Turbulence and internal waves in tidal flow over topography

Gayen, Bishakhdatta

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Turbulence and Internal Waves in tidal flow over topography

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in

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by

Bishakhdatta Gayen

Committee in charge:

Professor Sutanu Sarkar, Chair
Professor Eric Lauga
Professor Robert Pinkel
Professor James W. Rottman
Professor William R. Young

2012
The dissertation of Bishakhdatta Gayen is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

______________________________________________
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Chair

University of California, San Diego

2012
Dedicated to my parents and God ...
EPIGRAPH

Neither money pays, nor name, nor fame, nor learning; It is CHARACTER that can cleave through adamantine walls of difficulties.

—Swami Vivekananda
TABLE OF CONTENTS

Signature Page ................................................................. iii
Dedication ................................................................. iv
Epigraph ................................................................. v
Table of Contents ........................................................ vi
List of Symbols ........................................................ ix
List of Figures ........................................................ x
List of Tables ........................................................ xvi
Acknowledgements ......................................................... xvii
Vita and Publications ......................................................... xix
Abstract of the Dissertation ................................................... xxii

Chapter 1  **Introduction** .................................................. 1
  1.1 Internal waves .................................................. 4
  1.2 Generation of internal waves by oscillating tide over topo-
  graphy ................................................................. 7
    1.2.1 Theory of IW generation .................................. 8
    1.2.2 Ocean observations of IW generation ................. 11
    1.2.3 Laboratory observations of internal wave generation 14
    1.2.4 Numerical simulations of internal wave generation 15
  1.3 Internal tidal beam interaction with a pycnocline ........ 18
    1.3.1 Theory ................................................... 19
    1.3.2 Ocean observations ......................................... 21
    1.3.3 Laboratory observations ................................. 23
    1.3.4 Numerical simulations ................................. 24

Chapter 2  **Problem setup** ............................................. 26
  2.1 Flow over nonsloping bottom without streamwise varia-
  tion in mean flow .................................................. 26
    2.1.1 Governing equations .................................. 28
    2.1.2 Numerical method ...................................... 29
    2.1.3 Boundary conditions ................................... 29
    2.1.4 Subgrid scale model ................................. 30
2.2 Flow over sloping topography without streamwise variation in mean flow

2.2.1 Governing equations

2.2.2 Numerical method

2.2.3 Boundary Conditions

2.2.4 Subgrid scale model

2.3 Flow over sloping topography including streamwise variation in mean flow

2.3.1 Governing equations

2.3.2 Numerical method

2.3.3 Pressure Poisson Equation

2.3.4 Boundary condition

2.3.5 Subgrid scale model

Chapter 3
Large eddy simulation of a stratified boundary layer under an oscillatory current

3.1 Introduction

3.2 Domain resolution & initialization

3.3 Selection of simulated cases

3.4 Results

3.4.1 Passive scalar case, $R_i = 0$

3.4.2 Overall thermal field

3.4.3 Velocity field

3.4.4 Internal Waves

3.5 Conclusions

Chapter 4
Turbulence during generation of an internal tide on a critical slope

4.1 Selection of the simulated case

4.2 Results

4.3 Conclusions

Chapter 5
Direct and large eddy simulations of internal tide generation at near-critical slope

5.1 Domain resolution & initialization

5.2 Results

5.2.1 Velocity field

5.2.2 Thermal bore

5.2.3 Wave Energetics

5.2.4 Turbulence energetics

5.3 Conclusions
<table>
<thead>
<tr>
<th>Chapter 6</th>
<th>Boundary mixing by density overturns in an internal tidal beam</th>
<th>116</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Selection of the simulated case</td>
<td>117</td>
</tr>
<tr>
<td>6.2</td>
<td>Results</td>
<td>119</td>
</tr>
<tr>
<td>6.3</td>
<td>Conclusion</td>
<td>125</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 7</th>
<th>Negative turbulent production during flow reversal in a stratified oscillating boundary layer on a sloping bottom</th>
<th>127</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>Selection of the simulated case</td>
<td>128</td>
</tr>
<tr>
<td>7.2</td>
<td>Results</td>
<td>129</td>
</tr>
<tr>
<td>7.3</td>
<td>Conclusion</td>
<td>132</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 8</th>
<th>Degradation of an internal wave beam by resonant triad interaction in an upper ocean pycnocline.</th>
<th>134</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1</td>
<td>Introduction</td>
<td>135</td>
</tr>
<tr>
<td>8.2</td>
<td>Problem formulation</td>
<td>137</td>
</tr>
<tr>
<td>8.3</td>
<td>Results</td>
<td>141</td>
</tr>
<tr>
<td>8.4</td>
<td>Conclusion</td>
<td>148</td>
</tr>
</tbody>
</table>

| Chapter 9 | Summary | 149 |

<table>
<thead>
<tr>
<th>Appendix A</th>
<th>Algorithm and Parallelization of CFD code</th>
<th>154</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1</td>
<td>Algorithm: Channel Geometry</td>
<td>155</td>
</tr>
<tr>
<td>A.2</td>
<td>Algorithm: Curvilinear Geometry</td>
<td>160</td>
</tr>
<tr>
<td>A.3</td>
<td>Code Parallelization</td>
<td>172</td>
</tr>
<tr>
<td>A.3.1</td>
<td>Shared memory architecture</td>
<td>173</td>
</tr>
<tr>
<td>A.3.2</td>
<td>Distributed memory architecture</td>
<td>174</td>
</tr>
</tbody>
</table>

| Appendix B | Decomposition of pressure and velocity | 179 |

| Bibliography | | 181 |
LIST OF SYMBOLS

\( \mathbf{u} \) Velocity in cartesian coordinate system
\( U_0, U_\infty \) Free stream velocity
\( u_\tau \) Friction velocity
\( \tau_w \) Wall shear stress
\( c_f \) Friction coefficient
\( \rho \) Total density
\( \rho^* \) Deviation of from the background density
\( \rho^b \) Background density
\( \rho_0 \) Reference density
\( \nu \) Molecular viscosity
\( \kappa \) Thermal diffusivity
\( \beta \) Slope angle
\( \theta \) Potential temperature
\( Ri_g \) Gradient Richardson number
\( B \) Buoyancy number
\( Re_S \) Reynolds number
\( Re \) Stokes Reynolds number
\( Ex \) Excursion number
\( \epsilon \) Criticality parameter
\( \delta_S \) Stokes layer thickness
\( Fr \) Froude number
\( Pr \) Prandtl number
\( \langle \cdot \rangle \) Reynolds average
\( \overline{\cdot} \) Filtered quantity
\( g \) Gravitational acceleration
\( \Omega \) Wave frequency
\( \Theta \) Wave characteristic angle
\( N \) Buoyancy frequency
LIST OF FIGURES

Figure 1.1: Schematic of the energy pathways in general circulation with energy sources and sinks based on Ferrari & Wunsch (2009). 2

Figure 1.2: Schematic of constant cone of Ω for three-dimensional internal waves. Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity. 4

Figure 1.3: Schematic of IW reflection from sloping topography: (a) off-critical reflection ($\Theta \gg \beta$) and (b) near-critical reflection ($\Theta \approx \beta$). Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity. 6

Figure 1.4: Schematic of the near-slope energy cascade during generation of internal waves at sloping topography. Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity. 7

Figure 1.5: Schematic of internal wave generation from model ridge topography. Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity. 8

Figure 1.6: Contours of kinetic energy, $E$, in a case that illustrates the formation of a beam of internal waves. Normalization is with respect to the barotropic kinetic energy, $E_f$. 9

Figure 1.7: Spatial distribution of the dissipation field over the Oregon continental slope. The most intense and the most extensive region of turbulence was observed along the slope that parallels the M2 tidal characteristic shown by the black line. 11

Figure 1.8: (top) Cross-slope section of (a) root mean square (rms) topographic height in the 200 – 1000 m waveband (Hrms, green shading). 12

Figure 1.9: (top) The mean kinetic energy density $\langle E \rangle_k$ at KC. An M2 internal tide ray (the magenta curve) computed using buoyancy frequency from full-depth HOTS profiles, is superimposed on the path of concentrated $\langle E \rangle_k$. 20

Figure 1.10: Spatial variation of total velocity variance, $\langle u^2 + w^2 \rangle$. Black lines are M2 ray slopes with the location of the surface reflections chosen by eye to best coincide with regions of largest velocity variance. 21

Figure 1.11: A 3D perspective looking northwestward has bathymetry in grey colors (100-m contours). A model M2 energy density. Energy density is enhanced along the beam originating from Sur Platform and below it in the offshore-directed beam. 22

Figure 2.1: (a) Schematic of the problem, (b) Nondimensional values of imposed pressure gradient and free-stream velocity as a function of tidal phase. 27
Figure 2.2: (a) Schematic of the problem. Background color shade in the figure indicates the stable density stratification. (b) Profiles of cross slope velocity, normalized with peak beam velocity $U_b$, as a function of normalized height $(z_c/l_h)$. .......................................................... 32
Figure 2.3: Schematic of the problem along with oscillatory tidal forcing $F_0(t_d)$ in streamwise direction. .......................................................... 34
Figure 2.4: Curvilinear grid in the computational domain. A sponge layer surrounds the domain at the left, right and top. .......................... 39

Figure 3.1: Ensemble-averaged profiles of the streamwise velocity in a semi-log plot. .......................................................... 49
Figure 3.2: Profiles of streamwise turbulence intensity $u_{rms}$ in the passive scalar case. Each profile is staggered by 0.2 units in the horizontal. 50
Figure 3.3: Time evolution of vertical profiles of plane-averaged temperature gradient: (a) $Ri = 500$ and (b) $Ri = 2500$. The lower and upper white lines show contours of $\partial \theta / \partial z = 0.3$ and 0.5, respectively. .......................................................... 51
Figure 3.4: Profiles of thermal field: (a) Temperature and (b) Temperature gradient. .......................................................... 52
Figure 3.5: Contours, as a function of $z$ and $t$, in case 3 with $Ri = 2500$ that illustrate the behavior of the thermal statistics over a half-cycle. 53
Figure 3.6: Profiles of streamwise velocity at different phases. Case with $Ri = 0, 500$ and 2500 are shown in light gray, dark gray and back lines, respectively. .......................................................... 54
Figure 3.7: Explanation of the velocity overshoot. .......................................................... 55
Figure 3.8: Profiles of the streamwise velocity in semi-log plot. Each profile is staggered by 30 units in the horizontal. .......................................................... 57
Figure 3.9: Magnitude of wall shear stress, $\tau_w$, as a function of tidal phase. .......................................................... 58
Figure 3.10: (a) Contour plot of the turbulent kinetic energy over a half-cycle, normalized by $u^{2}_{\tau, max}$ for $Ri = 0$. (b) Turbulent layer thickness, based on the TKE isocontour at 10% of the global maximum value. .......................................................... 60
Figure 3.11: Vertical profiles of production (dashed), dissipation (dash-dot) and transport (solid) in the outer part of the boundary layer shown for $\phi = 60^\circ, 90^\circ, 120^\circ$ and 150$^\circ$, respectively. Light gray and black are used for $Ri = 0$ and 2500, respectively. .......................................................... 60
Figure 3.12: (a) Integrated (over $z$) TKE normalized by $u^{2}_{\tau, max} \delta_s$ as function of phase. (b) Integrated production, integrated dissipation and integrated buoyancy flux over a half tidal cycle. .......................................................... 62
Figure 3.13: Vertical profiles of streamwise turbulence intensity $u_{rms}/U_0$. Each profile is staggered by 0.2 units in the horizontal. .......................................................... 64
Figure 3.14: Vertical profiles of vertical turbulence intensity $w_{rms}/U_0$. Each profile is staggered by 0.1 units in the horizontal. .......................................................... 65
Figure 3.15: Vertical profiles of Reynolds shear stress $\langle u'w' \rangle / U^2_0$. Each profile is staggered by $4 \times 10^{-3}$ units in the horizontal. 65

Figure 3.16: (a) Profiles of streamwise r.m.s turbulence normalized by friction velocity $u_\tau$ as a function of $z^+$ ($z/\delta_t$) at $\phi = 60^\circ$, $75^\circ$, $90^\circ$ and $105^\circ$. (b) Profiles replotted using the $z^+$ coordinate and semi-log axes. 66

Figure 3.17: Four panels show the profiles of production (solid line), dissipation (dash-dot line) and modified transport (dotted line) normalized by $u_\tau^4(\phi)/\nu$ as function of $z^+(z/\delta_t)$ at $\phi = 50^\circ, 60^\circ, 90^\circ$ and $120^\circ$, respectively. 67

Figure 3.18: Cartoon of internal wave generation from the thermocline excited by turbulent boundary layer. Here $C_g$ and $C_p$ are the group and the phase velocity, respectively. 68

Figure 3.19: Slice of $\partial w'/\partial z$ in the x-z plane for the case with $Ri = 500$ at different tidal phases: (a) $\phi = -5^\circ$, (b) $\phi = 90^\circ$, (c) $\phi = 180^\circ$ and (d) $\phi = -90^\circ$. 70

Figure 3.20: In (a)-(i), slices of $\partial w'/\partial z$ in the x-z plane along with streamwise velocity profiles are shown over a time period that spans an entire tidal cycle, shown in the background of the figure. 72

Figure 3.21: (a) Power spectra of the $\partial w'/\partial z(t)$ field (log-scale) as a function of frequency in the case with $Ri = 500$. In (b) the two dimensional power spectrum (log-scale) of $\partial w'/\partial z(x, t)$ at $z = 50$ plotted. 73

Figure 3.22: Same as figure 3.21 for $Ri = 2500$. 75

Figure 3.23: Vertical energy flux normalized by (a) the integrated turbulent dissipation, (b) the integrated production and (c) the integrated buoyancy flux at $\phi = 90^\circ$ for $Ri = 500$ and $2500$. 76

Figure 3.24: Variation of the averaged vertical energy flux normalized by $\rho_0 u_\tau^3_{max}$ over a phase for $Ri = 500$ and $2500$. 77

Figure 4.1: (a) Internal wave field visualized by a slice of $dw/dz$ field in x-z plane. (b) Power spectra of the baroclinic velocity $u_{bar}(x, y, z, t)$ field (semi log-scale). 83

Figure 4.2: The resonant generation of the internal tide is shown by the normalized kinetic energy of the flow field $|\langle u(x, z, t) \rangle|^2/U_0^2$ along with the slope topography in black color. 84

Figure 4.3: Visualization of spanwise instability. (a) Isosurface of density preceding wave breaking, (b) Isosurface of density at wave breaking ($t \simeq T + 1/4T$), (c) Streamwise vorticity at same time as part (b). 85

Figure 4.4: Contour of $Ri_g$, buoyancy flux, production and turbulent kinetic energy at $t = T + 1/4T$ in (a), (b), (c) and (d), respectively. 86
Figure 4.5: Streamwise velocity profile as a function of vertical height and phase at three different locations X=5m, 6.5m and 8m.

Figure 5.1: Time evolution of (a) along slope velocity $U_{sl}$ and (b) magnitude of friction velocity $|u_r|$ at a particular location which is inside the boundary layer at the midpoint of the slope denoted by point Q in the inset of figure 2.4 for different cases.

Figure 5.2: Profiles of along slope velocity as a function of wall normal distance at the midpoint of the slope for (a) $\phi = 0^\circ$ and (b) $\phi = 180^\circ$. (c) Profiles at $\phi = 0^\circ$ replotted after normalization with the maximal velocity amplitude, $U_{sl}^{max}$ and the beam width, $l_b$.

Figure 5.3: (a) Amplitude of the along slope velocity as function of slope length. Right hand inset is a replot of the data in log-log scale along with a linear least squares fit.

Figure 5.4: Jet width, $l_b$, as function of horizontal slope length is shown for both upslope and downslope boundary flow. Inset shows data replotted in log-log scale along with its linear least squares fit.

Figure 5.5: (a) Baroclinic velocity amplitude $u_0(x, z)$ and (b) phase $\Delta \phi_u(x, z)$ as function of space given in Eq. (5.8) for case 4.

Figure 5.6: Formation and propagation of thermal bore: shown at four different time instants.

Figure 5.7: (a) Power spectra, $Y$, based on the time series data of the baroclinic streamwise velocity. Line-average of the power spectra, $\langle Y \rangle_z$, is shown in (b). Dashed vertical line corresponds to the normalized buoyancy frequency.

Figure 5.8: (a) Power spectra of the baroclinic velocity field at locations a, b and c along a vertical line at point Q. (b) Profile of buoyancy flux, pressure transport and turbulent transport as function height above the bottom at location Q.

Figure 5.9: Energy distribution over harmonics 1-3 at a point on the slope and in the boundary layer is shown with absolute units ($m^2 s^{-2}$) in the bar chart of (a). Part (b) is same as (a) except that all quantities are in relative units.

Figure 5.10: Normalized area integrated kinetic energy density, $\langle \mathcal{E}_k \rangle$ and potential energy density, $\langle \mathcal{E}_p \rangle$, are shown as function of slope length. Normalization factor is $(1/2) \rho_0 U_0^2 \Gamma$.

Figure 5.11: Spatial distribution of cycle-averaged streamwise IW flux, $\langle \mathbf{f}^\parallel \rangle = \langle p_{bc} u_{bc} \rangle_{y,t}$ ($W m^{-2}$) is shown in x-z plane for case 5. Time averaging is done over the final 5 cycles of the simulation.

Figure 5.12: (a) Cycle averaged integrated energy flux in streamwise direction, $\langle F^\parallel \rangle$ ($W m^{-1}$), and vertical direction, $\langle F^\perp \rangle$ ($W m^{-1}$), as a function of slope length.
Figure 5.13: Logarithmic profiles of production, $\log_{10}|P|$, and dissipation, $\log_{10}|\varepsilon|$, as function of height above the bottom at location Q in Figure 2.4 inset.

Figure 5.14: Case 4 at a time corresponding to $\phi = 0^\circ$: (a) Contours of turbulent production, (b) Contours of turbulent dissipation.

Figure 5.15: (a) Turbulent kinetic energy profile as a function of height, $z^*$, above the bottom at the midpoint of the slope shown in inset of Figure 2.4. (b) Same TKE profiles replotted after normalization.

Figure 5.16: Time evolution in case 3: (a) Stream velocity is shown as a function of time. Temporal evolution of turbulent kinetic energy, production, dissipation, buoyancy flux and modified transport are shown in (b) and (c) at midpoint of the slope.

Figure 5.17: (a) Normalized value of integrated TKE, $\langle K \rangle$, as a function of slope length. (b) Normalized values of production $\langle P \rangle$, dissipation $\langle D \rangle$ and buoyancy flux $\langle B \rangle$.

Figure 6.1: (a) Horizontally averaged $k_x$ spectra of the streamwise velocity fluctuation, $u'_r = u_r - \langle u_r \rangle$ at three heights from the bottom slope during flow reversal from downslope to upslope at $t = 16$ hrs. (b) Same as (a) for $k_y$ spectra.

Figure 6.2: Bottom panel shows vertical $x$-$z$ slice of the density field (after subtracting $1000 \text{ kg m}^{-3}$) at 4 different times (phases) in a tidal cycle.

Figure 6.3: (a) Averaged along stream velocity (blue solid line) as function of time. (b) Temporal evolution of spanwise averaged TKE, along a height.

Figure 6.4: (a) Cycle evolution of depth-averaged values of the quantities in TKE-budget along with averaged streamwise velocity (dashed red line). (b) Temporal evolution of depth-averaged dissipation (black line) and buoyancy frequency $N$ (dashed blue line).

Figure 7.1: (a) Temporal evolution of averaged TKE profiles. (b) Cycle evolution of depth-averaged values of the quantities in TKE-budget along with averaged streamwise velocity (dashed red line) at height of $z^* = 0.6$ m.

Figure 7.2: Illustration of negative production mechanism during the flow reversal event from downslope to upslope flow.

Figure 7.3: (a) Wall normal profiles of mean streamwise velocity at $\phi = \pi/2 + \pi/10$.

Figure 8.1: Schematic of the problem along with internal wave beam maker at left hand side and sponge region at the right and bottom of the computational domain. IW beam path is denoted by red line.
Figure 8.2: Overview of simulated IW beam dynamic during the interaction with upper ocean pycnocline shown by streamwise velocity contours: (a) laminar response at $t = 10T$, (b) weakly nonlinear regime with trapped harmonics at $t = 15T$.

Figure 8.3: Evolution of the line averaged spectrum, $\langle \mathcal{Y}(\omega, t) \rangle$, for streamwise velocity measured along the two horizontal lines intersecting the beam inside the pycnocline.

Figure 8.4: Time series of horizontal velocity along a horizontal line of original data and band-passed in time frequency ranges, as indicated below each plot. Here, the line is taken at location 1 as shown in figure 8.2(c).

Figure 8.5: Evolution of the area averaged kinetic energy, $\langle E_{k,\text{sub}} \rangle$, (solid red line) and potential energy $\langle E_{p,\text{sub}} \rangle$ (dashed red line) of the subharmonic wave motion as function of day.

Figure 8.6: (a) Profiles of streamwise velocity, density and inverse gradient Richardson number, $R_i^{-1}$. (b) The two-dimensional power spectrum (log-scale). (d) is similar to (b) with the time series taken over a time span $10T < t < 20T$.

Figure A.1: Grid layout of the channel geometry in the wall-normal directions.

Figure A.2: Computational grid discretization, $(\zeta - \eta$ coordinate system) in the physical domain. Here, $x$ and $\xi$ are directed into the plane of the figure.

Figure A.3: Grid layout of the curvilinear geometry in $\zeta - \eta$ plane.

Figure A.4: An illustration of multithreading in OpenMP. Here the master thread distribute job A and job B into a number of threads which execute blocks of code in parallel.

Figure A.5: An illustration of distributed memory System. Here, processors are connected through a network hub.

Figure A.6: Domain decomposition for the MPI version of channel flow geometry.

Figure A.7: Domain decomposition for the MPI version of curvilinear geometry.

Figure A.8: Local data structures pre and post MPI `ALL TO ALL` based on the four processors with rank 0, 1, 2 and 3. Before MPI `ALL TO ALL` call, the initial array is first divided into four segments, denoted A,B,C, and D.
LIST OF TABLES

Table 3.1: Simulation parameters. ........................................ 47
Table 3.2: Comparison of the overall boundary properties between various stratified cases. The subscript \( \text{avg} \) denotes an average over the complete tidal cycle. Here, \( \Delta \tau \) is the phase lead of \( \tau_{\text{w}}^{\text{max}} \). ........ 57

Table 5.1: Dimensional parameters of the simulated cases. ........... 90
Table 5.2: Non-dimensional parameters and grid resolution of the simulated cases. The excursion number is chosen to be small as is typical for deep water topography and the slope angle is critical. ........ 90
Table 5.3: Comparison of the overall boundary properties between cases. Here, \( u_r \) is the friction velocity calculated based on the wall friction, \( \tau_{\text{w}} \), and \( c_f \) is the friction coefficient. ............... 93

Table A.1: Runge-Kutta parameters ........................................ 160
Table A.2: Essential MPI routines. For more detail see Gropp et al. (1999). 175
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VITA

2006  B. E. in Mechanical Engineering, Jadavpur University, Kolkata, India

2007  M. S (Engg.), in EMU, JNCASR, Bangalore, India

2010  M. S. in Engineering Sciences (Engineering Physics), University of California, San Diego, USA

2012  Ph. D. in Engineering Sciences (Engineering Physics), University of California, San Diego, USA

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ABSTRACT OF THE DISSERTATION

Turbulence and Internal Waves in tidal flow over topography

by

Bishakhdatta Gayen

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Professor Sutanu Sarkar, Chair

Direct Numerical Simulation (DNS) and Large Eddy Simulation (LES) are used to investigate and quantify the dynamical processes underlying turbulence formed during the generation of an internal wave beam and its subsequent interaction with a realistically-stratified upper ocean. As a part of the thesis a three-dimensional mixed spectral/finite difference code was developed, parallelized, validated and employed to study several geophysical problems relevant to internal tide generation and its nonlinear breaking.

The thesis research has four phases. In the first phase, a study of a stratified non-sloping bottom boundary layer under an oscillating tide was completed. The focus is on the boundary layer response to an external stratification based on LES. Flow instabilities and turbulence in the bottom boundary layer are found to excite internal gravity waves that propagate away into the ambient with phase angle varying over the tidal cycle. Subsequent studies as part of the second phase consider a stratified oscillating flow over a sloping bottom wall to mimic the generation of baroclinic internal waves (IW) from the tide-topography interaction at a model continental slope. The DNS study shows transition to turbulence which is present along the entire extent of the near-critical region of the slope in the regime of low background excursion number and Reynolds number. The transition is found to be
initiated by a convective instability which is closely followed by shear instability. The peak value of the near-bottom velocity is found to increase with increasing length of the critical region of the topography. The scaling law that is observed to link the near-bottom peak velocity to slope length is explained by an analytical boundary layer solution that incorporates an empirically obtained turbulent viscosity. As an extension of the second phase, the objective of the third phase work is to numerically model a near-bottom beam with a larger, more realistic width using LES and characterize its turbulence statistics. Maximum turbulent kinetic energy and dissipation rate are found just after the zero velocity point when flow reverses from downslope to upslope motion. The phasing and other characteristics of the turbulent mixing in the present simulations show remarkable similarity with that observed off Kaena Ridge in Hawaii taken during the hawaiian ocean mixing experiment (HOME), and may be explained by the beam-scale convective overturns found here. The objective of the final phase is to understand the interaction process between an IW beam and an upper ocean pycnocline and to further characterize the cascade to small scales in the context of IW beam degradation observed in the ocean.
Chapter 1

Introduction

The general ocean circulation, of crucial importance to the global climate, involves fluid motion on scales ranging from millimeters to more than 10,000 km. Surface fluxes (heat, water, momentum etc.) and tides act as primarily inputs to the system at large space and long time scales. In the presence of continuous large scale forcing, an equilibrium is maintained for large and mesoscale flows ($\sim 10 - 1000$ km) in the general circulation. Here, energy dissipation plays the crucial role of energy sink (forward cascade) for the system helping to establish a balance with the forcing. Without a dissipation mechanism, the inverse cascade in geostrophic turbulence would result in barotropic eddies the size of the ocean basins, that are not seen. Therefore, understanding the physics of various dissipative phenomenon in the context of ocean mixing is an important step towards improving the dynamical description of large-scale motions.

Figure 1.1 shows a schematic of the energy pathways in the general circulation where the tide and surface fluxes maintain the reservoir of potential energy. The large-scale flow and the resultant eddies are both, however, in approximate geostrophic balance; a balance between pressure gradients and rotational effects. An important aspect of turbulence in the context of such balanced dynamics is an inverse cascade of energy as shown by left hand side pathway in figure 1.1 (indicated by a blue arrow). To counterbalance this increase of energy, some processes must exist to drain energy from geostrophic eddy motions and arrest the cascade. Examples of these energy sinks with significant impact are: (a) bottom
drag and bottom turbulence (Wunsch & Ferrari, 2004; Arbic & Flierl, 2004), (b) loss of balance (Molemaker et al., 2005), (c) interactions with the internal wave field (Staquet & Sommeria, 2002), (d) continental margin scattering/absorption, and (e) suppression by wind work. As part of the present dissertation study, we will focus particularly on two cascade mechanisms: (i) bottom turbulence at a tidal boundary layer and (ii) generation of internal tides and further its nonlinear interaction.

A dissipative interaction with bottom topography can arrest the cascade. At the bottom of the ocean, the stress exerted by the sea-floor against the near-bottom currents creates a layer of enhanced shear and turbulence. The turbulence produced in the bottom boundary layer creates a mixed layer near the sea-floor after stirring up local fluid. The bottom boundary layer is an important source of drag on mean currents and mesoscale eddies, and is a location where diapycnal mixing of the density field is large. Recent observations (Wunsch & Ferrari, 2004; Sen et al., 2008) have estimated that about 0.1 – 0.2 terra watt (TW) of energy are dissipated by bottom boundary layer. In order to quantify the momentum and buoyancy fluxes associated with the bottom boundary layer, it is important to accurately represent the turbulent motions responsible for these fluxes.

Understanding the physics of the turbulent bottom boundary layer is re-
quired to further model the large-scale motions. Since large scale ocean models can not accurately resolve the the bottom boundary motions, parameterization is necessary to relate the influence of the bottom boundary layer on the ocean circulation. In this context, real ocean data are relatively scarce to establish the validity of the given parameterization. One of the goals of this study is to produce a database of turbulence-resolving numerical simulations of the bottom boundary layer in idealized conditions that can be used to test the performance of new parameterizations.

Energy conversion from the barotropic surface tide into internal gravity waves and small-scale turbulence has also proved to be an efficient mechanism for forward cascade (Egbert & Ray, 2001). The conversion of the barotropic tide to internal waves over rough topography provides 0.7 to 0.9 TW of the approximately 2.1 TW total energy input which is required to maintain the abyssal stratification (Munk & Wunsch, 1998). A portion of the energy as large as 30-50% of barotropic energy input (St. Laurent & Nash, 2004) is dissipated within turbulent boundary layers near the generation site. The result is the now familiar picture of hotspots of elevated mixing near rough topography (Polzin et al., 1997; St. Laurent & Garrett, 2002; St. Laurent et al., 2003; Rudnick et al., 2003; Nash et al., 2007) and such a pattern of tidal mixing hotspots has further influence on the global circulation (Simmons et al., 2004; Saenko et al., 2005).

But mixing hotspots near tidal conversion sites are only half the story. Between 50% and 95% of the available baroclinic energy escapes the near-field to propagate up to thousands of kilometers across ocean basins (St. Laurent & Nash, 2004). Recent ocean measurement by Klymak et al. (2006) suggest that of the estimated 25 GW extracted from the barotropic tide near Hawaii, only approximately 15% is dissipated locally, leaving the rest to propagate away as a low mode internal tide. Away from sharp topographic features, the internal tide is subject to nonlinear interactions with motions at other spatial scales and temporal frequencies (Müller et al., 1986; Staquet & Sommeria, 2002; MacKinnon & Winters, 2005). Therefore, not only the generation of baroclinic tide, but also its further interaction with upper ocean stratification away from the generation
site needs understanding to have a play complete picture of the energy cascade.

1.1 Internal waves

The atmosphere and ocean are continuously stratified due to changes in temperature, pressure and composition. These changes can lead to significant variations of fluid density in the vertical direction. For example, fresh light water from rivers can rest on top of sea water, and due to the small diffusivity of the water, the density contrast remains for a long time. Density stratification supports oscillatory motion, i.e., internal waves. Here, gravity plays the restoring force for internal waves. Internal waves can transport energy in the ocean or atmosphere over long distances. Propagation, energy transfer and other basic properties of IW can be explained staring from the system of equations governing wave motion of an incompressible fluid with continuous density stratification.

The linearized, inviscid form of the NS equations for perturbation velocities, $u = [u \ v \ w]$, under the Boussinesq approximation (density is constant everywhere
except in the gravity, or buoyancy term) are written as:

\[ \nabla \cdot \mathbf{u} = 0 \]  \hspace{1cm} (1.1)

\[ \frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla p^* - \frac{g}{\rho_0} \rho^* \mathbf{k} - f \mathbf{k} \times \mathbf{u} \]  \hspace{1cm} (1.2)

\[ \frac{\partial \rho^*}{\partial t} = \frac{\rho_0 N_{\infty}^2(z)}{g} \mathbf{w}. \]  \hspace{1cm} (1.3)

Here, \( p^* \) and \( \rho^* \) denote deviation from the background pressure and density, respectively. \( N_{\infty} = \sqrt{(g/\rho_0)(d\rho_b/dz)} \) is the local buoyancy frequency based on the local density gradient \( d\rho_b/dz \), reference density \( \rho_0 \) and gravity, \( g \). \( f \) is the Coriolis parameter. Equations (1.1)-(1.3) can be simplified to wave equation

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2 w}{\partial t^2} + N_{\infty}^2(z) \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] + f \frac{\partial^2 w}{\partial z^2} = 0. \]  \hspace{1cm} (1.4)

We assume spatial and temporal periodic solution,

\[ w(x, z, t) = \text{Re} \left[ \tilde{W}_0(k, z)e^{i(kx + ly + mz - \Omega t)} \right], \]  \hspace{1cm} (1.5)

where \( \tilde{W}_0 \) is an arbitrary constant and \( \mathbf{k} = [k \ l \ m] \) are the wave numbers in the streamwise \( (x) \), spanwise \( (y) \) and vertical \( (z) \) directions, respectively. Here, \( \Omega \) is the forcing frequency. Since the fluid motion is incompressible, \( \nabla \cdot \mathbf{u} = -i\mathbf{k} \cdot \mathbf{u} = 0 \), i.e., the velocity is orthogonal to the wave number vector. These waves are also classified as shear waves. After substituting solution of the vertical velocity \( w \) into (1.4), the dispersion relation between wave number vector and frequency of waves follows as

\[ \Omega^2 = \frac{f^2 m^2 + (k^2 + l^2) N_{\infty}^2}{k^2 + l^2 + m^2}. \]  \hspace{1cm} (1.6)

For internal waves, surfaces of constant frequency in the wavenumber space are the cones with \( \Theta= \text{constant} \) as illustrated in figure 1.2 for a non-rotating environment \( (f = 0) \). Here, \( \Theta = \sin^{-1}(\Omega/N_{\infty}) \) is often known as phase propagation angle of IW. The phase lines of the internal waves oriented at the angle, \( \Theta \), with respect to the horizontal, move with phase speed \( C_p = \Omega \mathbf{k}/|\mathbf{k}|^2 \). The energy of the internal waves travels with group speed \( C_g = \nabla_k \Omega \) which can easily be shown to be orthogonal to the phase velocity. When the group velocity has an
upward component, then the phase velocity has a downward component, and vice versa. Internal waves can be easily generated by oscillating a horizontal cylinder at a frequency smaller than the buoyancy frequency. This causes waves to be generated, so that energy moves away from the cylinder in four beams, with the crests and troughs (or phase) moving perpendicularly to the direction in which the beams carry energy away from the oscillating cylinder.

![Figure 1.3: Schematic of IW reflection from sloping topography: (a) off-critical reflection ($\Theta \gg \beta$) and (b) near-critical reflection ($\Theta \approx \beta$). Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity.](image)

The slope of topography plays a crucial role in wave energetics. When the slope angle, $\beta$, becomes close to the characteristic angle of the internal wave, $\Theta$, a resonant response occurs during the generation or reflection, leading to formation of a localized IW beam. These internal waves further energize the boundary flow formed in the vicinity of the topography. During the reflection, phase lines of reflected internal waves come close to the surface leading to intensified fluid velocity as shown in figure 1.3. This can lead to wave steepening as well as subsequent turbulent mixing, and has been studied in the laboratory (Ivey & Nokes, 1989) and by numerical simulations (Slinn & Riley, 1998).
1.2 Generation of internal waves by oscillating tide over topography

Tides in the ocean interact with bottom topography to result in energetic internal gravity waves, the so-called internal tides. It is thought that internal tides play an important role in deep ocean mixing (Polzin et al., 1997; Munk & Wunsch, 1998; Ledwell et al., 2000; Wunsch & Ferrari, 2004). The conversion to internal tides is enhanced by sea-mounts (Lueck & Mudge, 1997; Kunze & Toole, 1997), submarine ridges (Rudnick et al., 2003; Klymak et al., 2006), submarine canyons (Polzin et al., 1996b; Carter & Gregg, 2002), continental slopes (Cacchione et al., 2002; Moum et al., 2002; Nash et al., 2004, 2007) and deep rough topography (Polzin et al., 1997; St. Laurent et al., 2001). As part of the baroclinic response to an oscillating flow over sloping topography shown schematically in Fig. 1.4, energy propagates out as internal waves while some energy is locally confined as trapped internal wave motion and a bottom boundary layer. The locally confined energy may cascade to small-scale turbulence owing to nonlinear effects that can be especially large when the slope angle is near the critical value. The propagating internal waves can also break down to turbulence at sites remote from the generation region through a variety of mechanisms including reflection at a topo-

**Figure 1.4:** Schematic of the near-slope energy cascade during generation of internal waves at sloping topography. Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity.
graphic slope with critical angle. The case of critical and near-critical reflection of waves incident on a slope have been studied in the laboratory, by analysis and by direct numerical simulation (DNS) of the three-dimensional Navier-Stokes equations. Laboratory studies (Ivey & Nokes, 1989; Thorpe, 1992) and DNS studies (Slinn & Riley, 1998; Venayagamoorthy & Fringer, 2007) of internal wave reflection find that wave/slope interactions lead to a complex turbulent flow. Here we investigate a different problem, that of internal wave generation rather than internal wave reflection, using both DNS and large eddy simulation (LES).

![Figure 1.5: Schematic of internal wave generation from model ridge topography. Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity.](image)

### 1.2.1 Theory of IW generation

Generation of internal waves at sloping topography is governed by the following physical parameters as discussed by Garrett & Kunze (2007): frequency of tidal oscillation $\Omega$; the buoyancy frequency $N_\infty$; the Coriolis frequency $f$; the topographic height $h$ and horizontal length $l$; the depth of the ocean, $H$, and the amplitude of the deep-water barotropic tidal velocity $U_0$. Based on the dimensional fluid properties: molecular viscosity $\nu$, thermal diffusivity $\kappa$, three governing nondimensional parameters are: Reynolds number $Re$, Buoyancy parameter $B$, and Prandtl number $Pr$, where

$$Re \equiv \frac{a U_0}{\nu} = \frac{U_0^2}{\Omega \nu}, \quad B \equiv -g \frac{d \rho_b}{dz}{|}_\infty \frac{1}{\rho_0 \Omega^2} = \frac{N_\infty^2}{\Omega^2}, \quad Pr \equiv \frac{\nu}{\kappa}. \quad (1.7)$$

Here, $a = U_0 / \Omega$ and $N_\infty$ is the background value of buoyancy frequency based on the local density gradient $d \rho_b / dz$, reference density density, $\rho_0$ and gravity, $g$. 
In addition to those listed in (1.7), two nondimensional parameters are especially important in the context of a nonlinear response:

(I) the criticality parameter, \( \epsilon = \frac{\tan(\beta)}{\tan \theta} \),

(II) the excursion number, \( Ex = \frac{U_0}{\Omega l} \).

Here, \( \beta \) is the topographic slope angle, \( \frac{dh(x)}{dx} \) and \( \Theta = \sin^{-1}(\Omega/N_\infty) \) is characteristic angle of the internal waves as shown in figure 1.5. There can be strong nonlinear response when \( Ex \) is comparable to or larger than unity. However, values of \( Ex \) are typically \( << 1 \) in the ocean except for some coastal regions where the topography is steep and the barotropic tides are strong. Topography is said to be subcritical, critical or supercritical if \( \epsilon <, = \) or \( > 1 \), respectively.

The present study is restricted to near-critical slope angles with \( \epsilon \approx 1 \) and low excursion number \( Ex \ll 1 \). Under such circumstances, the internal wave energy that leaves the topography is concentrated into a tidal beam as shown in Fig. 1.6 and the near-bottom velocity is strongly intensified with respect to the barotropic tidal amplitude.

**Figure 1.6**: Contours of kinetic energy, \( E \), in a case that illustrates the formation of a beam of internal waves. Normalization is with respect to the barotropic kinetic energy, \( E_f \).
Linear theory of internal tides is well developed. A popular theoretical approach is based on the weak topography approximation (WTA), i.e., height changes in topography are small compared to the depth of the ocean and the topographic slope is small relative to the slope of internal wave phase lines. Bell (1975a,b) decomposed the topography into Fourier modes, introduced WTA, and computed the energy conversion rate in a uniformly stratified, infinitely deep ocean by linear superposition. St. Laurent & Garrett (2002) used the analysis of Bell to deduce that the conversion from barotropic to internal tide is significant at the Mid-Atlantic Ridge and, under the assumption of linear theory, found that most of the internal wave energy resides in low modes which propagate away without dissipating locally. Llewellyn Smith & Young (2002), Balmforth et al. (2002), Khatiwala (2004) further developed WTA to estimate the tidal conversion in other types of subcritical topography while allowing for finite depth and nonuniform stratification. Balmforth et al. (2002) performed a perturbative expansion in the parameter $\epsilon$ to estimate the influence of increasing slope steepness. Their solution showed a singularity in the solution when $\epsilon \geq 1$. All the aforementioned theory is accurate for small-amplitude subcritical topography. In a different approach, Baines (1974, 1982) developed an analytical model based on ray theory and wave characteristics. This model can deal with arbitrary topography including steep, supercritical topography as long as the regions with $\epsilon = 1$ can be approximated as isolated critical points. Llewellyn Smith & Young (2003) revisited the problem of steep topography using a Green’s function approach (Robinson, 1969) to deal with the singularity at $\epsilon = 1$. Other studies (St. Laurent et al., 2003; Pétrélis et al., 2006; Balmforth & Peacock, 2009) estimated the conversion rate for arbitrary shape topography, among them Pétrélis et al. (2006) calculated a finite value of conversion rate at near-critical topography although velocity and density were found to be singular. Griffiths & Grimshaw (2007) employed a modal approach where the flow field is expanded in terms of basis functions newly derived by the authors. The number of modes required for a converged solution was found to be small, less than 10, for $\epsilon < 1$. However, the solution for $\epsilon \geq 1$ contains singular beams so that finer structure is revealed with increasing number of modes.
1.2.2 Ocean observations of IW generation

Internal waves with tidal frequency, or internal tides that are generated by the barotropic tidal flow over topography, such as continental shelves, ridges, and seamounts, can propagate over large distances before eventually dissipating. The amount of energy transferred to internal tides is largest for topography occupying a large fraction of the water column, topographic slope equal to or exceeding internal wave slope, and strong barotropic tides (Holloway & Merrifield, 1999; Llewellyn Smith & Young, 2002; Khatiwala, 2004; Klymak et al., 2010). Internal tides are susceptible to dissipation and energy transfer to other frequencies (Korobov & Lamb, 2008), particularly at surface and bottom boundaries near seamounts (Lueck & Mudge, 1997; Toole et al., 1997), ridges (Polzin et al., 1997; Rudnick et al., 2003; Klymak et al., 2006; Aucan et al., 2006; Aucan & Merrifield, 2008) and continental slopes (Thorpe, 1990; Cacchione et al., 2002; Nash et al., 2004; Moum et al., 2002; Nash et al., 2006, 2007; Gayen & Sarkar, 2010b).

Observations of large mixing as well as enhanced fine scale shear and strain several hundreds of meters above the Mid-Atlantic Ridge (Polzin et al., 1996a) and its foothills (Ledwell et al., 2000), east of the Brazil Basin, clearly implicate
Figure 1.8: (top) Cross-slope section of (a) rms topographic height in the 200 – 1000 m waveband ($H_{rms}$, green shading). Figure 2(a) also shows the cross-slope distribution mean of dissipation, $(\langle \epsilon \rangle_{500})$ within 500 mab placed vertically in each station. Station types are indicated in the top portion of figure 2(c); M-, L- and X- respectively, represent time series averages from moored profilers, CTD, LADCP and XCP. Some stations were occupied on multiple occasions and with different instrumentation as indicated. (b) semidiurnal slope criticality (> 1000-m scales), (c) turbulent energy dissipation rate, and (d) turbulent diffusivity $K_p$. A semidiurnal characteristic is shown in Figures c and d to illustrate the slope criticality, and is not intended to represent a beam emanating from the shelf break. Only data near the slope are shown; an additional mooring and two CTD/LADCP stations extend to 126°150' W. Figure is taken from Nash et al. (2007)

internal gravity waves in generating turbulence remote from the boundary. High turbulence levels have been observed along characteristics of the internal tidal beam emanating from topographic variations (Lueck & Mudge, 1997; Lien & Gregg, 2001). Lien & Gregg (2001) observed intense turbulence in a ~ 100 m thick layer
of stratified water across the ridge of a sea fan during a microstructure survey in Monterey Bay of California. This midwater beam of strong turbulence emanating from the shelf break along the ray path of the semidiurnal M2 internal tide, was separated from the bottom. Therefore, the source of this turbulence clearly was not bottom friction. According to the microstructure measurement, the turbulent kinetic energy dissipation exceeded $10^{-8} W kg^{-1}$, and the diapycnal eddy diffusivity $K_\rho$ was $> 0.01 m^2 s^{-1}$ within the 50 m thick internal tidal beam.

Observations of mixing over the bathymetry of a large submarine landslide off Oregon’s continental slope made by Moum et al. (2002), using a towed body reveal a great lateral extent (several kilometers) of continuously turbulent fluid within a few hundred meters of the boundary at a depth of 1600 m. The largest turbulent dissipation rates were observed over a 5 km horizontal region along a bottom slope which is critical to the M2 internal tide, shown in Fig. 1.7. As part of the Hawaii Ocean Mixing Experiment (HOME) Aucan et al. (2006) performed an observation on the south flank of Kaena Ridge, Hawaii to examine tidal variations within 200 m of the steeply sloping bottom. They measured temperature fluctuations and current data over 3 months from deep moorings and inferred that vertical displacements and the horizontal current are dominated by the semidiurnal internal tide with amplitudes with $\sim 0.1 m s^{-1}$. Strong temperature inversions are detected with vertical scales of $\sim 100 m$. Based on Thorpe scale analysis of the overturns, they measured a time-averaged dissipation near the bottom of $1.2 \times 10^{-8} W kg^{-1}$, 10 – 100 times of that at similar depths in the ocean interior 50 km from the ridge.

Nash et al. (2007) report two deep ocean hotspots of turbulent mixing over the Oregon continental slope. In the top portion of Fig. 1.8(b), M-, L- and X- represent the location of the moored CTD, LADCP and XCP profilers, respectively, to collect time series data averages. Fig. 1.8(c) and (d) show the time averaged dissipation and eddy diffusivity field, respectively, over the southern part of the Oregon continental slope along with the IW beam path marked by black line. Fig. 1.8(b) shows the distribution of criticality (ratio between topographic slope and internal wave angle) with respect to M2 tide across the slope. Enhanced dissipation
was observed in the stratified bottom boundary layer at station L2.5, the 2200-m hotspot and station L4.3, the 1300-m hotspot. At both stations, slopes are critical with respect to internal wave characteristic as clearly shown in Fig. 1.8(b). Using Thorpe-scale analysis at both hotspots, they estimated time-averaged turbulent energy dissipation rates of $10^{-7} \text{W kg}^{-1}$ and eddy diffusivities of $K_\rho = 10^{-2} \text{m}^2/\text{s}$. At the 2200 m isobath, sustained $>100$ m high turbulent overturns occur in the stratified fluid which is several hundred meters above the bottom. RMS of topographic height, $H_{rms}$, calculated as square root of the variance of 2D high-pass filtered bottom depth, quantifies roughness of the topography. Recently, Klymak et al. (2008) have discussed measurements at Kaena Ridge performed as a part of HOME. They report strong dissipation and intense density overturns at height of 200 m from the topography due to quasi-deterministic breaking associated with large-amplitude, small-scale internal waves. These waves having vertical wavelengths of the order of 400 m, are triggered by tidal forcing consistent with lee-wave formation at the ridge break.

### 1.2.3 Laboratory observations of internal wave generation

Wave generation in the form of a localized beam has been studied in laboratory experiments that consider model continental slopes (Gostiaux & Dauxois, 2007; Zhang et al., 2008; Lim et al., 2010) and other underwater model topography (Echeverri et al., 2009).

Gostiaux & Dauxois (2007) in their laboratory experiment considered a steep continental shelf, for which the internal tide is shown to be emitted from the critical point. They observed that the width of the emitted beam was dependent on the local curvature of topography and fluid viscosity. They also modeled the resulting internal tidal beam by drawing an analogy with an oscillating cylinder in a stationary stratified fluid.

Zhang et al. (2008) focussed on the case with critical slope and found that the resonant wave/slope interaction led to a laminar oscillating boundary layer with intensified bottom velocity, an order of magnitude larger than the imposed oscillatory forcing. They predicted a scaling law for the amplitude of the bottom
boundary flow using an analytical result obtained by Dauxois & Young (1999) for viscous laminar flow. Echeverri et al. (2009) in their laboratory experiments studied internal tides generation by a two-dimensional ridge in a channel of finite depth over the regimes spanning from sub- to supercritical topography. For their topographic configuration, most of the linear baroclinic energy flux was dominated by the low mode ($< 4$) tide, and these results revealed that nonlinear behavior did not significantly affect the barotropic to baroclinic energy conversion in this regime.

All the laboratory studies of internal wave generation have been conducted at low Reynolds number ($Re \sim O(1)$) with the notable exception of recent laboratory experiments by Lim et al. (2010). They performed IW generation from a shelf/slope (supercritical) topography. In their experiments, the baroclinic response along with formation IW beam, boundary layer turbulence and upslope propagation of bores depended upon both the topographic critical parameter, $\epsilon$, and Reynolds number, $Re$, that characterized the boundary layer flow. They classified the observed flow regimes based on a generation parameter $G$, defined as the ratio of the Reynolds number to the topographic critical parameter. The estimated flow regime boundaries were: for $G < 3$ only a beam was observed, for $3 < G < 50$ there was a transitional regime with both a beam and a bolus observed, for $50 < G < 400$ there was another transitional regime with no beam but a bolus observed, and finally for the regime with $G > 400$ there was no bolus observed.

1.2.4 Numerical simulations of internal wave generation

Nonlinear ocean models have proven effective in studying internal tides in a realistic oceanic environment. Holloway & Merrifield (1999) used the nonlinear hydrostatic Princeton Ocean Model (POM) to demonstrate that the conversion into internal wave energy is stronger for flow across elongated features like ridges rather than symmetric features such as islands and seamounts. The POM calculations of Merrifield et al. (2001) identified key generation sites at the Hawaiian Ridge and showed multiple dynamical modes in the near field. Legg (2004) used the MIT model to perform three-dimensional simulations of generation from a continental slope in a regime with $Ex << 1$ but with steep topography including critical and
supercritical regions. Along-slope corrugations in the slope were identified as important to realize high-mode internal waves with potential for local mixing. Legg & Klymak (2008) performed two-dimensional calculations of flow over a tall steep ridge and showed that, at the top of the ridge, the intensified barotropic flow in conjunction with a large slope angle leads to overturning events associated with transient internal hydraulic jumps. Korobov & Lamb (2008) have examined the frequency content of the propagating internal wave field to show the generation of subharmonics, higher harmonics and interharmonics during tide/topography interaction. Recently, Klymak et al. (2010) have numerically studied two dimensional steep gaussian topography under near supercritical environment using a simple mixing scheme based on density overturning scales and background stratification. They have parameterized the tidal dissipation from nonlinear breaking of lee waves under different barotropic forcing and compared with observed dissipation from Kauai channel bathymetry.

The turbulent boundary layer on a non-sloping flat bottom under an oscillating current has been examined in the unstratified case by simulations that resolve turbulence. DNS studies (Spalart & Bladwin, 1987; Akhavan et al., 1991b; Vittori & Verzicco, 1998; Costamagna et al., 2003; Sakamoto & Akitomo, 2008) have paid attention to primarily the disturbed laminar and intermittently turbulent flow regimes that occur at moderate values of Reynolds number. The LES approach has allowed studies in the fully turbulent regime. The simulations of Salon et al. (2007) performed with a dynamic mixed model agreed well with the experimental results of Jensen et al. (1989) and provided new insights into the phase dependence of inner and outer-layer turbulence. Radhakrishnan & Piomelli (2008) have performed LES with various subgrid models and near-wall treatments to further extend the Reynolds number of the simulations.

Turbulence-resolving simulations of oceanic bottom boundary layers in a stratified fluid are scarce. Taylor & Sarkar (2008a) examined the thermal field in a stratified boundary layer using both DNS and LES, and Taylor & Sarkar (2008b) showed that stratification has a significant effect on boundary layer thickness and structure. Broadband bottom turbulence was found to lead to internal waves
which tended to cluster around 45° during propagation as discussed by Taylor & Sarkar (2007a). An oscillating boundary layer in a stratified fluid was examined through LES by Gayen et al. (2010a) who found that stratification increases the asymmetry in turbulence between accelerating and decelerating phases and also increases the height-dependent lag in the phase of maximum turbulent kinetic energy with respect to the peak free-stream velocity. Li et al. (1999) employed LES to study an estuarine tidal boundary layer where the horizontal density gradient associated with salinity is found to introduce a strong ebb-flood asymmetry in the turbulence.

Since all the theoretical investigations of internal tide generation are based on linear analysis, they cannot study the evolution of flow instabilities into turbulence. Previous numerical models of tide generation have proved useful to study some nonlinear aspects of the generation problem but the relatively coarse resolution and high values of viscosity in these simulations preclude resolution of turbulence dynamics. Recently, Gayen & Sarkar (2010b) performed a three-dimensional DNS of generation by a laboratory-scale slope in the regime of $Ex << 1$ and $\epsilon \simeq 1$ that shows transition to turbulence along the entire slope. The transition is found to be initiated by a convective instability which is closely followed by shear instability. More details will be given later in Chapter 4. Later, Gayen & Sarkar (2011a) have extended the work of Gayen & Sarkar (2010b) by examining internal wave energetics as well as the energetics of turbulence in the bottom boundary layer. In addition, the effect of increasing slope length, $l$, is quantified by employing a LES approach to access higher values of $l$. In Chapter 5, we also extend previous DNS/LES of bottom turbulence from the case of tidal flow over a non-sloping bottom to the situation with a sloping bottom where the baroclinic wave velocity dominates the barotropic tidal velocity.

From oceanic observations, it is clear that strong dissipation is found at the generation site of internal tidal beam along the continental slope. Large density overturns of order of $> 100 \text{ m}$ are also reported in previous ocean observations (Moum et al., 2002; Nash et al., 2007). Though laboratory (Lim et al., 2010) and numerical (Gayen & Sarkar, 2010b, 2011a) experiments have shown evidence of
strong turbulence over slope topography of order $O(1-10)$ m, larger scale simulations resolving turbulence are nonexistent. However, the beam was at laboratory scale (width smaller than 0.2 m) and did not allow the large separation in scales between the viscous boundary layer and the oscillating core of the beam that occurs in the ocean. The objective of the work (Gayen & Sarkar, 2011b,c) discussed in Chapter 6 and Chapter 7 is to numerically model a near-bottom beam with a larger, more realistic width and characterize its turbulence characteristics. We have also shown that the phase dependence of turbulence in the numerical model agrees with that observed by Aucan et al. (2006) off Kaena Ridge and try to explain the underlying mechanism.

1.3 Internal tidal beam interaction with a pycnocline

Internal waves are generated by small disturbances in a stably stratified fluid. Phase lines of the internal waves are oriented at an angle, $\Theta$, with respect to the horizontal. Here, defined by $\Theta = \sin^{-1}(\Omega/N_{\infty})$, where $\Omega$ is the forcing frequency and $N_{\infty} = \sqrt{(g/\rho_0)(d\rho_b/dz)}$ is the local buoyancy frequency based on the local density gradient $d\rho_b/dz$ and reference density density, $\rho_0$. Internal tide generation at abrupt topography has been the focus of a number of recent theoretical, numerical, laboratory, and observational work (Garrett & Kunze, 2007). Internal waves generated by topography under supercritical (Gostiaux & Dauxois, 2007; Echeverri et al., 2009) and near-critical environments (Zhang et al., 2008; Gayen & Sarkar, 2010b) form a tidal beam, which is composed of many spatial modes within a finite-width region. As a results, the near-bottom velocity is strongly intensified with respect to the barotropic tidal amplitude(Zhang et al., 2008; Gayen & Sarkar, 2010b). After generation, an IW beam usually traverses into the upper part of the ocean away from the topography. During its propagation through a non-uniform stratified environment, a beam can encounter a sharp density interface (pycnocline) in the real ocean (Martin et al., 2006; Cole et al., 2009; Johnston et al., 2010)) and evanescent regions in mesosphere (Fritts & Yuan, 1989; Walter-
1.3.1 Theory

Interaction of internal wave beams with a stratified environment has received considerable attention in theoretical, laboratory, observational and numerical studies. On the theoretical side, Delisi & Orlanski (1975) initiated a study of the impact of sudden change in density stratification on the propagation of a plane internal wave. They studied internal wave reflection from a density discontinuity. Later, based on linear theory, internal wave beam propagation into an arbitrary stratified fluid was investigated by Kistovich & Chacheshkin (1998). They considered both the effects of viscous dissipation and diffusion and calculated the amount of energy loss during the reflection process. Gerkema (2001) studied the local generation of solitary waves by internal tides in a two-layer flow configuration which consisted of a pycnoline as a density jump across the interface of a relatively shallow homogeneous fluid layer on top of a finite-depth stratified fluid. A time-harmonic barotropic current was imposed over two-dimensional bottom shelf topography. They argued that local generation of solitary waves was a result of two consecutive processes. First, an internal tidal beam originating at the topography, hits the moderately strong thermocline and, via an essentially linear mechanism, excites an appreciable long-wave disturbance there. Then, in the course of propagation along the thermocline, the disturbance triggered by the beam experiences the effects of dispersion and nonlinearity, thereby forming solitary waves. Later, Akylas et al. (2007) developed a nonlinear long-wave theory for the interaction of an internal tidal wave beam with the ocean thermocline. The theory predicts local generation of solitary waves in the context of a two-layer flow configuration: a homogeneous upper layer and an infinitely deep uniformly stratified fluid. In this context, interaction of plane internal waves with a sharp density gradient was studied analytically by Sutherland & Yewchuk (2004), using ray theory. This work was extended by Brown & Sutherland (2007). Later, based on nonhydrostatic effects, Nault & Sutherland (2007) studied IW transmission through a continuously stratified fluid. Recently, Echeverri & Peacock (2010)
have advanced the Green function approach to address the generation of internal tides by two-dimensional topography of arbitrary shape. They have employed the Wentzel-Kramers-Brillouin (WKB) approximation to consider the impact of non-uniform stratification on the propagation of the IW beam. This method is then applied to the study of two important internal tide generation sites, the Hawaiian and Luzon Ridges, where it captures key features of the generation process.

**Figure 1.9:** (top) The mean kinetic energy density $\langle E \rangle_k$ at KC. An M2 internal tide ray (the magenta curve) computed using buoyancy frequency from full-depth HOTS profiles, is superimposed on the path of concentrated $\langle E \rangle_k$. The ridge bathymetry is from the second across-ridge section west of Oahu and has horizontal resolution of 125 m. (bottom) The absolute ratio of topographic slope, $-r-$, computed from differences of boxcar-filtered bathymetry on scales of 1.2, 6, and 12 km to the slope of M2 internal tide rays. Figure is taken from Martin et al. (2006).
Figure 1.10: Spatial variation of total velocity variance, \((u'^2 + w'^2)\). Black lines are M2 ray slopes with the location of the surface reflections chosen by eye to best coincide with regions of largest velocity variance. (b) shows the contour of diapycnal eddy diffusivity \(\langle K_\rho \rangle\). All data are phase-averaged. Figure is taken from Cole et al. (2009).

1.3.2 Ocean observations

Oceanic observations of the IW beam interaction with a pycnocline have been performed either in smaller segment \(\leq 20 \text{ km}\) at high horizontal resolution (Lueck & Mudge, 1997; Lien & Gregg, 2001) or in larger scale of order \(\sim 50–150\text{ km}\) (Pingree & New, 1991; Martin et al., 2006; Cole et al., 2009; Johnston & Rudnick, 2009) with lower horizontal resolution.

Internal tidal beam propagation in the deep ocean from its generation site was studied by Pingree & New (1991) in the Bay of Biscay. They observed a marked change in energy and phase in the beam after its first reflection from the bottom abyssal plain. Nash et al. (2006) studied a process of generation and propagation of the semidiurnal internal wave beam tide from Kaena Ridge, Hawaii. A 20 km long transect was sampled every 3 h using expendable current profilers and an absolute velocity profiler. They observed two distinct beams originated from the right and left upper flanks of ridge. Repeated full-depth profiles of velocity and density were used to examine the barotropic-to-baroclinic tidal conversion, the semidiurnal kinetic and potential energies, phase propagation, and energy flux.

Martin et al. (2006) observed that the IW beam that originated from the Keana Ridge in Kauai channel of the Hawaiian island was disrupted after interaction with the upper pycnocline. Fig. 1.9(a) shows the contour of horizontal kinetic
energy, $E_k = (\rho_0/2)\langle u^2 + v^2 \rangle$ of the flow field based on the horizontal velocity components $u$ and $v$. Here, the variable $\rho_0$ is mean density and the angle brackets denote a phase averaging over an M2 period. Fig. 1.9(b) shows the distribution of the absolute ratio between topographic slope and the internal wave angle across the ridge at three different locations. The internal wave beam, originating from the north side of the ridge that has larger near-critical region shown by ray path, is more energetic compared to its southern counterpart. Clustering of energy surrounding the ray path of the beam extends up to $\sim 50 \text{ km}$ from the topography before it hits the upper surface. There appears to be no wave beam reflected back down from the upper surface. Another set of observations of the across-ridge structure of internal tides have been performed at Hawaiian ridge at Kauai Channel by Cole et al. (2009) using SeaSoar and a Doppler sonar over the upper 400 – 600 m of the ocean extending $152 \text{ km}$ on each side of the ridge. Two beams are observed from both sides of the ridge as shown by the ray path in Fig. 1.10. Total velocity

**Figure 1.11:** A 3D perspective looking northwestward has bathymetry in grey colors (100-m contours). A model M2 energy density. Energy density is enhanced along the beam originating from Sur Platform and below it in the offshore-directed beam. Modeled tidal ray paths are shown as black lines and are shifted 0.06° northward relative to ray paths in data plots. Note the zonal direction is stretched. This figure is taken from the observation from a submarine ridge near Monterey by Johnston et al. (2010).
variance, $\langle u'^2 + v'^2 \rangle$, based on across-ridge ($u'$) and along-ridge ($v'$) velocity fluctuations, is shown in Fig. 1.10(a). Velocity variance is largest on either side of the ridge crest ($x = 0 km$) along the ray path and approximately symmetric in pattern and magnitude about the ridge crest. Dissipation rate and the diapycnal eddy diffusivity, $K_\rho$, are parameterized by velocity shear. Diapycnal eddy diffusivity is measured at an elevated level along the beam shown in Fig. 1.10(b).

Recently, Johnston et al. (2010) have measured the spatial structure of velocity, density, and mixing in an internal tidal beam generated at a submarine ridge near Monterey Bay. A beam originates from a submarine ridge and emerges with diminished amplitude at the surface as shown by spatial contour of phase averaged total baroclinic energy density, $\langle E \rangle = \langle E_k \rangle + \langle E_p \rangle$ in Fig. 1.11. Here, $\langle E_k \rangle$ and $\langle E_p \rangle$ are the baroclinic kinetic and potential energy, respectively. Both the up-going and down-going beam show elevated turbulent dissipation of $\sim 10^{-7} W kg^{-1}$. Elevated turbulence has been directly measured only a few times along tidal beams (Lueck & Mudge, 1997; Lien & Gregg, 2001) and once across a tidal beam of considerable transverse extent (i.e., a tidal sheet) (Carter et al., 2006).

### 1.3.3 Laboratory observations

Laboratory experiments on the tunneling of internal waves through weakly stratified fluid patches were first done by Sutherland & Yewchuk (2004). They considered the partial reflection and transmission of internal waves through a mixed region bounded by discontinuities in the density profile. Their results reveal a linear resonance between vertically propagating internal waves and interfacial waves that exist on either flank of the mixing region. The resonance permits perfect transmission of internal waves that would otherwise strongly reflect from the weakly stratified region. Recently, Mathur & Peacock (2009) have performed a set of experiments on both plane IW and localized IW beam propagation through variable density interfaces. Their experiments indicate that nonuniform stratification disrupts a beam, ducts energy, and affects beam transmission and reflection. They also show that wave energy ducting can occur without the primary requirement of evanescent layers and thus may relate to the ocean observation (Martin et al.,
Numerical simulations

Numerical process studies have generally assumed a uniform stratification and, therefore, beam structure is maintained after surface and bottom reflections. An exception is the analytical/numerical work of Gerkema (2001) where a beam incident on the thermocline forms nonlinear internal waves but even here the model is limited to weak nonlinearity. Recently, Grisouard et al. (2011) have studied the interaction of the IW beam with the pycnocline using two dimensional numerical simulation. They have observed the generation of internal solitary waves and higher harmonics during the interaction process. Most of the harmonics with frequency higher than the buoyancy frequency of the lower medium cannot propagate into the lower medium and locally are trapped inside the pycnocline. Existing tidal models partially describe internal tidal beam structure and low-mode propagation into the far-field (Carter et al., 2008; Carter, 2010), but these models rely on turbulence parameterizations. There are no turbulence resolving simulations of this problem.

Considerable beam degradation occurs after the first surface reflection (Martin et al., 2006; Cole et al., 2009; Johnston et al., 2010), though the actual dissipation is unclear. Possible non-dissipative processes leading to beam degradation away from its generation sites include: radial dispersion, dephasing of the beam in realistic stratification because the wavelengths are not integer multiples as they are in linear stratification and beam-beam interactions (Cole et al., 2009). In the work of Gayen & Sarkar (2012), we have conducted numerical investigations of the interaction of an internal-wave beam with realistic upper ocean stratification: i.e., thermocline, transition layer (Johnston & Rudnick, 2009), and mixed layer. In contrast to previous numerical studies, we have performed simulations that account for highly nonlinear responses including turbulence. There are observations of turbulence along beams deeper in the water column, but to our knowledge there are no comparable observations or suitable modeling in the strongly-stratified upper ocean. Our primary objective is to understand beam attenuation in the upper ocean via turbulence, modal defocusing/decoupling in nonuniform stratification,
and nonlinear interactions. A more detailed discussion will be given later in Chapter 8.
Chapter 2

Problem setup

The present computational study is based on direct numerical simulation (DNS) and large eddy simulation (LES) using a spectral/finite difference (pseudo spectral) code developed in the UCSD CFD Lab (Gayen et al., 2010a; Gayen & Sarkar, 2011a). In the following sections, we will discuss the governing differential equations, discretization method, numerical solution technique and boundary conditions used for three different problems: (1) Flow over a nonsloping bottom without streamwise variation in mean flow, (2) Flow over sloping topography without streamwise variation in mean flow, and (3) Flow over sloping topography including streamwise variation in mean flow. Note that all three problems involve an instantaneous flow that is unsteady and three-dimensional.

2.1 Flow over nonsloping bottom without streamwise variation in mean flow

The near-bottom flow resulting from a current oscillating with the $M_2$ tidal period of 12.4 hrs. on a flat bottom is illustrated in Fig. 2.1 (a). The bottom is adiabatic while there is a background thermal stratification with constant buoyancy frequency, $N_\infty$. The freestream velocity,

$$U_{\infty,d}(t_d) = U_{0,d} \sin(\omega_d t_d),$$

(2.1)
is forced by an imposed pressure gradient,

\[ \frac{dp_d}{dx_d}(t_d) = -\rho_0 U_0 \omega_d \cos(\omega_d t_d). \]  \hspace{1cm} (2.2)

Here, subscript \( d \) denotes dimensional variables. The phase variation of the freestream velocity, Fig. 2.1 (b), shows that it is antisymmetric, \( U_{\infty,d}(-\phi) = -U_{\infty,d}(\phi) \). Furthermore, the flow accelerates during \( 0 < \phi < \pi/2 \) in response to a positive pressure force and decelerates during \( \pi/2 < \phi < \pi \) when the pressure force is negative. Owing to the symmetry in the problem, it is sufficient to consider the phase variation of flow statistics during a half-cycle, \( 0 < \phi < \pi \) that spans 6.2 hrs; the response in the other half-cycle is either the same or the mirror image. The rotation of earth is neglected in the present work so as to focus on the effect of stratification on the turbulent velocity and thermal boundary layers. The coordinates \( x, y, z \) denote streamwise, spanwise (cross-stream) and vertical directions, respectively, while \( u, v \) and \( w \) are the corresponding velocity components.
2.1.1 Governing equations

Large-eddy simulation (LES) is used to obtain the filtered (denoted by overbar) velocity and temperature fields by numerical solution of the Navier-Stokes equations under the Boussinesq approximation, written in dimensional form as:

\[ \nabla \cdot \overline{u}_d = 0 \]  
\[ \frac{\partial \overline{u}_d}{\partial t} + \overline{u}_d \cdot \nabla \overline{u}_d = -\frac{1}{\rho_0,d} \nabla \overline{p}_d + U_{0,d} \omega_d \cos(\omega_d t_d) \mathbf{i} + \nu_d \nabla^2 \overline{u}_d \]  
\[ + g_d \beta_d \theta'_d \mathbf{k} - \nabla \cdot \tau_d \]  
\[ \frac{\partial \overline{\theta}_d}{\partial t} + \overline{u}_d \cdot \nabla \overline{\theta}_d = \kappa_d \nabla^2 \overline{\theta}_d - \nabla \cdot \lambda_d. \]  
\[ (2.3) \]

Here \( \overline{p}_d \) denotes deviation from the background pressure. The quantities \( \tau_d \) and \( \lambda_d \) which are the sub-grid-scale stress tensor and density flux vector, respectively, require models for closure.

The resulting non-dimensional form of the governing equations is

\[ \nabla \cdot \overline{u} = 0 \]  
\[ \frac{D\overline{u}}{Dt} = -\nabla \overline{p} + \cos(t) \mathbf{i} + \frac{1}{Re} \nabla^2 \overline{u} + Ri \overline{\theta} \mathbf{k} - \nabla \cdot \tau \]  
\[ \frac{D\overline{\theta}}{Dt} = \frac{1}{Re Pr} \nabla^2 \overline{\theta} - \nabla \cdot \lambda \]  
\[ (2.5) \]

The flow is governed by three nondimensional parameters: the Reynolds number \( Re \), Richardson number \( Ri \), and Prandtl number \( Pr \), where

\[ Re \equiv \frac{a U_{0,d}}{\nu_d} = \frac{U^2_{0,d}}{\omega_d \nu_d}, \quad Ri \equiv \beta_d g_d \frac{d \theta_d}{dz_d} |_{\infty} \frac{1}{\omega_d^2} = \frac{N^2}{\omega_d^2}, \quad Pr \equiv \frac{\nu_d}{\kappa_d}. \]  
\[ (2.6) \]
Here, $a = U_0 d / \omega_d$ and $N_\infty$ is the background value of buoyancy frequency, assumed constant. The following Reynolds number,

$$Re_s = \frac{U \delta_s}{\nu_d} = \sqrt{2Re}$$

(2.7)

based on the Stokes boundary layer thickness, $\delta_S = \sqrt{2 \nu_d / \omega_d}$, is a commonly used alternative to $Re$. We employ $Re_s$ rather than $Re_d$ to denote the Stokes Reynolds number since, in geophysical boundary layers, the latter expression is often used for definitions involving the friction velocity.

### 2.1.2 Numerical method

The simulations use a mixed spectral/finite difference algorithm, see Appendix A.1 for details. Derivatives in the horizontal directions are treated with a pseudo-spectral method, the grid is staggered in the vertical, and derivatives in the vertical direction are computed with second-order finite differences. A low storage third-order Runge-Kutta-Wray method is used for time-stepping, and viscous terms are treated implicitly with the Crank-Nicolson method. The eddy viscosity and diffusivity coefficients, $\nu_T$ and $\kappa_T$ defined by Eqs. (2.8) and (2.9), are computed using current values of velocity and temperature. Then, the subgrid eddy fluxes involving $\nu_T$ and $\kappa_T$ are treated with the Crank-Nicolson method. Variable time stepping with a fixed CFL number 1.1 is used. The code has been parallelized using MPI. Detailed descriptions regarding the algorithm and the parallelization of the present code is discussed in Appendix A.

### 2.1.3 Boundary conditions

Periodicity is imposed in the horizontal $x$ and $y$ directions. The bottom boundary, $z = 0$, has zero velocity and zero temperature gradient. The top boundary is an artificial boundary corresponding to the truncation of the domain in the vertical direction. Rayleigh damping or a ‘sponge’ layer (see e.g. Klemp & Durran (1983)) is used so as to minimize spurious reflections from the artificial boundary into the ‘test’ section of the computational domain. The velocity and
scalar fields are relaxed towards the background state in the sponge region by adding damping functions $-\sigma(z) [u(x,t) - U_\infty(t)]$ and $-\sigma(z) [\theta(x,t) - \theta_\infty(z)]$ to the right hand side of the momentum and scalar equations, respectively. Here $\theta_\infty(z) = zd\theta/dz|_\infty$ is the background temperature profile and $\sigma(z)$ increases exponentially from $\sigma(z = 70\delta_s) = 0$ to $\sigma(z = 90\delta_s) = 20$. The pressure boundary conditions are $p' = 0$ at the bottom wall and $\partial p'/\partial z = 0$ at the top of computational domain.

2.1.4 Subgrid scale model

The mixed model (Zang et al., 1993; Vreman et al., 1997) that is used here for the subgrid scale (SGS) stress tensor, $\tau$, has a scale-similarity part and a eddy viscosity part. The SGS heat flux, $\lambda$, is obtained using a dynamic eddy diffusivity model, Armenio & Sarkar (2002). The expressions for the SGS models are as follows:

$$\tau_{ij} = -2\nu_T \hat{S}_{ij} + \hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j, \quad \nu_T = C\Delta^2|\hat{S}|$$

(2.8)

and

$$\lambda_j = -\kappa_T \hat{\theta} \partial \hat{\theta} / \partial x_j, \quad \kappa_T = C_\theta \Delta^2|\hat{S}|.$$  

(2.9)

Here, $C$ and $C_\theta$ are the Smagorinsky coefficients evaluated through a dynamic procedure introduced by Germano et al. (1991). Averaging over horizontal planes is employed to prevent excessive back scattering owing to large local fluctuations (Cabot, 1991). The dynamic procedure involves the introduction of an additional test filter denoted by $\hat{\cdot}$. The model coefficient, $C$, in the SGS stress model is given by

$$C = \frac{\langle M_{ij}(L_{ij} - H_{ij}) \rangle}{\langle M_{kl}M_{kl} \rangle},$$

(2.10)

where

$$L_{ij} = \hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j, \quad M_{ij} = 2\Delta^2|\hat{S}|\hat{S}_{ij} - 2\Delta^2|\hat{S}|\hat{S}_{ij},$$

(2.11)

$$H_{ij} = \hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j - \left( \hat{u}_i \hat{u}_j - \hat{u}_j \hat{u}_i \right).$$

(2.12)
The model coefficient, $C_\theta$, in the SGS heat flux model is given by

$$
C_\theta = \frac{\langle M_0^\theta L_0^\theta \rangle}{\langle M_j^\theta M_j^\theta \rangle},
$$

(2.13)

where

$$
L_i^\theta = \hat{\theta} u_i - \hat{\theta} \hat{u}_i, \quad M_i^\theta = 2\Delta^2 [\hat{S}] \frac{\partial \hat{\theta}}{\partial x_i} - 2\Delta^2 [\hat{S}] \frac{\partial \hat{\theta}}{\partial x_i}.
$$

(2.14)

The test filter, denoted by $\hat{\cdot}$, and grid filter, denoted by $\langle \cdot \rangle$, are applied over only the horizontal directions using a trapezoidal interpolation rule. For instance, application of the explicit filters to a LES variable, $\overline{\Psi}_i$, at node $i$ is given by

$$
\hat{\overline{\Psi}}_i = \frac{1}{4} \left[ \overline{\Psi}_{i-1} + 2 \overline{\Psi}_i + \overline{\Psi}_{i+1} \right]
$$

(2.15)

$$
\overline{\Psi}_i = \frac{1}{8} \left[ \overline{\Psi}_{i-1} + 6 \overline{\Psi}_i + \overline{\Psi}_{i+1} \right]
$$

(2.16)

The filter width ratio $\hat{\Delta}/\Delta$ is taken as $\sqrt{6}$, recommended by Lund (1997) to be the optimal choice for filters evaluated using the trapezoidal rule.

### 2.2 Flow over sloping topography without streamwise variation in mean flow

The boundary-conforming grid, curvilinear coordinates, and streamwise inhomogeneity in our studies (Gayen & Sarkar, 2010b, 2011a) enabled accurate simulation of internal wave beam formation at a slope in response to the imposed barotropic forcing and subsequent formation of turbulence. It would be impractical to perform a turbulence-resolving simulation in a streamwise non-periodic domain with the large slope length of several hundred meters that is required for beam generation with $l_b = 60$ m. To perform this study, we assume that there is a beam with a given width, $l_b$, and peak near-bottom velocity, $U_b$, on a slope and simulate phase-dependent turbulence in a small patch of the beam. This problem setup allows us to use a rectangular domain with homogeneity in both the spanwise and the streamwise direction.
2.2 Governing equations

Large-eddy simulation (LES) is used to obtain the filtered velocity and temperature fields by numerical solution of the Navier-Stokes equations under the Boussinesq approximation, written in rotated coordinates \([x_r, y_r, z_r]\) in dimensional form as:

\[
\nabla_{r} \cdot \mathbf{u}_r = 0
\]

\[
\frac{D\mathbf{u}_r}{Dt} = -\frac{1}{\rho_0} \nabla_r p^* + \nu \nabla^2_{r} \mathbf{u}_r - \frac{g \rho^*}{\rho_0} [\sin \beta \mathbf{i} + \cos \beta \mathbf{k}] - \nabla \cdot \tau
\]

\[
\frac{D\rho^*}{Dt} = \kappa \nabla^2_{r} \rho^* - (u_r \sin \beta + w_r \cos \beta) \frac{d\rho^b}{dz} - \nabla \cdot \lambda
\]  

(2.17)

Here, \(p^*\) and \(\rho^*\) denote deviation from the background pressure and density, respectively. The quantity \(\tau\) that denotes the subgrid scale stress tensor and \(\lambda\) that is the subgrid density flux are modeled as described by (Gayen et al., 2010a).
2.2.2 Numerical method

We have adopted the numerical methods which are similar to section 2.1.2, to solve the velocity in rotated coordinates $[u_r, v_r, w_r]$ and the deviations from background density and pressure.

2.2.3 Boundary Conditions

Periodicity is imposed in the spanwise, $y_r$, and streamwise, $x_r$, direction. Zero velocity and a zero value for wall-normal total density flux i.e. $d\rho/dz_r = 0$ are imposed at the bottom. This implies

$$u_r = 0, \frac{\partial \rho}{\partial z} = 0, \Rightarrow \frac{\partial \rho^*}{\partial z} = -\frac{d\rho^b(z_r)}{dz_t} \cos \beta \quad \text{at} \quad z_r = 0 \quad (2.18)$$

2.2.4 Subgrid scale model

A dynamic eddy viscosity model and a dynamic eddy diffusivity model are used for this problem to calculate $\tau$ and $\lambda$. The coefficient is averaged over the homogeneous directions (slope-parallel plane). Detailed description of this model is given in section 2.1.4.

2.3 Flow over sloping topography including streamwise variation in mean flow

The near-bottom flow resulting from a current oscillating on an inclined surface is illustrated in figure 2.3. The bottom is adiabatic while there is a background thermal stratification with constant buoyancy frequency, $N_\infty$. The flow is forced by an imposed pressure gradient,

$$F_0(t_d) = \rho_0 U_0 \Omega \cos(\Omega t_d) \cdot (2.19)$$

in the horizontal direction that results in a background barotropic current, $U(x) \sin(\phi)$, where $\phi$ is the tidal phase. In the figure, coordinates $x,y$ and $z$ denote the horizontal, spanwise and vertical directions and $u$, $v$ and $w$ are the corresponding
Figure 2.3: Schematic of the problem along with oscillatory tidal forcing $F_0(t_d)$ in streamwise direction.

velocity components, while $\xi$, $\zeta$ and $\eta$ are curvilinear coordinates employed in the simulation. In contrast to the configuration of section 2.2, the mean flow is allowed to evolve spatially in the streamwise direction.

2.3.1 Governing equations

The Navier-Stokes equations, under the Boussinesq approximation, which are numerically solved here are written as follows with dimensional quantities denoted by subscript $d$:

$$
\nabla \cdot u_d = 0 \quad (2.20a)
$$

$$
\frac{Du_d}{Dt_d} = -\frac{1}{\rho_0}\nabla p_d^* + \frac{F_0(t_d)}{\rho_0}\mathbf{i} + \nu \nabla^2 u_d - \frac{g \rho_d^*}{\rho_0} \mathbf{k} - \nabla \cdot \tau_d \quad (2.20b)
$$

$$
\frac{D\rho_d}{Dt_d} = \kappa \nabla^2 \rho_d - \nabla \cdot \lambda_d. \quad (2.20c)
$$

Here, $p_d^*$ denotes deviation from the background hydrostatic pressure and $\rho_d^*$ denotes the deviation from the linear background state, $\rho_d^b(z_d)$. In LES mode, $u_d$
and $\rho_d$ are to be interpreted in the equations as filtered quantities, i.e., we drop the overbar conventionally used to denote filtering. $\tau_d$ and $\lambda_d$ which are the subgrid-scale stress tensor and density flux vector, respectively, require models for closure in LES. In DNS cases, $\tau_d$ and $\lambda_d$ are zero. An evolution equation for $\rho_d^*$, the deviation from the linear background state $\rho_b^*(z_d)$, is written as

$$\frac{D\rho_d^*}{Dt_d} = \kappa \nabla^2 \rho_d^* - w_d \frac{dp_d^b}{dz_d} - \nabla \cdot \lambda_d. \quad (2.21)$$

The dimensional quantities in the problem are the free-stream velocity amplitude $U_0$, tidal frequency $\Omega$, background density gradient $d\rho_b/dz|_\infty$, and the fluid properties: molecular viscosity, $\nu$, thermal diffusivity, $\kappa$, and density, $\rho$.

The variables in the problem are nondimensionalized as follows:

$$t = t_d \Omega, \quad x = (x_d, y_d, z_d) = \left( \frac{x_d, y_d, z_d}{U_0/\Omega} \right), \quad p = \frac{\rho_d}{\rho_0 U_0^2}, \quad u = \left( \frac{u_d, v_d, w_d}{U_0} \right), \quad \rho^* = \frac{\rho_d}{\frac{U_0 d\rho_d}{dz_d}|_\infty}. \quad (2.22)$$

The resulting non-dimensional form of the governing equations is

$$\nabla \cdot u = 0 \quad (2.23a)$$

$$\frac{Du}{Dt} = -\nabla p^* + \cos(t)i + \frac{1}{Re} \nabla^2 u - B \rho^* - \nabla \cdot \tau \quad (2.23b)$$

$$\frac{D\rho^*}{Dt} = \frac{1}{Re Pr} \nabla^2 \rho^* + w - \nabla \cdot \lambda \quad (2.23c)$$

The governing equations have three nondimensional parameters: Reynolds number $Re$, Buoyancy parameter $B$, and Prandtl number $Pr$, where

$$Re \equiv \frac{aU_0}{\nu} = \frac{U_0^2}{\Omega \nu}, \quad B \equiv -g \frac{d\rho_b^b}{dz_d}|_\infty \frac{1}{\rho_0 \Omega^2} = \frac{N_\infty^2}{\Omega^2}, \quad Pr \equiv \frac{\nu}{\kappa} \quad (2.24)$$

The slope geometry is given by the slope angle, $\beta$, and the slope length in $x$-direction, $l$. The angle of the internal wave phase lines with the horizontal is given in a non-rotating environment by $\theta = \sin^{-1}(\Omega/N)$. Thus, in addition to those listed in (2.24), there are three other non-dimensional parameters: the excursion parameter $Ex = U_0/(l\Omega)$, the slope angle $\beta$ and the slope criticality parameter, $\epsilon = \tan(\beta)/\tan(\theta)$. 
The Navier-Stokes equations are written in the following coordinates:

\[ \xi = \xi(x, z), \eta = \eta(x, z), \zeta = \zeta(y), \quad (2.25) \]

where, at the slope, \( \xi \) points parallel to and across the slope while \( \eta \) is normal to the slope as shown in figure 2.3. Now (2.23) is transformed as described by Fletcher (1991) to the form of a strong-conservation law as

\[
\begin{align*}
\frac{\partial U_c^j}{\partial \xi_j} &= 0 \quad (2.26a) \\
\frac{\partial (J^{-1}u_i)}{\partial t} + \frac{\partial F_{ij}}{\partial \xi_j} &= J^{-1}\cos(t)\delta_{1i} - J^{-1}B\rho^*\delta_{3i} \quad (2.26b) \\
\frac{\partial (J^{-1}\rho^*)}{\partial t} + \frac{\partial H_j}{\partial \xi_j} &= J^{-1}w \quad (2.26c)
\end{align*}
\]

where the fluxes are

\[
F_{ij} = U_c^ju_i + J^{-1}\frac{\partial \xi_j}{\partial x_i}p^* - \frac{1}{Re}uG_{jm}^i\frac{\partial u_i}{\partial \xi_m} + J^{-1}\frac{\partial \xi_j}{\partial x_m}\tau_{im} \quad (2.27)
\]

\[
H_j = U_c^j\rho^* - \frac{1}{Re\; Pr}G_{jm}^i\frac{\partial \rho}{\partial \xi_m} + J^{-1}\frac{\partial \xi_j}{\partial x_m}\lambda_m \quad (2.28)
\]

Here \( J^{-1} \), the inverse of the the determinant of the Jacobian, is the volume of the cell in physical space; \( U_c^j \) is the volume flux (contravariant velocity multiplied by \( J^{-1} \)) normal to the surface of constant \( \xi_j \); and \( G_{jm}^i \) is called the “mesh skewness tensor”. These quantities are

\[
U_c^j = J^{-1}\frac{\partial \xi_j}{\partial x_i}u_i \quad (2.29)
\]

\[
J = \det \left( \frac{\partial \xi_j}{\partial x_i} \right) \quad (2.30)
\]

\[
G_{jm}^i = J^{-1}\frac{\partial \xi_j}{\partial x_m}\frac{\partial \xi_m}{\partial x_n} \quad (2.31)
\]

### 2.3.2 Numerical method

Boundary conforming grid generation based on transfinite interpolation (TFI) has been used. In this method, the domain boundary points are specified
through four sets of parametric equations,

\[ x_b(\xi), \quad x_t(\xi), \quad 0 \leq \xi \leq 1 \]  
\[ x_l(\eta), \quad x_r(\eta), \quad 0 \leq \eta \leq 1. \]  

(2.32a)  
(2.32b)

Here subscripts \( b \), \( t \), \( l \) and \( r \) of \( x = [x, z] \) denote bottom, top, left and right boundaries, respectively. The interior grid is created from knowledge of the boundary points by using TFI technique as follows,

\[ x(\xi, \eta) = (1 - \eta)x_b(\xi) + \eta x_t(\xi) + (1 - \xi)x_l(\eta) + \xi x_r(\eta) \]
\[ -\xi\eta x_t(1) - \xi(1 - \eta)x_b(1) - \eta(1 - \xi)x_l(0) \]
\[ -(1 - \xi)(1 - \eta)x_b(0). \]  

(2.33)

After grid generation by the TFI method, the grid is non-orthogonal. However, at the bottom boundary conforming to the topography, an orthogonal grid is convenient to impose accurately the condition of zero normal heat flux. So points on the two rows of points just above the bottom boundary are shifted sideways to make the \( \eta \) coordinate lines perpendicular to the \( \xi \) coordinate lines at the boundary. This simple process ensures grid orthogonality at the bottom boundary. The physical domain boundaries at top, left and right are such that grids are orthogonal at those boundaries.

The simulations use a mixed spectral/finite difference algorithm, see Appendix A.2 for detail. Derivatives in the spanwise direction are treated with a pseudo-spectral method and derivatives in the vertical and streamwise directions are computed with second-order finite differences. A low-storage third-order Runge-Kutta-Wray method is used for time stepping, except for the viscous terms which are treated implicitly with the alternating direction implicit (ADI) method. The eddy viscosity and diffusivity coefficients, \( \nu_T \) and \( \kappa_T \) defined later by (2.8) and (2.9), are computed using current values of velocity and temperature. The subgrid eddy fluxes involving \( \nu_T \) and \( \kappa_T \) are included in the time advance with the ADI method. Variable time stepping with a fixed CFL number 0.5 is used. The code has been parallelized using both MPI and OpenMP. Detail descriptions regarding the algorithm and the parallelization of the present code is discussed in Appendix A.
2.3.3 Pressure Poisson Equation

The fractional step method used here leads to the following Poisson equation for the pressure correction,

\[ \frac{\partial}{\partial \xi_i} \left( G_{ij} \frac{\partial \phi^{n+1}}{\partial \xi_j} \right) = \frac{\partial U_{j}^{c(n)}}{\partial \xi_j}. \] (2.34)

Here \( U_{j}^{c(n)} = J^{-1} \frac{\partial \xi_i}{\partial \xi_j} u_i^{(n)} \) is an intermediate volume flux and superscripts \( n, n+1 \) denote current and advanced time level. Finally, velocity and pressure are corrected as

\[ U_{j}^{c(n+1)} = U_{j}^{c(n)} - G^{jm} \frac{\partial \phi^{n+1}}{\partial \xi_m} \] (2.35)

\[ p^{*(n+1)} = p^{*(n)} + C_1 \phi^{n+1}. \] (2.36)

Here \( C_1 \) is a factor that depends on the time step in the RK substep. Eq. (2.34) is solved by a two-dimensional multigrid method developed by Zeeuw (1990) and based on sawtooth multigrid cycling (i.e. one smoothing-sweep after each coarse grid correction) with smoothing by incomplete line LU-decomposition, weighted 9-point prolongation and restriction, and Galerkin approximation of coarse grid matrices.

2.3.4 Boundary condition

Periodicity is imposed in the spanwise \((\zeta = \zeta(y))\) direction on velocity and density \( \rho^* \) and pressure, \( p^* \).

The bottom boundary, \( \eta = 0 \), has zero velocity and zero temperature gradient. Grids are forced to be orthogonal near the boundary so that

\[ \frac{\partial \rho}{\partial \eta} = 0 \Rightarrow \frac{\partial \rho^*}{\partial \eta} = \cos(\beta) \quad \text{at} \quad \eta = 0, \] (2.37)

where \( \beta = \tan^{-1}(h_x) \). At the top of the domain, \( \partial u/\partial \eta = 0; v, w = 0 \), and \( \rho^* = 0 \). At the left and right sides, \( \partial u/\partial \xi = 0; v, w = 0 \), and \( \rho^* = 0 \). To match the boundary condition for the density deviation, \( \rho^* \), between the left and the bottom (similarly, the right and the bottom) boundaries, \( \partial \rho^*/\partial \eta \) is set to 0 at both the left and right ends of the bottom boundary, then it gradually reaches to the value
Figure 2.4: Curvilinear grid in the computational domain. A sponge layer surrounds the domain at the left, right and top. Right inset shows an area $\Gamma$ demarcated at the top by the line $z = z_2$, at the left by $x = x_1$, at the right by $x = x_2$, at the bottom by $z = h(x)$. Quantities integrated over the area $\Gamma$ will be used to quantify turbulence and internal waves at the slope. In the inset, three points are on the slope are shown: Q at the middle, and P and R at opposite edges of the slope.

given by (2.37) within 1 m from the both ends and it is fixed at this value for the remaining extent of the bottom boundary. The pressure boundary conditions are $\partial p^*/\partial \eta = 0$ at the bottom and top wall and $p^* = 0$ at the left and right of computational domain.

Rayleigh damping or a ‘sponge’ layer is used at the left, right and top boundaries of the computational domain as shown in figure 2.4 so as to minimize spurious reflections from the artificial boundary into the ‘test’ section of the computational domain. The velocity and scalar fields are relaxed towards the background state in the sponge region by adding damping functions $-\sigma(\xi, \eta) [u_i(x, t) - 0]$ (i=2,3) and $-\sigma(\xi, \eta) [\rho^*(x, t) - 0]$ to the right hand side of the momentum and scalar equations, respectively. The value of $\sigma(\xi, \eta)$ is zero everywhere except in a region close to the top, left and right boundaries where it increases exponentially.
2.3.5 Subgrid scale model

The dynamic eddy viscosity model is used for this problem. Detailed description of this model is given in section 2.1.4.
Chapter 3

Large eddy simulation of a stratified boundary layer under an oscillatory current

The research of the present chapter constitutes the first phase of the dissertation and is based on Gayen et al. (2010a). Three-dimensional large eddy simulations are performed with the computational model as discussed in Chapter 2: section 2.1 to investigate a bottom boundary layer on nonsloping bottom under an oscillating tidal current. There are no topographically generated waves owing to the nonsloping nature of the bottom boundary. The focus is on the boundary layer response to an external stratification.

3.1 Introduction

At the bottom of the ocean, a turbulent mixed layer develops as near-bottom currents flow over the sea floor. These currents are often oscillatory, e.g. barotropic and internal tides, surface and internal gravity waves, inertial oscillations. The complex interplay of time-dependent currents, rotation, stratification and bottom topography determines the properties of the bottom boundary layer. The present LES of a boundary layer on a flat, non-sloping bottom under an oscillating current in a uniformly stratified fluid is designed to focus on the interaction
of stratification and oscillation in the absence of other complicating factors. The following literature survey shows that this fundamental problem, especially the phase-dependent aspects, is not well understood.

In the context of oscillatory flows, it is useful to distinguish between pulsatile flows with non-zero mean and purely oscillatory flows with zero mean; the latter is the subject of the present work. For oscillatory flow over a smooth surface, the Reynolds number, \( Re_s = U_0 \delta_s / \nu \) based on the Stokes thickness, \( \delta_s = \sqrt{2\nu / \omega} \), where \( \omega \) is the tidal frequency, and peak external velocity, \( U_0 \), is often used to classify the flow into qualitatively different regimes. Previous investigators such as Hino et al. (1976), Hino et al. (1983), Sleath (1987) Jensen et al. (1989), Akhavan et al. (1991a) and Sarpkaya (1993) have distinguished four different flow regimes for the unstratified zero time-mean oscillatory boundary layer based on the value of Reynolds number, \( Re_s \): (I) Laminar flow when \( Re_s < 100 \); (II) Disturbed laminar flow for \( 100 < Re_s < 550 \), where good agreement between velocity traces and laminar theory is found except during the accelerating phase of the cycle; (III) Intermittently turbulent (IT) flow for \( Re_s > 550 \), where turbulent bursts appear and three dimensional nature of the turbulence is visible in contrast to the earlier two dimensional behaviour in the disturbed laminar regime (for further details consult Blondeaux & Seminara (1979), Vittori & Verzicco (1998) and Akhavan et al. (1991b)); and (IV) Fully Turbulent (FT) flow at sufficiently large Reynolds number. While \( Re_s \) must be quite large for turbulence to persist throughout the entire cycle, Jensen et al. (1989) and Salon et al. (2007) found that turbulence is present for most of the cycle when \( Re_s \) is approximately 1800 or larger.

Unstratified oscillating flow has been the subject of numerical studies as summarized below. Direct numerical simulation (DNS) studies, e.g. Spalart & Bladwin (1987); Akhavan et al. (1991b) ; Vittori & Verzicco (1998); Costamagna et al. (2003); Sakamoto & Akitomo (2008) (their case of pure oscillation) have primarily studied the disturbed laminar and intermittently turbulent flow regimes. Spalart & Bladwin (1987) performed DNS of oscillating Stokes flow over a range of Reynolds numbers up to \( Re_s = 1200 \), observed a change from disturbed laminar to intermittent turbulence when \( Re_s = 600 – 800 \), identified a log-law over a portion
of the cycle at $Re_s = 1200$, and proposed a new algebraic turbulence model. Akhavan et al. (1991b) in their DNS study of oscillating flow in a channel focused on transition to turbulence and explained features of the transition process observed in their laboratory experiment (Akhavan et al., 1991a) as a secondary instability of two-dimensional Tollmein-Schlichting waves. Vittori & Verzicco (1998) performed DNS in the disturbed laminar and the IT regime taking wall imperfections into account. They found that wall imperfections induce transition to turbulence and have a strong effect on the time evolution of the turbulent kinetic energy in the disturbed laminar regime. Costamagna et al. (2003) have examined the role of coherent boundary layer structures in their DNS of the IT regime, and identify instability of low-speed streaks as important for the generation and the sustenance of turbulence in oscillating Stokes flow.

LES has been used recently to extend the scope to higher Reynolds numbers where the fully turbulent regime applies. LES has been shown to handle turbulence where there is a transition from laminar flow to turbulence in pulsating flows, for example, by Scotti & Piomelli (2001). The problem of purely oscillatory boundary layers has been studied with LES recently by Hsu et al. (2000), Lohmann et al. (2006), Salon et al. (2007) and Radhakrishnan & Piomelli (2008). Hsu et al. (2000) performed LES for Reynolds numbers up to $Re_s = 894$ that corresponds to the IT regime using a subgrid eddy viscosity model with the dynamic procedure, and a RANS calculation of both the IT and FT regime that employed a $k - \Omega$ model. Their simulations captured the transition from disturbed laminar to intermittently turbulent regime, and showed that the phase advance of peak shear stress with respect to peak velocity changed from its laminar value of 45° to about 10° in the IT regime. Lohmann et al. (2006) simulated a case with $Re_s = 3464$ in the FT region with LES using the standard Smagorinsky SGS model. Their results for temporal evolution of the wall shear stress and first and second order statistics did not reproduce the experimental results found by Jensen et al. (1989) in their test case 10 at the same Reynolds number. The discrepancy illustrates the unsuitability of the simple Smagorinsky SGS model for unsteady flow problems. Salon et al. (2007) performed LES using the Dynamic Mixed Model (DMM). The authors studied a
case with $Re_s = 1790$, reproduced the experimental results of test case 8 by Jensen et al. (1989), and provided insight into the phase dependent variation of near-wall and outer-layer turbulence over a complete cycle. Recently, Radhakrishnan & Piomelli (2008) have performed LES with various subgrid models and near-wall treatments for simulations in the FT regime. The experimental results by Jensen et al. (1989) of test cases 10 ($Re_s = 3464$) and 13 (same flow conditions as test case 10 but with a rough wall) were successfully matched. These authors demonstrate the superiority of the dynamic Smagorinsky model over the standard one as well as the necessity of an additional wall layer model if the eddies responsible for near-wall turbulence are unresolved.

DNS and LES studies of an oscillating boundary layer subject to stratification are scarce. Sakamoto & Akitomo (2006) and Sakamoto & Akitomo (2009) report a DNS of a stratified tidal bottom Ekman layer for various values of the Rossby number, $Ro$. The case of pure oscillation ($Ro = \infty$) in the study by Sakamoto & Akitomo (2009) was at $Re_s = 1000$, and all cases had a weak stratification, $N^2_\infty/\omega^2 = 47.5$. The authors mainly focused on the mixed layer growth along with energy transfer between potential and kinetic energy. They also studied the applicability of different turbulent scales for the mixed and the interfacial layers. There have also been studies of stratified tidal boundary layers using the Reynolds-averaged Navier Stokes (RANS) equations, e.g. by Richards (1982), Davies et al. (1997), Burchard et al. (1998). Calculations using the RANS equations typically require additional models, for example stability functions based on Richardson number, to incorporate stability effects.

Burchard et al. (1998) report observations at a location in the Irish sea that show a lag of dissipation rate with respect to the current. Lorke et al. (2002) in their measurements in a stratified basin under a low-speed oscillating flow also observed a lag in the dissipation rate with respect to the free stream current. The authors also found a considerable difference between the dissipation rate estimated from the law of wall and that obtained using microstructure measurements. Thorpe et al. (2008) observe a lag of dissipation with respect to the tidal velocity at a location in the Irish Sea (water depth 43.5 m). They also observe upward propagating bursts
that reach the surface to form boils. Measurements in the stratified boundary layer over the Oregon shelf have been employed to ascertain the effect of stratification on the log law by Perlin et al. (2005) and on Ekman veering by Perlin et al. (2007). Luznik et al. (2007) have obtained PIV measurements in the near-bottom unstratified part of a boundary layer on the continental shelf off the coast of South Carolina and Georgia, and examined turbulence spectra and isotropy. Recently, Lozovatsky et al. (2008a), Lozovatsky et al. (2008b) examined the phase variation of the velocity profiles, kinetic energy, dissipation rate as well as peak shear at three locations in the northwestern East China Sea, each having a different barotropic tidal environment.

Compared to oscillatory flows, the effect of stratification in steady wall-bounded flows has received considerable attention. Open channel flow with fixed temperature difference $\Delta T$ was studied using DNS by Nagaosa & Saito (1997). Armenio & Sarkar (2002) studied closed channel flow with a fixed temperature difference $\Delta T$ across the channel with LES. Applying a fixed temperature difference across the channel leads to a heat flux at the upper and lower boundaries. However, over most of the ocean, the sea floor can be well-approximated by an adiabatic boundary condition. This motivated the study of Taylor et al. (2005) who performed LES of open channel flow with an adiabatic bottom and constant heat flux at the top surface. More recently, a stratified bottom Ekman layer was studied using LES by Taylor & Sarkar (2007b) who focus on the properties of internal gravity waves forced by turbulence and by Taylor & Sarkar (2008c) who discuss stratification effects on the boundary layer characteristics. Taylor & Sarkar (2008c) propose a buoyancy related modification of the log-law profile as an alternative to Monin-Obukhov theory (which is inapplicable when the wall buoyancy flux is zero) and show that the modification leads to good agreement between the friction velocity estimated from the profile method and the true value.

After surveying the existing literature, it is clear that the effect of an external stratification on an oscillatory boundary layer has not been the subject of systematic study. While Sakamoto & Akitomo (2009) reported a single simulation with low stratification in the context of a broader study, they did not consider a
variety of stratification levels. In contrast, the present study focuses on the effect of stratification on the phase-dependent properties of turbulence and internal waves in a tidal boundary over a range of stratifications, \(0 < N_\infty^2/\omega^2 < 2500\). We select \(R_{\text{e}_s} = 1790\), which has been studied in detail when \(N_\infty^2 = 0\) using LES (Salon et al. (2007)) and laboratory experiments (Jensen et al. (1989)) and should result in active near-wall turbulence over most of the cycle.

### 3.2 Domain resolution & initialization

The computational domain size in the horizontal directions is \(l_x = 50\delta_s\) and \(l_y = 25\delta_s\). The vertical domain size is \(l_z = 70\delta_s\), significantly larger than the boundary layer thickness (15\(\delta_s\) for the unstratified case), so as to allow sufficient vertical space for propagation of internal waves. The sponge region spans \(70\delta_s - 90\delta_s\). The computational grid has \(64 \times 64 \times 360\) points in the \(x\), \(y\) and \(z\) directions, respectively, leading to grid steps of \(\Delta x^+ = 60\), \(\Delta y^+ = 30\), \(\Delta z_{\text{min}}^+ = 2\) and \(\Delta z_{\text{max}}^+ = 20\) in viscous units, \(\nu/u_\tau\). The resolution is sufficient to resolve the near-wall eddies that carry the Reynolds stress; thus, the present simulation is a resolved LES that does not require an additional near-wall model. The domain size and resolution chosen here is the same as in the \(C_2\) case of Salon et al. (2007). Since the present simulation has spectral accuracy in the horizontal directions, the larger horizontal grid size, \(96 \times 96\), of the \(C_4\) case needed by the second-order accurate finite difference method of Salon et al. (2007) is not required here.

The flow is statistically homogeneous in the horizontal and the \(x - y\) plane average is used to compute the time dependent mean, \(\langle A \rangle_{xy}(z, t)\), as follows:

\[
\langle A \rangle_{xy}(z, t) = \frac{1}{l_xl_y} \int_0^{l_x} \int_0^{l_y} A(x, y, z, t)\,dx\,dy
\]

(3.1)

The Reynolds average, \(\langle A \rangle(\phi)\), which is a function of the height above the bottom, \(z\), and the tidal phase, \(\phi\), is calculated by a further ensemble average of the \(x - y\) plane averages taken at an interval of \(\pi\). Thus,

\[
\langle A \rangle(z, \phi) = \frac{1}{N} \frac{1}{l_xl_y} \sum_{n=1}^{N} \int_0^{l_x} \int_0^{l_y} A(x, y, z, \phi + n\pi)\,dx\,dy
\]

(3.2)
Table 3.1: Simulation parameters.

<table>
<thead>
<tr>
<th>Case</th>
<th>$Re_s$</th>
<th>$Ri$</th>
<th>$Pr$</th>
<th>$l_x$</th>
<th>$l_y$</th>
<th>$l_z$</th>
<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1790</td>
<td>0</td>
<td>0.7</td>
<td>50$\delta_s$</td>
<td>25$\delta_s$</td>
<td>70$\delta_s$</td>
<td>64</td>
<td>64</td>
<td>360</td>
</tr>
<tr>
<td>2</td>
<td>1790</td>
<td>500</td>
<td>0.7</td>
<td>50$\delta_s$</td>
<td>25$\delta_s$</td>
<td>70$\delta_s$</td>
<td>64</td>
<td>64</td>
<td>360</td>
</tr>
<tr>
<td>3</td>
<td>1790</td>
<td>2500</td>
<td>0.7</td>
<td>50$\delta_s$</td>
<td>25$\delta_s$</td>
<td>70$\delta_s$</td>
<td>64</td>
<td>64</td>
<td>360</td>
</tr>
</tbody>
</table>

The ensemble average over $\mathcal{N}$ half cycles, calculated after accounting for a sign change if any, takes advantage of the fact that the flow statistics repeat in either symmetric or antisymmetric fashion after every half-cycle; for example, the mean velocity is antisymmetric, $\langle u \rangle(-\phi, z) = -\langle u \rangle(\phi, z)$. The velocity statistics are obtained by ensemble averaging after at least 15 cycles and over a period of 10, 15, and 15 cycles for cases 1, 2 and 3, respectively, so that the initial transient (spin up to a mixed layer from the initial linear thermal profile) of the thermal field is excluded. The thermal statistics are generally computed using plane averages since the mixed layer grows, albeit at a small rate.

3.3 Selection of simulated cases

The physical parameters imposed in our present numerical study are given in Table 3.1. All cases have a Reynolds number, based on the Stokes boundary layer thickness of $Re_s = 1790$, corresponding to the flow studied by Salon et al. (2007) and also to test case 8 from the experimental study of Jensen et al. (1989). Turbulence is present during most of the cycle at $Re_s = 1790$. This choice also allows us to perform a comparison with the results of Jensen et al. (1989); Salon et al. (2007) and thus validate the computational model. The focus of the current study is the assessment of stratification effects on the oscillatory boundary layer and, therefore, $Ri$ is varied between cases from 0 (temperature is a passive scalar in this case) to a high value of $Ri = 2500$. The Prandtl number, $Pr = 0.7$, is kept constant between cases. This choice, lower than the value of $Pr \simeq 5$ for heat transport in water, is motivated by our desire to keep the computational cost manageable; note that the ratio of smallest scale of the thermal field to that of the
velocity field is proportional to $Pr^{-1/2}$ when $Pr = O(1)$ or $Pr \gg 1$. At high values of $Pr$, the thermocline that caps the turbulent mixed layer could be stronger owing to reduced molecular diffusion of temperature and the characteristics of stratified turbulence in that region could be different than the $Pr = 1$ case considered here. The effect of $Pr$ in a stratified bottom boundary layer is deserving of a separate systematic study.

The choice of simulation parameters can be placed in the context of the oceanic bottom boundary layer as follows. Take the viscosity of water to be $\nu = 10^{-6} m^2/s$, amplitude of current velocity as $U_0 = 1.5 cm/s$ and take $\omega = 1.407 \times 10^{-4} rad/s$ corresponding to the $M_2$ tidal period of 12.4 hrs. The Stokes boundary layer thickness is $\delta_s = \sqrt{2\nu/\omega} = 0.119 m$. The Stokes Reynolds number becomes $Re_s = U_0\delta_s/\nu = 1788$, very close to the value chosen here, while $Re = Re_s^2/2 = 1.6 \times 10^6$. The computational domain size of $50\delta_s \times 25\delta_s \times 70\delta_s$ is equivalent to $5.95m \times 2.975m \times 8.925m$. A range for the typical buoyancy time period in the ocean is 1 hr to 15 min corresponding to $153.7 < Ri < 2460$ for the environmental parameter, $Ri = N_\infty^2/\omega^2$. The choices of $Ri$ in Table 3.1 correspond to a low and a high value in the expected range.

### 3.4 Results

#### 3.4.1 Passive scalar case, $Ri = 0$

Results from the $Ri = 0$ case are summarized here with the objective of setting up the context for discussing stratification effects and validating the numerical model by direct comparison with results from the numerical study of Salon et al. (2007) and from the laboratory study of Jensen et al. (1989) at the same $Re = 1790$.

The streamwise velocity $<u^+>(z^+)$ is plotted in semi-log coordinates at an intervals of $30^\circ$, along with previous data, in figure 3.1. The log-law, $u^+ = (1/k)log(z^+) + B$, applicable to the steady case, is also shown. Here $k$ is the von Kármán constant taken to be 0.41, and $B$ is the intercept with the $u^+$ axis taken as 5.2. A log-law is observed between $50^\circ - 140^\circ$. A significant asymmetry is observed
Figure 3.1: Ensemble-averaged profiles of the streamwise velocity in a semi-log plot. The present simulation with $Ri = 0$ is shown in a light gray solid line; case C4 of Salon et al. (2007) in filled circles; experimental results of Jensen et al. (1989) in unfilled circles. The straight dashed line shows the log-law with $\kappa = 0.41$ and $B = 5.2$. (a) $\phi = 0^\circ$, $30^\circ$ and $60^\circ$; (b) $\phi = 90^\circ$, $120^\circ$ and $150^\circ$. Each profile is staggered by 30 units in the horizontal.

between the accelerating and decelerating phases of the half-cycle, for example, the range of the log-law is larger during the decelerating phase. Fig. 3.1 shows that, over the central span where well-developed turbulence exists, the current result (in grey line) is in excellent agreement with both previous results. In the remaining portion of the cycle, the agreement is still very good especially with respect to the previous laboratory data of Jensen et al. (1989).

Figure 3.2 shows profiles of the streamwise turbulence intensity $u_{rms}$ at several phases. The peak values occur close to the bottom while, at large $z$, the turbulence dies down to zero (not shown in the figure). The peak value of $u_{rms}$ occurs just past $\phi = 90^\circ$, the point of maximum freestream velocity. The agreement with previous results is very good throughout the cycle. There is a pronounced outer-layer bulge of the profiles during $120^\circ < \phi < 180^\circ$, similar to experimental observations, that occurs in response to the adverse pressure gradient. Turbulent intensities in the spanwise and vertical directions, that also agree well with data
Figure 3.2: Profiles of streamwise turbulence intensity $u_{rms}$ in the passive scalar case. Each profile is staggered by 0.2 units in the horizontal.

from the previous studies, are not shown here.

3.4.2 Overall thermal field

Mixed layer growth and Entrainment

The development of the plane-averaged temperature gradient as a function of time is shown in figure 3.3. The mixed layer (indicated by dark shading) is separated from the outer stratified layer by a thermocline where the temperature gradient changes rapidly exhibiting an overshoot before approaching the background value. Fig. 3.3(a) shows that the mixed layer exhibits a small but persistent growth after an initially rapid transient. An interesting phenomenon of periodic modulation of mixed layer growth is also observed, a point that we will return to later. When the stratification level is intensified, the mixed layer height exhibits a substantial decrease as shown by comparison of figure 3.3(a) and 3.3(b).

The mixed layer height, $h_m$, is defined by the location where $\partial \langle \theta \rangle_{xy} / \partial z = 0.1$. The gradient Richardson number, defined by

$$Ri_g = \frac{N_d^2}{S_d^2} = \frac{Ri \cdot N^2}{S^2},$$

(3.3)
is often used to demarcate regions of mixing. It is found here that $h_m$ is smaller, by approximately 10%, relative to the height of the $Ri_g = 0.25$ location. Profiles of the plane-averaged thermal field are shown in figure 3.4. Part (b) of the figure illustrates the mixed layer with small temperature gradient as well as a capping thermocline. The unsteadiness of the thermal statistics is clearly shown by comparison of the dashed lines at $t = 15$ with the solid lines at $t = 50$. The mixed layer height, indicated by circles in Figure 3.4, increases with time with the amount of increase strongly inhibited in the $Ri = 2500$ case with respect to the $Ri = 500$ case. The thermocline thickens as time goes on and, correspondingly, the overshoot of temperature gradient decreases. Examination of the data shows that both molecular and turbulent diffusion contribute to decreasing the overshoot in the simulation.
Figure 3.4: Profiles of thermal field: (a) Temperature and (b) Temperature gradient. In both plots, lines and symbols in grey correspond to $Ri = 500$ and those in black to $Ri = 2500$. Dashed lines correspond to $t = 15$ and solid lines to $t = 50$. The symbol $\circ$ denotes the location of $\partial \langle \theta \rangle_{xy}(z,t)/\partial z = 0.1$ while the symbol $\triangle$ locates $Ri_g(z,t) = 0.25$.

Behavior of thermal field over a tidal cycle

The periodic modulation of the thermal field that occurs over a tidal cycle is examined here. Since the mixed layer height increases continually, plane-averaged quantities are plotted instead of an average over an ensemble with the same phase but different times. Figure 3.5 (a) shows that the tidal phase has a strong influence over the mixing of the thermal field and, in particular, the mixed layer height, $h_m$. During most of the deceleration stage, whose extent is from $t = 111.55$ ($\phi = -90^\circ$) to $t = 113.1$ ($\phi = 0^\circ$), the mixed layer height increases. This is linked to an increase of TKE in the outer region of the boundary layer during this time span (see Figure 3.5(b)). After reaching its peak value at about $t = 112.82$ ($16^\circ$ before the zero velocity point), the value of $h_m$ decreases consistent with a sharp drop of outer layer TKE. The isotherms at $z \sim 7 - 10$ show periodic modulation; they are compressed during the decelerating phase when turbulence moves into the outer layer and then relax when the turbulence level plummets towards the end of the
Figure 3.5: Contours, as a function of $z$ and $t$, in case 3 with $Ri = 2500$ that illustrate the behavior of the thermal statistics over a half-cycle. The symbol $\circ$ denotes the location of $\partial \langle \theta \rangle_{xy}(z,t)/\partial z = 0.1$ while the symbol $\triangle$ locates $Rig(z,t) = 0.25$. The two upper insets shows the background velocity with two downward arrows that show the locations of the local maxima of the mixed layer height, $h_m$. (a) Plane-averaged temperature; (b) Plane-averaged turbulent kinetic energy normalized by $U_0^2$; (c) Plane-averaged vertical heat flux, $\langle \theta'w' \rangle_{xy}(z,t)$ normalized by $U_0^2 d\theta/dz|_{\infty}/\omega$; (d) Eddy diffusivity, $\kappa_T(z,t)$ normalized by molecular diffusivity $\kappa$.

deceleration and early acceleration.

The plane-averaged vertical heat flux is shown in figure 3.5 (c). The region of intense thermal flux (dark black) is patchy in space/time and occurs in the vicinity of the location of mixed layer height and during the deceleration stage. Interestingly, the upper boundary of this region corresponds to the location of $Rig = 0.25$. Finally, the eddy diffusivity $\kappa_T$, defined by

$$\kappa_T = -\frac{\langle \theta'w' \rangle_{xy}}{\partial \langle \theta \rangle_{xy}/\partial z},$$

is plotted in 3.5 (d). Large values of $\kappa_T$ occur close to the bottom corresponding to
the location of large TKE; however, since the fluid is already mixed in that region, the heat flux is not large.

3.4.3 Velocity field

Statistics of the velocity field are examined here to investigate the dependence on tidal phase and, in particular, the influence of stratification.

Mean velocity

Vertical profiles of streamwise velocity at different phases are shown in figure 3.6. During the acceleration stage, the velocity increases rapidly throughout the boundary layer, and the profiles become progressively fuller until $\phi = 90^\circ$. Later, during the deceleration stage with adverse pressure gradient, the velocity decreases and the profile become flatter. A phase with zero wall shear stress occurs and, eventually, reverse near-wall flow commences, for instance at $\phi \simeq 175^\circ$ for $Ri = 500$ (although not plotted here, see the bottom panel in Fig. 3.19(a) for a similar
Figure 3.7: Explanation of the velocity overshoot. (a) Contribution of Reynolds shear shear stress and viscous shear stress to the overshoot, see Eq. (3.6), at $\phi = 90^\circ$ for $\text{Ri} = 0$ (shown in light gray shade) and $\text{Ri} = 2500$ (black shade), (b) Variation of various terms in the $x$-momentum equation at a location $z_0$ as a function of phase.

(example), and a new boundary layer forms in the opposite direction. The mean velocity exhibits an overshoot with respect to the prevailing freestream value. The position of overshoot occurs close to the wall at the beginning of the acceleration stage and progressively moves upward. At $\phi = 90^\circ$, the velocity exceeds unity, the freestream amplitude. The outer layer ‘jet’ at $\phi = 90^\circ$ is especially prominent in the $\text{Ri} = 2500$ case. The velocity overshoot and its sensitivity to stratification can be explained by the following analysis. Consider the $x-$momentum equation and integrate it with respect to phase from $\phi = \phi_0$ to $\phi = \phi_t$ to obtain,

$$
\int_{\phi_0}^{\phi_t} \frac{\partial}{\partial t} \langle U \rangle (z, \phi') d\phi' = \int_{\phi_0}^{\phi_t} \langle dp/dx \rangle (\phi') d\phi' - \int_{\phi_0}^{\phi_t} \left[ \frac{\partial}{\partial z} \langle u'w' \rangle (z, \phi') - \frac{1}{\text{Re}} \frac{\partial^2}{\partial z^2} \langle U \rangle (z, \phi') \right] d\phi' 
$$

$$
\langle U \rangle (z, \phi_t) - \langle U \rangle (z, \phi_0) = \langle U \rangle_\infty (z, \phi_t) - \langle U \rangle_\infty (z, \phi_0) - \int_{\phi_0}^{\phi_t} \left[ \frac{\partial}{\partial z} \langle u'w' \rangle (z, \phi') - \frac{1}{\text{Re}} \frac{\partial^2}{\partial z^2} \langle U \rangle (z, \phi') \right] d\phi' 
$$

(3.5)
The second line of Eq. (3.5) follows from the first after using the relationship, \( \langle dp/dx \rangle = -d\langle U \rangle_{\infty}/dt \). Denote the velocity overshoot at an arbitrary phase by \( \langle \delta U \rangle(z, \phi) = \langle U \rangle(z, \phi) - \langle U \rangle_{\infty}(z, \phi) \). Since the streamwise velocity behaves like an odd function of \( \phi \), it follows that \( \langle \delta U \rangle(z, -\phi) = -\langle \delta U \rangle(z, \phi) \) for any given \( \phi \).

Now evaluate (3.5) for \( \phi_t = \phi \) and \( \phi_0 = -\phi \) and use the relation, \( \langle \delta U \rangle(z, -\phi) = -\langle \delta U \rangle(z, \phi) \), to obtain the following expression for the velocity overshoot:

\[
\langle \delta U \rangle(z, \phi) = -\frac{1}{2} \int_{-\phi}^{\phi} \left[ \frac{\partial}{\partial z} \langle u'w' \rangle(z, \phi') - \frac{1}{Re} \frac{\partial^2}{\partial z^2} \langle U \rangle(z, \phi') \right] d\phi'
\]

(3.6)

Here \( \langle \delta U \rangle_{\text{Rey}}(z, \phi) \) and \( \langle \delta U \rangle_{\text{vis}}(z, \phi) \) are velocity overshoots owing to Reynolds shear stress and to viscous stress, respectively.

The overshoot in streamwise velocity profile at \( \phi = 90^\circ \) is explained by figure 3.7 (a) where \( \langle \delta U \rangle_{\text{Rey}} \) and \( \langle \delta U \rangle_{\text{vis}} \), evaluated using their definitions in (3.6), and their sum, \( \langle \delta U \rangle_{\text{tot}} \), are plotted for cases 1 and 3. The total overshoot \( \langle \delta U \rangle_{\text{tot}}(z, \phi) \) for both cases is consistent with the overshoot observed in the streamwise velocity at \( \phi = 90^\circ \) in figure 3.6. The maximum overshoot occurs at \( z \sim 8.2 \) when \( Ri = 2500 \) while, for the passive case, it occurs higher at \( z \sim 14.5 \) owing to a thicker turbulent boundary layer. Other than a region very close to the wall, \( \langle \delta U \rangle_{\text{vis}} \) is negligible compared to \( \langle \delta U \rangle_{\text{Rey}} \). It is the momentum flux provided by the Reynolds shear stress that, when integrated over the cycle, provides a net positive acceleration to give the overshoot. Furthermore, the gradient of \( \langle u'w' \rangle \) is larger in the presence of stratification, leading to a larger overshoot compared to the passive case and the formation of a jet at the top of the boundary layer.

In order to better understand the net positive acceleration provided over a cycle, the different \( x \)-momentum flux terms are evaluated over the range \(-90^\circ \leq \phi \leq 90^\circ \) and plotted in figure 3.7 (b). The evaluations are performed at locations, \( z_o = 8.2 \) and \( z_o = 14.5 \), corresponding to cases \( Ri = 2500 \) and \( Ri = 0 \), respectively, in order to focus on the location of the maximum overshoot at \( \phi = 90^\circ \) for each case. The pressure gradient force (solid line with triangles), responsible for the acceleration of the background velocity, behaves as \( \cos(\phi) \). At the wall, it is always balanced by the viscous force; however, since the chosen \( z_o \) are away
Table 3.2: Comparison of the overall boundary properties between various stratified cases. The subscript \( \text{avg} \) denotes an average over the complete tidal cycle. Here, \( \Delta_\tau \) is the phase lead of \( \tau^\text{max}_w \).

<table>
<thead>
<tr>
<th>Case</th>
<th>Ri</th>
<th>Log-law zone</th>
<th>( c_f )</th>
<th>( \Delta_\tau )</th>
<th>( c_f, \text{avg} )</th>
<th>( \int \epsilon dz, \text{avg} / U_0 u^+_w, \text{avg} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>40° − 140°</td>
<td>0.0043</td>
<td>17°</td>
<td>0.0023</td>
<td>0.2554</td>
</tr>
<tr>
<td>2</td>
<td>500</td>
<td>60° − 130°</td>
<td>0.0045</td>
<td>23°</td>
<td>0.0024</td>
<td>0.2438</td>
</tr>
<tr>
<td>3</td>
<td>2500</td>
<td>80° − 125°</td>
<td>0.0048</td>
<td>25°</td>
<td>0.0025</td>
<td>0.2275</td>
</tr>
</tbody>
</table>

Laminar · · · 45° · ·

Figure 3.8: Profiles of the streamwise velocity in semi-log plot. Each profile is staggered by 30 units in the horizontal.

From the wall, the viscous contribution (dashed lines) is small. The flux due to \( \langle u'w' \rangle \) (solid black curve for the \( Ri = 2500 \) case) is larger than the viscous term in magnitude and it is strongly positive during the deceleration stage, almost zero during most of the acceleration stage, and has a short region in the vicinity of \( \phi = 90^\circ \) where it is negative. The dashed curve with circles, corresponding to the sum \( \partial \langle \tau \rangle(z, \phi)/\partial z \) of all the component fluxes, shows the asymmetry as well as the pronounced positive excess with respect to the pressure gradient which, after integration over the half-cycle, leads to the observed overshoot of mean velocity at \( \phi = 90^\circ \).
Figure 3.9: Magnitude of wall shear stress, $\tau_w$, as a function of tidal phase.

The effect of stratification on the log-law is shown in figure 3.8. Although a logarithmic law is present in the stratified cases too, the range of phases where the logarithmic law holds becomes shorter with increasing stratification. Table 5.3 shows that, during a half-cycle of $180^\circ$, the log-law holds for a range of $45^\circ$ at $Ri = 2500$ instead of $100^\circ$ as in the passive scalar case. Figure 3.8 shows that $u^+ = z^+$ in the viscous sublayer ($z^+ < 5$) for all cases. However, there is a noticeable variation of the wall shear stress given by

$$\tau_w = \mu \left[ \left( \frac{\partial \langle u \rangle}{\partial z} \right)^2 + \left( \frac{\partial \langle v \rangle}{\partial z} \right)^2 \right]^{1/2}_{z=0}.$$

The wall stress leads in phase with respect to free stream velocity as shown in figure 3.9. $\tau_w$ commences a rapid increase at $\phi \sim 30^\circ$, attains its maximum value before the mean velocity does and, during the decelerating phase, it decreases almost linearly until it becomes zero at $\phi \sim 160^\circ$, signifying commencement of reverse flow. Finally, the wall stress grows slowly along with the development of a new boundary layer in the reverse direction during the late decelerating stage and during the early accelerating phase of the next half-cycle. The maximum value of $\tau_w$ leads the maximum value of freestream velocity by $17^\circ$ in the passive scalar case. With intensifying stratification, this lead increases to approximately $25^\circ$ for the case with $Ri = 2500$. The laminar value of the phase lead is $45^\circ$. The skin
friction coefficient given by

\[ c_f = \frac{\tau_{w,\text{max}}}{(1/2)\rho U_0^2} \]  \hspace{1cm} (3.8)

has been evaluated. The values, listed in table 5.3, show that there is a small increase in \( c_f \) with increasing \( Ri \).

**Turbulent kinetic energy (TKE)**

TKE contours, shown in figure 3.10(a) for the passive scalar case, show the phase dependence of vertical profiles of TKE. Note that, owing to the problem symmetry, the TKE repeats at a period of 180°. At \( \phi = 90° \), the maximum value of TKE occurs very close to the wall. The contours, that slant upward and to the right after \( \phi \sim 90° \), show that the location of maximum TKE in a given vertical profile shifts upward. The TKE near the bottom decreases during the late deceleration phase, becomes negligible at about 170° (the point of zero mean wall shear stress), and remains small until \( \phi = 200° \), equivalently 20°. Nevertheless, during this late-deceleration and early-acceleration stage, there is significant TKE in the outer layer, associated with residual large-scale turbulence from earlier, that decays slowly. Small-scale turbulence is generated after approximately 30° when the near-wall shear becomes sufficiently large and a new cycle of turbulence production commences.

The strong variation of mean velocity within a cycle, including an overshoot with respect to the freestream, complicates the definition of a representative boundary layer thickness. One possibility is the definition of a turbulent layer thickness (TLT) \( \delta_t \) based on the turbulent kinetic energy. The upper boundary of the turbulent layer, defined by the region where the TKE is 10% of the maximum of its global value, is shown by the dark dotted thick line in figure 3.10 (a). The behavior of \( \delta_t(\phi) \) is compared between cases in figure 3.10(b). Stratification leads to substantial reduction in the value of \( \delta_t \) by suppressing the length scale over which turbulence, generated at the wall, can mix momentum. Furthermore, the residual outer-layer turbulence, that occurs during \( 135° < \phi < (180 + 25)° \) in the passive scalar case, is diminished by stratification, and there is a sudden collapse of large-scale structures once the flow proceeds beyond the stage of turbulence.
Figure 3.10: (a) Contour plot of the turbulent kinetic energy over a half-cycle, normalized by $u_{r,max}^2$ for $\tilde{Ri} = 0$. (b) Turbulent layer thickness, based on the TKE isocontour at 10% of the global maximum value, shown as a function of phase for $\tilde{Ri} = 0, 500$ and 2500.

Figure 3.11: Vertical profiles of production (dashed), dissipation (dash-dot) and transport (solid) in the outer part of the boundary layer shown for $\phi = 60^\circ, 90^\circ, 120^\circ$ and $150^\circ$, respectively. Light gray and black are used for $\tilde{Ri} = 0$ and 2500, respectively. Here all terms are normalized by $u_{r,max}^4/\nu$. 
production by near-wall shear. It is worth noting that the maximum value of $\delta_t$ occurs during the decelerating stage.

From the preceding discussion it is clear that, during the decelerating stage, there is a strong increase of TKE in the outer layer. To better understand this phenomenon, the TKE equation given below is investigated.

$$\frac{\partial k}{\partial t} = P - \epsilon + B - \frac{\partial T}{\partial z}.$$  (3.9)

Here $k$ is TKE defined by $\frac{1}{2} \langle u'_i u'_j \rangle$. The term $\partial T / \partial z$ denotes the transport of the TKE containing pressure transport, turbulent transport, viscous transport and subgrid scale (SGS) transport where

$$T \equiv \frac{1}{\rho_0} \langle p'w' \rangle + \frac{1}{2} \langle u'_i u'_i w' \rangle - \frac{1}{Re} \frac{\partial k}{\partial z} + \langle \tau'_{ij} u'_i \rangle.$$

$P$ is the production term defined as

$$P \equiv -\langle u'_i u'_j \rangle \langle S_{ij} \rangle - \langle \tau_{ij} \rangle \langle \overline{S}_{ij} \rangle,$$

where the last term is the SGS production. The turbulent dissipation rate, $\epsilon$, is defined as the sum of the resolved and SGS components:

$$\epsilon \equiv \frac{1}{Re} \left\langle \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} \right\rangle - \langle \tau_{ij} \overline{S}_{ij} \rangle.$$

Finally, $B$ is the buoyancy flux defined as

$$B \equiv Ri \langle \theta w' \rangle.$$

The terms in the TKE transport equation are evaluated, and the production, dissipation and transport plotted in figure 3.11. At $\phi = 60^\circ$, significant values of all three terms are confined to near the wall. In the passive scalar case, the vertical span of all terms extends progressively upward as $\phi$ increases to $90^\circ$ and then to $150^\circ$. Thus, the increased outer layer TKE is a consequence of enhanced turbulent transport as well as enhanced shear production in the outer layer as a response to the adverse pressure gradient. At larger $z$, transport eventually dominates. The $Ri = 2500$ curves in black show that the upward extension of the transport, production and dissipation during the stage of deceleration is substantially less
Figure 3.12: (a) Integrated (over z) TKE normalized by $u_{τ, max}^2 δ_s$ as function of phase. (b) Integrated production, integrated dissipation and integrated buoyancy flux over a half tidal cycle. Both are non-dimensionalized by $u_{τ, max}^2 U_0$. Light gray, dark gray and black are used for $Ri = 0$, 500 and 2500, respectively. (c) TKE (normalized by $u_{τ, max}^2$), production and dissipation as a function of phase at $z = 1.1δ_s$ for $Ri = 0$ (Light gray) and $Ri = 2500$ (black). Both $P$ and $ε$ are made non-dimensional by $u_{τ, max}^4/ν$. (d) Same as in (c) for $z = 10δ_s$.

The overall strength of turbulence can be assessed through a depth-integrated value of TKE where TKE is integrated from $z = 0$ to $z = l_z$ where $l_z = 70δ_s$ is the height of the computational domain. The behavior of integrated TKE as a function of phase is shown in figure 3.12(a) and that of integrated production, dissipation and buoyancy flux in figure 3.12(b). The maximum value of $\int TKE dz$ occurs during the deceleration stage at a phase that decreases from $φ = 135°$ at
$Ri = 0$ to $\phi = 100^\circ$ at $Ri = 2500$. Although there is a strong suppression of TKE with increasing $Ri$ as clearly shown in figure 3.12(a), the integrated production and dissipation terms are relatively unaffected since the near-wall turbulence does not feel the overlying stratification. The integrated dissipation and production are negligible during $0^\circ < \phi < 25^\circ$ after which there is a sharp increase owing to the formation of small-scale near-wall turbulence until a peak is attained at approximately $\phi \sim 90^\circ$. Subsequently, during the deceleration stage, $\int P dz$ and $\int \epsilon dz$ decrease. The depth-integrated buoyancy flux, $\int B dz$, strongly depends on the phase. Although the integrated heat flux is always downward, its magnitude is large when the integrated production is small, that is, when the mean velocity is small. This observation is consistent with the larger turbulent heat flux and larger mixed layer height during the decelerating phase.

The behavior of TKE, production and dissipation at a location near the bottom boundary is illustrated by a plot of their phase variation at $z = 1.1 \delta_s$ in figure 3.12(c). At this location, $TKE$, $P$ and $\epsilon$ attain their peak values at $\phi \sim 90^\circ$ without any phase lag with respect to the maximum of the freestream velocity. However, in the outer layer ($z = 10 \delta_s$), all of these quantities show a significant lag, approximately $\delta \phi \sim 40^\circ - 50^\circ$, with respect to the freestream peak as shown in figure 3.12(d). Thus, the observed phase lag of the peak $TKE$ and other related quantities depend on the measurement location from the bottom surface. Such a phase lag has been observed in the tidal boundary layer. For instance, Burchard et al. (1998) show observations of dissipation in the Irish sea that show a lag of 1 hr, equivalently $\delta \phi = 29^\circ$, between dissipation and current at a height 12 m above the sea-bed. It is worth noting that the phase lag that prevails away from the wall for both $P$ and $\epsilon$ does not impact their depth-integrated values, because the regions of large production and dissipation are confined to a small region close to the wall throughout the cycle. On other hand, a significant amount of $TKE$ is transported away from the wall into the outer layer during the decelerating phase (see figure 3.10(a)) so that the observed outer layer lag gives rise to a corresponding phase lag in the depth-integrated value $\int TKE dz$. 
Reynolds stresses

Vertical profiles of Reynolds stresses are discussed in this section. As shown by figure 3.13, during the fully turbulent phases (φ ~ 60° – 120°), the streamwise intensities in the near-wall region behave similarly among cases. The maximum value of $u_{rms}$ for $Ri = 2500$ is somewhat greater than the corresponding value in the passive case owing to a somewhat higher wall stress. In the outer layer, where the flow is dominated by buoyancy, $u_{rms}$ is significantly lower in the stratified cases. At φ = 150° and φ = 0°, phases with low wall stress, the maximum value $u_{rms}$ for all cases occurs away from the bottom wall.

The vertical turbulent intensity, $w_{rms}$, is shown as a function of phase and height in figure 3.14. The suppressing effect of buoyancy in the outer region is also present in the $w_{rms}$ profiles and in the profiles of spanwise turbulence intensity (not shown here).

Figure 3.15 shows profiles of the Reynolds shear stress $\langle u'w' \rangle (\phi, z)/U_0^2$ at various phases during the tidal cycle. At φ = 90°, the maximum value of $\langle u'w' \rangle /U_0^2$ occurs close to the wall with the value for the stratified cases slightly larger than in the passive scalar case. In the outer layer, $\langle u'w' \rangle /U_0^2$ decreases more rapidly in the stratified cases compared to the passive scalar case. During the late deceleration...
Figure 3.14: Vertical profiles of vertical turbulence intensity $w_{rms}/U_0$. Each profile is staggered by 0.1 units in the horizontal.

Figure 3.15: Vertical profiles of Reynolds shear stress $\langle u'w' \rangle /U_0^2$. Each profile is staggered by $4 \times 10^{-3}$ units in the horizontal.
Figure 3.16: (a) Profiles of streamwise r.m.s turbulence normalized by friction velocity $u_\tau$ as a function of $z^*$ ($z/\delta_t$) at $\phi = 60^\circ$, 75°, 90° and 105°. (b) Profiles replotted using the $z^+$ coordinate and semi-log axes. Each profile is staggered by 4.0 units in the horizontal.

stage, there is significant Reynolds shear stress away from the wall owing to large-scale turbulent structures. During the early acceleration stage, $0^\circ < \phi < 30^\circ$, the Reynolds shear stress away from the wall decreases associated with the collapse of these large-scale structures while that near the wall increases as a consequence of progressively increasing near-wall shear.

Scaling of turbulence profiles

In figure 3.16(a), we have replotted the streamwise turbulent intensity in the fully turbulent phases using a different normalization: the turbulent boundary layer thickness, $\delta_t(\phi)$, for the vertical coordinate and the friction velocity $u_\tau(\phi)$ for $u_{rms}$. The profiles tend to collapse between the two stratified cases. The $Ri = 0$ profile agrees well with the other two cases for $z/\delta_t < 1$; however there is some difference at larger values of height over the bottom. This normalization cannot be used during the less energetic phases since $\delta_t$ and $u_\tau$ become very small and may drop to zero during these phases. The applicability of inner layer scaling is assessed in figure 3.16 (b). In the viscous sublayer and the buffer layer that span $z^+ < 20$, the profiles are almost indistinguishable between cases. When $z^+$ is
Figure 3.17: Four panels show the profiles of Production (solid line), dissipation (dash-dot line) and modified transport (dotted line) normalized by $u_4^4(\phi)/\nu$ as function of $z^*(z/\delta_t)$ at $\phi = 50^\circ, 60^\circ, 90^\circ$ and $120^\circ$, respectively. We have used light gray, dark gray and black for $\text{Ri} = 0, 500$ and $2500$, respectively.

not too large, the differences between cases is small during the acceleration stage and increases somewhat during deceleration. Production, dissipation rate and transport, the three leading terms in the turbulent kinetic energy equation, given in (7.1), are plotted in figure 3.17 as a function of $z^* = z/\delta_t$ for different phases. The normalization factor is $u_4^4/\nu$, customary for wall-bounded flows. Importantly, the use of $\delta_t$ as the appropriate length scale enables good collapse of the profiles among all $\text{Ri}$ cases over a wide range of phases $\phi \sim 55^\circ - 110^\circ$ when the turbulence is energetic. The viscous dissipation rate peaks at the wall where it is balanced by viscous transport, $\frac{1}{Re} (\partial^2 k/\partial z^2)$. Production reaches it maximum value within the buffer layer, a location where the transport term is negative signifying that it is a sink transporting $TKE$ away to other locations. At the edge of the boundary layer where production and dissipation are negligible, the buoyancy flux and the vertical energy flux, $\langle p'w' \rangle$ due to internal wave activity, become important in the stratified cases as will be shown later. In the region with significant turbulence, the balance is, for the most part, between production and dissipation. Since $u_\tau$ and $\delta_t$ become very small during the less energetic phases, this normalization cannot be used for those phases.
3.4.4 Internal Waves

Flow instabilities and turbulence in the bottom boundary layer can lead to internal gravity waves that propagate in the overlying stratified fluid. For the boundary layer under a steady freestream (Taylor & Sarkar (2008c)) internal waves are generated by vertical modulation of the thermocline by a broad spectrum of eddies in the boundary layer. The lines of constant phase are inclined towards the direction of the source velocity relative to the free stream velocity, similar to waves generated by flow over corrugated surfaces like hills and mountains (Lighthill (1990)). Figure 3.18 provides a schematic of internal gravity waves in a steady current. The group velocity $\mathbf{c}_g$, relative to the free stream, is aligned with the phase line, is orthogonal to the wave number vector, and transfers energy upward from the boundary layer to the background.

The source of internal waves is the turbulent flow in the mixed layer that consists of a broad range of scales. The inclination of the phase line of the internal wave radiated by an eddy depends on the relative velocity of that eddy and the combination of all those phase lines gives rise to the observed wave pattern. For the steady case, the relative velocities of the eddies with respect to free stream current have mostly the same sign whose value determines the tilt of the phase lines with respect to the vertical: the tilt is forward when the sign of the relative velocity is positive and backward when the sign is negative. In the oscillating flow considered here, the direction of the source velocity with respect to the free stream...
is phase dependent leading to a more complex wave pattern. This is illustrated by figure 3.19(a)-(d) that shows $x-z$ slices of the $\partial w'/dz$ field in a frame moving with free stream velocity at four different phases of the tidal cycle. The upper panels in the figure show the far-field waves while the middle panel shows turbulence as well as the near-field waves adjacent to the generation region. Clearly, the slope of the phase lines depend on tidal phase and the slope differs between far- and near-fields. The bottom panel, that shows mean velocity profiles at the corresponding times with black showing values positive with respect to the free stream and grey showing negative values, helps to explain the observed phase lines. Although, it is difficult to exactly demarcate the wave source, the generation region is bounded from above by the location of $\partial \theta(z,t)/\partial z = 0.1$, the top of the mixed layer. The snap in figure 3.19(a) corresponds to a late decelerating phase ($\phi \sim -5^\circ$) where a new boundary layer with reverse flow in the bottom relative to the free stream has been formed. The flow velocity, relative to the free stream, is positive in the source region as shown in the bottom panel of figure 3.19(a) leading to phase lines inclined to the front as shown in the middle panel of figure 3.19(a). These forward inclined waves do not have enough time to propagate into the outer region and the backward-inclined phase lines in the outer region, shown in the upper panel of figure 3.19(a), correspond to internal waves generated during the early-accelerating stage of the previous cycle, for example, those in the middle panel of figure 3.19(c)-(d). When the flow progresses in time, a near-bottom region of negative velocity with respect to the free stream, similar to the conventional steady boundary layer, progressively develops. Figure 3.19(b) shows $\phi = 90^\circ$ where, as shown by the middle panel, waves are generated from the source with no preferred inclination of the phase lines, see middle panel, but the outer region shown in the top panel exhibits forward inclined waves that were generated previously, for example at $\phi = -5^\circ$ shown in part (a) of the figure. Now, examination of the velocity profile at $\phi = 90^\circ$ reveals that most of the mixed layer has negative relative velocity compared to a small portion with positive relative velocity. Therefore, one would expect generation of backward waves from the source region and, indeed, such phase lines were observed in the steady stratified Ekman boundary layer by Taylor.
Figure 3.19: Slice of $\partial w'/\partial z$ in the x-z plane for the case with $Ri = 500$ at different tidal phases: (a) $\phi = -5^\circ$, (b) $\phi = 90^\circ$, (c) $\phi = 180^\circ$ and (d) $\phi = -90^\circ$. Each part is divided into three panels: top panel shows the waves in the far-field, $z = 35 - 86$; middle panel shows the turbulent source and near-field waves; bottom panel shows the streamwise velocity profile with black and light gray indicating positive and negative signs of the velocity relative to the free steam. The symbols on the velocity profiles show the location of $\partial \theta(z,t)/\partial z = 0.1$ to demarcate the mixed layer.
& Sarkar (2007b) where the streamwise velocity profile has a shape similar to that seen here at $\phi = 90^\circ$. However, the history of the mean flow is important in the oscillating case; forward phase lines emitted by the predominantly positive velocity at an earlier nearby phase remain adjacent to the boundary layer at $\phi = 0^\circ$ if $c_{g,z}$ is sufficiently small and, consequently, both forward and backward phase lines are observed in the middle panel of figure 3.19(b). Thus, owing to history effects, arriving at a conclusion based on steady currents that, at $\phi = 90^\circ$ in the present oscillating flow, the phase lines would tilt backward is clearly erroneous. During the following deceleration stage, the source is dominated by fluid with negative relative velocity giving rise to waves with phase lines inclined towards the back (middle part of the figure 3.19 (c)) while the outer region still has the forward phase lines of internal waves generated in the previous acceleration stage. In figure 3.19 (d), the phase of $-90^\circ$ corresponding to negative free-stream velocity is shown. Here, similar to the phase of $90^\circ$ shown in (b), there is no clear direction of phase lines adjacent to the boundary layer. However, the phase lines in the far field tilt backward.

The generation of internal gravity waves from the source over a tidal cycle is shown more elaborately in figure 3.20 through a sequence of nine snaps of the vertical strain field. The corresponding streamwise velocity profile plotted in two different colors (discussed before) is shown above each snap. The sequence begins with (a), corresponding to $\phi = -90^\circ$, and ends at (i), a time corresponding to approximately the same phase as (a). In figure 3.20(a)-(b), the wave source has both positive and negative values of relative velocity and therefore the phase lines have no directional preference. In figure 3.20(c)-(d), the wave source is dominated by positive relative velocity and, consequently, generates waves inclined towards the front. At $\phi = 90^\circ$, there is an overshoot of velocity and formation of a jet as discussed previously, leading to a small region of positive relative velocity in addition to the negative velocity of near-bottom fluid. Therefore, both backward and forward phase lines are generated from the source in (e) due to combined effect of the relative velocity condition during the current phase and previous nearby phases. In 3.20(f), phase lines start bending towards the back and stay backward.
Figure 3.20: In (a)-(i), slices of $\partial w'/\partial z$ in the x-z plane along with streamwise velocity profiles are shown over a time period that spans an entire tidal cycle, shown in the background of the figure. Each slice ($x = 0 - 50$ and $z = 5 - 25$) shows the generation region and near field. In the velocity profile, circles give the location of $\partial \theta(z,t)/\partial z = 0.1$. Black and light gray portions of the velocity profiles indicate positive and negative sign of the local velocity with respect to free stream velocity, respectively.
Figure 3.21: (a) Power spectra of the $\partial w'/\partial z(t)$ field (log-scale) as a function of frequency in the case with $Ri = 500$. The time series is taken in a frame moving with free stream velocity at $z = 50$ and $x = 10$ over a time span $40.95 < t < 43.9$ which corresponds to $-175^\circ < \phi < -5^\circ$, and the resulting spectra at different spanwise locations are averaged. The dotted vertical line indicates $N_\infty$. In (b) the two dimensional power spectrum (log-scale) of $\partial w'/\partial z(x,t)$ at $z = 50$ plotted as a function of the streamwise wave number and intrinsic frequency space. Spanwise averaging is used again. Here, the wave number has been normalized with $\delta_s$. 
at the phases shown in (g) and (h). During the remainder of the acceleration stage, the fluid region very close to wall is retarded and has slower speed than the free stream but the outer layer develops an overshoot in speed leading to (i), similar to the situation back in (a), where there are phase lines of both tilts.

In order to quantify the properties of the internal wave field, we have performed a power spectrum analysis of the $\partial w'/dz$ field. Unless otherwise mentioned, the $\partial w'/dz$ field is taken in a frame moving with the free stream velocity. Figure 3.21(a)-(b) show one and two-dimensional power spectra, respectively, for the $Ri = 500$ case. The intrinsic frequency or frequency in a frame moving with the freestream velocity $U_\infty$ is defined as

$$\Omega = \omega_{app} - \mathbf{U} \cdot \mathbf{k}$$

(3.10)

where $\mathbf{k} = (k_x, k_y)$ is the horizontal wave number vector and $\omega_{app}$ is the apparent frequency in the fixed frame. The time span of the analyzed data is taken to be $40.95 < t < 43.9$, a range over which phase lines in the outer layer have a backward tilt. Turbulence generated internal waves are characterized by broadband spectra in both wave number and frequency. But eventually, in the far-field the internal waves occupy a narrow band of frequencies as shown in 3.21 (a). This frequency range corresponds to phase lines whose angle with the vertical span $40^\circ < \Theta < 62^\circ$. Similar narrow-band internal wave propagation has been observed in laboratory experiments by Sutherland & Linden (1998), Dohan & Sutherland (2003) and Dohan & Sutherland (2005). More recently, Taylor & Sarkar (2008c) in their numerical study have also observed a similar narrow band of angles for the steady Ekman bottom boundary layer and have offered a frequency-selective viscous decay model to explain this phenomenon. Similar results are obtained for the strongly stratified case as shown in figure 3.22.

The relative importance of the energy flux, $\langle p'w' \rangle$, carried by the internal waves at a particular level, say $z = z'$, with respect to the other energy fluxes can be obtained by integrating the TKE equation up to height $z'$ and measuring the terms in the resulting equation 3.11. Here $z'$ is taken to be sufficiently larger than the boundary layer height $\delta_t$ so that viscous and turbulent transport are
The inverse of the three terms of Eq. (3.11) quantifies the internal wave energy flux relative to the integrated dissipation, production and buoyancy flux, respectively. These quantities are shown for $Ri = 500$ and 2500 at $\phi = 90^\circ$ in figure 3.23. For both cases, the vertical energy flux is less than 1% compared to the integrated dissipation and production indicating that the internal waves have negligible direct effect on the boundary layer turbulence. However, as shown by figure 3.23 (c), the integrated buoyancy flux and the vertical energy flux are comparable suggesting that the energy carried away by internal waves that could then potentially cause non-local mixing (for more complicated background conditions then the one considered here) is comparable to the local mixing of density. The average vertical energy flux, defined by

$$
\langle p'w' \rangle_{\text{avg}}(\phi) = \frac{1}{l_z - h_m} \int_{h_m}^{l_z} \langle p'w' \rangle(z', \phi)dz',
$$

where $h_m$ is the height of the mixed layer as defined previously, also varies over the phase as shown in figure 3.24. The average vertical energy flux peaks during the laminar phase ($\phi \sim 0^\circ$) of the cycle because of the buildup of internal waves

**Figure 3.22:** Same as figure 3.21 for $Ri = 2500$. 
generated during the previous turbulent portion of the cycle. According to the profile of \( \langle p'w' \rangle / Ri \int_0^z \langle \theta' w' \rangle dz' \) at \( \phi \sim 0^\circ \) (not shown here), its value is somewhat larger in the outer region, approximately 0.3 compared to 0.2 at \( \phi = 90^\circ \).

### 3.5 Conclusions

We have used large eddy simulation (LES) to investigate the dynamics of a stratified bottom boundary layer under an oscillating current driven by a pressure gradient that oscillates at a low frequency, 12.4 hrs, corresponding to a \( M_2 \) tide. A dynamic mixed model is used, buoyancy related stability functions are not necessary, and the near-bottom turbulence is resolved at the moderate Reynolds number, \( Re_s = 1790 \), considered here. The effect of increasing the background stratification, measured by \( Ri = N_\infty^2 / \omega^2 \), on the flow evolution is systematically studied with particular attention to the dependence on tidal phase.

The fluid has a uniform background stratification while the bottom boundary is adiabatic. A bottom mixed layer forms and grows with an entrainment rate which is reduced by the external stratification, and with a periodic modulation owing to the tidal oscillation. The mixed layer height decreases during the ac-
Figure 3.24: Variation of the averaged vertical energy flux normalized by $\rho_0 u_{r,max}^3$ over a phase for $Ri = 500$ and 2500.

celeration stage when the production of turbulence is confined to the wall region whereas, during the deceleration stage, the height increases due to the upward spread of the turbulence in response to an adverse pressure gradient. The turbulent mixed layer is separated from the stratified outer layer by a thermocline with an overshoot in temperature gradient that gradually weakens over time due to both turbulent and molecular diffusion. The turbulent heat flux has significant modulation over the tidal cycle: its depth-integrated value is low when the free-stream velocity is high and increases rapidly during the early decelerating phase.

Stratification has a strong influence on the flow statistics. When stratification increases, the range of phases where the log law is applicable shortens, and the thickness of the log layer decreases. Stratification changes the wall stress, $\tau_w$, leading to a small increase of the friction coefficient, $c_f$. During a portion of the tidal cycle, the mean velocity exhibits an appreciable overshoot with respect to the prevailing free stream current and, in particular, the peak velocity exceeds the peak value of the external velocity. By examination of terms in the mean momentum equation, it is shown that the velocity overshoot is caused by the asymmetric dependence on tidal phase of the Reynolds shear stress gradient.
The overshoot is intensified by stratification since the gradient of the Reynolds shear stress increases owing to the significant reduction in vertical length scale at high stratification. Stratification leads to a pronounced suppression of turbulence intensities in the outer layer although the inner, near-bottom values show little effect.

In oscillatory flow, flow statistics can have a noticeable phase lag or lead with respect to the external current. Stratification substantially changes the observed phase difference. The maximum value of the wall shear stress, $\tau_{w,max}$, leads (occurs before) the maximum freestream current by a phase of up to $25^\circ$ compared to $17^\circ$ when stratification is absent. Turbulent kinetic energy, $TKE$, in the outer layer lags the current for all cases because of the increase of outer layer turbulence during the decelerating part of the cycle when an adverse pressure gradient acts on the flow. Similarly, there is considerable phase lag of turbulent production and dissipation, as much as $60^\circ$ (equivalently 2 hrs.), depending on the height above bottom. Phase lags of peak turbulent dissipation rate with comparable magnitude have been reported in field data but it is worth emphasizing that the present simulations show that the value of the lag depends critically on the measurement location with respect to the boundary layer height. The depth-integrated production and dissipation attain their peak values at approximately the same time as the external current; however, the depth-integrated buoyancy flux and turbulent kinetic energy have a substantial phase lag.

A definition of boundary layer height based on mean velocity is difficult given the change in the shape of velocity profiles over a cycle. One definition of the turbulent boundary layer thickness, $\delta_t$, is based on the location of a specific $TKE$ contour, taken here to be 10% of the maximum $TKE$. The value of $\delta_t(\phi)$ increases continuously from its minimum value in the early acceleration stage to reach a maximum during the late deceleration stage followed by an abrupt decrease corresponding to a collapse of turbulence. The value of $\delta_t$ is suppressed when stratification increases; its maximum value at $Ri = N_\infty^2/\omega^2 = 2500$ decreases to about 0.6 of the corresponding unstratified value. The turbulent boundary layer height, $\delta_t(\phi)$, is found to be a suitable length scale for profiles of the outer layer
statistics. The friction velocity $u_r(\phi)$ is found to be a good scale for the velocity statistics in both the inner and the outer layer. However, the scales, $u_r(\phi)$ and $\delta_t(\phi)$ cannot be used when their values approach zero.

Fine-scale turbulence is initiated close to the wall during the late acceleration phase of the cycle ($45^\circ < \phi < 90^\circ$) and achieves its maximum level during the early deceleration stage ($90^\circ < \phi < 135^\circ$). Visualizations show that, during the late deceleration stage ($135^\circ < \phi < 180^\circ$), fine-scale turbulence is advected away from the wall and larger-scale eddies develop consistent with the larger intensity of the $TKE$ field in the outer layer that was shown in figure 3.10(a). Thorpe et al. (2008) observed a similar development of large-scale eddies from fine-sale near-wall turbulence in a tidal current in a weakly stratified, shallow-water region of the Irish Sea. Turbulence was found by the authors to move upwards in the form of bursts which, on reaching the sea surface, formed boils.

Turbulence in the bottom boundary layer excites internal gravity waves that propagate upward into the overlying stratified fluid. The source of the generation of the internal waves is bounded from above by the top of the mixed layer. The tilt of the phase lines of the internal waves with respect to the vertical depends of the relative velocity of the eddies with respect to the external background current. The tilt changes as a function of phase and between near and far fields because, first, the predominant sign of the relative velocity changes with phase and, second, there is a history effect, i.e., the phase lines observed at current time are emitted from the boundary layer at a prior time when a different mean flow and state of turbulence prevails. Although broadband with $\Omega < N_\infty$ in the region above and close to the boundary layer, the internal waves span a narrower band in the far field with phase lines clustered within $40^\circ < \Theta < 62^\circ$. More energetic waves are observed during the late decelerating and the early accelerating stages of the tidal cycle. The internal wave flux is comparable to the buoyancy flux but substantially smaller than the depth-integrated turbulent dissipation or production.

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Chapter 4

Turbulence during generation of an internal tide on a critical slope

The present chapter is a part of the second phase of the dissertation and is based on Gayen & Sarkar (2010b). In the contrast to Chapter 3, the tidal flow studied here oscillates over sloping topography (see figure 4.1) leading to internal waves, the so-called internal tides. When the slope angle is critical, there is a resonant response as discussed in the section 1.2. Three-dimensional direct numerical simulations are performed to examine nonlinear processes during such resonant generation of internal tides on a model slope. Details of numerical method were discussed in Chapter 2, section 2.3.

4.1 Selection of the simulated case

An oscillatory pressure gradient, \( F_0 = -U_0 \Omega \cos(\Omega t) \) forces tidal flow over a slope and leads to a barotropic velocity, \( U(x) \sin \phi \), where \( \phi \) is the tidal phase. As shown in Fig. 4.1(a), the bottom topography is irregular: a slope of length \( l = 3.5 \text{ m} \) between \( x = 4.5 \text{ m} \) and \( x = 8.0 \text{ m} \) is smoothly connected to horizontal sections before and after it. For simplicity, we do not include rotation.

For all cases, \( N_\infty = 8 \text{ rad/s} \) and \( \Omega = 1 \text{ rad/s} \), which gives the wave angle \( \theta \approx 7^\circ \). The kinematic viscosity, \( \nu = 10^{-6} \text{ m}^2/\text{s} \), is that of water. The Prandtl number is chosen to be \( Pr = 1 \), smaller than the value of \( Pr = 5 \) for thermal
transport in water, so as to avoid an unnecessary increase in computational grid points. Simulations at subcritical slopes \( \beta < 7^\circ \) did not result in turbulence for the range, \( Ex = U_0/\left(\Omega l\right) < 0.1 \), studied here. The Reynolds number is a key parameter and is measured by \( Re = aU_0/\nu \) with \( a = U_0/\Omega \), or by \( Re_s = U_0\delta_s/\nu = \sqrt{2Re} \), based on the Stokes boundary layer thickness, \( \delta_s = \sqrt{2\nu/\Omega} \). The salient features of generation at a critical slope will be brought out by discussion of the following case: slope length, \( l = 3.5 \) m; current velocity, \( U_0 = 0.125 \) m/s; \( Ex = 0.036 \); \( \epsilon = 1 \); \( Re = 15,625 \); and \( Re_s = 177 \).

We now place the numerical experiment in context by comparing nondimensional parameters with those corresponding to an oceanic slope. An example of conditions at a continental slope in deep water is as follows: a tidal amplitude of \( U_0 = 0.025 \) m/s, a tidal frequency of \( \Omega = 1.4 \times 10^{-4} \) rad/s corresponding to the \( M_2 \) tidal period of 12.4 hrs, a low latitude with \( f = 3.5 \times 10^{-5} \) rad/s and \( N = 1 \) cph = \( 1.74 \times 10^{-3} \) rad/s. A representative slope length of 5 km leads to an excursion number of 0.036 that matches our choice of \( Ex = 0.036 \). Our critical slope angle \( \beta = 7^\circ \) is slightly larger than the value of \( \beta = 4.5^\circ \) in the ocean example. The Stokes Reynolds number, \( Re_s = 177 \), of our case is smaller than the value of \( Re_s = 2975 \) in the oceanic example but still sufficiently large to exhibit turbulence in the case of critical slope as will be demonstrated. Details of numerical method are discussed in Chapter 2 under section 2.3. The numerical domain consists of a rectangular box of 13m length, 4m height and 1m width whose bottom boundary is coincident with the slope topography. The grid size is \( 260\times260\times64 \) in the x,z and y directions, respectively, with stretching in x and z directions. The grid spacing is chosen to resolve the viscous turbulence scales and the flow resolution is confirmed by examining the spanwise spectrum. Each simulation is computationally intensive because of the small time step that is required and the large number of cycles that are simulated. Periodicity is imposed in the spanwise, \( y \), direction. Zero velocity and a zero value for wall-normal density flux are imposed at the bottom. The upper boundary and the two \( x = \) constant boundaries are artificial boundaries where Rayleigh damping or a ‘sponge’ layer is used.
Figure 4.1: (a) Internal wave field visualized by a slice of $\frac{dw}{dz}$ field in x-z plane. (b) Power spectra of the baroclinic velocity $u_{\text{bar}}(x, y, z, t)$ field (semi log-scale) at the two locations A and B, marked in part (a). The time series is taken over 15 cycles.

4.2 Results

The remarkably complex wave pattern, shown in Fig. 4.1(a), includes energetic internal wave beams coincident with the slope angle, beams at steeper angles, and internal waves generated by boundary-layer turbulence that have a wide range of phase lines. The internal wave field is assessed by a power spectrum analysis of the baroclinic velocity field given by $u_{\text{bar}}(x, y, z, t) = u(x, y, z, t) - u_{\text{baro}}(x, y, t)$, where the barotropic component is the free-stream value of the velocity. Fig. 4.1(b) shows the power spectra at two locations A and B at different heights. The spectra show several temporal harmonics ($n\Omega$, $n \in \mathbb{N}$), subharmonics $\omega \in [0, \Omega)$, and inter-harmonics ($\omega_{\alpha} + n\Omega, \omega_{\alpha} \in [0, \Omega)$) having significant energy. The discrete spectral peak at the barotropic tidal frequency, $\Omega$, in Fig. 4.1(b), corresponds to an energetic linear response which, in physical space, corresponds to the strong beams
Figure 4.2: The resonant generation of the internal tide is shown by the normalized kinetic energy of the flow field $|\langle u(x, z, t) \rangle|^2/U_0^2$ along with the slope topography in black color at time $t=8T$, $8T+1/4T$ and $8T+1/2T$ in (a), (b) and (c) respectively. The arrows indicate the instantaneous velocity field. Bottom inset in each top panel figures indicates the corresponding tidal phase. In (d)-(f), vertical turbulence intensity is shown at a time corresponding to that of the figure just above it.

(upward propagation in black and downward in white) shown in Fig. 4.1(a). The spectrum at point A shows discrete peaks at the second and third harmonics as well as a significant band of waves with $\omega > N_\infty$. These super-$N$ waves are generated by high-frequency turbulence inside the boundary layer as in the studies of Taylor & Sarkar (2007a); Gayen et al. (2010a). The spectrum at point B, further away from the slope, has the same energy at the fundamental and second harmonic observed at point A, as well as an additional discrete peak at the first subharmonic. The range of super-$N$ waves in the spectrum is much smaller at point B relative to that at A, since the background does not support freely propagating waves with $\omega > N_\infty$. Resonant wave/slope interactions lead to kinetic energy as large as 15 times that of the barotropic current as shown in Fig. 4.2(a)-(c). The intense boundary layer flow becomes turbulent via the cumulative effect of convective and shear instabilities, as will be shown later. The turbulence interacts with the internal wave beams so that they buckle and twist as shown in
Fig. 4.2(b), a phenomenon not observed in recent experimental studies Gostiaux & Dauxois (2007); Zhang et al. (2008) at low Reynolds number. The r.m.s turbulent velocity can be quite large, as much as 50% of the external barotropic tidal velocity. The spatial distribution of turbulence is illustrated by the vertical component, $w_{rms}$, in Fig. 4.2(d)-(f). Turbulence is convected by the beam to the top of the slope in Fig. 4.2(e). The patches of turbulence at the top corner of the slope in Fig. 4.2(f) correspond to residual turbulence generated during the previous phase. There is a significant amount of turbulence observed even outside the boundary layer. We have identified two mechanisms, convective and shear instability, that cause transition to turbulence. The intensified velocity during the upslope motion becomes large enough to steepen the wave near the bottom boundary. Eventually, a spanwise instability develops as illustrated by Fig. 4.3.

The density isosurface which is smooth in Fig. 4.3(a) develops spanwise corrugations during the instability as shown in Fig. 4.3(b). These coherent corrugations are associated with pairs of counter-rotating streamwise vortices as shown in Fig. 4.3(c). The wavelength of the spanwise instability is intrinsic to the flow and is not observed to change upon changing the spanwise domain length. Spanwise instability has been implicated in other examples of breaking (Winters & D’Asaro, 1994; Koudella & Staquet, 2006) of free internal waves. Terms calculated to understand the energetics of the observed instability are show in Fig. 4.4 along with
isopycnals. The overturned isopycnals show the kinematically unstable structure of the wave due to wave steepening. by Fig. 4.4(b). The buoyancy flux transfers energy from the potential mode to the kinetic mode resulting in the corresponding patch of turbulent kinetic energy (TKE) observed in Fig. 4.4(d). The diagnostics show that a convective (buoyancy driven) instability is clearly present. Nevertheless, the role of shear instability cannot be overlooked. Fig. 4.4(a) shows the gradient Richardson number $Ri_g = N^2 / S^2$ with $N$ and $S$ the local values of mean buoyancy frequency and mean shear, respectively. A substantial region with low $Ri_g$ can be seen. There is significant shear production by turbulence (although smaller than the peak buoyancy flux at this time) corresponding to that region as shown by Fig. 4.4(c) as well as corresponding patches of TKE in Fig. 4.4(d). The coherent counter-rotating vortex pairs interact and, after a short time, there is breakdown into fine-scale turbulence. During subsequent cycles, turbulence levels

Figure 4.4: Contour of $Ri_g$, buoyancy flux, production and turbulent kinetic energy at $t = T + 1/4T$ in (a), (b), (c) and (d), respectively. Arrows indicate the vector field at a certain position at that instant of time. Isopycnals shown to indicate wave breaking via wave steepening.
Figure 4.5: Streamwise velocity profile as a function of vertical height and phase at three different locations \( X = 5 \text{m}, 6.5 \text{m} \) and \( 8 \text{m} \) shown in Fig. 4.1 (a) at \( t = 8T, 8T + 1/4T, 8T + 1/2T \) and \( 8T + 3/4T \) in (a), (b), (c) and (d) respectively.

increase and decrease corresponding to changes in the near-bottom velocity.

The internal tide causes the mean velocity to have a complex behavior as can be seen in Fig. 4.5. Velocity profiles are shown at 3 locations: \( x = 5 \text{ m} \) at the beginning of the slope, \( x = 6.5 \text{ m} \) at mid-slope, and \( x = 8 \text{ m} \) at the end of the slope. Fig. 4.5 shows that, near the bottom, there is a large increase of horizontal velocity with respect to the external (barotropic) current and, furthermore, this velocity shows significant variation among the 3 locations. At the mid-slope location, \( x = 6.5 \text{ m} \), the near-bottom velocity reaches its maximum value at \( \phi = 0^\circ \) and \( \phi = 180^\circ \) when the barotropic tide is minimum and, consequently, the turbulence level on most of the slope is maximum when the barotropic current is minimum.

4.3 Conclusions

Oscillating flow over a long slope (low excursion number) is examined using three-dimensional numerical simulations. In the critical case, the generated internal wave propagates at an angle identical to the slope angle. We show numerically
that, owing to the resonant wave/slope interaction in the critical case, the boundary flow is strongly intensified as in the laboratory experiment of Zhang et al. (2008) and, furthermore, at the higher Reynolds number of the numerical simulations relative to the laboratory cases, the flow becomes turbulent. Transition to turbulence involves wave steepening at the slope that is followed by a convective (buoyancy driven) instability that leads to three-dimensional fluctuations in the form of coherent spanwise vortices. In addition, there is shear instability associated with subcritical Richardson numbers in the intensified boundary flow. Recent observations Moum et al. (2002); Nash et al. (2004, 2007) find hot spots of turbulence in the vicinity of near-critical slopes, and critical reflection of incident internal waves has been suggested in the literature as a possible explanation. Our results imply that the mechanism of critical generation must also be considered as a potential explanation for enhanced turbulence near slopes.

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Chapter 5

Direct and large eddy simulations of internal tide generation at near-critical slope

The present chapter constitutes the second phase of the dissertation research and is based on the paper by Gayen & Sarkar (2011a). Numerical simulations are performed to investigate non-linear processes during internal wave generation by the oscillation of a background barotropic tide over a sloping bottom. In Chapter 4 it was found that, at the near-critical case where the slope angle is equal to the natural internal wave propagation angle there is a resonant wave response that leads to an intense boundary flow. In the present chapter, the previous work is extended by examining the energetics of turbulence in the bottom boundary layer in addition to internal wave energetics. Further more, the effect of increasing slope length, $l$, is quantified by employing an LES approach to access higher values of $l$. The present work also extends previous DNS/LES of bottom turbulence from the case of tidal flow over a non-sloping bottom (Gayen et al., 2010a; Sakamoto & Akitomo, 2009) to the situation with a sloping bottom where the baroclinic wave velocity dominates the barotropic tidal velocity. Details of numerical method were discussed in Chapter 2, section 2.3.
Table 5.1: Dimensional parameters of the simulated cases.

<table>
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<th>Case</th>
<th>$U_0$ ms$^{-1}$</th>
<th>$N_\infty^2$ s$^{-2}$</th>
<th>$\Omega^2$ s$^{-2}$</th>
<th>$\nu$ m$^2$s$^{-1}$</th>
<th>$l$ m</th>
<th>$l_x$ m</th>
<th>$l_y$ m</th>
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<td>10</td>
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<td>1.0</td>
<td>$10^{-6}$</td>
<td>12.0</td>
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</tr>
<tr>
<td>5</td>
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<td>131.6</td>
<td>1.0</td>
<td>$10^{-6}$</td>
<td>25.0</td>
<td>60</td>
<td>1</td>
<td>5.5</td>
<td>LES</td>
</tr>
</tbody>
</table>

Table 5.2: Non-dimensional parameters and grid resolution of the simulated cases. The excursion number is chosen to be small as is typical for deep water topography and the slope angle is critical.

<table>
<thead>
<tr>
<th>Case</th>
<th>$Re_s$</th>
<th>$\frac{\Omega^2}{N_\infty^2}$</th>
<th>$Pr$</th>
<th>$Ex$</th>
<th>$\epsilon$</th>
<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_z$</th>
<th>$\Delta x_{\text{min}}^+$</th>
<th>$\Delta y^+$</th>
<th>$\Delta z_{\text{min}}^+$</th>
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<tr>
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<td>0.0735</td>
<td>1.0</td>
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<td>256</td>
<td>260</td>
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</tr>
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<td>0.0357</td>
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<td>256</td>
<td>260</td>
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<td>10</td>
<td>2.0</td>
</tr>
<tr>
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<td>177</td>
<td>0.0076</td>
<td>1.0</td>
<td>0.01736</td>
<td>1.0</td>
<td>260</td>
<td>256</td>
<td>260</td>
<td>40</td>
<td>25</td>
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</tr>
<tr>
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<td>256</td>
<td>260</td>
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<td>260</td>
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</table>

5.1 Domain resolution & initialization

The computational domain length in the horizontal directions are given by $l_x$ and $l_y$. The vertical domain length is $l_z$. Five different numerical experiments are performed in a parametric study of the influence of slope length, $l$, as shown in tables 5.1-5.2. Case 1 and case 2 are DNS with $\Delta x_{\text{min}}^+ \leq 20$, $\Delta y^+ \leq 10$ and $\Delta z_{\text{min}}^+ \leq 2$ in terms of the viscous wall unit $\nu/u_r$. Cases 3-5 correspond to a resolved-LES mode with a dynamic eddy viscosity model.

The flow is statistically homogeneous in the spanwise direction and a $y$-average is used to compute the time dependent mean, $\langle A \rangle_y(x, z, t)$, as follows:

$$\langle A \rangle_y(x, z, t) = \frac{1}{l_y} \int_0^{l_y} A(x, y, z, t) dy$$  \hspace{1cm} (5.1)

Table 5.1 gives the dimensional parameters of the simulations. The nondi-
imensional parameters in Table 5.2 show that each case is critical ($\epsilon \sim 1$) and has a low excursion number. These nondimensional parameters can be compared with an oceanic example of tidal flow over sloping topography in deep water: a tidal amplitude of $U_0 = 0.025$ m/s, a tidal frequency of $\Omega = 1.4 \times 10^{-4}$ rad/s corresponding to the $M_2$ tidal period of 12.4 hrs, a low latitude with $f = 3.5 \times 10^{-5}$ rad/s, and $N = 1$ cph = $1.74 \times 10^{-3}$ rad/s. A representative slope length of 5 km leads to an excursion number of 0.04. In the simulations, $Ex$ is below this value except case 1 where $Ex = 0.0735$. The critical slope angle $\beta = 5^\circ$ is in the range, $\beta = 4^\circ - 5^\circ$, typical of the ocean. The Stokes Reynolds number, $Re_s = 151.6$, of the simulated cases is smaller than the value of $Re_s = 2975$ in the oceanic example but still sufficiently large in the case of critical slope to exhibit turbulence, owing to a substantial increase in near-bottom velocity, as will be demonstrated.

5.2 Results

5.2.1 Velocity field

The baroclinic response at a near-critical slope results in intensification of the near bottom velocity. In the following discussion, ‘across-slope’ velocity denoted by $U_{sl}$ refers to the slope-parallel velocity pointing in the $\xi$ direction.

The time evolution of across slope velocity at a point, Q, adjacent to the slope midpoint, is shown in figure 5.1 (a). As this location is very close to the slope, the mean flow velocity is dominantly parallel to the slope of the topography. Initially, the amplitude of near-bottom velocity increases rapidly with time due to resonant buoyant forcing. Nonlinear effects become important and there is transition to turbulence as discussed by Gayen & Sarkar (2010b). Shortly after, viscous dissipation becomes important and leads to amplitude saturation. After a couple of initial cycles, the flow field achieves a quasi-steady state where the variation of the velocity amplitude is slight. With increasing slope length, the resonance area increases resulting in enhancement of the baroclinic tidal velocity response in case 5 ($l = 25\text{ m}$) relative to case 1($l = 1.7\text{ m}$). However, longer slopes requires more time to adjust between the higher resonant forcing and frictional
Figure 5.1: Time evolution of (a) along slope velocity $U_{sl}$ and (b) magnitude of friction velocity $|u_\tau|$ at a particular location which is inside the boundary layer at the midpoint of the slope denoted by point Q in the inset of figure 2.4 for different cases. (c) Viscous dissipation calculated as an average over the region $\Gamma$ in figure 2.4 inset.

force before reaching a quasi-steady state. Here, case 5 requires five tidal cycles compared to three and two cycles for the smaller domains in case 3 and case 1, respectively.

Two measures of frictional effects are shown: (i) frictional velocity, $u_\tau$ at point Q in figure 5.1 (b), and (ii) viscous dissipation $\langle \varepsilon \rangle_{xyz}(t)$, calculated as an average over an area $\Gamma$ adjacent to the slope, in figure 5.1 (c). Here, the friction
Table 5.3: Comparison of the overall boundary properties between cases. Here, \( u_\tau \) is the friction velocity calculated based on the wall friction, \( \tau_w \), and \( c_f \) is the friction coefficient. The subscript \( \text{avg} \) denotes an average over at least five complete tidal cycles and \( \text{max} \) denotes the amplitude of the oscillatory wall stress, calculated as an average of the cycle peak values.

<table>
<thead>
<tr>
<th>Case</th>
<th>( u_{\tau,\text{max}} )</th>
<th>( u_{\tau,\text{avg}} )</th>
<th>( c_{f,\text{max}} )</th>
<th>( c_{f,\text{avg}} )</th>
<th>( \tau_{w,\text{max}} )</th>
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</tr>
<tr>
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<td>0.0504</td>
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<td>0.00122</td>
</tr>
<tr>
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<td>0.0130</td>
<td>0.0556</td>
<td>0.0202</td>
<td>0.00118</td>
</tr>
</tbody>
</table>

The friction velocity is calculated based on wall friction, \( \tau_w \),

\[ u_\tau = \sqrt{\frac{\tau_w}{\rho_0}}, \quad \tau_w = \rho_0 \nu \left[ \left( \frac{\partial (w)_y}{\partial x} \right)^2 + \left( \frac{\partial (u)_y}{\partial z} \right)^2 + \left( \frac{\partial (v)_y}{\partial x} \right)^2 + \left( \frac{\partial (v)_y}{\partial z} \right)^2 \right]^{1/2} \]  

(5.2)

The term,

\[ \langle \varepsilon \rangle_{xyz}(t) = \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle_{xyz} \]  

(5.3)

is viscous dissipation (more accurately pseudo-dissipation) and includes both mean and turbulent velocity. Both quantities exhibit a rapid increase during an initial transient followed by an approximately quasi-steady stage.

Time averaged quantities for different cases are calculated based on quasi steady-state data. Temporally averaged amplitude as well as time averaged frictional velocity for different cases are tabulated in table 5.3. Both statistics increases monotonically with slope length. The friction coefficients based on average wall friction and maximum wall friction velocity are calculated by

\[ c_{f,\text{avg}} = \frac{u_{\tau,\text{avg}}^2}{(1/2)U_0^2}, \quad c_{f,\text{max}} = \frac{u_{\tau,\text{max}}^2}{(1/2)U_0^2} \]  

(5.4)

and also exhibit an increase with increasing slope length. If \( U_{sl,\text{max}} \) is chosen to calculate friction factor instead of \( U_0 \), its behavior shows a reverse trend of a decrease with increasing slope length. In this context, it is worth noting that local Reynolds number increases with slope length. Therefore, the reverse trend
is similar to the well-known decrease in the value of $c_f$ with increasing Reynolds number, $Re$, in a turbulent flow.

Figure 5.2: Profiles of along slope velocity as a function of wall normal distance at the midpoint of the slope for (a) $\phi = 0^\circ$ and (b) $\phi = 180^\circ$. (c) Profiles at $\phi = 0^\circ$ replotted after normalization with the maximal velocity amplitude, $U_{sl}^{max}$ and the beam width, $l_b$.

Profiles of $U_{sl}(z_p, \phi)$, are shown in figure 5.2(a) and 5.2(b) at $\phi = 0^\circ$ and $\phi = 180^\circ$, respectively. In all cases, velocities in the bottom boundary layer are significantly larger compared to the external (barotropic) current. There is a $90^\circ$ phase lead of the baroclinic response with respect to the barotropic current. Velocity profiles become fuller with increasing slope length. The velocity profile has a spatial oscillation that is associated with internal wave modes. Figure 5.2(c) shows nondimensional velocity profiles during maximal upslope flow using the following quantities for normalization: the maximal amplitude, $U_{sl}^{max}$, over the velocity profile and the beam width, $l_b$, defined as the distance from the bottom of the profile in figure 5.2(a) up to the height where the velocity is 15% of $U_{sl}^{max}$. The profiles for the different cases tend to collapse into a single curve.

The maximal velocity amplitude, $U_{sl}^{max}$, increases with slope length as
shown in figure 5.3(a). The maximal downslope velocity amplitude is somewhat larger than the corresponding upslope value for all cases. In the right inset of figure 5.3(a), \( U_{sl}^{\text{max}} \) is shown as a function of slope length in a log-log plot. The linear fit in the inset shows that \( U_{sl}^{\text{max}} \approx l^{0.45} \). In an earlier experiment, Zhang et al. (2008) found a different scaling law of \( U_{sl}^{\text{max}} \approx l^{2/3} \) in the regime of laminar flow. Zhang et al. (2008) were able to provide theoretical justification based on an analytical result obtained by Dauxois & Young (1999) for a laminar oscillatory boundary layer,

\[
\frac{U_{sl}^{\text{max}}}{U_0} \sim \frac{\Omega}{N_\infty} \left[ \frac{\sqrt{N_\infty^2 - \Omega^2}}{\nu} \right]^{1/3} l^{2/3} \quad (5.5)
\]

As a first approximation, the analysis can be extended to turbulent flow by replacing \( \nu \) in Eq. (5.5) by \( \nu_{\text{tot}} = \nu + \nu_T \) and \( \nu_T = -\langle u'w' \rangle/d\langle u \rangle/dz \) taken to be independent of \( z \). The height-averaged value of \( \nu_T \) is calculated at midslope from the simulation data and plotted in figure 5.3(b) to find the dependence of \( \nu_T \) on slope length. In the right inset of figure 5.3(b), \( \nu_T/\nu \) is replotted as function of slope length in log-log scale and the linear fit suggests that \( \nu_T \) scales as \( l^{0.55} \). Therefore, (5.5) extended to the turbulent regime becomes,

\[
\frac{U_{sl}^{\text{max}}}{U_0} \sim \frac{\Omega}{N_\infty} \left[ \frac{\sqrt{N_\infty^2 - \Omega^2}}{\nu_T + \nu} \right]^{1/3} l^{2/3} \quad \Rightarrow \quad \frac{U_{sl}^{\text{max}}}{U_0} \sim \frac{\Omega}{N_\infty} \left[ \frac{\sqrt{N_\infty^2 - \Omega^2}}{\nu_T} \right]^{1/3} l^{2/3} \quad \text{as } \nu_T > 10\nu \\
\Rightarrow \quad \frac{U_{sl}^{\text{max}}}{U_0} \sim \frac{\Omega}{N_\infty} \left[ \frac{N_\infty^2 - \Omega^2}{\nu_T} \right]^{1/6} l^{0.48} \quad (5.6)
\]

The dependence on \( l \) in (5.6) is quite close to the relation, \( U_{sl}^{\text{max}} \sim l^{0.45} \), inferred previously by a direct fit to the simulation data.

The beam width, \( l_b \), that is plotted in figure 5.4 exhibits a monotone increase with increasing slope length. The beam width in the case of upslope flow is larger than in downslope flow for all five cases. The inset in the figure shows power law fits to the dependence of beam width on slope length.

The full velocity field computed in the simulation can be decomposed as

\[
u(x, t) = \nu_{ba}(x, z, t) + \nu_{bc}(x, t), \quad (5.7)\]
Figure 5.3: (a) Amplitude of the along slope velocity as function of slope length. Right hand inset is a replot of the data in log-log scale along with a linear least squares fit. (b) Turbulent viscosity normalized by molecular viscosity is shown as function of slope length. Inset is a replot in log-log scale.

where $u_{ba}$ and $u_{bc}(x, t)$ are respectively barotropic and baroclinic responses. The decomposition of the velocity is based on Nash et al. (2004, 2006) as further discussed in Appendix A. The baroclinic velocity $u_{bc}(x, t)$, is further decomposed into a mean wave velocity $\langle u_{bc} \rangle_y(x, z, t)$ obtained by a spanwise average, and a turbulent fluctuation $u'(x, t)$ with respect to the mean.

The baroclinic streamwise velocity can be approximated by a sinusoidal form

$$\langle u_{bc} \rangle_y(x, z, t) = u_0(x, z) \sin(\Omega t + \Delta \phi_u(x, z))$$  \hspace{1cm} (5.8)

with a space dependent amplitude $u_0(x, z)$ and space dependent phase, $\Delta \phi_u(x, z)$, with respect to the background barotropic tidal flow. The parameters, $u_0(x, z)$ and $\Delta \phi_u(x, z)$, determined by a least squares method from the time series data taken over 6 cycles, are shown in figure 5.5 (a) and 5.5 (b), respectively, for case 4. The amplitude distribution illustrates internal wave beam strength and spatial structure. The region with high amplitude adjacent to the slope corresponds to a narrow and strong internal wave beam. The amplitude increases to $\sim 0.81 \text{ms}^{-1}$ at the slope and remains relatively constant at the slope. When the beam travels away from its generation zone (along the slope) into the fluid, it gradually widens and weakens. At a particular position in the center of the beam, 1m away from the right
Figure 5.4: Jet width, $l_b$, as function of horizontal slope length is shown for both upslope and downslope boundary flow. Inset shows data replotted in log-log scale along with its linear least squares fit.

edge of topography slope, the amplitude decreases to $0.45 \text{ m s}^{-1}$. The spreading of the wave beam is caused by viscous diffusion and also by dispersion since waves with smaller wave length and, therefore, lower phase velocity may be locally dissipated leaving larger wavelength modes in the propagating wave. Spreading of the wave beam is also observed in observations at Kaena Ridge, Hawaii by Nash et al. (2006) and from a model topography in a laboratory experiment by Echeverri et al. (2009).

Figure 5.5 (b) shows that the oscillatory velocity exhibits significant spatial variability in the phase, $\phi$, with respect to the barotropic tidal velocity. There is phase variation along the sloping boundary associated with internal wave propagation. As will be shown later in section 5.2.3, the internal wave flux associated with the beam is such that the energy propagation is outward from the sloping topography. There is also an area of recirculation between the lower edge of the beam and the flat top of the topography. Other simulated cases also show similar spatial variability of phase and amplitude.

5.2.2 Thermal bore

The baroclinic wave response is intensified at a region of critical slope and leads to energy concentration into a beam as discussed in the previous section. An
upslope moving tidal bore or bolus may also form as observed by Lim et al. (2010) in laboratory experiments of internal tide generation. In the problem of reflection at a critical slope, upslope propagation of a thermal front has been observed in the laboratory (Thorpe (1992)), and upslope propagation of a bolus and on to the shelf has been identified in simulations (Venayagamoorthy & Fringer (2007)). We find in the present simulations of the generation problem that an upslope bore results as part of the baroclinic response. The thermal front has sufficient energy to move rightward as a gravity current along the top horizontal portion of the topography into the stable stratified region. The on-slope propagation of the thermal front is shown in figure 5.6 using a three dimensional visualization of a density iso-surface. The thermal front is unstable and undergoes spanwise corrugations as shown in figure 5.6(b). This is similar to the classical lobe and cleft instability found in the case of gravity currents. Finally, isolated fluid patches detach from the on-slope propagating turbulent bore/bolus as shown by figure 5.6(c,d) and dissipate locally due to the combined effect of turbulent and molecular diffusion.
Figure 5.6: Formation and propagation of thermal bore: shown at four different time instants. The velocity vectors, taken along a vertical line at a point R in the inset of figure 2.4, are shown by arrows. Here, the color scale refers to contour value of total density with an arbitrary reference value.

5.2.3 Wave Energetics

A complex wave pattern containing an energetic beam and internal waves with a wide range of phase angles emerges as shown by Gayen & Sarkar (2010b).
Here, we quantify the energy transport including the contribution of higher harmonics relative to the fundamental, i.e., the frequency of the barotropic tidal forcing.

The baroclinic streamwise velocity field over an area containing the slope of the topography is subjected to spectral analysis. In figure 5.7(a), the power spectrum at a location Q, midway on the slope, and close to the bottom, is shown for cases 1 and 3. The spectra show energy at several temporal harmonics \((n\Omega, n \in \mathbb{N})\), subharmonics \(\omega \in [0, \Omega)\) and interharmonics \((\omega_\alpha + n\Omega, \omega_\alpha \in [0, \Omega))\). The discrete spectral peak at the barotropic tidal frequency, \(\Omega\), in figure 5.7a, corresponds to an energetic linear response which, in physical space, corresponds to the strong beam parallel to the slope in figure 1.6. The spectrum shows discrete peaks at the second and third harmonics as well as significant strength at frequencies \(\omega > N_\infty\). The energy content at harmonics, interharmonics and subharmonics is higher for larger slope length. Significantly larger amplitude of the continuous part of the spectrum (especially at high frequencies) is observed at the longer slope length of case 3 relative to case 1. The continuous spectrum is associated with nonlinear interactions and, at high frequencies, reflects the broadband multiscale nature of
turbulence. With increasing slope length, turbulence (quantified in the the next section) is enhanced due to the increase of boundary velocity. Therefore, the relative magnitude of the continuous spectrum increases with increasing slope length.

The power spectra averaged over a vertical line at location Q, defined in figure 2.4 inset, are shown in figure 5.7(b) for both cases. The line extends from bottom of the wall to the height of the control area \( \Gamma \) so that the averaging procedure includes points both inside and outside the boundary layer. The discrete peaks at the higher harmonics and the interharmonics are present in the averaged spectrum too.

To illustrate the shift from turbulence inside the boundary layer to internal waves propagating outside, we choose three locations (a)-(c) at different heights on a vertical line at point Q midway on the slope. Position a (height with respect to the bottom, \( z^* = 0.01 \text{ m} \)) is well inside the boundary layer, position b (\( z^* = 0.15 \text{ m} \)) is outside the boundary layer and position c (\( z^* = 0.4 \text{ m} \)) is well outside. Power spectra of the baroclinic velocity field at those locations are shown in figure 5.8a. For all positions, the global spectral peak occurs at the fundamental tidal frequency. Due to strong turbulence activity, position a located inside the boundary layer has a continuous spectrum with significant magnitude that obscures discrete peaks at harmonics of the tidal frequency. Positions b and c are external to the
boundary layer and are locations with little turbulence. Nevertheless, there is a significant continuous spectrum. Boundary layer turbulence on a non-sloping bottom generates an externally propagating wave field without discrete peaks as shown in our previous work on a steady boundary layer (Taylor & Sarkar, 2007a) and an oscillating boundary layer (Gayen et al., 2010a). Such turbulence generated waves and, in addition, wave-wave interactions of the topography generated waves lead to a continuous spectrum at points b and c in the present problem. It is worth noting that there is a sharp decay in the amplitude of frequencies larger than the buoyancy frequency, $N_\infty$, at points b and c since the background does not support freely propagating waves with $\omega > N_\infty$. Also, at locations b and c the peaks at the tidal harmonics increase in strength relative to the continuous part of the spectrum. This is likely because the continuous spectrum is associated with smaller length scale waves that suffer higher viscous dissipation. Profiles of buoyancy flux, pressure transport and turbulent transport are plotted in figure 5.8b at maximal upslope flow. The location of observation points b and c are shown on the same figure by dashed horizontal lines. At these locations, buoyancy flux and pressure transport dominate. Turbulent production and dissipation as well as

**Figure 5.9:** Energy distribution over harmonics 1-3 at a point on the slope and in the boundary layer is shown with absolute units ($m^2s^{-2}$) in the bar chart of (a). Bar 1 in dark black corresponds to the fundamental at the tidal frequency. Bars 2 and 3 shown in dark gray and light gray, respectively, correspond to the second and third harmonics. Part (b) is same as (a) except that all quantities are in relative units.
turbulent transport are insignificant confirming that there is little contribution by turbulence to fluctuations at points $b$ and $c$.

A comparative study of energy distribution among the harmonics is performed for different slope lengths. In figure 5.9(a), the line averaged energy, $\langle Y \rangle_z$, is shown for the first three harmonics. For all cases, the fundamental dominates. The energy content at the higher harmonics decreases with increasing frequency. The energy at all harmonics increases with the slope length. The relative importance of the energy at higher harmonics with respect to the fundamental is shown as a function of slope length in figure 5.9(b). For each case, the fundamental is assigned as 100% and the energy in other harmonics is expressed as a relative percentage.

Energy contained in higher harmonics relative to the fundamental increases with increasing slope length. For example, the energy of the second harmonic in case 1 with slope length 1.7 m is 10% of the energy at the fundamental, a value that increases to 15% for the largest slope length of 25 m in case 5.

![Figure 5.10](image)

**Figure 5.10**: Normalized area integrated kinetic energy density, $\langle \mathcal{E}_k \rangle$ and potential energy density, $\langle \mathcal{E}_p \rangle$, are shown as function of slope length. Normalization factor is $(1/2)\rho_0 U_0^2 \Gamma$.

The energy density, $E_{IW}$, in the internal wave field is decomposed into
kinetic energy, $E_k$, and potential energy, $E_p$, as follows

$$E_k = \frac{1}{2}\rho_0 (\langle u_{bc}\rangle_y^2 + \langle v_{bc}\rangle_y^2 + \langle w_{bc}\rangle_y^2),$$

(5.9)

$$E_p = \frac{g^2 \langle \rho_{bc}\rangle_y^2}{2\rho_0 N^2_\infty}.$$  

(5.10)

We choose an area $\Gamma$ containing the topography and a time record of length $nT$ where $T$ is the time period of a tidal cycle and $n = 5$. The energies in this space-time section are normalized as follows,

$$\langle E_k \rangle = \frac{1}{nT U_0^2} \int_{nT} \int_{\Gamma} (\langle u_{bc}\rangle_y^2 + \langle v_{bc}\rangle_y^2 + \langle w_{bc}\rangle_y^2) dAdt$$

(5.11)

$$\langle E_p \rangle = \frac{1}{nT U_0^2} \int_{nT} \int_{\Gamma} \frac{g^2 \langle \rho_{bc}\rangle_y^2}{\rho_0^2 N^2_\infty} dAdt.$$  

(5.12)

The values of $\langle E_k \rangle$ and $\langle E_p \rangle$ for different cases are shown in figure 5.10. In most cases, the kinetic energy is greater than potential energy. In a linear plane wave, energy is equipartitioned between kinetic and potential modes. In more general situations, there can be deviations from linear theory as in the viscous, nonlinear waves simulated in the present work.

Streamwise and vertical wave fluxes are defined by $f^\parallel = \langle p_{bc} u_{bc}\rangle_y$ and $f^\perp = \langle p_{bc} w_{bc}\rangle_y$, respectively. Here $p_{bc}$ is the pressure anomaly calculated in Appendix A. Now the spanwise averaged total energy equation for a plane internal wave can be written in linearized form,

$$\frac{\partial (E_k + E_p)}{\partial t} + U(x) \sin(\Omega t) \frac{\partial (E_k + E_p)}{\partial x} = - \frac{\partial f^\parallel}{\partial x} - \frac{\partial f^\perp}{\partial z} + q(x, z)$$  

(5.13)

where $q(x, z)$ is a source (sink) term. Eq. (5.13) is integrated over the area $\Gamma$ to give

$$\frac{\partial}{\partial t} \int_{\Gamma} [E_k + E_p] dA = - \int_{\Gamma} \left[ U(x) \sin(\Omega t) \frac{\partial E_k + E_p}{\partial x} \right] dA$$

$$- \left[ \int_{h(x_2)}^{z_2} f^\parallel|_{x_2} - \int_{h(x_1)}^{z_2} f^\parallel|_{x_1} \right] dz - \int_{x_1}^{x_2} f^\perp|_{z_2} dx$$

$$+ \int_{\Gamma} q(x, z) dA.$$  

(5.14)

The integration area $\Gamma$ is bounded at the top by the line $z = z_2$, at the left by
Figure 5.11: Spatial distribution of cycle-averaged streamwise IW flux, \( \langle f || \rangle = \langle p_{bc} u_{bc} \rangle_{y,t} \ (W \ m^{-2}) \) is shown in x-z plane for case 5. Time averaging is done over the final 5 cycles of the simulation. Here, the vertical dashed lines given by \( x = x_1 \) and \( x = x_2 \) from the left, indicate the vertical integration boundaries used for obtaining the net streamwise flux.

\( x = x_1 \), at the right by \( x = x_2 \), and \( z = h(x) \) at the bottom. After averaging (5.14) over a time span of length \( nT \), where \( n=5 \), transient terms vanish and it gives

\[
0 = \langle G \rangle - \langle F || \rangle - \langle F \perp \rangle + \langle Q \rangle ,
\]

(5.15)

where, the cycle averaged net (sum of the outward fluxes at the left and right boundaries of the integration area \( \Gamma \)) horizontal flux, \( \langle F || \rangle \), and net (the outward fluxes at top boundary of the integration area \( \Gamma \)) vertical flux, \( \langle F \perp \rangle \), are defined by

\[
\langle F || \rangle = \frac{1}{nT} \int_{nT}^{x_2} \int_{h(x_2)}^{h(x_1)} f ||_{x_2} - f ||_{x_1} \, dz \, dt
\]

(5.16)

\[
\langle F \perp \rangle = \frac{1}{nT} \int_{nT}^{x_2} \int_{x_1}^{x_2} f \perp_{z} \, dx \, dt
\]

(5.17)

After normalizing by \( (\pi/4)\rho_0 U_0^2 h^2 N_\infty \) (a quantity that appears in linear theory of conversion to internal tides), Eq. (5.15) is rewritten as

\[
0 = \langle G \rangle - \langle F || \rangle - \langle F \perp \rangle + \langle Q \rangle .
\]

(5.18)

Here, normalized values are defined as below:

\[
\langle G \rangle = \frac{4\langle G \rangle}{\pi \rho_0 U_0^2 h^2 N_\infty}, \langle F || \rangle = \frac{4\langle F || \rangle}{\pi \rho_0 U_0^2 h^2 N_\infty}, \langle F \perp \rangle = \frac{4\langle F \perp \rangle}{\pi \rho_0 U_0^2 h^2 N_\infty}, \langle Q \rangle = \frac{4\langle Q \rangle}{\pi \rho_0 U_0^2 h^2 N_\infty}
\]

(5.19)
Figure 5.12: (a) Cycle averaged integrated energy flux in streamwise direction, $\langle F \parallel \rangle (W \ m^{-1})$, and vertical direction, $\langle F \perp \rangle (W \ m^{-1})$, as a function of slope length. (b) Same as for normalized energy fluxes. Here, the normalization factor is $(\pi/4)\rho_0 U_0^2 h^2 N_\infty$ where $h$ is the height of the sloping topography. Time averaging is done over the final 5 cycles of the simulation.

Note that, in the normalization factor (Pétrélis et al., 2006), $h$ is the height of the topography, $U_0$ is the barotropic tidal amplitude and $N_\infty$ is the background value of the buoyancy frequency.

After a couple of initial cycles, the flux reaches a quasi-steady state with an approximately constant amplitude, similar to the results found for other flow statistics. The spatial distribution of streamwise flux, $\langle p_{bc} u_{bc} \rangle_{y,t}$ averaged over five tidal cycles, is shown for case 5 in figure 5.11. Positive values of the flux correspond to rightward propagating energy and vice-versa. In order to evaluate the net streamwise flux carried by the beam, the flux is integrated over the vertical dashed lines in the figure. The net energy flux is outward at these vertical boundaries; rightward at the right boundary and leftward at the left boundary. Above and to the right of the slope, the internal wave flux is strong and concentrated into a beam. At the lower left portion, the beam widens and the internal wave flux weakens owing to dissipation associated with interaction with the bottom flat portion. Consequently, the magnitude of the rightward streamwise flux, $\int_{h(x_2)}^{z_2} F\parallel_{|x_2|} \, dz$ is substantially larger relative to the magnitude of the leftward streamwise flux $\int_{h(x_1)}^{z_2} F\parallel_{|x_1|} \, dz$. 


A comparative study is done among the five cases based upon the net (sum of the outward fluxes at the left and right boundaries) horizontal flux, \( \langle F^\parallel \rangle \), and net (the outward flux at top boundary) vertical flux \( \langle F^\perp \rangle \), and the results plotted in figure 5.12(a). The horizontal energy flux, \( \langle F^\parallel \rangle \), dominates the vertical component. Both energy fluxes increase with increasing slope length. Linear theory of internal tides leads to the result that the internal wave flux is proportional to \((\pi/4) \rho_0 U_0^2 h^2 N_\infty \) with \( h \) the slope height. Therefore, the energy flux increases with increasing slope length. The net normalized horizontal flux (\( \langle F^\parallel \rangle \)), that uses the linear scaling for normalization, is plotted as function of slope length in figure 5.12(b). The normalized value initially decreases and then seems to saturate at higher values of slope length. The initial decrease occurs because conversion to turbulence increases as will be quantified in the next section. The saturation occurs because eventually, for high enough Reynolds number, a constant fraction of the linear estimate of the internal wave flux is lost to local turbulence and locally trapped waves. This explanation assumes that nonlinearity and turbulence do not fundamentally alter the scaling law of internal wave generation as deduced from linear theory. The vertical energy flux, \( \langle F^\perp \rangle \), which is small compared to \( \langle F^\parallel \rangle \), shows a similar dependence on slope length.

### 5.2.4 Turbulence energetics

Turbulence statistics are computed using spanwise averages. The turbulent kinetic energy, \( K = 1/2 \langle u'_i u'_i \rangle_y \) also denoted by TKE, represents the energy in fluctuations with respect to the mean velocity and satisfies the following evolution equation:

\[
\frac{\partial K}{\partial t} + \langle u \rangle_y \frac{\partial K}{\partial x} + \langle w \rangle_y \frac{\partial K}{\partial z} = P - \varepsilon + B - \frac{\partial T_x}{\partial x} - \frac{\partial T_z}{\partial z} \tag{5.20}
\]

Here, \( \partial T_x / \partial x \) and \( \partial T_z / \partial z \) that correspond to the transport of TKE consist of pressure transport, turbulent transport, viscous transport and subgrid scale (SGS) transport,

\[
T_x \equiv \frac{1}{\rho_0} \langle p' u'_i \rangle_y + \frac{1}{2} \langle u'_i u'_i u'_i \rangle_y - \nu \frac{\partial K}{\partial x} + \langle \tau'_{i1} u'_i \rangle_y,
\]

\[
T_z \equiv \frac{1}{\rho_0} \langle p' w' \rangle_y + \frac{1}{2} \langle u'_i u'_i w' \rangle_y - \nu \frac{\partial K}{\partial z} + \langle \tau'_{i3} u'_i \rangle_y.
\]
Figure 5.13: Logarithmic profiles of production, Log\(_{10}|P|\), and dissipation, Log\(_{10}|\varepsilon|\), as function of height above the bottom at location Q in figure 2.4 inset. Parts (a)-(b) correspond to \(\phi = 0^\circ\) when the upslope bottom flow peaks while parts (c) and (d) correspond to \(\phi = 180^\circ\) when the downslope bottom flow peaks.

\(P\) is the production term defined as

\[ P \equiv -\langle u'_i u'_j \rangle_y \langle S_{ij} \rangle_y - \langle \tau_{ij} \rangle_y \langle S_{ij} \rangle_y. \]

where the last term is the SGS production. The turbulent dissipation rate, \(\varepsilon\), is defined as the sum of the resolved and SGS components:

\[ \varepsilon \equiv \nu \left\langle \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_j} \right\rangle_y - \langle \tau_{ij} S_{ij} \rangle_y. \]

Finally, \(B\) is the buoyancy flux defined as

\[ B \equiv -\frac{g}{\rho_0} \langle \rho' w' \rangle_y. \]

Profiles of turbulent production and dissipation are plotted at location Q on the slope as a function of height above the bottom in figure 5.13. Results for cases 1, 3 and 5 corresponding to slope lengths of 1.7 m, 7.2 m, and 25.0 m are shown at two phases: \(\phi = 0^\circ\), corresponding to peak value of upward bottom velocity and \(\phi = 180^\circ\) corresponding to peak value of downward velocity. At \(\phi = 0^\circ\), the turbulent production shown in figure 5.13 (a) peaks near the bottom.
boundary while the turbulent dissipation shown in figure 5.13 (b) peaks at the bottom boundary as is typical for boundary layer turbulence. Both, production and dissipation increase with slope length. The turbulent production and dissipation at $\phi = 180^\circ$, shown in figure 5.13 (c)-(d) exhibit profiles that have faster decay rates relative to those at $\phi = 0^\circ$. The peak dissipation at the bottom boundary during peak downslope flow is about an order of magnitude smaller than the corresponding value at $\phi = 0^\circ$. This occurs despite the fact that mean shear is higher during the downslope motion. In the present problem, shear alone does not play the key role for determining the turbulence intensity. During the downslope flow, stratification increases and acts opposite to the shear effect by suppressing near wall turbulence.

The spatial distribution of production and dissipation is shown in figure 5.14 (a,b), respectively, for case 4 at a time that corresponds to $\phi = 0^\circ$. Although the largest values of production are found in the the tidal beam and the associated boundary layer at the slope, a significant amount of production is also found outside
of the boundary layer where it is associated with other temporal harmonics that propagate out at angles that are larger with respect to the beam angle. Unlike production, the maximum value of dissipation ($\sim 1 \text{ m}^2\text{s}^{-3}$) is limited to the slope. Nevertheless, there are some patches of dissipation on the flat topography adjacent to the slope.

Profiles of turbulent kinetic energy, $K$, at a location midway up the slope are shown in figure 5.15(a) at a time which corresponds to $\phi \sim 0^\circ$. TKE reaches its maximum value close to the bottom boundary and then decreases with increasing height from the bottom. With increasing length of slope, the magnitude of $K$ is enhanced. The location of peak $K$ also shifts upward with increasing slope length. Figure 5.15 (b) shows profiles of $K$ at the same location and time, plotted after normalization with the peak mean velocity $U_{sl,max}$ and the beam thickness, $l_b$. This normalization significantly decreases variation between cases.

Figure 5.16 (b) shows the temporal evolution of each term in the $K$-budget along with turbulent kinetic energy for case 3 at the midpoint of slope and at a location close to the bottom. In order to illustrate the phasing of these quantities, the evolution of streamwise velocity at the same location is given in figure 5.16 (a). The production increases shortly after the onset of the upslope boundary flow to enhance the turbulent kinetic energy, clear visible in the curve corresponding
to TKE during this phase and reaches its maximum value soon after the maximal upward boundary flow. Peak production is followed closely by peak dissipation that occurs during the decelerating phase of upslope flow. Both turbulent production and dissipation have much smaller values during downslope flow. The buoyancy flux and transport terms are significant and correspond to turbulence-generated internal waves propagating away from the bottom as was discussed extensively by Gayen et al. (2010a) in the context of an oscillating boundary layer over a non-sloping bottom.

Figure 5.16: Time evolution in case 3: (a) Stream velocity is shown as function of time at a location midpoint of a slope and 1 cm above the bottom. Temporal evolution of turbulent kinetic energy, production, dissipation, buoyancy flux and modified transport are shown in (b) and (c) at midpoint of the slope at locations 1 cm and 2 cm above the bottom. Units for TKE is $m^2 s^{-2}$, whereas units of other terms in energy budget are $m^2 s^{-3}$. 
The behavior of production, dissipation, buoyancy flux and transport term at a location twice as far from the bottom surface is illustrated as a function of time in figure 5.16 (c). At this location, the dissipation is insignificant while production, buoyancy flux and transport dominate. The turbulent production as well as the buoyancy flux serve to increases the TKE during the decelerating phase of upslope flow. Higher up in the boundary layer, turbulent kinetic energy, production and dissipation lag in phase with respect to near-boundary values, similar to the case of a non-sloping bottom reported by Gayen et al. (2010a).

![Figure 5.17](image)

**Figure 5.17**: (a) Normalized value of integrated TKE, $\langle K \rangle$, as a function of slope length. (b) Normalized values of production $\langle P \rangle$, dissipation $\langle D \rangle$ and buoyancy flux $\langle B \rangle$. Here, $\langle K \rangle$ is normalized by $(1/2) \rho_0 U_0^2\Gamma$, whereas the normalization for $\langle P \rangle$, $\langle D \rangle$ and $\langle B \rangle$ is $(\pi/4) \rho_0 U_0^2 h^2 N_\infty$.

Time-averaged and area-integrated TKE, dissipation, production and buoyancy flux are denoted by $\langle K \rangle$, $\langle D \rangle$, $\langle B \rangle$ and $\langle P \rangle$,

\[
\langle K \rangle = \frac{\rho_0}{2nT} \int_{-nT}^{nT} \int_{\Gamma} \left[ \langle u'^2 \rangle_y + \langle v'^2 \rangle_y + \langle w'^2 \rangle_y \right] dAdt \tag{5.21}
\]

\[
\langle P \rangle = -\frac{\rho_0}{nT} \int_{-nT}^{nT} \int_{\Gamma} \left[ \langle u'_i u'_j \rangle_y \frac{\partial \langle u_i \rangle_y}{\partial x_j} + \langle \tau_{ij} \rangle_y \langle S_{ij} \rangle_y \right] dAdt \tag{5.22}
\]

\[
\langle D \rangle = \frac{\rho_0}{nT} \int_{-nT}^{nT} \int_{\Gamma} \left[ \nu \left( \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_i} \right)_y - \langle \tau_{ij} S_{ij} \rangle_y \right] dAdt \tag{5.23}
\]

\[
\langle B \rangle = -\frac{\rho_0}{nT} \int_{-nT}^{nT} \int_{\Gamma} \left[ \frac{g}{\rho_0} \langle \rho' w' \rangle_y \right] dAdt \tag{5.24}
\]
Normalized values are defined as below:

\[
\langle K \rangle = \frac{2\langle K \rangle}{\rho_0 U_0^2 \Gamma}, \quad \langle P \rangle = \frac{4\langle P \rangle}{\pi \rho_0 U_0^2 h^2 N_\infty}, \quad \langle D \rangle = \frac{4\langle D \rangle}{\pi \rho_0 U_0^2 h^2 N_\infty}, \quad \langle B \rangle = \frac{4\langle B \rangle}{\pi \rho_0 U_0^2 h^2 N_\infty}.
\] (5.25)

The normalization of turbulent kinetic energy, \( \langle K \rangle \), is the same as that of the kinetic energy density, \( \langle E_k \rangle \), discussed previously while that for \( \langle P \rangle \) and \( \langle D \rangle \) is the same as was used for the internal wave flux, \( \langle F \rangle \). Figure 5.17 shows that \( \langle K \rangle \), production \( \langle P \rangle \), dissipation \( \langle D \rangle \) and buoyancy production \( \langle B \rangle \) increase with increasing slope length. The curves also become flatter with increasing slope length. The integrated turbulent production is larger than the integrated dissipation. The integrated buoyancy flux is also significant. The data shown in figure 5.17(b) can be compared to the streamwise internal wave flux, \( \langle F^\parallel \rangle \), that was discussed earlier in figure 5.12. For instance, in case 5 with slope length of 25 m, the integrated turbulent production is approximately 18 % and the integrated turbulent dissipation is approximately 12 % of the streamwise wave flux.

### 5.3 Conclusions

Direct numerical simulation (DNS) and large eddy simulation (LES) approaches have been used to investigate the dynamics of a stratified flow over a sloping bottom under an oscillating tidal flow. The three-dimensional, unsteady simulations are performed in coordinates that conform to the bottom topography. A dynamic eddy viscosity model is used for LES and the near-bottom turbulence is resolved at the moderate Reynolds number considered here. The background stratification is such that the critical slope angle is 5°; a small value as is typical in the ocean. The slope length, \( l \), is varied between 1.7 m and 25 m to quantify its influence. The excursion number is chosen to be small in all cases as is typical for bottom topography in deep water.

Resonant generation of internal waves at the near-critical slope simulated here leads to an internal wave beam (where the kinetic energy and wave energy flux are concentrated) that leaves the slope and to a boundary layer under the intensified wave velocity. There is transition to turbulence in the boundary flow.
leading to enhanced viscous dissipation that regularizes the resonant response so that an oscillating boundary layer in quasi-steady state forms on the slope. The velocity profile, with a strong near-bottom jet (corresponding to the internal wave beam) and spatial oscillations as a function of vertical coordinate, is qualitatively different from the oscillating boundary layer on a non-sloping bottom. The peak velocity is found to increase as approximately $l^{0.45}$ in the turbulent regime which is different from that in the laminar regime due to a length dependence of the effective turbulent viscosity. An analytical expression that incorporates the observed length dependence of turbulent viscosity leads to a scaling law for the peak bottom velocity which is in good agreement with the scaling observed in the simulations. The width of the beam also increases according to a power law as a function of slope length. It is worth noting that the peak velocity and width of the beam cannot increase indefinitely because the region with uninterrupted critical slope is finite in realistic topography. The baroclinic velocity shows significant temporal and spatial variability. The internal wave response leads to an upslope moving tidal front which propagates shoreward as a turbulent gravity current.

The velocity spectrum on the slope and inside the boundary layer shows discrete peaks at the fundamental, harmonics, subharmonic and interharmonics that correspond to topography generated waves superposed on a continuous spectrum associated with broadband turbulence. The energy content in higher harmonics relative to that in the fundamental increases with slope length so that the second harmonic contains about 15% of the energy in the fundamental when the slope length is $l = 25$ m, the largest value in the present simulations. The velocity spectrum at points outside the boundary layer and with little turbulence show more prominent discrete peaks as well as a continuous spectrum. Turbulence generated waves and wave-wave interactions among topographic waves are responsible for the continuous spectrum at these points.

Turbulent kinetic energy and dissipation rate increase substantially and the locations of their peaks move upward with increasing slope length, $l$. Normalization of the profiles with the beam width and the maximum slope velocity substantially reduces the variation between cases. The integrated normalized value of turbulent
production is larger than the corresponding value of turbulent dissipation. In the case with the longest slope of \( l = 25 \, \text{m} \), the integrated production and turbulent dissipation are found to be 18 % and 12 %, respectively, of the energy flux associated with the internal tide. The profiles of turbulent production and dissipation depend on the phase of the internal tide and are significantly fuller than profiles observed in the boundary layer below an oscillating current without internal wave generation.

The dimensional baroclinic energy flux increases with slope length. The nondimensional baroclinic energy flux, \( \langle F_{\parallel} \rangle \), has also been computed with a normalization factor, \( (\pi/4) \rho_0 U_0^2 h^2 N_\infty \) taken from linear theory. The value of \( \langle F_{\parallel} \rangle \) shows a maximum value of 0.4 (or 40%) for the smallest slope, \( l = 1.7 \, \text{m} \), decreases with increasing \( l \), and appears to approach a saturation value, \( \langle F_{\parallel} \rangle \simeq 0.25 \). The substantial increase of conversion to turbulence with increasing slope length (turbulent production as high as 18 % of the internal wave flux when \( l = 25 \, \text{m} \)) is a likely contributor to the decrease in the normalized flux, \( \langle F_{\parallel} \rangle \), with increasing slope length and its potential saturation.

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Chapter 6

Boundary mixing by density overturns in an internal tidal beam

The present chapter constitutes the third phase of the dissertation and is based on Gayen & Sarkar (2011b). Three-dimensional large eddy simulation (LES) is performed with the computational model of section 2.2 to investigate near-bottom mixing processes in an internal wave beam over a critical slope. During the generation of internal tides at critical slopes, high vertical wave number modes are formed that combine to give internal tidal beams. Recently, laboratory experiments (Lim et al., 2010) and our numerical experiments, namely, DNS at slope length $l \sim 1\, \text{m}$ presented in Chapter 4 (also Gayen & Sarkar (2010b)) and LES at slope length upto 25 m presented in Chapter 5 (also Gayen & Sarkar (2011a)) have shown evidence of turbulence in tidal beams over critical slope topography. However, the beam was at laboratory scale where beam width was of the order of 0.2 m and did not allow the large separation in scales between the boundary layer and the oscillating core of the beam that occurs in the ocean. The main focus of the present chapter is to numerically model a near-bottom beam with a larger, more realistic width at geophysical scale. As will be seen, these results also lead to a numerical representation of the phase dependent mixing observed by Aucan et al. (2006) at a deep oceanic flank of the Kaena Ridge and a potential explanations.
6.1 Selection of the simulated case

The test domain, excluding the sponge region, consists of a rectangular box of 30 m length, 160 m height and 10 m width whose bottom boundary is coincident with the slope topography as shown in Figure 8.1(a). The grid size in the test domain is $128 \times 620 \times 64$ in the $x_r$, $z_r$ and $y_r$ directions, respectively, with stretching in $z_r$ direction. The grid spacing ($\Delta x_r = 0.234$ m, $\Delta y_r = 0.157$ m, $\Delta z_{r,\text{min}} = 0.0037$ m, $\Delta z_{r,\text{max}} = 3.5$ m) is sufficient for LES as is confirmed by examining the spanwise and the streamwise spectra as shown in Figure 6.1. The sponge region, with damping to the background state, contains 15 points, extends from 160 m to 200 m, and has a maximum $\Delta z_r = 5.59$ m.

Gayen & Sarkar (2011a) found that the beam width, $l_b$, and velocity amplitude, $U_b$, increased with increasing slope length, following a power-law dependence. The beam in the present simulation is initialized by taking advantage of the approximately self-similar velocity structure, when normalized with $l_b$ and $U_b$, that was previously found. At $t = 0$, there are neither velocity nor density fluctuations. In Gayen & Sarkar (2010b), an oscillating barotropic tide with constant amplitude was prescribed on a near-critical slope, and the barotropic-to-baroclinic energy conversion compensated for the losses due to turbulent dissipation leading to a boundary jet with constant amplitude, $U_b$. In the current simulation, the amplitude, $U_b$, is maintained by adding a relaxation term, $-\sigma_f(z)[u_r(x,t) - u_{r,f}(z)\cos(\Omega t)]$, to the right-hand side of the $x_r$-momentum equation, with the target shape function, $u_{r,f}(z)$, plotted in Figure 8.1(b). The shape of $\sigma_f(z)$ is such that the forcing is limited to the beam core, $z_r = 20 \, m - 60 \, m$, allowing the bottom boundary layer to develop independently. The imposed forcing is not shear unstable since the minimum gradient Richardson number, $Ri_g$, based on background velocity and stratification exceeds 0.5, larger than the critical value of $Ri_g = 0.25$. Owing to periodicity in the $x_r$ direction, mixed fluid recirculates in the computational domain so that turbulence at the flanks of the beam (caused by density overturns as will be shown) decreases after three tidal cycles. A simple way to overcome this limitation is to force $\rho$ towards the linear background during a short period (0.5 hour) when the downslope boundary flow is maximum by adding the function
to the evolution equation for $\rho^\ast$. 

Figure 6.1: (a) Horizontally averaged $k_x$ spectra of the streamwise velocity fluctuation, $u'_r = u_r - \langle u_r \rangle$ at three heights from the bottom slope during flow reversal from downslope to upslope at $t = 16$ hrs. (b) Same as (a) for $k_y$ spectra.

In the present simulation, $N_\infty = 1.6 \times 10^{-3}$ rad s$^{-1}$ and $\Omega = 1.4076 \times 10^{-4}$ rad s$^{-1}$, which gives the wave angle $\theta = \sin^{-1}(\Omega/N_\infty) \approx 5^\circ$. Slope angle $\beta = 5^\circ$ sets the criticality parameter, $\epsilon = \tan \beta/\tan \theta \approx 1$. The kinematic viscosity, $\nu = 10^{-6}$ m$^2$/s, is that of water. The Prandtl number is chosen to be $Pr = 7$. The beam velocity amplitude is chosen to be $U_b = 0.125$ m/s and the beam width is $b_b = 60$ m. Cycle-averaged and maximum values of turbulent Reynolds number are $Re_T \approx 4000$ and $\sim 10^5$, respectively. Here, $Re_T$ is based on the velocity scale, $u_T = (1/b_b) \int_{z=0}^{b_b} \sqrt{2K} dz$, where $K = 1/2\langle u'_r u'_r \rangle$ is the turbulent kinetic energy, the length scale, $b_b$, and the molecular viscosity, $\nu$. Variable time stepping with a fixed CFL number 1.2 is used leading to $\Delta t \approx \mathcal{O}(1)$ sec. One tidal cycle takes approximately 1000 CPU hours.

The simulation starts with the phase of peak downslope velocity. There is transition to turbulence within approximately 3 hrs in the upper flank of the beam and in approximately 6 hrs at the bottom wall, followed by distinct turbulent mixing events that repeat periodically at different phases of the internal tide cycle as explained below.

Figure 6.2 illustrates the phase variability of density and velocity by showing
snapshots at four different times.

### 6.2 Results

![Image of Figure 6.2](image)

**Figure 6.2**: Bottom panel shows vertical $x$-$z$ slice of the density field (after subtracting 1000 $kg$ $m^{-3}$) at 4 different times (phases) in a tidal cycle. Top panel shows spanwise-averaged streamwise velocities, $u(z,t)$ m/s and density profiles, $\rho(z,t,x=0)$ at corresponding phases. Background linear density (dashed blue line) profile is also shown. Note that $u$, $x$ are horizontal while $z$ is vertical. Parts (a) and (e) correspond to maximum downward boundary flow, parts (b) and (f) correspond to flow reversal from down to up, parts (c) and (g) correspond to peak up-slope velocity and, finally, parts (d) and (e) correspond to flow reversal from up to down. The four different times are indicated by the four red color circles in Figure 6.3a that follows. Here, arrows indicate the flow structures.

Figure 6.2(a) at $t = 12.8$ hrs corresponds to a phase when the downslope flow is near its peak. The corresponding density deviation, $\rho^*$, is small and spans a short region, $0 < z < 20$ m. Inspection of Figure 6.2(a)-(d) show that $\rho^*$ lags $u$ by $90^\circ$; it is maximum when $u$ is minimum and vice-versa. At $t = 12.8$ hrs, the near-wall shear is large and, in the sheared zone of $0 < z < 20$ m, the corresponding density field in the bottom panel, Figure 6.2(e), is indicative of both
shear instability and fine scale turbulence. From $t = 12.8$ hrs until $t = 16$ hr when the velocity becomes approximately zero, the downward flow continues to bring water from above. Consequently, the region between 5 m and 30 m which, at $t = 12.8$ hrs was occupied by water with density shown by green in Figure 6.2(e), is replaced by lighter, warmer water at $t = 16$ hrs, shown by red and yellow in in Figure 6.2(f). Furthermore, this body of relatively lighter, warmer water passes underneath the colder water above it that, being at the edge of the beam, has low velocity. The density profile that results at $t = 16$ hrs indicates a density inversion as shown in Figure 6.2(b). The corresponding density field in Figure 6.2(f) shows a mushroom shaped plume suggesting convective instability. The shear at this time is near zero. Note that the density inversion is even more prominent a little earlier at $t = 14.5$ hrs. Later in time, the large-scale overturns collapse and break into smaller structures. The shear starts increasing and at $t = 18.7$ hrs, the velocity profile in Figure 6.2(c) shows maximum upslope velocity and, correspondingly, the near wall flow is again susceptible to shear instability as shown in Figure 6.2(g). The velocity in the next snaphsot, Figure 6.2(d), is almost zero and corresponds to $t = 21.4$ hrs when the flow reverses from up to down. Between $t = 18.7$ and 21.4 hrs, the flow decelerates but continues to move upslope and heavy, cold fluid associated with the central portion of the internal tide beam surges up beneath the lighter, low-velocity fluid above. The cumulative effect of this upsurge is to strengthen the stratification in the upper flank of the beam as shown by the the region, $20 < z < 60$ m, in the density profile of Figure 6.2(d). A the same time, the upsurge overtakes low-velocity, lighter fluid near the wall leading to a density inversion with light fluid beneath dense fluid. Therefore, during flow reversal from up to down, a convective instability with the formation of a mushroom shaped structure occurs in the lower part of the flow as shown in Figure 6.2(h). The height of the density overturn ($< 15$ m) is significantly smaller than that observed earlier at $t = 16$ hr during flow reversal from down to up.

Figure 6.3 (b)-(d) show the time evolution of vertical profiles of some turbulence statistics. $\langle \phi \rangle$, denotes an average of the quantity $\phi$ calculated by averaging over an $x_r - y_r$ plane parallel to the slope and we will show the dependence,
Figure 6.3: (a) Averaged along stream velocity (blue solid line) as function of time. Density (green shaded color) is shown in same figure as function of time at different elevations from the sloping bottom, with darkest line corresponding to smallest elevation, \( z^* = 20 \) m and lightest one showing for \( z^* = 60 \) m. Flow reversal zone is shown by vertical grey colored bin. (b) Temporal evolution of spanwise averaged TKE, along a height. Same as (b) buoyancy flux (c), and turbulent dissipation (d).
\langle \phi \rangle (z^*, t), \text{ on height measured in meters above bottom (mab) }, z^*, \text{ and time, } t. \text{ Figure 6.3 (a) shows streamwise velocity at 15 mab, to locate the phase of the internal tidal cycle. In the same figure, temperature is shown as a function of time at different elevations. It can be seen in Figure 6.3 (a) that, during downward motion, e.g. } t = 10 \text{ hrs to } t = 14 \text{ hrs, the temperature at each } z^* \text{ increases owing to warmer water coming down the slope from above. The rate of temperature increase is largest at } z^* = 20 \text{ mab which is near the center of the beam and progressively decreases for higher elevations. Therefore, the temperature (density) gradient in the region } 20 < z^* < 60 \text{ progressively decreases in magnitude and, after } t \simeq 14 \text{ hrs, the gradient reverses sign. The vertical span of the region with negative temperature gradient increases but such an unstable profile cannot be maintained for long and there is convective instability that develops into dissipative turbulence.}

Figure 6.3(b) shows turbulent kinetic energy (TKE) defined by \( K = 1/2 \langle u'_i u'_i \rangle \), where \( u'_i = u_i - \langle u_i \rangle \). Figures 6.3 (c)-(d) show analogous plots for the buoyancy flux, \( B = -\left( g/\rho_0 \right) \langle \rho' w' \rangle \) and the turbulent dissipation rate, \( \varepsilon \), calculated as the sum of resolved and subgrid dissipation rates. The combined effect of boundary velocity and density fields results in significant vertical variability of turbulent kinetic energy over a tidal cycle. Maximum TKE occurs away from the boundary and during flow reversal mixing from down to up (we will refer to such mixing events \( FRM - DU \)) at various time spans, \( \Delta t \sim 3.1 - 4.8, 15.5 - 17.2 \) and \( 28 - 29.5 \) hrs during the tidal cycle as shown by vertical grey colored bins in Figure 6.3(a). Large positive buoyancy flux indicative of energy transfer from potential to kinetic energy occurs only during \( FRM - DU \) time intervals as show in Figure 6.3(c). Turbulent production, \( P \), as well as dissipation, \( \varepsilon \), are also large at that time. It is noteworthy that during these time intervals, the regions of large TKE and \( \varepsilon \) are detached from the wall and extend up to a height of \( \sim 60 - 70 \text{ m} \).

Shear production results in significant levels of turbulence when the upslope and down slope velocity are large; these mixing events are termed as \( SM-U \) and \( SM-D \), respectively. During late deceleration (e.g. \( 8 < t < 10 \)), the turbulent dissipation peak also moves up and extends up a height of 20 m. Such an enhancement of dissipation away from the immediate vicinity of the bottom was not
found by Gayen et al. (2010a) and occurs here because of an additional physical mechanism, namely, convective instability during flow reversal mixing from up to down (FRM – UD) which was visualized earlier in Figure 6.2(h).

Figure 6.4: (a) Cycle evolution of depth-averaged values of the quantities in $TKE$-budget along with averaged streamwise velocity (dashed red line) at height of $z^* = 25$ m. Here the averaging region extends from the bottom slope to $z^* = 80$ m. (b) Temporal evolution of depth-averaged dissipation (black line) and buoyancy frequency $N$ (dashed blue line).

Figure 6.4 (a) shows the temporal evolution of depth-averaged values of each term in the $TKE$-budget over a tidal period spanning $t = 26 - 38$ hrs. The depth-averaged buoyancy flux, $\langle B \rangle$, starts increasing at $t = 27.5$ at the onset of the FRM – DU event just prior to the zero-velocity phase, reaches its maximum value, and then plummets to zero at the end of the density overturn. Significant amount of negative production $\langle P \rangle$ occurs during the flow reversal event signaling the energy transfer from velocity fluctuations to the mean flow. The negative production, also seen in a companion DNS in a smaller domain, occurs because the turbulence is initiated by a positive buoyancy flux (not mean shear) and the
mean shear changes sign after $u$ passes through zero. Soon after peak $\langle B \rangle$, the dissipation, $\langle \varepsilon \rangle$, increases. Production, $\langle P \rangle$, attains positive values immediately after the $FRM - DU$ event. During the $SM - U$ event as shown in the inset of the figure, $\langle P \rangle$ and $\langle \varepsilon \rangle$ are the only dominant terms in the $TKE$-budget and both peak at the same time during the decelerating phase of up slope flow similar to the oscillatory boundary layer on a non-sloping bottom. At the end of the $SM - U$ event and during the beginning of the $FRM - UD$ event, the buoyancy flux, $\langle B \rangle$ regains positive values which are smaller relative to those during the $FRM - DU$ event.

Figure 6.4(b) quantifies the phase dependence of turbulent dissipation rate, $\varepsilon$, through a depth-averaged value. The evolution of $\varepsilon$ at $z^* = 40$ m, a location at the upper flank of the internal tide beam, is also given to isolate the contribution of detached mixing events. $\varepsilon(z^* = 40, t)$ shows pronounced peaks, shown by the downward pointing arrows in red, that occur during $FRM - DU$. The elevated levels of dissipation occur over a time interval of approximately 1.5 hrs. The depth-averaged $\varepsilon$ also shows peaks during $FRM - DU$. It is worth noting that at the onset of $FRM - DU$, the depth-averaged buoyancy frequency is minimum. Figure 6.4(b) shows additional time periods of elevated values of depth-averaged dissipation: one that occurs over a period of 1.5 hrs during the deceleration of up slope flow and is associated with $SM - U$ followed by $FRM - UD$, and another that occurs during shear mixing during down slope flow, $SM - D$.

To place these simulation results in context, we compare with the bottom mooring observations of Aucan et al. (2006) taken during HOME. There is remarkable agreement between their observations and our simulations: (i) Density (potential temperature) records at different elevations converge at the $t \simeq 11$ point and then overturn in Figure 8 (a) of Aucan et al. (2006) similar to the shaded regions in Figure 6.3(a), (ii) There is a large peak in turbulence and decrease in buoyancy frequency at the time corresponding to density overturns in both Figure 8(b) of Aucan et al. (2006) and well as in our Figure 6.3(b), and (iii) Peak dissipation occurs when the across-flow velocity reverses from downslope to up slope, compare Figure 8(c) of Aucan et al. (2006) with blue curve in Figure 6.4(a).
Gemmrich & Haren (2001) have observed thermal fronts associated with rapid fall of temperature above the continental slope in the Bay of Biscay. Their Figure 4 has similarities with the sudden decrease of temperature after a plateau that occurs towards the end of FRM-DU events (shaded) in all the near-bottom temperature records shown in our Figure 6.3(a).

6.3 Conclusion

The interaction of an internal tide beam with a bottom slope has been examined using LES. The slope angle is taken to be $5^\circ$, and a beam with width of 60 m and a peak velocity of 0.125 m/s that propagates parallel to and across the slope is considered. The fine grid LES performed here allows space-time description of small-scale processes and enables quantification of turbulence and associated fluxes. Turbulent mixing events are found to repeat at specific phases in the internal tide cycle. Immediately after the zero velocity point when the flow reverses from down to upslope, there is a burst of turbulence with large dissipation that spans the beam and lasts for about 1.5 hours. This burst is initiated by a convective instability detached from the bottom and is accompanied by a large positive buoyancy flux. There are also phases corresponding to peak tidal velocity when there is turbulence closer to the bottom that is driven by bottom shear. The present study considered a internal wave beam with a velocity profile corresponding to an internal tide generation site. Nevertheless, it is expected that an internal wave beam generated nearby that grazes the bottom will also lead to similar bottom mixing processes. The phasing and other characteristics of the beam-scale convectively-driven mixing in the present simulations show remarkable similarity to observations of Aucan et al. (2006) at a bottom mooring in the path of a internal wave beam generated at a nearby critical slope. The bottom mixing process, mediated by an internal tide beam during generation or propagation from a nearby generation site, is likely to be important in other situations with rough topography and the parametric dependence of this process on environmental parameters is worth future study.
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Chapter 7

Negative turbulent production during flow reversal in a stratified oscillating boundary layer on a sloping bottom

The present chapter is also part of the third phase of the dissertation and is based on Gayen & Sarkar (2011c). Chapter 6 we considered the scaled up the generation problem for a beam of width 60 m and a peak velocity of 0.125 m/s using large eddy simulation (LES). Theses LES results showed negative shear productions of turbulent kinetic energy at a specific tidal phase, an intriguing aspect that is uncommon in turbulent shear flow. This finding motivates the present DNS study which does not have any uncertainty associated with subgrid parameterizations. The DNS considers an internal wave beam with a reduced width of 6 m that, while larger compared to the beam width in our previous inhomogeneous studies (Gayen & Sarkar, 2010b, 2011a) so that the near-wall turbulence is separated from the beam core, is small enough to permit the DNS approach. All terms in the transport equation of turbulent kinetic energy are computed without recourse to turbulence models. The computational model discussed in section 2.2 of Chapter 2 is used here.
7.1 Selection of the simulated case

In the present simulation, \( N_\infty = 1.6 \times 10^{-3} \text{ rad. s}^{-1} \) and \( \Omega = 1.4076 \times 10^{-4} \text{ rad. s}^{-1} \) giving a wave angle \( \theta = \sin^{-1}(\Omega/N_\infty) \approx 5^\circ \). For slope angle, \( \beta = 5^\circ \), the criticality parameter is given by \( \epsilon = \tan \beta / \tan \theta \approx 1 \). The kinematic viscosity, \( \nu = 10^{-6} \text{ m}^2/\text{s} \), is that of water. The Prandtl number is chosen to be \( Pr = 7 \). The beam velocity amplitude is chosen to be \( U_b = 0.0125 \text{ m/s} \) and the beam width is \( l_b = 6 \text{ m} \). Cycle-averaged and maximum values of turbulent Reynolds number are \( Re_T \sim 1500 \) and \( \sim 6000 \), respectively. Here, \( Re_T \) is based on the velocity scale, \( u_T = (1/l_b) \int_{z_r=0}^{l_b} \sqrt{2Kdz_r} \) where \( K = 1/2\langle u_i'u_i' \rangle \) is the turbulent kinetic energy, \( l_b \) is length scale, and \( \mu \) is the molecular viscosity. Another Reynolds number is \( Re_S = U_b \delta_s/\nu \sim 1500 \) based on the Stokes boundary layer thickness \( \delta_S = \sqrt{2\nu/\Omega} \). Variable time stepping with a fixed CFL number 1.2 is used leading to \( \Delta t \approx O(1) \text{ sec} \). One tidal cycle takes approximately 1500 CPU hours.

![Figure 7.1](image.png)

**Figure 7.1:** (a) Temporal evolution of averaged TKE profiles. (b) Cycle evolution of depth-averaged values of the quantities in TKE-budget along with averaged streamwise velocity (dashed red line) at height of \( z^* = 0.6 \text{ m} \). Here the averaging region extends from the bottom slope to \( z^* = 10 \text{ m} \).
7.2 Results

The velocity profile, see schematic of figure 8.1, corresponds approximately to an oscillating wall jet in a stratified fluid. The phase variation of turbulence over a single tidal period was discussed in our previous LES Gayen & Sarkar (2011b) and is summarized below for the current DNS. Throughout the tidal cycle, the deviation density $\rho^*$ lags the velocity by approximately $\pi/2$, resulting in the maximum of the density deviation during the phase of minimum velocity and vice-versa. The cycle starts with negative peak velocity corresponding to $\phi = 0$ with little deviation of the density field from the background state. Soon after, during the decelerating phase of the downslope flow spanning $0 < \phi < \pi/2$, the density field changes in response to warmer fluid moving down from upslope to replace the cold fluid previously inside the jet. It is noted that, away from the jet core, the deformation of the density field decreases especially at the top edge of the beam due to relatively low jet velocity. As a result, a density inversion is observed in the upper flank of the beam. Later in time, the large-scale overturns collapse and break into smaller structures. Similarly, during flow reversal from upslope to downslope flow, a density inversion of heavier fluid over lighter fluid occurs inside the lower flank of the jet spanning $0 < z_r < 1$.

The velocity and the density structures formed during the flow reversal event have strong impact on turbulent statistics, calculated here as averages over $x_r - y_r$ planes parallel to the slope. The turbulent kinetic energy, $K = 1/2\langle u'_i u'_i \rangle$ also denoted by TKE, represents the energy in fluctuations with respect to the mean velocity and satisfies the following evolution equation:

$$\frac{\partial K}{\partial t} = P - \varepsilon + B - \frac{\partial T}{\partial z_r}.$$  \hspace{1cm} (7.1)

Here, $\partial T/\partial z_r$ denotes the transport of the TKE consisting of pressure transport, turbulent transport and viscous transport. $P$, $\varepsilon$ and $B$ are the production, dissipation and the buoyancy flux, respectively. Fig. 7.1(a) shows the evolution of turbulent kinetic energy, $K$. Peak TKE that extends up to a height of $\sim 8$ m above bottom, occurs during the flow reversal due to the large density inversion in the upper flank of the beam. Fig. 7.1(b) shows the temporal evolution of
Figure 7.2: Illustration of negative production mechanism during the flow reversal event from downslope to upslope flow.

depth-averaged values of each term in the $TKE$-budget over a tidal period. The depth-averaged buoyancy flux, $\langle B \rangle$ shows positive values during the flow reversal event, that correspond to the large density overturn and associated transfer of energy from potential form to kinetic form. Significant negative production $\langle P \rangle$ occurs just after the flow reversal. The event of negative $\langle P \rangle$ spans $\Delta t \sim 1 \text{ hrs}$ and signals energy transfer from velocity fluctuations to the mean flow. Soon after peak $\langle B \rangle$, the dissipation, $\langle \varepsilon \rangle$, increases. Production regains its positive values
after the beam becomes energetic in the upslope direction. Elevated amounts of $TKE$, production and dissipation are observed due to the wall shear during the upslope flow. $\langle P \rangle$ and $\langle \epsilon \rangle$ are the only dominant terms in the $TKE$-budget during this phase.

**Negative** turbulent production is an unusual occurrence in boundary layer turbulence. A primary goal of this note is to obtain an explanation for negative production. The mechanism responsible for the observed negative production is explained with the help of the illustration in Fig. 7.2 and confirmed with the DNS data in Fig. 7.3. During the decelerating phase of the downslope motion, the downward flow continues to bring water from above in the form of a jet as previously discussed. This creates a density inversion of heavier fluid on top of lighter fluid at the upper flank of the jet as shown in Fig. 7.2(a). At this phase, the streamwise velocity is small in magnitude but still in the downslope direction.

Soon after, the corresponding unstable density field forms mushroom shaped plumes similar to the Rayleigh-Taylor type instability problem as shown in Fig. 7.2(b). Structures containing lighter/warmer (heavier/colder) fluid tend to have upward motion i.e. $w > 0$ (downward motion i.e. $w < 0$), owing to buoyancy.
Thus, velocity fluctuations (mostly up-down vertical motion) are created by positive buoyancy flux during the density inversion, not by background shear associated with the small velocity at this phase. After the flow reversal, the jet starts to move in the upslope direction with a negative \( (d\langle u_r \rangle/dz_r < 0) \) velocity gradient in the upper flank of the jet as shown in Fig. 7.2(c). This sudden applied shear boosts the inclined fluctuating motions of the structures formed before the zero crossing, by providing the rightward streamwise motion for the downward moving structures and vice-versa. This results in negative values of the product of the two velocity components \( (u'_r, w'_r) \). The negative Reynolds stress, \( \langle u'_r w'_r \rangle \) acts on the negative shear in the upper portion of the jet to give negative turbulent production.

To verify this mechanism, snapshots of fluctuating fields are shown in Fig. 7.3 along with averaged streamwise velocity profile at \( \phi = \pi/2 + \pi/10 \) at a time immediately after flow reversal from down to upslope motion. Leftward inclined density structures are shown along with their relative motion by the arrows in Fig. 7.3(b). Figs. 7.3(c) and 7.3(d) show snaps of the streamwise fluctuating velocity, \( u_r' \) and wall normal fluctuating velocity, \( w_r' \), respectively. It is clear that the structures of cold heavier fluid (in blue contour values and white arrows) shown in Fig. 7.3(b) have positive streamwise velocity in Fig. 7.3(c) and negative wall normal velocity in Fig. 7.3(d). Similarly, the lighter fluid structure (in yellow contour values and black arrow) in Fig. 7.3(b) has negative streamwise and positive wall normal velocity. Consequently, the product of the two components of fluctuating velocity has negative values along the inclined buoyant structures as shown by the white dashed lines in Fig. 7.3(e) and hence yields a negative value for \( P = -\langle u'_r w'_r \rangle \partial \langle u_r \rangle / \partial z_r \) after multiplication with the negative shear in the upper flank of the jet. The duration of negative production is set by the time taken for the convectively driven structures to dissipate.

### 7.3 Conclusion

The interaction of an internal tide beam with a bottom slope has been examined using direct numerical simulation. Immediately after the zero velocity...
point when the flow reverses from down to upslope, there is a burst of turbulence with large dissipation that spans the beam and lasts for about 1.5 hours. This burst is initiated by a convective instability detached from the bottom and is accompanied by a large positive buoyancy flux. Negative turbulent production spanning about 1 hr, occurs during this turbulence episode indicating the transfer of energy from the fluctuating field to the mean field. It is shown that inclined turbulent structures initiated by buoyancy (not shear) are distorted by the non-zero mean shear that occurs after the velocity passes through its zero value, resulting in negative production.

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Chapter 8

Degradation of an internal wave beam by resonant triad interaction in an upper ocean pycnocline.

The present chapter constitutes the fourth phase of the dissertation and is based on Gayen & Sarkar (2012). The prior chapters have addressed formation of internal wave beams and associated near bottom turbulence. However, the beams that propagate into the ocean interior are initially linearly evolving object in our simulations. Observations (Martin et al., 2006; Cole et al., 2009; Johnston et al., 2010) show that internal beams may suffer degradation during their propagation into the high stratified pycnocline present in the upper ocean. These observations motivate the present three-dimensional numerical simulations, performed to investigate nonlinear effect during the semidiurnal internal wave beam interaction with the upper ocean pycnocline.
8.1 Introduction

Internal waves are generated by small disturbances in a stably stratified fluid. Internal tide generation at abrupt topography has been the focus of a number of recent theoretical, numerical, laboratory, and observational studies (Garrett & Kunze, 2007). Internal waves generated by topography under supercritical (Gosstiaux & Dauxois, 2007) and near-critical environments (Zhang et al., 2008; Gayen & Sarkar, 2010b) form a tidal beam, which is composed of many spatial modes within a finite-width region. After generation, an IW beam may traverse into the upper part of the ocean away from the topography. During its propagation through a non-uniform stratified environment, a beam often encounters a sharp density interface (pycnocline) in the ocean (Martin et al., 2006; Cole et al., 2009; Johnston et al., 2010) and evanescent regions in the mesosphere (Broutman et al., 2009).

Breaking of internal waves plays a significant role in deep-ocean mixing (Garrett, 2003) so as to maintain the vertical stratification observed in the mid-ocean. Mixing in the deep ocean conveys heat from the upper layer of the ocean to the abyssal cold waters, and, thus, it helps to maintain the meridional overturning circulation. Apart from internal wave breaking near topography (Polzin et al., 1997; Nash et al., 2007; Gayen & Sarkar, 2010b, 2011b), there can be nonlinear interaction between groups of propagating internal waves (Müller et al., 1986) to generate further spatial scales and temporal frequencies. A single internal wave of frequency $\omega_0$ and wave vector $K_0$ is subjected to an instability under certain circumstances and the primary wave will decay by transferring energy into two small-scale waves of lower frequency, such that $K_0 = K_1 + K_2$ and $\omega_0 = \omega_1 + \omega_2$ are satisfied. This process, known as the resonant subharmonic instability (RSI). Energy transfer is maximum under $\omega_1 = \omega_2 \approx \omega_0/2$ in the limit of high wave numbers which represents an special class of resonant triad interactions (Staquet & Sommeria, 2002) known as parametric subharmonic instability (PSI). In present studies we have observed energy partitioning in unequal frequencies which also was found before for case of the internal tide generation (Korobov & Lamb, 2008).

Rainville & Pinkel (2006) measured subharmonic signal in velocity and
displacement data in the near field and the far field (450 km offshore) at strong internal tide generation site at Keana Ridge during the Hawaii Ocean Mixing Experiment (HOME). At both sites, the energy fluxes, computed by cross-spectral analysis, in the diurnal frequency band is 15%-20% of the semi-diurnal flux. In far field, independent of any diurnal forcing, the energy flux associated with diurnal frequency varies in accord with the fortnightly cycle of the barotropic semidiurnal tide, suggesting direct energy transfer from the low-vertical-mode M2 internal tide to higher-mode internal waves at frequencies (1/2)M2. Later, Carter & Gregg (2006) observed cross-frequency energy transfer in a narrow depth range (525-595 m) from M2 tide to M1 via subharmonic instability for an IW wave beam close to Keana Ridge based on the one month data from a deep-profiling shipboard Doppler sonar arrangement.

Recent numerical studies (Hibiya et al., 2002; MacKinnon & Winters, 2005; Hazewinkel & Winters, 2011) have found that a significant amount of energy is transferred to (1/2)M2 from M2 by nonlinear interactions for plane monochromatic waves around critical latitude (28.9°N). Gerke et al. (2006) in their numerical experiment have showed qualitative agreement with the ocean observations of Carter & Gregg (2006) by showing breaking of a IW beam close to the supercritical topography. Radiation of subtidal beam like structures along with the fundamental from sloping topography are observed in recent simulations of a ocean ridge (Korobov & Lamb, 2008) and continental slope (Gayen & Sarkar, 2010b) during the generation of internal tide.

In spite of near topography beam dissipation due to turbulence which is often caused by wave overturning (Gayen & Sarkar, 2010b) and subharmonic instability (Carter & Gregg, 2006; Gerke et al., 2006), internal wave beam can still survive with higher modes in energy density (Johnston et al., 2010) and go toward the upper ocean. Degradation of IW beams originating from the Keana Ridge in Kauai channel, was observed by Martin et al. (2006) after interaction with the upper pycnocline. Clustering of energy surrounding the ray path of the beam extends up to ~ 50 km from the topography before it hits the upper surface. There appears to be no wave beam reflected back down from the upper surface. Another
set of observations of the across-ridge structure of internal tides have been obtained at the Hawaiian ridge at Kauai Channel by Cole et al. (2009) using SeaSoar and a Doppler sonar over the upper 400 − 600 m of the ocean extending 152 km on each side of the ridge. Recently, Mathur & Peacock (2009) have performed a set of experiments on both plane IW and localized IW beam propagation through variable density interfaces. Their experiments, limited to the laminar regime, indicate that nonuniform stratification disrupts a beam, ducts energy, and affects beam transmission and reflection.

Numerical process studies have generally assumed a uniform stratification and, therefore, beam structure is maintained after surface and bottom reflections. An exception is the analytical/numerical work of Gerkema (2001) and Grisouard et al. (2011) where a beam incident on the thermocline forms nonlinear internal waves but even here the model is limited to weak nonlinearity. Considerable beam degradation occurs after the first surface reflection (Martin et al., 2006; Cole et al., 2009; Johnston et al., 2010), though the actual dissipation is unclear. Possible non-dissipative processes leading to beam degradation away from its generation sites include: radial dispersion, dephasing of the beam in realistic stratification because the wavelengths are not integer multiples as they are in linear stratification and beam-beam interactions (Cole et al., 2009). In the present work, we will conduct numerical investigations of the interaction of an internal-wave beam with a model upper ocean stratification: i.e., pycnocline, transition layer and mixed layer. In contrast to previous numerical studies, we will perform simulations at high resolution that account for highly nonlinear responses.

8.2 Problem formulation

A linear, two-dimensional wave beam propagates upwards in a medium of constant density gradient, $N_1$. Then it encounters the upper ocean with a continuously varying density gradient in the vertical direction, $z$: a mixed layer on top of a pycnocline of strength $N_2$. Two different hyperbolic tangent functions are
used to model background stratification,

\[
N^2(z) = \begin{cases} 
\frac{N_2^2}{2} \frac{\tanh \left( \frac{z - z_2}{L_2} \right)}{\tanh \left( \frac{z - z_1}{L_1} \right)} + \frac{N_1^2}{2}, & z > h_p \\
\frac{N_2^2 - N_1^2}{2} \frac{\tanh \left( \frac{z - z_2}{L_2} \right)}{\tanh \left( \frac{z - z_1}{L_1} \right)} + \frac{N_2^2 + N_1^2}{2}, & z \leq h_p 
\end{cases}
\]  

(8.1)
as shown in figure 8.1. Here, \( z_1 = 0 \) and \( L_1 = 10 \) m \((z_2 = 40 \) m and \( L_2 = 10 \) m) are respectively the location and thickness of \( N_1 \to N_2 \) \((N_2 \to 0)\) transition layer. Eq. (8.1) leads to a transition thickness, \( l_{tr} \) (in figure 8.1) of 80 m. Both hyperbolic functions meet at \( z = h_p \). The location \( z = z_c \), corresponds to a caustic, i.e. the local buoyancy frequency, \( N(z_c) \), is equal to frequency of oscillation, \( \Omega \).

**Figure 8.1:** Schematic of the problem along with internal wave beam maker at left hand side and sponge region at the right and bottom of the computational domain. IW beam path is denoted by red line. Here, phase speed, \( C_p \) and group speed, \( C_g \) are shown on the ray path before and after the reflection from the caustic, \( N(z_c) = \Omega \). Background color shade in the figure indicates the stable density gradient. In left inset background density, \( \rho_b(z) \) and buoyancy profile, \( N(z) \) as a function of vertical water depth, where \( N_1 \) and \( N_2 \) being buoyancy frequency of the lower part of the domain and at the pycnocline, respectively.

In the present simulation, \( N_1 = 2.8152 \times 10^{-4} \text{ rad s}^{-1} \) and \( \Omega = 1.4076 \times 10^{-4} \text{ rad s}^{-1} \), which gives incoming wave angle \( \theta = \sin^{-1}(\Omega/N) \approx 30^\circ \). The pycnocline parameters are: degree of in homogeneity, \( \Gamma = N_2/N_1 = 4 \), and transition layer thickness, \( l_{tr} = 80 \text{ m} \). The kinematic viscosity, \( \nu = 10^{-6} \text{ m}^2/\text{s} \), is that of water. The Prandtl number is chosen to be \( Pr = 7 \). The beam velocity amplitude is chosen to be \( u_0 = 0.1 \text{ cm/s} \) and the beam width is \( l_b = 50 \text{ m} \). The Froude
number of the incoming beam based on $u_0$, $N_1$ and $b$ is $Fr = u_0/(N_1b) = 0.05$. The effects of the Earth’s rotation are ignored in this section. Rotational effects will modify but not change the existence of resonant triads during the interaction process. Variable time stepping with a fixed CFL number 1.2 is used leading to $\Delta t \simeq O(1) \text{ sec}$. One tidal cycle takes approximately 1000 CPU hours to simulate.

The NS equations are numerically solved to obtain the velocity $[u, v, w]$ in cartesian coordinates $[x, y, z]$ and the deviations from background density and pressure using a mixed spectral/finite difference algorithm. Derivatives in the spanwise direction are treated with a pseudo-spectral method and derivatives in the vertical and streamwise directions are computed with second-order finite differences. A low-storage, third-order Runge-Kutta-Wray method is used for time stepping, except viscous terms which are treated implicitly with the alternating direction implicit (ADI) method. Pressure correction is done by fractional step method. The test domain, excluding the sponge region, consists of a rectangular box of 1200 m length, 300 m height and 20 m width. The grid size in the test domain is $890 \times 500 \times 64$ in the $x$, $z$ and $y$ directions, respectively, with stretching in $x$, $z$ directions. The grid spacing ($\Delta x_{\text{min}} = 0.5 \text{ m, } \Delta x_{\text{max}} = 1.2 \text{ m, } \Delta y = 0.3 \text{ m, } \Delta z_{\text{min}} = 0.25 \text{ m, } \Delta z_{\text{max}} = 1 \text{ m}$) is sufficient to resolve smaller waves created during the subharmonic instability. Periodicity is imposed in the spanwise, $y$ direction.

Inlet flow condition is forced at the left hand side of the computational domain based on the analytical value of the parallel, transverse velocity components and buoyancy fields of the Thomas-Stevenson profiles (Thomas & Stevenson, 1972) at a location such that vertical distance between the pycnocline and the generation site is sufficiently large to avoid any direct effect of the generation site on the pycnocline as shown in figure 8.1. The profiles always have zero integrated flux at the left boundary. Thus, there is a rational linear beam that, after generated at bottom topography, travels into the computational domain. At the top of the domain, a free surface condition with the rigid-lid approximation is imposed with zero gradient value for the density. The other two boundaries have an artificial boundary corresponding to the truncation of the domain. Rayleigh damping or a sponge layer is used at right and bottom of computation domain, shown in figure
Figure 8.2: Overview of simulated IW beam dynamics during the interaction with the upper ocean pycnocline shown by streamwise velocity contours: (a) laminar response at $t = 10T$, (b) weakly nonlinear regime with trapped harmonics at $t = 15T$, (c) Nonlinear response with formation of subharmonics (see enclosed region) at $t = 20T$. Here, $T$ is the time period, $2\pi/\Omega$. All snaps are taken at same phase.

8.1, so as to minimize spurious reflections from the artificial boundary into the test section of the computational domain. The sponge region at the right boundary
contains 25 points and extends from 1200 m to 1500 m, while the bottom sponge with 15 points extends from $z = -180$ m to $-250$ m.

### 8.3 Results

An internal wave beam arrives at the upper pycnocline, after entering from the left hand boundary, in approximately two days. During the interaction of the IW beam with the pycnocline, as time progresses, various nonlinear phenomena take place such as formation of harmonics, subharmonic-resonance and further convective overturning, as will be explained below. Our main goal of the present study is to examine the driving mechanism behind the wave beam degradation and breaking suggested by recent ocean observations (Cole et al., 2009; Johnston et al., 2010).

Figure 8.2 illustrates the evolution of the horizontal velocity field by showing snapshots in the x-z plane at three different times. A profile of the background Brunt-Väisälä frequency, $N(z)$ is plotted in the right inset of figure 8.2(a) to show the location of the pycnocline. An internal wave beam enters the domain from the left hand side at $z \sim -130$ m and then it enters into the pycnocline at $z = -15$ m. The beam narrows due to enhancement of the vertical wave number during the transmission process through the lower transition layer, $N_1 \rightarrow N_2$ of the pycnocline. Here, the variation of the vertical wave number is governed by dispersion relation as

$$m^2(z) = k^2 \left[ \frac{N_2^2(z)}{\Omega^2} - 1 \right], \quad (8.2)$$

derived from linear wave theory. During the first transmission process through the $N_1 \rightarrow N_2$ layer, the characteristic of the internal wave beam gradually becomes shallower due to increasing stratification. In the later half of the pycnocline ($N_2 \rightarrow 0$ transition layer), the internal beam beam becomes steeper owing to dropping of the stratification and the vertical length scale increases ($m$ decreases) as it approaches to the caustic $z = z_c$ according to (8.2). Maximum amplification of the vertical wave velocity occurs slightly below the caustic and decreases exponentially above it. Later, after being reflected from the caustic, the IW beam returns back
to the bottom medium through the transmission layers in a similar fashion. Before
time $t = 10T$, the wave behavior is linear. As time progresses, higher harmonics
($\omega = n\Omega, n \in \mathbb{N}$) originate after the beam reflection from the caustic at $x \sim 580$
m. A remarkably complex wave pattern associated with the main beam along with
several harmonics during the interaction is shown in figure 8.2(b). The reflected
beam strength diminishes noticeably due to significant amount of energy transfer
to the higher temporal harmonics. The wave at the second harmonic ($2\Omega$) that
propagates down inside the pycnocline is clearly visible. As the bottom medium
cannot support any harmonics, unlike the main reflected IW beam, these harmonics
are unable to transmit into the bottom medium. They reflect back from lower
transition layer $N_1 \rightarrow N_2$ at $x \sim 780$ m. There is a similar reflection from their
respective caustic ($N = 2\Omega$) at $x \sim 950$ m. This process continues and repeats
for other higher harmonics resulting in the trapping of higher harmonics inside
the pycnocline. The phenomenon of extracting beam energy through formation
of higher harmonics was recently studied by Grisouard et al. (2011) in their two-
dimensional numerical simulation.

**Figure 8.3**: Evolution of the line averaged spectrum, $\langle Y \rangle(\omega, t)$, for streamwise
velocity measured along the two horizontal lines intersecting the beam inside the
pycnocline. The lines for (a) and (b) are indicated by 1 and 2, respectively, in
figure 8.2(c) as shown by dashed lines.

As time progresses, the internal wave field becomes more complex under
resonant subharmonic instability (RSI). At later time, subharmonic internal waves
(\omega \in [0, \Omega]) are excited inside the transition layer of the pycnocline owing to a non-linear energy transfer from the incoming beam to subharmonic motions. During this process the incoming beam structure is distorted due to the superposition of newly formed subharmonic waves which have shallower characteristic angle compared to the fundamental beam as shown in figure 8.2(c). The subharmonic waves with negative group velocity, eventually propagate in the leftward direction and encounter the left domain boundary. To avoid any spurious reflection from the boundary and absorb the leftward moving waves, a triangular shaped sponge at the top left hand corner is placed without impacting the dynamics of the problem.

\[ \text{Figure 8.4: Time series of horizontal velocity along a horizontal line of original data and band-passed in time frequency ranges, as indicated below each plot. Here, the line is taken at location 1 as shown in figure 8.2(c).} \]

To analyze the energy transfer among the higher harmonics and the subharmonics we have calculated the spectrogram \( Y(\omega, t) \) which represents the evolution of the spectrum \( Y(\omega) \) with respect to time. Here, temporal data of horizontal ve-
locity are used to calculate spectra for two locations along two horizontal segments as marked by the gray dash line in figure 8.2(c). The time window for the spectrum is chosen to be 10 T to accurately capture internal wave signals at lower frequency. The spectrum time is the centre of the time intervals of the corresponding window. Temporal evolution of the line averaged spectrogram, $⟨Y⟩(ω, t)$ is shown in figure 8.3. Figure 8.3(a) shows that, before $t = 120 \text{ hr}$, the velocity field of the incident wave has a signal frequency with a dominant peak at $ω = Ω$. Soon after the spectral signal reveals two additional peaks in the subharmonic regime $(0, Ω]$ as an effect of RSI in the incident beam which is already illustrated in figure 8.2(c). Between the subharmonic signals, one peaks at $ω_1 \sim 0.3Ω$ and another shows a peak at $ω_2 \sim 0.7Ω$. The two new born waves packets together with the forced internal wave beam compose a resonant triad ($ω_1 + ω_2 = Ω$). These two additional peaks start gaining energy from the forcing IW beam of frequency $Ω$ and weaken the main wave beam. A noticeable difference in the spectra is observed in the vicinity of the reflecting beam as shown in figure 8.3(b). Spectrogram data at this location shows peaks at different higher harmonics ($nΩ, n ∈ \mathbb{N}$). Most of them are finally trapped inside the pycnocline as the bottom medium is unable to support any harmonics with frequency higher than the background frequency, $N_1$.

So far resonant triad has been observed in the wave frequency space. Based on three discrete frequencies, $[ω_1, ω_2, Ω]$, participating in the triad, the velocity data has been processed using three bandpass filters. Time evolution of the original data for the streamwise velocity, measured along a horizontal line intersecting the incident beam as shown in figure 8.2(c), is plotted in figure 8.4(a). During the initial period before $t=180 \text{ hrs}$, the phase line are well organized and inclined towards right which indicates positive horizontal wave number and rightward energy propagation in the internal wave beam. Soon after, the incident wave beam experiences subharmonic instability forming additional internal wave packets with different phase angles and frequencies including leftward propagation. These waves later interact and overlap with the main IW beam resulting in complex beam structure as revealed in the $x - t$ diagram after $t = 230 \text{ hr}$. After band passing the original data into the forcing frequency, $ω \sim 0.95 - 1.05Ω$, the temporal data series
in figure 8.4(b) show the forcing IW wave field with rightward inclined phase over the entire time span. The wave amplitude for the fundamental beam drops significantly after $t = 210$ hr. The dominant wave number for a given signal, frequency is obtained by using the slope of the x-t curves to infer the phase speed, $C_p$ using $\omega = C_p k$. The dominant horizontal wave number for incident beam is approximately equal to $k_{x,0} = 0.05 \, m^{-1}$ which is consistent with the main beam structure in figures 8.2(a)-(b). Space-time diagrams of the two IW wave packets after band passing around $\omega = 0.7 \, \Omega$ and $\omega = 0.3 \, \Omega$ are shown in figures 8.4(c) and 8.4(d), respectively. The internal wave with frequency $\omega \sim 0.7\Omega$ has smaller phase speed, and a larger wave number ($k_{x,1} \sim 0.08 \, m^{-1}$) compared to the main beam. We emphasize that the smallest horizontal scale ($l_x \sim 78.5 \, m$) of the wave produced during RSI is well resolved with the existing grid resolution in the horizontal direction. For the IW having frequency, $\omega = 0.3 \, \Omega$, phase lines in the $x - t$ plane are oriented in leftward direction resulting in negative phase velocity. This packet of waves travels in leftward direction, which is clearly evident in spatial structure of the beam in the vicinity of the incident zone in figure 8.2(c). This wave has a wave number $k_{x,2} \sim -0.3 \, m^{-1}$ showing a perfect resonant triad condition in the
horizontal wave number space, i.e. \( k_{x,0} = k_{x,1} + k_{x,2} \).

The growth rate associated with the subharmonic resonance is quantitatively calculated by looking at temporal evolution of their energy. After extracting the subharmonic signals from the field data using bandpass filter, area averaged kinetic energy, \( \langle E_{k,\sub} \rangle = 1/2\rho_0 \langle u_{\sub} \cdot u_{\sub} \rangle \) and potential energy \( \langle E_{p,\sub} \rangle = \frac{1}{(2\rho_0)}(g\rho_{\sub}^*/N)^2 \) is calculated and plotted as function of day in figure 8.5. Growth for both the energy is negligible for initial 6 days. During the time interval, \( \Delta t \sim 7 - 9.5 \) day, it grows exponentially with a growth rate of \( \lambda_t \sim 1/\langle E_{\sub} \rangle d\langle E_{\sub} \rangle/dt \sim 2/3 \) per day. In the same figure, the area averaged kinetic energy, \( \langle E_{k,f} \rangle \), and potential energy, \( \langle E_{p,f} \rangle \), associated with the M2 wave beam are shown to illustrate the impact of RSI over the incident beam energetic. During the initial seven days, \( \langle E_{k,f} \rangle \) and \( \langle E_{p,f} \rangle \) are mostly constant over time. In response to RSI which transfers energy to the subharmonic motions, both the kinetic and the potential energy start dropping after \( t \sim 7 \) day.

The waves of smaller length scale, originating from RSI, may be susceptible to convective overturning. To examine wave overturning, the profiles of streamwise velocity, density and inverse Richardson number, \( R_i g^{-1} \) are plotted as a function of vertical distance at location \( x = 400 \) m for time \( t = 8T \) and \( t = 13T \) in figures 8.6(a) and 8.6(c), respectively. At early time, there is no evidence of wave steepening in the density field. Higher values of \( R_i g \) (\( R_i g^{-1} \ll 4 \)) over the profile also suggest little possibility of any shear driven instability. In order to quantify the scales and the frequencies associated with of the internal wave field, two-dimensional power spectrum \( Y(\omega, k_z) \) analysis is performed on the streamwise velocity measured at location \( x = 400 \) m. Spectrum \( Y(\omega, k_z) \), in figure 8.6(b) shows dominant peak at the forcing frequency, \( \Omega \) and \( k_z \approx 0.33 \) m\(^{-1}\), which suggests that the beam is mostly linear and unaffected by RSI. Later time velocity profile at \( t = 13T \) reveals the existence of additional smaller vertical scales associated with RSI as shown in figure 8.6(c). The density profile at this time indicates a density inversion with negative value of \( R_i g \) around \( z \sim 7 \) m as marked by a grey color vertical bin in the figure. Two-dimension spectrum during the late time illustrates the clear evidence of the wave packets with larger vertical wave number, \( k_x \approx 0.08 \) m\(^{-1}\).
Figure 8.6: (a) Profiles of streamwise velocity, density and inverse gradient Richardson number, $Ri^{-1}$ as a function of vertical coordinate at a location $x = 400$ m and time, $t = 8T$. (b) The two-dimensional power spectrum (log-scale) of streamwise velocity at $x = 400$ m plotted as a function of the vertical wave number and time frequency. The time series is taken over a time span $3 < t < 13T$. Here, the wave frequency is normalized by forcing frequency, $\Omega$. (c) is the same as (a) for $t = 14T$. (d) is similar to (b) with the time series taken over a time span $10T < t < 20T$.

associated with frequency close to $\omega \sim 0.7\Omega$. Simulations are terminated after $t = 20T$ because the present grid is not adequate to resolve the three-dimensional fine scale turbulence and measure the dissipation accurately. Significantly larger grid resolution or a large eddy simulation (LES) will be required in this regard which we have left for future investigation.
8.4 Conclusion

The interaction of an internal tide beam (width of 50 m and peak velocity 0.1 cm/s) with an upper ocean pycnocline has been examined using three-dimensional simulation. The fine grid simulation performed here allows space-time description of small-scale waves during the interaction. The internal wave field becomes complex inside the pycnocline with formation of harmonics due to the nonlinear interaction of the IW beam with the caustic. Most of the harmonics created inside the pycnocline cannot be supported by the bottom medium due to diminished stratification. Those harmonics undergo several reflections and are finally trapped inside the pycnocline resulting in significant energy absorption from IW beam. At a later time, incident beam becomes strongly nonlinear as it travels upward through the pycnocline, driven by resonant subharmonic instability. During the RSI, rapid energy transfer occurs to subharmonic motions leaving weaker incident wave beam. The RSI generated waves having smaller vertical scales which cause wave steepening which could evolve into turbulence.

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Chapter 9

Summary

A suite of numerical simulations has been performed to systematically study certain geophysical problems. The study begins with stratified tidal flow over a flat bottom and progressively increases the degree of complexity by adding sloping topography at the bottom and including effects owing to variable background stratification. A few of the important aspects of the dissertation research are summarized here.

Chapter 2 explains the numerical methods and the problem setup to study the various problems related to this dissertation work. The computational fluid dynamics code, developed as part of this thesis, is described including the algorithm and the parallelization methodology. The code is written in generalized coordinates and employs mixed a spectral/finite-difference algorithm to study flow over complex topography. Derivatives in the spanwise direction are treated with a pseudo-spectral method and derivatives in the vertical and streamwise directions are computed with second-order finite differences. A low-storage third-order Runge-Kutta-Wray method is used for time stepping, except for the viscous terms, which are treated implicitly with the alternating direction implicit (ADI) method. The code includes advanced turbulence models for large-eddy simulation, passive and/or active scalar advection, and has been parallelized using MPI.

Turbulent bottom boundary layer under an oscillating current at a low frequency, 12.4 h, corresponding to an M2 tide is studied in Chapter 3. The geometry of this flow is idealized with a flat nonsliping bottom. A dynamic mixed eddy vis-
cosity model is used for LES and the near-bottom turbulence is resolved at the moderate Reynolds number, $Re_s = 1790$ considered here. To assess the performance of the present large-eddy simulations for this type of highly unsteady boundary layer flow, LES results of an unstratified case are validated with results from the numerical study of Salon et al. (2007) and from the laboratory study of Jensen et al. (1989).

The focus of Chapter 3 is to study the effects of stratification on a tidally oscillating boundary layer. A bottom mixed layer forms and grows with an entrainment rate which is reduced by the external stratification, and with a periodic modulation owing to the tidal oscillation. The turbulent mixed layer is separated from the stratified outer layer by a thermocline with an overshoot in the temperature gradient that gradually weakens over time due to both turbulent and molecular diffusion. Flow statistics in oscillating flow are observed to have a noticeable phase lag or lead with respect to the external current. The observed phase lag of the peak TKE, production and dissipation depended on the measurement location from the bottom surface. A similar phenomenon is observed by Burchard et al. (1998) for the measured dissipation in the Irish Sea that showed a lag of 1 h, equivalently $\delta \phi = 29^\circ$, between dissipation and current at a height 12 m above the seabed. Flow instabilities and turbulence in the bottom boundary layer excite internal gravity waves that propagate away into the ambient. Several previous studies observed internal gravity waves generated by turbulence for a variety of flow types including shear layers, gravity currents, and stratified wakes. Many of these studies reported that the frequency and angle of propagation associated with the turbulence-generated internal gravity waves were confined to a relatively narrow range. A similar observation is made in the present study where the turbulence-generated internal waves in the waves in the outer layer are associated with angles of propagation clustered around $45^\circ$. Unlike the steady case (Taylor & Sarkar, 2007a), the phase lines of the internal waves change direction during the tidal cycle and also from near to far field.

When a M2 tide flows over simple sloping topography, the boundary layer response dramatically changes with formation of strong energetic baroclinic in-
ternal tides compared to cases with a flat bottom. To simulate this problem, (results discussed in Chapter 4-5), a new code is developed in the generalized coordinate system based on mixed spectral/finite-difference algorithm (numerical details given in Chapter 2). Though the excursion and Stokes Reynolds number of the barotropic background flow are very small, a strongly intensified boundary layer flow is created owing to the resonant wave-slope interaction in the critical case when angle of the slope matches with the characteristic angle of the internal wave. Transition to turbulence takes place through wave steepening at the slope that is followed by a convective (buoyancy driven) instability which further leads to three dimensional fluctuations in the form of coherent streamwise vortices. To our knowledge, these are the first turbulence-resolving simulations of an internal tide generation at near-critical slope, which implies that the mechanism of critical generation, not just critical reflection, must also be considered as a potential explanation for enhanced hotspots of mixing in the vicinity of near-critical slopes (Moum et al., 2002; Aucan et al., 2006; Nash et al., 2007).

In Chapter 5 we have discussed internal wave energetics as well as the energetics of turbulence in the bottom boundary layer at critical slope. In addition, the effect of increasing slope length is quantified by employing an LES approach to access higher values of slope length. Magnitude of the near-bottom jet increase with slope length as approximately $l^{0.45}$ in the turbulent regime which is different from that in the laminar regime (Zhang et al., 2008). The effective turbulent viscosity, which is strongly dependent on length of the slope, increases with slope length changing the scale law from the laminar regime. An analytical expression that incorporates the observed length dependence of turbulent viscosity leads to a scaling law for the peak bottom velocity which is in good agreement with the scaling observed in the simulations. The simulations show that the velocity profiles of different cases, tend to collapse into a single curve when normalized using the beam peak velocity and the beam width. The energy content in higher harmonics relative to that in the fundamental increases with slope length. Baroclinic energy flux, turbulent kinetic energy and dissipation rate increase substantially with increasing slope length, $l$. The nondimensional baroclinic energy flux, $\langle F^\parallel \rangle$, shows
a maximum value for the smallest slope, decreases with increasing \( l \), and appears to approach a saturation value, \( \langle \mathcal{F}^\parallel \rangle \simeq 0.25 \). The substantial increase of conversion to turbulence with increasing slope length (turbulent production as high as 18% of the internal wave flux for the largest slope length) is a likely contributor to the decrease in the normalized flux, \( \langle \mathcal{F}^\parallel \rangle \), with increasing slope length and its potential saturation.

During the generation of internal tides at critical slopes, we have observed that high vertical number modes are formed which combine to give internal tidal beams. Our numerical experiments have shown evidence of turbulence in tidal beams over slope topography with length of the order \( O(1-30) \) m. However, the beam is at laboratory scale and did not allow the large separation in scales between the viscous boundary layer and the oscillating core of the beam that occurs in the ocean. Also, it would be impractical to perform a turbulence-resolving simulation in a streamwise nonperiodic domain with the large slope length of several hundred meters that is required to generate a beam with width of \( O(100) \) m. Therefore, based on the scaling law in Chapter 5, we modeled a beam with a given width, \( l_b = 60 \) m, and peak near-bottom velocity, \( u_b = 0.125 \) m/s, on a slope and simulate phase-dependent turbulence in a small patch of the beam. Selection of the small patch of allows us to use a streamwise-periodic solver that is computationally less demanding. The self-similarity of the velocity profiles found in Chapter 5 was used to construct the initial velocity profile. A dynamic eddy viscosity model is used for LES to simulate the flow at large Reynolds number. One important finding presented in Chapter 6 is that strong turbulent mixing events are found to repeat at specific phases in the internal tide cycle. Immediately after the zero velocity point when the flow reverses from down to upslope, there is a burst of turbulence with large dissipation that spans the beam and lasts for about 1.5 hours. This burst is initiated by a convective instability detached from the bottom and is accompanied by a large positive buoyancy flux. There is remarkable agreement between our simulation and recent ocean observations (Aucan & Merrifield, 2008) of convectively-driven mixing off Kaena Ridge at Hawaii and our results could explain the underlying mechanism of ocean hotspots of turbulent mixing at near-
critical regions (Nash et al., 2007) that occur at wave generation sites. A significant amount of negative turbulent production spanning about 1 h occurs during this turbulence episode indicating the transfer of energy from the fluctuating field to the mean field. In Chapter 7 it is illustrated that inclined turbulent structures initiated by buoyancy (not shear) are distorted by the non-zero mean shear that occurs after the velocity passes through its zero value, resulting in negative production.

After generation, an IW beam often traverses into the upper part of the ocean away from the topography. During its propagation through a non-uniform stratified environment, a beam encounters a sharp density interface (pycnocline). Chapter 8 presents results from numerical investigations of the interaction of an internal-wave beam with a model upper ocean stratification, i.e., pycnocline, transition layer and mixed layer. Initially, the internal beam response is linear. During transmission through the transition layer, the IW beam narrows owing to enhancement of the vertical wave number and finally it reflects from the caustic where the frequency of the internal wave beam matches with background buoyancy frequency. As time progresses, nonlinear effects lead to the formation of higher harmonics. The harmonics which have frequency higher than the buoyancy frequency of the lower medium are unable to propagate into the lower medium. These harmonics undergo multiple reflections inside the pycnocline and are subsequently trapped inside it resulting in significant energy absorption from the IW beam. At a later time, inside the pycnocline, the incoming beam undergoes nonlinear parametric subharmonic instability (PSI) resulting in a significant energy transfer to subharmonic motions. A resonant triad is identified both in the wave frequency and in wave number space during the PSI. The smaller vertical scales that originate during the subharmonic resonance lead to wave steepening and eventually a convective instability. This work offers a possible mechanism for IW beam degradation and breaking during its propagation through the upper ocean.
Appendix A

Algorithm and Parallelization of CFD code

We numerically solve the Navier-Stokes equations for incompressible flow under the Boussinesq approximation in the following form:

\[ \frac{\partial u_j}{\partial x_j} = 0, \]

\[ \frac{\partial u_i}{\partial t} + \frac{\partial u_j u_i}{\partial x_j} = \nu \frac{\partial^2 u_i}{\partial x_j^2} - \frac{\partial p}{\partial x_i} - \delta_{ij} P_i(x, t) + F_i, \]

where \( F_i \) is a body force exerted on the fluid and \( P_i \) is an imposed pressure gradient. A discrete set of points in space is used to numerically represent spatial variation of the flow, and, to characterize temporal evolution, time advancement is done in discrete time steps. It is efficient to march in large time steps with as few spatial points as possible without compromising the accuracy (in both space and time) and stability of the simulation. We have adopted an algorithm (see Bewley (2012) for details) based on a mixed spectral/finite difference method. A low-storage third-order Runge-Kutta-Wray method is used for time stepping, except for the viscous terms which are treated implicitly. In this section we briefly discuss code structure for the channel geometry, (periodic in streamwise and spanwise direction, walls in lateral direction) adopted from the numerical algorithm of Taylor (2008), and the new code written for curvilinear geometry.
A.1 Algorithm: Channel Geometry

Here, the algorithm for channel flow with periodic boundary conditions for the velocity in the $x_1$ (x) and $x_3$ (z) directions and walls bounding the flow in the $x_2$ (y) direction is shown schematically. Derivatives in the wall-normal direction are treated with second order, central finite differences, while the $x_1$ (x) and $x_3$ (z) directions are treated with a pseudo-spectral method. Time-stepping is accomplished with a mixed implicit/explicit strategy with all terms involving wall-normal derivatives stepped with Crank-Nicolson and all other terms treated with a low storage 3rd order Runge-Kutta method. The right hand side of the momentum equations $\partial u_i/\partial t = ...$ are stored in $R_i$ while the Runge-Kutta terms are stored in $F_i$ and saved for the next R-K substep. $\hat{u}_i, \hat{R}_i$, etc. denote the Fourier space representations. To minimize the memory size, the physical and Fourier space arrays are assigned the same location in memory in an efficient way. In order to clarify the operations, the intermediate and final velocity will both be denoted by $u_i$. It is implied that each step is done over all grid points in physical space or modes in Fourier space.

In the above algorithm, the base grid ($G$ in Fig. A.1) is denoted by $Y$ and the fractional grid ($G_{1/2}$ in Fig. A.1) is denoted by $Y_F$. The grid spacing centered at the base and fractional grids are denoted by $\Delta Y$ and $\Delta Y_F$, respectively, where

$$\Delta Y(j) = Y_F(j) - Y_F(j - 1), \Delta Y_F(j) = Y(j + 1) - Y(j).$$  \hspace{1cm} (A.1)

Grid stretching is used in the wall-bounded directions in order to resolve small-scale turbulence near the wall. The $G_{1/2}$ cells are located exactly halfway between neighboring $G$ points. The staggering is done so that neighboring pressure values are coupled. The location of the walls as marked by cross-hatching in Fig. A.1, are chosen to coincide with wall-parallel velocity points. Interpolation from the $G$ grid to the $G_{1/2}$ grid is then accomplished by taking the average of neighboring values which is second-order accurate. For example, interpolation of the horizontal
The wall-normal velocity is stored at $G$ points (open circles), all other variables are stored at $G_F$ points (closed circles). Note that $G$ here stands for $G_X$, $G_Y$, and/or, $G_Z$, depending on which directions are wall-bounded.

Velocity components to the base grid is given by

$$
\overline{u_1}(i, j, k) = \frac{1}{2} (u_1(i, j, k) + u_1(i, j - 1, k)), \quad (A.2)
$$

$$
\hat{u}_1(i, j, k) = \frac{1}{2\Delta Y(j)} (\Delta Y_F(j)u_1(i, j, k) + \Delta Y_F(j - 1)u_1(i, j - 1, k)). \quad (A.3)
$$

The computational domain is discretized in the horizontal directions with a uniform grid-spacing and co-located variables, allowing these directions to be transformed efficiently to and from Fourier space. The discrete Fourier transforms are calculated using the freely available FFTW software (see www.fftw.org).

Below is the outline of an algorithm based on the channel flow geometry.
For $t = 1 \ldots \text{(\# of time steps)}$
For $rk = 1 \ldots 3$

1. Start building the right hand side array with the previous velocity in Fourier space ($\hat{u}_i$)
   $$\hat{R}_i = \hat{u}_i,$$
2. If $(rk > 1)$ then add the term from the previous $rk$ step
   $$\hat{R}_i = \hat{R}_i + \zeta_{rk} \hat{F}_i$$
3. Add the pressure gradient to the RHS
   $$\hat{R}_1 = \hat{R}_1 - \hbar_{rk} \hat{k}_x \hat{P}$$
   $$\hat{R}_2(k_x, k_z, j) = \hat{R}_2(k_x, k_z, j) - \hbar_{rk} \frac{\hat{P}(k_x, k_z, j) - \hat{P}(k_x, k_z, j-1)}{\Delta Y(j)}$$
   $$\hat{R}_3 = \hat{R}_3 - \hbar_{rk} \hat{k}_z \hat{P}$$
4. Add $P_x$, the background pressure gradient that drives the flow
   $$\hat{R}_1(k_x = 0, k_z = 0, j) = \hat{R}_1(k_x = 0, k_z = 0, j) - \hbar_{rk} P_x$$
5. Create a storage variable $F$ that will contain all Runge-Kutta terms and start with the viscous terms involving horizontal derivatives.
   $$\hat{F}_i = -\nu (k_x^2 + k_z^2) \hat{u}_i,$$
6. Convert the velocity to physical space
   $$\hat{u}_i \to u_i$$
7. Add the nonlinear terms involving horizontal derivatives to $\hat{F}$
   $$\hat{F}_1 = \hat{F}_1 - i \hat{k}_x u_1 \hat{u}_1 - i \hat{k}_z u_1 \hat{u}_3$$
   $$\hat{F}_2 = \hat{F}_2 - i \hat{k}_x u_1 \hat{u}_2 - i \hat{k}_z u_3 \hat{u}_2$$
   $$\hat{F}_3 = \hat{F}_3 - i \hat{k}_x u_1 \hat{u}_3 - i \hat{k}_z u_3 \hat{u}_3$$
   (Note that we need 5 independent FFTs here)
8. Now, we are done building the Runge-Kutta terms, add to the right hand side. We will need to keep $\hat{F}_i$ for the next $rk$ step, so it should not be overwritten below this point.
   $$\hat{R}_i = \hat{R}_i + \beta_{rk} \hat{F}_i$$
9. Convert the right hand side arrays to physical space
\[ \hat{R}_i \rightarrow R_i, \]

10. Compute the vertical viscous terms and add to the RHS as the explicit part of Crank-Nicolson.
\[
R_1(i, j, k) = R_1(i, j, k) + \frac{\nu \bar{h}_{rk}}{2\Delta Y_F(j)} \left( \frac{u_1(i, j + 1, k) - u_1(i, j, k)}{\Delta Y(j + 1)} \right) - \frac{\nu \bar{h}_{rk}}{2\Delta Y_F(j)} \left( \frac{u_1(i, j, k) - u_1(i, j - 1, k)}{\Delta Y(j)} \right)
\]
\[
R_2(i, j, k) = R_2(i, j, k) + \frac{\nu \bar{h}_{rk}}{2\Delta Y_F(j)} \left( \frac{u_2(i, j + 1, k) - u_2(i, j, k)}{\Delta Y_F(j)} \right) - \frac{\nu \bar{h}_{rk}}{2\Delta Y_F(j)} \left( \frac{u_2(i, j, k) - u_2(i, j - 1, k)}{\Delta Y_F(j - 1)} \right)
\]
\[
R_3(i, j, k) = R_3(i, j, k) + \frac{\nu \bar{h}_{rk}}{2\Delta Y_F(j)} \left( \frac{u_3(i, j + 1, k) - u_3(i, j, k)}{\Delta Y(j + 1)} \right) - \frac{\nu \bar{h}_{rk}}{2\Delta Y_F(j)} \left( \frac{u_3(i, j, k) - u_3(i, j - 1, k)}{\Delta Y(j)} \right)
\]

11. Compute the nonlinear terms involving vertical derivatives and add to the RHS.
\[
S_1 = \bar{u}_1 * u_2
\]
\[
R_1(i, j, k) = R_1(i, j, k) - \bar{h}_{rk} \left( S_1(i, j + 1, k) - S_1(i, j, k) \right) / \Delta Y_F(j)
\]
\[
S_2 = \bar{u}_2 * u_2
\]
\[
R_2(i, j, k) = R_2(i, j, k) - \bar{h}_{rk} \left( S_1(i, j + 1, k) - S_1(i, j, k) \right) / \Delta Y_F(j)
\]
\[
S_3 = \bar{u}_3 * u_2
\]
\[
R_3(i, j, k) = R_3(i, j, k) - \bar{h}_{rk} \left( S_1(i, j + 1, k) - S_1(i, j, k) \right) / \Delta Y_F(j)
\]

12. Solve the tridiagonal system for the intermediate wall-normal velocity for each \( i, j \):
\[
u \bar{h}_{rk} \left( \frac{u_2(i, j + 1, k) - u_2(i, j, k)}{\Delta Y_F(j)} - \frac{uu_2(i, j, k) - u_2(i, j - 1, k)}{\Delta Y_F(j - 1)} \right) / \Delta Y(j)
\]
\[= R_2(i, j, k)\]
13. Similarly, solve the tridiagonal system for the intermediate $u_1$ and $u_3$.

\[
u \frac{\nabla_{rk}}{2} \left( \frac{u_1(i, j + 1, k) - u_1(i, j, k)}{\Delta Y(j + 1)} - \frac{u_1(i, j, k) - u_1(i, j - 1, k)}{\Delta Y(j)} \right) / \Delta Y_F(j) = R_1(i, j, k)
\]

\[
u \frac{\nabla_{rk}}{2} \left( \frac{u_3(i, j + 1, k) - u_3(i, j, k)}{\Delta Y(j + 1)} - \frac{u_3(i, j, k) - u_3(i, j - 1, k)}{\Delta Y(j)} \right) / \Delta Y_F(j) = R_3(i, j, k)
\]

14. Convert the intermediate velocity to Fourier space

\[u_i \rightarrow \hat{u}_i\]

15. Solve the tridiagonal system for the pressure correction:

\[-(k_x^2 + k_z^2)\hat{\phi}(k_x, k_z, j) + \left( \frac{\hat{\phi}(k_x, k_z, j + 1) - \hat{\phi}(k_x, k_z, j - 1)}{\Delta Y(j)} \right) / \Delta Y_F(j) = \hat{k}_x\hat{u}_1(k_x, k_z, j) + \hat{k}_z\hat{u}_3(k_x, k_z, j) + (\hat{u}_2(k_x, k_z, j + 1) - \hat{u}_2(k_x, k_z, j)) / \Delta Y_F(j)\]

(Note that in order to avoid an extra storage array, we can store $\phi$ in $R_1$ which is no longer needed for this $rk$ step. Also notice that a factor of $\nabla_{rk}$ has been absorbed into $\phi$)

16. Now, use the gradient of the pressure correction to obtain a divergence-free velocity field.

\[\hat{u}_1^{k+1} = \hat{u}_1 - i\hat{k}_x\hat{\phi}\]

\[\hat{u}_2^{k+1} = \hat{u}_2(k_x, k_z, j) - \left( \frac{\hat{\phi}(k_x, k_z, j) - \hat{\phi}(k_x, k_z, j - 1)}{\Delta Y(j)} \right) / \Delta Y_F(j)\]

\[\hat{u}_3^{k+1} = \hat{u}_3 - i\hat{k}_z\hat{\phi}\]

(In order to avoid an extra storage array, only one set of velocity arrays are defined, and this update is done in place.)

17. Finally, update the pressure field using $\phi$

\[\hat{P} = \hat{P} + \hat{\phi} / \nabla_{rk}\]

(We need to divide by $\nabla_{rk}$ since this constant had been absorbed into $\phi$ in
In all, we have 14 FFT calls per Runge-Kutta substep, and 11 full-sized storage arrays.

The parameters used in the third order Runge-Kutta algorithm given in Table A.1.

<table>
<thead>
<tr>
<th>R-K Step</th>
<th>$h$</th>
<th>$\beta$</th>
<th>$\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$8/15 \Delta t$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$2/15 \Delta t$</td>
<td>$25/8$</td>
<td>$-17/8$</td>
</tr>
<tr>
<td>3</td>
<td>$1/3 \Delta t$</td>
<td>$9/4$</td>
<td>$-5/4$</td>
</tr>
</tbody>
</table>

**Figure A.2:** Computational grid discretization, ($\zeta - \eta$ coordinate system) in the physical domain. Here, $x$ and $\xi$ are directed into the plane of the figure. Volume flux, $U_{i}^{\xi}$ is located at the face center of grid cell parallel to the grid line and velocity, $u_{i}^{\xi}$ is located at the cell center in direction to physical coordinate system.

### A.2 Algorithm: Curvilinear Geometry

Here, the algorithm for curvilinear flow with periodic boundary condition for the velocity in the $x_1$ (x) directions and walls bounding the flow in the $x_2$
(y) and $x_3$ (z) directions is sketched in the figure A.2. Derivatives in the wall-normal directions are treated with second order, central finite differences, while the $x_1$ (x) direction is treated with a pseudo-spectral method. Time-stepping is accomplished with a mixed implicit/explicit strategy with all terms involving wall-normal derivatives stepped with Alternating Direction Implicit (ADI) method and all other terms treated with a low storage 3rd order Runge-Kutta method. The right hand side of the momentum equations $\partial u_i / \partial t = ...$ are stored in $R_i$ while the Runge-Kutta terms will be stored in $F_i$ and saved for the next R-K substep. $\hat{u}_i$, $\hat{R}_i$, etc. denote the Fourier space representations. To minimize the memory size, the physical and Fourier space arrays are assigned the same location in memory in an efficient way. In order to clarify the operations, the intermediate and final velocity will both be denoted by $u_i$. It is implied that each step is done over all grid points in physical space or modes in Fourier space.

In the above algorithm, a non-staggered grid system is used with grid lines aligning with the physical boundary. Cartesian velocity, $u_i$ and pressure are defined at the center and the volume flux, $U_i^c$ at the face center of the corresponding control volume as shown in figure A.2. Here, volume flux, $U_j^c = J^{-1}\partial \xi_j u_i / \partial x_i$ which is defined based on contravariant velocity multiplied by $J^{-1}$, is normal to the surface of constant $\xi_j$. $J^{-1}$, the inverse of the the determinant of the Jacobian, is the volume of the cell in physical space. After discretization with the $\zeta$ and $\eta$ lines, we can transform physical domain into $\zeta - \eta$ plane. In this regard, NS equation has to be rewritten to the form of a strong-conservation law as described in section 2.3.1. Typical grid structure in the transformed $(\zeta - \eta)$ plane are shown in figure A.3. Here control volume is surrounded by four faces. Physical grid information is stored in the the transformed parameter, $\partial x_i / \partial \xi_j$ evaluated as

\begin{align}
z_\zeta &\approx z_{k+1,j} - z_{k,j} & (A.4) \\
z_\eta &\approx z_{k,j+1} - z_{k,j} & (A.5) \\
y_\zeta &\approx y_{k+1,j} - y_{k,j} & (A.6) \\
y_\eta &\approx y_{k+1,j} - y_{k,j} & (A.7)
\end{align}

Here, the mapping from the $y - z$ plane to $\zeta - \eta$ is performed in such a way
Figure A.3: Grid layout of the curvilinear geometry in $\zeta - \eta$ plane. The velocity, pressure and inverse of Jacobian, $J^{-1}$ are stored at the cell center points (closed circles), mesh skewness tensor tensor, $G^{ij}$, Volume flux, $U^c_i$, and vector surface area, $C_{ij}$ are stored at the face center points (opened circles).

That in the transformed coordinate, distance between two adjacent grid lines is always unity ie. $\Delta \zeta = \Delta \eta = 1$. This helps to calculate derivative with respect to $\zeta$ or $\eta$ very easily (Fletcher, 1991). Vector surface area, $C_{ij}$ and for second derivatives mesh skewness tensor, $G^{ij}$ defined to evaluating the gradient and laplacian of the velocity as

$$J^{-1} \left[ \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \right] u_i = \left[ \frac{\partial}{\partial \zeta} J^{-1}(\zeta + \eta) + \frac{\partial}{\partial \eta} J^{-1}(\zeta + \eta) + \frac{\partial}{\partial \zeta} J^{-1}(\zeta + \eta) + \frac{\partial}{\partial \eta} J^{-1}(\zeta + \eta) \right] u_i \quad (A.8)$$

$$J^{-1} \left[ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right] u_i = \left[ \frac{\partial}{\partial \zeta} J^{-1}(\zeta^2 + \eta^2) + \frac{\partial}{\partial \eta} J^{-1}(\zeta^2 + \eta^2) + \frac{\partial}{\partial \zeta} J^{-1}(\zeta^2 + \eta^2) + \frac{\partial}{\partial \eta} J^{-1}(\zeta^2 + \eta^2) \right] u_i \quad (A.9)$$

$J^{-1}$ are stored at cell center points where as mesh skewness tensor tensor, $G^{ij}$,
Volume flux, $U_i$, and vector surface area, $C_{ij}$ are stored at face center points (Zang et al., 1994). Therefore, we have introduced additional third index for both $C_{ij}$ and $G_{ij}$ to specify face location; 1 for horizontal face and 2 for vertical face as shown in figure A.3. This results in total storage of 8 matrices per control volume. The computational domain is discretized in the $x_1$ direction with a uniform grid-spacing and co-located variables, allowing these directions to be transformed efficiently to and from Fourier space. The discrete Fourier transforms are calculated using the freely available FFTW software.

Below is the outline of an algorithm based on the curvilinear flow geometry.

For $t = 1 \ldots \#$ of time steps
For $rk = 1 \ldots 3$

1. Start building the right hand side array with the previous velocity in Fourier space ($\hat{u}_i$)
   $$\hat{R}_i = J^{-1}\hat{u}_i,$$

2. If ($rk > 1$) then add the term from the previous $rk$ step
   $$\hat{R}_i = \hat{R}_i + \zeta_{rk}\beta_{rk}\hat{F}_i$$

3. Add the pressure gradient to the RHS
   $$\hat{R}_1 = \hat{R}_1 - J^{-1}\hat{h}_{rk}\hat{k}_x\hat{P}$$
   $$\hat{S}_1 = \frac{1}{2}\left[ C_{12}^{(k+1,j,2)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k + 1, j)) - C_{12}^{(k,j,2)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k - 1, j)) + C_{22}^{(k,j+1,1)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k, j + 1)) - C_{22}^{(k,j,1)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k, j - 1))\right]$$
   $$\hat{R}_2 = \hat{R}_2 - \hat{h}_{rk}\hat{S}_1 \quad \text{add} \quad \left[ \frac{d(J^{-1}\zeta_y P)}{d\zeta} + \frac{d(J^{-1}\eta_y P)}{d\eta} \right]$$
\[
\hat{S}_1 = \frac{1}{2} \left[ C_{11}^{(k+1,j,2)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k + 1, j)) - C_{11}^{(k,j,2)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k - 1, j)) + C_{21}^{(k,j+1,1)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k, j + 1)) - C_{21}^{(k,j,1)}(\hat{P}(k_x, k, j) + \hat{P}(k_x, k, j - 1)) \right]
\]

\[
\hat{R}_3 = \hat{R}_3 - \hat{h}_{rk}\hat{S}_1 \quad \text{add} \quad \left[ \frac{d(J^{-1}\zeta P)}{d\zeta} + \frac{d(J^{-1}\eta P)}{d\eta} \right]
\]

4. Add \(P_x\), the background pressure gradient that drives the flow

\[
\hat{R}_1(k_x = 0, k, j) = \hat{R}_1(k_x = 0, k, j) - J^{-1}\hat{h}_{rk}P_x
\]

5. Create a storage variable \(F\) that will contain all Runge-Kutta terms and start with the viscous terms involving horizontal derivatives.

\[
\hat{F}_1 = -\nu J^{-1}k_x^2\hat{u}_i,
\]

6. Convert the velocity to physical space, here \(\hat{U}_i^c\) is from step \# 14 except for first time \(\hat{U}_i^c\) is calculated from step \# 14 before step \# 6

\[
\hat{u}_i \to u_i \\
\hat{U}_i^c \to U_i^c
\]

7. Add the nonlinear terms involving horizontal derivatives to \(\hat{F}\)

\[
S_1 = \frac{1}{2}(U_x^c(i, k, j + 1)(u_1(i, k, j) + u_1(i, k, j + 1)) - U_x^c(i, k, j)(u_1(i, k, j) + u_1(i, k, j - 1)) + \frac{1}{2}(U_x^c(i, k + 1, j)(u_1(i, k, j) + u_1(i, k + 1, j)) - U_x^c(i, k, j)(u_1(i, k, j) + u_1(i, k - 1, j))
\]

\[
\hat{F}_1 = \hat{F}_1 - J^{-1}\hat{i}k_x\hat{u}_i\hat{u}_1 - \hat{S}_1 \quad \text{add} \quad \left[ \frac{dU1u1}{dx} + \frac{dU2u1}{d\eta} + \frac{dU3 * u1}{d\zeta} \right]
\]
\[ S_1 = \frac{1}{2}(U_2^*(i, k, j + 1)(u_2(i, k, j) + u_2(i, k, j + 1)) - U_2^*(i, k, j)(u_2(i, k, j) + u_2(i, k, j - 1)) \]
\[ + \frac{1}{2}(U_3^*(i, k, j + 1)(u_2(i, k, j) + u_2(i, k + 1, j)) - U_3^*(i, k, j)(u_2(i, k, j) + u_2(i, k - 1, j)) \]
\[ \hat{F}_2 = \hat{F}_2 - J^{-1}ik_xu_2u_1 - \hat{S}_1 \quad \text{add} - \left[ \frac{dU_1u_2}{dx} + \frac{dU_2u_2}{d\eta} + \frac{dU_3u_2}{d\zeta} \right] \]
\[ S_1 = \frac{1}{2}(U_2^*(i, k, j + 1)(u_3(i, k, j) + u_3(i, k, j + 1)) - u_3(i, k, j)(u_3(i, k, j) + u_3(i, k, j - 1)) \]
\[ + \frac{1}{2}(U_3^*(i, k, j + 1)(u_3(i, k, j) + u_3(i, k + 1, j)) - U_3^*(i, k, j)(u_3(i, k, j) + u_3(i, k - 1, j)) \]
\[ \hat{F}_3 = \hat{F}_3 - J^{-1}ik_xu_3u_1 - \hat{S}_1 \quad \text{add} - \left[ \frac{dU_1u_3}{dx} + \frac{dU_2u_3}{d\eta} + \frac{dU_3u_3}{d\zeta} \right] \]

(Note that we need 3 independent FFTs here)

8. Add the mixed derivative terms from the diffusion terms to \( \hat{F} \)

\[ S_1 = \frac{1}{4}\nu[G_{12}^{(k+1,j,2)}\{u_1(i, k+1, j+1) + u_1(i, k, j+1) - u_1(i, k, j-1) - u_1(i, k+1, j-1)\}] \]
\[ - G_{12}^{(k,j,2)}\{u_1(i, k-1, j+1) + u_1(i, k, j+1) - u_1(i, k, j-1) - u_1(i, k-1, j-1)\}] \]
\[ S_2 = \frac{1}{4}\nu[G_{12}^{(k,j+1,2)}\{u_1(i, k+1, j+1) + u_1(i, k+1, j) - u_1(i, k-1, j-1) - u_1(i, k-1, j)\}] \]
\[ - G_{12}^{(k,j,2)}\{u_1(i, k+1, j) + u_1(i, k+1, j-1) - u_1(i, k-1, j) - u_1(i, k-1, j-1)\}] \]
\[ \hat{F}_1 = \hat{F}_1 + \hat{S}_1 + \hat{S}_2 \quad \text{add} \quad \left[ \nu \frac{d}{d\eta} \left( G_{12} \frac{d}{d\zeta} u_1 \right) + \nu \frac{d}{d\zeta} \left( G_{12} \frac{d}{d\eta} u_1 \right) \right] \]
$$S_1 = \frac{1}{4} \nu [G_{12}^{(k+1,j,2)} \{u_2(i, k+1, j+1) + u_2(i, k, j+1) \\ - u_2(i, k, j-1) - u_2(i, k+1, j-1) \} \\ - G_{12}^{(k,j,2)} \{u_2(i, k-1, j+1) + u_2(i, k, j+1) \\ - u_2(i, k, j-1) - u_2(i, k-1, j-1) \}]$$

$$S_2 = \frac{1}{4} \nu [G_{12}^{(k,j+1,2)} \{u_2(i, k+1, j+1) + u_2(i, k+1, j) \\ - u_2(i, k-1, j-1) - u_2(i, k-1, j) \} \\ - G_{12}^{(k,j,2)} \{u_2(i, k+1, j) + u_2(i, k+1, j-1) \\ - u_2(i, k-1, j) - u_2(i, k-1, j-1) \}]$$

$$\hat{F}_2 = \hat{F}_2 + [S1 + S2] \text{ add } \left[ \nu \frac{d}{d\eta} \left( G_{12} \frac{du_2}{d\zeta} \right) + \nu \frac{d}{d\zeta} \left( G_{12} \frac{du_2}{d\eta} \right) \right]$$

$$S_1 = \frac{1}{4} \nu [G_{12}^{(k+1,j,2)} \{u_3(i, k+1, j+1) + u_3(i, k, j+1) \\ - u_3(i, k, j-1) - u_3(i, k+1, j-1) \} \\ - G_{12}^{(k,j,2)} \{u_3(i, k-1, j+1) + u_3(i, k, j+1) \\ - u_3(i, k, j-1) - u_3(i, k-1, j-1) \}]$$

$$S_2 = \frac{1}{4} \nu [G_{12}^{(k,j+1,2)} \{u_3(i, k+1, j+1) + u_3(i, k+1, j) \\ - u_3(i, k-1, j-1) - u_3(i, k-1, j) \} \\ - G_{12}^{(k,j,2)} \{u_3(i, k+1, j) + u_3(i, k+1, j-1) \\ - u_3(i, k-1, j) - u_3(i, k-1, j-1) \}]$$

$$\hat{F}_3 = \hat{F}_3 + [S1 + S2] \text{ add } \left[ \nu \frac{d}{d\eta} \left( G_{12} \frac{du_3}{d\zeta} \right) + \nu \frac{d}{d\zeta} \left( G_{12} \frac{du_3}{d\eta} \right) \right]$$

9. Now, we are done building the Runge-Kutta terms, add to the right hand side. We will need to keep $\hat{F}_i$ for the next $rk$ step, so it should not be overwritten below this point.

$$\hat{R}_i = \hat{R}_i + \beta_{rk} \hat{T}_{rk} \hat{F}_i$$

10. Convert the right hand side arrays to physical space

$$\hat{R}_i \rightarrow R_i$$

11. Using ADI method each RK step is subdivided into two equal time steps.

**ADI step-I:** Splitting is done in $x_2 -$ direction at time $n$ and $n + 1/2$. Solve
the tridiagonal system for the velocity at intermediate time step \( n + 1/2 \) for each \( i, j \) using top and bottom boundary conditions:

\[
J^{-1}u_1^{n+1/2}(i, j, k) - \frac{\nu h_{rk}}{2} \left[ G_{22}^{(k,j+1,1)} \left( u_1^{n+1/2}(i, k, j + 1) - u_1^{n+1/2}(i, k, j) \right) \\
- G_{22}^{(k,j,1)} \left( u_1^{n+1/2}(i, k, j) - u_1^{n+1/2}(i, k, j - 1) \right) \right] = R^n_1(i, k, j) + \frac{\nu h_{rk}}{2} \left( G_{11}^{(k+1,j,1)}(u_1^n(i, k + 1, j) - u_1^n(i, k, j)) \\
- G_{11}^{(k,j,1)}(u_1^n(i, k, j) - u_1^n(i, k - 1, j)) \right)
\]

**ADI step-II:** Splitting is done in \( x_3 \)-direction at time \( n + 1/2 \) and \( n + 1 \). Solve the tridiagonal system for the velocity at time step \( n + 1 \) for each \( i, j \) using left and right boundary conditions:

\[
J^{-1}u_1^n(i, j, k) - \frac{\nu h_{rk}}{2} \left[ G_{11}^{(k+1,j,1)} \left( u_1^n(i, k + 1, j) - u_1^n(i, k, j) \right) \\
- G_{11}^{(k,j,1)} \left( u_1^n(i, k, j) - u_1^n(i, k - 1, j) \right) \right] = J^{-1}u_1^{n+1/2}(i, k, j) + \frac{\nu h_{rk}}{2} \left( G_{22}^{(k,j+1,1)}(u_1^n(i, k, j + 1) - u_1^n(i, k, j)) \\
- G_{22}^{(k,j,1)}(u_1^n(i, k, j) - u_1^n(i, k, j - 1)) \right)
\]

12. Now that we have the new intermediate horizontal velocity, \( u_1 \), solve the tridiagonal system for the intermediate \( u_2 \) and \( u_3 \) using this new velocity.

**ADI step-I:**
\[ J^{-1}u_2^{n+1/2}(i, j, k) \]
\[ - \frac{\nu h_{rk}}{2} \left[ G_{22}^{(k,j+1,1)} \left( u_2^{n+1/2}(i, k, j + 1) - u_2^{n+1/2}(i, k, j) \right) \right. \]
\[ - \left. G_{22}^{(k,j,1)} \left( u_2^{n+1/2}(i, k, j) - u_2^{n+1/2}(i, k, j - 1) \right) \right] \]
\[ = R_n^1(i, k, j) \]
\[ + \frac{\nu h_{rk}}{2} \left( G_{11}^{(k+1,j,1)}(u_2^n(i, k + 1, j) - u_2^n(i, k, j)) \right. \]
\[ - \left. G_{11}^{(k,j,1)}(u_2^n(i, k, j) - u_2^n(i, k - 1, j)) \right) \]

**ADI step-II:**

\[ J^{-1}u_2^n(i, j, k) \]
\[ - \frac{\nu h_{rk}}{2} \left[ G_{11}^{(k+1,j,1)} \left( u_2^n(i, k + 1, j) - u_2^n(i, k, j) \right) \right. \]
\[ - \left. G_{11}^{(k,j,1)} \left( u_2^n(i, k, j) - u_2^n(i, k - 1, j) \right) \right] \]
\[ = J^{-1}u_2^{n+1/2}(i, k, j) \]
\[ + \frac{\nu h_{rk}}{2} \left( G_{22}^{(k,j+1,1)}(u_2^n(i, k, j + 1) - u_2^n(i, k, j)) \right. \]
\[ - \left. G_{22}^{(k,j,1)}(u_2^n(i, k, j) - u_2^n(i, k, j - 1)) \right) \]

Similarly for and \( u_3 \),

**ADI step-I:**
\[ J^{-1} u_3^{n+1/2}(i, j, k) \]
\[ - \frac{\nu h_{rk}}{2} \left[ G_{22}^{(k, j+1, 1)} \left( u_3^{n+1/2}(i, k, j + 1) - u_3^{n+1/2}(i, k, j) \right) \right. \]
\[ - \left. G_{22}^{(k, j, 1)} \left( u_3^{n+1/2}(i, k, j) - u_3^{n+1/2}(i, k, j - 1) \right) \right] \]
\[ = R_1^n(i, k, j) \]
\[ + \frac{\nu h_{rk}}{2} \left( G_{11}^{(k+1, j, 1)}(u_3^n(i, k + 1, j) - u_3^n(i, k, j)) \right. \]
\[ - \left. G_{11}^{(k, j, 1)}(u_3^n(i, k, j) - u_3^n(i, k - 1, j)) \right) \]

**ADI step-II:**

\[ J^{-1} u_3^n(i, j, k) \]
\[ - \frac{\nu h_{rk}}{2} \left[ G_{11}^{(k+1, j, 1)} \left( u_3^{n+1/2}(i, k + 1, j) - u_3^{n+1/2}(i, k, j) \right) \right. \]
\[ - \left. G_{11}^{(k, j, 1)} \left( u_3^{n+1/2}(i, k, j) - u_3^{n+1/2}(i, k - 1, j) \right) \right] \]
\[ = J^{-1} u_3^{n+1/2}(i, k, j) \]
\[ + \frac{\nu h_{rk}}{2} \left( G_{22}^{(k, j+1, 1)}(u_3^n(i, k, j + 1) - u_3^n(i, k, j)) \right. \]
\[ - \left. G_{22}^{(k, j, 1)}(u_3^n(i, k, j) - u_3^n(i, k - 1, j)) \right) \]

13. Convert the intermediate velocity to Fourier space

\[ u_i \rightarrow \hat{u}_i \]

14. Calculate contravariant velocity required to define at cell faces

\[ \hat{U}_2^c = \frac{1}{2} \left[ C_{22}^{k,j,1} (\hat{u}_2(k_x, k, j) + \hat{u}_2(k_x, k, j - 1)) \right. \]
\[ + \left. C_{21}^{k,j,1} (\hat{u}_3(k_x, k, j) + \hat{u}_3(k_x, k, j - 1)) \right] \]
\[ \hat{U}_3^c = \frac{1}{2} \left[ C_{11}^{k,j,2} (\hat{u}_3(k_x, k, j + 1) + \hat{u}_3(k_x, k - 1, j)) \right. \]
\[ + \left. C_{12}^{k,j,2} (\hat{u}_2(k_x, k, j) + \hat{u}_2(k_x, k - 1, j)) \right] \]
15. Solve the 2nd partial differential system for the pressure correction using 2-D multigrid:

\[ D_{zz} = \frac{1}{4} \left( (G_{11}^{k,j,2} + G_{11}^{k+1,j,2} + G_{11}^{k,j,1} + G_{11}^{k,j+1,1}) \right) \]

\[ D_{yz} = \frac{1}{2} \left( G_{12}^{k,j,2} + G_{12}^{k+1,j,2} + G_{12}^{k,j,1} + G_{12}^{k,j+1,1} \right) \]

\[ D_{yy} = \frac{1}{4} \left( G_{22}^{k,j,2} + G_{22}^{k+1,j,2} + G_{22}^{k,j,1} + G_{22}^{k,j+1,1} \right) \]

\[ D_z = G_{11}^{k+1,j,2} - G_{11}^{k,j,2} + G_{12}^{k,j,1} - G_{12}^{k,j+1,1} \]

\[ D_y = G_{22}^{k,j+1,1} - G_{22}^{k,j,1} + G_{12}^{k+1,j,2} - G_{12}^{k,j,2} \]

\[- J^{-1} k_x^2 \hat{\phi}(k_x, k, j) \]

\[ + D_y \left[ \hat{\phi}(k_x, k, j + 1) - \hat{\phi}(k_x, k, j) \right] + D_z \left[ \hat{\phi}(k_x, k + 1, j) - \hat{\phi}(k_x, k, j) \right] \]

\[ + D_{yz} \left[ \left( \hat{\phi}(k_x, k + 1, j + 1) - \hat{\phi}(k_x, k + 1, j) \right) - \left( \hat{\phi}(k_x, k, j) - \hat{\phi}(k_x, k, j) \right) \right] \]

\[ + D_{yy} \left[ \left( \hat{\phi}(k_x, k, j + 1) - \hat{\phi}(k_x, k, j) \right) - \left( \hat{\phi}(k_x, k, j) - \hat{\phi}(k_x, k, j - 1) \right) \right] \]

\[ + D_{zz} \left[ \left( \hat{\phi}(k_x, k + 1, j) - \hat{\phi}(k_x, k, j) \right) - \left( \hat{\phi}(k_x, k, j) - \hat{\phi}(k_x, k - 1, j) \right) \right] \]

\[ = J^{-1} k_x u_1(k_x, k, j) + \left[ \hat{U}_2^c(k_x, k, j + 1) - \hat{U}_2^c(k_x, k, j) \right] \]

\[ + \left[ \hat{U}_3^c(k_x, k + 1, j) - \hat{U}_3^c(k_x, k, j) \right] \]

(Note that in order to avoid an extra storage array, we can store \( \phi \) in \( R_1 \) which is no longer needed for this \( rk \) step. Also notice that a factor of \( h_{rk} \) has been absorbed into \( \phi \))
16. update contravariant velocities

\[
\begin{align*}
\hat{U}_2^c(k, x, j) &= \hat{U}_2^c(k, x, j) - C_{22}^{k,j,1}\left(\hat{\phi}(k, x, j) + \hat{\phi}(k, x, j - 1)\right) \\
&\quad - \frac{1}{4} C_{12}^{k,j,1}\left(\hat{\phi}(k, x, k + 1, j) + \hat{\phi}(k, x, k + 1, j - 1)\right) \\
&\quad - \hat{\phi}(k, x, k - 1, j) - \hat{\phi}(k, x, k - 1, j - 1) \\
\hat{U}_2^c(k, x, j) &= \hat{U}_2^c(k, x, j) - C_{11}^{k,j,2}\left(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k - 1, j)\right) \\
&\quad - \frac{1}{4} C_{12}^{k,j,2}\left(\hat{\phi}(k, x, k, j + 1) + \hat{\phi}(k, x, k - 1, j + 1)\right) \\
&\quad - \hat{\phi}(k, x, k, j - 1) - \hat{\phi}(k, x, k - 1, j - 1)
\end{align*}
\]

17. Now, use the gradient of the pressure correction to obtain a divergence-free velocity and contravariant velocity field.

\[
\begin{align*}
\hat{u}_1^{r+1}(k, x, j) &= \hat{u}_1(k, x, j) - \hat{k} x \hat{\phi} \\
\hat{u}_2^{r+1}(k, x, j) &= \hat{u}_2^{r+1}(k, x, j) - \frac{1}{2} J^{-1}\left[ C_{12}^{(k+1,j,2)}(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k + 1, j)) \\
&\quad + C_{22}^{(k,j,2)}(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k - 1, j)) \\
&\quad + C_{22}^{(k,j+1,1)}(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k, j + 1)) \\
&\quad + C_{22}^{(k,j+1,1)}(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k, j - 1)) \right] \\
\hat{u}_3^{r+1}(k, x, j) &= \hat{u}_3^{r+1}(k, x, j) - \frac{1}{2} J^{-1}\left[ C_{12}^{(k+1,j,2)}(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k + 1, j)) \\
&\quad + C_{21}^{(k,j+1,1)}(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k, j + 1)) \\
&\quad + C_{21}^{(k,j+1,1)}(\hat{\phi}(k, x, k, j) + \hat{\phi}(k, x, k, j - 1)) \right]
\end{align*}
\]

(In order to avoid an extra storage array, only one set of velocity arrays are defined, and this update is done in place.)

18. Finally, update the pressure field using \( \hat{\phi} \)

\[
\hat{P} = \hat{P} + \hat{\phi}/\bar{h}_{rk}
\]

(We need to divide by \( \bar{h}_{rk} \) since this constant had been absorbed into \( \hat{\phi} \) in
the steps above.)

The parameters used in the third order Runge-Kutta algorithm given in table A.1.

A.3 Code Parallelization

Parallel computing is a form of computation by which many calculations are carried out simultaneously based on the principle that large problems can often be divided into smaller ones, which are then solved concurrently. The main goal of the parallel computing for fluid dynamics is to allow us to solve larger set of problems on several computers (or several processors within one computer) more efficiently instead of doing on a single computer for longer period of time. For direct numerical simulations (DNS) at large Reynolds numbers, parallel computation often becomes a necessity owing both to speed and memory limitations. Solving the Navier-Stokes equations involves performing the same set of operations at a large number of grid points, an indication that parallel computing may be effective for this problem. There are two basic approaches to parallel programming depending on the type of hardware, specifically whether memory is shared or distributed among the processors. It is now becoming common for new PCs to use CPUs with multiple cores on a single silicon chip, or to have multiple CPUs within one computer. These are examples of shared memory systems. It has become less common for supercomputers to share memory among all processors, but often processors within a single node are able to share memory. An example of a distributed memory system would be a cluster of PCs connected via a network, or separate nodes of a supercomputer. There are two basic levels of parallelization based on the memory arrangement:

- Shared memory architecture.
- Distributed memory architecture.
A.3.1 Shared memory architecture

For shared memory systems, a set of tasks is split into multiple threads each of which is able to access all of the address space. This effectively eliminates the need for communication between threads. The main task of the programmer is then to distribute the computational tasks among the threads with the goal of achieving an adequate load balance and minimizing the amount of time that any thread must wait for the others to complete a task. Since the threads utilize the same address space, it is important to ensure that they do not interfere with each other. In order to prevent this from occurring, synchronization (or barrier) commands are used to make sure that all threads have reached the specified point before proceeding further. A standard low-level set of thread commands is known as POSIX. This is commonly used in system-level programming and gives the programmer full control over the threads, but also requires significant modifications to a serial code.

The OpenMP standard provides a higher-level set of constructs for programming among a variety of shared memory architectures/platforms. OpenMP is an implementation of multithreading, a method of parallelization whereby the master ”thread” (a series of instructions executed consecutively) ”forks” a specified number of slave ”threads” and a task is divided among them. The threads then run concurrently, with the runtime environment allocating threads to different processors. An example of a structure of OpenMP compiler directives is shown in Figure A.4. Before any task which is required in parallel computing the master
thread then creates a team of parallel threads. Each thread has an "id" attached to it which can be obtained using a function (called \texttt{omp\_get\_thread\_num()}). The thread id is an integer, and the master thread has an id of "0". After the execution of the parallelized code, the threads "join" back into the master thread, which continues onward to the end of the program. Here is an example of a loop parallelized with OpenMP:

\begin{verbatim}
! On an OpenMP compatible compiler, the DO I loop will be distributed among the active threads
!
!$OMP DO
DO I=1,NX
!Parallel section executed by all threads
.
END DO
!$OMP END DO
\end{verbatim}

For more details about parallel computing with OpenMP see Chapman \textit{et al}. (2008).

### A.3.2 Distributed memory architecture

\textit{Distributed memory} defines as a multiple-processor computer system in which each processor has its own private memory and other processes are unable to access it directly. Computational tasks can only operate on local data, and if remote data is required, the computational task must communicate with one or more remote processors. An example of this type of system is called ‘beowulf cluster’ which consists of a number of personal computers connected through a network as shown in figure A.5. Many modern supercomputers use a combination of distributed and shared memory with memory distributed over a number of nodes each consisting of several processors in a shared memory configuration. There are several models for parallel computing on distributed memory systems.

The most common and useful programming construct for distributed memory systems is the Message Passing Interface (MPI) which is a standardized library
Figure A.5: An illustration of distributed memory System. Here, processors are connected through a network hub.

Table A.2: Essential MPI routines. For more detail see Gropp et al. (1999).

<table>
<thead>
<tr>
<th>Subroutine</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>MPI_INIT</td>
<td>Initialize MPI</td>
</tr>
<tr>
<td>MPI_COMM_SIZE</td>
<td>Get the total number of processes</td>
</tr>
<tr>
<td>MPI_COMM_RANK</td>
<td>Obtain the rank of the current process</td>
</tr>
<tr>
<td>MPI_SEND</td>
<td>Send data to one other process</td>
</tr>
<tr>
<td>MPI_RECV</td>
<td>Receive data from another process</td>
</tr>
<tr>
<td>MPI_BCAST</td>
<td>Broadcasts a message from rank ”0” to all other process</td>
</tr>
<tr>
<td>MPI_REDUCE</td>
<td>Reduces values on all processes to a single value</td>
</tr>
<tr>
<td>MPI_FINALIZE</td>
<td>Terminate MPI</td>
</tr>
</tbody>
</table>

of subroutines that allow the user to direct the communication between processes. MPI is user friendly due to its portability and usability on a variety of platforms. Some useful subroutines which are listed in table A.2, are used for parallelization of our CFD codes.

While designing a distributed memory parallel algorithm, it is required to distribute the data among the processes which is commonly known as domain decomposition. Since both codes use pseudo spectral method, FFT is often executed inside the RK sub-step to transform velocity field from physical to Fourier space and vice-versa. The Fourier transform is a very difficult algorithm to parallelize, since it requires a large amount of non-local communication. Therefore, all information regarding data has to be stored locally for any FFT calls. In the channel flow code, FFTs are needed in two directions (x and z directions) and this requires the data information along whole $x - z$ plane to be local for each processor. There-
Figure A.6: Domain decomposition for the MPI version of channel flow geometry.

Therefore, the domain is decomposed into \( x - z \) slabs as shown in Figure A.6. The grid indexing on each processor for both the fractional and base grids is chosen similar to figure A.1. Tridiagonal system in the \( y \)-direction at each \( x \) and \( z \) location forms after discretization of the implicit terms (mainly from viscous contribution) in each RK sub-step. Solving for the intermediate velocity involves solving a tridiagonal system in the \( y \)-direction. Since the data is decomposed along the \( y \)-axis, this solve process is nonlocal and must be parallelized. In this regard, a pipelined Thomas algorithm is used to solve the tridiagonal system in \( y \)– direction for each data point in \( x - z \) plane. A detailed description on the forward and the backward sweeps performed in a parallel environment as part of the pipelined Thomas algorithm, is found in Bewley (2012) and Taylor (2008). During pressure correction step, another tridiagonal system in the \( y \)– direction is solved using similar pipelined Thomas algorithm. Ghost cell communication is another important subroutine required to exchange data residing between processor interface. At the start of each Runge-Kutta sub-step and before the pressure correction step the ghost cells on interior nodes are filled by obtaining the data from neighboring nodes.

A different strategy is implemented for parallelization of curvilinear geometry. In the case of curvilinear geometry which is an extension to a duct geometry, FFTs are only needed in one direction (\( x \)-direction). In each RK3 sub-step the
viscous terms are treated implicitly with the alternating direction implicit (ADI) method. Finally, two-dimensional multigrid routine is executed for pressure correction in fractional step method. Since the grid and smoothing operations in multigrid are communication intensive, it is desirable to keep this operation location on each process which requires that data should be decomposed in the $y-z$ plane. The data remains decomposed in the $y-z$ plane during the ADI splitting process. With this configuration the computational cost of pipelined Thomas algorithm can be avoided. Because a parallel algorithm is slower than the serial version due to additional communication overhead. Now the present domain decomposition is inefficient during any FFT call. Therefore, apart from any FFT and inverse FFT operation inside the RK sub-step, computational domain is decomposed in $y-z$ plane only. For any data in the real space, domain decomposition is similar to the schematic as shown in figure A.7(a), where schematic in figure A.7(c) is applicable for the data in the Fourier space. It is noticeable that for a real data, number of $x-$ grids per processor is two times larger than that for the same data in Fourier space. Now during the data transformation from the physical to the Fourier space or vise versa, it is necessary to alter present domain decomposition at the cost of an additional two data operations. Before calling FFT in the $x$-direction, the data must be made local in the $x$-direction. This can be accomplished by an MPI\textsc{All-To-All} transpose to distribute the data in $x-y$ planes. MPI\textsc{All-To-All} is one of the collective data movement routines (Gropp \textit{et al.}, 1999). A schematic of data locations before and after MPI\textsc{All-To-All} is shown in figure A.8 based on the four processors. The data interchange is done

\textbf{Figure A.7}: Domain decomposition for the MPI version of curvilinear geometry.
The figure shows the local data structures pre and post MPI\_ALL\_TO\_ALL based on the four processors with rank 0, 1, 2 and 3. Before MPI\_ALL\_TO\_ALL call, the initial array is first divided into four segments, denoted A, B, C, and D. Where each segment is stored on a separate processor. As part of the MPI\_ALL\_TO\_ALL process each segment is then broken into four smaller sub-segments of equal length depending on the number of processors. These sub-segments are denoted 0, 1, 2 and 3. Each numbered sub-segment is sent to the processor with its corresponding rank and placed in the section corresponding to the sending processor rank.

In full duplex, so the total upstream and downstream bandwidth is fully utilized at all times. Even with the full utilization of bandwidth the overhead involved with MPI\_ALL\_TO\_ALL, as with any other collective data movement routine is significant. After MPI\_ALL\_TO\_ALL call, the domain is decomposed into $x - y$ slabs with processors lying long $z$-direction as shown in figure A.7(b). At this point, FFTs are performed on the velocity to transform it from physical space into Fourier space. Soon after the FFTs, another MPI\_ALL\_TO\_ALL transpose is called to redistribute the data (in the Fourier space) in $y - z$ planes. An inverse FFT transform (Fourier space to physical space) also follows similar process at the cost of two MPI\_ALL\_TO\_ALL transpose operations. Ghost cell communication subroutine is necessary for the parallelization implemented on curvilinear geometry.
Appendix B

Decomposition of pressure and velocity

Decomposition into barotropic and baroclinic (internal tide) components can be performed in different ways (Holloway, 1996; Kunze et al., 2002; Khatiwala, 2004; Nash et al., 2004, 2006; Gerkema & van Haren, 2007). Here, we follow Nash et al. (2004) and Nash et al. (2006). The baroclinic wave velocity is defined by

\[ u_{bc}(x, t) = u(x, t) - \hat{u}(x) - u_b(t, x, y) \]  \hspace{1cm} (B.1)

where the residual velocity \( \hat{u}(x) = 1/(5T) \int_t^{5T+t} u(x, t)dt \) is a cycle-averaged mean and \( u_b(t, y, z) \) is calculated by enforcing baroclinicity:

\[ \int_{h(x)}^{h(x)} u_{bc}(x, t)dz = 0. \]  \hspace{1cm} (B.2)

Here, \( h(x) \) is height of the slope topography with respect to the left flat bottom. Note that, for the present problem with small slope angle, \( u_b(t, y, z) \) is approximately equal to the oscillatory current resulting from the imposed oscillatory pressure gradient and its dependence on \( y \) and \( z \) is weak while the residual velocity \( \hat{u}(x) \) is very small.

The density anomaly is estimated as \( \rho_{bc}(x, t) = \rho(x, t) - \hat{\rho}(x) - \rho_b(t, x, y) \), where \( \hat{\rho}(x) \) is the average over five tidal cycles. \( \rho_b(t, x, y) \) is calculated by enforcing baroclinicity similar to the baroclinic wave velocity in (B.2). The pressure anomaly
is calculated imposing hydrostatic balance,

\[ p_{bc}(x, t) = p_s(t, y, z) + \int_z^{l_z} \rho_{bc}(x, t) g dz'. \]  

(B.3)

The surface contribution, \( p_s(t, y, z) \), of the baroclinic motion is computed using the constraint of zero depth-average of baroclinic pressure perturbation, i.e.,

\[ \int_{h(x)}^{l_z} p_{bc}(x, t) dz = 0. \]  

(B.4)
References


