results that involve computation on the spaces, especially when homotopy associativity and homotopy inverse operations are involved. First, we prove the result in the special case that \( X = G \) is a simply-connected topological group. Here, we will use Kono and Kozima’s definitions, which allow direct calculation of the maps
Com and Ad without needing to work in the free loop space with \( \hat{\text{Com}} \) and \( \hat{\text{Ad}} \). In a topological group, we have the advantage of an associative multiplication, and a strict inverse \( i(g) = g^{-1} \). Nevertheless, we will include additional parentheses in our calculations so that they resemble the ones in the proofs for HA-spaces, in order to make the arguments in the HA-space case easier to understand.

Once we transition over to HA-spaces, our first step is to prove a similar equation involving the maps \( \hat{\text{Com}} \) and \( \hat{\text{Ad}} \). Unlike \( \text{Com} \) and \( \text{Ad} \) for HA-spaces, the maps \( \hat{\text{Com}} \) and \( \hat{\text{Ad}} \) have pointwise definitions similar to Kono and Kozima’s definitions of \( \text{Com} \) and \( \text{Ad} \), and our calculations will actually resemble the ones in the proofs for topological groups. Essentially, we will be working in the free loop space for half of the proof. For the second part of the proof, we need to go from the free loop space (with \( \hat{\text{Com}} \) and \( \hat{\text{Ad}} \)) to the loop space (with \( \text{Com} \) and \( \text{Ad} \)). To do this, we use Theorems 3.2.3 and 3.2.4, along with properties of the inclusion map \( j : \Omega X \to \Lambda X \), to obtain the original equation in the theorem or lemma statement.

**Proof.** We will focus on proving that \( \text{Ad} \circ i_1 \) is nullhomotopic; the proof for the other two compositions is similar.

First, suppose that \( X = G \) is also a topological group. Then \( i(g) = g^{-1} \) and
\(Ad(g, l)(t) = (gl(t))g^{-1}\). Hence for any \(t \in [0, 1]\),

\[
Ad \circ i_1(g)(t) = Ad(g, l_0)(t) \\
= (gl_0(t))g^{-1} \\
= (gg_0)g^{-1} \\
= gg^{-1} \\
= g_0 \\
= l_0(t),
\]

so \(Ad \circ i_1\) is equal to the constant map from \(G\) to \(\Omega G\) that takes any element \(g\) to the constant loop \(l_0\).

Now suppose \(X\) is an HA-space. We have

\[
\hat{Ad} \circ i_1(x)(t) = \hat{Ad}(x, l_0)(t) \\
= (xl_0(t))i(x) \\
= (xx_0)i(x) \\
= xi(x),
\]

so the composition \(\hat{Ad} \circ i_1\) is nullhomotopic. If we let \(k_0 : X \to \Omega X\) and \(k'_0 : X \to \Lambda X\) be trivial maps (they both take \(x \in X\) to the constant loop \(l_0 \in \Omega X \subset \Lambda X\)), then we have

\[
\hat{Ad} \circ i_1 \simeq k'_0. \quad (3.2.11)
\]
By definition, \( j \circ Ad \simeq \widehat{Ad} \) and \( j \circ k_0 = k'_0 \) (since \( l_0 \in \Omega X \subset \Lambda X \)), so equation 3.2.11 becomes
\[
j \circ Ad \circ i_1 \simeq j \circ k_0,
\]
so by Lemma 3.1.4,
\[
Ad \circ i_1 \simeq k_0,
\]
so \( Ad \circ i_1 \) is nullhomotopic.

Recall that we defined \( ad \) in terms of \( com \) so that
\[
ad = \mu (com \times p_2^X) \Delta_{X \times X}.
\] (3.2.12)

We can find a similar relationship between \( \text{Com} \) and \( Ad \):

**Theorem 3.2.8.** Let \( X \) be a simply-connected HA-space. The maps \( \text{Com} \) and \( Ad \) are related as follows:
\[
Ad \simeq \mu_\Omega (\text{Com} \times p_2^\Omega) \Delta_{X \times \Omega X}.
\] (3.2.13)

That is, the following diagram commutes:
\[
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{Ad} & \Omega X \\
\Delta_{X \times \Omega X} \downarrow & & \downarrow \mu_\Omega \\
(X \times \Omega X) \times (X \times \Omega X) & \xrightarrow{\text{Com} \times p_2^\Omega} & \Omega X \times \Omega X
\end{array}
\] (3.2.14)

**Proof.** Let \((G, \mu)\) be a simply-connected topological group with a strict inverse \( i(x) =\)
We want to prove that

\[ Ad = \mu_{\Omega}(\text{Com} \times p_2^\Omega) \Delta_{G \times \Omega G} \tag{3.2.15} \]

(note the equals sign). For any \( t \in [0, 1] \), we have

\[ Ad(g, l)(t) = (gl(t))g^{-1}, \]

while

\[
\mu_{\Omega}(\text{Com} \times p_2^\Omega) \Delta_{G \times \Omega G}(g, l)(t) = \mu_{\Omega}(\text{Com} \times p_2^\Omega)(g, l, g, l)(t) = ((gl(t))(l(t)g)^{-1})l(t). \tag{3.2.16}
\]

In a topological group, \((l(t)g)^{-1} = g^{-1}l(t)^{-1}\), so equation 3.2.16 becomes

\[
\mu_{\Omega}(\text{Com} \times p_2^\Omega) \Delta_{G \times \Omega G}(g, l)(t) = \left( (gl(t)) \left( g^{-1}l(t)^{-1} \right) \right) l(t) = \left( (gl(t)) \left( g^{-1} \right) \right) (l(t)^{-1}l(t)) = \left( (gl(t)) \left( g^{-1} \right) \right) = Ad(g, l)(t),
\]

so equation 3.2.15 holds. \(\square\)

In the previous proof, notice that we used the following property of strict
inverses:

\[(gh)^{-1} = h^{-1}g^{-1}.\]

Recall that Lemma 2.3.4 is the corresponding property for homotopy inverse operations. We will need it to prove Theorem 3.2.8 for HA-spaces.

**Proof.** Let us first prove that

\[
\widehat{Ad} \simeq \mu_{\Lambda}(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}. \tag{3.2.17}
\]

Pointwise, we have

\[
\widehat{Ad}(x,l)(t) = (xl(t))i(x),
\]

and

\[
\mu_{\Lambda}(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}(x,l)(t) = \left(\widehat{Com}(x,l)(t)\right)[l(t)] = ((xl(t))i(l(t)x)) [l(t)]. \tag{3.2.18}
\]

By Lemma 2.3.4, \(\mu_{\Lambda}(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}\) is homotopic to a map which takes \((x,l)\) to a loop given pointwise by

\[
(x,l(t)) = ((xl(t))(i(x)i(l(t)))) [l(t)]. \tag{3.2.19}
\]

By homotopy associativity, \(\mu_{\Lambda}(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}\) is homotopic to a map
which takes \((x,l)\) to a loop given pointwise by

\[
(x,l(t)) = ((xl(t))i(x)) (i(l(t))l(t)) = \Ad(x,l)(t)(i(l(t))l(t)).
\]  

(3.2.20)

By applying the homotopy inverse operation, this map is homotopic to one that takes \((x,l)\) to a loop given by

\[
(xl(t))i(x) = \Ad(x,l)(t),
\]

so equation 3.2.17 holds. By definition,

\[
j \circ \Ad \simeq \Ad,
\]

\[
j \circ \Com \simeq \Com,
\]

and by diagram 3.1.14,

\[
jp_2^\Omega = p_2(\Id_{\Delta_X} \times j),
\]

so equation 3.2.17 becomes

\[
j \circ \Ad \simeq \mu_{\Lambda}(j \circ \Com) \times (jp_2^\Omega)_{\Delta_X \times \Omega_X}
\]

\[
\quad = \mu_{\Lambda}(j \times j) (\Com \times p_2^\Omega)_{\Delta_X \times \Omega_X}.
\]

(3.2.21)
By Lemma 3.1.3, \( j \) is an H-map, so

\[ \mu_A(j \times j) = j \mu_\Omega, \]

and equation 3.2.21 becomes

\[ j \circ Ad \simeq j \mu_\Omega(Com \times p^\Omega_2)\Delta_{X \times \Omega X}. \]

Therefore, by Lemma 3.1.4,

\[ Ad \simeq \mu_\Omega(Com \times p^\Omega_2)\Delta_{X \times \Omega X}. \tag{3.2.22} \]

\[ \square \]

We also have the following formula involving \( Ad \) that generalizes part (i) of Proposition 2.2 of [21].

**Lemma 3.2.9.** Let \( X \) be a simply-connected HA-space. Then

\[ Ad(id_X \times Ad) \simeq Ad(\mu \times id_{\Omega X}) \tag{3.2.23} \]

as maps from \( X \times X \times \Omega X \) to \( \Omega X \). In other words, the following diagram commutes:

\[ \begin{array}{ccc}
X \times X \times \Omega X & \xrightarrow{id_X \times Ad} & X \times \Omega X \\
\downarrow{\mu \times id_{\Omega X}} & & \downarrow{Ad} \\
X \times \Omega X & \xrightarrow{Ad} & \Omega X
\end{array} \tag{3.2.24} \]
Let us prove this lemma for topological groups.

**Proof.** Let \((G, \mu)\) be a simply-connected topological group with strict inverse \(i(g) = g^{-1}\). At any \(t \in [0, 1]\), we have

\[
Ad(id_G \times Ad)(g, h, l)(t) = g \left((hl(t))h^{-1}\right) g^{-1} = ((gh)l(t))(h^{-1}g^{-1}) = ((gh)l(t))(gh)^{-1} = Ad(\mu \times id_{\Omega G})(g, h, l)(t).
\]

Hence \(Ad(id_G \times Ad) = Ad(\mu \times id_{\Omega G})\). Notice that we actually have equality when \(G\) is a topological group. \(\square\)

**Remark.** If \(G\) is a simply-connected topological group, then the homotopy relations in Lemmas 3.2.6 and 3.2.9 become equalities, and hence \(Ad : G \times \Omega G \to \Omega G\) defines a group action of \(G\) on \(\Omega G\).

In order to prove this lemma for HA-spaces, we will need to define another map as well.

**Proof.** Let \((X, \mu)\) be a simply-connected HA-space. Define \(\widehat{Ad} : X \times \Lambda X \to \Lambda X\) so that at any \(t \in [0, 1]\),

\[
\widehat{Ad}(x, \varphi)(t) = (x\varphi(t))i(x).
\]

(3.2.25)
Notice that if $l \in \Omega X \subset \Lambda X$, then

$$\widehat{Ad}(x,l)(t) = (xl(t))i(x)$$

$$= \widehat{Ad}(x,l)(t),$$

so by definition,

$$\widehat{Ad}(id_X \times j) = \widehat{Ad} \quad (3.2.26)$$
as maps from $X \times \Omega X$ to $\Lambda X$.

First, let us show that

$$\widehat{Ad}(id_X \times \widehat{Ad}) \simeq \widehat{Ad}(\mu \times id_{\Omega X}) \quad (3.2.27)$$
as maps from $X \times X \times \Omega X$ to $\Lambda X$. Pointwise, at any $t \in [0,1]$,

$$\widehat{Ad}(id_X \times \widehat{Ad})(x, y, l)(t) = (x ((yl(t))i(y))) i(x),$$

and

$$\widehat{Ad}(\mu \times id_{\Omega X})(x, y, l)(t) = ((xy)l(t)) i(xy).$$

By Lemma 2.3.4, we see that $\widehat{Ad}(\mu \times id_{\Omega X})$ is homotopic to a map which takes $(x, y, l)$ to

$$((xy)l(t)) (i(y)i(x)).$$
By using homotopy associativity, we can conclude that this map is homotopic to $\widehat{\text{Ad}} \left( \text{id}_X \times \text{Ad} \right)$.

To finish the proof of the lemma, recall that $j \circ \text{Ad} \simeq \widehat{\text{Ad}}$. Then on the left side of equation 3.2.27, we have

$$
\widehat{\text{Ad}} \left( \text{id}_X \times \text{Ad} \right) \simeq \widehat{\text{Ad}} \left( \text{id}_X \times (j \circ \text{Ad}) \right)
$$

$$
= \widehat{\text{Ad}} \left( \text{id}_X \times j \right) \left( \text{id}_X \times \text{Ad} \right).
$$

(3.2.28)

By equation 3.2.26, we can simplify the previous equation and obtain

$$
\widehat{\text{Ad}} \left( \text{id}_X \times \text{Ad} \right) \simeq \widehat{\text{Ad}} \left( \text{id}_X \times \text{Ad} \right)
$$

$$
\simeq j \circ \text{Ad} \left( \text{id}_X \times \text{Ad} \right).
$$

(3.2.29)

Meanwhile, on the right side of equation 3.2.27, we have

$$
\widehat{\text{Ad}} \left( \mu \times \text{id}_{\Omega X} \right) \simeq j \circ \text{Ad} \left( \mu \times \text{id}_{\Omega X} \right).
$$

(3.2.30)

Therefore, by combining equations 3.2.27, 3.2.29, and 3.2.30, we obtain

$$
j \circ \text{Ad} \left( \text{id}_X \times \text{Ad} \right) \simeq j \circ \text{Ad} \left( \mu \times \text{id}_{\Omega X} \right).
$$

(3.2.31)

so by applying Lemma 3.1.4, we can conclude that $\text{Ad} \left( \text{id}_X \times \text{Ad} \right)$ and $\text{Ad} \left( \mu \times \text{id}_{\Omega X} \right)$ are homotopic maps. \qed
Let us find a relationship between $\text{com}$ and $\widehat{\text{Com}}$ using the evaluation maps.

**Lemma 3.2.10.** Let $X$ be a simply-connected HA-space. Then

$$\text{com}(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X}) \simeq \hat{\varepsilon}(id_{S^1} \times \text{Com})$$

as maps from $S^1 \times X \times \Omega X$ to $X$.

**Proof.** Given $(t,l) \in S^1 \times \Omega X$, by definition,

$$\hat{\varepsilon}_f(id_{S^1} \times \widehat{\text{Com}})(t,x,l) = l(t)$$

$$= \hat{\varepsilon}(t,l),$$

so the following diagram commutes:

$$
\begin{array}{ccc}
S^1 \times X \times \Omega X & \xrightarrow{id_{S^1} \times \text{Com}} & S^1 \times \Omega X \\
\downarrow id_{S^1} \times \widehat{\text{Com}} & & \downarrow id_{S^1} \times j \\
S^1 \times X \times \Omega X & \xrightarrow{id_{S^1} \times \text{Com}} & S^1 \times \Lambda X \\
\downarrow & & \downarrow \hat{\varepsilon}_f \\
& & X
\end{array}
$$

Pointwise, we have

$$\hat{\varepsilon}_f(id_{S^1} \times \widehat{\text{Com}})(t,x,l) = \text{com}(x,l(t))$$

$$= \text{com}(id_X \times \hat{\varepsilon})(x,t,l)$$

$$= \text{com}(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X})(t,x,l),$$
so therefore,

\[
\text{com}(\text{id}_X \times \hat{\varepsilon})(T_{S^1,X} \times \text{id}_{\Omega X}) \simeq \hat{\varepsilon}(\text{id}_{S^1} \times \text{Com}).
\]

Additional properties of \(\text{ad}, \text{com}, \text{Ad},\) and \(\text{Com}\) (generalizing those found in Kono and Kozima's paper) that were not used in our results will be presented and proven in Section 3.5. For convenience, let us summarize the properties we proved so far for a simply-connected HA-space \(X:\)

1. Let \(i_2 : \Omega X \to X \times \Omega X\) be inclusion in the second factor. Then

\[
\text{Ad} \circ i_2 \simeq \text{id}_{\Omega X}.
\]

(3.2.32)

2. Let \(i_1 : X \to X \times \Omega X\) and \(i_2 : \Omega X \to X \times \Omega X\) be inclusions. Then the compositions \(\text{Ad} \circ i_1 : X \to \Omega X, \text{Com} \circ i_1 : X \to \Omega X,\) and \(\text{Com} \circ i_2 : \Omega X \to \Omega X\) are nullhomotopic.

3. The maps \(\text{Com}\) and \(\text{Ad}\) are related as follows:

\[
\text{Ad} \simeq \mu_{\Omega}(\text{Com} \times p_2^{\Omega})\Delta_{X \times \Omega X}.
\]

(3.2.33)

4. As maps from \(X \times X \times \Omega X\) to \(\Omega X\), we have

\[
\text{Ad}(\text{id}_X \times \text{Ad} \simeq \text{Ad}(\mu \times \text{id}_{\Omega X})
\]

(3.2.34)
5. As maps from $S^1 \times X \times \Omega X$ to $X$,
\[ \text{com}(id_X \times \hat{\epsilon})(T_{S^1,X} \times id_{\Omega X}) \simeq \hat{\epsilon}(id_{S^1 \times \text{Com}}). \]

### 3.3 Induced Homomorphisms

Now let us study the cohomology and homology of a finite simply-connected HA-space using our maps. We will find formulas (Theorem 3.3.3 and Theorem 3.3.5) for the induced homomorphisms of our maps. To do this, we will take the results from the previous section on relationships between the maps and apply cohomology and homology to them.

Let us begin by listing the induced homomorphisms in cohomology and their domains and codomains. We have:

\[ \text{com}^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p), \]

\[ \text{ad}^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p), \]

\[ \text{Com}^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p), \]

\[ \text{Ad}^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p). \]
In homology, we have the following induced linear transformations:

\[ \text{com}_* : H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p), \]

\[ \text{ad}_* : H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p), \]

\[ \text{Com}_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p), \]

\[ \text{Ad}_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p). \]

Let us apply cohomology and homology to Lemmas 3.2.6, 3.2.7, 3.2.9, 3.2.10, along with Theorem 3.2.8 from the previous section to obtain properties of, and relationships between, these induced homomorphisms and linear transformations. Recall that for any \( x \in H^*(X; \mathbb{F}_p), \ t \in H^*(\Omega X; \mathbb{F}_p), \)

\[ i_1^* (x \otimes t) = \begin{cases} 
0, & |t| > 0 \\
x, & t = 1
\end{cases} \]

\[ i_2^* (x \otimes t) = \begin{cases} 
0, & |x| > 0 \\
t, & x = 1
\end{cases} \]

and for any \( \bar{x} \in H_*(X; \mathbb{F}_p), \ \bar{t} \in H_*(\Omega X; \mathbb{F}_p), \)

\[ i_1*(\bar{x}) = \bar{x} \otimes 1 \in H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p), \]
$$i_2^*(t) = 1 \otimes \bar{t} \in H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p).$$

**Proposition 3.3.1.** We have the following properties of \(com^*, ad^*, Com^*, \) and \(Ad^*:\)

1. As homomorphisms from \(H^*(\Omega X; \mathbb{F}_p)\) to \(H^*(\Omega X; \mathbb{F}_p),\)

\[i_2^* \circ Ad^* = id_{\Omega X}^*,\]

so for any \(t \in H^*(\Omega X; \mathbb{F}_p),\)

\[i_2^* \circ Ad^*(t) = t.\]

Hence \(Ad^*(t) = 1 \otimes t + \ldots\)

2. As homomorphisms from \(H^*(\Omega X; \mathbb{F}_p)\) to \(H^*(X; \mathbb{F}_p),\)

\[i_1^* \circ Ad^* = i_1^* \circ Com^* = 0,\]

and as a homomorphism from \(H^*(\Omega X; \mathbb{F}_p)\) to \(H^*(\Omega X; \mathbb{F}_p),\)

\[i_2^* \circ Com^* = 0.\]

In particular, for any \(t \in H^*(\Omega X; \mathbb{F}_p),\)

\[Com^*(t) \in \tilde{H}^*(\Omega X; \mathbb{F}_p) \otimes \tilde{H}^*(\Omega X; \mathbb{F}_p).\]
3. As homomorphisms from $H^*(\Omega X; \mathbb{F}_p)$ to $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$,

$$Ad^* = \Delta^*[X \times \Omega X] (Com^* \otimes p^\Omega_2) \mu^*_\Omega.$$

4. As homomorphisms from $H^*(\Omega X; \mathbb{F}_p)$ to $H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$,

$$(1 \otimes Ad^*) Ad^* = (\mu^* \otimes 1) Ad^*.$$

5. As homomorphisms from $H^*(X; \mathbb{F}_p)$ to $H^*(S^1; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$,

$$(T^*_{S^1, X} \otimes 1)(1 \otimes \hat{\varepsilon}^*) com^* = (1 \otimes Com^*) \hat{\varepsilon}^*.$$ 

In cohomology, there is a generator $s \in H^1(S^1; \mathbb{F}_p)$ such that if $x$ is an element of $H^*(X; \mathbb{F}_p)$, then

$$\hat{\varepsilon}^*(x) = s \otimes \sigma^*(x),$$

$$(T^*_{S^1, X} \otimes 1) (1 \otimes \hat{\varepsilon}^*) (com^*)(x) = s \otimes Com^*(\sigma^*(x)). \quad (3.3.1)$$

If we write

$$com^*(x) = \sum a \otimes b \in H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p),$$

then since $T^*_{S^1, X}(a \otimes s) = (-1)^{|a||s|}(s \otimes a) = (-1)^{|a|}(s \otimes a),$

$$Com^*(\sigma^*(x)) = (-1)^{|a|} \sum a \otimes \sigma^*(b). \quad (3.3.2)$$
We have the following properties of $\text{com}_*$, $\text{ad}_*$, $\text{Com}_*$, and $\text{Ad}_*$:

6. As linear transformations from $H_*(\Omega X; \mathbb{F}_p)$ to $H_*(\Omega X; \mathbb{F}_p)$,

$$\text{Ad}_*i_{2*} = id_{\Omega X*},$$

so for any $\tilde{t} \in H_*(\Omega X; \mathbb{F}_p)$,

$$\text{Ad}_*i_{2*}(\tilde{t}) = \text{Ad}_*(1 \otimes \tilde{t}) = \tilde{t}.$$

7. As linear transformations from $H_*(X; \mathbb{F}_p)$ to $H_*(\Omega X; \mathbb{F}_p)$,

$$\text{Ad}_*i_{1*} = \text{Com}_*i_{1*} = 0,$$

so for any $\tilde{x} \in H_*(X; \mathbb{F}_p)$,

$$\text{Ad}_*(\tilde{x} \otimes 1) = \text{Com}_*(\tilde{x} \otimes 1) = 0.$$

As a linear transformation from $H_*(\Omega X; \mathbb{F}_p)$ to $H_*(\Omega X; \mathbb{F}_p)$,

$$\text{Com}_*i_{2*} = 0,$$
so for any $\bar{t} \in H_\ast(\Omega X; \mathbb{F}_p),$

$Com_\ast i_{2\ast}(\bar{t}) = Com_\ast(1 \otimes \bar{t}) = 0.$

8. As linear transformations from $H_\ast(X; \mathbb{F}_p) \otimes H_\ast(\Omega X; \mathbb{F}_p)$ to $H_\ast(\Omega X; \mathbb{F}_p),$

$Ad_\ast = \mu_{\Omega \ast}(Com_\ast \otimes p_{2\ast}) \Delta_{X \times \Omega X}.$

9. As linear transformations from $H_\ast(X; \mathbb{F}_p) \otimes H_\ast(X; \mathbb{F}_p) \otimes H_\ast(\Omega X; \mathbb{F}_p)$ to $H_\ast(\Omega X; \mathbb{F}_p),$

$Ad_\ast(1 \otimes Ad_\ast) = Ad_\ast(\mu_\ast \otimes 1).$

Hence given $\bar{x}, \bar{y} \in H_\ast(X; \mathbb{F}_p),$ and $\bar{t} \in H_\ast(\Omega X; \mathbb{F}_p),$

$Ad_\ast(\bar{x} \otimes Ad_\ast(\bar{y} \otimes \bar{t})) = Ad_\ast(\bar{x}\bar{y} \otimes \bar{t}).$

10. As linear transformations from $H_\ast(S^1; \mathbb{F}_p) \otimes H_\ast(X; \mathbb{F}_p) \otimes H_\ast(\Omega X; \mathbb{F}_p)$ to $H_\ast(X; \mathbb{F}_p),$

$com_\ast(1 \otimes \hat{\varepsilon}_\ast)(T_{S^1, X} \otimes 1) = \hat{\varepsilon}_\ast(1 \otimes Com_\ast).$

Remark. Parts 6 and 9 of Proposition 3.3.1 tell us that $Ad_\ast$ gives $H_\ast(\Omega X; \mathbb{F}_p)$ the structure of a module over $H_\ast(X; \mathbb{F}_p).$

By our definition of $com,$ we can show that the induced homomorphism of $com$
in \( H^*(X; \mathbb{F}_p) \) is a homomorphism \( \text{com}^*: H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \) which can be written as a composition:

\[
\text{com}^* = \Delta_{X \times X}^* (\mu^* \otimes (T^* \mu^*)) (1 \otimes i^*) \mu^*.
\]

(3.3.3)

**Example 3.3.2.** Suppose \( x \in H^*(X; \mathbb{F}_p) \) is primitive. Then \( \mu^*(x) = x \otimes 1 + 1 \otimes x \), and \( i^*(x) = -x \). Hence \( \text{com}^*(x) = 0 \).

Let us find an elementwise formula for \( \text{com}^* \) in terms of the coproduct \( \mu^* \). To do this, we need to determine a choice of generators for \( H^*(X; \mathbb{F}_p) \) that will facilitate calculation. In particular, since \( \text{com}^* \) is a composition involving \( \mu^* \), we would like to pick generators based on their coproducts. If \( X \) is a finite simply-connected HA-space, the set \( S \) defined after Theorem 1.2.4 will suffice: if an odd generator \( x \) has reduced coproduct \( \sum x' \otimes x'' \), we want each \( x' \) to be an element of \( B \). Recall that \( B \) is primitively generated, so by Theorem 1.1.4, \( B \) is cocommutative: if \( b \in B \), then

\[
\mu^*(b) = T^*_X \mu^*(b).
\]

(3.3.4)

With this, we can find a formula for \( \text{com}^*(x) \) when \( x \in S \):

**Theorem 3.3.3.** Let \( X \) be a finite simply-connected HA-space. Let \( x \in S \). Then

\[
\text{com}^*(x) = \bar{\mu}^*(x) - T^*_X \bar{\mu}^*(x).
\]

(3.3.5)

**Proof.** If \( x \) is a primitive generator, then \( \text{com}^*(x) = 0 \). Thus, we will compute
com*(x) where x is a generator of $H^*(X; \mathbb{F}_p)$ that is not primitive. Write $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$, where each $c \in B$. Next, we apply $(1 \otimes i^*)$. By equation 2.3.11, we obtain

$$x \otimes 1 + 1 \otimes i^*(x) + \sum c \otimes i^*(d) = x \otimes 1 - 1 \otimes x - \sum 1 \otimes c[i^*(d)] + \sum c \otimes i^*(d). \tag{3.3.6}$$

Now we need to apply $\Delta^*_X \otimes_X (\mu^* \otimes (T^*_X \mu^*))$. Notice that the summand $\sum c \otimes i^*(d)$ is in $B \otimes H^*(X)$, so since

$$\mu^*(c) = T^*_X \mu^*(c), \tag{3.3.7}$$

for these terms, we have

$$\Delta^*_X \otimes_X (\mu^* \otimes (T^*_X \mu^*)) = \Delta^*_X ((T^*_X \mu^*) \otimes (T^*_X \mu^*)) = (T^*_X \mu^*) \Delta^*_X. \tag{3.3.8}$$

Since $\mu^*(1) = T^*_X \mu^*(1) = 1 \otimes 1$, the above equation also works on the terms $-1 \otimes x - \sum 1 \otimes c[i^*(d)]$. Thus, when we apply $\Delta^*_X \otimes_X (\mu^* \otimes (T^*_X \mu^*))$ to the right hand side of equation 3.3.6, we get

$$\Delta^*_X \otimes_X (\mu^* \otimes (T^*_X \mu^*))(x \otimes 1) + (T^*_X \mu^*) \Delta^*_X (-1 \otimes x - \sum 1 \otimes c[i^*(d)] + \sum c \otimes i^*(d))$$

$$= \mu^*(x) - T^*_X \mu^*(x) - (T^*_X \mu^*)(\sum (-c[i^*(d)] + c[i^*(d)]))$$

$$= \mu^*(x) - T^*_X \mu^*(x). \tag{3.3.9}$$

Since the summand $x \otimes 1 + 1 \otimes x$ in $\mu^*(x)$ is invariant under $T^*_X$ (due to the zero
degree factors), we can replace the coproduct with the reduced coproduct $\bar{\mu}^*(x)$:

$$com^*(x) = \bar{\mu}^*(x) - T^*_X \bar{\mu}^*(x). \quad (3.3.10)$$

Recall that we defined $ad = \mu(com \times p_2^X)\Delta_{X \times X}$. Let us use this to compute $ad^*(x)$ when $x \in S$:

**Lemma 3.3.4.** Let $X$ be a finite simply-connected HA-space. Let $x \in S$. Then

$$ad^*(x) = 1 \otimes x + com^*(x). \quad (3.3.11)$$

**Proof.** We have

$$ad^* = \Delta_{X \times X}^*(com^* \otimes p_2^{X^*})\mu^*.$$

If $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$, with each $c \in B$, then

$$(com^* \otimes p_2^{X^*})\mu^*(x)$$

$$= com^*(x) \otimes p_2^{X^*}(1) + 1 \otimes 1 \otimes p_2^{X^*}(x) + \sum com^*(c) \otimes p_2^{X^*}(x)$$

$$= com^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes x + \sum com^*(c) \otimes 1 \otimes d,$$

so

$$ad^*(x) = com^*(x) + 1 \otimes x + \sum com^*(c)(1 \otimes d). \quad (3.3.12)$$
Note that if we expand \( c = \sum c_1 \ldots c_n \) as a sum of products of primitive generators \( c_j \) in \( B \), then we have \( \text{com}^*(c) = \sum \prod_{j=1}^{n} \text{com}^*(c_j) = 0 \) since \( \text{com}^* \) vanishes on primitive elements. Hence equation 3.3.12 becomes

\[
ad^*(x) = \text{com}^*(x) + 1 \otimes x + \sum \text{com}^*(c)(1 \otimes d)
= 1 \otimes x + \text{com}^*(x).
\]

In particular, \( ad^*(x) = 1 \otimes x = p_2^X(x) \) if and only if \( \text{com}^*(x) = 0 \). \qed

Now that we have a formula for \( \text{com}^* \) on \( S \), we can find a formula for

\[
\text{Com}^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)
\]
on suspension elements in \( H^*(\Omega X; \mathbb{F}_p) \):

**Lemma 3.3.5.** Let \( X \) be a finite simply-connected HA-space. Let \( x \in S \). Then

\[
\text{Com}^*(\sigma^*(x)) = (1 \otimes \sigma^*)(\bar{\mu}^*(x)). \tag{3.3.13}
\]

**Proof.** By Lemma 3.2.10, we have

\[
\text{com}(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X}) \simeq \hat{\varepsilon}(id_{S^1} \times \text{Com}).
\]

This implies that in cohomology, there is a generator \( s \in H^1(S^1; \mathbb{F}_p) \) such that if \( x \) is
a element of $H^*(X; \mathbb{F}_p)$, then

$$\tilde{\varepsilon}^*(x) = s \otimes \sigma^*(x),$$

$$(T^*_{S^1,X} \otimes 1) (1 \otimes \tilde{\varepsilon}^*) (com^*(x)) = (1 \otimes Com^*) (\tilde{\varepsilon}^*(x)). \quad (3.3.14)$$

When we try to get a formula in $H^*(\Omega X; \mathbb{F}_p)$ with $\sigma^*(x)$ and without $s \in H^1(S^1; \mathbb{F}_p)$, we need to be mindful of $T^*_{S^1,X}$, which causes a sign change on terms in $com^*(x)$ whose left factors have odd degree: if $a \in H^*(X; \mathbb{F}_p)$ has odd degree, then since $|s| = 1$,

$$T^*_{S^1,X}(a \otimes s) = (-1)^{|a||s|} s \otimes a$$

$$= (-1)^{|a|} s \otimes a.$$ 

Consequently, if we write

$$com^*(x) = \sum a \otimes b \in H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p),$$

then

$$(-1)^{|a|} \sum a \otimes \sigma^*(b) = Com^*(\sigma^*(x)). \quad (3.3.15)$$

Hence if no term in the expansion of $com^*(x)$ has the form $a \otimes b$ where $a$ has odd
degree and $b$ is not in the kernel of $\sigma^*$, then equation 3.3.15 simplifies to

$$(1 \otimes \sigma^*)\text{com}^*(x) = \text{Com}^*(\sigma^*(x)).$$

(3.3.16)

Let $x \in S$. We can show that no term in the expansion of $\text{com}^*(x)$ has the form $a \otimes b$ where $a$ has odd degree and $b$ is not in the kernel of $\sigma^*$. If $x$ has even degree, $x$ is primitive, so $\text{com}^*(x) = 0$, and there are no terms in $\text{com}^*(x)$ with the form $a \otimes b$ where $a$ has odd degree and $b$ is not in the kernel of $\sigma^*$. If $x$ has odd degree, then if $x$ is primitive, $\text{com}^*(x) = 0$. If $x$ is not primitive, we can write its reduced coproduct as

$$\bar{\mu}^*(x) = \sum b \otimes r,$$

where each $b$ has even degree and each $r$ has odd degree. Hence by Theorem 3.3.3,

$$\text{com}^*(x) = \sum b \otimes r - \sum r \otimes b.$$

The terms in the expansion of $\text{com}^*(x)$ with an odd degree element in their left factor are all of the form $r \otimes b$, but since each $b$ has even degree, each $b$ is in the kernel of $\sigma^*$ by Lemma 2.2.3, so in the end, equation 3.3.16 holds for any element of $S$.

Now let us prove the lemma. If $x$ is an even degree generator in $S$, then $x$ is primitive, so $\bar{\mu}^*(x) = 0$. Furthermore, since $x$ has even degree, by Lemma 2.2.3,
\[ \sigma^*(x) = 0. \] Hence by Theorem 3.3.3,

\[
\begin{align*}
\text{Com}^*(\sigma^*(x)) &= \text{Com}^*(0) \\
&= 0 \\
&= (1 \otimes \sigma^*)(0) \\
&= (1 \otimes \sigma^*)(\bar{\mu}^*(x)) \quad (3.3.17)
\end{align*}
\]

Now suppose \( x \) is an odd degree generator in \( S \). We have that

\[
\begin{align*}
\text{Com}^*(\sigma^*(x)) &= (1 \otimes \sigma^*)\text{com}^*(x). \\
&= (1 \otimes \sigma^*)[\bar{\mu}^*(x) - T^*_X \bar{\mu}^*(x)]. \quad (3.3.18)
\end{align*}
\]

Since \( \bar{\mu}^*(x) \in B \otimes H^*(X; \mathbb{F}_p) \), \( T^*_X \bar{\mu}^*(x) \in H^*(X; \mathbb{F}_p) \otimes B \). By definition, every element of \( B \) is either an even degree generator in \( S \), or a product of even degree generators. Thus, by Lemma 2.2.3, when we apply \( (1 \otimes \sigma^*) \) to \( [\bar{\mu}^*(x) - T^*_X \bar{\mu}^*(x)] \), the terms from \(-T^*_X \bar{\mu}^*(x)\) will vanish. Therefore,

\[
\text{Com}^*(\sigma^*(x)) = (1 \otimes \sigma^*) \circ [\bar{\mu}^*(x)]. \quad (3.3.19)
\]

\[ \square \]

By Theorem 3.2.8, \( Ad \simeq \mu_\Omega(\text{Com} \times p^\Omega_2) \Delta_{\Omega \times X} \), so since \( \sigma^*(x) \) is primitive, by Lemma
3.3.5, we have:

\[
\text{Ad}^*(\sigma^*(x)) = \Delta^*_{X \times \Omega X}(\text{Com}^* \otimes p^\Omega_2)\mu^*_\Omega(\sigma^*(x)) \\
= \Delta^*_{X \times \Omega X}(\text{Com}^* \otimes p^\Omega_2)(\sigma^*(x) \otimes 1 + 1 \otimes \sigma^*(x)) \\
= \Delta^*_{X \times \Omega X}(\text{Com}^*(\sigma^*(x)) \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma^*(x)) \\
= \text{Com}^*(\sigma^*(x)) + 1 \otimes \sigma^*(x) \\
= 1 \otimes \sigma^*(x) + (1 \otimes \sigma^*)\bar{\mu}^*(x). \tag{3.3.20}
\]

In general, we have this relationship between the triviality of \text{Ad}^* and \text{Com}^*:

**Lemma 3.3.6.** Let \( X \) be a finite simply-connected HA-space. In \( H^*(\Omega X; \mathbb{F}_p) \), \( \text{Ad}^* = p^\Omega_2 \Longleftrightarrow \text{Com}^* = 0 \).

**Proof.** By part 3 of Proposition 3.3.1,

\[
\text{Ad}^* = \Delta^*_{X \times \Omega X}(\text{Com}^* \otimes p^\Omega_2)\mu^*_\Omega,
\]

so in cohomology, for any \( y \in H^*(\Omega X; \mathbb{F}_p) \), if we write \( \mu^*_\Omega(y) = y \otimes 1 + 1 \otimes y + \sum y' \otimes y'' \), then we have

\[
\text{Ad}^*(y) = \text{Com}^*(y) + 1 \otimes y + \sum (\text{Com}^*(y'))(1 \otimes y''). \tag{3.3.21}
\]
Suppose that $\text{Com}^* = 0$. Then by equation 3.3.21,

$$
\text{Ad}^*(y) = \text{Com}^*(y) + 1 \otimes y + \sum (\text{Com}^*(y'))(1 \otimes y'')
$$

$$
= 1 \otimes y
$$

$$
= \mathcal{p}_2^\Omega^*(y).
$$

(3.3.22)

Now suppose that $\text{Ad}^* = \mathcal{p}_2^\Omega^*$. We will show that $\text{Com}^* = 0$ by induction on degree. First, choose $y$ so that $y$ has positive degree and there are no elements of $\tilde{H}^*(\Omega X; \mathbb{F}_p)$ with degree less than $y$. Then $y$ is primitive, and hence by equation 3.3.21,

$$
\text{Ad}^*(y) = \text{Com}^*(y) + 1 \otimes y,
$$

and by our hypothesis,

$$
\text{Ad}^*(y) = \mathcal{p}_2^\Omega^*(y) = 1 \otimes y,
$$

so

$$
\text{Com}^*(y) + 1 \otimes y = 1 \otimes y,
$$

and hence $\text{Com}^*(y) = 0$.

For our induction, suppose $\text{Ad}^* = \mathcal{p}_2^\Omega^*$, and $\text{Com}^*(z) = 0$ for all $z \in \tilde{H}(\Omega X; \mathbb{F}_p)$ with degree less than $n$. Consider any class $y$ with degree $n$. We have

$$
\text{Ad}^*(y) = \text{Com}^*(y) + 1 \otimes y + \sum (\text{Com}^*(y'))(1 \otimes y''),
$$
and by our hypothesis,

\[ Ad^*(y) = p_2^{Ω^*}(y) = 1 \otimes y. \]

Since each \( y' \) has positive degree less than \( n \), \( Com^*(y') = 0 \), so

\[ Com^*(y) + 1 \otimes y + \sum (Com^*(y'))(1 \otimes y'') = Com^*(y) + 1 \otimes y, \]

so

\[ Com^*(y) + 1 \otimes y = 1 \otimes y, \]

and thus \( Com^*(y) = 0 \). Hence \( Com^* = 0 \) by induction on degree.

The computation of \( Com^* \) and \( Ad^* \) on elements of \( H^*(ΩX; \mathbb{F}_p) \) which are not in the image of the suspension (or in the subalgebra generated by suspension elements) is still an open problem when the integral cohomology of \( X \) has \( p \)-torsion. See [10] for some examples using the exceptional Lie groups. Nevertheless, we can use these results to find new ways of characterizing finite simply-connected HA-spaces whose integral homology has \( p \)-torsion.

### 3.4 Homology Algebras and HA-space Commutator Maps

Given a finite simply-connected HA-space, we will find a connection between the commutator map \( com \) and commutativity of \( H_*(X; \mathbb{F}_p) \) and then prove Theorem
3.0.4.

As stated in the introduction to this chapter, many of the classical compact simply-connected Lie groups have commutative homology algebras (with coefficients in $\mathbb{F}_p$, $p$ odd). The key to finding a relationship between homotopy commutativity of the group (or HA-space) and commutativity of its homology is not to look at $\text{com}$ itself, but rather at the induced homomorphism $\text{com}^*$.

Throughout this section, $X$ will be a finite simply-connected HA-space.

**Theorem 3.4.1.** Let $X$ be a finite simply-connected HA-space. The homomorphism $\text{com}^*: H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ is trivial iff $H_*(X; \mathbb{F}_p)$ is commutative.

**Proof.** If $H_*(X; \mathbb{F}_p)$ is commutative, then $H^*(X; \mathbb{F}_p)$ is cocommutative, so if $x$ is any element of $S$,

$$\mu^*(x) = T^*_X \mu^*(x).$$

By Theorem 3.3.3,

$$\text{com}^*(x) = \mu^*(x) - T^*_X \mu^*(x) = 0,$$

so $\text{com}^*$ vanishes on the generating set, and hence on $H^*(X; \mathbb{F}_p)$.

If $H_*(X; \mathbb{F}_p)$ is not commutative, then $H^*(X; \mathbb{F}_p)$ is not primitively generated [16]. Thus, in $S$, there is an odd element $x$ whose reduced coproduct is $\sum c \otimes d \neq 0$, where each $c$ has even degree and each $d$ has odd degree. Then by Theorem 3.3.3,

$$\text{com}^*(x) = \sum c \otimes d - \sum d \otimes c.$$
No term of the form $c \otimes d$ could equal any term of the form $d \otimes c$ for degree reasons: the left factor of $c \otimes d$ has even degree, while the left factor of $d \otimes c$ has odd degree. Hence $com^*(x) \neq 0$. \hfill \Box

Now let us turn to Theorem 3.0.4. Zabrodsky has shown that $H^*(X; \mathbb{F}_p)$ is primitively generated if and only if $H_*(X; \mathbb{Z})$ has no $p$-torsion in Theorem D of [41], and Kane has shown that $H^*(X; \mathbb{F}_p)$ is primitively generated if and only if $H_*(X; \mathbb{F}_p)$ is commutative in Theorem 1.1 of [16], so by Theorem 3.4.1, we have the equivalence of statements (ii), (iii), and (v) in Theorem 3.0.4. Let us finish things by showing that statements (iii) and (vi) are equivalent:

**Theorem 3.4.2.** Let $X$ be a finite simply-connected HA-space. The Hopf algebra $H^*(X; \mathbb{F}_p)$ is primitively generated if and only if the homomorphism $Com^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ is trivial.

**Proof.** Suppose that $H^*(X; \mathbb{F}_p)$ is primitively generated. Zabrodsky shows that since $X$ is an HA-space, then $H^*(X; \mathbb{F}_p)$ being primitively generated implies $H^*(X; \mathbb{F}_p)$ must be a free algebra [41]. Since $H^*(X; \mathbb{F}_p)$ is finite dimensional, it must be an exterior algebra on odd degree generators. Consequently, by the Leray-Samelson theorem (quoted from [19]), since $H_*(X; \mathbb{F}_p)$ is associative (due to the fact that $X$ is an HA-space), it must also be an exterior algebra on odd degree generators.

Let us use part 9 of Proposition 3.3.1 to show that if $\bar{x}$ is a positive degree element of $H_*(X; \mathbb{F}_p)$, and $\bar{u} \in H_*(\Omega X; \mathbb{F}_p)$, then $Ad_*(\bar{x} \otimes \bar{u}) = 0$. By Lemma 2.2.3, $H_*(\Omega X; \mathbb{F}_p)$ has no odd degree elements, so any element $\bar{u} \in H_*(\Omega X; \mathbb{F}_p)$ has
even degree. Thus, if \( \bar{x} \in H_s(X; \mathbb{F}_p) \) has odd degree, then \( Ad_s(\bar{x} \otimes \bar{u}) \) has odd degree, so it must be zero. Otherwise, if \( \bar{x} \) has even degree, then since \( H_s(X; \mathbb{F}_p) \) is an exterior algebra on odd degree elements, \( \bar{x} \) must be a product of two elements \( \bar{x} = \bar{y} \bar{z} = \mu_*(\bar{y} \otimes \bar{z}) \) where \( \bar{z} \) has odd degree (note that \( \bar{y} \) and \( \bar{z} \) might be decomposables as well). Then

\[
Ad_s(\bar{x} \otimes \bar{u}) = Ad_s(\mu_* \otimes 1)(\bar{y} \otimes \bar{z} \otimes \bar{u})
= Ad_s(1 \otimes Ad_s)(\bar{y} \otimes \bar{z} \otimes \bar{u})
= Ad_s(\bar{y} \otimes Ad_s(\bar{z} \otimes \bar{u}))
\]

and since \( Ad_s(\bar{z} \otimes \bar{u}) \) would have odd degree, it must be zero, so \( Ad_s(\bar{x} \otimes \bar{u}) = 0 \) for any \( \bar{x} \in H_s(X; \mathbb{F}_p) \) with positive degree. On the other hand, by part 6 of Proposition 3.3.1, \( Ad_s(1 \otimes \bar{u}) = \bar{u} \). In comparison, we have

\[
p^\Omega_{2s}(\bar{x} \otimes \bar{u}) = \begin{cases} 
0 & |\bar{x}| > 0 \\
\bar{u} & \bar{x} = 1
\end{cases}
\]

so \( Ad_s = p^\Omega_{2s} \). Dually, in cohomology, we have \( Ad^* = p^\Omega_{2s}^* \), so by Lemma 3.3.6, \( Com^* \) is trivial on \( H^*(\Omega X; \mathbb{F}_p) \).

Now suppose \( H^*(\Omega X; \mathbb{F}_p) \) is not primitively generated. Then there exists a generator \( z \in S \), with

\[
\tilde{\mu}^*(z) = \sum b \otimes r \neq 0,
\]
and since any even degree element of $S$ is primitive, $z$ must have odd degree, so each $b$ has even degree, and each $r$ is a generator of $H^*(\Omega X; \mathbb{F}_p)$. Since $z$, and each $r$, are odd degree generators, by Theorem 2.2.4, $\sigma^*(z) \neq 0$, and $\sigma^*(r) \neq 0$ for each $r$. Hence

$$Com^*(\sigma^*(x)) = (1 \otimes \sigma^*) \circ [\bar{\mu}^*(x)] = \sum b \otimes \sigma^*(r) \neq 0.$$ 

Hence $Com^*$ is not trivial on $H^*(\Omega X; \mathbb{F}_p)$.

With this result, we have shown that Kono, Kozima, and Iwase’s results do not require a space to be homotopy equivalent to a topological group. That is, we do not need the entire structure of a finite simply-connected topological group: a finite simply-connected HA-space will be enough.

### 3.5 Other Properties of the Maps

Let us prove some other properties of the maps $ad$, $com$, $Ad$, and $Com$, generalizing the results from Kono and Kozima’s paper [21] (specifically from Section 2 of their paper). These properties were not used in our results from this chapter, but will be needed in a few examples in Chapter 4.

**Remark.** Part 3 of Proposition 2.2 from Kono and Kozima’s paper [21] states that $Ad$ commutes with diagonal maps, so we will not prove it ourselves. The material
involving maximal tori of Lie groups is not applicable to HA-spaces, so they will not be generalized.

**Proposition 3.5.1.** Let $X$ be a simply-connected HA-space. The maps $\text{Com}$ and $\text{Ad}$ are related as follows:

$$\text{Com} \simeq \mu_\Omega(\text{Ad} \times i_\Omega)(\text{id}_X \times \Delta_{\Omega X}).$$ (3.5.1)

That is, the following diagram commutes:

\[
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\text{Com}} & \Omega X \\
\downarrow_{\text{id}_X \times \Delta_{\Omega X}} & & \uparrow_{\mu_\Omega} \\
X \times \Omega X \times \Omega X & \xrightarrow{\text{Ad} \times i_\Omega} & \Omega X \times \Omega X
\end{array}
\] (3.5.2)

**Proof.** First, suppose $X = G$ is also a topological group, so $\text{Ad}(g, l)(t) = (gl(t))g^{-1}$, $\text{Com}(g, l)(t) = (gl(t))(l(t)g)^{-1}$, and $i_\Omega(l)(t) = l(t)^{-1}$. Then

\[
\begin{align*}
\mu_\Omega(\text{Ad} \times i_\Omega)(\text{id}_G \times \Delta_{\Omega G})(g, l)(t) & = \mu_\Omega(\text{Ad} \times i_\Omega)(g, l, l)(t) \\
& = ((gl(t)g^{-1})l(t)^{-1} \\
& = (gl(t))(g^{-1}l(t)^{-1}) \\
& = (gl(t))(l(t)g)^{-1} \\
& = \text{Com}(g, l)(t).
\end{align*}
\]
Now suppose $X$ is only known to be an HA-space. Let us first prove that

$$
\mu_\Lambda(id_{\Lambda X} \times j)(\hat{\text{Ad}} \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq \hat{\text{Com}}
$$

(3.5.3)

as maps from $X \times \Omega X$ to $\Lambda X$. We have

$$
\mu_\Lambda(id_{\Lambda X} \times j)(\hat{\text{Ad}} \times i_\Omega)(id_X \times \Delta_{\Omega X})(x,l)(t) = ((xl(t))i(x))i(l(t)).
$$

By homotopy associativity, $\mu_\Omega(\text{Ad} \times i_\Omega)(id_X \times \Delta_{\Omega X})$ is homotopic to a map which takes $(x,l)$ to a loop given by

$$(xl(t))(i(x)i(l(t))).$$

By Lemma 2.3.4, this map is homotopic to one that takes $(x,l)$ to a loop given by

$$(xl(t))i(l(t)x) = \hat{\text{Com}}(x,l)(t).$$

Hence

$$
\mu_\Lambda(id_{\Lambda X} \times j)(\hat{\text{Ad}} \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq \hat{\text{Com}}.
$$

By definition, $j \circ \text{Ad} \simeq \hat{\text{Ad}}$ and $j \circ \text{Com} \simeq \hat{\text{Com}}$, so

$$
\mu_\Lambda(j \times j)(\text{Ad} \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq j \circ \text{Com}.
$$
Since \( j \) is an H-map, \( \mu_A(j \times j) \simeq j \mu_\Omega \), so

\[ j \mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq j \circ \text{Com,} \]

so by Lemma 3.1.4,

\[ \mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq \text{Com.} \]

\[ \square \]

**Proposition 3.5.2.** Let \( X \) be a simply-connected HA-space. Let \( D : X \times \Omega X \times \Omega X \to X \times \Omega X \times X \times \Omega X \) be defined as

\[
D(x, l_1, l_2) = (x, l_1, x, l_2)
\]

\[
= (id_X \times T_{\Omega X, X} \times id_{\Omega X})(\Delta_X \times id_{\Delta_{\Omega X} \times \Omega X})(x, l_1, l_2).
\]

Then

\[ \mu_\Omega(Ad \times Ad)D \simeq Ad(id_X \times \mu_{\Omega X}). \]

That is, the following diagram commutes:

\[
\begin{array}{ccc}
X \times \Omega X \times \Omega X & \xrightarrow{D} & X \times \Omega X \times X \times \Omega X \\
\downarrow{id_X \times \mu_\Omega} & & \downarrow{Ad \times Ad} \\
X \times \Omega X & \xrightarrow{Ad} & \Omega X \\
\end{array}
\]
Proof. Suppose $X = G$ is also a topological group. Then

$$
\mu_\Omega(\text{Ad} \times \text{Ad})D(g, l_1, l_2)(t) = \mu_\Omega(\text{Ad} \times \text{Ad})(g, l_1, g, l_2)(t)
$$

$$
= ((gl_1(t))g^{-1}) (gl_2(t))g^{-1})
$$

$$
= ((gl_1(t))(g^{-1}g))(l_2(t)g^{-1})
$$

$$
= ((gl_1(t))(l_2(t)g^{-1})
$$

$$
= (g(l_1(t)l_2(t)))g^{-1}
$$

$$
= Ad(g, \mu_\Omega(l_1, l_2))(t)
$$

$$
= Ad(id_G \times \mu_\Omega)(g, l_1, l_2)(t).
$$

Now suppose $X$ is only known to be an HA-space. We will first prove that

$$
\mu_\Lambda(\widehat{\text{Ad}} \times \widehat{\text{Ad}})D \simeq \widehat{\text{Ad}}(id_X \times \mu_{\Omega X})
$$

as maps from $X \times \Omega X \times \Omega X$ to $\Lambda X$. We have

$$
\mu_\Lambda(\widehat{\text{Ad}} \times \widehat{\text{Ad}})D(x, l_1, l_2)(t) = ((xl_1(t))i(x))((xl_2(t))i(x)).
$$

By homotopy associativity, $\mu_\Lambda(\widehat{\text{Ad}} \times \widehat{\text{Ad}})D$ is homotopic to a map which takes $(x, l_1, l_2)$ to a loop given by

$$
((xl_1(t))(i(x)x)) (l_2(t)i(x)).
$$

By definition of the homotopy inverse operation, this map is homotopic to one that
takes \((x, l_1, l_2)\) to a loop given by

\[ ((xl_1(t))) (l_2(t)i(x)). \]

By homotopy associativity, this map is homotopic to one that takes \((x, l_1, l_2)\) to a loop given by

\[ (x(l_1(t)l_2(t)))i(x) = \widehat{Ad}(id_X \times \mu_\Omega)(x, l_1, l_2)(t). \]

Hence

\[ \mu_\Lambda(\widehat{Ad} \times \widehat{Ad})D \simeq \widehat{Ad}(id_X \times \mu_\Omega X). \]

By definition, \(j \circ Ad \simeq \widehat{Ad}\), so

\[ \mu_\Lambda(j \times j)(Ad \times Ad)D \simeq j \circ Ad(id_X \times \mu_\Omega X). \]

Since \(j\) is an H-map, \(\mu_\Lambda(j \times j) \simeq j \mu_\Omega\), so

\[ j \mu_\Omega(Ad \times Ad)D \simeq j \circ Ad(id_X \times \mu_\Omega X), \]

so by Lemma 3.1.4,

\[ \mu_\Omega(Ad \times Ad)D \simeq Ad(id_X \times \mu_\Omega X). \]

\[ \square \]

**Proposition 3.5.3.** Let \(X\) be a simply-connected HA-space. Let \(D : X \times \Omega X \times \Omega X \to \)
$X \times \Omega X \times X \times \Omega X$ be defined as

$$D(x, l_1, l_2) = (x, l_1, x, l_2)$$

$$= (id_X \times T_{X, \Omega X} \times id_{\Omega X})(\Delta_X \times id_{\Omega X \times \Omega X})(x, l_1, l_2).$$

Then

$$\mu_{\Omega}(\text{Com} \times \text{Com})D \simeq \text{Com}(id_X \times \mu_{\Omega X}).$$

That is, the following diagram commutes:

\[
\begin{array}{ccc}
X \times \Omega X \times \Omega X & \xrightarrow{D} & X \times \Omega X \times X \times \Omega X \\
\downarrow{id_X \times \mu_{\Omega}} & & \downarrow{\mu_{\Omega}} \\
X \times \Omega X & \xrightarrow{\text{Com}} & \Omega X
\end{array}
\]

Proof. Unlike the other properties proven in this section, we will immediately work with HA-spaces, and we will not need the free loop space. We will exploit the fact that $\Omega X$ is a homotopy commutative HA-space:

$$\mu_{\Omega}T_{\Omega X} \simeq \mu_{\Omega}.$$  

By Proposition 3.5.1, $\mu_{\Omega}(\text{Com} \times \text{Com})D$ is homotopic to

$$\mu_{\Omega}((\mu_{\Omega}(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X})) \times (\mu_{\Omega}(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X})))D.$$
Elementwise, this map takes \((x, l_1, l_2)\) to an element of \(\Omega X\) given (in terms of \(Ad\)) by

\[
(Ad(x, l_1)i_\Omega(l_1)) (Ad(x, l_2)i_\Omega(l_2))
\]

(here, concatenation indicates multiplying using \(\mu_\Omega\)). By homotopy commutativity and homotopy associativity (of \(\Omega X\)), this map is homotopic to one that takes \((x, l_1, l_2)\) to

\[
(Ad(x, l_1)Ad(x, l_2)) (i_\Omega(l_2)i_\Omega(l_1)),
\]

and by Lemma 2.3.4, this map is homotopic to one that takes \((x, l_1, l_2)\) to

\[
(Ad(x, l_1)Ad(x, l_2)) i_\Omega(l_1 l_2).
\]  \hspace{1cm} (3.5.5)

Now, elementwise, we have

\[
\mu_\Omega(Ad \times Ad) D(x, l_1, l_2) = Ad(x, l_1)Ad(x, l_2),
\]

and by Proposition 3.5.2, this is homotopic to \(Ad(id_X \times \mu_{\Omega X})\), which is given elementwise by

\[
Ad(id_X \times \mu_{\Omega X})(x, l_1, l_2) = Ad(x, l_1 l_2).
\]

Thus, the map described in equation 3.5.5 is homotopic to one that takes \((x, l_1, l_2)\) to

\[
(Ad(x, l_1 l_2)) i_\Omega(l_1 l_2).
\]
By Proposition 3.5.1, this map is homotopic to one that takes \((x, l_1, l_2)\) to

\[ \text{Com}(x, l_1 l_2) = \text{Com}(\text{id}_X \times \mu_{\Omega X})(x, l_1, l_2). \]

Therefore,

\[ \mu_{\Omega}(\text{Com} \times \text{Com})D \simeq \text{Com}(\text{id}_X \times \mu_{\Omega X}). \]

Let us apply cohomology and homology to the previous propositions to find properties of the induced homomorphisms and induced linear transformations:

**Proposition 3.5.4.** We have the following properties of \(\text{com}^*, \text{ad}^*, \text{Com}^*, \text{and Ad}^*:\)

1. In \(H^*(\Omega X; \mathbb{F}_p)\),

\[ \text{Com}^* = (1 \otimes \Delta^*_\Omega X)(\text{Ad}^* \otimes i^*_\Omega)\mu^*_\Omega. \]

2. In \(H^*(\Omega X; \mathbb{F}_p)\),

\[ D^*(\text{Ad}^* \otimes \text{Ad}^*)\mu_{\Omega*} = (1 \otimes \mu^*_{\Omega X})\text{Ad}^*. \]

3. In \(H^*(\Omega X; \mathbb{F}_p)\),

\[ D^*(\text{Com}^* \otimes \text{Com}^*)\mu_{\Omega*} = (1 \otimes \mu^*_{\Omega X})\text{Com}^*. \]

We have the following properties of \(\text{com}_*, \text{ad}_*, \text{Com}_*, \text{and Ad}_*:\)

4. In \(H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)\),

\[ \text{Com}_* = \mu_{\Omega*}(\text{Ad}_* \otimes i_{\Omega*})(1 \otimes \Delta_{\Omega X*}). \]
5. In $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$,

$$\mu_{\Omega_*}(Ad_* \otimes Ad_*)D_* = Ad_* (1 \otimes \mu_{\Omega X_*})$$

6. In $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$,

$$\mu_{\Omega_*}(Com_* \otimes Com_*)D_* = Com_* (1 \otimes \mu_{\Omega X_*})$$

Let us end with a relationship between $Ad$, $Com$, and H-maps.

**Lemma 3.5.5.** Let $(X, \mu_X)$ and $(Y, \mu_Y)$ be simply-connected HA-spaces. Let $f : X \to Y$ be an H-map. Then the following diagrams commute:

\[
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{Ad_X} & \Omega X \\
\downarrow f \times \Omega f & & \downarrow \Omega f \\
Y \times \Omega Y & \xrightarrow{Ad_Y} & \Omega Y
\end{array}
\]

\[
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{Com_X} & \Omega X \\
\downarrow f \times \Omega f & & \downarrow \Omega f \\
Y \times \Omega Y & \xrightarrow{Com_Y} & \Omega Y
\end{array}
\]

(we will use subscripts $X$ and $Y$ to distinguish between the respective maps for each space). For reference, $\Omega f : \Omega X \to \Omega Y$ is pointwise application of $f$; if $l \in \Omega X$,

$$(\Omega f)(l)(t) = f(l(t)),$$
and similarly, \( \Lambda f : \Lambda X \to \Lambda Y \) is pointwise application of \( f \); if \( \varphi \in \Lambda X \),

\[
(\Lambda f)(\varphi)(t) = f(\varphi(t)).
\]

**Proof.** Let us first prove this lemma for the special case that \((G, \mu_G)\) and \((H, \mu_H)\) are simply-connected topological groups, and \(f : G \to H\) is a group homomorphism. For any \( g \in G \) and \( l \in \Omega G \), if \( t \in [0, 1] \),

\[
(\Omega f)Ad_G(g, l)(t) = f((gl(t))g^{-1}) \in H. \tag{3.5.6}
\]

Since \( f \) is a group homomorphism, 3.5.6 becomes

\[
(\Omega f)Ad_G(g, l)(t) = (f(g)f(l(t)))f(g)^{-1}
= Ad_H(f(g), (\Omega f)(l))(t)
= Ad_H(f \times \Omega f)(g, l)(t).
\]

The proof for the second diagram is similar.

Now suppose that \((X, \mu_X)\) and \((Y, \mu_Y)\) are simply-connected HA-spaces with homotopy inverse operations given by \( i_X \) and \( i_Y \) respectively, and \( f : X \to Y \) is an
H-map. We will first prove that the following diagram with free loop spaces commutes:

\[
\begin{array}{c}
X \times \Omega X \xrightarrow{Ad_X} \Lambda X \\
\downarrow f \times \Omega f \\
Y \times \Omega Y \xrightarrow{Ad_Y} \Lambda Y
\end{array}
\]

(3.5.7)

For reference, here are the maps involved:

\[
\begin{array}{c}
X \times \Omega X \xrightarrow{Ad_X} \Omega X \\
\downarrow j_X \\
X \times \Omega X \xrightarrow{Ad_X} \Lambda X
\end{array}
\]

\[
\begin{array}{c}
Y \times \Omega Y \xrightarrow{Ad_Y} \Omega Y \\
\downarrow j_Y \\
Y \times \Omega Y \xrightarrow{Ad_Y} \Lambda Y
\end{array}
\]

For any \( x \in X \) and \( l \in \Omega X \), if \( t \in [0, 1] \),

\[
(\Lambda f) \widehat{Ad}_X(x, l)(t) = f((xl(t))i(x)) \in Y.
\]

Since \( f \) is an H-map, \((\Lambda f) \widehat{Ad}_X\) is homotopic to a map which takes \((x, l)\) to a loop given by

\[
(f(x)f(l(t))) f(i_X(x)).
\]

By Lemma 2.4.4, \( fi_X \simeq i_Y f \), so \((\Lambda f) \widehat{Ad}_X\) is homotopic to a map which takes \((x, l)\)
to a loop given by

\[
(f(x)f(l(t)))i_Y(f(x)) = \widehat{Ad}_Y(f(x), (\Omega f)(l))(t)
\]

\[
= \widehat{Ad}_Y(f \times \Omega f)(x, l)(t).
\]

Therefore, diagram 3.5.7 commutes.

By definition, \(j_X Ad_X \simeq \widehat{Ad}_X\) and \(j_Y Ad_Y \simeq \widehat{Ad}_Y\). Meanwhile, if \(l \in \Omega X \subset \Lambda X\), then \((\Omega f)(l) \in \Omega Y \subset \Lambda Y\), so

\[
j_Y(\Omega f)(l) = (\Omega f)(l)
\]

\[
= (\Lambda f)(j_X(l)),
\]

so \(j_Y(\Omega f) = (\Lambda f)j_X\). Since we have proven that

\[
\Lambda f \circ \widehat{Ad}_X \simeq \widehat{Ad}_Y(f \times \Omega f),
\]

then since \(j_X Ad_X \simeq \widehat{Ad}_X\) and \(j_Y Ad_Y \simeq \widehat{Ad}_Y\), we must have

\[
\Lambda f \circ j_X Ad_X \simeq j_Y Ad_Y(f \times \Omega f),
\]

so since \(j_Y(\Omega f) = (\Lambda f)j_X\),

\[
j_Y \Omega f \circ Ad_X \simeq j_Y Ad_Y(f \times \Omega f),
\]
and therefore, by Lemma 3.1.4,

\[ \Omega f \circ Ad_X \simeq Ad_Y (f \times \Omega f), \]

so the diagram is commutative:

\[
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{Ad_X} & \Omega X \\
f \times \Omega f & \downarrow & \Omega f \\
Y \times \Omega Y & \xrightarrow{Ad_Y} & \Omega Y
\end{array}
\]

The proof for the second diagram with \( Com_X \) and \( Com_Y \) is similar.

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4 The Free Loop Space of an HA-space

One of Kono and Kozima’s motivations for introducing the adjoint action came from their study of the free loop space $(\Lambda G, \mu_\Lambda)$ of a Lie group $(G, \mu)$. One can ask how the group structure on $(\Lambda G, \mu_\Lambda)$ differs from the product group structure on $(G \times \Omega G, \mu_{G \times \Omega G})$ where

$$\mu_{G \times \Omega G} = (\mu \times \mu_\Omega)(id_G \times T_{\Omega G,G} \times id_{\Omega G}).$$

Kono and Kozima show that there is a homeomorphism of topological spaces $\Phi : \Omega G \times G \to \Lambda G$ where $\Phi(l, g)$ is the loop given by $l(t)g$. If this homeomorphism is also an H-map, then the following diagram would commute:

$$
\begin{array}{ccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{(\mu_\Omega \times \mu)(id_{\Omega G} \times T \times id_G)} & \Omega G \times G \\
\downarrow \Phi \times \Phi & & \downarrow \Phi \\
\Lambda G \times \Lambda G & \xrightarrow{\mu_\Lambda} & \Lambda G
\end{array}
$$
Kono and Kozima present a modified diagram that changes the maps in the top row. They define a map

$$\omega : \Omega G \times G \times \Omega G \times G \to \Omega G \times \Omega G \times G \times G$$

given by the composition

$$\omega = (id_{\Omega G} \times T_{G, \Omega G} \times id_G) (id_{\Omega G \times G} \times Ad \times id_G) (id_{\Omega G} \times \Delta_G \times id_{\Omega G \times G});$$

for any $t \in [0, 1]$,

$$\omega(l_1, g_1, l_2, g_2)(t) = (l_1, Ad(g_1, l_2), g_1, g_2)(t)$$

$$= (l_1(t), g_1 l_2(t) g_1^{-1}, g_1, g_2).$$

Then the following diagram commutes whether or not $\Phi$ is a group homomorphism:

$$\begin{array}{ccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{\omega} & \Omega G \times \Omega G \times G \times G \\
\Phi \times \Phi & & \mu_{\Omega \times \mu} \\
\Lambda G \times \Lambda G & \xrightarrow{\mu_\Lambda} & \Lambda G
\end{array}$$

An application of this diagram would be the computation of products in the algebra $H_s(\Lambda G, \mu_\Lambda; \mathbb{F}_p)$ using three ingredients: the product structures of $H_s(G, \mu; \mathbb{F}_p)$ and $H_s(\Omega G, \mu_\Omega; \mathbb{F}_p)$, and knowledge of the induced linear transformation $Ad_\mu$. However, the furthest that anyone has gone in the past is Hamanaka, who computed the map...
Ad_\ast : H_\ast(G; F_2) \otimes H_\ast(\Omega G; F_2) \to H_\ast(\Omega G; F_2) for the cases when G is an exceptional Lie group other than E_8 [9]. Hamanaka proceeded to apply homology to the previous diagram to provide a formula for \mu_{\Lambda_\ast} in terms of the maps \mu_\ast, \mu_{\Omega_\ast}, and Ad_\ast, but is only able to provide an example for G = G_2, remarking that the other cases were too complicated to write down.

In this chapter, we present another approach to computing products in the algebra H_\ast(\Lambda X, \mu_{\Lambda}; F_p), where X is a finite simply-connected HA-space. Our method stems from the H-deviation of another homotopy equivalence, h_1 : \Lambda X \to X \times \Omega X. In the case that X is a topological group, we can write down h_1:

\[ h_1(\varphi) = (\varphi(0), l) \]

where \( l \in \Omega X \) is a loop given pointwise by

\[ l(t) = \varphi(0)^{-1}\varphi(t). \]

However, if X is only known to be an HA-space, we will need the free loop fibration to define h_1.

This chapter starts with an introduction to the free loop space \( \Lambda X \) and its relation to the product space \( X \times \Omega X \). We will focus on the homotopy equivalence \( h_1 : \Lambda X \to X \times \Omega X \) and its H-deviation. Our most important finding is a formula for the H-deviation of \( h_1 \) which involves the map Com : \( X \times \Omega X \to \Omega X \) (from Chapter 3). We then use the H-deviation of \( h_1 \) to find a twisted H-structure on \( X \times \Omega X \), obtaining
a diagram similar to Kono and Kozima’s diagram 4.0.1 for HA-spaces. From here, we
can use the induced homomorphisms (in cohomology) of $h_1$ and $Com$, along with the
twisted H-structure, to find another characterization of finite simply-connected HA-
spaces whose homology over $\mathbb{F}_p$ is commutative, and compute products in $H_\ast (\Lambda X; \mathbb{F}_p)$
using $\mu_\ast$, $\mu_{\Omega_\ast}$, and $Com_\ast$.

Throughout this chapter, we will let $p_1^\Omega : X \times \Omega X \to X$, $p_2^\Omega : X \times \Omega X \to \Omega X$,
$p_1^\Lambda : X \times \Lambda X \to X$, $p_2^\Lambda : X \times \Lambda X \to \Lambda X$ be projections. Recall that the following
diagrams commute strictly:

\[
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{p_1^\Omega} & X \\
\downarrow{id_X \times j} & & \downarrow{id_X} \\
X \times \Lambda X & \xrightarrow{p_1^\Lambda} & X
\end{array}
\quad (4.0.2)
\]

\[
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{p_2^\Omega} & \Omega X \\
\downarrow{id_X \times j} & & \downarrow{j} \\
X \times \Lambda X & \xrightarrow{p_2^\Lambda} & \Lambda X
\end{array}
\quad (4.0.3)
\]

4.1 The Homotopy Equivalence and its H-deviation

We have noticed a few mentions of the homotopy equivalence of $\Lambda X$ and
$X \times \Omega X$ with various conditions on what kind of space $X$ needs to be throughout the
literature, but we were unable to find a proof. In particular, given a simply-connected
HA-space $X$, we would like a map $h_1 : \Lambda X \to X \times \Omega X$ which is a homotopy equiva-
ience. We will define $h_1$ in this section, and prove that it is a homotopy equivalence.

First, let us introduce and define $h_1$ for the special case when $X = G$ is a
simply-connected topological group and prove that it is a homeomorphism in this case using elementwise calculations. This will prepare the reader for the more general case when $X$ is a simply-connected HA-space when additional tools such as the free loop fibration are needed due to an inability to write down $h_1$ pointwise.

**Definition 4.1.1.** Let $G$ be a simply-connected topological group. We define a map $h_1 : \Lambda G \to G \times \Omega G$ as follows. Given a loop $\varphi \in \Lambda G$, at $t \in [0,1]$, let

$$h_1(\varphi)(t) = (\varphi(0), \varphi(0)^{-1}\varphi(t)).$$

Notice that at $t = 0$, the second component is

$$\varphi(0)^{-1}\varphi(0) = x_0,$$

so the loop given by $\varphi(0)^{-1}\varphi(t)$ is indeed an element of $\Omega G$. We also define a map $h_2 : G \times \Omega G \to \Lambda G$ so that given $(g, l) \in G \times \Omega G$, at $t \in [0,1]$,

$$h_2(g, l)(t) = gl(t).$$

**Theorem 4.1.2.** Let $X$ be a simply-connected topological group. Then $h_2 h_1 = id_{\Lambda G}$ and $h_1 h_2 = id_{G \times \Omega G}$, so $h_1$ and $h_2$ are homeomorphisms.
Proof. Let us compute $h_2h_1$ and $h_1h_2$ elementwise. At $t \in [0,1]$, for any $\varphi \in \Lambda G$,

$$h_2h_1(\varphi)(t) = h_2(\varphi(0), \varphi(0)^{-1}\varphi(t))$$
$$= \varphi(0) (\varphi(0)^{-1}\varphi(t))$$
$$= (\varphi(0) \varphi(0)^{-1}) \varphi(t)$$
$$= \varphi(t),$$

so elementwise, $h_2h_1 = id_{\Lambda G}$. Meanwhile, for any $g \in G$ and $l \in \Omega G$,

$$h_1h_2(g,l)(t) = h_1(gl(t))$$
$$= (gl(0), (gl(0))^{-1}(gl(t)))$$
$$= (gg_0, (g_0^{-1}g^{-1})(gl(t)))$$
$$= (g, (g^{-1})(gl(t)))$$
$$= (g, (g^{-1}g) l(t))$$
$$= (g, l(t)), $$

so elementwise, $h_1h_2 = id_{G \times \Omega G}$. \hfill \Box

Now let us transition to the case when $X$ is a simply-connected HA-space. We cannot simply copy the definition of $h_1$ for topological groups, because a loop $l$ given by

$$l(t) = i(\varphi(0))\varphi(t)$$
has its basepoint at \(i(\varphi(0))\varphi(0)\), which may not equal \(x_0\), so this loop might not be an element of \(\Omega X\). Just like in Chapter 3 when defining \(Com\) and \(Ad\), we will need the free loop fibration.

Recall that we have a fibration sequence

\[
\begin{array}{ccc}
\Omega X & \xrightarrow{j} & \Lambda X \\
\downarrow & & \downarrow \varepsilon_0 \\
X & & X
\end{array}
\]

where \(j : \Omega X \to \Lambda X\) is the inclusion

\[j(l) = l,\]

and \(\varepsilon_0 : \Lambda X \to X\) is evaluation at \(t = 0\):

\[\varepsilon_0(\varphi) = \varphi(0).\]

Recall that the map \(j\) is an H-map, and it has unique lifts (see Section 3.1). We will need these facts when defining the homotopy equivalences.

**Definition 4.1.3.** Given an HA-space \(X\), its loop space \(\Omega X\), and its free loop space \(\Lambda X\), let us define a map \(h'_1 : \Lambda X \to X \times \Lambda X\) so that at \(t \in [0, 1]\), \(h'_1(\varphi)\) is given by

\[h'_1(\varphi)(t) = (\varphi(0), i(\varphi(0))\varphi(t)).\]
Notice that

\[ p_1^\Lambda h'_1(\varphi)(t) = \varphi(0) \]
\[ = \varepsilon_0(\varphi). \quad (4.1.1) \]

When we apply \( \varepsilon_0 p_2^\Lambda \), we get

\[ \varepsilon_0 p_2^\Lambda h'_1(\varphi) = p_2^\Lambda h'_1(\varphi)(0) \]
\[ = i(\varphi(0))\varphi(0), \quad (4.1.2) \]

so the composition \( \varepsilon_0 p_2^\Lambda h'_1 : \Lambda X \to X \) is nullhomotopic. Hence there is a map \( h''_1 : \Lambda X \to \Omega X \) such that the diagram commutes:

\[ h''_1 \]
\[ \Lambda X \quad \xrightarrow{p_2^\Lambda h'_1} \quad \Lambda X \]
\[ \downarrow \varepsilon_0 \quad \downarrow j \]
\[ \varepsilon_0 \downarrow \quad \Omega X \]

From here, we can define \( h_1 : \Lambda X \to X \times \Omega X \) as

\[ h_1 = (\varepsilon_0 \times h''_1) \Delta_{\Lambda X}. \quad (4.1.4) \]

Then by equation 4.1.1 and diagram 4.1.3, we have the following commutative dia-
Unlike the case when $X$ is a topological group, we cannot write down a pointwise definition of $h_1$.

Now let us define a map $h'_2 : X \times \Lambda X \to \Lambda X$ as follows: $h'_2(x, \varphi)$ is a loop given by

$$h'_2(x, \varphi)(t) = \mu(x, \varphi(t)) = x\varphi(t).$$

We define $h_2 : X \times \Omega X \to \Lambda X$ to be the restriction of $h'_2$ to $X \times \Omega X$. Then the following diagram commutes:

Let us show that $h_1$ is a homotopy equivalence:

**Proposition 4.1.4.** The spaces $\Lambda X$ and $X \times \Omega X$ are homotopy equivalent as topological spaces.

**Proof.** We will prove that the compositions $h_2h_1 : \Lambda X \to \Lambda X$ and $h_1h_2 : X \times \Omega X \to X \times \Omega X$ are homotopic to the identity maps $id_{\Lambda X}$ and $id_{X \times \Omega X}$ respectively.

To prove that $h_2h_1 \simeq id_{\Lambda X}$, let us look at the composition $h'_2h'_1 : \Lambda X \to \Lambda X$. 

\[
\begin{array}{ccc}
\Lambda X & \xrightarrow{h'_1} & X \times \Lambda X \\
\downarrow^{h_1} & & \downarrow^{id_{X \times \Omega X} \circ h_2} \\
X \times \Omega X & \xrightarrow{\circ h_2} & X \times \Lambda X \\
\downarrow^{id_{X \times \Omega X} \circ h_1} & & \downarrow^{h'_1} \\
\Lambda X & \xrightarrow{h'_1} & X \times \Lambda X \\
\end{array}
\]
Elementwise, it is a loop given by

\[ h_2' h_1' (\varphi)(t) = \varphi(0) (i(\varphi(0)) \varphi(t)). \] (4.1.7)

By homotopy associativity, this map is homotopic to one that takes \( \varphi \) to a loop given by

\[ (\varphi(0) i(\varphi(0))) \varphi(t). \]

By using the homotopy inverse operation, this map is homotopic to one that takes \( \varphi \) to a loop given by \( \varphi(t) \), so

\[ h_2' h_1' \simeq id_{\Delta X}. \]

Hence by diagram 4.1.6,

\[ h_2 h_1 \simeq id_{\Delta X}. \] (4.1.8)

To prove that \( h_1 h_2 \simeq id_{X \times \Omega X} \), let us look at \( h_1' h_2 : X \times \Omega X \to X \times \Lambda X \).

Pointwise, it is given by

\[ h_1' h_2 (x, l)(t) = (xl(0), i(xl(0))[xl(t)]) \]

\[ = (x, i(x)[xl(t)]). \]

Let us use projection maps to study each component of \( h_1' h_2 (x, l)(t) \). In particular,
we have

\[ p_1^Ah_1'h_2(x, l) = x \]
\[ = p_1^A(id_X \times j)(id_{X \times \Omega X})(x, l), \quad (4.1.9) \]

while at any \( t \in [0, 1] \),

\[ p_2^Ah_1'h_2(x, l)(t) = i(x)[xl(t)], \quad (4.1.10) \]

so by homotopy associativity, \( p_2^Ah_1'h_2 \) is homotopic to a map which takes \((x, l)\) to the loop given by

\[ (i(x)x)l(t), \]

and this map, in turn, is homotopic to one that takes \((x, l)\) to the loop given by

\[ l(t) = j(l)(t) \]
\[ = p_2^A(id_X \times j)(id_{X \times \Omega X})(x, l)(t). \quad (4.1.11) \]

Consequently, by equations 4.1.10 and 4.1.11,

\[ p_2^Ah_1'h_2 \simeq p_2^A(id_X \times j)id_{X \times \Omega X} \quad (4.1.12) \]

and therefore, by equations 4.1.9 and 4.1.12,

\[ h_1'h_2 \simeq (id_X \times j)id_{X \times \Omega X}. \quad (4.1.13) \]
Thus, by diagram 4.1.6, \((id_X \times j)h_1' \simeq h_1\), so we can rewrite the previous equation as

\[(id_X \times j)h_1h_2 \simeq (id_X \times j)id_{X\times\Omega X}. \quad (4.1.14)\]

We need to show that \(h_1h_2 \simeq id_{X\times\Omega X}\). We will proceed by proving that

\[p_1^\Omega h_1h_2 \simeq p_1^\Omega id_{X\times\Omega X}
\]

and

\[p_2^\Omega h_1h_2 \simeq p_2^\Omega id_{X\times\Omega X}.
\]

When we apply \(p_1^\Lambda\) on both sides of equation 4.1.14, we get

\[p_1^\Lambda(id_X \times j)h_1h_2 \simeq p_1^\Lambda(id_X \times j)id_{X\times\Omega X}. \quad (4.1.15)\]

and by diagram 4.0.2,

\[p_1^\Lambda(id_X \times j) \simeq id_Xp_1^\Omega,
\]

so we obtain

\[id_Xp_1^\Omega h_1h_2 \simeq id_Xp_1^\Omega id_{X\times\Omega X}.
\]

and hence

\[p_1^\Omega h_1h_2 \simeq p_1^\Omega id_{X\times\Omega X}. \quad (4.1.16)\]
When we apply $p_2^\Lambda$ on both sides of equation 4.1.14 and use diagram 4.0.3, we obtain

$$jp_2^\Omega h_1h_2 \simeq jp_2^\Omega id_{X \times \Omega X}.$$ 

so by Lemma 3.1.4,

$$p_2^\Omega h_1h_2 \simeq p_2^\Omega id_{X \times \Omega X}. \quad (4.1.17)$$

Therefore, by combining equations 4.1.16 and 4.1.17, we obtain

$$h_1h_2 \simeq id_{X \times \Omega X}. \quad (4.1.18)$$

Hence, we can conclude that $h_1$ is a homotopy equivalence. \hfill \Box

In particular, the induced homomorphisms

$$h_1^* : H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p)$$

and

$$h_2^* : H^*(\Lambda X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$$

are algebra isomorphisms. Hence any element of $H^*(\Lambda X; \mathbb{F}_p)$ has the form

$$h_1^*(x \otimes y)$$

where $x \in H^*(X; \mathbb{F}_p)$ and $y \in H^*(\Omega X; \mathbb{F}_p)$. 
An immediate application of the homotopy equivalence $h_1$ is that we can use it to help us compute the homomorphism on cohomology induced by $\varepsilon_0$. By equation 4.1.4, the following diagram commutes:

\[
\begin{array}{c}
\Lambda X \\
\downarrow h_1 \downarrow \varepsilon_0 \\
X \times \Omega X \\
\downarrow h_2 \\
\downarrow p^1_1 \\
X
\end{array}
\]

Hence if $x \in H^*(X; \mathbb{F}_p)$, then

\[
\varepsilon_0^*(x) = h_1^*p^1_{1*}(x) = h_1^*(x \otimes 1). \tag{4.1.20}
\]

Now that we have introduced the homotopy equivalence $h_1$, we can proceed to discuss when it is an $H$-map, and examine its $H$-deviation, a map $D_{h_1} : \Lambda X \times \Lambda X \to X \times \Omega X$. We will find a way to write $D_{h_1}$ as a composition of maps, including $Com$.

One may ask how the homotopy equivalences interact with the HA-space structures. Recall that we give $X \times \Omega X$ a product HA-space structure with

\[
\mu_{X \times \Omega X} = (\mu \times \mu_\Omega)(id_X \times T_{\Omega X,X} \times id_{\Omega X}) : X \times \Omega X \times X \times \Omega X \to X \times \Omega X. \tag{4.1.21}
\]

Pointwise, given $(x_1, l_1), (x_2, l_2) \in X \times \Omega X$, at $t \in [0, 1],

\[
\mu_{X \times \Omega X}((x_1, l_1), (x_2, l_2))(t) = (x_1x_2, l_1(t)l_2(t)). \tag{4.1.22}
\]
Thus, what we would like to know is if $h_1$ and $h_2$ are $H$-maps. If they are $H$-maps, then in particular, the induced algebra isomorphism in cohomology

$$h_1^* : H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p)$$

would be a Hopf algebra isomorphism, and the induced linear transformation in homology

$$h_{2*} : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Lambda X; \mathbb{F}_p)$$

is a vector space isomorphism which is also an algebra isomorphism. Thus knowing the coproducts in $H^*(X; \mathbb{F}_p)$ and $H^*(\Omega X; \mathbb{F}_p)$ would allow easy computation of coproducts in $H^*(\Lambda X; \mathbb{F}_p)$: we compute the coproducts in $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ and apply $h_1^*$ to get coproducts in $H^*(\Lambda X; \mathbb{F}_p)$. Dually, we can find the product of two elements in $H_*(\Lambda X; \mathbb{F}_p)$ by multiplying their preimages under $h_{2*}$ in $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ and then applying $h_{2*}$.

However, $h_1$ might not be an $H$-map. Instead, we will determine how it fails to preserve the multiplication by using its $H$-deviation:

$$D_{h_1} : \Lambda X \times \Lambda X \to X \times \Omega X,$$

$$D_{h_1} = \mu_{X \times \Omega X} ((h_1 \circ \mu_\Lambda) \times ((i \times i_\Omega) \circ (\mu_{X \times \Omega X} \circ (h_1 \times h_1))) \Delta_{\Lambda X \times \Lambda X}.$$  

In order to study $D_{h_1}$, we will also need the $H$-deviation of $h_1'$, which is given as
follows. First, we need the multiplication map on \( X \times \Lambda X \):

\[
\mu_{X \times \Lambda X} = (\mu \times \mu_\Lambda)(id_X \times T_{\Lambda X,X} \times id_{\Lambda X}): X \times \Lambda X \times X \times \Lambda X \to X \times \Lambda X.
\]

Pointwise, given \((x_1, \varphi_1), (x_2, \varphi_2) \in X \times \Lambda X\), at \( t \in [0, 1]\),

\[
\mu_{X \times \Lambda X}((x_1, \varphi_1), (x_2, \varphi_2))(t) = (x_1 x_2, \varphi_1(t) \varphi_2(t)). \quad (4.1.23)
\]

From there, we define the H-deviation of \( h'_1 \) as a map

\[
D_{h'_1} : \Lambda X \times \Lambda X \to X \times \Lambda X,
\]

\[
D_{h'_1} = \mu_{X \times \Lambda X}((h'_1 \circ \mu_\Lambda) \times ((i \times i_\Lambda) \circ (\mu_{X \times \Lambda X} \circ (h'_1 \times h'_1)))) \Delta_{\Lambda X \times \Lambda X}. \quad (4.1.24)
\]

We can write out \( D_{h'_1} \) pointwise. Given \( \varphi_1, \varphi_2 \in \Lambda X \), \( D_{h'_1}(\varphi_1, \varphi_2) \in X \times \Lambda X \) is given as follows: its first component is

\[
(\varphi_1(0) \varphi_2(0)) i (\varphi_1(0) \varphi_2(0)) ;
\]

and its second component is a loop (in \( \Lambda X \)) given by

\[
(i (\varphi_1(0) \varphi_2(0)) (\varphi_1(t) \varphi_2(t))) i ((i (\varphi_1(0)) \varphi_1(t)) (i (\varphi_2(0)) \varphi_2(t))).
\]
Recall that according to diagram 4.1.5,

\[(id_X \times j) h_1 \simeq h'_1.\] (4.1.25)

We can use this to find a formula involving \(D_{h_1}\) and \(D_{h'_1}\). Note that \(id_X \times j : X \times \Omega X \to X \times \Lambda X\) is an H-map (the proof is similar to that of Lemma 3.1.3). From there, by applying Proposition 1.4.2 from Zabrodsky's book [40], we see that

\[(id_X \times j) D_{h_1} \simeq D_{h'_1},\] (4.1.26)

Let us combine equation 4.1.26 with diagrams 4.0.2 and 4.0.3; the resulting diagrams in the following proposition will be used in the proof of Lemma 4.1.6.

**Proposition 4.1.5.** The following diagrams commute:

\[
\begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{D_{h'_1}} & X \times \Lambda X \\
\downarrow^{id_X \times j} & & \downarrow^{p_1^\Lambda} \\
X \times \Omega X & \xrightarrow{D_{h_1}} & X
\end{array}
\] (4.1.27)

\[
\begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{D_{h'_1}} & X \times \Lambda X \\
\downarrow^{id_X \times j} & & \downarrow^{p_1^\Lambda} \\
X \times \Omega X & \xrightarrow{D_{h_1}} & X
\end{array}
\] (4.1.28)
We will show how the H-deviation of $h_1$ allows us to examine coproducts in $H^*(\Lambda X; \mathbb{F}_p)$. The first part of Lemma 4.1.6 is a restatement of Lemma 2.4.8 with $h_1^*$, but the second part of the lemma is specific to $D_{h_1}$, so the proof following the lemma will focus on proving equation 4.1.30.

Lemma 4.1.6. Given any $x \otimes y \in H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$, we have:

$$
\mu^*_\Lambda(h_1^*(x \otimes y)) = \Delta^*_{\Lambda X \times \Lambda X} \left[ (D_{h_1}^* \otimes [(h_1^* \otimes h_1^*)(\mu_{X \times \Omega X})^*]) \mu_{X \times \Omega X}^*(x \otimes y) \right]. \quad (4.1.29)
$$

In addition, we can write the map $D_{h_1}$ as the composition

$$
D_{h_1} \simeq j_2 \circ \text{Com} \circ (i \times id_{\Omega X}) \circ (\varepsilon_0 \times (p^\Omega_2 \circ h_1)) \circ T_{\Lambda X} \quad (4.1.30)
$$

where $j_2 : \Omega X \to X \times \Omega X$ is defined by $j_2(l) = (x_0, l)$.

This means that the coproduct structure in $H^*(\Lambda X; \mathbb{F}_p)$ can be determined from the coproduct structure of $H^*(X; \mathbb{F}_p)$ and $H^*(\Omega X; \mathbb{F}_p)$, along with knowledge of the map $\text{Com}^*$.

We will prove equation 4.1.30 for topological groups first, and then prove it for HA-spaces. We will use the definition of $h_1$ from Definition 4.1.1.

Proof. Let $X = G$ be a finite simply-connected topological group. We will prove that

$$
D_{h_1} = j_2 \circ \text{Com} \circ (i \times id_{\Omega G}) \circ (\varepsilon_0 \times (p^\Omega_2 \circ h_1)) \circ T_{\Lambda G}
$$
(notice the equality sign). We can write

\[ D_{h_1} = \mu_{G \times \Omega G} \left( (h_1 \circ \mu_A) \times ((i \times i_\Omega) \circ (\mu_{G \times \Omega G}) \circ (h_1 \times h_1)) \right) \Delta_{AG \times AG} \]

pointwise: for any \( \varphi_1, \varphi_2 \in \Lambda G \), \( D_{h_1}(\varphi_1, \varphi_2) \in G \times \Omega G \) is given as follows: its first component is

\[
(\varphi_1(0)\varphi_2(0)) (\varphi_1(0)\varphi_2(0))^{-1}
\]

\[ = g_0, \quad (4.1.31) \]

and its second component is a loop in \( \Omega G \) given by

\[
\left( (\varphi_1(0)\varphi_2(0))^{-1} (\varphi_1(t)\varphi_2(t)) \right) \left( (\varphi_1(0)^{-1}\varphi_1(t)) (\varphi_2(0)^{-1}\varphi_2(t)) \right)^{-1}
\]

\[ = (\varphi_2(0)^{-1} \varphi_1(0)^{-1}) \varphi_1(t) \left( \varphi_2(t) \left( \varphi_2(0)^{-1} \varphi_2(t)^{-1} (\varphi_1(0)^{-1}\varphi_1(t))^{-1} \right) \right)
\]

\[ = (\varphi_2(0)^{-1} \varphi_1(0)^{-1}) \varphi_1(t) \left( \varphi_2(t) \left( \varphi_2(0)^{-1} \varphi_2(t)^{-1} \right) \left( \varphi_2(0) (\varphi_1(0)^{-1}\varphi_1(t))^{-1} \right) \right)
\]

\[ = (\varphi_2(0)^{-1} \varphi_1(0)^{-1}) \varphi_1(t) \left( \varphi_2(0) (\varphi_1(0)^{-1}\varphi_1(t))^{-1} \right)
\]

\[ = (\varphi_2(0)^{-1} \varphi_1(0)^{-1}) \varphi_1(t) \left( \varphi_2(0) (\varphi_1(0)^{-1}\varphi_1(t))^{-1} \right)
\]

\[ = \text{Com}(\varphi_2(0)^{-1}, \varphi_1(0)^{-1}\varphi_1(t)). \quad (4.1.32) \]

By definition,

\[ i(\varepsilon_0(\varphi_2)) = \varphi_2(0)^{-1} \]
and

\[ \varphi_1(0)^{-1}\varphi_1(t) = p_2^\Omega h_1(\varphi_1)(t), \]

so \( D_{h_1}(\varphi_1, \varphi_2) \) is given by

\[
D_{h_1}(\varphi_1, \varphi_2) = (g_0, \text{Com}(i(\varepsilon_0(\varphi_2)), p_2h_1(\varphi_1))
\]

\[
= j_2 \circ \text{Com} \circ (i \times i(\varepsilon_0(\varphi_2))) \circ (p_2^\Omega \circ h_1) \circ T_{\Lambda G}(\varphi_1, \varphi_2).
\]

Hence

\[
D_{h_1} = j_2 \circ \text{Com} \circ (i \times i(\varepsilon_0(\varphi_2))) \circ (p_2^\Omega \circ h_1) \circ T_{\Lambda G}.
\]

When we prove Lemma 4.1.6 and equation 4.1.30 for finite simply-connected HA-spaces, we will need to work with \( h_1' \) and \( D_{h_1'} \), since these maps have pointwise formulas that we can use for calculations. We will also need to define another map \( \text{Com} : X \times \Lambda X \to \Lambda X \) similar to \( \text{Ad} \) from Lemma 3.2.9.

**Proof.** Let us start by writing out \( D_{h_1'} \) again. This time, we will need to define a map \( \text{Com} : X \times \Lambda X \to \Lambda X \), where for any \( t \in [0, 1] \),

\[
\text{Com}(x, \varphi)(t) = \text{com}(x, \varphi(t))
\]

\[
= (x \varphi(t))i(\varphi(t)x). \quad (4.1.33)
\]
Notice that

\[ \widehat{\text{Com}}(id_X \times j) = \widehat{\text{Com}}. \] (4.1.34)

Let us first prove that \( p_1^\Lambda D_{h_1'} \) is nullhomotopic. Note that

\[ p_1^\Lambda D_{h_1'}(\varphi_1, \varphi_2) = (\varphi_1(0)\varphi_2(0)) i (\varphi_1(0)\varphi_2(0)), \]

so \( p_1^\Lambda D_{h_1'} \) is nullhomotopic. Now let us prove that

\[ p_2^\Lambda D_{h_1'} \simeq \widehat{\text{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X}. \]

The verification of this equation will imitate the calculations in equations 4.1.31 and 4.1.32 from the previous proof. By definition, \( p_2^\Lambda D_{h_1'}(\varphi_1, \varphi_2) \in \Lambda X \) is a loop in \( \Lambda X \) given by

\[ (i (\varphi_1(0)\varphi_2(0)) (\varphi_1(t)\varphi_2(t))) i ((i (\varphi_1(0)) \varphi_1(t))(i (\varphi_2(0)) \varphi_2(t))). \]

By using homotopy associativity, along with Lemma 2.3.4 (see equation 4.1.32 for a similar calculation), we can see that \( p_2^\Lambda D_{h_1'} \) is homotopic to each of the following
maps (defined by their image of $(\varphi_1, \varphi_2)$):

$$(i(\varphi_1(0)\varphi_2(0))(\varphi_1(t)\varphi_2(t)))i((i(\varphi_1(0))\varphi_1(t))(i(\varphi_2(0))\varphi_2(t)))$$

$$(i(\varphi_2(0))(i(\varphi_1(0)))\varphi_1(t))(\varphi_2(t)(i(\varphi_2(0))\varphi_2(t))(i(\varphi_1(0))\varphi_1(t))))$$

$$(i(\varphi_2(0))(i(\varphi_1(0))\varphi_1(t)))(\varphi_2(t)((i(\varphi_2(t))\varphi_2(0))(i(\varphi_1(0))\varphi_1(t))))$$

$$(i(\varphi_2(0))(i(\varphi_1(0))\varphi_1(t)))(\varphi_2(0)i(i(\varphi_1(0))\varphi_1(t))))$$

$$(i(\varphi_2(0))(i(\varphi_1(0))\varphi_1(t)))i((i(\varphi_1(0))\varphi_1(t))\varphi_2(0))$$

and therefore, $p^A_2Dh'$ is homotopic to a map $D_2: \Lambda X \times \Lambda X \to X \times \Omega X$ which takes $(\varphi_1, \varphi_2)$ to an element of $X \times \Lambda X$ given pointwise by

$$D_2(\varphi_1, \varphi_2)(t) = [i(\varphi_2(0))(i(\varphi_1(0))\varphi_1(t))][i[(i(\varphi_1(0))\varphi_1(t))i(\varphi_2(0))].$$

Now, recall that

$$p^A_2h'_1(\varphi_1) = i(\varphi_1(0))\varphi_1(t) \in \Lambda X,$$

so

$$D_2(\varphi_1, \varphi_2)(t) = [i(\varphi_2(0))(i(\varphi_1(0))\varphi_1(t))][i[(i(\varphi_1(0))\varphi_1(t))i(\varphi_2(0))]
= \widehat{\text{Com}}(i(\varphi_2(0)), p^A_2h'_1(\varphi_1)).$$

(4.1.35)
Elementwise, we have

\[
(i(\varphi_2(0)), p_2^\Lambda h_1'(\varphi_1)) = (i \times \text{id}_X)(\varphi_2(0), p_2^\Lambda h_1'(\varphi_1))
= (i \times \text{id}_X)(\varepsilon_0 \times (p_2^\Lambda h_1'))(\varphi_2, \varphi_1)
= (i \times \text{id}_X)(\varepsilon_0 \times (p_2^\Lambda h_1'))T_\Lambda(\varphi_1, \varphi_2),
\]

so we can expand equation 4.1.35 as

\[
\widehat{\text{Com}}(i(\varphi_2(0)), p_2^\Lambda h_1'(\varphi_1)) = \widehat{\text{Com}}(i \times \text{id}_X)(\varepsilon_0 \times (p_2^\Lambda h_1'))T_\Lambda(\varphi_1, \varphi_2).
\]

Hence

\[
p_2^\Lambda D_{h_1'} \simeq p_2^\Lambda D_2 = \widehat{\text{Com}}(i \times \text{id}_X)(\varepsilon_0 \times (p_2^\Lambda h_1'))T_\Lambda.
\]

Now let us prove that

\[
jp_2^\Omega D_{h_1} \simeq p_2^\Lambda D_{h_1'}
\]

\[
\simeq \widehat{\text{Com}}(i \times \text{id}_X)(\varepsilon_0 \times (p_2^\Lambda h_1'))T_\Lambda
\]

\[
\simeq j \circ \text{Com}(i \times \text{id}_\Omega X)(\varepsilon_0 \times (p_2^\Omega h_1))T_\Lambda.
\]

According to diagram 4.1.28,

\[
jp_2^\Omega D_{h_1} \simeq p_2^\Lambda D_{h_1'}.
\]
Meanwhile, by combining diagrams 4.1.5 and 4.0.3,

\[ p_2^{A}h_1' \simeq j \circ p_2^{\Omega}h_1, \]

so we see that the right side of equation 4.1.37 becomes

\[ \widehat{\text{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^{A}h_1'))T_{\Lambda X} \simeq \widehat{\text{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (j \circ p_2^{\Omega}h_1))T_{\Lambda X}. \quad (4.1.39) \]

We need to move \( j \) to the left end of the composition. Given \( \varphi_1, \varphi_2 \in \Lambda X \), the map on the right can be expanded elementwise:

\[
\widehat{\text{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (j \circ p_2^{\Omega}h_1))T_{\Lambda X}(\varphi_1, \varphi_2)
= \widehat{\text{Com}}(i(\varphi_2(0)), j \circ p_2^{\Omega}h_1(\varphi_1))
= \widehat{\text{Com}}(id_X \times j)(i \times id_{\Omega X})(\varphi_2(0), p_2^{\Omega}h_1(\varphi_1))
= \widehat{\text{Com}}(id_X \times j)(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega}h_1))T_{\Lambda X}(\varphi_1, \varphi_2),
\]

and by equation 4.1.34, \( \widehat{\text{Com}}(id_X \times j) = \widehat{\text{Com}} \), so equation 4.1.39 becomes

\[ \widehat{\text{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^{A}h_1'))T_{\Lambda X} \simeq \widehat{\text{Com}}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega}h_1))T_{\Lambda X}. \quad (4.1.40) \]
Since \( j \circ \text{Com} \simeq \hat{\text{Com}} \), we can simplify equation 4.1.40 to

\[
\hat{\text{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^{\Lambda} h_1′)) T_{\Lambda X} \\
\simeq \hat{\text{Com}}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1)) T_{\Lambda X} \\
\simeq \hat{\text{Com}}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1)) T_{\Lambda X}.
\] (4.1.41)

Thus, when we combine equations 4.1.38 and 4.1.41, we obtain

\[
j p_2^{\Omega} D_{h_1} \simeq j \circ \text{Com}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1)) T_{\Lambda X},
\]

so by Lemma 3.1.4,

\[
p_2^{\Omega} D_{h_1} \simeq \text{Com}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1)) T_{\Lambda X}.
\] (4.1.42)

To conclude the proof, recall that \( p_1^{\Lambda} D_{h_1′} \) is nullhomotopic, so by diagram 4.1.27, \( p_1^{\Omega} D_{h_1} \) is nullhomotopic as well. Since

\[
D_{h_1}(\varphi_1, \varphi_2) = (p_1^{\Omega} D_{h_1}(\varphi_1, \varphi_2), p_2^{\Omega} D_{h_1}(\varphi_1, \varphi_2)),
\]

\[
(j_2 \circ p_2^{\Omega}) \circ D_{h_1}(\varphi_1, \varphi_2) = (x_0, p_2^{\Omega} D_{h_1}(\varphi_1, \varphi_2)),
\]

and \( p_1^{\Omega} D_{h_1} \) is nullhomotopic, we have \( D_{h_1} \simeq (j_2 \circ p_2^{\Omega}) \circ D_{h_1} \). Therefore, by equation 4.1.42,

\[
D_{h_1} \simeq j_2 \circ \text{Com}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1)) T_{\Lambda X}.
\]
We can use this lemma for some applications. First, we can find a twisted H-structure on $X \times \Omega X$ using $h_1$ and $D_{h_1}$. Second, we can look at multiplication in the homology algebra $H_*(\Lambda X; \mathbb{F}_p)$ using information from the product and coproduct structures of $H_*(X; \mathbb{F}_p)$ and $H_*(\Omega X; \mathbb{F}_p)$, along with knowledge of $\text{Com}_*$. After that, we can look for a relationship between $h_1^*$ being a Hopf algebra isomorphism and commutativity of the homology over $\mathbb{F}_p$ of the original space $X$ itself.

### 4.2 Twisted H-structures and $\Lambda X$

Let $X$ be a simply-connected HA-space. This section will restate the results of Section 4.2 in the context of twisted H-structures. We will show how to obtain a twisted H-structure on $X \times \Omega X$ using the H-deviation of the homotopy equivalence $h_1 : \Lambda X \rightarrow X \times \Omega X$.

Although $\Lambda X$ and $X \times \Omega X$ have the same homotopy type, it may not be true that the homotopy equivalence $h_1$ is an H-map. Instead, we can use the H-deviation and Lemma 4.1.6 to find a map $\omega : X \times \Omega X \rightarrow \Omega X$ and an $\omega$-twisted multiplication structure $\tilde{\mu}$ on $X \times \Omega X$ such that the following diagram commutes:

$$
\begin{array}{cccc}
\Lambda X \times \Lambda X & \xrightarrow{h_1 \times h_1} & X \times \Omega X \times X \times \Omega X & \\
\mu_X \downarrow & & \downarrow \tilde{\mu} & \\
\Lambda X & \xrightarrow{h_1} & X \times \Omega X & \\
\end{array}
$$

(4.2.1)
or equivalently,

\[
\begin{array}{c}
\Lambda X \times \Lambda X \\
\downarrow \mu \downarrow \mu \\
\Lambda X \\
\end{array}
\xleftarrow{h_2 \times h_2}
\xrightarrow{h_2}
\begin{array}{c}
X \times \Omega X \times X \times \Omega X \\
\downarrow \mu \\
X \times \Omega X \\
\end{array}
\tag{4.2.2}
\]

**Theorem 4.2.1.** Let \( X \) be a simply-connected HA-space. Let \( \omega : X \times \Omega X \to \Omega X \) be given by

\[
\omega(x, l) = \text{Com} \circ (i \times \text{id}_\Omega X)(x, l)
= \text{Com}(i(x), l).
\]

Then the following choice of \( \tilde{\mu} : X \times \Omega X \times X \times \Omega X \to X \times \Omega X \) will make diagram 4.2.1 commute:

\[
\tilde{\mu} = (\text{id}_X \times \mu_\Omega)(\mu \times \omega \times \mu_\Omega)(\text{id}_X \times \Delta X \times \Delta _\Omega X \times \mu_\Omega X)(\text{id}_X \times T_{\Omega X, X} \times \text{id}_\Omega X),
\]

\[
\tilde{\mu}(x_1, l_1, x_2, l_2) = (\mu(x_1, x_2), \mu_\Omega \left( \text{Com}(i(x_2), l_1), \mu_\Omega (l_1, l_2) \right)).
\]

Note that this theorem implies that \( h_1 \) is a \( \mu_\Lambda - \tilde{\mu} \) H-map.

**Proof.** We will prove that diagram 4.2.1 commutes as follows. Recall that

\[
(\mu_{X \times \Omega X}) \circ (D_{h_1} \times [(\mu_{X \times \Omega X}) \circ (h_1 \times h_1)])[\Delta_{AX \times AX}] \simeq h_1 \circ \mu_\Lambda,
\]
so it suffices to show that

\[(\mu_{X \times \Omega X} \circ (D_{h_1} \times [(\mu_{X \times \Omega X} \circ (h_1 \times h_1)]) \circ \Delta_{\Lambda X \times \Lambda X} \simeq \tilde{\mu}(h_1 \times h_1).\]

Throughout this proof, given \(x_1, x_2 \in X\) and \(l_1, l_2 \in \Omega X\), we will write

\[\mu(x_1, x_2) = x_1x_2,\]

\[\mu_{\Omega}(l_1, l_2) = l_1l_2.\]

Given \(\varphi_j \in \Lambda X\), we will write

\[h_1(\varphi_j) = (y_j, \alpha_j)\]

where \(y_j \in X\), \(\alpha_j \in \Omega X\).

First, let \(i_1 : X \to X \times \Omega X\) and \(i_2 : \Omega X \to X \times \Omega X\) be inclusions. Let us verify that the compositions \(\omega i_1\) and \(\omega i_2\) are nullhomotopic. For any \(x \in X\) and \(l \in \Omega X\),

\[\omega i_1(x) = Com(i \times id_{\Omega X})(x, l_0) = Com(i(x), l_0) = Com \circ i_1 \circ i(x),\]

and by Lemma 3.2.7, \(Com \circ i_1\) is nullhomotopic, so therefore \(\omega i_1\) is nullhomotopic.
Similarly, we have

\[
\omega i_2(l) = \text{Com}(i \times id_{\Omega X})(x_0, l)
\]

\[
= \text{Com}(i(x_0), l)
\]

\[
= \text{Com}(x_0, l)
\]

\[
= \text{Com} \circ i_2(l),
\]

and by Lemma 3.2.7, \(\text{Com} \circ i_2\) is nullhomotopic, so therefore \(\omega i_2\) is nullhomotopic.

Let us compute \(\tilde{\mu}(h_1 \times h_1)\). For any \(\varphi_1, \varphi_2 \in \Lambda X\),

\[
\tilde{\mu}(h_1 \times h_1)(\varphi_1, \varphi_2) = \tilde{\mu}(h_1(\varphi_1), h_1(\varphi_2))
\]

\[
= \tilde{\mu}(y_1, \alpha_1, y_2, \alpha_2)
\]

\[
= (id_X \times \mu)(\mu \times \omega \times \mu)(y_1, y_2, y_2, \alpha_1, \alpha_1, \alpha_2)
\]

\[
= (id_X \times \mu)(y_1y_2, \text{Com}(i(y_2), \alpha_1), \alpha_1\alpha_2)
\]

\[
= (y_1y_2, \mu(\text{Com}(i(y_2), \alpha_1), (\alpha_1\alpha_2))). \quad (4.2.3)
\]

Now let us write out

\[
(\mu_{X \times \Omega X}) \circ (D_{h_1} \times [(\mu_{X \times \Omega X}) \circ (h_1 \times h_1)]) \circ \Delta_{\Lambda X \times \Lambda X}
\]
elementwise. For any \( \varphi_1, \varphi_2 \in \Lambda X \),

\[
\begin{align*}
(\mu_{X \times \Omega X})(D_{h_1} \times [(\mu_{X \times \Omega X})(h_1 \times h_1)])\Delta_{\Lambda X \times \Lambda X}(\varphi_1, \varphi_2) \\
= (\mu_{X \times \Omega X})(D_{h_1}(\varphi_1, \varphi_2), (\mu_{X \times \Omega X})(y_1, \alpha_1, y_2, \alpha_2)) \\
= (\mu_{X \times \Omega X})(D_{h_1}(\varphi_1, \varphi_2), (y_1 y_2, \alpha_1 \alpha_2)) \\
= (\mu(p_1^{\Omega}D_{h_1}(\varphi_1, \varphi_2), p_1^{\Omega}(y_1 y_2, \alpha_1 \alpha_2)), \mu_{\Omega}(p_2^{\Omega}D_{h_1}(\varphi_1, \varphi_2), p_2^{\Omega}(y_1 y_2, \alpha_1 \alpha_2))) \\
= (\mu(p_1^{\Omega}D_{h_1}(\varphi_1, \varphi_2), y_1 y_2), \mu_{\Omega}(p_2^{\Omega}D_{h_1}(\varphi_1, \varphi_2), \alpha_1 \alpha_2)).
\end{align*}
\]

By Lemma 4.1.6, \( p_1^{\Omega}D_{h_1} \) is nullhomotopic, while

\[
p_2^{\Omega}D_{h_1} \simeq Com(i \times i_{d_{\Omega X}})(\varepsilon_0 \times (p_2^{\Omega}h_1))T_{\Lambda X}.
\]

Let us write out the map on the right side elementwise. Recall that

\[
p_1^{\Omega}h_1(\varphi_j) = \varepsilon_0(\varphi_j),
\]

so according to the notation of this proof,

\[
\varepsilon_0(\varphi_j) = y_j,
\]

\[
h_1(\varphi_j) = \alpha_j.
\]
Hence

\[
Com(i \times id_{\Omega X})(\varepsilon_0 \times (p^\Omega_1 h_1)) T_{\Lambda X}(\varphi_1, \varphi_2)
\]

\[
= \text{Com}(i(y_2), \alpha_1).
\]

Thus, \((\mu_{X \times \Omega X})(D_{h_1} \times [(\mu_{X \times \Omega X})(h_1 \times h_1)]) \Delta_{\Lambda X \times \Lambda X}\), and hence \(h_1 \circ \mu_\Lambda\), is homotopic to a map which takes \((\varphi_1, \varphi_2)\) to

\[
(y_1 y_2, \mu_\Omega(\text{Com}(i(y_2), \alpha_1), (\alpha_1 \alpha_2))) = \tilde{\mu}(h_1 \times h_1)(\varphi_1, \varphi_2).
\]

Therefore, \(h_1 \circ \mu_\Lambda \simeq \tilde{\mu}(h_1 \times h_1)\).

\[
\square
\]

In the next section, we will use this theorem to compute \(\mu_\Lambda \ast\) in \(H_\ast(\Lambda X; \mathbb{F}_p)\) using \(\mu_\ast\), \(\mu_{\Omega \ast}\), and \(\text{Com}_\ast\).

## 4.3 Multiplication in the Homology of the Free Loop Space

Let \(X\) be a simply-connected HA-space. Since \(h_1\) and \(h_2\) are homotopy equivalences, their induced linear transformations in homology,

\[
h_{1 \ast} : H_\ast(\Lambda X; \mathbb{F}_p) \to H_\ast(X; \mathbb{F}_p) \otimes H_\ast(\Omega X; \mathbb{F}_p)
\]
and

\[ h_{2*} : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Lambda X; \mathbb{F}_p) \]

are vector space isomorphisms (for which \( h_{1*} h_{2*} \) and \( h_{2*} h_{1*} \) are identity linear transformations), but not necessarily algebra isomorphisms. We will use Theorem 4.2.1 to give a formula for multiplying two elements in \( H_*(\Lambda X; \mathbb{F}_p) \) in terms of the multiplication structures on \( H_*(X; \mathbb{F}_p) \) and \( H_*(\Omega X; \mathbb{F}_p) \), along with the linear transformation

\[ \text{Com}_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p). \]

Since \( h_{2*} \) is a vector space isomorphism, any element of \( H_*(\Lambda X; \mathbb{F}_p) \) has the form

\[ h_{2*}(\bar{x} \otimes \bar{t}) \]

where \( \bar{x} \in H_*(X; \mathbb{F}_p) \) and \( \bar{t} \in H_*(\Omega X; \mathbb{F}_p) \).

**Remark.** We must stress that \( h_{2*} \) and \( h_{1*} \) may not be algebra isomorphisms. For example, in \( H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \), if \( \bar{x} \in H_*(X; \mathbb{F}_p) \) and \( \bar{t} \in H_*(\Omega X; \mathbb{F}_p) \) (\( \bar{t} \) has even degree),

\[ \bar{x} \otimes \bar{t} = \mu_{X \times \Omega X*}(\bar{x} \otimes 1 \otimes 1 \otimes \bar{t}) \]

\[ = (\bar{x} \otimes 1)(1 \otimes \bar{t}) \]

\[ = (1 \otimes \bar{t})(\bar{x} \otimes 1) \]

\[ = \mu_{X \times \Omega X*}(1 \otimes \bar{t} \otimes \bar{x} \otimes 1) \]

and we will see that in \( H_*(\Lambda X; \mathbb{F}_p) \), if \( \text{Com}_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p) \)
is not trivial, then there exists choices of \( \bar{x} \in H_*(X; \mathbb{F}_p) \) and \( \bar{t} \in H_*(\Omega X; \mathbb{F}_p) \) such that

\[
h_{2*}(\bar{x} \otimes \bar{t}) \neq \mu_{\Lambda_*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{x} \otimes 1)).
\]

To avoid confusion, we will never suppress \( \mu_{\Lambda_*} \) using concatenation, but for convenience, for any \( \bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p) \), \( \bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p) \), we will write:

\[
\mu_*(\bar{x} \otimes \bar{y}) = \bar{x} \bar{y},
\]

\[
\mu_{\Omega_*}(\bar{t} \otimes \bar{u}) = \bar{t} \bar{u},
\]

\[
\mu_{X \times \Omega X_*}(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}) = (\bar{x} \otimes \bar{t})(\bar{y} \otimes \bar{u}) = \bar{x} \bar{y} \otimes \bar{t} \bar{u}.
\]

Let us use Theorem 4.2.1 to examine the multiplication of \( H_*(AX; \mathbb{F}_p) \):

**Theorem 4.3.1.** Let \( X \) be a simply-connected HA-space. Let \( \bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p) \), \( \bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p) \), and the coproducts of \( \bar{y} \) and \( \bar{t} \) are as follows:

\[
\Delta_{X_*}(\bar{y}) = \bar{y} \otimes 1 + 1 \otimes \bar{y} + \sum_\beta \bar{y}_\beta \otimes \bar{y}_\beta',
\]

\[
\Delta_{\Omega X_*}(\bar{t}) = \bar{t} \otimes 1 + 1 \otimes \bar{t} + \sum_\gamma \bar{t}_\gamma \otimes \bar{t}_\gamma'.
\]
Then if \( \bar{y} \) and \( \bar{t} \) have positive degree,

\[
\mu_{\Lambda^*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u})) \\
= h_{2*}[\bar{x} \otimes \text{Com}_*(i_* (\bar{y} \otimes \bar{t})) \bar{u} \\
+ \sum_{\beta} \bar{x} \bar{y}''_{\beta} \otimes \text{Com}_*(i_* (\bar{y}''_{\beta} \otimes \bar{t})) \bar{u} \\
+ \sum_{\gamma} \bar{x} \otimes \text{Com}_*(i_* (\bar{y} \otimes \bar{t}')) \bar{u}''_{\gamma} \\
+ \sum_{\beta, \gamma} \bar{x} \bar{y}''_{\beta} \otimes \text{Com}_*(i_* (\bar{y}''_{\beta} \otimes \bar{t}')) \bar{u}''_{\gamma} \\
+ \bar{x} \bar{y} \otimes \bar{t} \bar{u}].
\]

(4.3.1)

If either \( \bar{y} = 1 \) or \( \bar{t} = 1 \),

\[
\mu_{\Lambda^*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u})) = h_{2*}(\bar{x} \bar{y} \otimes \bar{t} \bar{u}).
\]

(4.3.2)

**Proof.** When we apply homology to diagram 4.2.2, we see that if \( \bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p) \), \( \bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p) \), then

\[
\mu_{\Lambda^*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u})) \\
= h_{2*}(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}),
\]

(4.3.3)

where

\[
\tilde{\mu}_* = (1 \otimes \mu_{\Omega^*})(\mu_* \otimes \omega_* \otimes \mu_{\Omega^*})(1 \otimes \Delta_{X^*} \otimes \Delta_{\Omega X^*} \otimes 1)(1 \otimes T_{\Omega X^*, X^*} \otimes 1),
\]
\[ \omega_* = (\text{Com}_*)(i_* \otimes 1). \]

By applying \( h_{1*} \) on equation 4.3.3, we get

\[
\begin{align*}
  h_{1*} \mu_{\Lambda_*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u})) \\
  &= h_{1*} h_{2*} \tilde{\mu}_* (\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}) \\
  &= \tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}), \quad (4.3.4)
\end{align*}
\]

so to prove the theorem, it suffices to compute \( \tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}) \) and then apply \( h_{2*} \).

At some point, we will need to evaluate \( (\Delta_{X*} \otimes \Delta_{\Omega X*})(\bar{y} \otimes \bar{t}) \). Because of this, we will break the argument into four cases, depending on whether \( \bar{y} = 1 \) or not, and whether \( \bar{t} = 1 \) or not.

First, suppose that both \( \bar{y} \) and \( \bar{t} \) have positive degree. Before we compute \( \tilde{\mu}_* \), let us write down \( (\Delta_{X*} \otimes \Delta_{\Omega X*})(\bar{y} \otimes \bar{t}) \):

\[
(\Delta_{X*} \otimes \Delta_{\Omega X*})(\bar{y} \otimes \bar{t}) \\
= \bar{y} \otimes 1 \otimes \bar{t} \otimes 1 + \bar{y} \otimes 1 \otimes 1 \otimes \bar{t} + \sum_{\gamma} \bar{y} \otimes 1 \otimes \bar{t}_\gamma \otimes \bar{t}'_\gamma \\
  + 1 \otimes \bar{y} \otimes \bar{t} \otimes 1 + 1 \otimes \bar{y} \otimes 1 \otimes \bar{t} + \sum_{\gamma} 1 \otimes \bar{y} \otimes \bar{t}'_\gamma \otimes \bar{t}'_\gamma \\
  + \sum_{\beta} \bar{y}'_\beta \otimes \bar{y}'_\beta \otimes \bar{t} \otimes 1 + \sum_{\beta} \bar{y}'_\beta \otimes \bar{y}'_\beta \otimes 1 \otimes \bar{t} \\
  + \sum_{\beta, \gamma} \bar{y}'_\beta \otimes \bar{y}'_\beta \otimes \bar{t}_\gamma \otimes \bar{t}'_\gamma. \quad (4.3.5)
\]

First, we apply \((1 \otimes T_{\Omega X, X*} \otimes 1)\) to \( \bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u} \). Since \( H_*(\Omega X ; \mathbb{F}_p) \) has no nonzero
elements of odd degree, \( \tilde{t} \) has even degree, so

\[
(1 \otimes T_{\Omega X,X} \otimes 1)(\tilde{x} \otimes \bar{y} \otimes \bar{u}) = \tilde{x} \otimes \bar{y} \otimes \bar{t} \otimes \bar{u}.
\]

Next, we apply \((1 \otimes \Delta_{X*} \otimes \Delta_{\Omega X*} \otimes 1)\). By equation 4.3.5, we get

\[
(1 \otimes \Delta_{X*} \otimes \Delta_{\Omega X*} \otimes 1)(1 \otimes T_{\Omega X,X} \otimes 1)(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})
\]

\[
= \bar{x} \otimes \bar{y} \otimes 1 \otimes \bar{t} \otimes 1 \otimes \bar{u} + \bar{x} \otimes \bar{y} \otimes 1 \otimes 1 \otimes \bar{t} \otimes \bar{u} + \sum_{\gamma} \bar{x} \otimes \bar{y} \otimes 1 \otimes \bar{t}_{\gamma} \otimes \bar{t}_{\gamma} \otimes \bar{u}
\]

\[
+ \bar{x} \otimes 1 \otimes \bar{y} \otimes \bar{t} \otimes 1 \otimes \bar{u} + \bar{x} \otimes 1 \otimes \bar{y} \otimes 1 \otimes \bar{t} \otimes \bar{u} + \sum_{\gamma} \bar{x} \otimes 1 \otimes \bar{y} \otimes \bar{t}_{\gamma} \otimes \bar{t}_{\gamma} \otimes \bar{u}
\]

\[
+ \sum_{\beta} \bar{x} \otimes \bar{y}_{\beta} \otimes \bar{y}_{\beta} \otimes \bar{t} \otimes 1 \otimes \bar{u} + \sum_{\beta} \bar{x} \otimes \bar{y}_{\beta} \otimes \bar{y}_{\beta} \otimes 1 \otimes \bar{t} \otimes \bar{u}
\]

\[
+ \sum_{\beta, \gamma} \bar{x} \otimes \bar{y}_{\beta} \otimes \bar{y}_{\beta} \otimes \bar{t}_{\gamma} \otimes \bar{t}_{\gamma} \otimes \bar{u}.
\]  

(4.3.6)

Now we apply \((\mu_* \otimes \omega_* \otimes \mu_{\Omega*})\). By part 7 of Proposition 3.3.1, we see that for any \( \bar{z} \in H_*(X; \mathbb{F}_p) \), \( \bar{v} \in H_*(\Omega X; \mathbb{F}_p) \),

\[
Com_*(\bar{z} \otimes 1) = Com_*(1 \otimes \bar{v}) = 0,
\]

and since \(Com_*\) is a graded linear transformation,

\[
Com_*(1 \otimes 1) = 1,
\]
so we get

\[
(\mu_* \otimes \omega_* \otimes \mu_{\Omega^*})(1 \otimes \Delta_{X_*} \otimes \Delta_{\Omega X_*} \otimes 1)(1 \otimes T_{\Omega X_* \otimes 1})(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})
\]

\[
= \bar{x} \bar{y} \otimes 1 \otimes \bar{t} \bar{u} + \bar{x} \otimes \text{Com}_*(i_*(\bar{y}) \otimes \bar{t}) \otimes \bar{u} + \sum_{\gamma} \bar{x} \otimes \text{Com}_*(i_*(\bar{y}) \otimes \bar{t}_\gamma) \otimes \bar{t}_\gamma \bar{u} + \sum_{\beta} \bar{x} \bar{y}_\beta \otimes \text{Com}_*(i_*(\bar{y}_{\beta}) \otimes \bar{t}) \otimes \bar{u} + \sum_{\beta, \gamma} \bar{x} \bar{y}_\beta \otimes \text{Com}_*(i_*(\bar{y}_{\beta}) \otimes \bar{t}_\gamma) \otimes \bar{t}_\gamma \bar{u}.
\] (4.3.7)

Finally, we apply \((1 \otimes \mu_{\Omega^*})\), which gives us \(\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})\):

\[
\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})
\]

\[
= \bar{x} \bar{y} \otimes \bar{t} \bar{u} + \bar{x} \otimes \text{Com}_*(i_*(\bar{y}) \otimes \bar{t}) \bar{u} + \sum_{\gamma} \bar{x} \otimes \text{Com}_*(i_*(\bar{y}) \otimes \bar{t}_\gamma) \bar{t}_\gamma \bar{u} + \sum_{\beta} \bar{x} \bar{y}_\beta \otimes \text{Com}_*(i_*(\bar{y}_{\beta}) \otimes \bar{t}) \bar{u} + \sum_{\beta, \gamma} \bar{x} \bar{y}_\beta \otimes \text{Com}_*(i_*(\bar{y}_{\beta}) \otimes \bar{t}_\gamma) \bar{t}_\gamma \bar{u}.
\] (4.3.8)

We obtain the formula in the theorem by applying \(h_{2*}\). This proves the main part of the theorem. Now we proceed to the special cases where one of \(\bar{y}\) or \(\bar{t}\) has degree zero.
Suppose $\bar{y} = 1$ and $\bar{t}$ still has positive degree. Then

$$(\Delta_{X^*} \otimes \Delta_{\Omega X^*})(1 \otimes \bar{t}) = 1 \otimes 1 \otimes \bar{t} \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \bar{t} + \sum_{\gamma} 1 \otimes 1 \otimes \bar{t}_\gamma \otimes \bar{t}_\gamma' ,$$

so

$$\hat{\mu}_*(\bar{x} \otimes \bar{t} \otimes 1 \otimes \bar{u})$$

$$= (1 \otimes \mu_{\Omega^*})(\bar{x} \otimes Com_*(1 \otimes \bar{t}) \otimes \bar{u} + \bar{x} \otimes Com_*(1 \otimes 1) \otimes \bar{t} \bar{u}$$

$$+ \sum_{\gamma} \bar{x} \otimes Com_*(1 \otimes \bar{t}_\gamma) \otimes \bar{t}_\gamma' \bar{u})$$

$$= \bar{x} \otimes \bar{t} \bar{u} .$$

Now suppose $\bar{t} = 1$ and $\bar{y}$ has positive degree. Then

$$(\Delta_{X^*} \otimes \Delta_{\Omega X^*})(\bar{y} \otimes 1) = \bar{y} \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \bar{y} \otimes 1 \otimes 1 + \sum_{\beta} \bar{y}_\beta \otimes \bar{y}_\beta' \otimes 1 \otimes 1 ,$$

so

$$\hat{\mu}_*(\bar{x} \otimes 1 \otimes \bar{y} \otimes \bar{u})$$

$$= (1 \otimes \mu_{\Omega^*})(\bar{x} \bar{y} \otimes Com_*(1 \otimes 1) \otimes \bar{u} + \bar{x} \otimes Com_*(i_*\bar{y} \otimes 1) \otimes \bar{u}$$

$$+ \sum_{\beta} \bar{x} \bar{y}_\beta \otimes Com_*(i_*\bar{y}_\beta' \otimes 1) \otimes \bar{u})$$

$$= \bar{x} \bar{y} \otimes \bar{u} .$$
Finally, if $\bar{y} = 1$ and $\bar{t} = 1$, then

$$(\Delta_{X^*} \otimes \Delta_{\Omega X^*})(1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1,$$

so

\begin{align*}
\tilde{\mu}_*(\bar{x} \otimes 1 \otimes 1 \otimes \bar{u}) &= (1 \otimes \mu_{\Omega})(\bar{x} \otimes \text{Com}_*(1 \otimes 1) \otimes \bar{u}) \\
&= \bar{x} \otimes \bar{u}.
\end{align*}

We obtain the formulas in the theorem by applying $h_{2*}$.

We have the following corollary about the linear transformations $h_{1*}$ and $h_{2*}$ and how they affect generators and decomposables.

**Corollary 4.3.2.** The linear transformation $h_{2*}$ sends decomposables of $H_*(X; F_p) \otimes H_*(\Omega X; F_p)$ to decomposables of $H_*(\Lambda X; F_p)$. In addition, if $\{\bar{c}_1, \ldots, \bar{c}_r\}$ is a generating set for $H_*(X; F_p) \otimes H_*(\Omega X; F_p)$, then a possible generating set for $H_*(\Lambda X; F_p)$ is a subset of $\{h_{2*}(\bar{c}_1), \ldots, h_{2*}(\bar{c}_r)\}$.

**Proof.** A decomposable in $H_*(X; F_p) \otimes H_*(\Omega X; F_p)$ will be a sum of one of the following kinds of elements:

1) An element of the form $\bar{x}\bar{y} \otimes 1 = (\bar{x} \otimes 1)(\bar{y} \otimes 1)$ where $\bar{x}, \bar{y} \in H_*(X; F_p)$ have positive degree.
2) An element of the form $1 \otimes \bar{u} = (1 \otimes \bar{t})(1 \otimes \bar{u})$ where $\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)$ have positive degree.

3) An element of the form $\bar{x} \otimes \bar{t} = (\bar{x} \otimes 1)(1 \otimes \bar{t})$ where $\bar{x} \in H_*(X; \mathbb{F}_p)$ and $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$ have positive degree.

By Theorem 4.3.1 (specifically equation 4.3.2), we have

$$h_2^*(\bar{x}\bar{y} \otimes 1) = \mu_{\Lambda}^*(h_2^*(\bar{x} \otimes 1) \otimes h_2^*(\bar{y} \otimes 1)),$$

$$h_2^*(1 \otimes \bar{t}\bar{u}) = \mu_{\Lambda}^*(h_2^*(1 \otimes \bar{t}) \otimes h_2^*(1 \otimes \bar{u})),$$

$$h_2^*(\bar{x} \otimes \bar{t}) = \mu_{\Lambda}^*(h_2^*(\bar{x} \otimes 1) \otimes h_2^*(1 \otimes \bar{t})),$$

so $h_2^*(\bar{x}\bar{y} \otimes 1)$, $h_2^*(1 \otimes \bar{t}\bar{u})$, and $h_2^*(\bar{x} \otimes \bar{t})$ are decomposables as well. Hence $h_2^*$ sends decomposables to decomposables.

For the rest of this proof, let us abbreviate $C = H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$. Since $h_2^*$ sends decomposables to decomposables, we have an induced linear transformation on the module of indecomposables,

$$h_2^Q : Q(H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)) \rightarrow QH_*(\Lambda X; \mathbb{F}_p),$$

$$h_2^Q(\bar{c} + \bar{C}\bar{C}) = h_2^*(\bar{c}) + \bar{H}_*(\Lambda X; \mathbb{F}_p)\bar{H}_*(\Lambda X; \mathbb{F}_p).$$
Note that $h^{Q}_{2s}$ is surjective: given

$$
\bar{z} + \bar{H}_s(\Lambda X; \mathbb{F}_p)\bar{H}_s(\Lambda X; \mathbb{F}_p) \in QH_s(\Lambda X; \mathbb{F}_p),
$$

we have

$$
h^{Q}_{2s}(h_{1s}(\bar{z}) + \bar{C}\bar{C}) = h_{2s}h_{1s}(\bar{z}) + \bar{H}_s(\Lambda X; \mathbb{F}_p)\bar{H}_s(\Lambda X; \mathbb{F}_p)
$$

$$
= \bar{z} + \bar{H}_s(\Lambda X; \mathbb{F}_p)\bar{H}_s(\Lambda X; \mathbb{F}_p).
$$

However, $h_{2s}$ may send generators to decomposables, so $h^{Q}_{2s}$ may not be injective.

Consequently, if

$$
\mathcal{B} = \{\bar{c}_1 + \bar{C}\bar{C}, \ldots, \bar{c}_r + \bar{C}\bar{C}\}
$$

is a basis of $Q(H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p))$, then the set

$$
h^{Q}_{2s}(\mathcal{B}) = \{h_{2s}(\bar{c}_1) + \bar{H}_s(\Lambda X; \mathbb{F}_p)\bar{H}_s(\Lambda X; \mathbb{F}_p), \ldots, h_{2s}(\bar{c}_r) + \bar{H}_s(\Lambda X; \mathbb{F}_p)\bar{H}_s(\Lambda X; \mathbb{F}_p)\}
$$

will span $QH_*(\Lambda X; \mathbb{F}_p)$, but it may not be linearly independent. Then a subset of $h^{Q}_{2s}(\mathcal{B})$ will be a basis for $QH_*(\Lambda X; \mathbb{F}_p)$. Therefore, if $\{\bar{c}_1, \ldots, \bar{c}_r\}$ is a generating set for $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$, then a subset of $\{h_{2s}(\bar{c}_1), \ldots, h_{2s}(\bar{c}_r)\}$ will be a generating set for $H_*(\Lambda X; \mathbb{F}_p)$.

We also have the following corollary about powers in $H_*(\Lambda X; \mathbb{F}_p)$. 

\[ \square \]
Corollary 4.3.3. Let $\bar{x} \in H_\ast(X; \mathbb{F}_p)$, $\bar{t} \in H_\ast(\Omega X; \mathbb{F}_p)$ and $n \geq 1$. Then

$$[h_2(\bar{x} \otimes 1)]^n = h_2(\bar{x}^n \otimes 1),$$

where $[h_2(\bar{x} \otimes 1)]^n$ means to take the power using $\mu\Lambda_\ast$, and

$$[h_2(1 \otimes \bar{t})]^n = h_2(1 \otimes \bar{t}^n).$$

Proof. Let us use induction on $n$. Suppose

$$[h_2(\bar{x} \otimes 1)]^{n-1} = h_2(\bar{x}^{n-1} \otimes 1).$$

Then by equation 4.3.2,

$$h_2(\bar{x}^n \otimes 1) = \mu\Lambda_\ast(h_2(\bar{x}^{n-1} \otimes 1) \otimes h_2(\bar{x} \otimes 1))$$

$$= \mu\Lambda_\ast([h_2(\bar{x} \otimes 1)]^{n-1} \otimes h_2(\bar{x} \otimes 1))$$

$$= [h_2(\bar{x} \otimes 1)]^n.$$ 

Hence equation 4.3.9 holds by induction. The proof of equation 4.3.10 is similar. 

Example 4.3.4. Let us illustrate Theorem 4.3.1. Let $\bar{x}, \bar{y} \in H_\ast(X; \mathbb{F}_p)$, $\bar{t}, \bar{u} \in H_\ast(\Omega X; \mathbb{F}_p)$. We will use the same notation as in the theorem for their coproducts.
According to Theorem 4.3.1, if $\bar{t} = \bar{u} = 1$, we have

$$\mu_{\Lambda^*}(h_{2*}(\bar{x} \otimes 1) \otimes h_{2*}(\bar{y} \otimes 1)) = h_{2*}(\bar{x}\bar{y} \otimes 1),$$

(4.3.11)

and if instead $\bar{x} = \bar{y} = 1$, we have

$$\mu_{\Lambda^*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(1 \otimes \bar{u})) = h_{2*}(1 \otimes \bar{t}\bar{u}),$$

(4.3.12)

and if $\bar{y} = 1$ and $\bar{t} = 1$, we have already seen that

$$\mu_{\Lambda^*}(h_{2*}(\bar{x} \otimes 1) \otimes h_{2*}(1 \otimes \bar{u})) = h_{2*}(\bar{x} \otimes \bar{u}),$$

(4.3.13)

so in these cases, $h_{2*}$ acts like an algebra homomorphism on $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$.

Now, let us look at $\mu_{\Lambda^*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes 1))$ with the additional assumption that $\bar{y}$ and $\bar{t}$ are primitive. By Corollary 2.3.6, $i_*(\bar{y}) = -\bar{y}$, so by Theorem 4.3.1, we have

$$\mu_{\Lambda^*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes 1)) = h_{2*}(1 \otimes Com_*(i_*(\bar{y}) \otimes \bar{t}) + \bar{y} \otimes \bar{t})$$

$$= h_{2*}(-1 \otimes Com_*(\bar{y} \otimes \bar{t}) + \bar{y} \otimes \bar{t}).$$

(4.3.14)
Note that if $Com_*(\bar{y} \otimes \bar{t})$ vanishes (for example, if $\bar{y}$ has odd degree), then

$$\mu_{\Lambda_*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes 1)) = h_{2*}(1 \otimes Com_*(i_* \bar{y} \otimes \bar{t}) + \bar{y} \otimes \bar{t})$$

$$= h_{2*}(-1 \otimes Com_*(\bar{y} \otimes \bar{t}) + \bar{y} \otimes \bar{t})$$

$$= h_{2*}(\bar{y} \otimes \bar{t}). \quad \text{(4.3.15)}$$

Let us see what happens if $Com_*(\bar{y} \otimes \bar{t})$ does not vanish in a concrete example in $H_*(\Lambda F_4; \mathbb{F}_3)$. We will use Theorems 1 and 2 from Hara and Hamanaka’s paper [10] for data about coproducts in $H_*(\Omega F_4; \mathbb{F}_3)$ and computations of $Ad_*$. We will use $\bar{x}$ to denote elements in $H_*(F_4; \mathbb{F}_3)$ (instead of $y$) and $\bar{t}$ to denote elements in $H_*(\Omega F_4; \mathbb{F}_3)$ (instead of $t$). Note that Hara and Hamanaka use $ad$ (with a lower case) to denote the map Kono and Kozima call $Ad$. For reference, a generating set of $H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3)$ would be

$$\{\bar{x}_3 \otimes 1, \bar{x}_7 \otimes 1, \bar{x}_8 \otimes 1, 1 \otimes \bar{t}_2, 1 \otimes \bar{t}_6, 1 \otimes \bar{t}_{10}, 1 \otimes \bar{t}_{14}, 1 \otimes \bar{t}_{22}\}.$$

According to Hara and Hamanaka, in $H_*(\Omega F_4; \mathbb{F}_3)$, we have the following calculation of $Ad_* : H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3) \to H_*(\Omega F_4; \mathbb{F}_3)$:

$$Ad_* (\bar{x}_8 \otimes \bar{t}_2) = \bar{t}_{10}.$$

According to part 4 of Proposition 3.5.4, if $\bar{t} \in H_*(\Omega F_4; \mathbb{F}_3)$ is primitive $(\Delta_{\Omega X_*}(\bar{t}) = \ldots$
\( \bar{t} \otimes 1 + 1 \otimes \bar{t} \), then for any \( \bar{x} \in H_*(F_4; \mathbb{F}_3) \),

\[ \text{Com}_*(\bar{x} \otimes \bar{t}) = \mu_{\Omega^*}(\text{Ad}_* \otimes i_{\Omega^*})(\bar{x} \otimes \Delta_{\Omega X^*}(\bar{t})) \]

\[ = \mu_{\Omega^*}(\text{Ad}_* \otimes i_{\Omega^*})(\bar{x} \otimes \bar{t} \otimes 1 + \bar{x} \otimes 1 \otimes \bar{t}) \]

\[ = \mu_{\Omega^*}(\text{Ad}_*(\bar{x} \otimes \bar{t}) \otimes 1 + \text{Ad}_*(\bar{x} \otimes 1) \otimes \bar{t}) \]

\[ = \text{Ad}_*(\bar{x} \otimes \bar{t}) \]

(4.3.16)

since \( \text{Ad}_*(\bar{x} \otimes 1) = 0 \) (by part 7 of Proposition 3.3.1). Thus, in \( H_*(\Omega F_4; \mathbb{F}_3) \), we have

\[ \text{Com}_*(\bar{x}_8 \otimes \bar{t}_2) = \bar{t}_{10}. \]

Therefore, in \( H_*(\Lambda F_4; \mathbb{F}_3) \), by equation 4.3.13,

\[ \mu_{\Lambda^*}(h_{2*}(\bar{x}_8 \otimes 1) \otimes h_{2*}(1 \otimes \bar{t}_2)) = h_{2*}(\bar{x}_8 \otimes \bar{t}_2), \]

while by equation 4.3.14,

\[ \mu_{\Lambda^*}(h_{2*}(1 \otimes \bar{t}_2) \otimes h_{2*}(\bar{x}_8 \otimes 1)) = h_{2*}(1 \otimes \text{Com}_*(i_*(\bar{x}_8) \otimes \bar{t}_2) + \bar{x}_8 \otimes \bar{t}_2) \]

\[ = h_{2*}(-1 \otimes \text{Com}_*(\bar{x}_8 \otimes \bar{t}_2) + \bar{x}_8 \otimes \bar{t}_2) \]

\[ = -h_{2*}(1 \otimes \bar{t}_{10}) + h_{2*}(\bar{x}_8 \otimes \bar{t}_2) \]

\[ \neq \mu_{\Lambda X^*}(h_{2*}(\bar{x}_8 \otimes 1) \otimes h_{2*}(1 \otimes \bar{t}_2)). \]
Hence there is a nontrivial commutator \([h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_2)]\) in \(H_*(\Lambda F_4; \mathbb{F}_3)\),

\[
[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_2)] = h_{2*}(1 \otimes \bar{t}_{10})
= h_{2*}(1 \otimes \text{Ad}_*(\bar{x}_8 \otimes \bar{t}_2)).
\] (4.3.17)

Notice that this means \(h_{2*}(1 \otimes \bar{t}_{10}) \in H_*(\Lambda F_4; \mathbb{F}_3)\) is a decomposable; recall that \(1 \otimes \bar{t}_{10} \in H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3)\) is a generator.

Let us continue our calculation of products and commutators in \(H_*(\Lambda F_4; \mathbb{F}_3)\).

If \(j = 3\) or \(7\), and \(k\) is either \(2, 6, 10, 14,\) or \(22\), then for any \(\bar{u} \in H_*(\Omega F_4; \mathbb{F}_3)\) (so \(\bar{u}\) has even degree),

\[
\text{Com}_*(\bar{x}_j \otimes \bar{u}) = 0,
\]

so by Theorem 4.3.1,

\[
\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_k) \otimes h_{2*}(\bar{x}_j \otimes 1)) = h_{2*}(\bar{x}_j \otimes \bar{t}_k)
= \mu_{\Lambda*}(h_{2*}(\bar{x}_j \otimes 1) \otimes h_{2*}(1 \otimes \bar{t}_k));
\]

and consequently,

\[
[h_{2*}(\bar{x}_j \otimes 1), h_{2*}(1 \otimes \bar{t}_k)] = 0.
\]

Next, we look at \(\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_6) \otimes h_{2*}(\bar{x}_8 \otimes 1))\). The element \(\bar{t}_6 \in H_*(\Omega F_4; \mathbb{F}_3)\) is not primitive; its coproduct (again using data from Hara and Hamanaka’s paper [10]) is

\[
\Delta_{\Omega X*}(\bar{t}_6) = \bar{t}_6 \otimes 1 + 1 \otimes \bar{t}_6 - \bar{t}_2^2 \otimes \bar{t}_2 - \bar{t}_2 \otimes \bar{t}_2^2.
\]
In order to use $Ad_*(\bar{x}_8 \otimes \bar{t}_6)$ to compute $Com_*(\bar{x}_8 \otimes \bar{t}_6)$ (with part 5 of Proposition 3.5.4), we will also need the following: $i_{\Omega_*}(\bar{t}_2^2)$, $i_{\Omega_*}(\bar{t}_6)$, $Ad_*(\bar{x}_8 \otimes \bar{t}_2)$, and $Ad_*(\bar{x}_8 \otimes \bar{t}_6^2)$.

Let us compute $i_{\Omega_*}(\bar{t}_6)$. Since $i_{\Omega_*}(\bar{t}_2) = -\bar{t}_2$, by applying homology to Lemma 2.3.4,

$$i_{\Omega_*}(\bar{t}_2) = i_{\Omega_*}(\mu_{\Omega_*}(\bar{t}_2 \otimes \bar{t}_2))$$
$$= \mu_{\Omega_*}(i_{\Omega_*} \otimes i_{\Omega_*})(T_{\Omega X_*}(\bar{t}_2 \otimes \bar{t}_2))$$
$$= \mu_{\Omega_*}(-\bar{t}_2 \otimes (-\bar{t}_2))$$
$$= \bar{t}_2^2,$$

so by Corollary 2.3.6,

$$i_{\Omega_*}(\bar{t}_6) = -\bar{t}_6 + i_{\Omega_*}(\bar{t}_2^2)\bar{t}_2 + i_{\Omega_*}(\bar{t}_2)\bar{t}_2^2$$
$$= -\bar{t}_6 + \bar{t}_2^3 + (-\bar{t}_2)\bar{t}_2^2$$
$$= -\bar{t}_6.$$

We already know that $Com_*(\bar{x}_8 \otimes \bar{t}_2) = \bar{t}_{10} = Ad_*(\bar{x}_8 \otimes \bar{t}_2)$ and $Ad_*(\bar{x}_8 \otimes \bar{t}_6) = \bar{t}_{14} - \bar{t}_{10}\bar{t}_2^2$.

Before we compute $Com_*(\bar{x}_8 \otimes \bar{t}_6)$, we also need $Ad_*(\bar{x}_8 \otimes \bar{t}_6^2)$, which can be calculated
using part 5 of Proposition 3.5.4:

\[
\text{Ad}_s(\bar{x}_8 \otimes \bar{t}_2^2) = \mu_{\Omega}(\text{Ad}_s(\bar{x}_8 \otimes \bar{t}_2), \text{Ad}_s(1 \otimes \bar{t}_2)) \\
+ \mu_{\Omega}(\text{Ad}_s(1 \otimes \bar{t}_2), \text{Ad}_s(\bar{x}_8 \otimes \bar{t}_2)) \\
= \bar{t}_{10}\bar{t}_2 + \bar{t}_2\bar{t}_{10} \\
= -\bar{t}_{10}\bar{t}_2.
\]

Hence

\[
\text{Com}_s(\bar{x}_8 \otimes \bar{t}_6) \\
= \mu_{\Omega s}(\text{Ad}_s \otimes i_{\Omega s})(\bar{x}_8 \otimes \Delta_{\Omega X_s}(\bar{t}_6)) \\
= \mu_{\Omega s}(\text{Ad}_s \otimes i_{\Omega s})(\bar{x}_8 \otimes \bar{t}_6 \otimes 1 + \bar{x}_8 \otimes 1 \otimes \bar{t}_6 - \bar{x}_8 \otimes \bar{t}_2^2 \otimes \bar{t}_2 - \bar{x}_8 \otimes \bar{t}_2 \otimes \bar{t}_2^2) \\
= \mu_{\Omega s}(\text{Ad}_s(\bar{x}_8 \otimes \bar{t}_6) \otimes 1 - \text{Ad}_s(\bar{x}_8 \otimes 1) \otimes \bar{t}_6 \\
+ \text{Ad}_s(\bar{x}_8 \otimes \bar{t}_2^2) \otimes \bar{t}_2 - \text{Ad}_s(\bar{x}_8 \otimes \bar{t}_2) \otimes \bar{t}_2^2) \\
= \mu_{\Omega s}((\bar{t}_{14} - \bar{t}_{10}\bar{t}_2^2) \otimes 1 - 0 - \bar{t}_{10}\bar{t}_2 \otimes \bar{t}_2 - \bar{t}_{10} \otimes \bar{t}_2^2) \\
= \bar{t}_{14} - \bar{t}_{10}\bar{t}_2 - \bar{t}_{10}\bar{t}_2^2 - \bar{t}_{10}\bar{t}_2^2 \\
= \bar{t}_{14}.
\]
Therefore, by Theorem 4.3.1,

\[ \mu_{\Lambda_4}(h_2^*(1 \otimes \bar{t}_6) \otimes h_2^*(\bar{x}_8 \otimes 1)) \]

\[ = h_2^*[1 \otimes \text{Com}_*(i_*(\bar{x}_8) \otimes \bar{t}_6) - 1 \otimes \text{Com}_*(i_*(\bar{x}_8) \otimes \bar{t}_2^3)\bar{t}_2 \]

\[ - 1 \otimes \text{Com}_*(i_*(\bar{x}_8) \otimes \bar{t}_2)\bar{t}_2^2 + \bar{x}_8 \otimes \bar{t}_6]. \tag{4.3.18} \]

By part 6 of Proposition 3.5.4,

\[ \text{Com}_*(\bar{x}_8 \otimes \bar{t}_2^2) = \mu_{\Omega}(\text{Com}_*(\bar{x}_8 \otimes \bar{t}_2), \text{Com}_*(1 \otimes \bar{t}_2)) \]

\[ + \mu_{\Omega}(\text{Com}_*(1 \otimes \bar{t}_2), \text{Com}_*(\bar{x}_8 \otimes \bar{t}_2)) \]

\[ = 0. \]

Hence equation 4.3.18 becomes

\[ \mu_{\Lambda_4}(h_2^*(1 \otimes \bar{t}_6) \otimes h_2^*(\bar{x}_8 \otimes 1)) = h_2^*[-1 \otimes \bar{t}_14 + 1 \otimes \bar{t}_10\bar{t}_2^2 + \bar{x}_8 \otimes \bar{t}_6]. \]

Thus, we have another commutator:

\[ [h_2^*(\bar{x}_8 \otimes 1), h_2^*(1 \otimes \bar{t}_6)] = h_2^*(1 \otimes \bar{t}_{14} - 1 \otimes \bar{t}_{10}\bar{t}_2^2) \]

\[ = h_2^*(1 \otimes \text{Ad}_*(\bar{x}_8 \otimes \bar{t}_6)). \tag{4.3.19} \]

Since the remaining generators of \( H_*(\Omega F_4; \mathbb{F}_3) \), namely \( \bar{t}_{10}, \bar{t}_{14}, \) and \( \bar{t}_{22} \), are primitive, by calculations similar to the one for \( \mu_{\Lambda F_4}(1 \otimes \bar{t}_2 \otimes \bar{x}_8 \otimes 1) \) in Example
4.3.4, we have

\[
Com_\ast(\bar{x}_8 \otimes \bar{t}_{10}) = Ad_\ast(\bar{x}_8 \otimes \bar{t}_{10})
\]

\[
= 1 \otimes \kappa \bar{t}_6^3
\]

where \( \kappa = \pm 1 \in \mathbb{F}_3 \),

\[
Com_\ast(\bar{x}_8 \otimes \bar{t}_{11}) = Ad_\ast(\bar{x}_8 \otimes \bar{t}_{11})
\]

\[
= 1 \otimes \bar{t}_{22},
\]

\[
Com_\ast(\bar{x}_8 \otimes \bar{t}_{22}) = Ad_\ast(\bar{x}_8 \otimes \bar{t}_{22})
\]

\[
= -1 \otimes \bar{t}_1^2,
\]

so

\[
\mu_{\Lambda_\ast}(h_2(1 \otimes \bar{t}_{10}) \otimes h_2(\bar{x}_8 \otimes 1)) = h_2(1 \otimes Com_\ast(i_\ast(\bar{x}_8) \otimes \bar{t}_{10}) + \bar{x}_8 \otimes \bar{t}_{10})
\]

\[= -h_2(1 \otimes \kappa \bar{t}_6^3) + h_2(\bar{x}_8 \otimes \bar{t}_{10}),\]

\[
\mu_{\Lambda_\ast}(h_2(1 \otimes \bar{t}_{14}) \otimes h_2(\bar{x}_8 \otimes 1)) = h_2(1 \otimes Com_\ast(i_\ast(\bar{x}_8) \otimes \bar{t}_{14}) + \bar{x}_8 \otimes \bar{t}_{14})
\]

\[= -h_2(1 \otimes \bar{t}_{22}) + h_2(\bar{x}_8 \otimes \bar{t}_{14}),\]
\[ \mu_{\Lambda_2}(h_2^*(1 \otimes \bar{t}_{22}) \otimes h_2^*(\bar{x}_8 \otimes 1)) = h_2^*(1 \otimes \text{Com}_*(i_*(\bar{x}_8) \otimes \bar{t}_{22}) + \bar{x}_8 \otimes \bar{t}_{22}) = h_2^*(1 \otimes \bar{t}_{10}^3) + h_2^*(\bar{x}_8 \otimes \bar{t}_{22}) \]

At this point, we have seen generators of \( H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \) whose image under \( h_2^* \) is a decomposable in \( H_*(\Lambda X; \mathbb{F}_p) \). By Corollary 4.3.2, there is a subset of

\[
\{ h_2^*(\bar{x}_3 \otimes 1), h_2^*(\bar{x}_7 \otimes 1), h_2^*(\bar{x}_8 \otimes 1), h_2^*(1 \otimes \bar{t}_2), \\
h_2^*(1 \otimes \bar{t}_6), h_2^*(1 \otimes \bar{t}_{10}), h_2^*(1 \otimes \bar{t}_{14}), h_2^*(1 \otimes \bar{t}_{22}) \}
\]

which is generating set for \( H_*(\Lambda F_4; \mathbb{F}_3) \). Based on our calculations, we could take

\[ \mathcal{B} = \{ h_2^*(\bar{x}_3 \otimes 1), h_2^*(\bar{x}_7 \otimes 1), h_2^*(\bar{x}_8 \otimes 1), h_2^*(1 \otimes \bar{t}_2), h_2^*(1 \otimes \bar{t}_6) \} \]

We can use equations 4.3.11, 4.3.12, and 4.3.15 to show that these five elements of \( \mathcal{B} \) are indeed generators; there is no way to write any of these five elements as (sums of) products of each other. Since the heights of \( \bar{x}_3, \bar{x}_7, \bar{x}_8 \in H_*(F_4; \mathbb{F}_3) \) are 2, 2, 3 respectively, \( \bar{t}_2 \in H_*(\Omega F_4; \mathbb{F}_3) \) has height 3, and \( \bar{t}_6 \in H_*(\Omega F_4; \mathbb{F}_3) \) has infinite height, by Corollary 4.3.3, the heights of the elements of \( \mathcal{B} \) are 2, 2, 3, 3, and infinity respectively. Furthermore, we have these commutators involving \( \bar{x}_8 \otimes 1 \) and \( 1 \otimes \bar{t}_j \) for \( j = 2, 6, 10, 14, 22 \) in \( H_*(\Lambda X; \mathbb{F}_p) \):

\[ [h_2^*(\bar{x}_8 \otimes 1), h_2^*(1 \otimes \bar{t}_2)] = h_2^*(1 \otimes \bar{t}_{10}), \]
\[ [h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_6)] = h_{2*}(1 \otimes \bar{t}_{14} - 1 \otimes \bar{t}_{10}t_2^2), \]

\[ [h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_{10})] = \pm h_{2*}(1 \otimes \bar{t}_6^2), \]

\[ [h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_{14})] = h_{2*}(1 \otimes \bar{t}_{22}), \]

\[ [h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_{22})] = -h_{2*}(1 \otimes \bar{t}_{10}^3). \]

The generating set \( \mathcal{B} \), along with information on heights and commutators, give us a complete picture of the algebra \( H_*(\Lambda F_4; \mathbb{F}_3) \). We see that \( h_{2*} \) takes \( 1 \otimes \bar{t}_{10} \), \( 1 \otimes \bar{t}_{14} \), and \( 1 \otimes \bar{t}_{22} \) (which are generators of \( H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3) \)) to decomposables in \( H_*(\Lambda F_4; \mathbb{F}_3) \). We note that \( \bar{t}_{10}, \bar{t}_{14}, \) and \( \bar{t}_{22} \) are in the image of \( \text{Com}_* \). In addition, for \( j = 2, 6, 10, 14, 22, \)

\[ [h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_j)] = h_{2*}(1 \otimes Ad_*(\bar{x}_8 \otimes \bar{t}_j)). \]

We are currently exploring further connections between commutators in \( H_*(\Lambda F_4; \mathbb{F}_3) \) and the linear transformation \( Ad_* \).

Let us end this section by briefly looking at what happens in cohomology. We can use Theorem 4.2.1 and Lemma 3.3.5 to compute coproducts of elements in the image of

\[ h_1^p \bar{p}_2^{\Omega*} \sigma^* : H^*(X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p). \]

**Theorem 4.3.5.** Let \( X \) be a finite simply-connected HA-space. In \( H^*(X; \mathbb{F}_p) \), let \( x \in S \) (see Theorem 1.2.4 and the discussion afterward for the definitions of \( S \)).
Then

$$\mu_\Lambda^*(h_1^*(1 \otimes \sigma^*(x))) = h_1^* \otimes h_1^*[1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x)$$

$$+ 1 \otimes (T_{\Omega X,X}^*(\iota^* \otimes \sigma^*)(\bar{\mu}^*(x))) \otimes 1].$$

Proof. When we apply cohomology to diagram 4.2.1, we see that

$$\mu_\Lambda^* h_1^* = (h_1^* \otimes h_1^*)\tilde{\mu}^*,$$

where

$$\tilde{\mu}^* = (1 \otimes T_{\Omega X,X}^* \otimes 1)(1 \otimes \Delta^* \otimes \Delta_{\Omega X}^* \otimes 1)(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*),$$

$$\omega^* = (i^* \otimes 1)Com^*.$$

Recall that $\sigma^*(x) \in PH^*(\Omega X; \mathbb{F}_p)$, so

$$\mu_\Omega^*(\sigma^*(x)) = \sigma^*(x) \otimes 1 + 1 \otimes \sigma^*(x),$$

and by Lemma 3.3.5,

$$Com^*(\sigma^*(x)) = (1 \otimes \sigma^*)(\bar{\mu}^*(x)).$$
so

\[
\omega^*(\sigma^*(x)) = (i^* \otimes 1)Com^*(\sigma^*(x)) = (i^* \otimes 1)(1 \otimes \sigma^*)(\bar{\mu}^*(x)) = (i^* \otimes \sigma^*)(\bar{\mu}^*(x)). \tag{4.3.20}
\]

Let us compute \(\tilde{\mu}^*(1 \otimes \sigma^*(x))\). We start with \((1 \otimes \mu_\Omega^*)\):

\[
(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x)) = 1 \otimes \sigma^*(x) \otimes 1 + 1 \otimes 1 \otimes \sigma^*(x).
\]

Next, we apply \((\mu^* \otimes \omega^* \otimes \mu_\Omega^*)\) and use equation 4.3.20:

\[
(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x))
= \mu^*(1) \otimes \omega^*(\sigma^*(x)) \otimes \mu_\Omega^*(1) + \mu^*(1) \otimes \omega^*(1) \otimes \mu_\Omega^*(\sigma^*(x))
= 1 \otimes 1 \otimes (i^* \otimes \sigma^*)(\bar{\mu}^*(x)) \otimes 1 \otimes 1 \\
+ 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma^*(x) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma^*(x).
\]

To apply \((1 \otimes \Delta_X^* \otimes \Delta_{\Omega X}^* \otimes 1)\), we multiply the second and third factors, and multiply
the fourth and fifth factors.

\[(1 \otimes \Delta_X^* \otimes \Delta_{\Omega X}^* \otimes 1)(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x))\]

\[= 1 \otimes (i^* \otimes \sigma^*)(\bar{\mu}^*(x)) \otimes 1 + 1 \otimes 1 \otimes \sigma^*(x) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x).\]

Finally, we apply \((1 \otimes T_{\Omega X}^* \otimes 1)\):

\[(1 \otimes T_{\Omega X}^* \otimes 1)(1 \otimes \Delta_X^* \otimes \Delta_{\Omega X}^* \otimes 1)(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x))\]

\[= \bar{\mu}^*(1 \otimes \sigma^*(x))\]

\[= (1 \otimes T_{\Omega X}^* \otimes 1)(1 \otimes (i^* \otimes \sigma^*)(\bar{\mu}^*(x)) \otimes 1)\]

\[+ 1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x).\]

Therefore, when we apply \((h_1^* \otimes h_1^*)\), we obtain

\[\mu^*_\Lambda(h_1^*(1 \otimes \sigma^*(x))) = (h_1^* \otimes h_1^*)\bar{\mu}^*(1 \otimes \sigma^*(x))\]

\[= (h_1^* \otimes h_1^*)[1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x)]\]

\[+ 1 \otimes T_{\Omega X}^*[((i^* \otimes \sigma^*)(\bar{\mu}^*(x))] \otimes 1].\]

Overall, it will be worthwhile to continue learning about the maps \(Ad\), \(Com\), and their induced homomorphisms and linear transformations in order to examine co-products in \(H^*(\Lambda X; \mathbb{F}_p)\), and multiplication and commutators in \(H_*(\Lambda X; \mathbb{F}_p)\). When
Hara and Hamanaka made their calculations of $Ad_*$ in [10], they needed to use the fact that for the Lie groups $G = F_4, E_6, E_7, E_8$, for any positive even integer $k$,

$$\dim Q^k H^*(G; \mathbb{F}_3) \leq 1.$$ 

Future efforts will need to find ways of bypassing this requirement.

## 4.4 The Cohomology of the Free Loop Space and $h_1$

For this section, we will restrict ourselves to finite simply-connected spaces. Recall Iwase’s observation that the homeomorphism $\Phi : \Omega G \times G \to \Lambda G$ from the introduction to this chapter will be a group homomorphism if $G$ is abelian. In that case, there would be an isomorphism of Hopf algebras $\Phi^* : H^*(\Lambda G; \mathbb{F}_p) \to H^*(\Omega G; \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p)$. In this section, we will show that a weaker condition than (homotopy) commutativity will suffice for showing that our homotopy equivalence $h_1$ will induce a Hopf algebra isomorphism, and this condition applies to finite simply-connected HA-spaces as well.

Given a finite simply-connected HA-space $X$, we can characterize commutativity of $H_*(X; \mathbb{F}_p)$ using $h_1^*$ by applying Lemma 4.1.6 and Theorem 3.4.2.

**Theorem 4.4.1.** The algebra $H_*(X; \mathbb{F}_p)$ is commutative iff the induced homomorphism $h_1^*$ is a Hopf algebra isomorphism.

**Proof.** By Theorem 3.4.2, commutativity of $H_*(X; \mathbb{F}_p)$ is equivalent to $Com^*$ being
trivial.

If $\text{Com}^* = 0$, then by Lemma 4.1.6,

$$D_{h_1}^* = T_{AX}^* \circ (\epsilon_0^* \otimes (h_1^* \circ \Omega^*)) \circ (i^* \otimes 1) \circ \text{Com}^* \circ j_2^* = 0,$$

so $D_{h_1}^* = 0$, and hence by equation 4.1.29,

$$\mu_\Lambda^* \circ h_1^*(x \otimes y) = (h_1^* \otimes h_1^*) \circ (\mu_{X \times \Omega X}^*)^*(x \otimes y).$$

Hence $h_1^*$ is not only an algebra isomorphism, it also preserves coproducts, so it is a Hopf algebra isomorphism. This implies that $H^*(AX; \mathbb{F}_p)$ (with coproduct $\mu_\Lambda^*$) and $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ (with coproduct $\mu_{X \times \Omega X}^*$) are isomorphic as Hopf algebras.

Now suppose that $h_1^*$ is a Hopf algebra isomorphism. Then by Lemma 2.4.8, $D_{h_1}^* = 0$, so by Lemma 4.1.6,

$$D_{h_1}^* = T_{AX}^* \circ (\epsilon_0^* \otimes (h_1^* \circ \Omega^*)) \circ (i^* \otimes 1) \circ \text{Com}^* \circ j_2^* = 0.$$

We will prove that if $D_{h_1}^* = 0$, then $\text{Com}^* = 0$. Now, note that $i^*$, $\epsilon_0^*$, $\Omega^*$, $h_1^*$, and $T_{AX}^*$ are all injective homomorphisms:

1) By Lemma 2.3.4, $i^* \circ i^* = id_{AX}^*$, so $i^*$ is a bijection.

2) By equation 4.1.20, $\epsilon_0^*(x) = x \otimes 1$.

3) Recall that $\Omega^*(y) = 1 \otimes y$.

4) The maps $h_1$ and $T_{AX}$ are homotopy equivalences, so their induced homo-
morphisms are algebra isomorphisms.

Although $j_2^*$ is not injective on $\mathcal{H}^*(X; \mathbb{F}_p) \otimes \mathcal{H}^*(\Omega X; \mathbb{F}_p)$, we have $j_2^*(1 \otimes y) = y$, so $j_2^*$ is injective on $\{1\} \otimes \mathcal{H}^*(\Omega X; \mathbb{F}_p)$ and surjective onto $\mathcal{H}^*(\Omega X; \mathbb{F}_p)$. Therefore, in order for $D_{h_1}^* = 0$, we must have $\text{Com}^* = 0$.

Thus, if the algebra $H_*(X; \mathbb{F}_p)$ is commutative, then $h_1^*$ is a Hopf algebra isomorphism. Consequently, if the coproduct structures of $\mathcal{H}^*(X; \mathbb{F}_p)$ and $\mathcal{H}^*(\Omega X; \mathbb{F}_p)$ are known, then the coproduct structure of $\mathcal{H}^*(\Lambda X; \mathbb{F}_p)$, and dually the multiplication in $H_*(\Lambda X; \mathbb{F}_p)$ are easily determined in this case. As we have seen, if $H_*(X; \mathbb{F}_p)$ is not commutative and $\text{Com}_*$ is not trivial, then the multiplication in $H_*(\Lambda X; \mathbb{F}_p)$ can look very different from the tensor product multiplication structure on $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$.

As a corollary, we have the following connection between commutativity of $H_*(X; \mathbb{F}_p)$ and commutativity of $H_*(\Lambda X; \mathbb{F}_p)$:

**Corollary 4.4.2.** The algebra $H_*(X; \mathbb{F}_p)$ is commutative iff the algebra $H_*(\Lambda X; \mathbb{F}_p)$ is commutative.

**Proof.** If the algebra $H_*(X; \mathbb{F}_p)$ is commutative, then by Theorem 4.4.1, $h_1^*$ is a Hopf algebra isomorphism, and equivalently $h_{1*}$ is an algebra isomorphism. Thus, $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ and $H_*(\Lambda X; \mathbb{F}_p)$ are isomorphic as algebras. Since $H_*(X; \mathbb{F}_p)$ and $H_*(\Omega X; \mathbb{F}_p)$ are commutative (see Definition 1.2.7), this implies $H_*(\Lambda X; \mathbb{F}_p)$ is commutative as well.

If the algebra $H_*(X; \mathbb{F}_p)$ is not commutative, then $H^*(X; \mathbb{F}_p)$ is not primitively
generated, so there is a generator \( z \in S \) with \( \tilde{\mu}^*(z) \neq 0 \). Since even degree elements in
\( S \) are primitive by definition, this generator \( z \) must have odd degree (and \( |z| > 1 \) since
we assume \( X \) is simply-connected), so by Theorem 2.2.4, \( \sigma^*(z) \neq 0 \). By Theorem
4.3.5, the coproduct of \( h_1^*(1 \otimes \sigma^*(z)) \) in \( H^*(\Lambda X; \mathbb{F}_p) \) is
\[
\mu^*_\Lambda(h_1^*(1 \otimes \sigma^*(z))) = (h_1^* \otimes h_1^*)(1 \otimes \sigma^*(z) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(z) \\
+ 1 \otimes T^*_\Omega X, X[((i^* \otimes \sigma^*) (\tilde{\mu}^*(z))) \otimes 1].
\]
Let us write out the reduced coproduct. First, we write
\[
\tilde{\mu}^*(z) = \sum z' \otimes z'',
\]
where each \( z'' \) is an odd degree generator (again \( |z''| > 1 \), so \( \sigma^*(z'') \neq 0 \). Then
\[
\tilde{\mu}^*_\Lambda(h_1^*(1 \otimes \sigma^*(z))) = (h_1^* \otimes h_1^*)(1 \otimes T^*_\Omega X, X[((i^* \otimes \sigma^*) (\tilde{\mu}^*(z))) \otimes 1] \\
= (h_1^* \otimes h_1^*)(1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1].
\]
To compute \( T^*_\Lambda X \tilde{\mu}^*_\Lambda(h_1^*(1 \otimes \sigma^*(z))) \), we note that the following diagram commutes
strictly:
\[
\begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{h_1 \times h_1} & X \times \Omega X \times X \times \Omega X \\
T^*_\Lambda X \downarrow & & \downarrow T^*_{X \times \Omega X} \\
\Lambda X \times \Lambda X & \xrightarrow{h_1 \times h_1} & X \times \Omega X \times X \times \Omega X
\end{array}
\]
\[ T_{X \times \Omega X}(h_1 \times h_1)(\varphi_1, \varphi_2) = T_{X \times \Omega X}(h_1(\varphi_1), h_1(\varphi_2)) \]
\[ = (h_1(\varphi_2), h_1(\varphi_1)) \]
\[ = (h_1 \times h_1)(\varphi_2, \varphi_1) \]
\[ = (h_1 \times h_1)T_{\Lambda X}(\varphi_2, \varphi_1). \]

Hence

\[ T^*_\Lambda \overline{\mu}_\Lambda^* (h_1^*(1 \otimes \sigma^*(z))) = T^*_\Lambda (h_1^* \otimes h_1^*)[\sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1] \]
\[ = (h_1^* \otimes h_1^*)T^*_\Lambda (1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1) \]
\[ = (h_1^* \otimes h_1^*)[\sum i^*(z') \otimes 1 \otimes 1 \otimes \sigma^*(z'')] \].

In \( H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \), we have

\[ \sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1 \neq \sum i^*(z') \otimes 1 \otimes 1 \otimes \sigma^*(z'') \]

since the second factor of every term on the left has positive degree, while the second factor of every term on the right has degree zero. Since \((h_1^* \otimes h_1^*)\) is an algebra isomorphism, it is injective, so

\[ (h_1^* \otimes h_1^*)[\sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1] \neq (h_1^* \otimes h_1^*)[\sum i^*(z') \otimes 1 \otimes 1 \otimes \sigma^*(z'')] \],

and therefore

\[ \overline{\mu}_\Lambda^* (h_1^*(1 \otimes \sigma^*(z))) \neq T^*_\Lambda \overline{\mu}_\Lambda^* (h_1^*(1 \otimes \sigma^*(z))), \]
so $H^*(\Lambda X; \mathbb{F}_p)$ is not cocommutative, and hence $H_*(\Lambda X; \mathbb{F}_p)$ is not commutative. □

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5 Altering H-space Structures

Given an H-space $X$ with multiplication map $\mu$, we may ask if there is another multiplication map on $X$ which induces a different coproduct structure on its cohomology with desired properties. For example, suppose $X$ is homotopy associative with multiplication map $\mu$, and for some odd prime $p$, $H_\ast(X, \mu; \mathbb{F}_p)$ is not commutative. One may ask if there is a multiplication map which makes $H_\ast(X, \nu; \mathbb{F}_p)$ commutative, and furthermore, if the coproduct $\nu^\ast$ makes the cohomology algebra $H^\ast(X, \nu; \mathbb{F}_p)$ primitively generated.

For example, if $G$ is one of the classical compact simply-connected Lie groups $A_n$, $B_n$, $C_n$, and $D_n$, or the exceptional group $G_2$, $H_\ast(G; \mathbb{F}_3)$ is commutative, while the opposite is true if $G$ is one of the following groups: $F_4$, $E_6$, $E_7$, or $E_8$. The main difference between these two sets of Lie groups is that the homomorphism $com^\ast$ does not vanish on the cohomology over $\mathbb{F}_3$ of $F_4$, $E_6$, $E_7$, and $E_8$. Thus, one can ask if there is an alternative multiplication map on $F_4$, $E_6$, $E_7$, and $E_8$ which induces a commutative homology algebra over $\mathbb{F}_3$, and if $com$ or $com^\ast$ will play a role.

The idea of modifying the multiplication map of an H-space to produce desired properties in cohomology originated from Harper and Zabrodsky’s work [11]. They
used the fact that an H-space \((X, \mu)\) has a map \(\psi_\lambda : (X, \mu) \to (X, \mu)\) (where \(\lambda\) is an integer) which is analogous to raising an element to the \(\lambda\)th power in a group (because of this, Harper and Zabrodsky call \(\psi_\lambda\) a power map for \(X\)). In particular, \(\psi_\lambda\) induces multiplication by \(\lambda\) on \(QH^*(X; \mathbb{F}_p)\). Harper and Zabrodsky use the H-deviation of \(\psi_\lambda\) from a \(\mu - \mu\) H-map to construct new multiplication maps on \(X\) and produce primitively generated cohomology algebras under specific conditions.

While Harper and Zabrodsky focused on power maps where \(\lambda\) multiplicatively generates the group \(\mathbb{F}_p \setminus \{0\}\), we instead looked at a specific power map for \(\lambda = -1\): the homotopy inverse operation of an HA-space. Consequently, the multiplication maps we obtained (in terms of the H-deviation of the homotopy inverse operation) could actually be rewritten in terms of \(\text{com}\). Hence we present our formulas for new multiplication maps in terms of \(\text{com}\), and utilize our results from Chapter 3 to determine properties of the cohomology algebra using the new coproduct.

Since we will be dealing with multiple multiplication maps on the same space (different H-spaces with the same underlying topological space), from now on, we will write \((X, \mu)\) to indicate an H-space \(X\) with multiplication map \(\mu\). Consequently, we will write the cohomology of \((X, \mu)\) over \(\mathbb{F}_p\) with coproduct induced by \(\mu\) as \(H^*(X, \mu; \mathbb{F}_p)\); the homology with product induced by \(\mu\) will be denoted \(H_*(X, \mu; \mathbb{F}_p)\). As a warning, the homology might no longer be an associative algebra.
5.1 Cocommutative Hopf Algebra Structures

Let us begin by defining what it means to alter an H-structure. Basically, we start with one multiplication map on $X = (X, \mu)$, and use the binary operation from Section 2.3 to combine it with another map to produce a new multiplication map. Once we have introduced alterations of H-structures, we introduce iterated multiplication maps on $(X, \mu)$ in order to discuss multiplying more than two elements at a time, along with their induced homomorphisms, the iterated coproducts in $H^*(X, \mu; \mathbb{F}_p)$. Finally, given a finite simply-connected HA-space $(X, \mu)$, we will show how to construct a multiplication map $\nu : X \times X \to X$ using iterated multiplication maps such that $H_*(X, \nu; \mathbb{F}_p)$ is commutative. Throughout this chapter, the symbol “\text{"#"}” will always be defined using the original multiplication map $\mu$, and concatenation of elements will mean multiply using $\mu$. In addition, $T : X \times X \to X \times X$ will be given by

$$T(x, y) = (y, x).$$

**Definition 5.1.1.** Given an H-space $(X, \mu)$ an altered multiplication map (or altered H-structure) on $X$ is obtained by choosing a map $w : X \times X \to X$ which must factor through the smash product. That is, if $i_1, i_2 : X \to X \times X$ are inclusions and $k : X \to X$ is the constant map $k(x) = x_0$, then

$$w i_1 \simeq w i_2 \simeq k.$$

(5.1.1)
From here, we define a map

$$\nu : X \times X \to X$$

by

$$\nu = w \ast \mu = \mu (w \times \mu) \Delta_X. \quad (5.1.2)$$

Elementwise, we have

$$\nu(x, y) = w(x, y)(xy).$$

We need to check if $$(X, \nu)$$ has a homotopy identity and verify:

$$\nu i_1 \simeq \nu i_2 \simeq id_X.$$

To do this, we use the fact that $$(X, \mu)$$ is an H-space, so by Definition 1.2.2,

$$\mu i_1 = \mu i_2 = id_X. \quad (5.1.3)$$

By using equation 5.1.2, we have:

$$\nu i_1 = \mu (w \times \mu) \Delta_X i_1 = \mu (w \times \mu) (i_1 \times i_1) \Delta_X = \mu ((w i_1) \times (\mu i_1)) \Delta_X. \quad (5.1.4)$$
By equations 5.1.1 and 5.1.3, we have

\[ \nu_i \simeq \mu(k \times id_X)\Delta_X \]

\[ = \mu i_2 \]

\[ = id_X. \]

Similarly, we can show that

\[ \nu i_2 \simeq id_X, \]

so \((X, \nu)\) is an H-space (with a homotopy unit).

The process of creating \(\nu\) is also called altering \(\mu\).

A justification for this method of producing new multiplication maps on \(X\) comes from Theorem 1.4.3 from [40]. In fact, if \(\mu\) and \(\nu\) are two multiplication maps for a space \(X\), and \((X, \mu)\) is also homotopy associative and has homotopy inverse operation \(i\), then we can set \(w = \nu \ast (i \circ \mu)\) so that \(\nu = (\nu \ast (i \circ \mu)) \ast \mu \simeq w \ast \mu\). In other words, if we start with a homotopy associative multiplication map \(\mu\), we can obtain any other multiplication map (up to homotopy) by altering \(\mu\).

Before we continue, we need to determine an order of operations for using our binary operation on three or more maps.

**Definition 5.1.2.** Given an H-space \((X, \mu)\), we define the *iterated multiplication maps* inductively:

\[ \mu^2 = \mu(\mu \times id_X) : X^{\times 3} \to X, \]  

\[ (5.1.5) \]
and for \( n > 1 \),

\[
\mu^n = \mu^{n-1}(\mu \times id_X^{(n-1)}) : X^{\times(n+1)} \to X.
\]  

(5.1.6)

Essentially, when multiplying more than two elements, we multiply from the left. For example,

\[
\mu^n(x_1, x_2, \ldots, x_{n+1}) = (\ldots((x_1 x_2)x_3)\ldots x_n)x_{n+1}.
\]

We will call

\[
\mu^{\ast n} : H^\ast(X, \mu; \mathbb{F}_p) \to H^\ast(X, \mu; \mathbb{F}_p)^{\otimes(n+1)}
\]

the \( n \)th (iterated) coproduct on \( H^\ast(X, \mu; \mathbb{F}_p) \). We have

\[
\mu^{\ast n}(x) = (\mu^\ast \otimes 1^{\otimes(n-1)})\mu^{(n-1)^\ast}(x).
\]  

(5.1.7)

Given any space \( Y \), we define the iterated diagonal maps inductively:

\[
\Delta^2_Y = (\Delta_Y \times id_Y)\Delta_Y : Y \to Y^{\times 3},
\]  

(5.1.8)

and for \( n > 1 \),

\[
\Delta^n_Y = (\Delta_Y \times id_Y^{\times(n-1)})\Delta_{Y}^{n-1} : Y \to Y^{\times(n+1)}.
\]  

(5.1.9)

\[
\Delta^n_Y(y) = (y, y, \ldots, y).
\]

We successively apply \( \Delta_Y \) on the left each time. In \( H^\ast(Y; \mathbb{F}_p) \), we have the \( n \)th
iterated cup product

\[ \Delta_Y^n : H^*(Y; \mathbb{F}_p)^{\otimes n+1} \to H^*(Y; \mathbb{F}_p), \]

and

\[ \Delta_Y^n(y_1 \otimes y_2 \otimes \ldots \otimes y_{n+1}) = \Delta_Y^{n-1}(\Delta_Y^*(y_1 \otimes y_2) \otimes \ldots \otimes y_{n+1}) \]

\[ = \Delta_Y^{n-1}(y_1 y_2 \otimes \ldots \otimes y_{n+1}) \]

\[ = y_1 y_2 \ldots y_{n+1}. \quad (5.1.10) \]

Hence \( \Delta_Y^n \) takes \( y_1 \otimes y_2 \otimes \ldots \otimes y_{n+1} \) to the product \( y_1 y_2 \ldots y_{n+1} \).

Suppose \( f_1, \ldots, f_n \) are \( n \) maps from \( Y \) to \( X \) (\( n > 1 \)). Then the map

\[ f_1 * f_2 * \ldots * f_n : Y \to X \]

is defined by the composition

\[ f_1 * f_2 * \ldots * f_n = \mu^{n-1}(f_1 \times \ldots \times f_n) \Delta_Y^{n-1}. \quad (5.1.11) \]

In addition, we abbreviate

\[ f * \ldots * f \quad (n \text{ times}) \quad (5.1.12) \]

as \( n \cdot f \).
Let us determine what $\mu^*(x)$, the $n$th coproduct of $x \in H^*(X, \mu; \mathbb{F}_p)$, could look like. That is, we compute it up to a sum of subalgebras.

**Lemma 5.1.3.** Let $x \in S$, where $S$ is the generating set from Theorem 1.2.4. Then $\mu^*(x)$ has the form

$$
\sum_{i=0}^{n} 1^\otimes n-i \otimes x \otimes 1^\otimes i + \sum_{k=0}^{n-1} \left( \sum_{j=0}^{n-1-k} 1^\otimes j \otimes \bar{B} \otimes B^\otimes n-1-k-j \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^\otimes k \right).
$$

(5.1.13)

**Proof.** Since the iterated coproducts are defined inductively, we will prove the lemma using induction.

In order to perform induction, we need some facts about the sub Hopf algebra $B$. Let $b \in \bar{B}$. Since $B$ is a Hopf algebra, we have $\mu^*(b) \in B \otimes B$. We can be more specific and note that

$$
\mu^*(b) = b \otimes 1 + \sum b' \otimes b'' + 1 \otimes b,
$$

where $b', b'' \in B$ have positive degree (so they are in $\bar{B}$), and thus $\mu^*(b)$ will have the form

$$
\bar{B} \otimes B + 1 \otimes B.
$$

(5.1.14)

We will need this observation in an inductive proof of formula 5.1.13.

Now let $x$ be an element of the generating set $S$. Then $\mu^*(x)$ has the form

$$
x \otimes 1 + 1 \otimes x + B \otimes H^*(X, \mu; \mathbb{F}_p).
$$
Since the terms in the reduced coproduct must have factors which have positive degree, we can be more specific and write

\[ x \otimes 1 + 1 \otimes x + \tilde{B} \otimes H^*(X, \mu; F_p). \] (5.1.15)

Suppose that the \((n - 1)\)th coproduct of \(x\), \(\mu^{(n-1)*}(x)\), has the form

\[
\sum_{i=0}^{n-1} 1^{\otimes n-1-i} \otimes x \otimes 1^{\otimes i} \\
+ \sum_{k=0}^{n-2} \left( \sum_{j=0}^{n-2-k} 1^{\otimes j} \otimes \tilde{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; F_p) \otimes 1^{\otimes k} \right). \tag{5.1.16}
\]

By definition, the \(n\)th coproduct of \(x\) is

\[ \mu^{n*}(x) = (\mu^* \otimes 1^{\otimes (n-1)}) \mu^{(n-1)*}(x). \]

Let us apply \(\mu^* \otimes 1^{\otimes (n-1)}\) on each term of equation 5.1.16 individually. By equation 5.1.15, we have

\[
(\mu^* \otimes 1^{\otimes (n-1)}) \left( \sum_{i=0}^{n-1} 1^{\otimes n-1-i} \otimes x \otimes 1^{\otimes i} \right) \\
= \sum_{i=0}^{n-2} 1 \otimes 1^{\otimes n-1-i} \otimes x \otimes 1^{\otimes i} + (x \otimes 1) \otimes 1^{\otimes n-1} + (1 \otimes x) \otimes 1^{\otimes n-1} \\
+ (\tilde{B} \otimes H^*(X, \mu; F_p)) \otimes 1^{\otimes n-1} \\
= \sum_{i=0}^{n-2} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + x \otimes 1 \otimes 1^{\otimes n-1} + 1 \otimes x \otimes 1^{\otimes n-1} \\
+ \tilde{B} \otimes H^*(X, \mu; F_p) \otimes 1^{\otimes n-1}. \tag{5.1.17}
\]
The last term in equation 5.1.17 will give us the $k = n - 1$ term in equation 5.1.13.

When we examine the sum

$$\sum_{i=0}^{n-2} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + x \otimes 1 \otimes 1^{\otimes n-1} + 1 \otimes x \otimes 1^{\otimes n-1},$$

we see that it equals

$$\sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i},$$

since the $i = n - 1$ term is $1 \otimes x \otimes 1^{\otimes n-1}$ and the $i = n$ term is $x \otimes 1 \otimes 1^{\otimes n-1}$.

Consequently, we have

$$(\mu^* \otimes 1^{\otimes(n-1)}) \left( \sum_{i=0}^{n-1} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} \right)$$

$$= \sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + \bar{B} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes n-1}. \quad (5.1.19)$$

For the other term in 5.1.16, we can equation 5.1.14 when applying $\mu^* \otimes 1^{\otimes(n-1)}$. It will help to break up the inner sum into two parts: the $j = 0$ term (whose leftmost factor is $\bar{B}$), and the remaining terms $j = 1, \ldots, n - 2 - k$ (which have 1 as their
leftmost factor). We have

\[
\left( \mu^* \otimes 1^{\otimes (n-1)} \right) \left( \sum_{k=0}^{n-2} \left( \sum_{j=0}^{n-2-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \right)
\]

\[
= \left( \mu^* \otimes 1^{\otimes (n-1)} \right) \left( \sum_{k=0}^{n-2} \left( B \otimes B^{\otimes n-2-k-0} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) + \sum_{k=0}^{n-2} \left( \sum_{j=1}^{n-2-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \right)
\]

\[
= \sum_{k=0}^{n-2} \left( (B \otimes B) \otimes B^{\otimes n-2-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right)
\]

\[+ \left( 1 \otimes \bar{B} \right) \otimes B^{\otimes n-2-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \]

\[+ \sum_{k=0}^{n-2} \left( \sum_{j=1}^{n-2-k} 1 \otimes 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right). \tag{5.1.20}
\]

This will give us the \( k = 0, \ldots, n - 2 \) terms in equation 5.1.13. To see this, we will need to change the indexing of the inner sum in the last term:

\[
\sum_{j=1}^{n-2-k} 1 \otimes 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k}
\]

\[
= \sum_{j=1}^{n-2-k} 1^{\otimes j+1} \otimes \bar{B} \otimes B^{\otimes n-1-k-(j+1)} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k}
\]

will go from \( j = 2 \) to \( j = n - 1 - k \):

\[
\sum_{j=2}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k}.
\]
The terms in the second to last line of equation 5.1.13,

\[
\begin{align*}
\sum_{k=0}^{n-2} \left( \bar{B} \otimes B \otimes B^{\otimes n-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \\
+ 1 \otimes \bar{B} \otimes B^{\otimes n-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \\
= \sum_{k=0}^{n-2} \left( \bar{B} \otimes B^{\otimes n-1-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \\
+ 1 \otimes \bar{B} \otimes B^{\otimes n-1-k-1} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right)
\end{align*}
\]

will become the \( j = 0 \) and \( j = 1 \) terms respectively in the sum

\[
\sum_{k=0}^{n-2} \left( \sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right).
\] (5.1.21)

Consequently, we have

\[
\begin{align*}
(\mu^* \otimes 1^{\otimes (n-1)}) \left( \sum_{k=0}^{n-2} \left( \sum_{j=0}^{n-2-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \right) \\
= \sum_{k=0}^{n-2} \left( \sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right)
\end{align*}
\] (5.1.22)

Thus, by equations 5.1.19 and 5.1.22, the \( n \)th coproduct \( \mu^{n*}(x) \) will have the form

\[
\sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + \bar{B} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes n-1} \\
+ \sum_{k=0}^{n-2} \left( \sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right), \quad (5.1.23)
\]
and since

\[
\hat{B} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^\otimes n-1
\]

\[
+ \sum_{k=0}^{n-2} \left( \sum_{j=0}^{n-1-k} 1^\otimes j \otimes \hat{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^\otimes k \right)
\]

\[
= \sum_{k=0}^{n-1} \left( \sum_{j=0}^{n-1-k} 1^\otimes j \otimes \hat{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^\otimes k \right),
\]

we can simplify equation 5.1.23 to

\[
\sum_{i=0}^{n} 1^\otimes n-i \otimes x \otimes 1^\otimes i + \sum_{k=0}^{n-1} \left( \sum_{j=0}^{n-1-k} 1^\otimes j \otimes \hat{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^\otimes k \right).
\]

Therefore, by induction, we have formula 5.1.13.

The following result shows that any finite simply-connected HA-space can be given a new multiplication map that will result in an H-space whose cohomology over \( \mathbb{F}_p \) (with coproduct induced by the new multiplication map) is cocommutative.

**Theorem 5.1.4.** Let \((X, \mu)\) be a finite simply-connected HA-space with homotopy inverse \(i\). Let \(\nu = (\frac{p-1}{2} \cdot \text{com}) \ast \mu\) Then \((X, \nu)\) is an H-space for which \(H_\ast(X, \nu; \mathbb{F}_p)\) is a commutative (non-associative) algebra, and there is a generating set for \(H^\ast(X, \nu; \mathbb{F}_p)\) such that if \(x\) is an element of this set,

\[
\nu^\ast(x) = \frac{1}{2} (\mu^\ast(x) + T^\ast(\mu^\ast(x))).
\]

(5.1.24)
Proof. We have

\[ \nu = \left( \frac{p-1}{2} \cdot \text{com} \right) * \mu \]

\[ = \mu^{[(p-1)/2]}(\text{com}^{X(p-1)/2} \times \mu)\Delta_{X \times X}^{[(p-1)/2]} , \]

so when we apply cohomology, we obtain

\[ \nu^* = \Delta_{X \times X}^{[(p-1)/2]^*}((\text{com}^*)^{\otimes (p-1)/2} \otimes \mu^*)\mu^{[(p-1)/2]^*} . \]

Let \( x \in S \). By Lemma 5.1.3, \( \mu^{[(p-1)/2]^*}(x) \) has the form

\[ \sum_{i=0}^{(p-1)/2} 1^{\otimes [(p-1)/2] - i} \otimes x \otimes 1^{\otimes i} \]

\[ + \sum_{k=0}^{[(p-1)/2] - 1} \left( \sum_{j=0}^{[(p-1)/2] - 1 - k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes [(p-1)/2] - 1 - k - j} \right) \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} . \]  

(5.1.25)

Next, we apply \( (\text{com}^*)^{\otimes (p-1)/2} \otimes \mu^* \). Since \( B \) is generated by primitive even degree classes, by Example 3.3.2, \( \text{com}^* \) vanishes on \( \bar{B} \) (so the terms in the second and third lines of equation 5.1.25 vanish on \( (\text{com}^*)^{\otimes (p-1)/2} \otimes \mu^* \)). Hence, when we apply
$(com^*)^{(p-1)/2} \otimes \mu^*$ to the expression in 5.1.25, we obtain

\[
((com^*)^{(p-1)/2} \otimes \mu^*)\mu^{[(p-1)/2]^*}(x) \\
= \sum_{i=0}^{(p-1)/2-1} (1 \otimes 1)^{[(p-1)/2]-i} \otimes com^*(x) \otimes (1 \otimes 1)^{\otimes i} \\
+ (1 \otimes 1)^{(p-1)/2} \otimes \mu^*(x)
\]  
(5.1.26)

Finally, we apply an iterated cup product $\Delta^{[(p-1)/2]^*}$ on equation 5.1.26. For each term, we will obtain a product of elements (most of them will be $(1 \otimes 1)$) in $H^*(X, \mu; \mathbb{F}_p) \otimes H^*(X, \mu; \mathbb{F}_p)$. For example,

\[
\Delta^{[(p-1)/2]^*}((1 \otimes 1)^{(p-1)/2} \otimes \mu^*(x)) \\
= (1 \otimes 1) \ldots (1 \otimes 1)\mu^*(x) \\
= \mu^*(x).
\]

Hence when we apply $\Delta^{[(p-1)/2]^*}$ on equation 5.1.26, we obtain

\[
\nu^*(x) = \left( \sum_{i=0}^{(p-1)/2-1} com^*(x) \right) + \mu^*(x). \\
= \frac{p-1}{2} com^*(x) + \mu^*(x).
\]  
(5.1.27)
By Theorem 3.3.3, if \( x \in S \), \( \text{com}^*(x) = \mu^*(x) - T^*\mu^*(x) \), so

\[
\nu^*(x) = \frac{p-1}{2} (\mu^*(x) - T^*\mu^*(x)) + \mu^*(x) \\
= \frac{1}{2} (\mu^*(x) + T^*\mu^*(x)).
\]

Thus, when \( x \in S \),

\[
\nu^*(x) = \frac{1}{2} (\mu^* + T^*\mu^*) (x),
\]

and in particular,

\[
T^*\nu^*(x) = \frac{1}{2} (T^*\mu^* + \mu^*) (x) = \nu^*(x),
\]

and since \( S \) is a generating set, we can conclude that \( H^*(X, \nu; \mathbb{F}_p) \) is cocommutative.

\[\square\]

**Example 5.1.5.** Consider the exceptional Lie group \((F_4, \mu)\) with its usual Lie group multiplication map \( \mu \). Recall that

\[
H^*(F_4, \mu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{11}, x_{15}) \otimes \mathbb{F}_3[x_8]/(x_8^3).
\]

According to the previous theorem, if we want its mod 3 cohomology to be a cocommutative Hopf algebra we should replace \( \mu \) by \( \nu = \text{com} \ast \mu \). That is,

\[
\nu(g, h) = ghg^{-1}h^{-1}gh. \tag{5.1.28}
\]

Let us compute \( \nu^* \) on the generating set \( S \) for \( H^*(F_4, \mu; \mathbb{F}_3) \) (which is isomorphic to
$H^*(F_4, \nu; \mathbb{F}_3)$ as an algebra, but not as a Hopf algebra. The elements $x_3$, $x_7$, and $x_8$ are already primitive (their coproducts are symmetric), so $\mu^* = \nu^*$ on these elements. Instead, let us focus on $x_{11}$ and $x_{15}$.

We have $x_{11}$, whose coproduct is $\mu^*(x_{11}) = x_{11} \otimes 1 + x_8 \otimes x_3 + 1 \otimes x_{11}$. If we compute $\nu^*(x_{11})$ without using the previous theorem, the calculations would look like this:

$$\nu^*(x_{11}) = \text{com}^*(x_{11}) + \text{com}^*(x_8)\mu^*(x_3) + \mu^*(x_{11}) = (x_8 \otimes x_3 - x_3 \otimes x_8) + 0 + (x_{11} \otimes 1 + x_8 \otimes x_3 + 1 \otimes x_{11}) = x_{11} \otimes 1 - x_8 \otimes x_3 - x_3 \otimes x_8 + 1 \otimes x_{11}.$$  

Let us use the theorem on $\nu^*(x_{15})$:

$$\nu^*(x_{15}) = x_{15} \otimes 1 - x_8 \otimes x_7 - x_7 \otimes x_8 + 1 \otimes x_{15}.$$  

Thus, $\nu^* = T^* \nu^*$ on this generating set for $H^*(F_4, \nu; \mathbb{F}_3)$. Hence $H^*(F_4, \nu; \mathbb{F}_3)$ is cocommutative.

Now let us look at $(E_8, \mu)$. This time, we want a multiplication map $\nu$ so that $H^*(E_8, \nu; \mathbb{F}_3)$ is cocommutative. Recall that

$$H^*(E_8, \mu; \mathbb{F}_5) \cong \wedge (x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathbb{F}_5[x_{12}]/(x_{12}^5).$$
According to the theorem, we should take

$$\nu(g, h) = ghg^{-1}h^{-1}ghg^{-1}h^{-1}gh. \tag{5.1.29}$$

For example, in the generating set $S$, let us look at three non-primitive elements: $x_{15}$, $x_{27}$, and $x_{39}$. Their original coproducts are as follows:

$$\bar{\mu}^*(x_{15}) = x_{12} \otimes x_3,$$

$$\bar{\mu}^*(x_{27}) = x_{12} \otimes x_{15} + 3x_{12}^2 \otimes x_3,$$

$$\bar{\mu}^*(x_{39}) = x_{12} \otimes x_{27} + 3x_{12}^2 \otimes x_{15} + x_{12}^3 \otimes x_3,$$

and the new coproducts are:

$$\bar{\nu}^*(x_{15}) = 3x_{12} \otimes x_3 + 3x_3 \otimes x_{12},$$

$$\bar{\nu}^*(x_{27}) = 3x_{12} \otimes x_{15} + 4x_{12}^2 \otimes x_3 + 3x_{15} \otimes x_{12} + 4x_3 \otimes x_{12}^2,$$

$$\bar{\nu}^*(x_{39}) = 3x_{12} \otimes x_{27} + 4x_{12}^2 \otimes x_{15} + 3x_{12}^3 \otimes x_3 + 3x_3 \otimes x_{12}^3 + 3x_{27} \otimes x_{12} + 4x_{15} \otimes x_{12}^3 + 3x_3 \otimes x_{12}^3.$$

Although $H_*(X, \nu; \mathbb{F}_p)$ is now commutative, remember that $H_*(X, \mu; \mathbb{F}_p)$ was associative (by virtue of $(X, \mu)$ being an HA-space). In the next section, we explore
the associativity of $H_*(X, \nu; \mathbb{F}_p)$ and look at examples where $H_*(X, \nu; \mathbb{F}_p)$ is both commutative and associative.

5.2 Associativity and Primitive Generators

This section deals with questions of associativity, both for the new H-spaces as well as the homology algebras. We start with a method of checking when $H_*(X, \nu; \mathbb{F}_p)$ is not only commutative, but also associative, using a formula called a coassociator consisting of the original coproduct. We then give an example of computing coassociators. After that, we look at $(X, \nu)$ and give sufficient conditions for $(X, \nu)$ to not be an HA-space. Finally, we look at examples of $(X, \nu)$ for which $H_*(X, \nu; \mathbb{F}_p)$ is commutative, but not associative.

**Theorem 5.2.1.** Let $(X, \mu)$ be a finite simply-connected HA-space with homotopy inverse $i$. Let $\nu = (\frac{p-1}{2} \cdot \text{com}) \ast \mu$. The Hopf algebra $H_*(X, \nu; \mathbb{F}_p)$ is associative if and only if in $H^*(X, \mu; \mathbb{F}_p)$,

$$\frac{1}{4} ((\mu^* \otimes 1)T^*\mu^* + (T^*\mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^*\mu^* - (1 \otimes T^*\mu^*)\mu^*) = 0. \quad (5.2.1)$$

Finally, if $H_*(X, \nu; \mathbb{F}_p)$ is associative, then $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated as a Hopf algebra with coproduct $\nu^*$.

**Proof.** To check associativity of $H_*(X, \nu; \mathbb{F}_p)$, we can dualize and check coassociativity of $H^*(X, \nu; \mathbb{F}_p)$. This means that we need to compute the difference of the second
coproducts for $\nu^*$ and check if it equals zero:

$$(\nu^* \otimes 1)\nu^* - (1 \otimes \nu^*)\nu^*.$$

Taking the second coproduct on the left gives us

$$(\nu^* \otimes 1)\nu^* = \frac{1}{2} ((\mu^* + T^* \mu^*) \otimes 1) \circ \frac{1}{2} (\mu^* + T^* \mu^*)$$

$$= \frac{1}{4} ((\mu^* \otimes 1)\mu^* + (\mu^* \otimes 1)T^* \mu^* + (T^* \mu^* \otimes 1)\mu^* + (T^* \mu^* \otimes 1)T^* \mu^*),$$

while taking the second coproduct on the right gives us

$$(1 \otimes \nu^*)\nu^* = \frac{1}{2} (1 \otimes (\mu^* + T^* \mu^*)) \circ \frac{1}{2} (\mu^* + T^* \mu^*)$$

$$= \frac{1}{4} (((1 \otimes \mu^*)\mu^* + (1 \otimes \mu^*)T^* \mu^* + (1 \otimes T^* \mu^*)\mu^* + (1 \otimes T^* \mu^*)T^* \mu^*).$$

Since $\mu$ is homotopy associative, on $(X, \mu)$, we have

$$\mu(\mu \times id_X) \simeq \mu(id_X \times \mu).$$

Elementwise, we have

$$\mu(\mu \times id_X)(x, y, z) = (\mu T)((\mu T) \times id_X)(z, y, x)$$
and
\[ \mu(id_X \times \mu)(x, y, z) = (\mu T)(id_X \times (\mu T))(z, y, x), \]
so
\[ (\mu T)((\mu T) \times id_X) \simeq (\mu T)(id_X \times (\mu T)). \]

Therefore, in cohomology, \( \mu^* \) and \( T^* \mu^* \) must satisfy
\[ (\mu^* \otimes 1)\mu^* = (1 \otimes \mu^*)\mu^* \]
and
\[ (T^* \mu^* \otimes 1)T^\star \mu^* = (1 \otimes T^* \mu^*)T^\star \mu^*. \]

Therefore,
\[
(\nu^* \otimes 1)\nu^* - (1 \otimes \nu^*)\nu^* \\
= \frac{1}{4} \left( (\mu^* \otimes 1)T^* \mu^* + (T^* \mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^* \mu^* - (1 \otimes T^* \mu^*)\mu^* \right),
\]
so \( H^*(X, \nu; \mathbb{F}_p) \) is coassociative if and only if
\[
(\nu^* \otimes 1)\nu^* - (1 \otimes \nu^*)\nu^* \\
= \frac{1}{4} \left( (\mu^* \otimes 1)T^* \mu^* + (T^* \mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^* \mu^* - (1 \otimes T^* \mu^*)\mu^* \right) = 0.
\]
Finally, since $X$ is a finite space, a generating set for $H^*(X, \nu; \mathbb{F}_p)$ will have a finite number of even degree elements. Hence there are no $p$th powers in the homology ring $H_*(X, \nu; \mathbb{F}_p)$ [24]. Thus, if $H_*(X, \nu; \mathbb{F}_p)$ is associative, then $H_*(X, \nu; \mathbb{F}_p)$ is a commutative and associative algebra with no $p$th powers, so by Theorem 1.1.4, $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated as a Hopf algebra.

Remark. The expression in 5.2.1 is called the coassociator (of the coproduct $\nu$).

Let us compute some coassociators of generators in some of the exceptional Lie groups.

Example 5.2.2. Let us return to $(F_4, \nu)$. The coassociator from Theorem 5.2.1 vanishes on the generating set $S$, so $H_*(F_4, \nu; \mathbb{F}_3)$ is not only commutative, but also associative. Since $H_*(F_4, \nu; \mathbb{F}_3)$ has no 3rd powers, we can conclude that $H^*(F_4, \nu; \mathbb{F}_3)$ is primitively generated. Can we find a generating set of primitive elements for $H^*(F_4, \nu; \mathbb{F}_3)$? Our calculations have shown that the $-1$ characteristic generators (see Proposition 2.3.10) corresponding to the ones from $S$ will work:

$$\bar{x}_3 = x_3, \quad \bar{x}_7 = x_7, \quad \bar{x}_8 = x_8,$$

$$\bar{x}_{11} = x_{11} + x_8 x_3, \quad \bar{x}_{15} = x_{15} + x_8 x_7.$$
For example,

\[
\nu^*(\tilde{x}_{11}) = \nu^*(x_{11}) + \nu^*(x_8 x_3) \\
= x_{11} \otimes 1 - x_8 \otimes x_3 - x_3 \otimes x_8 + 1 \otimes x_{11} \\
+ x_8 x_3 \otimes 1 + x_8 \otimes x_3 + x_3 \otimes x_8 + 1 \otimes x_8 x_3 \\
= x_{11} \otimes 1 + x_8 x_3 \otimes 1 + 1 \otimes x_{11} + 1 \otimes x_8 x_3 \\
= \tilde{x}_{11} \otimes 1 + 1 \otimes \tilde{x}_{11}. \tag{5.2.2}
\]

Now, since $H^*(F_4, \nu; F_3)$ is associative, we can ask if $(F_4, \nu)$ is homotopy associative, or even associative. However, if it turned out to be an HA-space, then since $H^*(F_4, \nu; F_3)$ is primitively generated, Theorem 3.2 from [41] would require $H^*(F_4, \nu; F_3)$ to be a free algebra. Since $H^*(F_4, \nu; F_3)$ is finite-dimensional, this would mean that $H^*(F_4, \nu; F_3)$ must be an exterior algebra, which is false. Therefore, $(F_4, \nu)$ is an H-space that is not an HA-space.

In general, given a finite simply-connected HA-space $(X, \mu)$ with altered multiplication map $\nu = (\frac{p-1}{2} \cdot \text{com}) * \mu$, let us show when $(X, \nu)$ is not an HA-space:

**Theorem 5.2.3.** Given a finite simply-connected HA-space $(X, \mu)$ with altered multiplication map $\nu = (\frac{p-1}{2} \cdot \text{com}) * \mu$, suppose that $H_*(X, \mu; F_p)$ is not commutative. Then the H-space $(X, \nu)$ is not an HA-space.

**Proof.** There are two cases to consider: whether $H_*(X, \nu; F_p)$ is associative or not.

If $H_*(X, \nu; F_p)$ is not associative, then $(X, \nu)$ cannot be (homotopy equivalent
If $H_*(X,\nu;\mathbb{F}_p)$ is associative, then by Theorem 5.2.1, $H^*(X,\nu;\mathbb{F}_p)$ is primitively generated. Since $(X,\mu)$ is a finite simply-connected HA-space and the algebra $H_*(X,\mu;\mathbb{F}_p)$ is not commutative, the integral homology $H_*(X;\mathbb{Z})$ has $p$-torsion [17]. Hence $H^*(X;\mathbb{F}_p)$ is not an exterior algebra [5]. Then $H^*(X,\nu;\mathbb{F}_p)$ (as an algebra) must have even degree generators, and since $H^*(X,\nu;\mathbb{F}_p)$ has finite dimension, these even degree generators must produce truncated polynomial algebras, so $H^*(X\nu;\mathbb{F}_p)$ is not a free algebra.

Let us finish with a proof by contradiction. If $(X,\nu)$ is an HA-space, then since $H^*(X,\nu;\mathbb{F}_p)$ is primitively generated, by Theorem 3.2 from [41], $H^*(X\nu;\mathbb{F}_p)$ is a free algebra, contradicting our earlier observation that $H^*(X\nu;\mathbb{F}_p)$ is not a free algebra. Hence the H-space $(X,\nu)$ is not an HA-space. \hfill \Box

Now we turn to some examples of nontrivial coassociators.

**Example 5.2.4.** Let us look at the Lie group $(E_8,\mu)$. We will start with $H^*(E_8,\mu;\mathbb{F}_3)$ and then look at $H^*(E_8,\mu;\mathbb{F}_5)$. For reference,

$$H^*(E_8;\mathbb{F}_3) \cong \wedge (x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathbb{F}_p[x_8, x_{20}] / (x_8^3, x_{20}^3).$$

A generating set for $H^*(E_8,\mu;\mathbb{F}_3)$ will have an element $x_{35}$ whose coproduct is

$$\mu^*(x_{35}) = x_{35} \otimes 1 + 1 \otimes x_{35} + x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7.$$
Let
\[ \nu(g, h) = ghg^{-1}h^{-1}gh. \] (5.2.3)

Then in \( H^*(E_8, \nu; \mathbb{F}_3) \), we have

\[ \nu^*(x_{35}) = x_{35} \otimes 1 + 1 \otimes x_{35} - x_8 \otimes x_{27} + x_8^2 \otimes x_{19} - x_{20} \otimes x_{15} - x_8 x_{20} \otimes x_{7} \]
\[ -x_{27} \otimes x_8 + x_{19} \otimes x_8^2 - x_{15} \otimes x_{20} - x_7 \otimes x_8 x_{20}, \]

and the coassocciator on \( x_{35} \) is

\[ x_8 \otimes x_8 \otimes x_{19} + x_8 \otimes x_{20} \otimes x_{7} + x_{20} \otimes x_8 \otimes x_{7} \]
\[ -x_{19} \otimes x_8 \otimes x_8 - x_7 \otimes x_8 \otimes x_{20} - x_7 \otimes x_{20} \otimes x_8. \] (5.2.4)

For reference,

\[ x_8 \otimes x_8 \otimes x_{19} \text{ came from computing } (T^*\mu^* \otimes 1) \text{ on } -x_8^2 \otimes x_{19}, \]
\[ x_8 \otimes x_{20} \otimes x_{7} + x_{20} \otimes x_8 \otimes x_{7} \text{ came from } (T^*\mu^* \otimes 1) \text{ on } x_8 x_{20} \otimes x_{7}, \]
\[ -x_{19} \otimes x_8 \otimes x_8 \text{ came from } -(1 \otimes \mu^*) \text{ on } x_{19} \otimes x_8^2, \]
\[ -x_7 \otimes x_8 \otimes x_{20} - x_7 \otimes x_{20} \otimes x_8 \text{ came from } -(1 \otimes \mu^*) \text{ on } x_7 \otimes x_8 x_{20}, \]

so the main causes for the nonzero coassocciator are the terms \( -x_8^2 \otimes x_{19} + x_8 x_{20} \otimes x_{7} \) in \( \mu^*(x_{35}) \). Notice that their left factors are decomposables.
In fact, due to the presence of two even degree elements in $QH^*(E_8, \mu; \mathbb{F}_3)$ linked by a Steenrod operation $P^3$ (namely the elements represented by $x_8$ and $x_{20}$ in $S$), it is impossible to find a multiplication map on $E_8$ which induces a product structure on $H_*(E_8; \mathbb{F}_3)$ that is both commutative and associative [18].

Meanwhile, in $H^*(E_8, \nu; \mathbb{F}_5)$, where

$$\nu(g, h) = ghg^{-1}h^{-1}ghg^{-1}h^{-1}gh,$$

we have $\frac{1}{4} = 4 = -1$ in $\mathbb{F}_5$, so the coassosiator of $x_{27}$ is

$$4x_{12} \otimes x_{12} \otimes x_{3} + x_{3} \otimes x_{12} \otimes x_{12}. \quad (5.2.5)$$

These came from $(T^*\mu^* \otimes 1)(3x_{12}^2 \otimes x_{3})$ and $-(1 \otimes \mu^*)(3x_{3} \otimes x_{12}^2)$ respectively. The coassosiator for $x_{39}$ is

$$4x_{12} \otimes x_{12} \otimes x_{15} + 2x_{12}^2 \otimes x_{12} \otimes x_{3} + 2x_{12} \otimes x_{12}^2 \otimes x_{3}$$

$$+ x_{15} \otimes x_{12} \otimes x_{12} + 3x_{3} \otimes x_{12}^2 \otimes x_{12} + 3x_{3} \otimes x_{12} \otimes x_{12}^2$$

$$= -(T^*\mu^* \otimes 1)(3x_{12}^2 \otimes x_{15}) - (T^*\mu^* \otimes 1)(x_{12}^3 \otimes x_{3})$$

$$+(1 \otimes \mu^*)(3x_{15} \otimes x_{12}^2) + (1 \otimes \mu^*)(x_{3} \otimes x_{12}^3). \quad (5.2.7)$$

Again, the terms in the original reduced coproducts with decomposable factors cause nonzero terms in the coassosiator. Unlike $QH^*(E_8, \mu; \mathbb{F}_3)$, $QH^*(E_8, \mu; \mathbb{F}_5)$ only has one element with even degree. There may be a possibility of an entirely different
multiplication map on $E_8$ which induces an algebra structure on $H_*(E_8; \mathbb{F}_5)$ that is both commutative and associative.

**Example 5.2.5.** Our final example is $(E_7, \mu)$. Recall that

$$H^*(E_7, \mu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}) \otimes \mathbb{F}_3[x_8]/(x_8^3).$$

In $H^*(E_7, \mu; \mathbb{F}_3)$, we have a generator $x_{35}$ with

$$\mu^*(x_{35}) = x_{35} \otimes 1 + x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + 1 \otimes x_{35},$$

If we let

$$\nu(g, h) = ghg^{-1}h^{-1}gh, \quad (5.2.8)$$

then the coassociator of $x_{35}$ in $H^*(E_7, \nu; \mathbb{F}_3)$ is

$$x_8 \otimes x_8 \otimes x_{19} - x_{19} \otimes x_8 \otimes x_8$$

$$= (T^* \mu^* \otimes 1) (-x_8^2 \otimes x_{19}) - (1 \otimes \mu^*) (x_{19} \otimes x_8^2). \quad (5.2.9)$$

Notice that the nonzero terms in the coassociator of $x_{35}$ came from applying left or right coproducts on terms with decomposable factors in the original reduced coproduct of $x_{35}$.

Overall, if

$$\nu(g, h) = ghg^{-1}h^{-1}gh,$$
then $H^*(F_4, \nu; \mathbb{F}_3)$ and $H^*(E_6, \nu; \mathbb{F}_3)$ are primitively generated, while $H^*(E_7, \nu; \mathbb{F}_3)$ and $H^*(E_8, \nu; \mathbb{F}_3)$ are cocommutative, but not coassociative. One property that both $H^*(F_4, \nu; \mathbb{F}_3)$ and $H^*(E_6, \nu; \mathbb{F}_3)$ have in common is that the generators can be chosen (using Theorem 1.2.4) so that each of their original reduced coproducts consists only of sums of tensor products of primitive generators. On the other hand, a generating set for the algebras $H^*(F_4, \nu; \mathbb{F}_3)$, $H^*(E_6, \nu; \mathbb{F}_3)$ or $H^*(E_7, \nu; \mathbb{F}_3)$ would have only one even degree generator, while a generating set for $H^*(E_8, \nu; \mathbb{F}_3)$ would require two. It is an open question as to whether there is a multiplication map on $E_7$ that makes its mod 3 homology commutative and associative.
Bibliography


