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November 17, 1966
PARAMETERIZATIONS OF OFF-ENERGY-SHELL BEHAVIOR
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ABSTRACT

Earlier approaches to the calculation of off-energy-shell
two-body partial-wave transition amplitudes are reviewed, and the
general requirements that may be placed on representations or
approximations for such amplitudes are listed. A basis for the
parameterization of the off-energy-shell behavior of the \( t \) matrix
of the form \( t_{\ell}(p,p';k^2) = \left[ \sum_i f_{\ell_i}^i(p,p';k^2) \right] t_{\ell}(k^2) \) is displayed
for each approach to the calculation of off-shell transition
amplitudes. These representations separate the off-energy-shell
behavior from the better known and more experimentally accessible
on-shell behavior. They have the added advantage that the experimental on-shell \( t \) matrix expressed in terms of the phase shift can
be included (for \( k^2 > 0 \)) in calculations involving off-shell
transition amplitudes. Representations of the form
\[
t_{\ell}(p,p';k^2) = t_{\ell}(k^2) + \Delta_{\ell}(p,p';k^2)
\]
given, as are analogous representations for the half-off-shell amplitude \( t_{\ell}(p,k;k^2) \).
I. INTRODUCTION

There has been considerable recent interest in off-energy-shell two-body partial-wave transition amplitudes\(^1\) and their use as input to multiparticle scattering calculations.\(^2\)

In view of this interest and the subsequent applications, it seems useful to discuss the general requirements which a representation or approximation for the off-shell two-body partial-wave transition amplitude should satisfy. We can give four general requirements on the approximation or representation for the off-shell amplitude \(t_2(p,p';k^2)\):

(a) It must reduce to the known form \(t_2(k^2)\) on the energy shell;

(b) It must satisfy two-particle elastic unitarity on and off the energy shell;\(^3\)

(c) It should be an analytic function of \(k^2\), with the only singularities in \(k^2\) being the right-hand unitarity cut and bound state or resonance poles. In addition, it should be analytic in \(p(p')\) for \(p^2 > 0\) (\(p'^2 > 0\)), and it should have singularities for \(p^2 < 0\) and \(p'^2 < 0\) which become the left-hand or driving singularities of the on-shell amplitude,\(^4\) when \(p^2 = p'^2 = k^2\).

(d) It should satisfy time-reversal invariance or detailed balance,\(^5\) that is, \(t_2(p,p';k^2) = t_2(p',p;k^2)\).

It also seems valuable to have a representation or a parameterization for the off-shell partial-wave transition amplitude of the form \(t_2(p,p';k^2) = \left[ \sum_i f_i(p,p';k^2) \right] t_2(k^2)\). The value of
such a parameterization is twofold:

(a) It enables us to separate the off-shell behavior of the transition amplitude from the on-shell behavior, which is better known and more accessible to experiment;

(b) It allows us to include the experimental data from on-shell two-body scattering in calculations involving off-shell transition amplitudes by writing:

\[ t_\ell(k^2) = \frac{-2}{\pi k} \exp[i \delta_\ell(k^2)] \sin \delta_\ell(k^2). \]

Thus, we can take on-shell two-body scattering into account exactly in multiparticle scattering calculations.

We see easily that such a representation must satisfy

\[ \Sigma \ell_i f_\ell_i(k,k;k^2) = 1 \quad \text{for} \quad k^2 > 0, \]

and that time-reversal invariance demands

\[ \Sigma \ell_i f_\ell_i(p,p';k^2) = [\Sigma \ell_i f_\ell_i(p',p;k^2)]. \]

Such a representation is invalid when the on-shell transition amplitude is zero, such as when the phase shift goes through zero.

This representation allows us to gain insight into off-shell behavior by plotting the half-off-shell factor

\[ \Sigma \ell_i f_\ell_i(p,k;k^2) \]

vs \( p \), and the symmetrically off-shell factor

\[ \Sigma \ell_i f_\ell_i(p,p;k^2) \]

vs \( p \) at given values of \( k^2 \). Also, if we were to obtain, by any convenient method, a parameterization of the off-shell amplitude which agreed with experiment, we could use the resulting (numerical) information on \( t_\ell(p,p';k^2) \) as input to the Lippmann-Schwinger equation, and solve for the local potential \( V_\ell(p,p') \) which gives the experimental transition amplitude. This experimentally determined local potential might then be compared with the relevant limit of some more fundamental description of the forces between elementary particles.
In addition, the limitations of available electronic computers make it desirable to have a separable representation or approximation for the off-energy-shell transition amplitude for use in multiparticle scattering calculations. That is, we would like to have

\[ t_{\Delta}(p,p';k^2) = \sum_i \alpha_i(p;k^2) \sum_i \beta_i(p';k^2) r_{\Delta}(k^2), \]

so that the off-energy-shell transition amplitude would be separable in incident and outgoing relative momenta. We shall discuss several such representations or approximations herein, and of course we shall demand that they satisfy our four general requirements.

Another way to separate the model-dependent off-shell behavior from the experimental on-shell behavior is not as useful as the multiplicative separation for insertion into the integral equations of multiparticle scattering, but is given for completeness. It may be helpful for gaining insight into the off-shell behavior and it has the added advantage that it is valid when \( t_{\Delta}(k^2) = 0 \). This representation is

\[ t_{\Delta}(p,p';k^2) = t_{\Delta}(k^2) + \Delta_{\Delta}(p,p';k^2), \]

where \( \Delta_{\Delta}(p,p';k^2) = \Delta_{\Delta}(p',p;k^2) \) from time-reversal invariance and \( \Delta_{\Delta}(k,k;k^2) = 0 \).

We must stress that very little is known about the off-energy-shell behavior of transition amplitudes. Perhaps the best experimental test of the off-shell behavior of two body transition amplitudes is \( p - p \) bremsstrahlung, but the calculations made so far suffer from a
lack of gauge invariance and probably can't be considered wholly reliable. That is, we are probably not yet at a point where a comparison of theory and experiment has meaning, because of the incompleteness of present theoretical calculations.

Furthermore, potential theory, the most common approach to the calculation of off-shell transition amplitudes, is really just a parameterization of the on- and off-shell behavior of the amplitude. In this approach, either a local potential form or a nonlocal separable potential form is chosen and fitted to the on-shell data. Then the fact that the Lippmann-Schwinger equation defines a unique continuation of the transition amplitude off the energy shell is exploited to obtain the off-energy-shell partial-wave transition amplitude. The potential form chosen for the parameterization of the on-shell data determines the off-shell behavior of the transition amplitude.

The other commonly used approach to the calculation of off-shell transition amplitudes, which is the single-pole-dominance model, is somewhat more fundamental and has some theoretical justification. However, this latter approach is good only when the scattering channel is dominated by a single pole and then only at energies near the bound state or resonance energy.

Throughout the paper we use the conventions of Goldberger and Watson, and we use units in which $\hbar = 2m = 1$ so that $k^2 = s$, the c.m. energy. For simplicity, we consider only the
nonrelativistic single-channel case. All the developments in this paper can be repeated for the partial-wave reaction matrix or \( K \) matrix which satisfies the Lippmann-Schwinger integral equation

\[
K(p,p';k^2) = V(p,p') + P \int_0^\infty \frac{dq^2}{k^2 - q^2} V(p,q) K(q,p';k^2)
\]

where \( P \) indicates a principal-value integral.

II. EARLIER APPROACHES TO CALCULATION OF OFF-SHELL TRANSITION AMPLITUDES

Let us look at the earlier approaches to the calculation of off-shell transition amplitudes in greater detail.

In the potential theory approach, on- and off-shell unitarity is enforced by putting a potential into the Schrödinger equation or, equivalently, into the Lippmann-Schwinger integral equation for the transition amplitude. Time-reversal invariance is ensured by demanding that the potential have the symmetry \( V(p,p') = V(p',p) \).

The way this approach is used is as follows: A potential form for parameterizing the on-shell data is chosen and fitted numerically to the on-shell amplitude which can be expressed in terms of phase shifts. This potential is inserted into the Lippmann-Schwinger equation which then completely determines the off-shell behavior of the resulting transition amplitude. Thus, assuming a potential form for parameterizing the on-shell data implies assuming the off-shell behavior. Next, the off-shell transition amplitude obtained from
the assumed potential form is used in, for example, nuclear matter, multiparticle scattering, or p-p bremsstrahlung calculations to see if the off-energy-shell behavior of the transition amplitude agrees with experiment.

It is obvious that this is a highly arbitrary approach, since many potential forms (some of which may lead to very different off-shell behavior) can be used to fit the on-shell data. Therefore, all that can be done is to try various forms which fit the on-shell data and see which form gives an off-shell behavior which coincides most closely with experiment. This, at best, is a slow and tedious process. Finally, we repeat that, since we have no good reason to think that the interactions of elementary particles are really well described by potential theory, the use of a potential amounts only to a parameterization of the on- and off-shell transition amplitudes.

For a local potential, the transition amplitude must be obtained by solving, usually numerically, the Lippmann-Schwinger integral equation

\[ t_s(p,p';k^2) = V_s(p,p') + \int_0^\infty \frac{dq q^2}{k^2 - q^2 + i\epsilon} V_s(p,q) t_s(q,p';k^2). \]

However, in another common method wherein the potential is assumed to be a nonlocal separable potential, the Lippmann-Schwinger equation reduces to a set of algebraic equations and the solution can be displayed in closed form. In fact, if we consider a nonlocal separable potential of the form
where $\lambda_1$ and $\lambda_2$ are constant and $V_\ell(p,p')$ has the symmetry required for time-reversal invariance, it can be shown that the off-shell matrix is given by

$$t_\ell(p,p';k^2) = \frac{N_\ell(p,p';k^2)}{D_\ell(k^2)}, \quad (1)$$

where

$$N_\ell(p,p';k^2) = \lambda_1 g_\ell(p) g_\ell(p') [1 + \lambda_2 H_\ell(k^2)]$$

$$- \lambda_2 h_\ell(p) h_\ell(p') [1 - \lambda_1 G_\ell(k^2)]$$

$$- \lambda_1 \lambda_2 [g_\ell(p) h_\ell(p') + h_\ell(p) g_\ell(p')] M_\ell(k^2)$$

$$D_\ell(k^2) = (1 - \lambda_1 G_\ell(k^2)) (1 + \lambda_2 H_\ell(k^2)) + \lambda_1 \lambda_2 [M_\ell(k^2)]^2$$

and

$$G_\ell(k^2) = \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} [g_\ell(q)]^2$$

$$H_\ell(k^2) = \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} [h_\ell(q)]^2$$

$$M_\ell(k^2) = \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} [g_\ell(q) h_\ell(q)].$$
When the separable potential has only one separable term, so that
\[ V_{\ell}(p, p') = \lambda_{\ell} g_{\ell}(p) g_{\ell}(p'), \]
we find
\[ t_{\ell}(p, p'; k^2) = \frac{N_{\ell}(p, p')}{D_{\ell}(k^2)} \]
or
\[ t_{\ell}(p, p'; k^2) = \frac{\lambda_{\ell} g_{\ell}(p) g_{\ell}(p')}{{1 - \lambda_{\ell} \int_{0}^{\infty} dq \frac{q^2}{k^2 - q^2 + i\epsilon}} \left[ g_{\ell}(q) \right]^2}, \]
(2)
as can be seen by setting \( \lambda_2 = 0 \) in the equations above. In this case the relation of the off-shell behavior to the assumed separable potential form is very transparent.

If we consider an attractive single-term separable potential of the form
\[ V_{\ell}(p, p') = -\lambda g_{\ell}(p) g_{\ell}(p'), \]
we find
\[ \tan \vartheta_{\ell}(k^2) = \frac{\frac{\pi}{2} \lambda g_{\ell}^2(k)}{1 + \lambda \int_{0}^{\infty} dq \frac{q^2}{k^2 - q^2} g_{\ell}^2(q)} \]
where the $P$ indicates a principal-value integral. Here we see the main drawback of assuming that the interaction between two particles is given by a single separable potential—we cannot reproduce a phase shift which changes sign unless we make $\lambda$ energy dependent.

The other main approach to calculation of off-shell transition amplitudes is the single-pole dominance model wherein we assume that the $l$th partial wave is dominated by a single bound state or resonance pole. We say single-pole dominance because the introduction of more than one pole term in a given partial wave leads to a nonunitary transition amplitude in this approach.

Lovelace shows that the residues of bound state or resonance poles in the partial-wave transition amplitudes factor in the incident and outgoing relative momenta. Thus, in the vicinity of a bound state or resonance pole, we may write the transition amplitude in a separable form and, when a single pole dominates a given partial wave, it is frequently assumed that this separable form describes the transition amplitude for all energies in that partial wave. There are two ways in which this information can be used, when a single pole dominates a given partial wave, to give a separable $t$ matrix which is unitary and has correct threshold behavior.

First, we may use the facts that the residues of the poles of the partial wave $R$ matrix also factor in incident and outgoing relative momenta, and that the insertion of a Hermitian form for
the partial wave $R$ matrix in the algebraic equation relating the partial wave $t$ and $R$ matrices leads to a unitary $t$ matrix.\textsuperscript{7} The resulting unitary and separable $t$ matrix is

$$t_\ell(p,p';k^2) = \frac{g_\ell(p) g_\ell(p')}{E_R - k^2 + i \frac{\pi}{2} K g_\ell^2(k)} ,$$

where $E_R$ is the bound state or resonance energy. It can also be shown that in the case of a bound-state pole, the bound-state form factor $g_\ell(p)$ is related to the bound-state wave function $\psi_B^\ell(p)$ by

$$g_\ell(p) = (E_B - p^2) \psi_B^\ell(p) ,$$

where $E_B$ is the bound-state energy.

The second, and perhaps more general, approach is to write the on-shell transition amplitude in the $N/D$ form, in which

$$t_\ell(k^2) = \frac{N_\ell(k^2)}{D_\ell(k^2)}$$

and

$$D_\ell(k^2) = 1 - \int_0^\infty dq \rho(q) \frac{N_\ell(q)}{k^2 - q^2 + i\epsilon} , \quad (3)$$

where $\rho(q) = q^2$, and $N_\ell(k^2)$ has only the left-hand cuts but $D_\ell(k^2)$ has only the right-hand cuts. Lovelace, noting that the
off-shell amplitude has only a right-hand cut in $k^2$ and that the bound state and resonance form factors $g_p(p)$ satisfy dispersion relations in $p^2$ with only a left hand cut, proposed that the off-shell transition amplitude could be written as

$$t_\ell(p,p';k^2) = \frac{\sqrt{N_\ell(p)}\sqrt{N_\ell(p')}}{D_\ell(k^2)},$$

provided that the partial wave is dominated by a bound state or resonance. Thus, to fit the on-shell data in this version of the single-pole dominance model, we fit the numerator function $N_\ell(k^2)$ in the exact on-shell transition amplitude, in the vicinity of the pole.

It can easily be seen by comparing Eqs. (2), (3), and (4) that this approach is equivalent to the introduction of a single separable potential, but in this form it is a little more susceptible to generalization. This approach has the drawback that $N_\ell(k^2)$ must be of definite sign so that we do not introduce spurious cuts into the amplitude. Also, when we demand that $N_\ell(k^2)$ be of definite sign, we again find that we cannot reproduce a phase shift which changes sign.

If we choose a single separable potential form $g_\ell(k)$ such that $g_\ell^2(k) = B_\ell(k^2)$, where $B_\ell(k^2)$ is the Born term in the integral equation for $N_\ell(k^2)$, then the separable potential method leads to a form for $t_\ell(k^2)$ identical to that obtained in the determinantal approximation to the $N/D$ equations. However, if we
were to choose $g_\ell^2(k) = N_\ell(k^2)$ and then fit the resulting form for $t_\ell(k^2)$ to the on-shell experimental data, we would achieve a parameterization of $N_\ell(k^2)$ in the exact on-shell partial-wave scattering amplitude.

The single-pole dominance approach is also rather arbitrary in practice since, for example, the momentum-space wave functions of bound states (which we related to the bound state form factors) are not well known. Therefore, this approach, too, is only a parameterization of on- and off-shell behavior, since the choice of bound state and resonance form factors is to a great extent arbitrary at our present state of knowledge, as is the assumption that the form valid near a pole is valid for all energies.

III. SEPARATION OF ON- AND OFF-SHELL BEHAVIOR

Let us look at the common ways of calculating the partial-wave transition amplitude, and attempt to separate the on- and off-shell behavior for values of $k^2$ such that $k^2 > 0$ and $t_\ell(k^2) \neq 0$.

In local potential theory, Kowalski has shown that the off-energy-shell partial-wave transition amplitude may be obtained as the solution to a nonsingular integral equation obtained by Fredholm reduction from the Lippmann-Schwinger equation. This equation is

$$t_\ell(p,p';k^2) = \frac{V_\ell(p,k)}{V_\ell(k,k)} t_\ell(p',k;k^2)$$

$$+ \left[ \frac{V_\ell(p,p') - V_\ell(p,k) V_\ell(k,p')}{V_\ell(k,k)} \right]$$ (5)
where the nonsingular kernel is given by
\[
\Lambda_{q}(p,q;k^2) = \left[ V_{q}(p,q) - \frac{V_{q}(p,k)V_{q}(k,q)}{V_{q}(k,k)} \right] \frac{q^2}{k^2 - q^2}.
\]

Kowalski further shows that the half-off-shell amplitude \( t_{q}(p,k;k^2) \)
is given exactly by
\[
t_{q}(p,k;k^2) = \mathcal{F}_{q}(p,k) \times t_{q}(k^2),
\]
where \( \mathcal{F}_{q}(p,k) \) satisfies a nonsingular integral equation with the same kernel as Eq. (5), namely,
\[
\mathcal{F}_{q}(p,k) = \frac{V_{q}(p,k)}{V_{q}(k,k)} + \int_{0}^{\infty} dq \Lambda_{q}(p,q;k^2) \mathcal{F}_{q}(q,k).
\]

Note that if \( V_{q}(p,p') \) is a single separable potential, such that \( V_{q}(p,p') = \lambda g_{q}(p) g_{q}(p') \), then \( \Lambda_{q}(p,q;k^2) = 0 \) and \( \mathcal{F}_{q}(p,k) = g_{q}(p)/g_{q}(k) \). In addition, time-reversal invariance, that is \( t_{q}(p,p';k^2) = t_{q}(p',p;k^2) \), although not manifest, can be shown to hold in these equations if \( V_{q}(p,p') = V_{q}(p',p) \).

We can rewrite Eq. (5) as
\[ t_2(p, p'; k^2) = \frac{V_\ell(p, k)}{V_\ell(k, k)} \, f_\ell(p', k) \, t_2(k^2) \]

\[ + \left[ \frac{V_\ell(p, p')}{V_\ell(k, k)} - \frac{V_\ell(p, k) \, V_\ell(k, p')}{V_\ell(k, k)} \right] \]

\[ + \int_0^\infty dq \, \Lambda_\ell(p, q; k^2) \, t_2(q, p'; k^2). \]

If we next write

\[ t_2(p, p'; k^2) = X_2(p, p'; k^2) \, t_2(k^2), \]

we find the following nonsingular integral equation for the off-shell factor \( X_2(p, p'; k^2) \),

\[ X_2(p, p'; k^2) = \frac{V_\ell(p, k)}{V_\ell(k, k)} \, f_\ell(p', k) \]

\[ + \frac{1}{t_2(k^2)} \left[ V_\ell(p, p') - \frac{V_\ell(p, k) \, V_\ell(k, p')}{V_\ell(k, k)} \right] \]

\[ + \int_0^\infty dq \, \Lambda_\ell(p, q; k^2) \, X_2(q, p'; k^2). \]  \hspace{1cm} (6)

We see that the full off-shell factor \( X_2(p, p'; k^2) \) and the half-off-shell factor \( f_\ell(p, k) \) satisfy nonsingular integral equations with the same kernel.

In addition to the fact that \( \Lambda_\ell(p, k; k^2) = 0 \), which guarantees the nonsingularity of our integral equations, we have
\( \Lambda (k,q;k^2) = 0 \), which ensures that \( f_\ell(k,k) = 1 \) as demanded by the representation \( t_\ell(p,k;k^2) = f_\ell(p,k) t_\ell(k^2) \). Using this in our integral Eq. (6) for the off-shell factor \( X_\ell(p,p';k^2) \), we find

\begin{itemize}
  \item (a) \( X_\ell(k,k;k^2) = 1 \) as required by the representation \( t_\ell(p,p';k^2) = X_\ell(p,p';k^2) t_\ell(k^2) \), and
  \item (b) \( X_\ell(k,p';k^2) = f_\ell(p',k) \) as required by the equations \( t_\ell(k,p';k^2) = X_\ell(k,p';k^2) t_\ell(k^2) \) and \( t_\ell(k,p';k^2) = f_\ell(p',k) t_\ell(k^2) \).
  \item (c) Likewise,
\end{itemize}

\[ X_\ell(p,k;k^2) = \frac{V_\ell(p,k)}{V_\ell(k,k)} + \int_0^\infty dq \Lambda_\ell(p,q;k^2) X_\ell(q,k;k^2), \]

so that

\[ X_\ell(p,k;k^2) \equiv f_\ell(p,k). \]

As another starting point for the derivation of the off-shell factor \( X_\ell(p,p';k^2) \), we may take the following equation derived by Kowalski\textsuperscript{9}

\[ t_\ell(p,p';k^2) = f_\ell(p,k) t_\ell(k^2) f_\ell(p',k) \]

\[ + \left[ g_\ell(p,p';k^2) - f_\ell(p,k) \frac{V_\ell(p',k)}{V_\ell(k,k)} \right] V_\ell(k,k), \]

where \( g_\ell(p,p';k^2) \) satisfies

\[ g_\ell(p,p';k^2) = \frac{V_\ell(p,p')}{V_\ell(k,k)} + \int_0^\infty dq \Lambda_\ell(p,q;k^2) g_\ell(q,p';k^2). \]

We see that

\[ g_\ell(k,p';k^2) = V_\ell(k,p')/V_\ell(k,k), \]
\[ g_\varepsilon(p,k;k^2) = r_\varepsilon(p,k) \]

and

\[ g_\varepsilon(k,k;k^2) = 1. \]

Again we let

\[ t_\varepsilon(p,p';k^2) = X_\varepsilon(p,p';k^2) t_\varepsilon(k^2), \]

and we find that

\[
X_\varepsilon(p,p';k^2) = r_\varepsilon(p,k) f_\varepsilon(p',k) + \frac{V_\varepsilon(k,k)}{t_\varepsilon(k^2)} \left[ g_\varepsilon(p,p';k^2) - f_\varepsilon(p,k) \frac{V_\varepsilon(p',k)}{V_\varepsilon(k,k)} \right]
\]

or

\[
X_\varepsilon(p,p';k^2) = r_\varepsilon(p,k) \left[ f_\varepsilon(p',k) - \frac{V_\varepsilon(p',k)}{t_\varepsilon(k^2)} \right] + \frac{V_\varepsilon(k,k)}{t_\varepsilon(k^2)} g_\varepsilon(p,p';k^2)
\]

We see easily that the on-shell and half-on-shell reductions

\[ X_\varepsilon(k,p';k^2) = f_\varepsilon(p',k) , \quad X_\varepsilon(p,k;k^2) = f_\varepsilon(p,k) \quad \text{and} \quad X_\varepsilon(k,k;k^2) = 1 \]

still hold. This way of obtaining the off-shell factor \( X_\varepsilon(p,p';k^2) \) appears to be simpler than the first method, but it really involves the solution of three integral equations. The first method is probably numerically simpler since it only requires the solution of
two nonsingular integral equations. Of course, all the integral equations to be solved have the same nonsingular kernel which alleviates the difficulty somewhat.

There is yet another way (of determining the off-shell factor) which does not involve the on-shell t matrix explicitly but which does require the solution of a singular integral equation. Suppose we start with the Lippmann-Schwinger integral equation for the off-energy-shell partial-wave transition amplitude

\[ t_\ell(p,p';k^2) = V_\ell(p,p') + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} V_\ell(q,p) t_\ell(q,p';k^2). \]  

(7)

We also know that

\[ t_\ell(k,p';k^2) = V_\ell(k,p') + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} V_\ell(k,q) t_\ell(q,p';k^2), \]  

(8)

so if we multiply Eq. (8) by \( V_\ell(p,p')/V_\ell(k,p') \) and subtract the result from Eq. (7), we obtain

\[ t_\ell(p,p';k^2) = \frac{V_\ell(p,p')}{V_\ell(k,p')} t_\ell(k,p';k^2) \]

\[ + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} \left[ V_\ell(q,p) - \frac{V_\ell(p,p') V_\ell(k,q)}{V_\ell(k,p')} \right] t_\ell(q,p';k^2). \]
Now, if we note that
\[ t_\sigma(k, p'; k^2) = f_\sigma(p', k) t_\sigma(k^2) \]
and let
\[ t_\sigma(p, p'; k^2) = X_\sigma(p, p'; k^2) t_\sigma(k^2) , \]
we obtain
\[ X_\sigma(p, p'; k^2) = \frac{V_\sigma(p, p')}{V_\sigma(k, p')} \cdot f_\sigma(p', k) \]
\[ + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} \left[ \frac{V_\sigma(p, q)}{V_\sigma(k, p')} V_\sigma(k, q) \right] X_\sigma(q, p'; k^2) , \]
where the usual reductions \( X_\sigma(k, p'; k^2) = f_\sigma(p', k) , \)
\( X_\sigma(p, k; k^2) = f_\sigma(p, k) \) and \( X_\sigma(k, k; k^2) = 1 \) hold.

Noyes\(^9\) and Kowalski\(^9\) have gone on to write \( t_\sigma(p, p'; k^2) \)
as the sum of a term separable in incident and outgoing momenta and
a nonseparable term. The approximation of neglecting the nonseparable
term yields
\[ t_\sigma(p, p'; k^2) \approx f_\sigma(p, k) t_\sigma(k^2) f_\sigma(p', k) , \]
or equivalently,
\[ t_\sigma(p, p'; k^2) \approx t_\sigma(p, k; k^2) t_\sigma(k, p'; k^2)/t_\sigma(k^2) . \]
That is, in this approximation the full off-shell amplitude is in a sense the product of the half-off-shell amplitudes. However, Basdevant points out that this approximation has cuts for $k^2 < 0$ that are not found in the full off-energy-shell amplitude defined by the Lippmann-Schwinger equation, and that these cuts can cause disastrous consequences in multiparticle scattering calculations. In addition, numerical solution of the equations of Noyes and Kowalski for a Yukawa potential have shown that the term neglected in their approximation is of the same order of magnitude as the term retained when $p$ and $p'$ differ from $k$ by an amount comparable to $k$.

Recently, Wong and Zambotti, noting that the off-shell $t$ matrix can be written in local potential theory as

$$t_\ell(p,p';k^2) = \frac{F_\ell(p,p';k^2)}{D_\ell(k^2)},$$

where $F_\ell(p,p';k^2)$ has only a right cut in $k^2$, have expanded $F_\ell(p,p';k^2)$ as a sum of separable terms which preserve the correct analyticity structure for $F_\ell(p,p';k^2)$. They show that the resulting separable approximation amplitude gives results reasonably close to the corresponding local potential amplitude. We can make the separation we desire simply by writing

$$t_\ell(p,p';k^2) = \frac{F_\ell(p,p';k^2)}{F_\ell(k,k;k^2)} \frac{F_\ell(k,k;k^2)}{D_\ell(k^2)} = \frac{F_\ell(p,p';k^2)}{F_\ell(k,k;k^2)} t_\ell(k^2).$$
Let us now consider representations of the form

\[ t_{\phi}(p,p';k^2) = t_{\phi}(k^2) + \Delta_{\phi}(p,p';k^2) \]

or

\[ t_{\phi}(p,p';k^2) = \frac{2}{\pi} \frac{1}{k} e^{i \delta_{\phi}(k^2)} \sin \delta_{\phi}(k^2) + \Delta_{\phi}(p,p';k^2) . \]

In the usual local potential theory approach, it is probably simplest to obtain \( \Delta_{\phi}(p,p';k^2) \) in a straightforward way as

\[ \Delta_{\phi}(p,p';k^2) = t_{\phi}(p,p';k^2) - t_{\phi}(k^2) , \]

where \( t_{\phi}(p,p';k^2) \) and \( t_{\phi}(k^2) \) are obtained by solution of the Lippmann-Schwinger equation with the model potential. Of course, in the Wong-Zambotti approach, we readily see that we obtain

\[ \Delta_{\phi}(p,p';k^2) = [F_{\phi}(p,p';k^2) - F_{\phi}(k,k;k^2)]/D_{\phi}(k^2) . \]

We will now look at the separation of on- and off-shell behavior in the separable potential case. In this case, the separation is very easy for we have shown in Eq. (1) that we then have

\[ t_{\phi}(p,p';k^2) = N_{\phi}(p,p';k^2)/D_{\phi}(k^2) . \]
We get results like those in the Wong-Zambotti approach, namely

\[ t_{\ell}(p, p'; k^2) = \frac{N_{\ell}(p, p'; k^2)}{N_{\ell}(k, k; k^2)} \ t_{\ell}(k^2) \]  

(9)

and

\[ t_{\ell}(p, p'; k^2) = t_{\ell}(k^2) + \frac{N_{\ell}(p, p'; k^2) - N_{\ell}(k, k; k^2)}{D_{\ell}(k^2)} \]

It can be shown that in the general case of a two-term separable potential, the separation (9) is equivalent to two separable terms in an integral equation of Fadeev type. Note that this approach is no more complicated for computation than the usual separable potential method in which we use

\[ t_{\ell}(p, p'; k^2) = \frac{N_{\ell}(p, p'; k^2)}{D_{\ell}(k^2)} . \]  

(10)

Now we use

\[ t_{\ell}(p, p'; k^2) = \frac{N_{\ell}(p, p'; k^2)}{N_{\ell}(k, k; k^2)} \ t_{\ell}(k^2) \]

when \( k^2 > 0 \) and \( t_{\ell}(k^2) \neq 0 \), and we use Eq. (10) when \( t_{\ell}(k^2) = 0 \) or \( k^2 < 0 \).

In the case of a single separable potential, when

\[ V_{\ell}(p, p') = \lambda \ g_{\ell}(p) \ g_{\ell}(p') \]  

the multiplicative separation takes on the particularly appealing form
\[ t_{\lambda}(p,p';k^2) = \frac{\epsilon_{\lambda}(p)}{\epsilon_{\lambda}(k)} \cdot t_{\lambda}(k^2) \cdot \frac{\epsilon_{\lambda}(p')}{\epsilon_{\lambda}(k)} . \]

If we look next at what might be called the \( N/D \) approaches, we see that the single-pole dominance model leads to an off-shell amplitude of the form

\[ t_{\lambda}(p,p';k^2) = \sqrt{N_{\lambda}(p)} \cdot \frac{1}{D_{\lambda}(k^2)} \cdot \sqrt{N_{\lambda}(p')}. \]

Alternatively, this may be looked on as a basic form, satisfying all our requirements, that may be used for parameterizing the on- and off-shell behavior of the transition amplitude in those partial waves for which the phase shift does not change sign, so that \( N_{\lambda}(k^2) \) has a definite sign for \( k^2 > 0 \). Since, as we pointed out earlier, this approach is equivalent mathematically to the introduction of a single separable potential, we obtain the same separations as above, that is,

\[ t_{\lambda}(p,p';k^2) = \frac{\sqrt{N_{\lambda}(p)} \cdot \sqrt{N_{\lambda}(p')}}{N_{\lambda}(k)} \cdot t_{\lambda}(k^2) \]

and

\[ t_{\lambda}(p,p';k^2) = t_{\lambda}(k^2) - \frac{\sqrt{N_{\lambda}(p)} \cdot \sqrt{N_{\lambda}(p')}}{D_{\lambda}(k^2)} \cdot N_{\lambda}(k). \]

It might be interesting to see what restrictions we can put on possible generalizations of this approach. Suppose we write
\( t_{\ell}(k^2) \) in the N/D form, \( t_{\ell}(k^2) = \frac{N_{\ell}(k^2)}{D_{\ell}(k^2)} \) or

\[
t_{\ell}(k^2) = \frac{N_{\ell}(k^2)}{\left[ \sqrt{1 - \sum_{q} \frac{q^2}{k^2 - q^2 + i\epsilon N(q)}^2} \right]} , \quad (11)
\]

in the general case, where the phase shift can change sign and \( N_{\ell}(k^2) \) need not be of definite sign for \( k^2 > 0 \). Then suppose we seek a basis for the parameterization of the off-shell amplitude of the form \( t_{\ell}(p,p';k^2) = \sum_i f_{\ell_i}(p,p';k^2) t_{\ell}(k^2) \) for values of \( k^2 \) where \( t_{\ell}(k^2) \neq 0 \).

Now, since \( t_{\ell}(p,p';k^2) \) has only a right cut in \( k^2 \) and \( D_{\ell}(k^2) \) already contains this right cut, we see that we must have

\[
\sum_i f_{\ell_i}(p,p';k^2) = \sum_i \frac{\alpha_{\ell_i}(p,p')}{N_{\ell}(k^2)}
\]

for \( k^2 \) and to ensure that there are no other singularities in \( k^2 \).

Then we have

\[
t_{\ell}(p,p';k^2) = \left[ \sum_i \frac{\alpha_{\ell_i}(p,p')}{N_{\ell}(k^2)} \right] t_{\ell}(k^2) = \left[ \sum_i \frac{\alpha_{\ell_i}(p,p')}{D_{\ell}(k^2)} \right] ,
\]

where the second form would be valid even when \( t_{\ell}(k^2) = 0 \).

We require that \( \alpha_{\ell_i}(k,k) = N_{\ell}(k^2) \), and since \( N_{\ell}(k^2) \) is real for \( k^2 > 0 \), let us assume \( \sum_i \alpha_{\ell_i}(p,p') \) is real.

Let us look at what unitarity tells us about such a representation. The off-shell unitarity relation which reduces to the usual on-shell relation is
\[ \text{Im} \, t_\ell^\prime(p,p';k^2) = - \frac{\pi k}{2} \, t_\ell^\prime(p,k;k^2) \, t_\ell(k,p';k^2). \]

Now, let us see what restrictions this impose on a relation of the form

\[ t_\ell^\prime(p,p';k^2) = \left[ \sum_{i} \alpha_{\ell_i}(p,p') \right] / D_\ell(k^2). \]

Since we assume \( \sum_{i} \alpha_{\ell_i}(p,p') \) is real, we obtain

\[ \text{Im} \, t_\ell^\prime(p,p';k^2) = \left[ \sum_{i} \alpha_{\ell_i}(p,p') \right] \frac{1}{D(k^2) \, D^*(k^2)} \, \text{Im} \, D^*(k^2). \]

Then, using Eq. (11) and the facts that \( k^2 = s \) and

\[ \frac{1}{s - s' + i\epsilon} = P \frac{1}{s - s'} - i\pi \delta(s - s'), \]

we find

\[ \text{Im} \, D^*(k^2) = \frac{\pi}{2} \, k \, N_\ell(k^2). \]

Thus, the off-shell unitarity relation becomes

\[
\left[ \sum_{i} \alpha_{\ell_i}(p,p') \right] \frac{1}{D(k^2) \, D^*(k^2)} \left( - \frac{\pi}{2} \, k \, N_\ell(k^2) \right) = \\
- \frac{\pi k}{2} \left[ \sum_{i} \alpha_{\ell_i}(p,k) \right] \left[ \sum_{i} \alpha_{\ell_i}(k,p') \right] \frac{1}{D(k^2) \, D^*(k^2)}. \]
In order to satisfy this relation, we must have

\[ [\sum \alpha_{\ell i}(p,p')] N_\ell(k^2) = [\sum \alpha_{\ell i}(p,k)] [\sum \alpha_{\ell i}(k,p')] .\]

We know that if

\[ \sum \alpha_{\ell i}(p,p') = \sqrt{N_\ell(p)} \sqrt{N_\ell(p')} ,\]

off-shell unitarity is satisfied, but this can be used only when \( N_\ell(k^2) \) is of definite sign for \( k^2 > 0 \), or else we introduce spurious cuts in the amplitude. Now, suppose we ask that

\[ \sum \alpha_{\ell i}(p,p') = [\sum g_{\ell i}(p)] [\sum g_{\ell i}(p')] \]

where \( \sum g_{\ell i}(p) \) is real for \( p^2 > 0 \) and both sums must have the same functional form to preserve time-reversal invariance. We will then satisfy off-shell unitarity, providing

\[ \sum \alpha_{\ell i}(k,k) = [\sum g_{\ell i}(k)]^2 = N_\ell(k^2) .\]

We will also have a separable representation for the off-shell amplitude, which is very valuable for computer calculations in multi-particle scattering problems.

Therefore, suppose we desire a basis for the parameterization of \( t_\ell(p,p';k^2) \) in the general case where the \( N \) function and the phase shift can change sign. We have shown that such a basis must be of the form
where \( [\Sigma g_{i}^{2}(k)]^{2} = N_{s}(k^{2}) \), if it is to be separable, related to the on-shell \( N/D \) approach, and satisfy all our requirements including off-shell unitarity. Then, at points where \( t_{2}(k^{2}) \neq 0 \), we have

\[
t_{2}(p,p';k^{2}) = \frac{[\Sigma g_{i}^{2}(p)][\Sigma g_{i}^{2}(p')]}{N_{s}(k^{2})} t_{2}(k^{2}).
\]

We notice that if \( \Sigma g_{i}^{2}(k) \) has only one term, we will force \( N_{s}(k^{2}) \) to have definite sign and the best choice is the earlier one

\[
\Sigma g_{i}^{2}(p) = \sqrt{N_{s}(p)}. 
\]

However, if we choose \( \Sigma g_{i}^{2}(p) = a(p) - b(p) \), where the functions \( a(p) \) and \( b(p) \) (real for \( p^{2} > 0 \)) need not have definite sign, we have

\[
N_{s}(k^{2}) = [\Sigma g_{i}^{2}(k)]^{2} = [a(k)]^{2} + [b(k)]^{2} - 2 a(k) b(k),
\]

and \( N_{s}(k^{2}) \) can change sign. In this case, we would obtain

\[
t_{2}(p,p';k^{2}) = \frac{[a(p) - b(p)] a(p') - [a(p) - b(p)] b(p')}{D_{2}(k^{2})},
\]

which is equivalent to two separable terms in an integral equation of Fadeev type.

Again, the separation \( t_{2}(p,p';k^{2}) = t_{2}(k^{2}) + \Delta_{s}(p,p';k^{2}) \)
is easily made, with

\[
\Delta_{\ell}^{2}(p,p';k^2) = \frac{\left[ \Sigma g_{\ell}^{2}(p) \right] \left[ \Sigma g_{\ell}^{2}(p') \right] - N_{\ell}(k^2)}{D_{\ell}^{2}(k^2)}
\]

We have thus given limitations on a parameterization of the off-shell transition amplitude which is separable in incident and outgoing momenta, which satisfies all our general requirements, and which is related to the on-shell N/D approach.

IV. HALF-OFF-SHELL AMPHILITUDES

Let us consider the half-off-shell amplitudes \( t_{\ell}(p,k;k^2) \). Quantities like this enter into \( p-p \) bremsstrahlung calculations, and since \( t_{\ell}(p,k;k^2) = t_{\ell}(k,p;k^2) \) by time-reversal invariance, we need discuss only \( t_{\ell}(p,k;k^2) \).

Noyes and Kowalski have shown that in local potential theory, the half-off-shell amplitude \( t_{\ell}(p,k;k^2) \) satisfies the nonsingular integral equation

\[
t_{\ell}(p,k;k^2) = \frac{V_{\ell}(p,k)}{V_{\ell}(k,k)} t_{\ell}(k^2) + \int_{0}^{\infty} dq \ A_{\ell}(p,q;k^2) t_{\ell}(q,k;k^2),
\]

where

\[
A_{\ell}(p,q;k^2) = \frac{q^2}{k^2 - q^2} \left[ V_{\ell}(p,q) - \frac{V_{\ell}(p,k) V_{\ell}(k,q)}{V_{\ell}(k,k)} \right].
\]

Thus, at points where \( t_{\ell}(k^2) \neq 0 \), we may write
\[ t_{\xi}(p,k;k^2) = f_{\xi}(p,k) \ t_{\xi}(k^2) \], \text{ where } f_{\xi}(p,k) \text{ satisfies}

\[ f_{\xi}(p,k) = \frac{V_{\xi}(p,k)}{V_{\xi}(k,k)} + \int_0^\infty dq \ \Lambda_{\xi}(p,q;k^2) \ f_{\xi}(q,k). \]

Noyes also shows that any singularities due to the vanishing of \( V_{\xi}(k,k) \) cannot cause any essential difficulties.\textsuperscript{10}

We may also look for a representation of the form

\[ t_{\xi}(p,k;k^2) = t_{\xi}(k^2) + \vartheta_{\xi}(p,k;k^2), \]

and we shall find that we can obtain an integral equation for \( \vartheta_{\xi}(p,k;k^2) \) in local potential theory. First, we write the Lippmann-Schwinger equations for the half-off-shell amplitude and for the on-shell amplitude

\[ t_{\xi}(p,k;k^2) = V_{\xi}(p,k) + \int_0^\infty dq \ \frac{q^2}{k^2 - q^2 + i\varepsilon} \ V_{\xi}(p,q) \ t_{\xi}(q,k;k^2) \]

and

\[ t_{\xi}(k,k;k^2) = V_{\xi}(k,k) + \int_0^\infty dq \ \frac{q^2}{k^2 - q^2 + i\varepsilon} \ V_{\xi}(k,q) \ t_{\xi}(q,k,k^2). \]

Then, using \( t_{\xi}(k,k;k^2) = t_{\xi}(k^2) \) and

\[ \vartheta_{\xi}(p,k;k^2) = t_{\xi}(p,k;k^2) - t_{\xi}(k^2), \]

we find

\[ \vartheta_{\xi}(p,k;k^2) = V_{\xi}(p,k) - V_{\xi}(k,k) + \int_0^\infty dq \ \frac{q^2}{k^2 - q^2 + i\varepsilon} \ [V_{\xi}(p,q) - V_{\xi}(k,q)] \]

\[ \times \ t_{\xi}(q,k;k^2). \]
But now, since \( t_\ell(q,k;k^2) = t_\ell(k^2) + \theta_\ell(q,k;k^2) \), we have the following singular integral equation for \( \theta_\ell(p,k;k^2) \):

\[
\theta_\ell(p,k;k^2) = V_\ell(p,k) - V_\ell(k,k) + t_\ell(k^2) \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} [V_\ell(p,q) - V_\ell(k,q)] \\
+ \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} [V_\ell(p,q) - V_\ell(k,q)] \theta_\ell(q,k;k^2).
\]

We see that \( \theta_\ell(k,k;k^2) = 0 \) so that \( t_\ell(k,k;k^2) = t_\ell(k^2) \).

The separation can be made simply in the Wong-Zambotti, separable potential and \( N/D \) related approaches where we have

\[
t_\ell(p,k;k^2) = \frac{N_\ell(p,k;k^2)}{D_\ell(k^2)}.\]

Obviously, the separations are

\[
t_\ell(p,k;k^2) = \frac{N_\ell(p,k;k^2)}{N_\ell(k,k;k^2)} t_\ell(k^2)
\]

and

\[
t_\ell(p,k;k^2) = t_\ell(k^2) + \left[ \frac{N_\ell(p,k;k^2) - N_\ell(k,k;k^2)}{D_\ell(k^2)} \right].
\]

For a partial wave in which the phase shift does not change sign, and we can parameterize the amplitude in the form

\[
t_\ell(p,p';k^2) = \frac{\sqrt{N_\ell(p)} \sqrt{N_\ell(p')}}{D_\ell(k^2)},
\]
or, equivalently, with a single separable potential

\[ V_\lambda(p,p') = \lambda \, g_\lambda(p) \, g_\lambda(p') \],

we see that the half-off-shell factor

\[ f_\lambda(p,k) \]

is given by

\[ f_\lambda(p,k) = g_\lambda(p)/g_\lambda(k) \]

or

\[ f_\lambda(p,k) = \sqrt{N_\lambda(p)} / \sqrt{N_\lambda(k)} \],

where

\[ t_\lambda(p,k;k^2) = f_\lambda(p,k) \, t_\lambda(k^2) \]

Also, in the case of a two-term separable potential

\[ V_\lambda(p,p') = \lambda_1 \, g_\lambda(p) \, g_\lambda(p') - \lambda_2 \, h_\lambda(p) \, h_\lambda(p') \],

the half-off-shell factor

\[ f_\lambda(p,k) = N_\lambda(p,k;k^2)/N_\lambda(k,k;k^2) \]

is of the form

\[ f_\lambda(p,k) = g_\lambda(p) \, \alpha_\lambda(k^2) - h_\lambda(p) \, \beta_\lambda(k^2) \],

where \( \alpha \) and \( \beta \) are known functions of \( k^2 \). So if we knew the quasi-phase parameters discussed below, we could determine \( g_\lambda(p) \) and \( h_\lambda(p) \) by curve fitting.

Sobel\(^4\) has shown that, just as we may write the on-shell amplitude \( t_\lambda(k^2) \) as

\[ t_\lambda(k^2) = -\frac{2}{\pi} \frac{1}{k} \, e^{i \theta_\lambda(k^2)} \sin \theta_\lambda(k^2) \],

\[ (12) \]

the half-off-shell amplitude \( t_\lambda(p,k;k^2) \) may be parameterized in the form

\[ t_\lambda(p,k;k^2) = -\frac{2}{\pi} \frac{1}{k} \, e^{i \theta_\lambda(k^2)} \Delta_\lambda(p;k^2) \]

\[ (13) \]
where the quasi-phase parameter \( \Delta_2(p;k^2) \) is such that

\[
\Delta_2(k;k^2) = \sin \delta_2(k^2) .
\]

Although the half-off-shell nucleon-nucleon scattering amplitudes enter into the \( p - p \) bremsstrahlung process, it is difficult to extract the quasi-phase parameters from the experimental data, because the process also depends on the (poorly understood) photon emission from the internal lines of the \( p - p \) scattering diagrams. Nevertheless, it may be interesting to note the connection between the "experimental" quasi-phase parameters and the theoretical half-off-shell factors \( f_2(p,k) \) at an energy where \( \delta_2(k^2) \neq 0 \). We have \( t_2(p,k;k^2) = f_2(p,k) t_2(k^2) \), and if we use Eqs. (12) and (13), we find

\[
\Delta_2(p;k^2) = f_2(p,k) \sin \delta_2(k^2) .
\]

Since \( f_2(k,k) = 1 \), this explicitly shows that

\[
\Delta_2(k;k^2) = \sin \delta_2(k^2) .
\]

V. APPLICATIONS

In the application of these ideas to scattering calculations in which the off-energy-shell two-body partial wave transition amplitude is needed, the first step is to choose a method of calculating or representing the off-shell amplitude, making sure it fits the on-shell amplitude as well as possible. With the methods known at present, however, this really amounts only to choosing a basis for
the parameterization of the on-shell and off-shell amplitudes.

In the range $k^2 < 0$, where the on-shell amplitude is no more accessible to experiment than the off-shell amplitude (and, for the multiplicative separation, at values of $k^2$ where $t_L(k^2) = 0$), we use the method originally chosen. But for $k^2 > 0$, we may use one of the separations proposed herein and put into the calculation the experimental on-shell two-body data in the form of phase shifts by writing

$$t_L(k^2) = -\frac{2}{\pi} \frac{1}{k} e^{i\delta_L(k^2)} \sin \delta_L(k^2).$$

This allows us to take on-shell two-body effects into account exactly, and to study the off-shell behavior of amplitudes independently of their on-shell behavior.

Notice that the multiplicative full off-shell and half-off-shell factors $X_L(p,p';k^2)$ and $f_L(p,k)$ have a pole in $k^2$ at $k^2 = k_0^2$, where $t_L(k_0^2) = 0$. This is so because for $k^2$ near $k_0^2$,

$$t_L(k^2) \approx t_L(k_0^2) (k^2 - k_0^2)$$

and we know that

$$X_L(p,p';k^2) = t_L(p,p';k^2)/t_L(k^2)$$

while

$$f_L(p,k) = t_L(p,k;k^2)/t_L(k^2).$$
So, for \( k^2 \) near \( k_0^2 \), we have

\[
X_\ell(p,p';k^2) = \frac{t_\ell(p,p';k^2)}{t_\ell'(k_0^2)(k^2 - k_0^2)}
\]

and

\[
f_\ell(p,k) = \frac{t_\ell(p,k;k^2)}{t_\ell'(k_0^2)(k^2 - k_0^2)}
\]

In the case of the one-pole dominance model in the N/D approach of Lovelace, we see that the fit to the on-shell data really amounts to a fit to \( N_\ell(k^2) \), the numerator function in the exact on-shell transition amplitude, in the neighborhood of the pole. For this purpose, we would like to suggest functions of the form

\[
N_\ell(k^2) = \sum_i G_i \frac{1}{\pi k^2} Q_\ell \left( 1 + \frac{\mu_1^2}{2k^2} \right),
\]

where \( Q_\ell(x) \) is the Legendre function of the second kind. We note that

\[
\frac{1}{k^2} Q_\ell \left( 1 + \frac{\mu_1^2}{2k^2} \right)
\]

is real and of definite sign for \( k^2 > 0 \). So, this choice satisfies the two requirements that we put on \( N_\ell(k^2) \) in this approach. Also, this functional form has left cuts in \( k^2 \). We are thus approximating the function \( N_\ell(k^2) \), which is known to have left cuts, with a functional form which, indeed, has left cuts rather than the poles we would find if we took some other common forms. In fact, the \( \mu_1 \) can be chosen to ensure that these cuts.
start at the same points as the known cuts in \( t_\ell(k^2) \).

From the fit to the on-shell data, which (at least in the neighborhood of the dominant pole) is

\[
t_\ell(k^2) = -\frac{2}{\pi} \frac{1}{k} e^{i\delta_\ell(k^2)} \sin \delta_\ell(k^2) = \frac{N_\ell(k^2)}{1 - \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\varepsilon}}
\]

we see that the suggested functional form leads to an on-shell amplitude which goes to zero as \( k^2 \to \infty \). Also, the amplitude has the correct threshold behavior as \( k^2 \to 0 \), and the integrals appearing in the expression for \( t_\ell(k^2) \) can be done in closed form.

The suggested functional form also leads to a bound state or resonance form factor of the form

\[
\mathcal{G}_\ell(p) = \sqrt{N_\ell(p^2)} = \left[ \frac{\Sigma G_i}{i\pi} \frac{1}{p^2} - Q_\ell \left( 1 + \frac{\mu^2_\ell}{2p^2} \right) \right]^{\frac{1}{2}}
\]

Then the bound state and resonance form factors have the behavior which, according to Lovelace, \(^8\) they are known to possess. In particular, the form factors have only a left cut in \( p^2 \). Also, the form factors have a threshold behavior like \( p^2 \) as \( p \to 0 \), which is necessary for the correct centrifugal barrier.

We might also suggest the use of functional forms such as

\[
\mathcal{G}_\ell(p) = \left[ \frac{\Sigma G_i}{i\pi} \frac{1}{p^2} - Q_\ell \left( 1 + \frac{\mu^2_\ell}{2p^2} \right) \right]^{\frac{1}{2}}
\]

in separable potentials of the form
\[ V_2(p,p') = \lambda_1 \hat{g}_2(p) \hat{g}_2(p') - \lambda_2 \hat{h}_2(p) \hat{h}_2(p'). \]

Note that the on-shell Born approximation to the transition amplitude obtained from such a potential is the same as that arising from a superposition of Yukawa potentials. Also, Watson\(^\text{16}\) and Wong and Zambotti\(^\text{12}\) have shown that in potential theory \( t_2(p,p';k^2) \) can be written (for \( p^2 \) and \( p'^2 \) positive) as

\[ t_2(p,p';k^2) = \frac{N_2(p,p';k^2)}{D_2(k^2)}, \]

where \( N_2(p,p';k^2) \) and \( D_2(k^2) \) are, respectively, operator analytic and analytic for \( k^2 \) in the complex plane cut from 0 to \( \infty \). In addition, all the poles of \( t_2(p,p';k^2) \) in the variable \( k^2 \) for positive values of \( p^2 \) and \( p'^2 \) must appear as zeros of \( D_2(k^2) \), and \( N_2(p,p';k^2) \) will have cut singularities for negative \( p^2 \) or \( p'^2 \). In fact, if we use the suggested potential forms in a separable potential, we will find that the resulting amplitude has the desired analyticity properties as can be seen from Eq. (1).

VI. CONCLUSION

In conclusion, the basic idea of this paper is that a separation of the off-energy-shell behavior from the better known and more experimentally accessible on-energy-shell behavior of two-body partial-wave transition amplitudes may be valuable in attempting to understand the off-energy-shell behavior of transition amplitudes. Also, such a separation may allow the incorporation of experimental
on-shell two-body scattering data in the form of phase shifts into scattering calculations requiring the knowledge of off-energy-shell two-body amplitudes.
FOOTNOTES AND REFERENCES

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