Poincaré-Birkhoff-Witt Bases for Dioperads

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by

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ABSTRACT OF THE DISSERTATION

Poincaré-Birkhoff-Witt Bases for Dioperads

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We define a functorial way to turn a cyclic operad into a dioperad. Bergman’s Diamond Lemma and Hoffbeck’s PBW bases for operads are generalized to the setting of dioperads. We use these results to show that the induced dioperads from the cyclic operads $Com$ and $Lie$ are Koszul.
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Chapter 1

Introduction

Let $V$ be a finite dimensional $K$-vector space equipped with an algebraic structure. That is, $V$ is equipped with products $\mu_i : V^\otimes n_i \to V$ satisfying some relations. Furthermore, suppose there is a nondegenerate bilinear form $\eta : V \otimes V \to \mathbb{K}$. Then we obtain a coalgebraic structure on $V^*$ by taking the dual maps $\mu_i^*$, and hence, a coalgebraic structure on $V$. In this article, we study the compatibility relations between these structures, and in particular, we want to know if the algebraic structure $(V, \mu_i)$ is Koszul, then is the algebraic structure $(V, \mu_i, \mu_i^*)$ Koszul.

Studying algebraic structures of the form $(V, \mu_i, \mu_i^*)$ can be done using the theory of dioperads. The notion of a dioperad was defined in [6]. These objects can be used to describe certain algebraic structures which have operations involving multiple inputs and outputs. In [6], the author gave a criterion to determine if a dioperad generated by binary operations is Koszul.

The present article was motivated by a conjecture of Chas and Sullivan in [2]. According to Chas and Sullivan, the reduced equivariant homology $H$ of the free loop space of a smooth oriented manifold has the structures of a gravity algebra and gravity
coalgebra. In [2], the authors conjecture that the algebraic structure on $H$ generated by the gravity algebra brackets and gravity coalgebra cobrackets is Koszul dual to the “positive boundary” version of the algebraic structure of formal Frobenius manifolds. But, before one can prove this conjecture, one must understand this algebraic structure.

In [7] the author discusses the operad $Grav$ and its Koszul dual $Hycomm$ and proves that these operads are Koszul. Algebras over $Grav$ are called gravity algebras and algebras $Hycomm$ are called hypercommutative algebras. A gravity algebra is a graded vector space $A$ equipped with antisymmetric products $[x_1, \ldots, x_n] : A^\otimes n \to A$ of degree $2-n$ that satisfy the following relations: for $k > 2$ and $l \geq 0$ and $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$,

$$\sum_{1 \leq i < j \leq k} \pm[[a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k, b_1, \ldots, b_l] = \begin{cases} [[a_1, \ldots, a_k], b_1, \ldots, b_l] & \text{if } l > 0 \\ 0 & \text{if } l = 0 \end{cases}$$

One may think of a gravity algebra as a graded vector space equipped with a sequence of antisymmetric products that satisfy a “generalized” version of the Jacobi identity. A hypercommutative algebra is a graded vector space $A$ equipped with symmetric products $(x_1, \ldots, x_n) : A^\otimes n \to A$ of degree $2(n-2)$ that satisfy the following relations: for $a, b, c, x_1, \ldots, x_n \in A$ and $n \geq 0$,

$$\sum_{S_1 \sqcup S_2 = \{1, \ldots, n\}} \pm((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \sqcup S_2 = \{1, \ldots, n\}} \pm(a, (b, c, x_{S_1}), x_{S_2})$$

where if $S = \{s_1, \ldots, s_k\}$ is a finite set, $x_S$ means $x_{s_1}, \ldots, x_{s_k}$. One may think of a hypercommutative algebra as a graded vector space equipped with a sequence of symmetric products that satisfy a “generalized” version of the associativity condition.

The work in this paper is an attempt to answer what the algebraic structure is described in the conjecture by Chas and Sullivan. As a result, we provide a criterion to determine if a dioperad generated by operations of any arity is Koszul. In chapter 2 we recall the definition of a cyclic operad given in [8] and provide an alternative definition
of a dioperad using the language of digraphs. We then define a functor which turns a
cyclic operad into a dioperad. In chapter 3 we generalize Hoffbeck’s theory of PBW
bases for operads to the setting of dioperads and in chapter 4 we generalize Bergman’s
Diamond Lemma to the setting of dioperads. The final chapter provides a recipe to
determine if a dioperad is Koszul by using the Diamond Lemma to show the existence
of a PBW basis. We demonstrate this procedure on the induced dioperads of the cyclic
operads $Lie$ and its quadratic dual. For the induced dioperads $\widetilde{Com}$ and $\widetilde{Lie}$ we exhibit
an explicit PBW basis, and hence, providing another proof that these dioperads are
Koszul.
Chapter 2

Cyclic Operads and Dioperads

All vector spaces throughout this article are defined over a field $\mathbb{K}$ of characteristic zero.

2.1 Graphs

Definition 2.1 (See [8]). A graph $\Gamma$ consists of the following data:

- Finite sets $F(\Gamma)$ and $V(\Gamma)$ where elements of $F(\Gamma)$ are called flags and elements of $V(\Gamma)$ are called vertices.

- A function $p : F(\Gamma) \to V(\Gamma)$; we call the elements of $p^{-1}(v)$ the flags that meet at $v$.

- An involution $\sigma : F(\Gamma) \to F(\Gamma)$, where the fixed points of $\sigma$ are called the legs of $\Gamma$ and the remaining orbits are called the edges of $\Gamma$.

We shall denote by $L(\Gamma)$ the set of legs of $\Gamma$. 
2.1.1 The category $G_0$

Let $G_0$ be the category where $\text{Ob}G_0$ are graphs whose set of edges is empty. We denote an object $\Gamma$ of $G_0$ as $(L,V,p)$ where $L = L(\Gamma)$, $V = V(\Gamma)$, and $p$ is the map from $F(\Gamma)$ to $V(\Gamma)$ that sends a flag to the vertex at which it meets. We assume that the valency of valency at each vertex is at least 3. A morphism with domain $(L_1,V_1,p_1)$ and codomain $(L_2,V_2,p_2)$ is defined to be a triple $(\Gamma, \alpha, \beta)$ such that:

- $\Gamma$ is a graph with the connected components of $\Gamma$ simply connected, $F(\Gamma) = L_1$ and $V(\Gamma) = V_1$
- $\alpha : L_2 \to L(\Gamma)$ is bijective
- $\beta : V_2 \to \pi_0(\Gamma)$ is bijective where $\pi_0(\Gamma)$ denotes the set of connected components of $\Gamma$
- $(p_1 \circ \alpha)(f) \in (\beta \circ p_2)(f)$ for all $f \in L_2$

The morphisms of $G_0$ will be denoted by $\text{Mor}G_0$. We note that $G_0$ is a symmetric monoidal category with tensor product given by disjoint union.

Let $\Gamma_1 : (L_1,V_1,p_1) \to (L_2,V_2,p_2)$ and $\Gamma_2 : (L_2,V_2,p_2) \to (L_3,V_3,p_3)$ be any two morphisms in $G_0$. That is, $\Gamma_i$ is the triple $(\Gamma_i, \alpha_i, \beta_i)$. The composition $\Gamma = \Gamma_2 \circ \Gamma_1$ is defined as the triple $(\Gamma, \alpha, \beta)$ where $\alpha = \alpha_2, \beta = \beta_2$. We have: $F(\Gamma) = L_1, V(\Gamma) = V_1, p = p_1$, and the involution $\sigma : F(\Gamma) \to F(\Gamma)$ defined by:

$$
\sigma(f) = \begin{cases} 
  f & : f \text{ is an edge of } \Gamma_1 \\
  \alpha_1(\sigma_2(\alpha_1^{-1}(f))) & : f \text{ is a leg of } \Gamma_1
\end{cases}
$$

where $\sigma_2$ is the involution coming from $\Gamma_2$. In other words, we glue together pairs of legs of $\Gamma_1$ that correspond to edges of $\Gamma_2$. 

5
2.1.2 Example of a composition of morphisms

Let $\Gamma_1$ be the morphism

\[
\begin{array}{c}
\bullet x_1 \quad x_2 \\
\bullet y_1 \quad y_2 \\
\bullet z_1 \quad z_2 \\
\bullet u_1 \\
\end{array}
\quad \xrightarrow{\Gamma_1} 
\begin{array}{c}
\bullet a_1 \quad a_2 \quad a_3 \quad b_1 \quad b_2 \\
\bullet a_4 \quad a_5 \quad a_6 \quad b_3 \quad b_4 \\
\end{array}
\]

where $\Gamma_1$ is the graph

and $\Gamma_2$ be the morphism

\[
\begin{array}{c}
\bullet a_1 \quad a_2 \quad a_3 \quad b_1 \quad b_2 \\
\bullet b_3 \quad b_4 \\
\end{array}
\quad \xrightarrow{\Gamma_2} 
\begin{array}{c}
\bullet c_1 \quad c_2 \quad c_3 \quad c_4 \\
\bullet c_5 \quad c_6 \quad c_7 \quad c_8 \\
\end{array}
\]

where $\Gamma_2$ is the graph:

Then, their composition $\Gamma = \Gamma_2 \circ \Gamma_1$ is the graph

\[
\begin{array}{c}
\bullet z_1 \quad z_2 \\
\bullet u_1 \quad u_2 \quad u_3 \\
\bullet y_1 \quad y_2 \\
\bullet x_1 \quad x_2 \\
\bullet x_3 \quad x_4 \\
\bullet y_3 \quad y_4 \\
\bullet z_3 \quad z_4 \\
\bullet a_1 \quad a_2 \quad a_3 \quad b_1 \quad b_2 \\
\bullet a_4 \quad a_5 \quad a_6 \quad b_3 \quad b_4 \\
\end{array}
\]

Definition 2.2 An $n$-corolla in $\mathcal{G}_0$ is a graph isomorphic to the following graph:

\[
[n] := \begin{array}{c}
1 \\
\vdots \\
n \\
0
\end{array}
\]

Observe that a connected object $\Gamma \in \mathcal{G}_0$ with $|L(\Gamma)| = n + 1$ is an $n$-corolla in $\mathcal{G}_0$, and in general, objects of $\mathcal{G}_0$ are products of corollas.
2.2 Cyclic Operads

**Definition 2.3** Let \( C \) be a symmetric monoidal category. A cyclic operad is a symmetric monoidal functor \( \mathcal{P} : \mathcal{G}_0 \to C \).

In the succeeding sections, we will be working with augmented weight graded dioperads and the reduced bar construction for dioperads. In the setting of (cyclic) operads we refer the reader to [11] section 1 and [5] Chapters 3 and 5. By a weight grading on a vector space \( V \), we mean a vector space decomposition

\[
V = \bigoplus_{s=0}^{\infty} V(s).
\]

We say that the vectors in \( V(s) \) have weight \( s \). Moreover, from this point on, unless we note otherwise, the category \( C \) will denote the category of finite dimensional differential graded vector spaces equipped with a weight grading. In other words, an object \( V \) of \( C \) is a direct sum

\[
V = \bigoplus_i V_i,
\]

where each \( V_i \) comes equipped with a weight grading, and together with a differential \( d : V_i \to V_{i-1} \) such that \( d \) preserves weights.

### 2.2.1 Definition of a (partial) operad versus definition 2.3

In [13] (see also [12] section 5.3.7), the author define an operad as a sequence of vector spaces \( \{ \mathcal{P}(n) \} \), such that \( \mathcal{P}(n) \) has an \( S_n \)-action for each \( n \), and the operadic composition is given by the partial composition product, that is we have maps

\[
\mathcal{P}(n) \circ_i \mathcal{P}(m) \to \mathcal{P}(n + m - 1)
\]

for \( 1 \leq i \leq n \). This product is supposed to satisfy various axioms of associativity and equivariance. For cyclic operads, we require that the \( S_n \)-action on \( \mathcal{P}(n) \) extend to an
\(S_{n+1}\)-action and require a few more axioms to be satisfied (see [13] Proposition 42).

In the "partial definition" of an operad, we think of the operation \(o_i\) as grafting the zeroth leg (or root if dealing with operads without a cyclic structure) of an \(m\)-tree to the \(i\)th input of an \(n\)-tree. To see that these definitions are equivalent, let \(\mathcal{P}\) be a cyclic operad as in definition 2.3. For any connected objects \([n], [m] \in \mathcal{G}_0\) and \(1 \leq i \leq n\), there is a morphism \(\Gamma_i : [n] \sqcup [m] \to [n + m - 1]\), defined by joining the flag labelled 0 in \([m]\) to the flag labelled \(i\) in \([n]\). Setting \(\mathcal{P}([n]) = \mathcal{P}(n)\) for \(n \geq 2\), we see that we have a sequence of vector spaces equipped with an \(S_{n+1}\)-action given by \(\mathcal{P}(\text{Aut}([n]))\).

We define \(\mathcal{P}(1) := \mathbb{K}\). Moreover, as \(\mathcal{P}\) is a monoidal functor, we have

\[
\mathcal{P}([n] \sqcup [m]) = \mathcal{P}([n]) \otimes \mathcal{P}([m]) \xrightarrow{\mathcal{P}(\Gamma_i)} \mathcal{P}([n + m - 1]) = \mathcal{P}(n + m - 1)
\]

Since \(\mathcal{P}\) is required to satisfy various axioms, \(\mathcal{P}(\Gamma_i)\) must as well. These axioms give rise to the axioms that the operadic composition product must satisfy. For more details see [3] Section 2.

### 2.2.2 Example

Let \(M \in \mathcal{C}\) and \(\eta\) a nondegenerate bilinear form on \(M\). The cyclic endomorphism operad \(\text{End}_0(M, \eta)\) is defined by:

- For objects \([n] \in \mathcal{G}_0\), \(\text{End}_0(M, \eta)([n]) = M^\otimes n+1\)

- For morphisms \(\Gamma \in \mathcal{G}_0\), \(\text{End}_0(M, \eta)(\Gamma)\) is defined by contracting pairs of tensor factors via \(\eta\)

- The \(S_{n+1}\)-action is given by permuting the labels of the \(n + 1\) legs of an object
2.2.3 \( \mathcal{P} \)-algebras.

**Definition 2.4** A \( \mathcal{P} \)-algebra is a pair \((M, \eta)\) equipped with a natural transformation \(\rho : \mathcal{P} \to \text{End}_0(M, \eta)\).

A similar argument as in 2.2.1 shows that the previous definition of a \( \mathcal{P} \)-algebra is equivalent to the standard definition of an algebra over a cyclic operad \( \mathcal{P} \) (see [7] section 2.4). The following articles provide several standard examples of algebras over an (cyclic) operad \( \mathcal{P} \): [12] Chapter 13, [9] Chapter 2, and the seminal paper [10].

2.2.4 More Operadics

One can also define (cyclic) \( S \)-modules, (cyclic) suboperads, (cyclic) operadic ideals, and quotients in a similar fashion to definition 2.3. For our purposes, we will need the notions of cyclic suboperads, cyclic operadic ideals and quotients. A cyclic suboperad \( Q \) of \( \mathcal{P} \) is a functor on \( G_0 \) such that \( Q([n]) \subset \mathcal{P}([n]) \) is a \( S_n \)-submodule for every \( n \). This brings us to the notion of an ideal. Let \( f \in \mathcal{P}([n]), g \in \mathcal{P}([m]) \). A cyclic suboperad \( I \subset \mathcal{P} \) is an ideal if the image of \( f \otimes g \) under each of the maps \( \circ_i : \mathcal{P}([n]) \otimes \mathcal{P}([m]) \to \mathcal{P}([n + m - 1]) \) lies in \( I([n + m - 1]) \) whenever \( f \in I([n]) \) or \( g \in I([m]) \).

2.3 Digraphs

**Definition 2.5** A digraph is a graph \( \Gamma \) together with a partition \( F(\Gamma) = F_+(\Gamma) \sqcup F_-(\Gamma) \) of \( F(\Gamma) \) into incoming (+) and outgoing (-) flags, such that each edge has one outgoing and one incoming flag. We shall denote the legs as: \( L_\pm = L(\Gamma) \cap F_\pm(\Gamma) \). Elements of \( L_+ \) are called leaves and elements of \( L_- \) are called roots.
2.3.1 The category $\mathcal{D}_0$

Let $\mathcal{D}_0$ denote the category of digraphs without edges, and whose morphisms are simply connected digraphs. Composition of morphisms in $\mathcal{D}_0$ is defined the same way as it is in $\mathcal{G}_0$. Observe that $\mathcal{D}_0$ is a symmetric monoidal category with tensor product given by disjoint union.

**Definition 2.6** An $(m, n)$-spider in $\mathcal{D}_0$ is a digraph of the form

$$[m, n] := \begin{array}{ccc} 1 & \ldots & m \\ & \downarrow & \\
1 & & m \end{array}$$

Connected objects of $\mathcal{D}_0$ are isomorphic to $(m, n)$-spiders, and in general, objects of $\mathcal{D}_0$ are products of spiders.

2.4 Dioperads

**Definition 2.7** A dioperad is a symmetric monoidal functor $\mathcal{P} : \mathcal{D}_0 \to \mathcal{C}$.

2.4.1 Original definition of dioperads versus definition 2.7

In [6], the author defines a dioperad as a sequence $\{\mathcal{P}(m, n)\}$ of $(S_m, S_n)$-bimodules, where $m, n \in \mathbb{N}$, equipped with composition morphisms $i \circ_j : \mathcal{P}(m_1, n_1) \otimes \mathcal{P}(m_2, n_2) \to \mathcal{P}(m_1 + m_2 - 1, n_1 + n_2 - 1)$ for each $1 \leq i \leq n_1, 1 \leq j \leq m_2$, and a morphism $\eta : \mathbb{K} \to \mathcal{P}(1, 1)$. These morphisms are required to satisfy axioms of associativity and equivariance (see [6] section 1.1).

In a similar fashion to 2.2.1, we would like to point out why definition 2.7 is equivalent to the definition given in [6]. To see this, let $\mathcal{P}$ be a dioperad as in definition 1.3. Let $[m, n], [r, s]$ be any two connected objects in $\mathcal{D}_0$, and let $i, j \in \mathbb{N}$ with $1 \leq i \leq n, 1 \leq j \leq r$. Then, there is a morphism $\Gamma_i : [m, n] \otimes [r, s] \to [m + r - 1, n + s - 1]$. 
Setting $P([m,n]) = P(m,n)$, we get a sequence of vector spaces and these spaces come equipped with a left $S_m$-action and a right $S_n$-action given by $P(\text{Aut}([m,n]))$. And, since $P$ is a monoidal functor, we again have

$$P([m,n] \otimes [r,s]) = P([m,n]) \otimes P([r,s]) \xrightarrow{P(\Gamma)} P([m+r-1,n+s-1]) = P(m+r-1,n+s-1)$$

For more details on this see [3] Section 2.

### 2.4.2 Example

Let $M \in \mathcal{C}$. The endomorphism dioperad $\text{End}_{0}^{-}(M)$ is defined by:

- For objects $[m,n] \in D_0$, $\text{End}_{0}^{-}(M)(m,n) = M^{*} \otimes M^{n} \otimes M^{m}$, where $M^{*}$ denotes the vector space dual of $M$.
- For morphisms $\Gamma \in D_0$, $\text{End}_{0}^{-}(M)(\Gamma)$ is defined by contracting pairs of tensor factors from $M$ and $M^{*}$.
- The left $S_m$-action comes from permuting the labels of the outputs (i.e. the elements of $L_{-}$).
- The right $S_n$-action comes from permuting the labels of the inputs (i.e. the elements of $L_{+}$).

**Definition 2.8** A $\mathcal{P}$-algebra is an object $M \in \mathcal{C}$ equipped with a natural transformation $\rho : \mathcal{P} \to \text{End}_{0}^{-}(M)$.

### 2.4.3 $\mathcal{P}$-algebras

We would first like to point out that the previous definition is also equivalent to the definition of a $\mathcal{P}$-algebra given in section 1.2 [6] (via a similar argument used in 1.4.1). Since we can regard an operad as a dioperad by setting $\mathcal{P}(1,n) := \mathcal{P}(n)$
and \( P(m, n) := 0 \) for \( m > 1 \), we see that there are already several trivial examples of \( P \)-algebras. For nontrivial examples of \( P \)-algebras, see section 2 of [6] and [14] section 2.3.

### 2.4.4 More Dioperadics

One can define \((S_m, S_n)\)-bimodules (or \( S \)-bimodules for short), subdioperads, dioperadic ideals, and quotients. To do this, we simply follow the same procedure as in 1.2.4 except instead we consider the subcategory \( D_0^- \) of \( D_0 \) where \( \text{Ob} D_0^- = \text{Ob} D_0 \) and \( \text{Mor} D_0^- \) are isomorphisms of digraphs. These concepts were originally introduced in [6]. Using a similar arguments as in 2.2.1, one can show that the definition in [6] is equivalent to definition 2.7.

### 2.4.5 Codioperads

We will be working with the (reduced) bar construction of a quadratic dioperad. In order to make sense of this construction, the notion of a codioperad must be defined.

A codioperad is a symmetric monoidal functor \( D_0^{\text{op}} \rightarrow C \). A codioperad structure can be thought of the dual structure to that of a dioperad. This is a generalization of the notion of a cooperad. For more on this, see [5] 1.2.17 and [12] section 5.7.

**Definition 2.9** The augmentation ideal of a dioperad \( P \) is defined to be \( \tilde{P}(1, 1) := 0 \) and \( \tilde{P}(m, n) := P(m, n) \) for \( m + n \geq 3 \).

### 2.4.6 Dioperads equipped with a weight grading

A dioperad \( P \) is said to be equipped with a weight grading if each term \( P(m, n) \) is weight graded and the monoidal structure on \( P \) preserves the weight grading. In other
words, given homogenous elements $p \in \mathcal{P}_{(s)}(m,n), q \in \mathcal{P}_{(t)}(m',n')$ we have

$$p \circ_j q \in \mathcal{P}_{(s+t)}(m+m'-1, n+n'-1).$$

We say that a dioperad equipped with a weight grading is **connected** if

$$\mathcal{P}_{(0)}(m,n) = \begin{cases} \mathbb{K}.1 \text{ for } (m,n) = (1,1) \\ 0 \text{ otherwise} \end{cases}$$

Observe that a connected dioperad is augmented, where the augmentation ideal comes from the projection $\mathcal{P} \rightarrow \mathcal{P}_{(0)}$.

### 2.4.7 Free dioperad

An $(m,n)$-tree is a directed tree such that every vertex has at least one incoming flag, one outgoing flag, and the leaves are labelled by $\{1, \ldots, n\}$ and the roots are labelled by $\{1, \ldots, m\}$. Let $E = \{E(m,n)\}$ where $E(m,n)$ is a $(\mathbb{S}_m, \mathbb{S}_n)$-bimodule.

The **free dioperad generated by $E$**, denoted $\mathcal{F}(E)$ is defined as:

$$\mathcal{F}(E)(m,n) := \bigoplus_{(m,n) \text{ trees } T} E(T)$$

where $E(T) := \bigotimes_{v \in V} E(\text{Out}(v), \text{In}(v))$, $\text{Out}(v)$ is the number of outgoing edges of $v$ and $\text{In}(v)$ is the number of incoming edges of $v$. In the definition of $E(T)$ we are taking the tensor product over all vertices $v$ of the underlying tree $T$. We think of $\mathcal{F}(E)$ as being generated by all possible ways of composing trees via $i \circ_j$.

The free dioperad $\mathcal{F}(E)$ generated by $E$ has a natural weight grading given by the number of vertices of a tree. There is also a notion of a **cofree codioperad generated by $E$**, which we denote by $\mathcal{F}^c(E)$. As $\mathbb{S}$-bimodules $\mathcal{F}(E)$ and $\mathcal{F}^c(E)$ are isomorphic, however $\mathcal{F}^c(E)$ is generated by all possible ways of “de-composing” trees.
2.4.8 Homogeneous dioperadic ideals

A homogenous dioperadic ideal is a dioperadic ideal $I$ such that $I = \bigoplus I(s)$ where $I(s) = I \cap \mathcal{P}(s)$. The quotient of a dioperad $\mathcal{P}$ by a homogenous ideal $I$ induces a weight grading on the quotient dioperad given by $(\mathcal{P}/I)(s) = \mathcal{P}(s)/(\mathcal{P}(s) \cap I)$.

2.4.9 Quadratic dioperads

A quadratic dioperad is a dioperad $\mathcal{P} = \mathcal{F}(E)/I$ where $I$ is the dioperadic ideal generated by an $S$-submodule $R \subset \mathcal{F}(2)(E)$. Observe that this definition of a quadratic dioperad allows for generators with multiple inputs and multiple outputs. If a dioperad is generated by binary products and coproducts, that is $E(m, n) = 0$ for $(m, n) \neq (2, 1), (1, 2)$, then this definition of quadratic dioperad is equivalent to the definition given in [6] section 2.4.

Observe that a quadratic dioperad has a natural weight grading induced by the weight grading of the free dioperad. We will always assume $E(m, n) = 0$ if $m = 0$ or $n = 0$. Moreover, unless otherwise stated, we will always assume that only nonzero element of weight 0 is the unit element. Thus, we will be concerned with connected (and therefore augmented) dioperads. Observe that in this situation, we have the relations $\mathcal{P}(1)(m, n) = E(m, n)$ and $\mathcal{P}(2)(m, n) = \mathcal{F}(2)(E)/I$.

2.5 Reduced Bar Construction for Dioperads

2.5.1 Suspension and desuspension dioperads.

We use the definitions of the suspension and desuspension dioperads as defined in [6] section 1.4. The suspension dioperad $\Sigma$ is defined to be the endomorphism dioperad of $\mathbb{K}[1]$. This means $\Sigma(m, n)$ is a one dimensional vector space placed in degree $n - m$. 
As an $S$-bimodule, we equip $\Sigma(m, n)$ with the sign representations of $S_m$ and $S_n$. The suspension of a dioperad $P$ is $\Sigma P := \Sigma \otimes P$. The sheared suspension dioperad $\Lambda$ is defined by $\Lambda(m, n) := \Sigma(m, n)/(2 - 2m)$. This implies $\Lambda(m, n)$ is sitting in degree $m + n - 2$. The sheared suspension of a dioperad $P$ is $\Lambda P := \Lambda \otimes P$.

We define the desuspension dioperad $\Sigma^{-1}$ as the endomorphism dioperad of $\mathbb{K}[-1]$. Thus, $\Sigma^{-1}(m, n)$ is a one dimensional vector space in degree $m - n$ equipped with the sign representations of $S_n$ and $S_m$ and the desuspension of a dioperad $P$ is $\Sigma^{-1}P := \Sigma^{-1} \otimes P$. The sheared desuspension dioperad $\Lambda^{-1}$ is defined by $\Lambda^{-1}(m, n) := \Sigma(m, n)/(2m - 2)$. This implies $\Lambda^{-1}(m, n)$ is sitting in degree $2 - m - n$. The sheared desuspension of a dioperad $P$ is $\Lambda^{-1}P := \Lambda^{-1} \otimes P$.

### 2.5.2 Reduced Bar Construction

Given an augmented dioperad $P$ equipped with a weight grading, we define the reduced bar construction $\bar{B}(P) := F_c(\Sigma \tilde{P})$. Observe that $\bar{B}(P)$ has differential $d + \delta$ where $d$ is induced by the differential of $P$ and $\delta$ is induced by edge contractions (see [10] 3.2.3). In general, we may consider dioperads $P$ where $P$ is a dg-$S$-bimodule equipped with a weight grading. If $P$ is simply a weight graded $S$-bimodule, then $d$ is trivial.

Observe that $\bar{B}(P)$ comes equipped with a homological grading given by the number of vertices of a tree. That is, $\bar{B}_\bullet(P) = F_c^\bullet(\Sigma \tilde{P})$. Moreover, we get a chain complex

$$\cdots \xrightarrow{\delta} \bar{B}_s(P) \xrightarrow{\delta} \bar{B}_{s-1}(P) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bar{B}_1(P) \xrightarrow{\delta} \bar{B}_0(P)$$

We will use parentheses to distinguish the weight from the homological degree. In other words, given a weight graded dioperad $P$, its bar construction $\bar{B}(P)$ has a homological grading and a weight grading. An element $\alpha \in \bar{B}_s(P)(t)$ has homological degree $s$, that is $s$-many vertices, and weight $t$, that is the sum of the weights of the elements placed
at those vertices is $t$.

Given $m, n \geq 1$, we get the following complex for $\mathcal{B}_\bullet(P)(m, n)$:

where the leftmost term is placed in (homological) degree $m + n - 2$ and the rightmost term in (homological) degree 1. The total degree of an element in $\mathcal{B}_\bullet(P)(m, n)$ is the sum of its homological degree plus the sum of the weights of the elements placed at each vertex of an $(m, n)$ tree in $\mathcal{B}_\bullet(P)(m, n)$.

In [6] section 3, the author considers the cobar complex $CP$ of a dioperad $P$. In order to obtain the cobar complex in [6] from the reduced bar construction we simply require the generators of a dioperad to be binary and observe that we have

$$CP^*(m, n) = (\mathcal{B}_\bullet(P)(m, n))^* \otimes \text{Sgn}_n \otimes \text{Sgn}_m$$

From this, we can then obtain the cobar dual $DP$ of $P$ by shifting the degrees of the terms in $CP$.

### 2.6 Koszul dual of a dioperad

Let $P$ be a connected dg-dioperad equipped with weight. The Koszul dual is defined by

$$K(P)_{(\delta)}(m, n) = H_s(\mathcal{B}_\bullet(P)_{(\delta)}(m, n), \delta)$$

The Koszul dual $K(P)$ is a codioperad (see [5] 5.2.3).

**Lemma 2.10** Let $P$ be a weight graded connected dg-dioperad. Then we have the fol-
following equalities:
\[
\begin{cases}
\tilde{B}_s(P)_{(s)}(m,n) = \mathcal{F}_{(s)}(\Sigma \tilde{P}(1))(m,n) \\
\tilde{B}_d(P)_{(s)}(m,n) = 0 \text{ if } d > s
\end{cases}
\]

**Proof.** The first equality holds since given an \((m,n)\)-tree \(\alpha \in \tilde{B}_s(P)_{(s)}(m,n)\), we know that \(\alpha\) has \(s\)-many vertices and the elements of \(\Sigma \tilde{P}\) placed at each vertex have weight 1 (where the weight is the weight grading of \(P\)). In other words, \(\alpha \in \mathcal{F}_{(s)}(\Sigma \tilde{P}(1))(m,n)\). A similar argument shows the other direction. The second equality holds since by assumption we know that \(P\) is connected. This means that the only element of weight 0 is the unit. Thus, if \(d > s\) then for some vertex, we know that this vertex is labelled by some element of weight 0. If this element is not the unit, then it must be 0. 

This lemma gives us the following for \(K(P)_{(s)}(m,n)\):
\[
\cdots \xrightarrow{\delta} 0 \xrightarrow{\delta} \tilde{B}_s(P)_{(s)}(m,n) \xrightarrow{\delta} \tilde{B}_{s-1}(P)_{(s)}(m,n) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \tilde{B}_1(P)_{(s)}(m,n)
\]

Thus, \(K(P)_{(s)}(m,n) = \ker(\delta : \tilde{B}_s(P)_{(s)}(m,n) \to \tilde{B}_{s-1}(P)_{(s)}(m,n))\).

### 2.6.1 Quadratic Duality of dioperads

In this section, we will give the relationship between \(P\) and its Koszul dual \(K(P)\) and \(P\) and its quadratic dual \(P^!\) as defined in [6] section 2.5. Consider a connected dioperad \(P\) equipped with the canonical weight grading ([5] 5.1.2), that is:

\[
P_{(s)}(m,n) = \begin{cases} 
P(m,n) & \text{if } s = m + n - 2 \\
0 & \text{otherwise.}
\end{cases}
\]

Therefore, given a connected dioperad \(P\) equipped with the canonical weight grading, we can obtain \(P^!\) from \(K(P)\) via the following relationship (see [5] section 5.2.7)

\[
K(P)(m,n) = \Sigma^{m+n-2}P^!(m,n)^* \otimes \text{Sgn}_n \otimes \text{Sgn}_m.
\]
Definition 2.11 A connected dioperad $\mathcal{P}$ equipped with a weight grading is Koszul if the inclusion morphism $K(\mathcal{P}) \hookrightarrow \mathcal{B}(\mathcal{P})$ is a quasi-isomorphism.

2.7 Turning a Cyclic Operad into a Dioperad

Definition 2.12 Let $F : \mathcal{D}_0 \to \mathcal{G}_0$ be the forgetful functor defined on objects by $F([m,n]) = [n + m - 1]$ and on morphisms by “forgetting” the direction in the digraph. That is,

\[
\begin{array}{c}
1 \\
\vdots \\
\ell
\end{array} \quad \xleftarrow{F} \quad \begin{array}{c}
m \\
\vdots \\
\ell
\end{array}
\]

Therefore, given a cyclic operad $\mathcal{P}$, we obtain a dioperad $\tilde{\mathcal{P}} = \mathcal{P} \circ F$. Thus, we have:

\[
\begin{array}{ccc}
\mathcal{G}_0 & \xrightarrow{\mathcal{P}} & \mathcal{C} \\
\downarrow F & \Downarrow \tilde{\mathcal{P}} & \downarrow \rho \\
\mathcal{D}_0
\end{array}
\]

Proposition 2.13 Let $\mathcal{P}$ be a cyclic operad and $(M, \eta)$ a $\mathcal{P}$-algebra. Then, $(M, \eta)$ is a $\tilde{\mathcal{P}}$-algebra.

Proof. Given $\rho : \mathcal{P} \to \text{End}_0(M, \eta)$ we have the morphism $\tilde{\rho} : \tilde{\mathcal{P}} \to \text{End}_0(M, \eta)$.

To show $M$ is a $\tilde{\mathcal{P}}$-algebra it suffices to show that we have a natural isomorphism:

\[
\Theta : \text{End}_0(M, \eta) \longrightarrow \text{End}_0^\sim(M)
\]

where for any $X \in \mathcal{D}_0$, the map

\[
\Theta_X : \text{End}_0(M, \eta)(X) \longrightarrow \text{End}_0^\sim(M)(X)
\]

is defined by using $\eta$ to identify $M \cong M^*$. It is clear from the definition of $\Theta$ that we have a natural isomorphism of functors.
The functor in Definition 2.12 does not send a free cyclic operad to a free dioperad. Indeed the following morphisms in \( D_0 \) are identified under the functor

\[
\begin{align*}
\begin{array}{c}
1 \\
\mu \\
2 \\
1 \\
\end{array}
\quad &= 
\begin{array}{c}
1 \\
\mu \\
2 \\
2 \\
\end{array}
\quad \text{and} 
\begin{array}{c}
2 \\
\mu \\
1 \\
1 \\
\end{array}
\quad &= 
\begin{array}{c}
2 \\
\mu \\
1 \\
2 \\
\end{array}
\end{align*}
\]

2.7.1 Example.

Consider the (cyclic) quadratic operad \( \text{Com} \) (see [12] 13.1 and [10] 2.1.10). \( \text{Com} \) is generated by a symmetric product

\[
\begin{array}{c}
1 \\
\mu \\
2 \\
0 \\
\end{array}
\]

and has relations generated by

\[
\begin{align*}
\begin{array}{c}
1 \\
\mu \\
2 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
1 \\
\mu \\
3 \\
0 \\
\end{array}
\quad &= 
\begin{array}{c}
1 \\
\mu \\
2 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
1 \\
\mu \\
3 \\
0 \\
\end{array}
\quad = 0 \\
\begin{array}{c}
2 \\
\mu \\
1 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
2 \\
\mu \\
3 \\
0 \\
\end{array}
\quad &= 
\begin{array}{c}
2 \\
\mu \\
1 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
2 \\
\mu \\
3 \\
0 \\
\end{array}
\quad = 0
\end{align*}
\]

Then \( \widetilde{\text{Com}} \) is generated by a product \( \mu \), a coproduct \( \Delta \)

\[
\begin{array}{c}
1 \\
\mu \\
2 \\
0 \\
\end{array}
\quad \Delta 
\begin{array}{c}
1 \\
\mu \\
2 \\
0 \\
\end{array}
\]

and has relations generated by

\[
\begin{align*}
\begin{array}{c}
1 \\
\mu \\
3 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
1 \\
\mu \\
2 \\
0 \\
\end{array}
\quad &= 
\begin{array}{c}
1 \\
\mu \\
3 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
1 \\
\mu \\
2 \\
0 \\
\end{array}
\quad = 0 \\
\begin{array}{c}
2 \\
\mu \\
1 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
2 \\
\mu \\
3 \\
0 \\
\end{array}
\quad &= 
\begin{array}{c}
2 \\
\mu \\
1 \\
0 \\
\end{array}
\quad - 
\begin{array}{c}
2 \\
\mu \\
3 \\
0 \\
\end{array}
\quad = 0
\end{align*}
\]
2.7.2 Remark

The functor in definition 2.12 enlarges the class of algebras. For example, \( \mathcal{C}om \)-algebras are Frobenius algebras and hence are \( \widetilde{\mathcal{C}om} \)-algebras by Proposition 2.13. Now, suppose, \( A \) is a commutative algebra. Define a map \( \Delta : A \to A \otimes A \) given by \( \Delta(a) \equiv 0 \). Then, \( A \) is a \( \widetilde{\mathcal{C}om} \)-algebra that did not come from a Frobenius algebra.
Chapter 3

PBW bases for dioperads

We define PBW bases for dioperads which generalizes what Hoffbeck did in the setting of operads and Priddy in the setting of algebras (cf. [11] and [15]).

3.0.3 A basis for the free dioperad.

Let $E$ be an $S$-bimodule with an ordered basis $B^E$ (as a $K$-module) and such that $E(m,n) = 0$ whenever $m = 0$ or $n = 0$. For any tree $T$ we define a monomial basis $B_T^{F(E)}$ of $E(T)$. Recall from 2.4.7 that $E(T) = \bigotimes_{v \in V} E(\text{Out}(v), \text{In}(v))$. We say an element $\alpha = \bigotimes_{v \in V} x_v \in E(T)$ belongs to $B_T^{F(E)}$ if and only if each $x_v \in B^E$. We call $\alpha$ a treewise tensor. We set $B_T^{F(E)} = \bigsqcup_T B_T^{F(E)}$ and this defines our monomial basis for the free dioperad generated by $E$. In practice, we shall identify our basis elements as labelled trees. Thus, our monomial basis $B_T^{F(E)}$ is a basis of treewise tensors.

Let $\alpha \in E(T)$ and $e$ be an edge of $T$. Define the tree $T|_e$ to be the subtree of $T$ with edge $e$ and two vertices which are joined by $e$. The legs of $T|_e$ are those flags in $T$ which meet the two vertices that are joined by $e$. The restricted treewise tensor $\alpha_{T|_e}$ is the treewise tensor whose underlying tree is $T|_e$.

Let $T$ be any $(m,n)$-tree and $T'$ be any $(r,s)$-tree and $\alpha \in E(T), \beta \in E(T')$. 21
A pair of pointed shuffles for the composition $\alpha \circ_j \beta$ is a pair $(\sigma, \tau)$ of pointed shuffles (see [11] 3.1 or [4] Proposition 2) where $\sigma$ and $\tau$ act in the following way:

$\sigma$ is an $(m - 1, r - j)$-shuffle, that is:

- $\sigma(1) = 1, \ldots, \sigma(j) = j$
- $\sigma(j + 1) < \sigma(j + 2) < \cdots < \sigma(j + m - 1)$
- $\sigma(j + m) < \sigma(j + m + 1) < \cdots < \sigma(m + r - 1)$

and $\tau$ is an $(s - 1, n - i)$-shuffle, that is:

- $\tau(1) = 1, \ldots, \tau(i) = i$
- $\tau(i + 1) < \tau(i + 2) < \cdots < \tau(i + s - 1)$
- $\tau(i + s) < \tau(i + s + 1) < \cdots < \tau(n + s - 1)$.

In other words, $\sigma(\alpha \circ_j \beta) \tau$ is of the form

![Diagram](image-url)

3.0.4 Order on $B^{\mathcal{F}(E)}$.

We define an order on the monomial basis of $\mathcal{F}(E)(m, n)$ for every $m, n \in \mathbb{N}$ that verifies a generalization of the semigroup partial ordering (cf. [1]) defined in section 3.3 of [11]:

Let $\alpha, \alpha'$ be any treewise tensors with $m$ outputs and $n$ inputs and $\beta, \beta'$ be any
treewise tensors with \( r \) outputs and \( s \) inputs. We shall have

\[
\begin{align*}
\alpha \leq \alpha' & \implies \sigma(\alpha, \circ_j \beta).\tau \leq \sigma(\alpha', \circ_j \beta').\tau \\
\beta \leq \beta' & \implies \sigma(\alpha, \circ_j \beta).\tau \leq \sigma(\alpha', \circ_j \beta').\tau
\end{align*}
\]

for every \( 1 \leq i \leq n, 1 \leq j \leq r \) and every pair of pointed shuffles \((\sigma, \tau)\).

3.0.5 Example of a suitable order.

We generalize the orders defined in section 3.4 of [11]. Let \( T \) be any \((m, n)\)-tree and \( \alpha \in E(T) \). For each output \( j \) of \( \alpha \), there is a unique monotonic path of vertices from \( j \) to each input \( i \). The word \( a_{ji} \) is the word composed (from left to right) of the labels of these vertices (from bottom to top). Observe that for any output \( j \) of \( \alpha \), there may exist an input \( i \) of \( \alpha \) such that there is no monotonic path from \( j \) to \( i \). In this situation, we say \( a_{ji} = \emptyset \). Thus, to \( \alpha \) we associate a sequence of \( mn \) words \((a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, \ldots, a_{mn})\). For example, let \( \alpha \) be the treewise tensor

![Diagram of treewise tensor]\( \begin{array}{c}
1 \\
\text{xv1} \\
\text{xv2} \\
2
\end{array} \)

Then the word associated to \( \alpha \) is \((a_{11}, a_{12}, a_{21}, a_{22}) = (x_{v1}, x_{v1}, x_{v2}, \emptyset, x_{v2})\).

Recall \( E \) has an ordered basis \( B^E \). Now, to compare any two words \( a \) and \( b \), we first compare the length of the words, that is \( a < b \) if \( l(a) < l(b) \) where \( l \) is the length, and if \( l(a) = l(b) \), we compare \( a \) and \( b \) lexicographically (with each letter of the words \( a \) and \( b \) being in \( E \)). Whenever we have \( a_{ji} = \emptyset \) we set \( l(\alpha) = -\infty \). The order described above is the lexicographical order. The reverse-length lexicographical order is the same except we say \( a > b \) if \( l(a) < l(b) \) and when \( a = \emptyset \), we set \( l(a) = -\infty \).

Let \( T, T' \) be \((m, n)\)-trees, \( \alpha \in E(T) \), and \( \beta \in E(T') \). To compare \( \alpha \) and \( \beta \) we proceed as follows. We compare the sequence of words associated to \( \alpha \) with the sequence
of words associated to β by comparing \( a_{i1} \) with \( b_{i1} \), then \( a_{i2} \) with \( b_{i2} \) and so on. If at some point \( a_{ji} < b_{ji} \) for some \( j \) and \( i \), we say \( \alpha < \beta \). If the words associated to \( \alpha \) and \( \beta \) are equal, then, we say that under the (lexicographical) order, the treewise tensors \( \alpha \leq \beta \).

Observe that in the situation where \( \alpha \leq \beta \) and \( \beta \leq \alpha \), we may not necessarily have \( T = T' \). For example, let \( E \) be the \( S \)-bimodule where \( E(1,2) = \mathbb{K}, \mu, E(2,1) = \mathbb{K}, \Delta \) (equipped with trivial \( S_2 \)-action) and \( E(m,n) = 0 \) for \( (m,n) \neq (1,2) \) or \( (2,1) \). Let \( \alpha \) and \( \beta \) be the following treewise tensors respectively

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\mu & \mu & \mu & \mu \\
\Delta & & & \\
1 & 2
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
1 & 3 & 2 & 4 \\
\mu & \mu & \mu & \mu \\
\Delta & & & \\
1 & 2
\end{array}
\]

Then, if we call \( T \) the underlying tree of \( \alpha \) and \( T' \) the underlying tree of \( \beta \), we see that \( \alpha \leq \beta \) and \( \beta \leq \alpha \), however, \( T \neq T' \).

**Proposition 3.1** The lexicographical order defined above verifies the generalized semigroup partial ordering compatibility condition of section 3.0.4.

**Proof.** Let \( \alpha, \alpha' \) be \((m,n)\)-treewise tensors and \( \beta, \beta' \) be \((r,s)\)-treewise tensors where \( \alpha \leq \alpha' \) and \( \beta \leq \beta' \). Let \( a = (a_{11}, \ldots, a_{mn}) \), respectively \( a' = (a'_{11}, \ldots, a'_{mn}) \) be the words associated to \( \alpha \) respectively \( \alpha' \). Let \( b = (b_{11}, \ldots, b_{rs}) \), respectively \( b' = (b'_{11}, \ldots, b'_{rs}) \) be the words associated to \( \beta \) respectively \( \beta' \).
The word associated to $\alpha_i \circ_j \beta$ is the word

$$a_i \circ_j b = (\emptyset, \ldots, \emptyset, b_{11}, \ldots, b_{1s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}, \ldots, b_{j-1,s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{i-1,1}, \ldots, b_{i-1,s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}, \ldots, b_{1s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}, \ldots, b_{2s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}, \ldots, b_{j-1,s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}, \ldots, b_{1s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}, \ldots, b_{2s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}, \ldots, b_{j-1,s}, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}, \ldots, b_{1s}, \emptyset, \ldots, \emptyset),$$

and similarly, the word for associated to $\alpha_i' \circ_j \beta'$ has the form

$$a_i' \circ_j b' = (\emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{i-1,1}', \ldots, b_{i-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset),$$

And, similarly, the word for associated to $\alpha_i' \circ_j \beta'$ has the form

$$a_i' \circ_j b' = (\emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{i-1,1}', \ldots, b_{i-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset),$$

$$a_i' \circ_j b' = (\emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{i-1,1}', \ldots, b_{i-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset),$$

$$a_i' \circ_j b' = (\emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{i-1,1}', \ldots, b_{i-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{21}', \ldots, b_{2s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{j-1,1}', \ldots, b_{j-1,s}', \emptyset, \ldots, \emptyset, \ldots, \emptyset, b_{11}', \ldots, b_{1s}', \emptyset, \ldots, \emptyset).$$
First, to compare $\alpha_i \circ_j \beta$ and $\alpha'_i \circ_j \beta'$, we begin by comparing $b_{kl}$ with $b'_{kl}$ where $1 \leq k \leq j - 1$ and $1 \leq l \leq s$. By assumption $\beta \leq \beta'$, thus $b_{kl} \leq b'_{kl}$. If this inequality is strict for some $k$ and $l$, then we are done and $\alpha_i \circ_j \beta \leq \alpha'_i \circ_j \beta'$. If the inequality is an equality, we move on and compare $(a_{11}, \ldots, a_{1,i-1})$ and $(a'_{11}, \ldots, a'_{1,i-1})$. By assumption, $\alpha \leq \alpha'$ and so $a_{1t} \leq a'_{1t}$. If this inequality is strict for some $t$, we are done, otherwise we have an equality and we move on and compare $a_{11} b_{jl}$ with $a'_{11} b'_{jl}$. Observe that comparing $a_{11} b_{jl}$ with $a'_{11} b'_{jl}$ is equivalent to comparing $b_{jl}$ with $b'_{jl}$. If for some $l$ the inequality is strict we are done, otherwise we move on and compare $(a_{1,i+1}, \ldots, a_{1,n+s-1})$ with $(a'_{1,i+1}, \ldots, a'_{1,n+s-1})$. If the inequality is strict, then we are done. Otherwise, we move on and in a similar fashion we first compare $(a_{u1}, \ldots, a_{u,i-1})$ and $(a'_{u1}, \ldots, a'_{u,i-1})$ starting with $u = 2$, where $2 \leq u \leq m$. If for a fixed $u$ we have an equality, we move on and compare $a_{u1} b_{jl}$ with $a'_{u1} b'_{jl}$, and if the inequality is an equality we move on and compare $(a_{u,i+1}, \ldots, a_{u,n+s-1})$ with $(a'_{u,i+1}, \ldots, a'_{u,n+s-1})$. If for each $u$ the inequality is an equality we move on and finish by comparing $b_{wl}$ with $b'_{wl}$ where $j + 1 \leq w \leq r$. Thus, we have $\alpha_i \circ_j \beta \leq \alpha'_i \circ_j \beta'$.

Next we must compare $\sigma(\alpha_i \circ_j \beta) \tau$ with $\sigma(\alpha'_i \circ_j \beta') \tau$. Observe that the word associated to $\sigma(\alpha_i \circ_j \beta) \tau$ is obtained from the word associated to $\alpha_i \circ_j \beta$ in the following way:

- A row in (*) of the form

$$\emptyset, \ldots, \emptyset, b_{i1}, b_{i2}, \ldots, b_{is}, \emptyset, \ldots, \emptyset$$

for $1 \leq i \leq j - 1$ gets changed by shuffling $b_{i2}, \ldots, b_{is}$ with those $\emptyset, \ldots, \emptyset$ at the end using $\tau$.

- A row in (*) of the form

$$a_{k1}, \ldots, a_{k,i-1}, a_{k,i} b_{jl}, a_{k,i} b_{jl}, \ldots, a_{k,i} b_{js}, a_{k,i+1}, \ldots, a_{k,n+s-1}$$
for $1 \leq k \leq m$ gets changed by shuffling $a_{k,i}b_{j_2}, \ldots, a_{k,i+s}$ with $a_{k,i+1}, \ldots, a_{k,n+s-1}$ using $\tau$.

- Rows of the form

$$a_{i_1}, \ldots, a_{i_{j-1}}, a_{i_j}b_{j_1}, \ldots, a_{i_j}b_{j_s}, a_{i_j+1}, \ldots, a_{i_l,n+s-1}$$

where $2 \leq l \leq m$ are shuffled with rows of the form

$$\emptyset, \ldots, \emptyset, b_{q_1}, \ldots, b_{q_s}, \emptyset, \ldots, \emptyset$$

where $j + 1 \leq q \leq r$ using $\sigma$.

Observe that the order we have defined on $B^F(E)$ looks at the sequence of words associated to each treewise tensor recursively. Moreover, since shuffles preserve the order of each set they act on, the order of the words associated to $\sigma(\alpha \circ_j \beta)\tau$ and $\sigma(\alpha' \circ_j \beta')\tau$ is the same as the order of the words associated to $\alpha \circ_j \beta$ and $\alpha' \circ_j \beta'$. It follows that

$$\sigma(\alpha \circ_j \beta)\tau \leq \sigma(\alpha' \circ_j \beta')\tau$$

We note that a similar proof shows that the reverse-length lexicographical order verifies the compatibility condition of section 3.0.4. Moreover, if $m = r = 1$, i.e. we are working with trees that represent $n$-ary and $s$-ary operations as in the case of operads, the above proof reduces to the situation as seen in [11] Proposition 3.5. Indeed, in this situation, there would be no shuffle on outputs and the shuffle on inputs would induce a shuffle between the set composed of the $a_{i_j}b_q$ and the set composed of $a_q$.

### 3.0.6 PBW basis.

Let $\mathcal{P} = F(E)/(R)$ be a quadratic dioperad. A **PBW basis** for $\mathcal{P}$ is a set $B^P \subset B^F(E)$ of elements representing a basis of the $K$-module $\mathcal{P}$ containing $1, B^E$ and
for every tree $T$, a subset $B_P^T \subset B_{F(E)}^T$ which verifies the following:

1. Let $\alpha \in B_P^T, \beta \in B_P^T$ and $(\sigma, \tau)$ a pair of pointed shuffles, then either $\sigma.(\alpha_i \circ_j \beta).\tau \in B_P^T(\alpha_i \circ_j \beta) \cdot \tau$ or the elements of the basis $\gamma \in B_P^T$ which we use to uniquely write $\sigma.(\alpha_i \circ_j \beta).\tau = \sum \gamma c_\gamma \gamma$, verify $\gamma < \sigma.(\alpha_i \circ_j \beta).\tau$ in $F(E)$.

2. Let $\alpha$ be a treewise tensor. Then we have $\alpha \in B_P^T$ if and only if for every internal edge $e$ of $T$, the restricted treewise tensor $\alpha|_{T_e}$ (see [11] sections 2.4 and 3.6) is in $B_P^T$.

We would like to point out that we use a different convention in the above definition of a PBW basis than in [11] section 3.7. In [11] the author requires that a pointed shuffle acting on a composition of treewise tensors is either in the PBW basis or it is written as a sum of elements from the PBW basis which are strictly larger than it.

3.0.7 $\bar{B}(F(E))$.

We follow [11] section 4.1. A treewise tensor of $\bar{B}(F(E))$ is a treewise tensor $\alpha$ where each vertex of $\alpha$ is labelled by a tree labelled by elements of $E$. Therefore, we can represent a generator of $\bar{B}(F(E))$ by a “large” tree $T$ labelled by elements of $E$. This “large” tree $T$ comes equipped with a splitting into subtrees $T_{comp}$, which are seen as connected components of $T$. The connected components $T_{comp}$ are separated by cutting edges. The set of cutting edges are the edges of the bar construction and will be denoted by $D$ and $D \subset I(T)$ where $I(T)$ is the set of edges of $T$. We also have the edges of the connected components $T_{comp}$ of the tree $T$. We denote this set by $S$. An element $e \in S$ will be called a marking edge. Therefore, we have $S \coprod D = I(T)$.

Let $\alpha$ be a treewise tensor of $\bar{B}(F(E))$. Since the underlying tree $T$ of $\alpha$ has a
splitting into connected components $T_{\text{comp}}$ we too have a splitting of $\alpha$ into connected components. In other words, $\alpha = \bigotimes \alpha_{\text{comp}}$ where $\alpha_{\text{comp}} = \alpha_{|T_{\text{comp}}}$. We can now identify an element of $\bar{B}(\mathcal{F}(E))$ by a pair $(\alpha, S)$, where $\alpha \in E(T)$.

**Example 3.2** Consider the treewise tensor of $\bar{B}(\mathcal{F}(E))$

![Diagram](image1)

where the dotted edges are the edges of $\bar{B}(\mathcal{F}(E))$ and

![Diagram](image2)

where $a_1, a_2, b_1, c_1, c_2, d_1 \in E$. Then, after substituting each $p_i$, we get the element $(\alpha, S) \in \mathcal{F}(E)$

![Diagram](image3)

where $S$ is the set of edges which connecting the vertices $c_1$ to $c_2$ and $a_1$ and $a_2$. In other words,

$$S = \{c_2 \rightarrow c_1, a_1 \rightarrow a_2\}$$

**3.0.8 Differential structure of $\bar{B}(\mathcal{F}(E))$.**

As defined in 2.5.2, the differential of the bar construction of a dioperad is given by edge contraction. Since a generator of $\bar{B}(\mathcal{F}(E))$ is of the form $(\alpha, S)$, it is
enough to describe the differential on such a generator. We have

$$\delta(\alpha, S) = \sum_{e \in I(T)-S} \pm (\alpha, S \prod \{e\})$$

For instance, in the previous example,

$$\delta(\alpha, S) = \pm (\alpha, S \prod \{d_1 b_1\}) \pm (\alpha, S \prod \{c_1 b_1\}) \pm (\alpha, S \prod \{b_1 a_1\})$$

Thus, $\delta$ would change each of these cutting edges into a marking edge and as an element of the bar construction, this corresponds to a contraction at the marking edge.

### 3.0.9 Basis for $\bar{B}(\mathcal{P})$.

We first consider a basis for $\bar{B}(\mathcal{F}(E))$. For any tree $T$, let $B_T^{\bar{B}(\mathcal{F}(E))}$ be a basis of treewise tensors labelled with elements of $B^{\mathcal{F}(E)}$. That is, $\alpha \in B_T^{\bar{B}(\mathcal{F}(E))}$ if and only if, each tree labelling a vertex of $\alpha$ is actually in $B^{\mathcal{F}(E)}$. As before, we set

$$B^{\bar{B}(\mathcal{F}(E))} = \prod_T B_T^{\bar{B}(\mathcal{F}(E))}.$$  

Since $\mathcal{P}$ is a quotient of $\mathcal{F}(E)$ we have that $\bar{B}(\mathcal{P})$ is a quotient of $\bar{B}(\mathcal{F}(E))$. Therefore, in $\bar{B}(\mathcal{P})$, two elements $(\alpha, S)$ and $(\alpha', S')$ are identified if and only if $S = S'$ and $\alpha_{\text{comp}} \simeq \alpha'_{\text{comp}}$ in $\mathcal{P}$.

We can now define a basis for the bar construction of a quadratic dioperad $\mathcal{P}$.

The basis $B^{\bar{B}(\mathcal{P})}$ is defined as a set of elements in $\bar{B}(\mathcal{F}(E))$ such that $(\alpha, S) \in B_T^{\bar{B}(\mathcal{P})}$ if and only for every factor $\alpha_{|T_{\text{comp}}}$ of $\alpha$, we have $\alpha_{|T_{\text{comp}}} \in B_{T_{\text{comp}}}^P$. Observe that $B_T^{\bar{B}(\mathcal{P})} \subset B_T^{\bar{B}(\mathcal{F}(E))}$. An edge $e$ is said to be admissible for $\alpha$ if the restricted treewise tensor $\alpha_{|T_e} \in B^P$ (see [11] section 4.2).

### 3.0.10 Filtration of $\bar{B}(\mathcal{P})$.

Following [11] section 4.4, we will define an increasing filtration on $\bar{B}(\mathcal{P})$ using the lexicographical order defined in 3.0.5. That is, an element $(\alpha, S) \in \bar{B}(\mathcal{P})(m, n)$ is in
\( \bar{B}(P)(m,n)_\lambda \) if and only if \( \alpha \leq \lambda \), where \( \lambda \in B_{P}(F(E)) \) for some \( T \). Therefore,

\[
\bar{B}(P)(m,n) = \bigcup_{\lambda \in F(E)(m,n)} \bar{B}(P)(m,n)_\lambda
\]

We first note that \( \bar{B}(P)(m,n)_\lambda \) is a subcomplex of \( \bar{B}(P)(m,n) \). To see this, let \( \alpha, S \in \bar{B}(P)(m,n)_\lambda \). Then \( \alpha \leq \lambda \), and \( \delta(\alpha, S) = \sum_{e \in I(T) - S} \pm (\alpha, S \coprod \{e\}) \). If \( (\alpha, S \coprod \{e\}) \notin B_{\bar{B}(P)} \) then \( (\alpha, S \coprod \{e\}) \) is a sum of elements \( (\alpha', S') \) in the PBW basis with \( \alpha' < \alpha \). Thus, in either situation, we have \( \delta(\alpha, S) \in \bar{B}(P)(m,n)_\lambda \) and so \( \bar{B}(P)(m,n)_\lambda \) is a subcomplex of \( \bar{B}(P)(m,n) \).

Let \( \delta^0 \) be the differential induced by \( \delta \) on the zeroth page of the spectral sequence

\[
E^0\bar{B}(P)(m,n)_\lambda = \frac{\bar{B}(P)(m,n)_\lambda}{\sum_{\lambda' < \lambda} \bar{B}(P)(m,n)_{\lambda'}}
\]

induced by the filtration \( \bar{B}(P)_\lambda \) on \( \bar{B}(P) \). Then,

\[
\delta^0(\alpha, S) = \begin{cases} 
(\alpha, S \coprod \{e\}) & \text{if } (\alpha, S \coprod \{e\}) \in B(\bar{P}) \\
0 & \text{Otherwise}
\end{cases}
\]

**Lemma 3.3** An edge \( e \notin S \) is admissible if and only if \( \delta^0(\lambda, S) \neq 0 \).

**Proof.** Clear \( \blacksquare \)

**Theorem 3.4** A quadratic dioperad \( P \) which has a PBW basis is Koszul.

**Proof.** We study the homology of \( E^0\bar{B}(P)(m,n)_\lambda \). Let \( Adm_\lambda \) be the set of admissible edges of \( \lambda \) and suppose \( |Adm_\lambda| = t \). Let \( A \) be the \( t \)-dimensional vector space over \( \mathbb{K} \) with basis \( Adm_\lambda \). Let \( (\lambda, S) \) be a pair with \( S = \{v_1, \ldots, v_r\} \subset Adm_\lambda \). Define the complex \( C := \bigoplus_{r=0}^{t} C^r \) where \( C^r := \bigwedge^r A \). We define a differential \( d \) on \( C \) as follows:

\[
d(v_1 \wedge \cdots \wedge v_r) := \sum_{a \in Adm_\lambda - S} a \wedge v_1 \wedge \cdots \wedge v_r \in C^{r+1}.
\]
Then, we extend $d$ linearly to a map $d : C^r \to C^{r+1}$ (by definition we have $d^2 = 0$).

Now, if $t = 0$, that is, there are no admissible edges of $\lambda$, then we get the complex

$$
\cdots \to 0 \to \mathbb{K} \to 0 \cdots
$$

and therefore the cohomology of this complex is concentrated in degree 0, that is

$$
H^r(C) = \begin{cases} 
\mathbb{K} & \text{if } r = 0 \\
0 & r \neq 0.
\end{cases}
$$

Now, suppose $t > 0$ and again consider $S = \{v_1, \ldots, v_r\}$. We claim that $C$ is exact.

Define a map $h$ as follows

$$
h(v_1 \wedge \cdots \wedge v_r) = \sum_{i=1}^{r} (-1)^{i-1} v_1 \wedge \cdots \hat{v_i} \cdots \wedge v_r \in C^{r-1}.
$$

Extending $h$ linearly, we get a map $h : C^r \to C^{r-1}$.

Consider the map $dh + hd$. Then, by the definitions of $d$ and $h$, we get:

$$
dh + hd = (r + 1)\text{Id}_{C^r} : C^r \to C^r
$$

Hence, for $a \in C^r$, if $d(a) = 0$, then,

$$
d(h(a)) = (r + 1)a
$$

which implies

$$
a = \frac{1}{r + 1}d(h(a)).
$$

Thus, $\ker(d : C^r \to C^{r+1}) \subset \text{im}(d : C^{r-1} \to C^r)$ and so $C$ is exact.

Therefore, we have just shown that the cohomology of the complex $C$ is trivial when $t > 0$. Now, when $t = 0$ we have that the cohomology, and hence by taking duals the homology, is concentrated in degree 0. In this situation, we have $\text{Adm}_\lambda = \emptyset$.

This implies that the components of $\lambda$ are the vertices, i.e., we only have cutting edges.
This is true if and only if the weight of $\lambda$ is equal to its homological degree in the bar construction. Thus, following [11] 4.6, we conclude that $H_*\mathcal{B}(\mathcal{P}) = 0$ whenever the weight differs from the homological degree. ■
Chapter 4

Diamond Lemma for Dioperads

We generalize the Diamond Lemma of [1] to the setting of dioperads.

4.1 Introduction

We will adopt the conventions, notations, and terminology discussed in the previous chapter and those discussed in [1]. Let \( \mathcal{P} \) be a dioperad with the presentation \( \mathcal{P} = \langle E, R \rangle \), where \( E \) is the \( S \)-bimodule of generators of \( \mathcal{P} \) and \( R \subset \mathcal{F}_S(E) \) is an \( S \)-submodule. Elements of \( R \) are called relations. We let \( (R) \) be the dioperadic ideal generated by \( R \). Suppose each relation \( r \in R \) can be written in the form \( \alpha_r = \gamma_r \), where \( \alpha_r \) is a treewise tensor and \( \gamma_r \) is a \( K \)-linear combination of treewise tensors (see section 2.0.5), i.e. \( r = \alpha_r - \gamma_r \).

In [1], the author considers the free semigroup \( \langle X \rangle \) generated by a set \( X \). We will consider the dioperadic analogue of \( \langle X \rangle \). Let \( E \) be an \( S \)-bimodule and \( X(m,n) \) a subset of \( E(m,n) \) whose elements are basis elements of \( E(m,n) \) which are stable under the left and right actions of the symmetric group. We can think of \( X = \{X(m,n)\} \) as the set of products and coproducts that the dioperad describes. We define the free
dioperad generated by $X$ in the category of sets as follows

$$\mathcal{F}_{\text{Set}}(X)(m, n) = \coprod_{(m,n)} \left( \prod_{v \in V} X(\text{Out}(v), \text{In}(v)) \right)$$

Therefore, an element of $\mathcal{F}_{\text{Set}}(X)(m, n)$ is an $(m, n)$-tree where the vertices are labelled by elements of the set $X$, and moreover, $\mathcal{F}_{\text{Set}}(X)(m, n)$ is a basis for $\mathcal{F}(E)(m, n)$.

Let $P$ be a dioperad with presentation $P = \langle E, R \rangle$. Let $Y = \{Y(m, n)\}$ be a basis for the $S$-submodule $R = \{R(m, n)\}$ of relations. For each relation $y \in Y$, we may write $y$ as $y = \alpha_y - \gamma_y$ with $\alpha_y \in \mathcal{F}_{\text{Set}}(X)$ and $\gamma_y \in \mathcal{F}(E)$. The element $\alpha_y$ is called the leading term of the relation $y$.

4.1.1 Reductions

Let $P$ be a dioperad and $y \in P$. We say $z \in \mathcal{F}(E)$ is obtained from $y$ by a shuffle composition if

$$z = \pi(y_i \circ_j y') \pi'$$

or

$$z = \pi(y'_{k} \circ_l y) \pi'$$

where $y' \in P$ and $(\pi, \pi')$ is a pair of pointed shuffles (see 2.0.2) for the composition $y_i \circ_j y'$ or $y'_{k} \circ_l y$ respectively.

Suppose $P = \langle E, R \rangle$ is a dioperad with $X \subseteq E$ a subset of basis elements stable under the left and right symmetric group actions and $Y$ a basis for $R$. Let $\mathcal{F}_{\text{Set}}(X)$ be the free dioperad generated by $X$ and $\mathcal{F}(E)$ the free dioperad generated by $E$ (as in section 1.4.7). Let $y_0 = \alpha_y, y_1, \ldots, y_r \in \mathcal{F}_{\text{Set}}(X)$ be a sequence such that $y_i$ is obtained from $y_{i-1}$ by a shuffle composition. Let $a \in \mathcal{F}_{\text{Set}}(X)$. We define a map
\( \Lambda_{y_0, y_1, \ldots, y_r} : \mathcal{F}_{\text{Set}}(X) \to \mathcal{F}(E) \) as follows:

\[
a \mapsto \begin{cases} 
  a & \text{if } a \neq y_r \\
  \bar{y}_r & \text{if } a = y_r
\end{cases}
\]

where \( \bar{y}_r \) is the element obtained by considering the same sequence \( y_0, y_1, \ldots, y_r \) of shuffle compositions except we begin the sequence with \( y_0 = \gamma_y \), instead of \( y_0 = \alpha_y \).

### 4.1.1.1 Example of a Reduction

Let \( y = \alpha_y - \gamma_y \in Y \) and consider the sequence \( y_0, y_1, y_2 \) where

\[
y_0 = \alpha_y \\
y_1 = \pi_1(y_0 \circ_j y'_1)\pi'_1 \\
y_2 = \pi_2(y_1 k \circ_l y'_2)\pi'_2
\]

with \( y'_1, y'_2 \in \mathcal{P} \). Then,

\[
\Lambda_{y_0, y_1, y_2}(a) = \begin{cases} 
  a & \text{if } a \neq y_2 \\
  \bar{y}_2 & \text{if } a = y_2
\end{cases}
\]

where \( \bar{y}_2 \) is obtained as follows

\[
\bar{y}_0 = \gamma_y \\
\bar{y}_1 = \pi_1(\bar{y}_0 \circ_j y'_1)\pi'_1 \\
\bar{y}_2 = \pi_2(\bar{y}_1 k \circ_l y'_2)\pi'_2
\]

The map \( \Lambda_{y_0, y_1, \ldots, y_r} \) is extended linearly to \( \mathcal{F}(E) \) and this map is called a reduction. Observe that \( \Lambda_{y_0, y_1, \ldots, y_r} : \mathcal{F}(E) \to \mathcal{F}(E) \) is a morphism of \( \mathbb{K} \)-modules.

Let \( a \in \mathcal{F}(E) \). We say a reduction \( \Lambda_{y_0, y_1, \ldots, y_r} \) acts trivially on \( a \) if the coefficient of \( y_r \) in \( a \) is zero when \( a \) is expressed as a linear combination of elements of \( \mathcal{F}_{\text{Set}}(X) \).
The element $a$ is said to be irreducible under $R$ if every reduction is trivial on $a$. Let $\mathcal{F}(E)_{\text{irr}}$ denote the set of all irreducible elements of $\mathcal{F}(E)$. Observe that $\mathcal{F}(E)_{\text{irr}}$ is a $\mathbb{K}$-submodule of $\mathcal{F}(E)$.

Let $\Lambda_1, \ldots, \Lambda_n$ be a finite sequence of reductions (here $\Lambda_i = \Lambda_{y_0, y_1, \ldots, y_r}$ and $y_0 = \alpha_{y_i}$). The sequence $\Lambda_1, \ldots, \Lambda_n$ will be called final on $a \in \mathcal{F}(E)$ if $\Lambda_n \cdots \Lambda_1(a) \in \mathcal{F}(E)_{\text{irr}}$. Now, consider infinite sequences $\Lambda_1, \Lambda_2, \ldots$ of reductions. An element $a \in \mathcal{F}(E)$ is said to be reduction-finite if for every infinite sequence of reductions $\Lambda_1, \Lambda_2, \ldots$, we have $\Lambda_i$ acts trivially on $\Lambda_{i-1} \cdots \Lambda_1(a)$ for all sufficiently large $i$. Moreover, if $a \in \mathcal{F}(E)$ is reduction-finite, then any sequence of reductions $\Lambda_i$ such that each $\Lambda_i$ acts nontrivially on $\Lambda_{i-1} \cdots \Lambda_1$ will be a finite sequence, and hence a final sequence. From the definition of a reduction-finite element of $\mathcal{F}(E)$, it readily follows that these elements form a $\mathbb{K}$-submodule of $\mathcal{F}(E)$. Furthermore, for any $a \in \mathcal{F}(E)$, this element will be called reduction-unique if it is reduction-finite and its images under all final sequences are equal. This value will be denoted by $\Lambda_R(a)$.

**Lemma 4.1** The set of reduction-unique elements of $\mathcal{F}(E)$ form a $\mathbb{K}$-submodule of $\mathcal{F}(E)$ and $\Lambda_R$ is a $\mathbb{K}$-module endomorphism of this submodule into $\mathcal{F}(E)_{\text{irr}}$

**Proof.** See [1] Lemma 1.1(i). ■

4.1.2 Ambiguities

Let $T$ be a tree with 3 vertices $v_1, v_2, v_3$ such that $v_1$ is connected to $v_2$ by an edge and $v_2$ is connected to $v_3$ by an edge. Let $T_1$ be the subtree with vertices $v_1, v_2$ and $T_2$ be the subtree with vertices $v_2, v_3$ (see [4] section 3.3). It then follows that $T$ can be obtained from $T_1$ (and $T_2$) by a shuffle composition. Recall in the preceding section we defined shuffle compositions for treewise tensors (i.e., labelled trees). The notion of
a tree $T$ being obtained from a tree $T'$ by a shuffle composition is defined analogously.

Define

$$F_{\text{Set}}(X)(T) = \{ \alpha \in F_{\text{Set}}(X) | T \text{ is the underlying tree of } \alpha \}.$$

Let $a \in F_{\text{Set}}(3)(X)(T)$. We say that $a$ is an ambiguity of $R$ if there exists $\alpha_y \in F_{\text{Set}}(X)(T_1)$ and $\alpha_y' \in F_{\text{Set}}(X)(T_2)$ such that $a$ can be obtained from $\alpha_y$ and $a$ can be obtained from $\alpha_y'$ by the same shuffle compositions where $T$ was obtained from $T_1$ and $T_2$ respectively.

### 4.1.2.1 Example of an ambiguity

Recall example 2.7.1. Let

$$\alpha_y = \begin{pmatrix} 1 & 2 \\ \mu & \mu \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \alpha_z = \begin{pmatrix} 1 & 3 \\ \mu & \mu \\ 0 & 0 \end{pmatrix}$$

Then

$$a = \begin{pmatrix} 1 & 3 \\ \mu & \mu \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 \\ \mu & \mu \\ 0 & 0 \end{pmatrix}$$

is an ambiguity with

$$a = \begin{pmatrix} 1 & 2 \\ \mu & \mu \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \mu & \mu \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ \mu & \mu \\ 0 & 0 \end{pmatrix}$$

Let $a \in F_{\text{Set}}(3)(X)(T)$ be an ambiguity, that is

$$\sigma(z' k \circ l \alpha_z)\sigma' = a = \pi(\alpha_y i \circ j y')\pi'$$

We say that $a$ is resolvable if there exists compositions of reductions $\Lambda, \Lambda'$ such that

$$\Lambda\left(\pi(\gamma y i \circ j y')\pi'\right) = \Lambda'\left(\sigma(z' k \circ l \gamma_z)\sigma'\right)$$
This says that the two ways of reducing $a$ are confluent (for examples see [12] section 8.4 and [4] chapter 4).

### 4.1.2.2 Example of resolving an ambiguity

We resolve the ambiguity $a$ of example 4.1.2.1.

We will consider an ordering $\leq$ on $\mathcal{F}_{\text{Set}}(X)$ which verifies the compatibility condition of section 2.0.3. (This is the dioperadic analogue of the semigroup partial order defined in [1].) This order is said to be compatible with $R$ if for every relation $r \in R$, we have that $\gamma_r$ is a linear combination of elements $< \alpha_r$. We will only be concerned with orders which are compatible with $R$, verify the compatibility condition of 2.0.3 and satisfy the descending chain condition (see [1] and [?] definition 2.1.1).

For any $a \in \mathcal{F}(E)$, let $<a>$ be the dioperadic ideal generated by $a$, that is

$$<a> = \bigcap_{a \in I} I$$

**Theorem 4.2** Let $R \subset \mathcal{F}_{(3)}(E)$ be an $S$-submodule of relations and $\leq$ an ordering on $\mathcal{F}_{\text{Set}}(X)$. If all ambiguities $a \in \mathcal{F}_{\text{Set}(3)}(X)$ of $R$ are resolvable then $\mathcal{F}(E)$ has a PBW basis.
Proof. First we observe that all the elements of $F_{Set}(X)$ are reduction-finite, and hence, every element of $F(E)$ are reduction-finite. Indeed, let $N = \{a \in F_{Set}(X)|a \text{ is not reduction-finite}\}$. Since $\leq$ satisfies the DCC, there is a minimal element $a_0 \in N$. Therefore, there exists a reduction $\Lambda_{y_0, y_1, \ldots, y_r}$ acting nontrivially on $a_0$, that is $\Lambda_{y_0, y_1, \ldots, y_r}(a_0) = \bar{y}_r$. But, $\bar{y}_r$ is a linear combination of elements which are $<a_0$ by assumption. Therefore $a_0$ is reduction-finite.

Second, we show that all elements in $F(E)$ are reduction-unique if and only if $F(E) = F(E)_{irr} \bigoplus (R)$ as $K$-modules. Suppose all elements in $F(E)$ are reduction-unique under $R$. By Lemma 3.1 we know that $\Lambda_R$ maps the $K$-submodule of reduction-unique elements into $F(E)_{irr}$. Let $a \neq 0 \in \ker(\Lambda_R)$. Then $\Lambda_R(a) = 0$ and since every reduction alters elements by elements of $(R)$ we then have that $a \in (R)$. Observe that $(R) = \langle Y \rangle$ and therefore, it suffices to show $\langle Y \rangle \subset \ker(\Lambda_R)$. Let $a \in \langle Y \rangle$. Then $a$ is a linear combination of elements which have been obtained by shuffle compositions starting from elements of $Y$, that is

$$a = \sum \ldots \pi(\beta_k \circ_l (\alpha_y \circ_j \beta')\pi' \ldots$$

Then,

$$\Lambda_R(a) = \sum \Lambda_R(\ldots \pi(\beta_k \circ_l \alpha_y \circ_j \beta')\pi' \ldots) - \sum \Lambda_R(\ldots \pi(\beta_k \circ_l \gamma_y \circ_j \beta')\pi' \ldots)$$

$$= \ldots \pi(\beta_k \circ_l \gamma_y \circ_j \beta')\pi' \ldots - \ldots \pi(\beta_k \circ_l \gamma_y \circ_j \beta')\pi' \ldots$$

$$= 0$$

Therefore, $a \in \ker(\Lambda_R)$ and $\ker(\Lambda_R) = (R)$. Now, we know $\Lambda_R$ is onto $F(E)_{irr}$ since $F(E)_{irr} \subset F(E)$. It follows that $F(E)_{irr} \cap (R) = 0$ and hence, $F(E) = F(E)_{irr} \bigoplus (R)$ as $K$-modules.

Conversely, suppose $F(E) = F(E)_{irr} \bigoplus (R)$. Let $a \in F(E)$ and suppose there
are reductions \( \Lambda_{y_0, y_1, \ldots, y_r} \) and \( \Lambda_{y'_0, y'_1, \ldots, y'_r} \) such that
\[
\Lambda_{y_0, y_1, \ldots, y_r}(a) = \bar{y}_r \in \mathcal{F}(E)_{\text{irr}}
\]
\[
\Lambda_{y'_0, y'_1, \ldots, y'_r}(a) = \bar{y}'_r \in \mathcal{F}(E)_{\text{irr}}
\]
Then, \( a - \bar{y}_r \in (R) \) and \( a - \bar{y}'_r \in (R) \). Hence,
\[
\bar{y}_r - \bar{y}'_r \in \mathcal{F}(E)_{\text{irr}} \cap (R) = 0
\]
Therefore, \( \bar{y}_r = \bar{y}'_r \) and so \( a \) is reduction-unique.

Now, suppose every ambiguity \( a \in \mathcal{F}_{\text{Set}(a)}(X) \) is resolvable. It suffices to show that all elements \( d \in \mathcal{F}_{\text{Set}}(X) \) are reduction-unique. To do so, we use induction. First, we observe that all the irreducible treewise tensors are already reduction-unique. Let \( d \in \mathcal{F}_{\text{Set}}(X) \) and suppose all elements \( < d \) are reduction-unique. We then have two cases:

Case 1: \( d \) has an ambiguity as a subtree. For convenience, suppose the ambiguity is of the form
\[
\sigma(z'_k \circ_l \alpha_z)\sigma' = a = \pi(\alpha_{y'_i} \circ_j y')\pi'
\]
There are reductions \( \Lambda_z, \Lambda_y \) such that \( \Lambda_z(a) = \sigma(z'_k \circ_l \gamma_z)\sigma' \) and \( \Lambda_y(a) = \pi(\gamma_{y'_i} \circ_j y')\pi' \). By assumption, we know that \( \Lambda_z(a), \Lambda_y(a) < a \) and so, \( \Lambda_z(d), \Lambda_y(d) < d \). Therefore, by the induction hypothesis, \( \Lambda_z(d) \) and \( \Lambda_y(d) \) are reduction-unique. We must show
\[
\Lambda_R(\Lambda_z(d)) = \Lambda_R(\Lambda_y(d))
\]
However, since all ambiguities are resolvable, there exists reductions \( \Lambda, \Lambda' \) such that
\[
\Lambda\left(\pi(\gamma_{y'_i} \circ_j y')\pi'\right) = \Lambda'\left(\sigma(z'_k \circ_l \gamma_z)\sigma'\right)
\]
Thus, we have $\Lambda(\Lambda_y(d)) = \Lambda'(\Lambda_z(d))$ and each of these elements is less than $d$ and hence reduction unique. It then follows that

$$\Lambda_R(\Lambda_y(d)) = \Lambda_R(\Lambda(\Lambda_y(d))) = \Lambda_R(\Lambda'(\Lambda_z(d))) = \Lambda_R(\Lambda_z(d))$$

Case 2: $d$ has disjoint leading terms of relations as subtrees. Let $\alpha_y, \alpha_z$ be leading terms of some relations $y, z \in Y$ and suppose $\alpha_y$ and $\alpha_z$ are subtrees from in $d$ which are disjoint (that is, there is no one edge in $d$ connected the subtrees $\alpha_z$ and $\alpha_y$).

Similar to Case 1, there are reductions $\Lambda_z, \Lambda_y$ such that these reductions replace the subtrees $\alpha_z, \alpha_y$ in $d$ by the subtrees $\gamma_z, \gamma_y$ in $d$ respectively, say $\Lambda_z(d) = d_{\gamma_z}$ and $\Lambda_y(d) = d_{\gamma_y}$. Then, $d_{\gamma_z}, d_{\gamma_y} < d$ and so each is reduction-unique. Now, we must show $\Lambda_R(d_{\gamma_z}) = \Lambda_R(d_{\gamma_y})$. However, since these $d_{\gamma_z}$ and $d_{\gamma_y}$ are reduction-unique, it doesn’t matter how we reduce each of these treewise tensors because after all reductions have been performed (regardless of the order) we end up with $\Lambda_R(d_{\gamma_z}) = \Lambda_R(d_{\gamma_y})$.

\[\blacksquare\]

4.1.3 Remark about Theorem 4.2

Observe that in the proof of Theorem 3.2, we proved that if all ambiguities in $\mathcal{F}_{\text{Set}}(X)$ of $R$ are resolvable then $\mathcal{F}(E) = \mathcal{F}(E)_{\text{irr}} \oplus (R)$. We claim that if $\mathcal{F}(E) = \mathcal{F}(E)_{\text{irr}} \oplus (R)$ then $\mathcal{P} = \mathcal{F}(E)/(R) \cong \mathcal{F}(E)_{\text{irr}}$ has a PBW basis. Indeed, let $B^P = \{\text{irreducible monomials of } \mathcal{F}_{\text{Set}}(X) \text{ which form a basis of } \mathcal{P}\}$. Clearly $B^P$ contains 1 and $B^E$. Let $\alpha, \beta \in B^P$ and $(\sigma, \tau)$ be a pair of pointed shuffles. If $\sigma.(\alpha \circ \beta).\tau \in B^P$ then this means $\sigma.(\alpha \circ \beta).\tau$ is an irreducible monomial of $\mathcal{F}_{\text{Set}}(X)$. Otherwise, $\sigma.(\alpha \circ \beta).\tau$ is reducible and since $\mathcal{F}(E) = \mathcal{F}(E)_{\text{irr}} \oplus (R)$ we can write $\sigma.(\alpha \circ \beta).\tau = \sum c_{\gamma}\gamma$ with each $\gamma < \sigma.(\alpha \circ \beta).\tau$ since we always write the leading
terms of relations as a sum of smaller terms. The next condition that \( \alpha \in B^P \) if and only if for each internal edge \( e \) of the underlying tree of \( \alpha \), the restriction \( \alpha_{T|e} \in B^P \) is obvious.
Chapter 5

Koszulity of Dioperads

In this chapter we will introduce a few new dioperads and prove that they are Koszul. This will be done by using the following recipe:

1. Given a quadratic cyclic operad $\mathcal{P} = \mathcal{F}(E)/(R)$, we consider the induced dioperad $\tilde{\mathcal{P}}$ (see 2.12).

2. We will choose a suitable order on the monomial basis of $\tilde{\mathcal{P}}$.

3. Under this order, we implement a rewriting rule on the relations by rewriting the lead, i.e. largest, term of a given relation as a linear combination of the smaller terms of that relation.

4. We will show that all the ambiguities of the relations of $\tilde{\mathcal{P}}$ are resolvable (i.e. confluent).

5. This will imply that $\tilde{\mathcal{P}}$ has a PBW basis (Theorem 4.2), which implies $\tilde{\mathcal{P}}$ is Koszul (Theorem 3.4).
5.1 Example: \( \tilde{\text{Com}} \)

By choosing a suitable order and following the above recipe, it is easy to see that \( \tilde{\text{Com}} \) is Koszul (in [6], the author proved that this dioperad was Koszul by using the “replacement rule”, see [6] Proposition 5.9). We use the lexicographical order (see Section 3.0.5) and order the generators as follows:

Under this order, we get the following rewriting rules for the relations:

5.1.1 PBW basis for \( \tilde{\text{Com}} \)

We note that using the diamond lemma only proves the existence of a PBW basis, which is still enough to show that a dioperad is Koszul (see Theorem 3.4). For the dioperad \( \tilde{\text{Com}} \) we will construct an explicit PBW basis to prove that this dioperad
is Koszul. Thus, we will provide another proof that $\widetilde{\text{Com}}$ is Koszul without using the Diamond Lemma (the first proof was provided in [6]).

**Proposition 5.1** Under the lexicographical order on $\widetilde{\text{Com}}$, the underlying tree of a tree-wise tensor $\widetilde{\text{Com}}(m,n)$, must have the following form:

$$
\begin{array}{c}
n \\
  \downarrow \\
  1 \\
  \vdots \\
  2 \\
  m-1 \\
  \downarrow \\
  m \\
  \end{array}
$$

where the path from root $m$ to leaf $n$ has $m + n - 2$ vertices, the labels of the leaves are written in descending order $n, n-1, \ldots, 1$ from left to right and the labels of the roots are written in ascending order $1, \ldots, m$ from left to right.

**Proof.** Let $\alpha \in \widetilde{\text{Com}}(m,n)$ and let $T$ be the underlying tree of $\alpha$. Suppose $T$ is not of the form (5.2). This gives two cases for $T$

1. $T$ is a permutation of a tree of the form in (5.2)

2. The path from root $m$ to leaf $n$ has less than $n + m - 2$ vertices in $T$

In the first case, we may rewrite this tree using the rewriting rules to obtain a tree of the form (5.2). In the second case, observe that there exists an $i$ and there exists a $p$ with $1 \leq p < n$ such that the path from root $m$ to leaf $p$ is of the form

$$
\begin{array}{c}
\text{root } m \leftarrow v_1 \leftarrow \cdots \leftarrow v_i \leftarrow w_1 \leftarrow \cdots \leftarrow w_s \leftarrow p
\end{array}
$$

where $s \geq 1$ and the vertices $w_k$ are not on the path from root $m$ to leaf $n$ for all $k$. Observe that the vertices $v_j, w_k$ are either labelled by $\mu$ or $\Delta$ for $1 \leq j \leq i$ and
1 ≤ k ≤ s. Consider the subtree with vertices \{v_i, w_1\}. This subtree is a leading term of one of the eight relations and can therefore be rewritten. Moreover, for any subtree with two vertices containing the vertices \(w_k\), we can rewrite this subtree so that after all possible rewritings have been performed, we get a tree of the form (5.2). ■

From definition 2.12, we know that \(\widetilde{\text{Com}}(m, n) = \text{Com}(m + n - 1)\) and hence, \(\text{dim}(\widetilde{\text{Com}}(m, n)) = 1\). By the preceding proposition, we know that there is one tree of the form (5.2) in \(\widetilde{\text{Com}}(m, n)\) and this is forms a PBW basis for \(\widetilde{\text{Com}}\). By Theorem 3.4, \(\widetilde{\text{Com}}\) is Koszul.

5.2 Example: \(\widetilde{\text{Lie}}\)

Consider the cyclic quadratic operad \(\text{Lie}\) (see [10] 2.1.20 and [12] 13.2).

\(\text{Lie}\) is generated by a skew-symmetric product

\[
\begin{array}{c}
1 \\
\downarrow \\
0
\end{array}
\begin{array}{c}
2 \\
\downarrow \\
0
\end{array}
\]

and has relations generated by

\[
\begin{array}{c}
1 \\
\downarrow \\
0
\end{array} + \begin{array}{c}
2 \\
\downarrow \\
0
\end{array} + \begin{array}{c}
3 \\
\downarrow \\
0
\end{array} = 0
\]

Then \(\widetilde{\text{Lie}}\) is generated by a skew-symmetric product \([,]\) and co-product \(\nu\)

\[
\begin{array}{c}
1 \\
\downarrow \\
1
\end{array} + \begin{array}{c}
2 \\
\downarrow \\
2
\end{array} = 0
\]

and has relations generated by

\[
\begin{array}{c}
1 \\
\downarrow \\
1
\end{array} + \begin{array}{c}
2 \\
\downarrow \\
2
\end{array} = 0
\]

\[
\begin{array}{c}
1 \\
\downarrow \\
1
\end{array} - \begin{array}{c}
2 \\
\downarrow \\
2
\end{array} = 0
\]

\[
\begin{array}{c}
1 \\
\downarrow \\
1
\end{array} - \begin{array}{c}
2 \\
\downarrow \\
2
\end{array} = 0
\]

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In this example, we will use the reverse-length lexicographical order on $\tilde{\text{Lie}}$. Thus, we get the following rewriting rules for the above relations:
Under these rewriting rules, we get the following ambiguities
Next, we show that the preceding ambiguities are resolvable. The first ambiguity is resolved as follows:
and
The second ambiguity is resolved like the first ambiguity. The third ambiguity is resolved as follows:
and
The fourth ambiguity is resolved as follows:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

The fifth ambiguity is resolved like the third ambiguity. The sixth ambiguity is resolved.
like the fourth ambiguity. The seventh ambiguity is resolved as follows:
The eight ambiguity is resolved like the third ambiguity. The ninth ambiguity is resolved like the fourth ambiguity. The tenth is resolved like the seventh ambiguity. The eleventh,
twelfth, and thirteenth ambiguities are resolved like the fourth ambiguity. Therefore, since all ambiguities are resolvable, the dioperad \( \widetilde{\text{Lie}} \) has a PBW basis and is hence Koszul.

### 5.2.1 PBW basis for \( \widetilde{\text{Lie}} \)

As was the case for \( \widetilde{\text{Com}} \), we construct an explicit PBW basis for \( \widetilde{\text{Lie}} \), thus providing a second proof that this dioperad is Koszul.

**Proposition 5.3** Under the reverse-lexicographical order on \( \widetilde{\text{Lie}} \), the underlying tree of a treewise tensor \( \widetilde{\text{Lie}}(m, n) \), must have the following form:

![Diagram of tree structure](5.4)

where the path from the root labelled 1 to the leaf labelled 1 has \((m + n - 2)\)-vertices. In other words, all the vertices of the underlying tree occur on the path from root 1 to leaf 1.

**Proof.** Suppose the path from root 1 to leaf 1 has less than \( m + n - 2 \) vertices, that is

\[
\text{root} 1 \leftarrow v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_r \leftarrow \text{leaf} 1
\]

where \( r < m + n - 2 \) and \( v_1, \ldots, v_r \) are the vertices along this path. Then, there exists an \( i \) and there exists a \( p > 1 \) such that the path from root 1 to leaf \( p \) is of the form

\[
\text{root} 1 \leftarrow v_1 \leftarrow \cdots \leftarrow v_i \leftarrow w_1 \leftarrow \cdots \leftarrow w_s \leftarrow p
\]
where $s \geq 1$ and $w_j$ are not on the path from root 1 to leaf 1 for all $j$.

Now, consider the subtree with vertices $\{v_i, w_1\}$. The vertex $w_1$ will either have 1 input and 2 outputs or 2 inputs and 1 output. In either case, this subtree can be rewritten using the rewriting rules of Section 5.2 so that after rewriting, the vertex $w_1$ is on the path from root 1 to leaf 1. Moreover, for any subtree with two vertices containing the vertices $w_j$, we can rewrite this subtree so that after all possible rewritings have been performed, we get that the path from root 1 to leaf 1 has $m + n - 2$ vertices. 

From definition 2.12, we know $\tilde{\text{Lie}}(m, n) = \text{Lie}(m + n - 1)$. It has been shown that $\dim(\text{Lie}(n)) = (n - 1)!$ (see [10] section 1.3.10 or [12] section 13.2.5). Therefore, $\dim(\tilde{\text{Lie}}(m, n)) = (m + n - 2)!$. Moreover, there are $(m + n - 2)!$ many trees of the form (5.4) since we have $(m + n - 2)!$ many bijections between the set of vertices labelled $\{v_1, \ldots, v_{m+n-2}\}$ and the collection of $m+n-2$ labels for the roots and leaves of the tree. Observe that trees of the form (5.4) are PBW elements, and hence, we have exhibited a PBW basis for $\tilde{\text{Lie}}$. By Theorem 3.4, $\tilde{\text{Lie}}$ is Koszul.

### 5.3 Example: $\tilde{\text{Lie}}^!$

Consider the quadratic dual $\tilde{\text{Lie}}^!$ of $\tilde{\text{Lie}}$ in the sense of [6]. The dioperad $\tilde{\text{Lie}}^!$ is generated by a commutative product $\mu$ and cocommutative coproduct $\Delta$

$$
\begin{align*}
\mu & \quad \Delta \\
1 & \quad 1 \\
\downarrow & \quad \downarrow \\
2 & \quad 2 \\
\end{align*}
$$

and has relations spanned by

$$
\begin{align*}
\mu & \quad \Delta \\
1 & \quad 1 \\
\downarrow & \quad \downarrow \\
2 & \quad 2 \\
\end{align*} - \begin{align*}
\mu & \quad \Delta \\
1 & \quad 1 \\
\downarrow & \quad \downarrow \\
2 & \quad 2 \\
\end{align*} = 0
$$
In this example, we will use the lexicographical order (see [11] Theorem 5.5). Thus, we get the following rewriting rules
Under these rewriting rules, we get the following ambiguities:

As was the case in the previous example, the preceding ambiguities are all resolvable.

Therefore, $\widetilde{\mathcal{L}ie}$ has a PBW basis and is thus a Koszul dioperad.
Bibliography


