On Renormalized Volume

By

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Abstract

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Renormalized volume $V_R$ is a quantity that gives a notion of volume for hyperbolic manifolds having infinite volume under the classical definition. Its study for convex co-compact hyperbolic 3-manifolds can be found in [KS08], while the geometrically finite case including rank 1-cusps was developed in [GMR]. In this dissertation we will answer what value is the infimum of the renormalized volume for a given geometrically finite hyperbolic manifold with incompressible boundary, and how a sequence converging to the infimum behaves. Partial results to this question were given in the acylindrical case by the author in [VPb] and in the convex co-compact case by [VPa] and [BBB] independently ([BBB] studied the gradient flow of the renormalized volume). In addition to finding the infimum value, we give the notion of additive geometric convergence in order to describe any sequence converging to the infimum. The study of local minima on the acylindrical case was done in parallel by [Mor] and [VPc], which [VPb] proves to be the infimum.

Given a hyperbolic manifold $(M, g)$ (as explained in Section 2.1), there is a family of equidistant surfaces parallel to $\partial M$ called Epstein surfaces, denoted by $\{Eps_r\}$ and that bound a interior region denoted by $\text{int}(Eps_r)$, such that $\lim_{r \to \infty} Eps_r$ converges to the hyperbolic metric of $\partial M$ compatible with the conformal structure that $(M, g)$ induces on $\partial M$. Then we can define $V_R$ of $(M, g)$ as

$$V_R(M, g) = \text{vol}(\text{int}(Eps_r)) - \frac{1}{4} \int_{Eps_r} H da + \pi r \chi(\partial M),$$
where $H$ is the mean curvature and $da$ is the volume form of the induced metric. Thanks to Section 2.1 this definition does not depend on $r$, and we will keep the dependence on $g$ implicit. The factor $-\frac{1}{4} \int_{E^r} H da + \pi r \chi(\partial M)$ is balancing the blow up of $\text{vol}(\text{int}(E^r))$ as $\text{int}(E^r)$ exhaust $M$. Given a infinite magnitude like $\text{vol}(M)$, the process of finding such factor as $-\frac{1}{4} \int_{E^r} H da + \pi r \chi(\partial M)$ is called renormalization.

In Section 2.4 we will defined the corrected renormalized volume $\overline{V}_R$ as

$$\overline{V}_R(M) = V_R(M) - \frac{1}{2} V_R(\partial M),$$

where $V_R(\partial M)$ denotes the sum of renormalized volumes of the covering hyperbolic manifolds obtained by the inclusion $\pi_1(\partial M) \hookrightarrow \pi_1(M)$ on each component of $\partial M$. $\overline{V}_R$ has the advantage of splitting additively under cutting, which is a volume-like property.

These are the main result of the dissertation. Theorem 4.2.1 describes the local behavior around a critical point of $V_R$ when $M$ is acylindrical.

**Theorem** (Theorem 4.2.1 [VPc, Theorem 2]). Let $M$ be a compact acylindrical 3-manifold with hyperbolizable interior such that $\partial M \neq \emptyset$. Then there is a unique critical point $c$ for the renormalized volume of $M$, where $c$ is the unique conformal class at the boundary that makes every component of the region of discontinuity a disk (a.k.a. the geodesic class). The Hessian at this critical point is positive definite.

Theorem 4.3.1 describes the local behavior of $\overline{V}_R$ around a critical point of both $\overline{V}_R$ and $V_R$, when $M$ is acylindrical.

**Theorem** (Theorem 4.3.1 [VPc, Theorem 6]). Let $M$ be a compact acylindrical 3-manifold with hyperbolizable interior, $\partial M \neq \emptyset$ without cusps, and $c \in T(\partial M)$ be the geodesic class. Then $c$ is a local minimum for the corrected renormalized volume of $M$.

Theorem 5.1.1 describes the notion of additive continuity. This means that on a sequence where $V_R$ converges, this limit can be found as the sum of renormalized volumes of geometric limits. We denote this collection of limits as the additive geometric limit.

**Theorem** (Theorem 5.1.1 [VPa, Theorem 6.1]). Let $M$ be a geometrically finite hyperbolic manifold with $\partial M \neq \emptyset$ incompressible. Let $M_n \in QF(M)$ be a sequence such that $V_R(M_n)$ converges. Then we can select finite many base points such that (possibly after taking a subsequence) $N_1, \ldots, N_k$ are the additive geometric limit corresponding to the base points (in the sense of Proposition 1.2.1) and

$$\lim_{n \to \infty} V_R(M_n) = \sum_{i=1}^{k} V_R(N_i).$$

Theorem 5.2.1 uses Theorem 5.1.1 to calculate the infimum of $V_R$ for a pared hyperbolic 3-manifold.
**Theorem** (Theorem 5.2.1 [VPa, Theorem 7.1]). Let \((M, P)\) be a pared compact hyperbolizable 3-manifold. Furthermore, assume that \((M, P)\) is incompressible, i.e. any compressing disk with boundary in \(\partial M \setminus P\) bounds a homotopically trivial curve. Then

\[
\inf_{QF(M, P)} V_R(M) = \frac{v_3}{2} \|DM\|
\]

where \(v_3\) is the volume of the regular ideal tetrahedron in \(\mathbb{H}^3\), \(DM\) is the double of the pair \((M, P)\) and \(\|\cdot\|\) denotes the Gromov norm of a manifold. Moreover, for any sequence \(\{M_n\}\) such that \(\lim_{n \to \infty} V_R(M_n) = \inf V_R(M)\), there exist a decomposition of \(M\) along essential cutting cylinders in components \(A_1 \sqcup \ldots \sqcup A_s \sqcup F_1 \sqcup \ldots \sqcup F_r\) (with \(A_1, \ldots, A_s\) acylindrical and \(F_1, \ldots, F_r\) fuchsian) such that \(A_1 \sqcup \ldots \sqcup A_s \sqcup F_1 \sqcup \ldots \sqcup F_r\) is the additive geometric limit of a subsequence of \(\{M_n\}\).

**Proposition** (Proposition 5.3.1). Let \(M\) be a convex co-compact hyperbolic manifold with \(\partial M \neq \emptyset\) incompressible. Let \(M_n \in QF(M)\) be a sequence such that \(\overline{V_R(M_n)}\) converges. Then we can select finite many base points such that (possibly after taking a subsequence) \(N_1, \ldots, N_k\) are the geometric limits corresponding to the base points (in the sense of Proposition 1.2.1) and

\[
\lim_{n \to \infty} \overline{V_R(M_n)} = \sum_{i=1}^{k} V_R(N_i) - \frac{1}{2} \sum_{l} V_R(S_l),
\]

where \(S_l\) are the additive limits obtained from the components \((\partial N_i)_{1 \leq i \leq n}\).
Dedico esta disertación a mi familia, quienes han sido mi soporte
y tendrán mi amor por siempre
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# Contents

1 Background .............................. 1  
1.1 Hyperbolic Geometry ................. 1  
1.2 Geometrically finite limits .......... 2  
1.3 Peripherally generic fundamental domains .... 6  
1.4 Small deformations of sequences .... 10  
1.5 Teichmüller space ..................... 12  

2 $V_R$ for convex co-compact hyperbolic 3-manifolds 16  
2.1 $W$-volume ............................ 16  
2.2 First variation formula ............... 18  
2.3 On the maximality of the $W$-volume among metrics of constant area .... 19  
2.4 Corrected $V_R$ ....................... 20  

3 $V_R$ for geometrically finite hyperbolic 3-manifolds 22  
3.1 Geodesic boundary defining functions ...... 22  
3.2 Epstein surfaces and finite volume cores .... 24  

4 Second variation of $V_R$ at critical points 29  
4.1 Fuchsian case .......................... 29  
4.2 General case and strict local minimums for acylindrical manifolds .... 31  
4.3 Local minima for Corrected $V_R$ .... 33  

5 Additive continuity ....................... 34  
5.1 Additive continuity for additive geometric convergent sequences .......... 34  
5.2 Global minima of $V_R$ ................ 44  
5.3 Minimizing and additive continuity for corrected $V_R$ ................. 51
Preface

The work here presented is organized as follows. Chapter 1 develops the necessary background in hyperbolic geometry and Teichmüller theory. Section 1.1 gives the basic definitions and properties of hyperbolic geometry in dimension 3. Section 1.2 introduces both algebraic and geometric convergence for hyperbolic 3-manifolds, as well as the notion of additive geometric convergence that we will use in Chapter 5. Sections 1.3, 1.4 study small deformations of Kleinian groups, which we will also use in Chapter 5 to perturb a given sequence that converges geometrically additively. At the end of Chapter 1 we have Section 1.5 which covers the relevant background in Teichmüller theory needed for the study of the first variation of $V_R$.

Chapter 2 gives the definition and basic properties of $V_R$ in the case of convex co-compact hyperbolic 3-manifold by the use of a family of equidistant surfaces. Section 2.1 defines $V_R$ by a renormalization process applied to the volume via the $W$-volume and the Epstein surfaces of the Poincaré metric at infinity. Section 2.2 establishes the first variation formula for $V_R$ that will be needed in order to understand the critical points. Section 2.3 discusses the maximality of the $W$-volume at the Poincaré metric by using Ricci flow. Section 2.4 defines $\overline{V}_R$, the modified version of $V_R$ that splits additively under cutting, which is a volume-like property.

Chapter 3 generalizes the definition of $V_R$ for geometrically finite 3-manifolds. Section 3.1 shows an alternative definition of $V_R$ that extends to the geometrically finite case, due to [GMR], via geodesic boundary definition functions. In this section we show via a direct computation that the two definitions of $V_R$ are equivalent. In past literature this has only appeared by showing that both definitions have the same derivative and coincide at one point. Section 3.2 shows that the approach via Epstein surfaces is valid as well for the geometrically finite case while considering an appropriate family of metrics, called admissible.

Chapter 4 studies the critical points of $V_R$ by calculating the Hessian at these points. Section 4.1 calculates the Hessian for fuchsian manifolds, which are the critical points of the quasifuchsian case. Section 4.2 uses this calculation to find the Hessian in the general case in terms of the skinning map $\sigma$ as in Equation (4.10)

$$\text{Hess}_R(v, w) = \frac{1}{16} \sum_{i=1}^{n} \langle v_i, w_i - d\sigma_i(w) \rangle,$$

where the tangent vectors $v, w$ are real parts of quadratic holomorphic differentials described
in Section 1.5. Thanks to a result of [McM90], we are able to use that $d\sigma$ is a contraction to show that the Hessian is strictly positive definite for acylindrical manifolds, as stated in Theorem 4.2.1. At the end of Chapter 4, Section 4.3 concludes that those critical points have also a strictly positive definite Hessian for $V_R$, calculated in Equation (4.14)

$$\text{Hess}_R(v, w) = \frac{1}{32} \langle v + d\sigma(v), w - d\sigma(w) \rangle,$$

so we also have the similar statement as Theorem 4.2.1 for $V_R$.

Since now critical points of a given topological type are understood, a natural next step is to describe sequences converging to the infimum. Chapter 5 addresses this and finds the infimum of $V_R$ for a given manifold $M$ in terms of topological informations about $M$. Section 5.1 shows that $V_R$ is continuous under additive geometric limits, meaning the the limit of $V_R$ is the sum of $V_R$ of the different components of the additive geometric limit. This is described in Theorem 5.1.1. Section 5.2 uses this together with the perturbation argument from Sections 1.3, 1.4 to show that $\inf V_R$ is calculated as the Thurston norm of the double of $M$, up to a known constant (the volume of the ideal hyperbolic tetrahedra). This result is described in Theorem 5.2.1. Finally, Section 5.3 shows that for $\inf V_R$, while we will also have an additive geometric limit and convergence, we lack the full characterization of the limit needed to calculate $\inf V_R$. Nevertheless, we have Proposition 5.3.1 as what we can say about this situation.
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Chapter 1

Background

1.1 Hyperbolic Geometry

We will use the well known notation of $\mathbb{H}^3$ for the three dimensional hyperbolic space, which is the model geometry for constant negative curvature in dimension three. In order to represent it, we will use either the Poincaré ball model $\{ \mathbf{x} = (x_1, x_2, x_3) | \|\mathbf{x}\| < 1 \}$ with metric $\frac{4||d\mathbf{x}||^2}{(1-\|\mathbf{x}\|^2)^2}$ or the Poincaré half-space model $\{ (x, y, z) | z > 0 \}$ with metric $\frac{dx^2 + dy^2 + dz^2}{z^2}$.

These models are isometric by a Cayley transform type of map that sends spheres tangent to a given point in the boundary of the ball model to horizontal planes in the half-space model. Hyperbolic space is homogeneous, which means that the group of isometries acts transitively. Moreover, acts transitively in the space of frames, where a frame is a orthonormal basis at some point of $\mathbb{H}^3$. Then any isometry is determined by its image on three different points.

All isometries act conformally (i.e. preserving angles) in the boundary of $\mathbb{H}^3$ (usually called boundary at infinity), identified as the unit sphere for the Poincaré ball model or as the compactified complex plane for the Poincaré half-space model (in both cases, angles are measured with respect to the euclidean metric). Thanks to this, we are able to identify the space of isometries with the space of conformal homeomorphism of the sphere, $\text{PSL}(2, \mathbb{C})$, also known as Möbius transformations. We will denote by $\overline{\mathbb{H}}^3$ the compactification of $\mathbb{H}^3$ by its boundary at infinity.

A hyperbolic 3-orbifold is the quotient of $\mathbb{H}^3$ under a discrete subgroup of isometries. This includes all complete and connected constant negative curvature 3-manifolds since the universal cover of such space is isometric to $\mathbb{H}^3$. Given that, we can describe a hyperbolic manifold by the discrete representation of its fundamental group $\pi_1$ into $\text{PSL}(2, \mathbb{C})$. There are three possibilities for an element in $\pi_1$ represented in $\text{PSL}(2, \mathbb{C})$ given its set of fixed points in $\mathbb{H}^3$. An isometry is called loxodromic if it fixes exactly two points, both in the boundary at infinity. An isometry is called parabolic if it fixes exactly one point, located in the boundary at infinity. An isometry is called elliptic if it fixes the geodesic joining two points in the boundary at infinity. Since we will be interested only in manifolds instead of orbifolds, we will assume that we are free from elliptics from now on.
A compact 3-manifold $M$ will be called hyperbolizable if its interior admits a hyperbolic metric $g$. Under such metric, we will denote by $C_{(M,g)}$ the convex core of $(M,g)$ (minimal convex submanifold isotopic to $M$). The metric $g$ is said to be convex co-compact if $C_M$ is compact. The space of convex co-compact metrics will be denoted by $QF(M)$. Since we will be working with boundary incompressible manifolds from now on (where boundary incompressible means that $\pi_1(\partial M) \rightarrow \pi_1(M)$ is injective for every component of $\partial M$). In this case, by the Ahlfors-Bers theorem [AB60] we know that $QF(M)$ is homeomorphic to $\mathcal{T}(\partial M)$, the Teichmüller space of $\partial M$.

For manifolds with designated parabolic loci, we will need to consider the concept of a pared manifold, as seen in [Thurston86, Section 7]

**Definition 1.1.1.** A pared manifold is a pair $(M, P)$, where $M$ is a compact 3-manifold, $P \subset \partial M$ is a (possibly empty) 2-dimensional submanifold with boundary such that

(a) The fundamental group of each of its components injects into the fundamental group of $M$.

(b) The fundamental group of each of its components contains an abelian subgroup with finite index.

(c) Any cylinder $C : (S^1 \times [0,1], \partial S^1 \times [0,1]) \rightarrow (M, P)$ such that $\pi_1(C)$ is injective is homotopic rel boundary to $P$.

(d) $P$ contains every component of $\partial M$ that has an abelian subgroup of finite index.

Analogous to the concept of convex co-compact, we say that a metric is geometrically finite is $C_M$ has finite volume. While convex-compact manifolds are made out exclusively by loxodromics, geometrically finite manifolds can admit parabolic elements. Due to classical Kleinian group theory, we know that parabolic elements will be peripheral, i.e. elements in $\pi_1$ induced by the inclusion $\partial M \hookrightarrow M$. Moreover, any maximal abelian subgroup including a parabolic is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$, hence we can label each of such maximal abelian subgroup by its rank, which is 1 or 2. Again, classical Kleinian group theory tell us that parabolic subgroups appear as cusps in $\partial M$, of either rank-1 (in which case appears as a cylinder) or rank-2 (in which case appears as a tori component). Moreover, if $P$ is the parabolic loci, then $(M, P)$ will be a pared manifold. Hence given a pared manifold $(M, P)$ we can define $QF(M, P)$ as the space of geometrically finite hyperbolic metrics in $M$ where $P$ corresponds to the parabolic loci. As with convex co-compact metrics, [AB60] tell us that $QF(M, P)$ is homeomorphic to $\mathcal{T}(\partial(M, P))$, the Teichmüller space of $\partial(M, P)$, which is the Teichmüller space of $\partial M$ while discounting the tori components and mandating cusps in the cylinder components of $P$.

### 1.2 Geometrically finite limits

Denote by $\Gamma \leq PSL(2, \mathbb{C})$ a discrete subgroup. We say that $\Gamma$ is non-elementary if $\Gamma$ is not virtually abelian (i.e. does not have a finite index abelian subgroup). Given a sequence of
discrete and faithful representations of $\Gamma$ into $\text{PSL}(2, \mathbb{C})$, denoted by $\rho_n : \Gamma \rightarrow \text{PSL}(2, \mathbb{C})$, we have the following notion of convergence.

**Definition 1.2.1.** We say that $(\rho_n)$ converges algebraically if the limit $\rho(\gamma) = \lim_{n \to \infty} \rho_n(\gamma)$ exists for all $\gamma \in \Gamma$. In such a case, we say that $\rho : \Gamma \rightarrow \text{PSL}(2, \mathbb{C})$ is the algebraic limit of $(\rho_n)$.

It is easy to say that if $\rho_n$ converges algebraically to $\rho$ then $\rho : \Gamma \rightarrow \text{PSL}(2, \mathbb{C})$ is a homomorphism. Moreover, from [Jør76, Theorem 1] and [JK82, Theorem] we know that $\rho$ will be also discrete.

While this convergence comes from the natural topology for $\text{PSL}(2, \mathbb{C})$ representations, it does not necessarily describes convergence properties at the geometric level. Let us define then the following convergence notion for a sequence $G_n$ of subgroups of $\text{PSL}(2, \mathbb{C})$.

**Definition 1.2.2.** We say that $G = \{g \in \text{PSL}(2, \mathbb{C}) \mid g = \lim_{n \to \infty} g_n, \ g_n \in G_n\}$ is the geometric limit of $G_n = \rho_n(\Gamma)$.

Observe that, once existing, the algebraic limit is a subgroup of the geometric limit. Given that the geometric limit is non-elementary and if we identify by $M_n = \mathbb{H}^3/G_n$, then $(M_n, 0)$ converges to $(M, 0)$ in the pointed Gromov-Hausdorff sense, where $M = \mathbb{H}^3/G$ and 0 is the equivalence class of 0 $\in \mathbb{H}^3$ in each quotient.

Nevertheless, this geometric description relies on the fixed points. If we were to choose different base points in the quotients (or equivalently, conjugate $G_n$ by $g_n \in \text{PSL}(2, \mathbb{C})$ as $g_n G_n g_n^{-1}$), we will have the same geometric objects in the sequence with possibly a different limit (this is because the pointed space sequence is different in principle). Let us describe this phenomenon in the case that all geometric limits are geometrically finite.

Let us make precise some of the terminology used by the proposition below. We say that a curve $\gamma \subset M$ is **drilled** if after the geometric limit a neighbourhood $U$ of $\gamma$ converges to $U \setminus \gamma$. We also say that $\gamma \subset \partial M$ is **pinched** if after the geometric limit a neighbourhood $V$ of $\gamma$ in $\partial M$ converges to $V \setminus \gamma$. We say that a collection of cylinders with union $\mathcal{C}$ are **essential cutting cylinders** if (1) each cylinder $(S^1 \times [0, 1])$ is a (disjoint) properly embedded cylinder in $(M, \partial M)$, and (2) taking base points in the complement of $\mathcal{C}$, the geometric limit is the corresponding component of $M \setminus \mathcal{C}$.

**Proposition 1.2.1.** Let $M_n \in QF(M, P)$ be a sequence such that $V_C(M_n)$ is uniformly bounded. Then we can select finitely many base points such that (possibly after taking a subsequence) $N_1, \ldots, N_k$ are the geometric limits corresponding those base points. These hyperbolic manifolds are geometrically finite and are obtained from $M$ by drilling curves into rank-2 cusps, pinching peripheral curves into rank-1 cusps or cutting essential cylinders forming one rank-1 cusp at each side. These cylinders are the division between distinct geometric limits $N_i$.

**Proof.** Start recalling (see for instance [Mar07]) that each boundary component of $C_{M_n}$ is path isometric to a surface of constant curvature, and say $S$ is one of such components. Then,
by Deligne-Mumford compactification (see [DM69] for the definition of the moduli stack $\mathcal{M}$, with compactifies Riemann surfaces by Riemann surfaces with nodes), we can assume that (after some relabelling) $S$ converges as Riemann surfaces with nodes to a (possibly disconnected) union of hyperbolic surfaces of finite type $S_1, \ldots, S_l$, obtained by pinching some disjoint essential curves of $S$. Let us denote by $S_j^n$ each component in $\partial M_n$ such that $S_j^n \to S_j$ geometrically. Let us also place a basepoint on each $S_j$ and then place a basepoint on each $S_j^n$ such that we have convergence of pointed spaces. Thus, since the path metric in $S$ bounds the metric in $M$, we will have that $S_j^n$ converges algebraically if we center at their basepoints, after possibly passing to a subsequence. Indeed, each element in the fundamental group of $S_j$ translates the basepoints of $S_j^n$ by a uniformly bounded distance in $\mathbb{H}^3$ when we take the universal cover. This uniform bound is given by the translation distance convergence from $S_j^n \to S_j$ and the fact that $S_j^n \hookrightarrow M_n$ is a path isometry. Then after having a point with bounded translation, we can always extract a subsequence to have convergence of the given element in $\pi_1(S_j)$. To have simultaneous convergence of all elements of $\pi_1(S_j)$ we just need to remember that this group is finitely generated. Since $S_j$ can not be an elementary Riemann surface, this algebraic limit will be non-elementary, so the same is true when we take the geometric limits. Do the same for all $j$ and all boundary components of $M$, and for each pair of basepoint sequences that stay at bounded distance, erase one of them (bounded distance basepoints give the same space as a limit just with a different basepoint). Each basepoint sequence gives a non-elementary geometric limit, which we will label by $N_1, \ldots, N_k$.

Take $2\epsilon$ to be both a 2 and 3 dimensional Margulis constant. Then, since $V_C(M_n)$ is uniformly bounded, every component of the $\epsilon$-thick part of $C_{M_n}, C'_M$, has finite diameter. Indeed, on each component choose an efficient cover of embedded balls of radius $2\epsilon$ (by efficient we mean that any two different centers are at least $2\epsilon$ apart). Then the balls with radius $\epsilon$ are disjoint, and either contained in $C_{M_n}$ or intersect its boundary. Since the volume of $C_{M_n}$ and the surface area of $\partial C_{M_n}$ are uniformly bounded (as well as for the $\epsilon$-neighbourhoods), the number of such balls is uniformly bounded as well. Each $N_j$ has associated one component of $C'_{M_n}$. Moreover, following [Mar07] (in particular Lema 4.3.1), if we pick base points in other components of $C'_{M_n}$ we can take a subsequence and assume that the geometric limit exists.

Consider any non-elementary geometric limit $N$ and assume that the limit basepoint lies on a closed geodesic of length greater than $2\epsilon$. Thanks to the uniform bounded diameter of the components of $C'_{M_n}$, we know that all closed geodesics are at bounded distance from the basepoint. Hence we can construct a polyhedra fundamental domain for $C_N'$ with finitely many sides that we can extend to a fundamental polyhedra for $N$ with the same number of sides, concluding that $N$ is geometrically finite. And if we take a neighborhood of $C_N$ (make $\epsilon$ small enough so it is connected) then it is the geometric limit of $C'_{M_n}$, or particularly of one the components that we considered from the start (with the appropriately chosen base points). So $\partial C_N$ can be match to components of $\partial C_M$, having in consideration that different components could be match with the same component, although as disjoint subsurfaces. Then the base point on the corresponding component $S_j \subseteq \partial C_M$ is at bounded distance,
hence \( N \) coincides with \( N_i \) for some \( 1 \leq i \leq k \).

From the previous paragraph we can also extrapolate that we can make \( \epsilon \) smaller if necessary so \( C'_{N_i} \) is connected, and that there is a uniformly bounded number of Margulis tubes that correspond only to cusps. This in turn says that different components of \( C'_{M_n} \) give different geometric limits since the distance between them goes to \( \infty \).

The type of cusp at the limit can be described as follows: If the Margulis tube \( T \) does not intersect the boundary of \( C_{M_n} \), then it corresponds to a rank-2 cusp. If \( T \) intersects the boundary of \( C_{M_n} \), it will correspond to rank-1 cusp in one or many geometric limits \( N_i \), depending on how it intersects the boundary. At every component \( S \) of \( \partial C_{M_n} \) intersected by \( T \), the corresponding peripheral curve shrinks to a parabolic along the sequence (if not, from algebraic convergence of each subcomponent \( S_1, \ldots, S_i \), the Margulis tube \( T \) will eventually stay away from \( S \)). If \( T \) only intersects one component \( S \), then either one rank-1 cusp is created by pinching the peripheral representative at \( S \), and \( T \) does not separate different geometric limits \( N_i \), or it corresponds to an original rank-1 cusp from \( M \). If intersects multiple boundary components, then the different components of \( \partial T \cap C_{M_n} \) are cylinders that locally separate geometric limits \( N_i \). Each cylinder will correspond to a rank-1 cusp at a geometric limit \( N_i \), and it will be called a cutting cylinder. Notice that the Margulis tube core could come from either a loxodromic, a rank-1 cusp or a rank-2 cusp.

Divide \( M_n \) along cutting cylinders as follows:

- If the Margulis tube \( T \) has a loxodromic core, draw a cylinder between its core geodesic and its peripheral geodesic representative at component of \( \partial C_{M_n} \) intersected by \( T \), making sure that such cylinders are disjoint. From the circles at \( \partial C_{M_n} \) follow the normal geodesic flow to have a collection of \( S^1 \times \mathbb{R} \) cylinders.

- If the Margulis tube \( T \) has a rank-1 cusp core, draw a cylinder that starts at its peripheral geodesic representative at component of \( \partial C_{M_n} \) intersected by \( T \) and goes into the cusp, and consider such cylinders to be disjoint. From the circles at \( \partial C_{M_n} \) follow the normal geodesic flow to have a collection of \( S^1 \times \mathbb{R} \) cylinders.

- If the Margulis tube \( T \) has a rank-2 cusp core, draw a cylinder that starts at its peripheral geodesic representative at component of \( \partial C_{M_n} \) intersected by \( T \) and goes into the cusp, and consider such cylinders to be disjoint. From the circles at \( \partial C_{M_n} \) follow the normal geodesic flow to have a collection of \( S^1 \times \mathbb{R} \) cylinders.

Doing this, we will have a collection of \( S^1 \times \mathbb{R} \) cylinders, whose disjoint union we name by \( \mathcal{C} \). Each component of \( M_n \setminus \mathcal{C} \) has assigned a geometric limit \( N_i \), so label these components as \( M_n^i \). Inside each \( M_n^i \) we have totally included \( \epsilon \)-Margulis tubes converging to rank-2 cusps, which could be some original rank-2 cusps from \( P \) or could have a loxodromic core, a situation described as drilling. We can also have in \( M_n^i \) the \( \epsilon \)-Margulis tubes intersecting \( \partial C_{M_n} \) uniquely at \( M_n^i \) converging to rank-1 cusps, which could be some original rank-1 cusps from \( P \) or could have a loxodromic core, situation described as pinching.
Definition 1.2.3. We say that $N_1 \sqcup \ldots \sqcup N_k$ is the additive geometric limit of a sequence \{M_n\} if they satisfy the properties described by Proposition 1.2.1.

Then an alternative formulation of 1.2.1 is that any sequence with bounded $V_C$ has a subsequence with an additive geometric limit. This notion of limit is related to Benjamini-Schramm convergence [BS01] and discussed for hyperbolic manifolds in [ABB+17, Section 3.9]. In their notion of limit, the basepoints are selected at random by a measure. Hence for our case, if we choose uniform measures on the convex cores, we will obtain $N_1 \sqcup \ldots \sqcup N_k$ with uniform measures on their convex cores, weighted out by their volumes.

1.3 Peripherally generic fundamental domains

For the present section, consider the projective Klein model of $\mathbb{H}^3$. In this, $\mathbb{H}^3$ is represented by the unit ball in $\mathbb{P}^3$, $\partial \mathbb{H}^3$ by $S^2$, geodesic lines and planes are the intersection of euclidean lines and planes with $\mathbb{H}^3$. While this model is not conformal, orthogonality can be described by the pole and polar duality. This duality pairs points with planes, lines with lines, and can be extended from $\mathbb{R}^3$ to $\mathbb{P}^3$ as an algebraic map. Then a line $L$ and a plane $P$ are orthogonal if $P$ contains the polar of $L$. This is equivalent to $L$ containing the polar of $P$.

In the following section we will show that given a geometrically finite hyperbolic manifold $M$, its Dirichlet fundamental domain base at $p \in \mathbb{H}^3$ will be peripherally generic for almost all $p$. By peripherally generic we mean that (except when we refer to hyperplanes of the same abelian subgroup) no two different hyperplanes involved in the fundamental domain are tangent nor that their intersection is tangent to $S^2$, and that non three hyperplanes share a common line, as well as their triple intersection not to lie in $\partial \mathbb{H}^3$ (note that this is equivalent to the fundamental region at $S^2$ to be generic in the usual sense). While in [Mar07] it is claim that generic Dirichlet fundamental regions are generic for most basepoints (not only peripherally generic), the proof in [JM90] is flawed as indicated in [DaU09], where they solve the Fuchsian case. The techniques of the present section extrapolate their approach to the boundary of $M$.

Say then that $T$ is a isometry of $\mathbb{H}^3$. Define the axis of $T$, $\text{Ax}(T)$, as the line between the two fixed points of $T$ at $S^2$ if $T$ is non parabolic, or the tangent line at the only fixed point of $T$ with direction tangent to any fiber of $T$, if $T$ parabolic. Observe then that $T$ sends $\text{Ax}(T)$ to itself in all cases, and since $T$ can be extended as a projective map from $\mathbb{P}^3$ to itself, then it must send the polar of $\text{Ax}(T)$, $\text{Ax}^*(T)$, to itself. This polar axis does not intersect the open unit ball, being only tangent to $S^2$ if $T$ is parabolic. Observe that unless $T$ is the identity or an involution, these are the only two lines preserved by $T$. One can also well define an isometry $\sqrt{T}$ such that has the same axis as $T$ and $\sqrt{T} \circ \sqrt{T} = T$.

For $p \in \mathbb{P}^3 \setminus \text{Ax}^*(T)$ define the plane $P_{T,p}$ as the unique plane containing $\sqrt{T}(p)$ and $\text{Ax}^*(T)$. Observe that $T$ takes $P_{T^{-1},p}$ to $P_{T,p}$, and for $p \in \mathbb{H}^3$, $p$ lies in between these planes. Then, if we define by $\mathcal{F}(L)$ to be the sheaf of planes containing a given line $L$, $P_{T,p} \in \mathcal{F}$ and the algebraic map $P_T : \mathbb{P}^3 \to \mathcal{F}(\text{Ax}^*(T))$ is not defined at $\mathbb{P}^3 \setminus \text{Ax}^*(T)$. A geometric description for $P_T$ is the following. Take first $P$ is a tangent plane to $S^2$ containing $\text{Ax}^*(T)$,
$P_T$ sends the points of $P$ to $P$ itself. Depending if $T$ is parabolic or not there could be one or two of these planes, but for planes in between $P_T$ uses $\text{Ax}^*(T)$ as an axis of rotation to find the target plane.

**Definition 1.3.1.** Let $T_1, T_2$ be two distinct isometries of $\mathbb{H}^3$. Define $L_{T_1,T_2}(p) = P_{T_1}(p) \cap P_{T_2}(p)$ for $p \in \mathbb{P}^3$ such that $P_{T_1}(p), P_{T_2}(p)$ are defined and different.

From the definition of $P_T(\cdot)$, we see that the codomain of $L_{T_1,T_2}$ should at least contain $\text{Ax}^*(T_1) \cup \text{Ax}^*(T_2)$, but could be greater if $P_{T_1}$ and $P_{T_2}$ are not generic for a given point. The following lemma answers this question.

**Lemma 1.3.1.** Let $T_1, T_2$ be two distinct isometries of $\mathbb{H}^3$ that don’t generate a elliptic isometry. Then the codomain of $L_{T_1,T_2}$ is equal to:

1. The tangent plane(s) to $S^2$ from $\text{Ax}^*(T_1)$, if $\text{Ax}^*(T_1) = \text{Ax}^*(T_2)$
2. $\text{Ax}^*(T_1) \cup \text{Ax}^*(T_1)$, if $\text{Ax}^*(T_1)$ and $\text{Ax}^*(T_1)$ are distinct and not coplanar.
3. $\text{Ax}^*(T_1) \cup \text{Ax}^*(T_1) \cup Q$, if $\text{Ax}^*(T_1)$ and $\text{Ax}^*(T_1)$ are distinct coplanar lines but the plane that contains them is not tangent to $S^2$. $Q$ is another line coplanar to each of them that contains $\text{Ax}^*(T_1) \cap \text{Ax}^*(T_2)$.
4. $P$, if $\text{Ax}^*(T_1)$ and $\text{Ax}^*(T_1)$ are distinct, coplanar and their common plane $P$ is tangent to $S^2$.

**Proof.** For case (1), the polar axes coincide if and only if the axes coincide. Then the tangent plane(s) are send to themselves by $P_{T_1}$ and $P_{T_2}$, so the intersection will no be generic. For all other points, $P_{T_1}$ and $P_{T_2}$ only intersect at $\text{Ax}^*(T_1)$.

For case (2), we see that since there is not common plane containing them, the intersection $P_{T_1} \cap P_{T_2}$ is always a line.

For case (3), the points $p$ that will give us trouble will be the ones such that $P_{T_1}(p)$ and $P_{T_2}(p)$ are the common plane $P_0$. Since $P_0$ is not tangent to $S^2$, then $P^{-1}_{T_1}(P_0)$ is a plane distinct from $P_0$ that contains $\text{Ax}^*(T_1)$ (similarly for $T_2$). Then the set of trouble points is $P^{-1}_{T_1}(P_0) \cap P^{-1}_{T_2}(P_0) = Q$, which is a line since $P^{-1}_{T_1}(P_0), P^{-1}_{T_2}(P_0)$ are both distinct to $P_0$. The last property of $Q$ follow easily.

For case (4), similar to case (3), the extra codomain comes from $P^{-1}_{T_1}(P) \cap P^{-1}_{T_2}(P)$, but in this case the are both equal to $P$.

Notice that in case (4), $T_1$ and $T_2$ have a common fix point in $S^2$, which is the point of tangency of the plane containing them.

As in [DaU09], $L_{T_1,T_2}$ is an algebraic map that could be extended to some of the points of its codomain as a map $\tilde{L}_{T_1,T_2} : \mathbb{P}^3 \rightarrow \mathcal{L}$, where $\mathcal{L}$ is the space of lines in $\mathbb{P}^3$. The new lemma answers the nature of this extension and what the codomain will be in 3 out of the 4 cases.
Lemma 1.3.2. Let $T_1, T_2$ be two distinct isometries of $\mathbb{H}^3$ that don’t generate a elliptic isometry. Moreover, assume they are in cases (1) – (3) of Lemma 1.3.1. Then the codomain of $\hat{L}_{T_1, T_2}$ is equal to:

1. $\emptyset$, if $Ax^*(T_1) = Ax^*(T_2)$. $\hat{L}$ is constant equal to $Ax^*(T_1)$.
2. $Ax^*(T_1) \cup Ax^*(T_2)$, if $Ax^*(T_1)$ and $Ax^*(T_2)$ are not coplanar.
3. $Ax^*(T_1) \cup Ax^*(T_2) \cup Q$, if $Ax^*(T_1)$ and $Ax^*(T_2)$ are distinct coplanar lines but the plane that contains them is not tangent to $S^2$. $Q$ is another coplanar line to them as described in Lemma 1.3.1.

Proof. Let us look to each case of 1.3.1:

1. Obvious since the map was already constant.
2. Notice that while approaching a point in $Ax^*(T_1)$ you can obtain any possible plane containing $Ax^*(T_1)$, but $P_{T_2}$ converge to a unique plane not containing $Ax^*(T_1)$. Then the map $\mathcal{L}$ cannot be extended for points in $Ax^*(T_1)$, and similarly for $Ax^*(T_2)$.
3. Since the plane containing $Ax^*(T_1), Ax^*(T_2)$ is not tangent to $S^2$ (hence not constant under $T_1, T_2$), the same argument as in the previous case proofs that $\mathcal{L}$ cannot be extend to $Ax^*(T_1) \cup Ax^*(T_2)$. And as for the line $Q$, approaching from the plane generated by $(Q, Ax^*(T_1))$ will conclude that the extension needs to be $Ax^*(T_2)$ but then we will have the same by swapping the indices 1, 2, which makes the extension impossible.

Definition 1.3.2. Let $T_1, T_2, T_3$ distinct non-elliptic isometries of $\mathbb{H}^3$. We denote by $A_{T_1, T_2, T_3}$ the subset:

$$\mathcal{A}_{T_1, T_2, T_3} = \{ p \in \mathbb{P}^3 | \hat{L}_{T_1, T_2}(p) = \hat{L}_{T_1, T_3}(p) \} \cup \text{codomain}(\hat{L}_{T_1, T_2}) \cup \text{codomain}(\hat{L}_{T_1, T_3})$$

It is not hard to see as in [DaU09]

Lemma 1.3.3. The set $\mathcal{A}_{T_1, T_2, T_3}$ is an algebraic subset of $\mathbb{P}^3$.

Similarly, we can define the set of basepoints that give non peripherally generic bisecting planes

Definition 1.3.3. Let $T_1, T_2, T_3$ distinct non-elliptic isometries of $\mathbb{H}^3$. We denote by $\mathcal{D}_{T_1, T_2}$, $\mathcal{D}_{T_1, T_2, T_3}$ the subsets:

$$\mathcal{D}_{T_1, T_2} = \{ p \in \mathbb{P}^3 | \hat{L}_{T_1, T_2}(p) \text{ is tangent to } S^2 \} \cup \text{codomain}(\hat{L}_{T_1, T_2})$$

$$\mathcal{D}_{T_1, T_2, T_3} = \{ p \in \mathbb{P}^3 \setminus A_{T_1, T_2, T_3} | (\hat{L}_{T_1, T_2}(p) \cap \hat{L}_{T_1, T_3}(p)) \in S^2 \} \cup A_{T_1, T_2, T_3}$$
The following lemma follows straight out of the definition.

**Lemma 1.3.4.** The sets $D_{T_1,T_2}$, $D_{T_1,T_2,T_3}$ are algebraic subsets of $\mathbb{P}^3$.

Since we are interested in cases when this algebraic sets are proper, let us describes the cases where they are not in the following lemmas.

**Lemma 1.3.5.** $D_{T_1,T_2} = \mathbb{P}^3$ if and only if $Ax^*(T_1) = Ax^*(T_2)$ is tangent to $S^2$. In particular $T_1, T_2$ are both parabolic with the same axis.

Proof. From Lema 1.3.1 we see that for cases (2), (3) and (4) there is a point $p \in \mathbb{P}^3$ such that both bisecting lines are exterior to $S^2$, then $L_{T_1,T_2}(p)$ is not tangent to $S^2$, so $D_{T_1,T_2}$ is a proper algebraic subset. Finally, case (1) clearly needs to be as described. \[\Box\]

From now on consider every pair of isometries of $\mathbb{H}^3$ not in case (4) of Lemma 1.3.1.

**Lemma 1.3.6.** $A_{T_1,T_2} = \mathbb{P}^3$ only if $Ax^*(T_1), Ax^*(T_2), Ax^*(T_3)$ all coincide or if pairwise they are in cases (3) of Lema 1.3.2. For case (3), $Ax^*(T_3) = Q$.

Proof. Indeed, the codomains of $\hat{L}_{T_1,T_2}, \hat{L}_{T_1,T_3}$ need to coincide since the algebraic functions are equal in a open set. \[\Box\]

**Lemma 1.3.7.** $D_{T_1,T_2,T_3} = \mathbb{P}^3$ only if $Ax^*(T_1), Ax^*(T_2), Ax^*(T_3)$ coincide or if pairwise they are in cases (3) of Lema 1.3.2. For case (3), $Ax^*(T_3) = Q$.

Proof. Indeed, in the cases where $A_{T_1,T_2} \neq \mathbb{P}^3$ and with generic basepoint outside the unit ball, the intersection point lies also outside the unit ball. \[\Box\]

**Definition 1.3.4.** Let $M$ be a geometrically finite hyperbolic 3-manifold. We say that a Dirichlet fundamental domain $F_p$ with center $p$ is peripherally generic if its intersection with $S^2$ is a union of generic cuspid polygons (disregard the rank-2 cusps). Here generic means that two consecutive edges intersect transversally unless they join at a rank-1 cusp, and a vertex is not shared by more than 2 edges.

**Proposition 1.3.1.** Let $M$ be a geometrically finite hyperbolic 3-manifold. Then for a dense set of $p \in \mathbb{H}^3$ the fundamental Dirichlet domain $F_p$ is peripherally generic.

Proof. Denote by $\Gamma = \pi_1(M) < \text{PSL}(2, \mathbb{C})$. Define $D$ as the union of all sets $D_{T_1,T_2}$ for all pairs $T_1, T_2 \in \Gamma$ except when they are parabolic elements with the same fixed point, and all sets $D_{T_1,T_2,T_3}$ for triples $T_1, T_2, T_3 \in \Gamma$, except when all three belong to the same abelian subgroup of $\Gamma$. Consider $p \notin D$. The proof (similar to [JM90] [DaU09]) divides into showing that $D_{T_1,T_2}, D_{T_1,T_2,T_3}$ are proper algebraic subsets (for the cases considered), and that this suffices for the fundamental polyhedron to be peripherally generic. The statement of the proposition follows from Baire’s theorem and these two facts.

For $D_{T_1,T_2}$ sets, we see clearly from Lema 1.3.5 that this set is proper. For $D_{T_1,T_2,T_3}$ sets, observes first that the isometries of $\Gamma$ with a common fix point form an abelian subgroup
(because \(\Gamma\) is discrete). Then the case (4) of Lemma \[1.3.1\] does not occur in our considerations. According to Lemma \[1.3.7\] we need to discard cases (3). For case (3) \(T^2_2\) will fix two planes that are not the tangents from \(Ax^*(T_2)\), implying that \(T_2\) is elliptic.

Remains to show that for a point \(p \notin D\), \(F_p\) is peripherally generic. Assume the contrary. Then \(F_p\) has a polygonal face \(\Pi\) in \(S^2\) that is not a generic cuspid polygon. By the definitions of \(D_{T_1,T_2}\), \(D_{T_1,T_2,T_3}\) and \(p \notin D\), non-generic edge intersections ain \(\Pi\) are not from the cases considered in \(D\). Let us look into each case.

- Two consecutives edges of \(\Pi\) are tangent: then the corresponding elements \(T_1, T_2 \in \Gamma\) associated to the edges must be parabolic with the same dual axis. Since they are appearing consecutively at a fundamental domain, they correspond to a rank-1 cusp, which is accounted in the definition of peripheral generic.

- A vertex of \(\Pi\) is shared by three edges: then the corresponding elements \(T_1, T_2, T_3 \in \Gamma\) associated to the edges must belong to the same abelian subgroup. The subgroup cannot be loxodromic since in that case all the bisecting planes intersect at the common dual axis, which lies outside \(S^2\). In the case that the subgroup is parabolic, the vertex will be then the common fix point. There can’t be three edges for a rank-1 cusp and rank-2 cusp dixed point do not appear in \(F_p\).

Then \(F_p\) is peripherally generic, which is the last part of the proof. \(\square\)

### 1.4 Small deformations of sequences

As in [Thua], let us understand a hyperbolic 3-manifold \(M\) as a \((\text{PSL}(2, \mathbb{C}, \mathbb{H}^3))\) manifold, with associated holonomy \(H : \pi_1(M) \to \text{PSL}(2, \mathbb{C})\). For \(M\) with finitely generated fundamental group (i.e. the interior of a compact 3 manifold, thanks to the Scott/Shalen compact core [Sco73]) the representation variety \(\text{Def}(M) = \text{Hom}(\pi_1(M), \text{PSL}(2, \mathbb{C}))/\text{PSL}(2, \mathbb{C})\) is a finite dimensional complex algebraic variety.

**Definition 1.4.1.** Let \(M\) be a geometrically finite hyperbolic 3-manifold. Then define

\[
\text{Def}_0(M) := \{H_0 \in \text{Def}(M)| \text{ if } \gamma \text{ is conjugated to a rank-1 cusp, } H_0(\gamma) \text{ is parabolic}\}
\]

As in Theorem 5.6 of [Thua], we have:

**Proposition 1.4.1.** The dimension of \(\text{Def}_0(M)\) (near \(H\)) is as great as the total dimension of the Teichmüller space of \(\partial(M)\), that is,

\[
\dim_{\mathbb{C}}(\text{Def}_0(M)) \geq \sum_{\chi(\partial M)_i < 0} 3|\chi(\partial M)_i| + \text{(number of rank-2 cusps)}
\]
Definition 1.4.2. (Dehn-filling notation) Let $M$ be a 3-manifold with $k$ tori boundary components. For fixed $\alpha_i, \beta_i$ meridian and longitude of the $i$-tori component, and $(d_1, \ldots, d_k) \in S^2 \times \ldots \times S^2$ ($S^2 = \mathbb{R}^2 \cup \{\infty\}$), denote by $M_{(d_1, \ldots, d_k)}$ as the result of gluing disks along the curves $a_i \alpha_i + b_i \beta_i$ ($d_i = a_i + b_i$) and then filling by 3-balls. In case $d_i = \infty$, we do not perform any filling at the $i$-tori component.

Following [Thur], [BO88] let us show:

Theorem 1.4.1 (Generalized Dehn-filling). Let $M$ be a geometrically finite hyperbolic 3-manifold. Fix $\alpha_i, \beta_i$ meridians and longitudes for the tori components of $\partial M$, and denote by $\tau \in \mathcal{T}(\partial M)$ the conformal class at infinity. Then exists a neighbourhood $U$ of $(\tau, \infty, \ldots, \infty)$ in $\mathcal{T}(\partial M) \times S^2 \times \ldots \times S^2$ such that for $(\tau_0, d_1, \ldots, d_k) \in U$ there exists a hyperbolic structure in $M_{(d_1, \ldots, d_k)}$ with conformal class at infinity $\tau_0$, which we will call by $M_{(\tau_0, d_1, \ldots, d_k)}$. Moreover, as $(\tau_0, d_1, \ldots, d_k) \to (\tau, \infty, \ldots, \infty)$, $M_{(\tau_0, d_1, \ldots, d_k)}$ converges geometrically to $M$.

Proof. Thanks to Proposition [1.3.1] take a fundamental domain $F_p$ that is peripherally generic. Then for elements $H_0 \in \text{Def}_0(M)$ sufficiently close to $H$ (the holonomy of $M$), the elements that shape the cusped polygons for $F_p$ also shape generic cuspid polygons under this deformation (note that rank-1 cusps stay parabolic). Then by developing these cuspid polygons we have projective structures for each component of $\partial M$, which we can restrict to an element $\tau(H_0) \in \mathcal{T}(\partial M)$. Define then the map:

$$T : U \to \mathcal{T}(\partial M) \times \mathbb{C}^k, T(H_0) = (\tau(H_0), Tr(H_0(\alpha_1)^2, \ldots, Tr(H_0(\alpha_k)^2))$$

(1.1)

were $U \subset \text{Def}_0(M)$ is the sufficiently small neighbourhood of $H$. Notice that $T$ is a holomorphic map were $\dim_{\mathbb{C}}(U) \geq \dim_{\mathbb{C}}(\mathcal{T}(M)) + k$, were $\dim_{\mathbb{C}}(\mathcal{T}(M)) + k$ is precisely the complex dimension of the range of $T$ in (1.1).

We claim that $T^{-1}(\tau, 4, \ldots, 4) = \{H\}$. Indeed, $Tr(H_0(\alpha_i)) = \pm 2$ are the equations for $\alpha_i$ to be parabolic. Then the deformation space is characterized entirely by the conformal structure at infinity, concluding our claim.

Hence the dimensions at (1.1) must coincide and $T$ is an open map. As in Theorem 5.8.2, for $(\tau_0, d_1, \ldots, d_k)$ close to $(\tau, \infty, \ldots, \infty)$ there is a hyperbolic structure in $M_{(d_1, \ldots, d_k)}$ with holonomy map $H_0$ that satisfies $\tau(H_0) = \tau_0$. Since this manifold is hyperbolic, $\tau_0$ coincides with the conformal class at infinity. To close the argument, as $(\tau_0, d_1, \ldots, d_k) \to (\tau, \infty, \ldots, \infty)$, $H_0 \to H$. Since $M$ is discrete, this garanties that $M_{(\tau_0, d_1, \ldots, d_k)}$ converges geometrically to $M$. 

Proposition 1.4.2. Let $M \in \text{QF}(M)$ be a sequence with additive geometric limit $N_1 \sqcup \ldots \sqcup N_k$ (as in Proposition [1.2.7]). Then for any sufficiently small deformation of $N_1, \ldots, N_k$ there exist a sequence $\tilde{M}_n \in \text{QF}(M)$ that has them as additive geometric limit.

Proof. Fix one component of the additive geometric limit $N_i$. Each rank-2 cusp in $N_i$ is the limit of a Margulis tube that stays inside $C_{M_n}$. For each Margulis tube that has a loxodromic core choose $\alpha_j, \beta_j$ meridian and longitude for the $j$ rank-2 cusp of $N_i$, so we can
describe the filling of this tube at \( M_n \) by \( d_{j,n} = (a_{j,n}, b_{j,n}) \). Hence \( d_{j,n} \to \infty \) and \( (N_i)_{d_{1,n}, \ldots, d_{j,n}} \) is homeomorphic to \( M_n' \), where we are undrilling the drilled rank-2 cusps. Then applying Theorem 1.4.1, \( (N_i)_{\tau, d_{1,n}, \ldots, d_{j,n}} \) is a sequence of hyperbolic manifolds homeomorphic to \( M_n' \), with only rank-1 cusps and geometric limit \( (N_i, \tau) \). Let us call this sequence \( (N_i)_{\tau, n} \) for simplicity.

Now, \( (N_1)_{\tau, n} \sqcup \ldots \sqcup (N_i)_{\tau, n} \) have pairings between some of their rank-1 cusps corresponding to cutting cylinders. Perform then Klein-Maskit combinations between these pairs of rank-1 cusps (see [Mas88], Chapter 7) for more details in Klein-Maskit combination theory) with smaller and smaller horodisks. The glued manifold will be homeomorphic to \( M \) minus the loxodromic cores of Margulis tubes giving cutting cylinders, and except for those cusps it has parabolics in the right places. In order to unpinch the rank-1 cusps, let us do a self Klein-Maskit combination to close them to a rank-2 cusp. Then we have a sequence \( (M_{\tau, n})_m \) with the following properties:

- \( (M_{\tau, n})_m \) is homeomorphic to \( M \) minus the loxodromic cores of Margulis tubes converging to ran-2 cusps. These rank-2 cusps together with the ones obtained after self combining the pinched rank-1 cusps are the only parabolic subgroups we need to fill-in.

- \( (N_1)_{\tau, n} \sqcup \ldots \sqcup (N_i)_{\tau, n} \) are all the possible geometric limits as \( m \to \infty \) and we move the base point around.

Use again Theorem 1.4.1 to fill-in the rank-2 cusps of \( (M_{\tau, n})_m \) by generalized Dehn-filling. These gives manifolds arbitrarily close to \( (M_{\tau, n})_m \), so by the diagonal argument we can name these parabolic free manifolds by \( (M_{\tau, n})_m \) and still have that \( (N_1)_{\tau, n} \sqcup \ldots \sqcup (N_i)_{\tau, n} \) are the additive geometric limit as \( m \to \infty \) and we move the base point around. In order to see that these manifolds are from the topological type desired, the undrilled rank-2 cusps are filled as they were in the original sequence, while for the additional cusps created from Klein-Maskit combinations we can select fillings that don’t change the topology. Indeed, each cusp has a simple curve in the torus boundary of its Margulis tube that is parallel to the boundary \( \partial M \) and a transversal simple curve. Choose \( \alpha, \beta \) meridian and longitude on the torus such that the curve filled in is \( \beta \) and the curve in \( \partial M \) is \( a\alpha + b\beta, a > 0 \). Then, if \( c, d \) are integers such that \( ac - bd = 1 \) (they exist since \( a\alpha + b\beta \) is a simple curve), it is an easy 3-manifold exercise to see that the filling \( d = (ad, ac + 1) \) fills-in a manifold homeomorphic to \( M \) and converges to a cusp (the key point is that there is a torus isomorphism that preserves the curve \( a\alpha + b\beta \in \partial M \) and sends \( \beta \) to \( ad\alpha + (ac + 1)\beta \).

By doing the diagonal argument one more time, we have constructed a sequence \( M_{\tau, n} \) of hyperbolic manifolds homeomorphic to \( M \) such that \( (N_i, \tau) \) are the additive geometric limit.

\[ \Box \]

1.5 Teichmüller space

Let \( \Sigma \) be a finite analytic type Riemann surface of genus \( g \) and \( n \) punctures. We will give a nice (an appropriate for our goals) description for \( T \) its Teichmüller space, \( B \) its space of
Beltrami differentials and $Q$ its space of quadratic holomorphic differentials. This can be found, for example, in [Gar87]. Let us start by setting our definition for a quasiconformal map.

**Definition 1.5.1.** For a simply connected domain in the plane $R$ with two disjoint arcs $\alpha_1, \alpha_2 \subset \partial R$, we define $m(R)$ as the quotient $a/b$, where $a, b$ are the horizontal and vertical sides of a rectangle $R^*$ where $R$ can be mapped conformally, onto, and in such a way that $\alpha_1, \alpha_2$ are mapped to the vertical sides of $R^*$. A orientation preserving homeomorphism $f : \Sigma \rightarrow \Sigma^*$ between two Riemann surfaces is called $K$-quasiconformal (or just quasiconformal) if $m(f(R)) \leq K m(R)$ for every quadrilateral $R \in \Sigma$. The smallest $K$ for a given quasiconformal map $f$ is denoted as the dilatation of $f$, $K(f)$.

A couple main properties are that for $f$ a $K-$quasiconformal map, then $f^{-1}$ is also $K-$quasiconformal, and for $f_1, f_2 \quad K_1-, K_2-$ quasiconformal maps, then and $f_2 \circ f_1$ is $K_1K_2-$quasiconformal.

**Definition 1.5.2.** $T = T(\Sigma)$ is defined as the space of all pairs $(f, \Sigma^*)$ of quasiconformal homeomorphisms $f : \Sigma \rightarrow \Sigma^*$ onto some Riemann surface $\Sigma^*$, quotient by the following equivalence relation

$$(f_0, \Sigma_0^*) \sim (f_1, \Sigma_1^*) \Leftrightarrow \exists c : \Sigma_0^* \rightarrow \Sigma_1^* \text{ conformal such that } c \circ f_0, f_1 \text{ are homotopic.}$$

We can put a metric topology in $T$, called the Teichmüller metric, defined as

$$d([f_0], [f_1]) = \frac{1}{2} \inf_{\tilde{f}_0 \in [f_0], \tilde{f}_1 \in [f_1]} \log(K(\tilde{f}_1 \circ \tilde{f}_0^{-1}))$$

where $K$ denotes the dilatation of a quasiconformal map.

If a quasiconformal map $f$ has continuous derivatives (or absolute continuous) we can define the following equivalent conditions of quasiconformality.

**Definition 1.5.3.** A orientation preserving homeomorphism $f : \Sigma \rightarrow \Sigma^*$ is $K-$ quasiconformal if

$$|f_\bar{z}| \leq k |f_z|$$

for $k = \frac{K-1}{K+1}$.

For a proof of the equivalence between both definition see [Ahl66] or [LV65]. We could also write (1.4) as

$$f_\bar{z}(z) = \mu(z)f_z(z)$$

for a $L^\infty$ function $\mu$ defined on coordinate charts. Such function is called a Beltrami differential. In order to define it precisely, take the covering map $\mathbb{H}^2 \rightarrow \mathbb{R}$ and denote by $\Gamma \subseteq \text{PSL}(2, \mathbb{R})$ the group of deck transformations. Then we can define:
Definition 1.5.4. $B = B(\Sigma)$ is the space of measurable $L^\infty$ functions $\mu$ defined in $\mathbb{H}^2$ such that
\[
\mu(A(z))A'(z) = \mu(z)A'(z), \forall A \in \Gamma.
\]
(1.6)

Similarly, we have the following definition for quadratic holomorphic differentials

Definition 1.5.5. $Q = Q(\Sigma)$ is the space of holomorphic functions $\phi$ in $\mathbb{H}^2$ such that
\[
\phi(A(z))A'(z)^2 = \phi(z), \forall A \in \Gamma
\]
(1.7)
and $\int\int_{\Sigma(\Gamma)} |\phi(z)|dx \wedge dy < \infty$, where $\Sigma(\Gamma)$ is any fundamental domain for $\Gamma$ in $\mathbb{H}^2$.

Observe that the hyperbolic metric on $\mathbb{H}^2$, $\frac{1}{y^2}dzd\bar{z}$, corresponds to $\rho(z) = \frac{1}{y^2}$ which has the property
\[
\rho(A(z))A'(z)A'(\bar{z}) = \rho(z), \forall A \in \text{PSL}(2, \mathbb{R}),
\]
and hence $\frac{\phi}{2\rho}$ satisfies (1.6). Because $\int\int_{R(\Gamma)} |\phi(z)|dx \wedge dy < \infty$, $\frac{\phi}{2\rho}$ is in $L^\infty$, and then we have a map $Q \to B, \phi \mapsto \frac{\phi}{2\rho}$. This cross section of Beltrami differentials defines a family of Beltrami differentials that are called harmonic Beltrami differentials.

Thanks to the theory of the Beltrami equation ($[Ahl66, AB60, LV65]$), we can find a solution of $f_\mu(z) = \mu(z)f_\mu(z)$ for $\|\mu\|_{\infty} < 1$ by a unique quasiconformal self-mapping $f^\mu$ of $\mathbb{H}^2$ that extends continuously to $\partial\mathbb{H}^2$, fixes 0, 1, $\infty$ and depends analytically on $\mu$ (f solves the Beltrami equation in the distributional sense). With this we have a map from the unit ball of $B$ to $T$, and by Teichmüller theory, the correspondence that sends $\phi \mapsto \frac{\phi}{2\rho}$ and then to the solution of the Beltrami equation for $\frac{\phi}{2\rho}$ defines a local homeomorphism between a neighbourhood of 0 in $Q$ to a neighbourhood of $T$.

Now, the solution to the Beltrami equation for $\mu = \frac{\phi}{2\rho}$, $(f^\mu)^*(\rho dzd\bar{z})$ is a local parametrization of hyperbolic metrics defined on the same space. We would like to compute the variation of this family of metrics at the origin. Because $f^\mu$ satisfies the Beltrami equation for $t\mu$, we have
\[
(f^\mu)^*(\rho dzd\bar{z})(z) = \rho(f^\mu(z))|f^\mu_z|^2dz + t\mu(z)dzd\bar{z}.
\]
(1.9)
Hence (here is implicit the analytic dependence of $f^\mu$ with respect to $\mu$)
\[
\frac{\partial}{\partial t} \bigg|_{t=0} (f^\mu)^*(\rho dzd\bar{z})(z) = \rho(z)\mu(z)dz^2 + \rho\mu(z)dz^2 + Edzd\bar{z},
\]
(1.10)
where $E$ groups the terms of the derivative that go together with $dzd\bar{z}$. We can then replace $\mu = \frac{\phi}{2\rho}$ to obtain
\[
\frac{\partial}{\partial t} \bigg|_{t=0} (f^\mu)^*(\rho dzd\bar{z})(z) = \text{Re}(\phi dz^2) + Edzd\bar{z}.
\]
(1.11)

Define $RQ$ as the space of real parts of quadratic holomorphic differentials. Since taking the real part defines an isomorphism $Q \to RQ$, we have a local homeomorphism from a neighbourhood of 0 in $RQ$ to a neighbourhood of $T$. Moreover, if we define $I_v$ to be the
hyperbolic metric \((f^\mu)^*(\rho dzd\bar{z})\), where \(\mu = \frac{\bar{z}}{z}\rho\) and \(\text{Re}(\phi dz^2) = v\), then (1.11) implies that at 0
\[
DI_v = v + Edzd\bar{z}.
\] (1.12)
Using the hyperbolic metric of \(R\) to define an inner product for tensors, \(DI_v\) projects orthogonally to \(v\) (because \(dz^2, d\bar{z}^2\) are pointwise orthogonal to \(dzd\bar{z}\)). In particular, for \(v, w \in RQ_c\) we have
\[
\langle DI_c(v), w \rangle = \langle v, w \rangle.
\] (1.13)
where \(\langle \cdot, \cdot \rangle\) is the mentioned inner product.

Observe finally that if we had chosen \(R\) with the same hyperbolic metric but with opposite orientation, the space \(RQ\) coincides with the one defined for the original Riemann surface structure.
Chapter 2

\( V_R \) for convex co-compact hyperbolic 3-manifolds

\[ \text{2.1 } W\text{-volume} \]

Given a convex co-compact hyperbolic 3-manifold \((M, g)\), Krasnov and Schlenker [KS08] defined its renormalized volume and calculated its first variation from the \( W\)-volume of a compact submanifold \( N \) ([KS08] Definition 3.1) as

\[
W(M, N) = V(N) - \frac{1}{4} \int_{\partial N} H da,
\]

where \( V \) denotes volume, \( H \) is the mean curvature and \( da \) is the area form of the induced metric.

Setting the notation used in [KS08], denote by \( I \) the metric induced on \( \partial N \), \( \bar{II} \) its second fundamental form (so \( \bar{II}(x, y) = I(x, By) \) and \( H = \frac{tr(B)}{2} \), where \( B \) is the shape operator) and \( \bar{III}(x, y) = I(Bx, By) \) its third fundamental form. Using a particular case of the volume variation formula calculated in [RS99], [KS08], Section 6] calculates the variation of the \( W\)-volume as

\[
\partial W(N) = \frac{1}{4} \int_{\partial N} \langle \partial \bar{II} - H \partial I, I \rangle da
\]

when the metric in \( N \) varies along hyperbolic metrics ([RS99] does the more general case when variation is along Einstein metrics).

If we further assume that \( N \) has convex boundary and that the normal exponential map (pointing towards the exterior of \( \partial N \)) defines a family of equidistant surfaces \( \{S_r\} \) that exhaust the complement of \( N \) \((S_0 = \partial N)\), then one can compute the \( W\)-volume of \( N_r \) (points on the interior of \( S_r \)) by using [2.2] as [Sch13 Lemma 3.6] to finding that

\[
\partial_r W(N_r) = \pi \chi(\partial N),
\]

so then
\[ W(N_r) = W(N) - \pi r \chi(\partial N). \] (2.4)

In this context, we will usually say that \((N_r)_r\) is an equidistant foliation.

We can then define the \(W\)-volume of an equidistant foliation as

\[ W(M, (N_r)_r) = W(M, N_r) + \pi r \chi(\partial M), \] (2.5)

which is well-defined (because of (2.4) this does not depend on \(r\)).

In order to give a definition of volume on \(M\) that does not depend on the equidistant foliation, one needs to make a canonically chosen among them. In order to do such thing, and as observed in [Sch13] Definition 3.2, Proposition 3.3, \(I^* = 4 \lim_{r \to \infty} e^{-2r} I_r\) (where \(I_r\) is the metric induced on \(S_r\), which is identified with \(S\) by the normal exponential map) it is a pointwise finite and lies in the conformal class of the boundary, albeit it could be only non-negative definite. The analogous re-scaled limits for \(II^*, III^*, B^*\) also exist and are denoted by \(II^*, III^*, B^*\). The reason to multiply by 4 is so \(I^* = g|_N\) in the case when \(N\) is a totally geodesic surface.

For the case of convex co-compact manifolds, any metric in the conformal class at infinity (given by the hyperbolic structure) can be obtained as the rescaled limit of the induced metrics of some explicit family of equidistant surfaces. Theorem 5.8 of [KS08] describes this by the use of Epstein surfaces (as established in [Eps84]) in the following way. For the upper-half space model of \(\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{\xi \geq 0}\) if the metric at infinity is given by \(h = e^\phi |dz|^2\) then the Epstein surface \(Eps_r = Eps_r(h)\) is given by the quotient of the disks \([Eps84]\):

\[ \xi = \frac{2\sqrt{2} e^{-r} e^{-\phi}}{2 + e^{-2r} e^{-\phi} |\phi|_z^2}, \] (2.6)

\[ \zeta = z + \phi z \frac{2e^{-2r} e^{-\phi}}{2 + e^{-2r} e^{-\phi} |\phi|_z^2}. \]

under the Kleinian group associated to \(M\). These surfaces are locally equidistant. Since \(M\) is convex-cocompact, all boundary components are compact, which allows us to conclude that, given \(h\) metric in the conformal class at infinity, \(Eps_r(h)\) are embedded for \(r\) large enough. We can define then the \(W\)-volume of \(h\) as the \(W\)-volume of the equidistant foliation \((N_r = \text{int}(Eps_r(h)))_r\)

\[ W(M, h) = W(M, (N_r)_r) = W(M, N_r) + \pi r \chi(\partial M), \] (2.7)

Then we only need to make a canonical choice among metrics in a given conformal class. Hence we can pick the constant curvature metric in that class and finally define the renormalized volume of \(M\) as

\[ V_R(M) = W(M, h_{\text{hyp}}), \] (2.8)
where $h_{\text{hyp}}$ is the metric in the conformal class at infinity that has constant curvature $-1$.

As an example, if $M$ has a totally geodesic convex core $C$, then the metric at infinity will be the metric at $\partial C$ which is constant curvature. Then

$$W(M, C) = V_R(M) = V_C(M). \tag{2.9}$$

It will be useful for later results to observe that $V_C(M)$ is half of the hyperbolic volume of the double of $M$, $DM$. This can be extended to non-hyperbolic manifolds as $tvp3\|DM\|$, where $v_3$ is the volume of the regular ideal tetrahedra and $\| \cdot \|$ is the simplicial volume of a manifold, defined as $\inf\{\sum_k |a_k| \mid \sum a_k \sigma_k \text{ represents the fundamental cycle } [M]\}$.

### 2.2 First variation formula

Krasnov-Schlenker [KS08] derived the variation formula of the $W$-volume in terms of the input at infinity (observe that because of the description by Epstein surfaces, $I^*$ determines $\mathbb{II}^*$ and $\mathbb{III}^*$) from the volume variation of Rivin-Schlenker [RS99]. Since for the rest of the paper we are going to write everything in terms of these limit tensors, let us omit the superscript $^*$. Then [KS08, Lemma 6.1] states

$$\partial W(N) = -\frac{1}{4} \int_{\partial N} \langle \partial \mathbb{II} - H \partial I, I \rangle da, \tag{2.10}$$

which can be equivalently written as ([KS08, Corollary 6.2])

$$\partial W(N) = -\frac{1}{4} \int_{\partial N} \partial H + \langle \partial I, \mathbb{II}_0 \rangle da, \tag{2.11}$$

where $\mathbb{II}_0 = \mathbb{II} - HI$ is the traceless second fundamental form at infinity.

This formula can be further reduce for $V_R$ since we are dealing with hyperbolic metrics. We do this by using the Gauss equation, or rather, its limit version at infinity. As [KS08, Remark 5.4] states, the limit of the Gauss equation gives the following equation

$$H = -\frac{K}{2}, \tag{2.12}$$

which means that the mean curvature at infinity is equal to minus the curvature of $I = I^*$. Hence the term $\partial H$ in (2.11) vanishes, since $-\frac{K}{2}$ is always equal to $\frac{1}{2}$. In order to write this reduced formula for the first variation of $V_R$, recall the notation done in section 1.5. Let us then fix $c \in T(\partial M)$ and some metric $I_c$ that represents it, so we can parametrize $T(\partial M)$ by $RQ_c$. Then for $v \in RQ_c$ we have the variation of $V_R$ at $I_c$ ([KS08, Corollary 6.2, Lemma 8.5])

$$DV_R(v) = -\frac{1}{4} \int_{\partial M} \langle DI_c(v), \mathbb{II}_0 \rangle da, \tag{2.13}$$

where the metric between tensors and the area form $da$ are defined from $I_c$ and $\mathbb{II}_0 = \mathbb{II} - \frac{1}{2}I$ is the traceless second fundamental form. This 2-form is (at each component of $\partial M$, after
taking quotient by the action of $\pi_1(M)$ the negative of the real part of the Schwarzian derivative of the holomorphic map between one component of the region of discontinuity and a disk ([KS08] Lemma 8.3). In particular (as we stated in (1.13) $\langle DL_c(v), I_0 \rangle = \langle v, I_0 \rangle$ pointwise. Then if we take $c$ to be a critical point (i.e. $DV_R(v) = 0$ at $I_c$ for every $v \in RQ_c$) $I_0$ must vanish at every point. This in turn implies that the holomorphic map between a component of the region of discontinuity and a disk has Schwarzian derivative identically zero, which means that the components are disks and the boundary of the convex core is totally geodesic.

2.3 On the maximality of the $W$-volume among metrics of constant area

In Section 7 of [KS08], Krasnov and Schlenker study the variation of the $W$-volume among metrics of the same conformal class while keeping the area constant. One way of showing this is by observing that the Ricci flow is a gradient-like flow for the $W$-Volume. We include this argument due to the connection to Ricci flow, since earlier proofs of this fact appeared in [GMR] (Proposition 7.1, which includes the cusped case) and previously in [GMS] (Proposition 3.11, convex co-compact case).

From [KS08] Corollary 6.2 we know that

$$\delta W = -\frac{1}{4} \int_{\partial M} \delta H + \langle \delta I, I_0 \rangle da,$$

where we omit the superscript $*$ and $H$ denotes the mean curvature at infinity.

Now, since we are taking a conformal variation, $\delta I = uI$ for some function $u : \partial M \to \mathbb{R}$. Moreover, since the volume is preserved, $\int_{\partial M} u da = 0$.

Remember that $I_0$ is the traceless part of the second fundamental form, so $\langle uI, I_0 \rangle = 0$ pointwise. Also ([KS08] Remark 5.4) $H = -K$ and hence we have

$$\delta W = \frac{1}{4} \int_{\partial M} \delta K da.$$  

But by the Gauss-Bonnet formula $\int_{\partial M} \delta K da + \int_{\partial M} K \delta(da) = 0$ and the equality $\delta(da) = \frac{1}{2} \langle \delta I, I \rangle da = u da$, we can reduce it to

$$\delta W = -\frac{1}{4} \int_{\partial M} K da,$$

from where we can recover that $K = \text{const.}$ is the unique critical point. If two points had different values of $K$, we can take $u$ supported around those points such that $\int_{\partial M} u da = 0$, but $-\int_{\partial M} K da > 0$.

Note that $u = -K + \frac{2\pi \chi(\partial M)}{\text{vol}(\partial M)}$ has integral equal to 0 ($\int_{\partial M} K da = 2\pi \chi(\partial M)$), and by Hölder inequality

$$\left( \int_{\partial M} K^2 da \right) \cdot \text{vol}(\partial M) \geq \left( \int_{\partial M} K da \right)^2,$$

19
giving \(-\frac{1}{4} \int_{\partial M} K\) \( \geq 0 \). Hence the \( W \)-volume is no decreasing under the Ricci flow in two dimensions. It is known (see, for example, [MSTI]) that this flow converges to the metric of constant curvature, proving that this metric is a global maximum. Since \( K = \text{const.} \) is the only critical point, the \( W \)-volume increases strictly under the flow, making this metric a strict global maximum.

2.4 Corrected \( V_R \)

Let \( M \) be a closed compact hyperbolic 3-manifold with an oriented incompressible surface \( \Sigma \) \((g > 2)\) that divides \( M \) into two components \( M_1, M_2 \). We can consider now hyperbolic structures \( N_1, N_2 \) on the interiors of \( M_1, M_2 \) such that we have the inclusions \( M_1 \subset N_1, M_2 \subset N_2 \) by taking coverings with respect to \( \pi_1(M_1), \pi_1(M_2) \), respectively. The fact that they glue along \( \Sigma \) tells us that the conformal classes at infinity of the open manifolds are each others skinning maps.

Now, for \( r \) sufficiently large, \( M_1 \) lives inside \( N_1^r \), the \( r \)-leaf of the foliation defined by the renormalized volume, and hence

\[
\text{vol}(M_1) = V_R(M_1, c_1) + r \pi \chi(\Sigma) - \text{vol}((\text{int}N_1^r \setminus M_1) + \frac{1}{4} \int_{N_1^r} H da, \quad (2.18)
\]

where \( c_1 \) is the conformal class at infinity for \( \text{int}(M_1) \).

We can take \( r \) even larger so that we have a similar statement for \( M_2 \). Observe that the regions \((\text{int}N_1^r \setminus M_1, (\text{int}N_2^r) \setminus M_2 \) glue along \( \Sigma \) and live inside the quasi-Fuchsian manifold obtained by gluing \( N_1 \setminus M_1, N_2 \setminus M_2 \) along \( \Sigma \). In particular the quasi-Fuchsian manifold \((N_1 \setminus M_1) \cup (N_2 \setminus M_2) \) has \((c_1, c_2)\) as a conformal class at infinity and \( N_1^r, N_2^r \) as the \( r \)-leaves for the renormalized volume, from where we conclude

\[
\text{vol}((\text{int}N_1^r \setminus M_1) \cup (\text{int}N_2^r) \setminus M_2) = V_R(\Sigma \times [0, 1], c_1, c_2) + \frac{1}{4} \int_{N_1^r} H da
\]

\[
+ \frac{1}{4} \int_{N_2^r} H da + 2r \pi \chi(\Sigma). \quad (2.19)
\]

This gives the following proposition.

**Proposition 2.4.1** ([VPc], Proposition 1). For \( M \) as above

\[
\text{vol}(M) = V_R(M_1, c_1) + V_R(M_2, c_2) - V_R(\Sigma \times [0, 1], c_1, c_2) \quad (2.20)
\]

In order to use this proposition to find a lower bound for the volume of \( M \) (and since \( c_1, c_2 \) are related by the skinning map), we want to show that the minimum of the corrected renormalized volume (defined as follows) is attained at the geodesic class.

**Definition 2.4.1** ([VPc], Definition 1). Let \( M \) be a compact acylindrical 3-manifold with hyperbolic interior, connected boundary of genus \( g > 1 \), and let \( c \in \mathcal{T}(\partial M) \) the element
that defines the conformal boundary at infinity. Then the corrected renormalized volume is defined as

\[ \widehat{V}_R(M) = V_R(M,c) - \frac{1}{2}V_R(\partial M \times [0,1],(c,\sigma(c))). \]  

(2.21)

For a simplified notation, we will refer to \( V_R(\partial M \times [0,1],(c,\sigma(c))) \) as \( V_R(\partial M) \)

Observe that Proposition 2.4.1 implies that, under cutting, the volume of \( M \) is equal to the sum of the corrected renormalized volumes of its parts. It is straightforward to extend this to the case when \( \Sigma \) has multiple components, and if \( M \) is open without cusps, we can replace \( \text{vol}(M) \) by \( \widehat{V}_R(M) \) in Proposition 2.4.1. Then if we consider the corrected renormalized volume to be an extension of the volume for closed hyperbolic manifolds, we obtain as a corollary of Proposition 2.4.1:

**Corollary 2.4.1** ([VPc], Corollary 2). \( \widehat{V}_R \) is additive under cutting.
Chapter 3

$V_R$ for geometrically finite hyperbolic 3-manifolds

3.1 Geodesic boundary defining functions

There is an alternative approach to $V_R$ defined in [GMR] that is really useful to prove continuity under limits, technique that we will adopt in Section 4.3. The equidistant foliation is interpreted by a function $\rho$ of $M$ (named geodesic boundary defining function) satisfying:

$$\left| \frac{d\rho}{\rho} \right|_g^2 = 1, \quad (\rho^2 g)_{\partial M} = \frac{h_{hyp}}{4} \quad (3.1)$$

near the boundary of $M$, where $h_{hyp}$ is the hyperbolic metric of $\partial M$ compatible with the conformal class at $\infty$. Note that $\rho$ needs to vanish at $\partial M$ for the second equation to be able to hold. Then the renormalized volume of $M$ can be calculated as the finite part at 0 of the meromorphic function $\int_M \rho^z d\text{vol}$, $z \in \mathbb{C}$, namely

$$V_R(M) = \text{FP}_{z=0} \int_M \rho^z d\text{vol}. \quad (3.2)$$

[GMR] also makes sense of this definition in the case where $(M, g)$ has parabolics.

To see equivalence between Definitions 2.8 and 3.2 for the convex co-compact case, [GMR] note that $\left| \frac{d\rho}{\rho} \right|_g^2 = 1$ is equivalent to the level sets of $\rho$ being equidistant. More precisely, since the gradient vector field $\nabla \rho$ (which is normal to the level sets of $\rho$) has norm equal to $\rho$, then $(N_r = \{\rho \geq e^{-r}\})_r$ is an equidistant foliation for $r$ large enough. On the other hand, $(\rho^2 g)_{\partial M} = \frac{h_{hyp}}{4}$ is equivalent to $I^* = h_{hyp}$ for the equidistant foliation $(N_r)_r$. Finally, [[GMR], page 9] establishes that for any compact set $K \subset M$

$$\text{FP}_{z=0} \int_M \rho^z d\text{vol} = \text{vol}(K) + \text{FP}_{z=0} \int_{M\setminus K} \rho^z d\text{vol}. \quad (3.3)$$

We can use 3.3 for $K = N_r = \{\rho \geq e^{-r}\}_r$ to have
\[
\text{FP}_{z=0} \int_M \rho^* \text{dvol} = \text{vol}(N_r) + \text{FP}_{z=0} \int_{M \setminus K} \rho^* \text{dvol} \\
= \text{vol}(N_r) + \text{FP}_{z=0} \int_{\infty}^{\infty} \int_{\partial N_s} e^{-sz} da_s ds
\]

(3.4)

where \( da_s \) is the volume form of the induced metric at \( \partial N_s \). Now by [KS08, Corollary 2.3]

\[
\int_{\partial N_s} da_s = \int_{\partial N_r} (\cosh^2(s - r) + \cosh(s - r) \sinh(s - r) H + \sinh^2(s - r) K_e) da_r
\]

(3.5)

where \( H \) and \( K_e \) correspond to half the trace (mean curvature) and determinant of the shape operator of \( \partial N_r \), respectively. We can then use (3.5) to simplify

\[
\int_{\infty}^{\infty} \int_{\partial N_s} e^{-sz} da_s ds = \int_{\infty}^{\infty} \int_{\partial N_r} \left( \cosh^2(s - r) + \frac{\sinh(2s - 2r)}{2} H + \sinh^2(s - r) K_e \right) da_r ds \\
= \int_{\infty}^{\infty} e^{-sz} \cosh^2(s - r) ds \int_{\partial N_r} da_r \\
+ \int_{\infty}^{\infty} e^{-sz} \frac{\sinh(2s - 2r)}{2} ds \int_{\partial N_r} H da_r \\
+ \int_{\infty}^{\infty} e^{-sz} \sinh^2(s - r) ds \int_{\partial N_r} K_e da_r \\
= \left( \frac{e^{-rz}}{4(z - 2)} + \frac{e^{-rz}}{4(z + 2)} - \frac{e^{-rz}}{2z} \right) \int_{\partial N_r} da_r \\
+ \left( \frac{e^{-rz}}{4(z - 2)} - \frac{e^{-rz}}{4(z + 2)} \right) \int_{\partial N_r} H da_r \\
+ \left( \frac{e^{-rz}}{4(z - 2)} + \frac{e^{-rz}}{4(z + 2)} + \frac{e^{-rz}}{2z} \right) \int_{\partial N_r} K_e da_r.
\]

(3.6)

where we have use that \( \int_{\infty}^{\infty} e^{-ks} ds = \frac{e^{-kr}}{k} \) for \( \text{Re}(k) > 0 \), so Equation (3.6) is for \( \text{Re}(z) > 2 \), which extends as a meromorphic function around 0 as the last equality of (3.6).

By replacing (3.6) in (3.4) and recalling that \( \text{FP}_{z=0}(\frac{e^{-rz}}{2z}) = -\frac{r}{2} \) we will have
\[ \text{FP}_{z=0} \int_M \rho^2 d\text{vol} = \text{vol}(N_r) + \left( -\frac{1}{8} + \frac{1}{8} + \frac{r}{2} \right) \int_{\partial N_r} da_r + \left( -\frac{1}{8} - \frac{1}{8} \right) \int_{\partial N_r} H da_r \\
+ \left( -\frac{1}{8} + \frac{1}{8} - \frac{r}{2} \right) \int_{\partial N_r} K_e da_r \\
= \text{vol}(N_r) - \frac{1}{4} \int_{\partial N_r} H da_r - \frac{r}{2} \int_{\partial N_r} (K_e - 1) da_r \\
= \text{vol}(N_r) - \frac{1}{4} \int_{\partial N_r} H da_r - \frac{r}{2} \int_{\partial N_r} K_g da_r \\
= \text{vol}(N_r) - \frac{1}{4} \int_{\partial N_r} H da_r - r\pi \chi(\partial M) = V_R(M) \]

where \( K_e - 1 = K_g \) is the Gauss equation (\( K_g \) denotes the induced metric on \( \partial N_r \)) and the last line uses Gauss-Bonnet and definition \[2.8\]. Moreover, by the same calculation, one can show that W-volume can be defined by \[3.2\] by taking \( 4(\rho^2 g)|_{\partial M} \) equal to our desired metric in the conformal class at infinity.

### 3.2 Epstein surfaces and finite volume cores

Following \[\text{[KS08, Theorem 5.8]}\] we have that, for the upper-half space model of \( \mathbb{H}^3 = \mathbb{C}_z \times \mathbb{R}_{\xi \geq 0} \), if the metric at infinity is given by \( e^\phi|dz|^2 \) then the Epstein surface \( \text{Eps}_r \) is given by \([\text{Eps84}]\):

\[
\xi = \frac{2\sqrt{2} e^{-r} e^{-\phi}}{2 + e^{-2r} e^{-\phi}|\phi_z|^2}, \\
\zeta = z + \phi_z \frac{2 e^{-2r} e^{-\phi}}{2 + e^{-2r} e^{-\phi}|\phi_z|^2}. \tag{3.8}
\]

Since for the convex co-compact case all boundary components are compact, one has that for \( r \) large enough \( \text{Eps}_r \) will be embedded. Moreover, after taking the quotient, it will bound a compact core of \( M \). One then notices that for geometrically finite manifolds a control over \( \phi \) would be required in each cusp, and the region \( \text{Eps}_r \) will bound in the quotient will not be compact but finite volume. Hence we are motivated to do the following definition.

**Definition 3.2.1.** A metric \( h \) in the conformal class at infinity of a geometrically finite hyperbolic 3-manifold \( M \) is said to be admissible if for each rank-1 cusp of \( M \) there are complex coordinates \( x + iy = z \) of \( \mathbb{C}_z = \partial \mathbb{H}^3 \) that positions the cusp at the origin and \( h \) can be written as \( \frac{1}{y^2} e^{2\psi}|dz|^2 \), where \( |\psi|, ||\psi||_{C^1}, y||\psi||_{C^2} \) are bounded near the origin, where \( ||\psi(z)||_{C^1} = \max_{|\beta|=i} \sup |D_\beta(\phi(z))| \) for \( \beta \) a multi-index.
Observe that if $\Psi$ is bounded away from 0, then the bounds for $||\psi||_{C^1}, y||\psi||_{C^2}$ are equivalent to the bounds for $||\Psi||_{C^1}, y||\Psi||_{C^2}$ for $e^{2\psi} = \Psi$. This follows from the formulas

$$D_\beta \Psi = 2\Psi \cdot D_\beta \psi, |\beta| = 1 \quad (3.9)$$

$$D_\beta \Psi = 2\Psi \cdot D_\beta \psi + \text{(lower order terms)}, |\beta| = 2. \quad (3.10)$$

Then we can show that metric is admissible if we show that the conformal factor $\Psi$ is bounded away from zero near the origin and we have the appropriate derivative bounds.

Examples of admissible metrics are metrics that are hyperbolic near a cusp. To see this, parametrize $\mathbb{C}$ such that the generator of the parabolic subgroup associated to the cusp is $z \mapsto z + 2\pi$ after we do the change of variables $z \mapsto \frac{1}{z}$. Then the map $z \mapsto e^{i\pi}$ is a local isometry (and covering map) from the cusp to the punctured disk ($\{0 < |z|^2 < \delta\}, \frac{|dz|^2}{|z|^2 \log^2 |z|}$) that send 0 to 0. By Uniformization, any other hyperbolic metric in ($\{0 < |z| < \delta\}$ is obtained by the pullback of a conformal map $f$ that sends 0 to 0. Since $z \mapsto \frac{1}{z}$ is an isometry of ($\{0 < |z| < \delta\}$, $\frac{|dz|^2}{|z|^2 \log^2 |z|}$) we can assume that $f$ is holomorphic. Then the singularity at 0 can be solved, and since $z \mapsto z^n$ is also an isometry of ($\{0 < |z| < \delta\}, \frac{|dz|^2}{|z|^2 \log^2 |z|}$), we can assume that $f'(z) \neq 0$. From this, we can write $f$ as $ze^g$, where $g$ is a holomorphic function in ($\{0 < |z| < \delta\}$.

The pullback metric $f^* \left( \frac{|dz|^2}{|z|^2 \log^2 |z|} \right)$ is equal to $\frac{|f'(z)|^2 |dz|^2}{|f(z)|^4 \log^2 |f(z)|}$, so the conformal factor with respect to $\frac{|dz|^2}{|z|^2 \log^2 |z|}$ is $\frac{|f'(z)|}{e^{2g}(|1 + \text{Re}(g(z))|)}$ (in ($\{0 < |z| < \delta\}$) coordinates).

Since $g$ is holomorphic at the origin, $f'(0) \neq 0$ and $\lim_{z \to 0} \log |z| = \infty$, the conformal factor $\frac{|f'(z)|}{e^{2g}(|1 + \text{Re}(g(z))|)}$ (in ($\{0 < |z| < \delta\}$) stays bounded away from 0 at the origin and it is a bounded function. Then it only remains to prove the $||.||_{C^1}, y||.||_{C^2}$ bounds. Thanks to the product rule, it is sufficient to prove these bounds for each multiplicative factor of $\frac{|f'(z)|}{e^{2g}(|1 + \text{Re}(g(z))|)}$.

And since $z \mapsto e^{i\pi}$ has bounded first and second order derivatives near the origin, chain rule gives us that bounded derivatives in ($\{0 < |z| < \delta\}$ coordinates imply bounded derivatives in $x + iy$ coordinates. Then we can clear the factors $|f'(z)|, \frac{1}{|e^{g}|}$ since they extend to the origin of ($\{0 < |z| < \delta\}$) as analytic functions. Then from the conformal factor $\frac{|f'(z)|}{e^{2g}(|1 + \text{Re}(g(z))|)}$ only rest to find the derivative bounds for $\frac{1}{(1 + \text{Re}(g(z)))^2}$, which is equivalent to finding the derivative bounds for $(1 + \frac{\text{Re}(g(z))}{\log |z|})^2$. As before, we can clear $\text{Re}(g(z))$, so we reduce the problem to $\frac{1}{\log |z|} = \frac{x^2 + y^2}{y}$. Since for the fundamental domain of the cusp we have that $x = O(y^2)$ as we approach the origin, then we have the desired bounds for $||\frac{x^2 + y^2}{y}||_{C^1}, y||\frac{x^2 + y^2}{y}||_{C^2}$.

Definition 3.2.1 can be thought as the set of metrics that are $C^2$-controlled by hyperbolic metrics (near the cusps). The main result of this subsection is to prove that $Eps_r$ of an admissible metric will be embedded and will bound a region of finite volume in $M$, for $r$ large enough. This will allows us to extend our definition of W-volume and $V_R$ for geometrically finite hyperbolic 3-manifolds.

25
Proposition 3.2.1. Let $M$ be a geometrically finite hyperbolic 3-manifold, and let $h$ be an admissible metric in the conformal class at infinity of $M$. Then the Epstein surfaces for $h$, $\text{Eps}_r$, are embedded for $r$ large enough. Moreover, their quotients in $M$ bound finite volume regions.

Proof. Since that $h$ is admissible, let us write the $z$-coordinate of the map 3.8 in terms of $r$, $y$ and $\psi$

$$f(z) = z + (-i + 2y\psi z) \frac{2e^{-2r}e^{-2\psi} y}{2 + e^{-2r}e^{-2\psi}|i + 2y\psi z|^2}.$$  \hspace{1cm} (3.11)

We can find the first derivatives of $f$ as

$$f_x(z) = 1 + 2y\partial_z\psi z \frac{2e^{-2r}e^{-2\psi} y}{2 + e^{-2r}e^{-2\psi}|i + 2y\psi z|^2} + (-i + 2y\psi) \frac{-4e^{-2r}\psi z e^{-2\psi} y}{2 + e^{-2r}e^{-2\psi}|i + 2\psi z|^2}$$

$$- (-i + 2y\psi z) \frac{\partial_z(e^{-2r}e^{-2\psi}|i + 2\psi z|^2) e^{-2r}e^{-2\psi} y}{(2 + e^{-2r}e^{-2\psi}|i + 2\psi z|^2)^2}.$$  \hspace{1cm} (3.12)

$$f_y(z) = 1 + (2\partial_y \psi z + 2\psi) \frac{2e^{-2r}e^{-2\psi} y}{2 + e^{-2r}e^{-2\psi}|i + 2\psi z|^2} + (-i + 2y\psi z) \frac{-4e^{-2r}\psi y e^{-2\psi} y + 2e^{-2r}e^{-2\psi}}{2 + e^{-2r}e^{-2\psi}|i + 2\psi z|^2}$$

$$- (-i + 2y\psi z) \frac{\partial_y(e^{-2r}e^{-2\psi}|i + 2\psi z|^2) e^{-2r}e^{-2\psi} y}{(2 + e^{-2r}e^{-2\psi}|i + 2\psi z|^2)^2}.$$  \hspace{1cm} (3.13)

Then we have the following estimatives

$$|f_x(z) - 1| \leq C_1\|\psi\|(y^2(\|\psi\|_C^1 + \|\psi\|_C^2) + y^2\|\psi\|_C^1\|\psi\|_C^2)$$

$$|f_y(z) - 1| \leq C_2\|\psi\|(y^2(\|\psi\|_C^1 + \|\psi\|_C^2) + y\|\psi\|_C^1\|\psi\|_C^2e^{-2r} + e^{-2r})$$  \hspace{1cm} (3.14)

were $C_{1,2,3}$ are constants that do not depend on $y, \psi$ (and we are considering $r > 0$). Considering then $y$ sufficiently small and $r$ sufficiently large, 3.14 implies that $|f_x(z) - 1|, |f_y(z) - 1| \leq \epsilon$.

Now for two points $z_0, z_1$ in the vicinity of the origin we have that, since a developing of a cusp neighbourhood is a disk in $\mathbb{C}$, we can join them by a segment $\gamma(t)_{t \in [0,1]}$ and apply the Fundamental Theorem of Calculus together with our bound for $df$

$$|f(z_2) - f(z_1)| = |\int_0^1 df(\gamma(t))\gamma'(t)dt| \geq (1 - \epsilon) \int_0^1 |\gamma'(t)|dt,$$  \hspace{1cm} (3.15)

concluding that under our conditions $\text{Eps}_r$ is an embedding near the cusps. Since the regions far from the cusps is a compact set, for also $r$ large enough $\text{Eps}_r$ will be embedded far from the cusps and the images will not intersect. Hence $\text{Eps}_r$ is an embedding for $r$ large enough.
To see that the region bounded has finite volume, observe that in \(3.8\) the vertical coordinate \(\xi\) is bounded below (near the cusp) by a uniform constant times \(y\|\psi\|_{C^2}\). Then the bound of the vertical component is linear in \(y\), which implies that our region is contained on a finite volume open set, all near the rank-1 cusps (see Figure 3.1). Since our region is compact outside this cusps neighbourhoods, then the volume will be finite.

Let us end this section verifying that for an incompressible hyperbolic pared manifold \((M, P)\) \(V_R\) and \(V_C\) are at uniformly apart, namely, the generalization of [BC, Theorem 1]

**Proposition 3.2.2.** Let \(M\) be a geometrically finite hyperbolic manifold and \(P\) is parabolic loci. Furthermore, assume that \((M, P)\) is boundary incompressible. Then

\[
V_C(M) - 9.185|\chi(\partial M)| \leq V_R(M) \leq V_C(M) \tag{3.16}
\]

**Proof.** The result is deduced as in [BC] from the following lemmas
Lemma 3.2.1 ([HMM05], Lemma 3.2). Let $\tau$ be the Thurston metric at infinity and $h_{\text{hyp}}$ the hyperbolic metric at infinity, also known as the Poincaré metric. Then

$$\frac{\tau}{2} \leq h_{\text{hyp}} \leq \tau$$ (3.17)

Lemma 3.2.2 ([Sch13], Proposition 3.12). Let $h, h'$ be two admissible non-positively curved metrics at infinity such that $h \leq h'$ and they have constant negative curvature for some neighbourhood of the cusps. Then $W(M, h) \leq W(M, h')$.

Lemma 3.2.3 ([Sch13], Lemma 4.1). For $\tau$ the Thurston metric at infinity. Then

$$W(M, \tau) = V_C(M) - \frac{1}{4}L(\beta_M)$$ (3.18)

where $\beta_M$ is the bending lamination of $C_M$ and $L(.)$ is its length.

Lemma 3.2.4 ([BC05], Theorem 3). If $M$ is an orientable hyperbolic 3-manifold with finitely generated, non-abelian fundamental group and $\partial M$ is incompressible in $M$, then

$$L(\beta_M) \leq \frac{\pi^3}{\sinh^{-1}(1)}|\chi(\partial M)|$$ (3.19)

It is well known that $\tau, h_{\text{hyp}}$ are non-positively curved and constant negative curvature near the cusps. Then by Lemma 3.2.1 and Lemma 3.2.2

$$W(M, \tau) + \ln(2)\pi \chi(\partial M) = W(M, \frac{\tau}{2}) \leq W(M, h_{\text{hyp}}) \leq W(M, \tau)$$ (3.20)

where we are using the linear property of $W(M, e^s h) = W(M, h) - s \pi \chi(\partial M)$. Then using Lemma 3.2.3 and Lemma 3.2.4 we have

$$V_C(M) - \left(\ln(2)\pi + \frac{\pi^3}{4\sinh^{-1}(1)}\right)|\chi(\partial M)| \leq V_R(M) \leq V_C(M),$$ (3.21)

which gives us the result.

Lemma 3.2.3 follows as done in [Sch13]. Lemmas 3.2.1 and 3.2.4 follow as in [HMM05] and [BC05] once we observe that the condition of $(M, P)$ being boundary incompressible means that the lift of every component of $\partial M \setminus P$ to the universal covering of $M$ is simply connected. Finally, for the proof of Lemma 3.2.2 once we write $h' = e^{2u}h$ with $u \geq 0$ we construct the family of metrics at infinity $h_t = e^{2ut}h$. Schlenker uses in [Sch13] the variational formula of the $W$-volume to show that $\partial_t W(M, h_t) \geq 0$ so then $W(M, h) \leq W(M, h')$. In our setup we need to verify that $h_t$ is still admissible. For that, recall that both $h, h'$ are admissible since they are hyperbolic near the cusp. Moreover, near the cusp they look like $h = \frac{2e^2|dz|^2}{y^2}$, $h' = \frac{2e'^2|dz|^2}{y^2}$, so the interpolation $h_t = e^{2ut}h$ is equal to $\frac{2e^{2t}|dz|^2}{y^2}$ near the origin for $\psi_t = (1-t)\psi + t\psi'$. Since $\psi_t$ is a convex interpolation, the admissibility of $h_t$ follows. $\square$
Chapter 4

Second variation of $V_R$ at critical points

In order to study local behavior around the critical points we want to compute the Hessian of $V_R$ at these points. Let $c$ be a critical point (i.e. $I_0 \equiv 0$) and $I_c$ a metric representing this class. For $v, w \in RQ_c$, we vary (6) with respect to $w$ to obtain

$$\text{Hess}V_R(v, w) = -\frac{1}{4} \int_{\partial M} \langle DI(v), D\Pi_0(w) \rangle da$$

(4.1)

and since $I_0$ vanishes identically, all the terms in parenthesis get canceled, so

$$\text{Hess}V_R(v, w) = -\frac{1}{4} \int_{\partial M} \langle DI(v), D\Pi_0(w) \rangle da.$$ (4.2)

Let us first understand the quasi-Fuchsian case (here we are referring to hyperbolic structures on the product $S \times \mathbb{R}$, where $S$ is a closed orientable surface of genus $g > 1$). In order to differentiate the two ends let us label them (as well as the tensors defined on each one) with + and −.

4.1 Fuchsian case

**Theorem 4.1.1** ([VPc], Theorem 1). Let $M$ be a Fuchsian manifold (i.e. the conformal classes at infinity $c_+, c_-$ both equal to say a conformal class $c$). Then the Hessian at $M$ of the renormalized volume is positive definite in the orthogonal complement of the diagonal of $RQ_c \times RQ_c \approx T_c \mathcal{T}(\partial M) = T_c^+ \mathcal{T}(\partial M^+) \times T_c^- \mathcal{T}(\partial M^-)$ (where we are using the parametrization by real parts of holomorphic quadratic differentials w.r.t. $(c, c)$ and that the real parts of holomorphic quadratic differentials are the same for both orientations). Moreover, the Hessian is equal to $\frac{1}{8}$ of the Weil-Petersson metric.
Proof. Recall by (4.2) that since $M$ is a critical point, then for $v = (v_+, v_-), w = (w_+, w_-)$ tangent vectors at $(c, c)$

$$4\text{Hess} V_R(v, w) = -\int_{\partial_+ M} \langle D I_+^+(v), D \Pi_0^+(w) \rangle da - \int_{\partial_- M} \langle D I_-^-(v), D \Pi_0^-(w) \rangle da. \quad (4.3)$$

As mentioned in [KS08] (Lemma 8.3) we know that $\Pi_0^+(c_+, c)$ is equal to $-\text{Re}(q_+(c_+, c))$, where $q_+(c_+, \cdot) : T_+ \to Q_{c_+}$ is the Bers embedding. In particular $D \Pi_0^+(v, 0)$ lands in $RQ_c$, and because $D \Pi_0^+(v, v) = 0$, then the whole image of $D \Pi_0^+$ lands in $RQ_c$. Since an analogous argument works for $D \Pi_0^-$, we can then reduce to

$$4\text{Hess} V_R(v, w) = -\int_{\partial_+ M} \langle v_+, D \Pi_0^+(w) \rangle da - \int_{\partial_- M} \langle v_-, D \Pi_0^-(w) \rangle da$$

$$= -\langle v, D \Pi_0(w) \rangle_{L^2}, \quad (4.4)$$

where $\langle \cdot, \cdot \rangle$ on forms denotes the $L^2$ scalar product defined by $I_c$.

Now $D \Pi_0$ is diagonalizable with orthogonal eigenvectors (is the expression of the Hessian in terms of the metric induced by $I(c, c)$). We can exploit this fact in the following lemma.

**Lemma 4.1.1**. $D \Pi_0(v, -v) = -\frac{1}{2}(v, -v)$

**Proof.** We prove first the following claim.

**Claim:** Let $M$ be a Fuchsian manifold with associated conformal class $c$. Then $D \Pi_0^+(v, 0)$ is orthogonal to $RQ_c$ for all $v \in RQ_c$.

Recall from [KS08] (as stated in the comments of Definition 5.3) that almost-Fuchsian manifolds are quasi-Fuchsian manifolds with the principal curvatures at infinity between $-1, 1$, and for those manifolds $\Pi$ is conformal to $I^-$, so $D \Pi_0^+(v, 0)$ is $I(c, c)$ multiplied by some function ($\Pi$ stays conformal to $I(c, c)$ when we only varied $c_+$). To see that those tensors are orthogonal to $RQ_c$ (note that for every tensor space we are taking the inner product induced by $I_c$ and integration), recall that if we take conformal coordinates, elements of $RQ_c$ are expressed in terms of $dz^2$ and $d\bar{z}^2$, where the metric is in terms of $dzd\bar{z}$, meaning that $D \Pi_0^+(v, 0)$ is even pointwise orthogonal to any element of $RQ_c$.

Going back to the proof of the lemma, it follows from [KS08] (Definition 5.3) that the first variations at $M$ satisfy

$$\delta \Pi_0 = \delta \Pi - \frac{1}{2} \delta I = I(\delta B \cdot, \cdot) \quad (4.5)$$

$$\delta \Pi_0 = \frac{1}{4} \delta I + I(\delta B \cdot, \cdot), \quad (4.6)$$

hence

$$D \Pi = D \Pi_0 + \frac{1}{4} D I. \quad (4.7)$$

In particular, by the claim, $D \Pi_0(v, 0) = -\frac{1}{2} v$, and so

$$D \Pi_0^+(v, -v) = D \Pi_0^+(2v, 0) + D \Pi_0^+(-v, -v) = -\frac{1}{4} D I^+(2v, 0) = -\frac{1}{2} v. \quad (4.8)$$
The lemma follows since $D\Pi_0$ is diagonalizable in the orthogonal complement of the diagonal (the diagonal is part of the 0-eigenspace).

This lemma implies that the orthogonal complement is the $-\frac{1}{2}$-eigenspace for $D\Pi_0$, concluding the last part of the theorem.

4.2 General case and strict local minimums for acylindrical manifolds

Now we will use this local behavior for quasi-Fuchsian manifolds to conclude our main result for acylindrical manifolds.

**Theorem 4.2.1** ([VPc], Theorem 2). Let $M$ be a compact acylindrical 3-manifold with hyperbolizable interior such that $\partial M \neq \emptyset$. Then there is a unique critical point $c$ for the renormalized volume of $M$, where $c$ is the unique conformal class at the boundary that makes every component of the region of discontinuity a disk (a.k.a. the geodesic class). The Hessian at this critical point is positive definite.

**Proof.** Since $DV_R(v) = -\frac{1}{4}\langle DI_c(v), \Pi_0\rangle = -\frac{1}{4}\langle v, \Pi_0\rangle$ we have

$$DV_R = 0 \iff \Pi_0 \equiv 0. \quad (4.9)$$

Now, $\Pi_0 \equiv 0$ at each boundary component corresponds to the nullity of the Schwarzian derivative between every component of the domain of discontinuity and a disk. This implies as we stated that every component of the region of discontinuity is a disk, and since our manifold is acylindrical, there is a unique such point in the Teichmüller space of the boundary. We can also observe that for this point the boundary of the convex core is completely geodesic.

To prove that it is a strict local minimum, it is sufficient to prove that the Hessian is positive definite at this point. Let $\partial M = S_1 \cup \ldots \cup S_n$, and $c = (c_1, \ldots, c_n)$ denote the geodesic class. To show that the Hessian is positive definite, take parametrization $RQ$ for $S_i$ based at $c_i$ and use the same metric to compute variations of $V_R$ for $M$ and for both ends of quasi-Fuchsian manifolds $S_i \times \mathbb{R}$.

Recall that, since $M$ is hyperbolic acylindrical, the subgroups associated to the components of $\partial M$ are quasifuchsian. Hence we have a map from $T(\partial M)$ (corresponding to hyperbolic metrics in $M$) to $T(\partial M) \times T(\partial M)$ (corresponding to the quasifuchsian subgroups). The first coordinate of this map is the identity on $T(\partial M)$, while the second coordinate $\sigma : T(\partial M) \rightarrow T(\partial M)$ is Thurston’s skinning map. Observe then that $I$ at $S_i$ coincides with $I^+(c_i, \sigma_i(c))$, where $\sigma_i$ is the image of the skinning map $\sigma$ corresponding to $S_i$. From the dependence of $\Pi$ in terms of $I$ (Remark 5.9 of [KS08]) we see that $\Pi_0$ at $S_i$ coincides with $\Pi_0^+(c_i, \sigma_i(c))$, and hence $DI(v), D\Pi_0(w)$ are equal to $D(I^+(v), d\sigma_i(v)), D(I_0^+(w), d\sigma_i(w))$ at $S_i$ (note that here $d\sigma$ is written in terms of our charts given by $RQ_c$, so it is essentially
the conjugation of the derivative of the skinning map, given our remark on how our charts
behave with an orientation change). Hence

\[
4\text{Hess}_{V_R}(v, w) = -\sum_{i=1}^{n} \langle v_i, D\Pi_0^+(w_i, d\sigma_i(w)) \rangle = \frac{1}{4} \sum_{i=1}^{n} \langle v_i, w_i - d\sigma_i(w) \rangle, \tag{4.10}
\]

which is greater than zero for \( v \neq 0 \) according to the following result:

**Theorem 4.2.2** (McMullen [McM90]). Under the conditions above, \( \|d\sigma\| < 1 \), where the norm \( \|d\sigma\| \) is calculated in terms of the Teichmüller metric.

Indeed, (4.10) tells us that \( d\sigma \) is diagonalizable, which together with McMullen’s theorem implies that all eigenvalues are less than 1. Then \( \text{Hess}_{V_R} \) is also diagonalizable with all eigenvalues positive.

McMullen’s result can be sketched as follows. The skinning map is holomorphic between Teichmüller spaces, in particular sending holomorphic disks to holomorphic disks. This implies that \( \|d\sigma\| \leq 1 \) with respect to the Kobayashi metric, which coincides with the Teichmüller metric. If \( \|d\sigma\| = 1 \) at a point, then \( \sigma \) would be an isometry on the extremal holomorphic disk, but an earlier result of Thurston states that \( \sigma \) is a strict contraction for the acylindrical case.

**Corollary 4.2.1.** Let \( c \in T(\partial M) \) be as in the previous theorem. Then \( d\sigma \) admits an orthonormal eigenbasis with respect to the \( L^2 \)-norm on \( RQ_c \), with all eigenvalues less than 1 in absolute value.

Observe that if we take holomorphic quadratic differentials for the tangent space, we have the conclusion for \( d\sigma \) after taking a complex conjugation.

This result was somehow expected thanks to a parallel between the skinning map and the Thurston map for postcritically finite rational maps, since the Thurston map is contracting for non-Lattè maps. Moreover, if \( f \) is a postcritically finite quadratic polynomial, there is a uniform spectral gap for the derivative at its unique critical point (more precisely, all eigenvalues are greater than \( 1/8 \) in norm. For this refer to [BEK]). It is unknown to the author if there is an analog for hyperbolic 3-manifolds, namely a family of manifolds for which there is some uniform espectral gap.

It is worth mentioning that it is still open that the geodesic class is the absolute minimum of \( V_R \) (for acylindrical manifolds). It is also open that \( V_R \) is strictly positive for quasi-Fuchsian manifolds outside the Fuchsian locus, although this is known for almost-Fuchsian manifolds [CM].
4.3 Local minima for Corrected $V_R$

Now, from the first variation of $V_R$ at $I = I^+$, we have

$$8D\nabla_R(v) = -\int_{\partial M^+} \langle DI^+(v, d\sigma(v)), \Pi_0^+(c, \sigma(c)) \rangle da^+ + \int_{\partial M^-} \langle DI^-(v, d\sigma(v)), \Pi_0^-(c, \sigma(c)) \rangle da^-,$$

(4.11)

where $da^+, da^-$ are the volume forms for $I^+, I^-$, respectively.

Observe that if we take $c$ to be the geodesic class, $\Pi_0^+, \Pi_0^-$ are both zero, and hence $c$ is also a critical point for the corrected renormalized volume. Moreover, the Hessian is expressed as

$$8\text{Hess}_R(v, w) = -\langle DI^+(v, d\sigma(v)), D\Pi_0^+(w, \sigma(w)) \rangle + \langle DI^-(v, d\sigma(v)), D\Pi_0^-(w, \sigma(w)) \rangle,$$

(4.12)

which by Lemma 4.1.1 is equal to

$$8\text{Hess}_R(v, w) = -\langle v, -\frac{1}{4}(w - d\sigma(w)) \rangle + \langle d\sigma(v), \frac{1}{4}(w - d\sigma(w)) \rangle.$$

(4.13)

Then

$$32\text{Hess}_R(v, w) = \langle v + d\sigma(v), w - d\sigma(w) \rangle,$$

(4.14)

and since all eigenvalues of $d\sigma$ are between $-1$ and $1$, we obtain the following result.

**Theorem 4.3.1** ([VPc], Theorem 6). *Let $M$ be a compact acylindrical 3-manifold with hyperboliable interior, $\partial M \neq \emptyset$ without cusps, and $c \in \mathcal{T}(\partial M)$ be the geodesic class. Then $c$ is a local minimum for the corrected renormalized volume of $M$.***
Chapter 5

Additive continuity

5.1 Additive continuity for additive geometric convergent sequences

Theorem 5.1.1 ([VPa], Theorem 6.1). Let $M$ be a geometrically finite hyperbolic manifold with $\partial M \neq \emptyset$ incompressible. Let $M_n \in QF(M)$ be a sequence such that $V_R(M_n)$ converges. Then we can select finite many base points such that (possibly after taking a subsequence) $N_1, \ldots, N_k$ are the additive geometric limit corresponding to the base points (in the sense of Proposition 1.2.1) and

$$\lim_{n \to \infty} V_R(M_n) = \sum_{i=1}^{k} V_R(N_i) \quad (5.1)$$

Proof. First, because of our discussion of ([BC], Theorem 1.1) for the geometrically finite case, we have that $V_C(M_n)$ is uniformly bounded. Then take into account the results of Proposition 1.2.1 as well as the definition for $M_n$. Then this theorem will follow from

$$\lim_{n \to \infty} \int_{M_n} \rho \, \text{dvol} = \int_{N_i} \rho \, \text{dvol} = V_R(M_n) \quad (5.2)$$

since we have the immediate additive formula for $V_R(M_n)$

$$\sum_{i=1}^{k} \int_{M_n} \rho \, \text{dvol} = \int_{M_n} \rho \, \text{dvol} = V_R(M_n). \quad (5.3)$$

In order to show (5.2) we will adapt [GMR]. The broad idea is to select neighbourhoods around the rank-1 cusps of $N_i$ and show the convergence of the integral at each cusp neighbourhood and on the complement of all of them. We will be using [GMR, Proposition 5.1] and [GMR, Corollary 5.3] in several steps of the proof, which are technical results for convergence of conformal factors after developing cusps in surfaces, which also applies in our situation. Let us examine then each possible scenario.
Near a rank-1 cusp obtained by pinching: Select the $\epsilon$ thin part of $C_{M_n}$ corresponding to the limit rank-1 cusp $C$. Then take the region obtained by normal exterior geodesics to $\partial C_{M_n}$ as the neighbourhood $U^C_n$ to analyze in $M_n$. Given the geometric convergence (i.e. select basepoints in $M_n$ for the possible geometric limit corresponding to $C$, giving all these sets parametrizations in $\mathbb{H}^3$), $U^C_n$ converges to $U^C$, the $\epsilon$ thin part corresponding to the rank-1 cusp. In order to calculate $V_R$ in these neighbourhoods, [GMR] parametrizes $U^C_n, U^C$ as follows.

Start considering the half upper-space model for $\mathbb{H}^3$. For a loxodromic transformation $\gamma$ with multiplier $e^{\ell(1+i\nu)}$ and fixed points $p, q \in \mathbb{C}$, consider the flow lines of $\gamma^t$ for $t \in \mathbb{R}$. Consider as well the hyperplanes between $p$ and $q$ that are image of half-spheres centered at 0 under a transformation that sends $\{0, \infty\}$ to $\{p, q\}$. Use the stereographic projection to parametrize the half-sphere of radius 1 at 0 (and hence also the corresponding hyperplane) by $\{z = v + iu, u > 0\}$. Since the flow lines identify the hyperplanes with one another, we can parametrize $\mathbb{H}^3$ by $(w, \zeta)$, where $w = \frac{1}{2}z\ell$, $\zeta = z\ell$ and $(0, z)$ is our first parametrized hyperplane. In these variables, $\gamma$ sends $(w, \zeta)$ to $(w + \frac{1}{2}, \zeta)$, $\{(w, i\ell), w \in \mathbb{R}\}$ parametrize the geodesic joining $p$ and $q$ and $\{-\frac{1}{2} \leq w \leq \frac{1}{2}, \zeta\}$ parametrizes the fundamental region for $\gamma$.

As in [GMR, Section 4] we have a neighbourhood of the Margulis tube in $N_t$ isometric to a neighbourhood of the manifold $(\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \mathbb{H}^2_{\zeta=v+iu}$ equipped with the metric

$$g_t = \frac{du^2 + dv^2 + ((1 + \nu^2)R^4 - 4\nu^2\ell^2u^2)dw^2 + 2\nu(R^2 - 2u^2)dwdv + 4\nu wvdudw}{u^2} \quad (5.4)$$

where $\exp^{\ell(1+i\nu)}$ is the multiplying factor of the geodesic of $M_n$ converging to a parabolic, and $R := \sqrt{u^2 + v^2 + \ell^2}$. These neighbourhoods are intersections with sufficiently (but uniformly) small half hemispheres in $\mathbb{H}^3$ and then take quotient.

The limit model at $N_t$ is the manifold $(\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \mathbb{H}^2_{\zeta=v+iu, g_L}$ equipped with the metric

$$g = \frac{du^2 + dv^2 + (1 + \nu^2)(u^2 + v^2)^2 dw^2 + 2\nu(v^2 - u^2)dwdv + 4\nu wvdudw}{u^2} \quad (5.5)$$

Near a rank-1 cusp obtained by cutting cylinders: Similarly as before, we have an isometry with a neighbourhood of $((\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \mathbb{H}^2_{\zeta=v+iu, g_L})$, except that in this case we also need to cut along the corresponding cylinder $C$. Since again we have the convergence of $U^C_t$ to $U^C$, the region carved out by the cylinder $C$ is contained in smaller and smaller balls around $u = v = 0$. Let us redefine then the cylinders $C$ such that the new cutting cylinders have a friendly description in $(w, v + iu)$ coordinates. For a cutting cylinder coming from a loxodromic core and a fixed $w$, take the lines (hyperbolic lines in $\mathbb{H}^2$) joining $i\ell$ with $a_\ell$ and $i\ell$ with $b_\ell$, denoted by $i\ell, a_\ell$ and $i\ell, b_\ell$ where $b_\ell < a_\ell$. In order to choose $a_\ell, b_\ell$, observe that since all the region carved out by $C$ collapses at $u = v = 0$, we can make a choice of $a_\ell, b_\ell$ for a given side of $C$. If we now see the other geometric limit adjacent to $C$ at the side of $a_\ell$, the boundary of the new cylinder is also of the type $\{-\frac{1}{4} \leq w \leq \frac{1}{4}\} \times i\ell, a_\ell$ in the coordinates of these geometric limit. This follows from the definition of the coordinates in terms of the loxodromic transformation $\gamma$. Then, we had selected one of $a_\ell, b_\ell$ for the
adjacent geometric limit, so we make an arbitrary choice for the remaining of $a_\ell, b_\ell$ and then move to the next adjacent geometric limit. The process ends when, after moving cyclically around the geodesic corresponding to $\gamma$, we arrive to the final possible geometric limit.

Then in this case $U^C_n$ is the neighbourhood of $u = v = 0$ described in the pinching case minus $C_\ell := \{(w, v + iu)| -\frac{1}{4} \leq w \leq \frac{1}{4}, \zeta \in \Delta_\ell\}$, where $\Delta_\ell$ is the region between $i\ell, a_\ell$ and $i\ell, b_\ell$.

The picture one should have in mind is Figure 5.1. In case that the cutting cylinder has a cusp core of either rank, since the cylinder goes into the cusp then the point $i\ell$ will be located in the sphere at infinity, and hence we can take it to be the origin. Hence the picture should be Figure 5.2.

The neighbourhood (prequotient) is the set between the halfspace previously mentioned and the pseudo-hyperplane obtained by lifting the cylinder $C$ to $\mathbb{H}^3$. This pseudo-hemisphere collapses to the parabolic fix point as $n \to \infty$, so the limit model for $N_i$ is the same as in the previous case 5.5.

**Limit far from the rank-1 cusps:** In this case the limit follows from showing convergence in compacts subsets of some geodesic boundary defining function $\rho_n$ of $M_n$ to the the geodesic boundary defining function $\rho$ of $N_i$. Say then that $\mathcal{K}$ is the complement of the rank-1 cusp neighbourhoods. Then $\rho$ is the solution of the Hamilton-Jacobi equation

$$\left| \frac{d\rho}{\rho} \right|^2_g = 1, \quad (\rho^2 g)|_{\partial M} = h_{hyp}. \quad (5.6)$$

Then we define an auxiliar geodesic boundary function $\hat{\rho}_n$ as the solution of

$$\left| \frac{d\hat{\rho}_n}{\hat{\rho}_n} \right|^2_{g_n} = 1, \quad \hat{\omega}_n|_{\rho=0} = 0 \quad (5.7)$$
where $\tilde{\rho}_n = e^{\hat{\omega}_n} \rho$.

The result follows as done in [GMR], so we will just cite their work. The main difference is that we have to apply to each component $M_i^n$ since our case doesn’t have to be connected. Next is [GMR, Lemma 6.3]

**Lemma 5.1.1.** There exists $\delta > 0$ such that for sufficiently large $n$, the Hamilton-Jacobi equation (5.7) has a solution $\tilde{\omega}_n$ in $\mathcal{K} \cap \{ \rho < \delta \}$ and $\tilde{\omega}_n$ converges to $0$ in $\mathcal{C}^k$-norms there for all $k$.

The relationship between $\tilde{\rho}_n$ and $\rho_n$ is

$$\rho_n = e^{\omega_n} \tilde{\rho}_n$$ (5.8)

where $\omega_n$ is the solution of

$$\left| \frac{d\rho_n}{\rho_n} \right|^2 |_{g_n} = 1, \quad \omega_n |_{\rho=0} = \varphi_n$$

and $\varphi_n$ is the uniformization factor such that $h_n^{\text{hyp}} := e^{2\varphi_n} h_n$ is hyperbolic if $h_n := (\rho^2 g_n) |_{\rho=0}$; The Hamilton-Jacobi equation (5.8) has a unique solution in $\mathcal{K}$ near $M$ and in particular one has $\omega|_{\mathcal{K} \cap M} = \varphi = 0$.

Then this result is used in order to show continuity far from rank-1 cusps.

**Proposition 5.1.1.** [GMR, Proposition 8.1] Let $\rho_n \in \mathcal{C}^\infty(M_n)$ be a geodesic boundary defining function such that $h_n := (\rho^2 g_n)|_{\partial M_n}$ is the unique hyperbolic metric in the conformal boundary ($\rho_n$ is uniquely defined near $\partial M$). Let $\rho \in \mathcal{C}^\infty(N_i)$ be a geodesic boundary defining function of $N_i$ with $h := (\rho^2 g)|_{\partial N_i}$ being the unique finite volume hyperbolic metric in the
conformal boundary ($\rho$ is uniquely defined near $N_i$). Let $\theta^i_n$ be a family of smooth functions on $M^i_n$ with support in $M^i_n$ and converging in all $C^k$-norms to $\theta_i$, a function in $N_i$ that vanishes in a neighborhood of the rank-1 cusps. The following limit holds

$$\lim_{n \to \infty} \left( \text{FP}_{z=0} \int_{M^i_n} \theta^i_n \rho^z_n \text{dvol}_{g_n} \right) = \text{FP}_{z=0} \int_{N_i} \theta_i \rho^z \text{dvol}_g.$$

We next study the behaviour of the renormalized volume in the regions containing the degeneration to rank 1-cusps. We notice that the main Theorem 5.1.1 follows from Proposition 5.1.1 and the following

**Proposition 5.1.2.** With the notations and assumptions of Proposition 5.1.1 and Theorem 5.1.1, we have

$$\lim_{n \to \infty} \text{FP}_{z=0} \int_{M^i_n} (1 - \theta^i_n) \rho^z_n \text{dvol}_{g_n} = \text{FP}_{z=0} \int_{N_i} (1 - \theta_i) \rho^0 \text{dvol}_{g_0}.$$

**Proof.** Let us start by assuming the Margulis tube forming the rank-1 cusp has a loxodromic core. We can further assume that $(1 - \theta^i_n)$ is supported in the region $U^C$ for each rank-1 cusp $C$, then we can assume that we have the parametrization $(w, \zeta = v + iu)$ of (5.4), where we have forgot the $n$ parameter and use rather $\ell$ with $\ell \to 0$, and $\nu = \nu(\ell)$ is converging to some limit $\nu_0$ as $\ell \to 0$. Then we are going to show equality at each rank-1 cusp appearing at $N_i$ from either a pinching curve or a cutting cylinder from $M^i_n$. The rest of the proof follows [GMR, Proposition 8.2] for both pinching and cutting cylinder.

First, using 5.4, we can calculate $\text{dvol}_{g_\ell}$ as

$$\text{dvol}_{g_\ell} = \frac{R^2 du dv dw}{u^3},$$

where $R^2 = u^2 + v^2 + \ell^2$. Then the results follows from showing

$$\lim_{\ell \to 0} \text{FP}_{z=0} \int_{(u,v,w,\ell) \in U_\ell} \rho^\ell \chi_\ell \frac{R^2 du dv dw}{u^3} = \text{FP}_{z=0} \int \rho^0 \chi_0 \frac{R^2 du dv dw}{u^3}$$

(5.9)

for the pinching case, and

$$\lim_{\ell \to 0} \text{FP}_{z=0} \int_{(u,v,w,\ell) \in U_\ell \setminus C_\ell} \rho^\ell \chi_\ell \frac{R^2 du dv dw}{u^3} = \text{FP}_{z=0} \int \rho^0 \chi_0 \frac{R^2 du dv dw}{u^3}$$

(5.10)

for the cutting cylinder case. In both limits $\rho_\ell = \rho_n$ is the function solving

$$\left| \frac{d \rho_\ell}{\rho_\ell} \right|_{g_\ell} = 1, \quad \rho_\ell = e^{\omega_\ell} U$$

for some $\omega_\ell$ satisfying $(\omega_\ell)_{U=0} = \varphi_\ell$ (5.11)

with $e^{2\varphi_\ell} h_\ell$ being hyperbolic if $h_\ell$ is given by $h_\ell = \frac{\partial g}{\partial z}$ (here we are also introducing the notation $U = \frac{\partial}{\partial R}$), and $\chi_\ell \in C^\infty_c (U_\ell)$ is equal to 1 near $u = v = 0$ and converges to
Figure 5.3: $w$-slice of $R_1(\ell), R_2(\ell), R_3(\ell)$

$\chi_0$. Note that for the convergence in (5.9), (5.10), we can choose $\chi$ independent to $\ell$ (as in [GMR]). We will study the convergence of (5.10), since the proof of (5.9) appears in [GMR]. Nevertheless, the limits to be calculated are analogous to one another, so the reader can follow both arguments simultaneously.

Let us start by dividing the integral (5.10) as the sets $R_1(\ell), R_2(\ell), R_3(\ell)$ defined by:

$R_1(\ell) = \{(w, \zeta = v + iu) \in U_\ell^C \mid u \geq |v|, \ u^2 + v^2 \geq \ell^2\}$  \hspace{1cm} (5.12)

$R_2(\ell) = \{(w, \zeta = v + iu) \in U_\ell^C \mid u^2 + v^2 \leq \ell^2\}$  \hspace{1cm} (5.13)

$R_3(\ell) = \{(w, \zeta = v + iu) \in U_\ell^C \mid |v| \geq u, \ u^2 + v^2 \geq \ell^2\}$.  \hspace{1cm} (5.14)

Compare with the proof of [GMR, Proposition 8.2] for the notation (see Figure 5.3).

We can write the $R_1(\ell)$-term as

$R_1(\ell) = \{(w, r, \theta) \mid \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}, \ \ell \leq r \leq 2\delta, \ -\frac{1}{4} \leq w \leq \frac{1}{4}\}$

where we use the following coordinates,

$u = r \sin \theta, \quad v = r \cos \theta, \quad w$.  \hspace{1cm} (5.15)

Restricted to this region, $\int \chi \frac{R^2 u^2 v^2}{u^3} = \int \chi \frac{1+(\ell^2/r^2)d\theta dr dw}{\sin^3 \theta}$ both integrals are finite and there is no need to renormalize. Thus,

$FP_{z=0} \int_{R_1(\ell) \setminus C_\ell} \rho^* \chi \frac{R^2 u^2 v^2}{u^3} = \int_{R_1(\ell) \setminus C_\ell} \chi(r \sin \theta, r \cos \theta, w) \left(1 + \frac{\ell^2}{r^2}\right) \sin^{-3} \theta d\theta dr dw$

where $R_1(\ell) = \{(w, V, u) \mid -\frac{1}{4} \leq w \leq \frac{1}{4}, -1 \leq V \leq 1, \ell \leq u \leq \delta\}$. We can use dominated
convergence (recall that $\frac{\ell}{r^2} \leq 1$) to deduce that

$$
\lim_{\ell \to 0} \text{FP}_{z=0} \int_{(w,r,\theta) \in R_1(\ell) \setminus C_\ell} \rho_\ell^z \chi \frac{1 + (\ell^2/r^2)d\theta dr dw}{\sin^3 \theta} = \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell}^{2\delta} \int_{\frac{\pi}{2}}^\frac{2\delta}{2} \chi(r \sin \theta, r \cos \theta, w) \frac{d\theta dr dw}{\sin^3 \theta} = \text{FP}_{z=0} \int_{R_1(0)} \rho_0^z \chi d\theta dr dw.
$$

(5.16)

Next, let’s look to the region $R_2(\ell)$

$$
R_2(\ell) := \left\{ (w,v,u) \mid 0 \leq u, \ u^2 + v^2 \leq \ell^2, \ -\frac{1}{4} \leq w \leq \frac{1}{4} \right\}
$$

(5.17)

Define then the change of coordinates:

$$
u = \frac{\ell \sin \theta (\ell^2 + \hat{v}^2)}{\cos \theta (\ell^2 - \hat{v}^2) + \ell^2 + \hat{v}^2}, \quad v = \frac{2 \cos \theta \ell^2 \hat{v}}{\cos \theta (\ell^2 - \hat{v}^2) + \ell^2 + \hat{v}^2}
$$

(5.18)

A geometric interpretation of these new variables is to parametrize $\mathbb{H}^2_{v+iu}$ by the geodesics from $i\ell$ to points $\hat{v} \in \mathbb{R} \subset \mathbb{H}^2$, where for any of such geodesics, $\cos \theta$ is the euclidean distance from the origen, after we have identified $\mathbb{H}^2$ with the unit disk ($i\ell \leftrightarrow 0$). Hence

$$
\mathbb{H}^2 = \{ v + iu \mid 0 \leq u \} = \{ (\hat{v}, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2} \},
$$

(5.19)

where $i\ell$ is identified with the line $\mathbb{R} \times \{\frac{\pi}{2}\}$, $\mathbb{R} \times 0$ is identified with itself by the identity, and the geodesic from $i\ell$ to $\hat{v}$ is identified with the line $\{\hat{v}\} \times [0, \frac{\pi}{2}]$. Also, the Jacobian can be easily calculated as

40
\[ \frac{\partial (v, u)}{\partial (\hat{v}, \hat{\theta})} = \frac{2 \cos \theta \ell^2 (\ell^2 + \hat{v}^2)}{(\cos \theta (\ell^2 - \hat{v}^2) + \ell^2 + \hat{v}^2)^2} \]  \tag{5.20}

In these new coordinates, the regions \( R_2(\ell) \) and \( C_\ell \) are defined by

\[
R_2(\ell) := \left\{ (w, \hat{v}, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, |\hat{v}| \leq \ell, -\frac{1}{4} \leq w \leq \frac{1}{4} \right\} \tag{5.21}
\]

\[
C_\ell := \left\{ (w, \hat{v}, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, b_\ell \leq \hat{v} \leq a_\ell, -\frac{1}{4} \leq w \leq \frac{1}{4} \right\}. \tag{5.22}
\]

The two limits we are looking at are:

\[
\lim_{\ell \to 0} \int_{R_2(\ell) \setminus C_\ell} \rho^z_{\ell, \chi} \frac{R^2 du dv dw}{u^3} = \lim_{\ell \to 0} \int_{R_3(\ell) \setminus C_\ell} \rho^z_{\ell, \chi} \frac{2 \ell^2 \cos \theta d\theta d\hat{v} dw}{\ell^2 + \hat{v}^2} = 0. \tag{5.23}
\]

\[
\lim_{\ell \to 0} \int_{R_3(\ell) \setminus C_\ell} \rho^z_{\ell, \chi} \frac{R^2 du dv dw}{u^3} = \lim_{\ell \to 0} \int_{R_3(\ell) \setminus C_\ell} \rho^z_{\ell, \chi} \frac{2 \ell^2 \cos \theta d\theta d\hat{v} dw}{\ell^2 + \hat{v}^2} = \int_{R_3(0)} \rho^z_0 \chi \frac{2 \cos \theta d\theta \hat{v} dw}{1 + \hat{v}^2}. \tag{5.24}
\]

Observe then that the following statements are sufficient to prove our result

\[
\lim_{\ell \to 0} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\alpha_\ell}^{\beta_\ell} \rho^z_{\ell, \chi} \frac{2 \ell^2 \cos \theta d\theta d\hat{v} dw}{\ell^2 + \hat{v}^2} = 0. \tag{5.25}
\]
for sequences \( |\alpha_\ell|, |\beta_\ell| \leq \ell \leq \kappa_\ell \) all with limit 0.

Rescale \( \hat{V} = \frac{\hat{v}}{\ell} \) so now:

\[
\text{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\ell^2 \cos \theta d\theta d\hat{v} d\hat{w}}{\ell^2 + \hat{v}^2} = \text{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\alpha_\ell/\ell}^{\delta_\ell/\ell} \int_{0}^{\frac{\pi}{2}} \rho_{\ell}^2 \chi \frac{2\ell \cos \theta d\theta d\hat{V} dw}{1 + \hat{V}^2}.
\]

As with \( R_1(\ell) \), the integral is finite and dominated by

\[
\text{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^{1} \int_{0}^{\frac{\pi}{2}} \frac{2\ell \cos \theta d\theta d\hat{V} dw}{1 + \hat{V}^2} = \frac{\pi \ell}{2},
\]

so (5.25) follows.

Finally, let us deal with the region \( R_3(\ell) \). Similarly to the previous coordinates, we will parametrize \( R_3(\ell) \) by geodesics joining \( \hat{v} \) and \( i\ell \). Hence the coordinates \((\theta, \hat{v})\) are defined by

\[
v = \frac{\hat{v}}{2} - \frac{\ell^2}{2\hat{v}} + \cos \theta \left( \frac{\hat{v}}{2} + \frac{\ell^2}{2\hat{v}} \right), \quad u = \sin \theta \left( \frac{\hat{v}}{2} + \frac{\ell^2}{2\hat{v}} \right)
\]

Here, \( \theta \) is the counterclockwise angle on the half-circle joining \( \hat{v} \) and \( i\ell \), as represented in Figure 5.6.

The Jacobian of the change of variables can be calculated by

\[
\frac{\partial (v, u)}{\partial (\hat{v}, \theta)} = \frac{\hat{v}}{4} \left( 1 + \frac{\ell^2}{\hat{v}^2} \right) \left( 1 - \frac{\ell^2}{\hat{v}^2} + \cos \theta \left( 1 + \frac{\ell^2}{\hat{v}^2} \right) \right),
\]

and the representation of \( R_3(\ell) \) as
\( R_2(\ell) = \{(w, \bar{v}, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, \ell \leq |\bar{v}| \leq 2\delta, \quad -\frac{1}{4} \leq w \leq \frac{1}{4}\}. \)

Here there is actually an extra region already accounted by \( R_1(\ell) \). Nevertheless, the result still follows while the notation is more simple.

Similar to the reasoning of (5.25), the desired limit will be

\[
\lim_{\ell \to 0} \text{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\kappa_{\ell}}^{2\delta} \int_{0}^{\frac{\ell}{2}} \rho \chi \left( 1 + \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 - \frac{\ell^2}{|z|^2} \right) \right) \left( 1 - \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 + \frac{\ell^2}{|z|^2} \right) \right) \, d\theta d\bar{v} dw \\
\frac{\sin^3 \theta \left( 1 + \frac{\ell^2}{|z|^2} \right)}{\sin^3 \theta \left( 1 + \frac{\ell^2}{|z|^2} \right)},
\]

for some sequence \( \kappa_{\ell} \geq \ell \) with limit equal to 0.

With the notation of (5.11), \( \rho \ell = e^{\omega_{\ell} \pi \frac{a}{R}} = e^{\omega_{\ell} \sin \theta \sqrt{\frac{(1 + \frac{\ell^2}{|z|^2})}{2(1 + \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 - \frac{\ell^2}{|z|^2} \right))}}}. \) And since \( \ell \leq \bar{v} \), then \( \left( \frac{(1 + \frac{\ell^2}{|z|^2})}{2(1 + \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 - \frac{\ell^2}{|z|^2} \right))} \right) \leq \frac{1}{2} \) and \( \omega_{\ell} \) has the expansion with respect to \( \theta \) (as proved in [GMR, Proposition 6.7]):

\[
\omega_{\ell} = a_0 + a_2 \left( \frac{a}{R} \right)^2 + O \left( \left( \frac{a}{R} \right)^3 \right) = a_0 + a_2 \sin^2 \theta \left( \frac{1 + \frac{\ell^2}{|z|^2}}{2 \left( 1 + \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 - \frac{\ell^2}{|z|^2} \right) \right)} \right) + O (\theta^3),
\]

where \( a_0, a_2 \) depend on \( \ell, \bar{v} \) and \( w \) but not on \( \theta \).

Then the finite part of (5.26) can be decomposed as \( I_1(\ell) + I_2(\ell) \), where:

\[
I_1(\ell) := \text{FP}_{z=0} \int_{R_3(\ell) \setminus C_{\ell}} \chi \sin^2 \theta \left( 1 + \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 - \frac{\ell^2}{|z|^2} \right) \right) \left( 1 - \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 + \frac{\ell^2}{|z|^2} \right) \right) \, d\theta d\bar{v} dw \\
\frac{\sin^3 \theta \left( 1 + \frac{\ell^2}{|z|^2} \right)}{\sin^3 \theta \left( 1 + \frac{\ell^2}{|z|^2} \right)}
\]

\[
I_2(\ell) := \text{res}_{\frac{1}{4}} \int_{R_3(\ell) \setminus C_{\ell}} \chi \omega_{\ell} \sin^2 \theta \left( 1 + \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 - \frac{\ell^2}{|z|^2} \right) \right) \left( 1 - \frac{\ell^2}{|z|^2} + \cos \theta \left( 1 + \frac{\ell^2}{|z|^2} \right) \right) \, d\theta d\bar{v} dw \\
\frac{\sin^3 \theta \left( 1 + \frac{\ell^2}{|z|^2} \right)}{\sin^3 \theta \left( 1 + \frac{\ell^2}{|z|^2} \right)}
\]

Similar to the parallel case in the proof of [GMR, Prop. 8.2], we can observe that for \( I_1(\ell) \)

\[
I_1(\ell) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\kappa_{\ell}}^{2\delta} q_1(\bar{v}, \ell, w) d\bar{v} dw
\]

43
for some smooth function $q_1$ independent from $\ell$. Then we can easily see that $\lim_{\ell \to 0} I_1(\ell) = I_1(0)$.

For $I_2(\ell)$, we can simplify by replacing $\omega_\ell$ with the first two terms of (5.27). Then

$$I_2(\ell) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\kappa_\ell}^{2\delta} a_0q_2(\tilde{v}, \frac{\ell}{\tilde{v}}, w) + a_2q_3(\tilde{v}, \frac{\ell}{\tilde{v}}, w)d\tilde{v}dw,$$

for some smooth functions $q_2, q_3$ independent from $\ell$, and $a_0, a_2$ given by (from [GMR, Proposition 6.7])

$$a_0(w, \tilde{v}) = \phi_\ell(w, \tilde{v})$$

$$a_2(w, \tilde{v}) = -\frac{1}{4} \frac{|d\phi_\ell|^2}{|\tilde{v}|^2} + C_1 \frac{\ell^2 + C_2 v \partial_\ell \phi_\ell}{(\ell^2 + v^2)} + C_3 v \partial_v \phi_\ell + \frac{1}{2},$$

where $C_1, C_2, C_3$ are constants smoothly depending on $\nu$. Then using [GMR, Proposition 5.1], [GMR, Corollary 5.3] for the integral convergence of $\phi_\ell, d\phi_\ell$ to $\phi_0, d\phi_0$, we can see that $\lim_{\ell \to 0} I_2(\ell) = I_2(0)$. To see that this concludes all the cases, see that the integral with limits $-2\delta \leq \tilde{v} \leq -\kappa_\ell$ follows by analogy and the integral with limits $\alpha_\ell \leq \tilde{v} \leq \beta_\ell$ converges to 0 for $\alpha_\ell, \beta_\ell \to 0$ either both greater than $\ell$ or smaller than $-\ell$. Then the proof of Proposition 5.1.2 is finished for a loxodromic core.

If the core was a cusp and we were analyzing the result for a cutting cylinder, we will need to analyze Figure 5.2. Note that in this case all metrics $g_n$ are equal to $g_0$, so we will have similar statements for $\rho_n, \rho_0$ as we had for $\rho_\ell, \rho_0$. This will give us the appropriate convergence when we decompose Figure 5.2 similar to the decomposition of Figure 5.3.

\[\square\] 5.2 Global minima of $V_R$

In order to describe the infimum of $V_R$, let us set some notation.

**Definition 5.2.1.** A pared manifold $(M, P)$ is a pared acylindrical manifold if $M$ is a compact irreducible 3-manifold and $P \subseteq \partial M$ is a collection of incompressible tori and annuli such that every essential cylinder in $(M, \partial M)$ is isotopic to a component of $P$.

Compare to [Mor84, Definition 4.8]. Moreover, using that exact same chapter (more precisely Theorems A and B [Mor84]), we know that every pared manifold with $\partial M \neq \emptyset$ different from the unit ball is hyperbolizable, with a geometrically finite metric and $P$ corresponding to the parabolic locus. Likewise, we can also use the term acylindrical for a (possibly cusped) hyperbolic manifold.

44
Definition 5.2.2. A geometrically finite hyperbolic manifold \( N \) is said to be acylindrical if \((N, P)\) is an acylindrical pared manifold, where \( P \) is the parabolic locus of \( N \) in \( \partial N \).

Then, as in [Mor84], if \( N \) is an acylindrical hyperbolic manifold then there is a hyperbolic metric in \( N \) with the same parabolic locus that has totally geodesic boundary.

Theorem 5.2.1 ([VPa], Theorem 7.1). Let \((M, P)\) be a pared compact hyperbolizable 3-manifold. Furthermore, assume that \((M, P)\) is incompressible, i.e. any compressing disk with boundary in \( \partial M \setminus P \) bounds a homotopically trivial curve. Then

\[
\inf_{QF(M,P)} V_R(M) = \frac{v_3}{2} \|DM\|
\]

where \( v_3 \) is the volume of the regular ideal tetrahedron in \( \mathbb{H}^3 \), \( DM \) is the double of the pair \((M, P)\) and \( \| \cdot \| \) denotes the Gromov norm of a manifold. Moreover, for any sequence \( \{M_n\} \) such that \( \lim_{n \to \infty} V_R(M_n) = \inf V_R(M) \), there exists a decomposition of \( M \) along essential cutting cylinders in components \( A_1 \sqcup \ldots \sqcup A_s \sqcup F_1 \sqcup \ldots \sqcup F_r \) (where \( A_1, \ldots, A_s \) are acylindrical with geodesic class at infinity and \( F_1, \ldots, F_r \) fuchsian) such that \( A_1 \sqcup \ldots \sqcup A_s \sqcup F_1 \sqcup \ldots \sqcup F_r \) is the additive geometric limit of a subsequence of \( \{M_n\} \).

Proof. Indeed, because of Proposition 1.2.1 and Theorem 5.1.1 any sequence \( M_n \in QF(M, P) \) where \( \lim_{n \to \infty} V_R(M_n) = \inf V_R(M) \) has a subsequence with additive geometric limit \( N_1 \sqcup \ldots \sqcup N_k \) and

\[
\inf V_R(M) = \lim_{n \to \infty} V_R(M_n) = \sum_{i=1}^k V_R(N_i).
\]

Because of Proposition 1.4.2 any small deformation of \( N_1 \sqcup \ldots \sqcup N_k \) is the additive geometric limit of another sequence \( \tilde{M}_n \in QF(M, P) \). Then \( N_1, \ldots, N_k \) are critical points for \( V_R \), which implies that their convex cores have totally geodesic boundaries. Hence each \( N_i \) is either acylindrical or fuchsian (depending if the convex core has non-empty interior or not), although the acylindrical components could arrive from pinching, drilling and cutting cylinders instead of just the latter as stated in the theorem. The next step is to notice that pinching and drilling increases \( V_R \). So let us assume that at least one curve gets pinched or drilled while converging to \( N_1, \ldots, N_k \).

Consider \( DN_i \), the double of the manifolds \( N_i \). Each acylindrical component doubles into a finite volume hyperbolic manifold by doubling along the geodesic boundary of \( C_{N_i} \). Each fuchsian component doubles into a Seifert fibered manifold, in fact as the product of a finite type surface \( S \) with \( S^1 \). The cusps from \( N_i \) give rank-2 cusp in the following pattern:

- A rank-2 cusp gives two rank-2 cusps in \( DN_i \), one per copy of \( C_{N_i} \) in \( DN_i \). Among them we have the rank-2 cusps originally from \((M, P)\) and the ones obtained from possibly drilling.

- A rank-1 cusp gives one rank-2 cusp in \( DN_i \), lying in the glued boundary of the two copies of \( C_{N_i} \). Among them we have the original rank-1 cusps from \((M, P)\) and the ones obtained by either pinching or a cutting cylinder.
Moreover, we can glue $N_1, \ldots, N_k$ along pared rank-1 cusps (pared by cutting cylinders) so we obtain a manifold $M^*$ that is $M$ with the added pinched rank-1 cusp and the drilled rank-2 cusps. To do this, identify the rank-1 cusp as they are pared by the cutting cylinders and fill in the geodesic if the core was a loxodromic. Then we can glue $DN_1, \ldots, DN_k$ along pared rank-2 cusps (again, pared by cutting cylinders) to obtain $DM^*$, which is $DM$ minus some curves (two for each rank-2 cusp obtained by drilling, and one for each rank-1 cusp that was obtained by pinching) in the following way:

- For pared cylinders coming from a rank-2 cusp, consider the following two dimensional picture Figure 5.7, which needs a $S^1$ factor to be the three dimensional case we are trying to represent. $A, B, C$ are the components $N_1, \ldots, N_k$ attached by the pasting region $P_1$ around the cusp represented at the center, were the cutting cylinders are represented in red. Then $DM^*$ around this Margulis tube can be obtained by gluing $DA, DB, DC$ along the red tori of $DP_1$.

- For pared cylinders coming from a loxodromic, we have Figure 5.8 similar Figure 5.7, where the difference is that we fill in the cusp in the pasting region $P_2$. Nevertheless, we still have $DM^*$ around this Margulis tube can be obtained by gluing $DA, DB, DC$ along the red tori of $DP_2$.

- For pared cylinders coming from a rank-1 cusp, we have Figure 5.9 similar Figure 5.7, where the difference is that $A, B, C$ do not close cyclically around the pasting region $P_3$ since the rank-1 cusp appears somewhere in the boundary. Nevertheless, we still have that $DM^*$ around this Margulis tube can be obtained by gluing $DA, DB, DC$ along the red tori of $DP_2$.

Observe that on all three cases the pasting regions $DP_j$ are circle bundles. Then the doubles $DN_1, \ldots, DN_k$ along with the pasting regions $DP_j$ is a decomposition of $DM^*$ along incompressible tori into finite-volume hyperbolic manifold or Seifert fibered manifolds, where now cutting cylinders can be seen as cutting tori and each component is (finite-volume) hyperbolic or Seifert fibered. Given Gromov’s theorem (as seen in [Thub], Theorem 6.2) we can relate the renormalized volume to the Gromov norm $\| \cdot \|$:

$$V_R(N_i) = \frac{v_3}{2} \|DN_i\|$$

(5.28)

where $v_3$ is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$. Indeed, when $N_i$ is acylindrical, $V_R(N_i)$ is half the hyperbolic volume of $DN_i$, which is equal to $v_3 \|DN_i\|$. When $N_i$ is Fuchsian both sides of the equality vanish. Similarly

$$\|DP_j\| = 0$$

(5.29)

since $DP_j$ are circle bundles.

Now, for fixed large $n$, consider $M_i$ equal to the double of $M_i^r$ along $\partial_0 M_i^r$. Noticing that cutting cylinders are glued into tori, we can paste along those tori to obtain $DM$
Figure 5.7: A, B, C and P for the rank-2 cusp case
Figure 5.8: A, B, C and P for the loxodromic case
Figure 5.9: A, B, C and P for the rank-1 cusp case
from $M_1, \ldots, M_k$. Now, we can divide each $M_n^i$ by essential cylinders until each component forms an acylindrical pair with the mentioned cylinders, hence hyperbolizable with totally geodesic convex core by the discussion after Definition 5.2.1. As before, these essential cylinders double into essential tori in $M_i$ that divide it into components that are either finite volume hyperbolic or Seifert fibered, depending if the convex core of the corresponding component had empty interior or not. Hence we have a decomposition of $M$ as in the statement of this theorem, so we will label the components as we did there. The decomposition $A_1, \ldots, A_s, F_1, \ldots, F_r$ is a subdecomposition of $M_n^1, \ldots, M_n^k$ where we are disregarding the pasting regions $DP_j$.

From [Thub, Proposition 6.5.2] and [Thub, Theorem 6.5.6], since $M_i$ can be obtained from $N_i$ by filling some cusps, we have

$$\|N_i\| \geq \|M_i\|$$

(5.30)

where the inequality is strict if we fill at least one cusp.

Also, by applying [Thub, Proposition 6.5.2,] and [Thub, Theorem 6.5.5] to each $M_i$ and then add them up, we have

$$\sum_{i=1}^{k} \|M_i\| = \sum_{j=1}^{s} \|A_j\| = \|DM\|$$

(5.31)

Putting (5.28), (5.29), (5.30) and (5.31) together and recalling that at least one curve was pinched or drilled

$$\sum_{i=1}^{k} V_R(N_i) > \frac{v_3}{2} \|DM\|$$

(5.32)

We will then contradict that $N_1, \ldots, N_k$ was obtained as the infimum sequence with at least one curve being pinched or drilled as soon as we observe that $A_1, \ldots, A_s, F_1, \ldots, F_r$ can be also obtained as limit. Indeed, as in the proof of Proposition 1.4.2, by doing Klein-Maskit combinations and generalized hyperbolic Dehn-fillings we can obtain a sequence of geometrically finite hyperbolic manifolds homeomorphic to $M$ with limit $A_1, \ldots, A_s, F_1, \ldots, F_r$.

From this we can easily see the following corollary for quasifuchsian manifolds

**Corollary 5.2.1.** Let $M$ be a quasifuchsian manifold. Then $V_R(M) \geq 0$ with equality if and only if $M$ is Fuchsian.

Moreover, any sequence such that $V_R \to 0$ must converge to a disjoint union of Fuchsian manifolds, since there cannot be a non-zero volume in Theorem 5.2.1.

Also, since for a acylindrical manifold $M$ there cannot be cutting cylinders, there is only one possible geometric limit under our conditions. Hence from Theorem 5.2.1 we have (also proved in [VPb])
Corollary 5.2.2. Let \((M, P)\) be a acylindrical pared hyperbolizable 3-manifold. Then any sequence \(M_n \in QF(M, P)\) such that \(\lim_{n \to \infty} V_R(M_n) = \inf V_R(M)\) converges geometrically to \(M_{tg} \in QF(M)\), the metric with convex core totally geodesic.

5.3 Minimizing and additive continuity for corrected \(V_R\)

In order to answer if we can obtain a global minimum for \(V_R\), we need to show first that \(V_R\) is bounded below. We will achieve this as we did for standard \(V_R\), by comparison against \(V_C\). Furthermore, this will allow us to start understanding sequences going to \(\inf V_R\). Thus, we have the following lemma.

Lemma 5.3.1. Let \(M\) be acylindrical. Then \(|V_R(M) - \frac{1}{2}V_C(M)| < \infty\).

Proof. Recall that \(|V_R(\cdot) - V_C(\cdot)| < \infty\) for both \(M\) and \(\partial M \times \mathbb{R}\). Then the result follows from showing

\[|V_C(M) - V_C(\partial M \times \mathbb{R})| < \infty.\]  (5.33)

Assume the contrary. Then we have a sequence \(M_n\) for which the difference in (5.33) goes to \(\infty\). After taking some subsequence, assume that \(M_n\) converges both algebraically and geometrically, to possibly different Kleinian groups \(H\) and \(G\). Assume the same for \(\partial M_n \times \mathbb{R}\). Take a compact core \(C\) representing the algebraic limit \(H\) in the geometric limit \(G\). Notice then that each end of \(C\) is the same as one of the top components of \((\partial M_n \times \mathbb{R}) \setminus \partial C\). Then in (5.33) for \(M_n\), the respective volumes placed in each of these ends cancel out, leaving the terms corresponding to the volume of \(C\) and the volume placed in the bottom components of \((\partial M_n \times \mathbb{R}) \setminus \partial C\), all converging to finite values. Then (5.33) follows.

Then we have the following corollary, analogous to the situation with \(V_R\).

Corollary 5.3.1. Let \(M\) be acylindrical. Then \(\overline{V_R}\) is bounded below for all geometrically finite hyperbolic metric in \(M\), and any sequence converging to \(\inf V_R\) has bounded \(V_C\).

For such sequence where both \(\overline{V_R}, V_C\) stay uniformly bounded, we can then use Proposition 1.2.1 to express the limits of \(V_R(M), V_R(\partial M)\) in \(\overline{V_R} = V_R(M) - \frac{1}{2}V_R(\partial M)\) as done by the following proposition.

Proposition 5.3.1. Let \(M\) be a convex co-compact hyperbolic manifold with \(\partial M \neq \emptyset\) incompressible. Let \(M_n \in QF(M)\) be a sequence such that \(\overline{V_R}(M_n)\) converges. Then we can select finite many base points such that (possibly after taking a subsequence) \(N_1, \ldots, N_k\) are the geometric limits corresponding to the base points (in the sense of Proposition 1.2.1) and

\[\lim_{n \to \infty} \overline{V_R}(M_n) = \sum_{i=1}^{k} V_R(N_i) - \frac{1}{2} \sum_{l} V_R(S_l)\]  (5.34)
where $S_l$ are the additive limits obtained from the components $(\partial N_i)_{1 \leq i \leq n}$.

Proof. From Theorem 5.1.1 we see that \( \lim_{n \to \infty} V_R(M_n) = \sum_{i=1}^{k} V_R(N_i) \) and \( \lim_{n \to \infty} V_R(\partial M_n) = \sum_{l} V_R(S_l) \). This completes the proof given the definition of $V_R = V_R(M) - \frac{1}{2} V_R(\partial M)$.
Bibliography

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