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On Dynamic Scheduling of a Parallel Server System with Certain Graph Structure

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics by Vladimir Pesic

Committee in charge:
Professor Ruth Williams, Chair
Professor Thomas Bewley
Professor Patrick Fitzsimmons
Professor Tara Javidi
Professor Jason Schweinsberg

2011
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The dissertation of Vladimir Pesic is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2011
DEDICATION

To my wife and family
Be the change you want to see in the world.

Mahatma Gandhi
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ABSTRACT OF THE DISSERTATION

On Dynamic Scheduling of a Parallel Server System with Certain Graph Structure

by

Vladimir Pesic

Doctor of Philosophy in Mathematics

University of California San Diego, 2011

Professor Ruth Williams, Chair

We consider a problem of dynamic scheduling for a parallel server system. This system consists of finitely many infinite capacity buffers (classes) for holding incoming jobs awaiting service and finitely many non-identical servers working in parallel. Jobs within each buffer are served on a first-in-first-out basis. Each job requires a single service before it leaves the system. Each server can work on at most one job at a time, but it may be capable of processing several different classes of jobs over time, and it may suspend service of a job to work on a job of another class. Jobs of a given class incur holding costs at a rate proportional to the number of jobs in the class at each instant of time. The system manager seeks to minimize a cumulative discounted holding cost by dynamically scheduling waiting jobs to available servers. Following a method introduced by Harrison, for this parallel server system in heavy traffic we approximate the scheduling control problem by a Brownian control problem (BCP), which can be reduced to an Equivalent Workload Formulation (EWF). We first prove that the server-buffer graph, consisting of servers and buffers linked by basic activities, is a forest of trees. Then we give sufficient conditions for a least control process to be the optimal solution of the EWF. Under these conditions, we propose a continuous review threshold-type control policy that exploits partial pooling of servers. We conjecture that this policy is asymptotically optimal for the original parallel server system in the heavy traffic limit. To illustrate the solution of the EWF, and our proposed control policy for the original network we give a three buffer, three server example. We prove that this control policy is asymptotically optimal for this example in the heavy traffic limit. This is the first instance of a proof of asymptotic optimality for a parallel server system with partial pooling is the usual heavy traffic regime.
Chapter 1

Introduction

We consider a dynamic scheduling problem for a parallel server system. This system may be viewed as a model for a manufacturing or computer system consisting of a bank of buffers for holding incoming jobs and a bank of servers for processing these jobs. Incoming jobs are classified into one of several different job classes (or buffers). Jobs within each class are served on a first-in-first-out basis by a server from a subset of the bank of parallel servers. Servers may have different, but overlapping capabilities, and so may be capable of serving more than one class. Jobs of each class incur linear holding costs while present within the system. The system manager seeks to minimize a cumulative discounted holding cost by dynamically scheduling waiting jobs to available servers.

The parallel server system is described in more detail in Chapter 2 below. Except in a few special cases, the dynamic scheduling problem for such a system cannot be analyzed exactly and it is natural to consider more tractable approximations. One class of such approximations are the so-called Brownian Control Problems (BCPs), which were first introduced by Harrison in [16]. These are formal heavy traffic approximations to stochastic network control problems. Various authors (see for example [9, 10, 24, 25, 29, 30, 31, 38]) have used the analysis of Brownian Control Problems, together with clever interpretations of their optimal (analytic) solutions, to suggest “good” policies for the original network control problem. In [22], Harrison and Van Mieghem showed that the Brownian Control Problem can be reduced to a typically lower dimensional Equivalent Workload Formulation (EWF). In performing this reduction one has to choose a workload matrix. In [20], Harrison proposed a mechanism for choosing this matrix in a “canonical” way. In [21], Harrison and Lopez identified a condition under which the workload is one dimensional and the parallel server system exhibits complete resource pooling (CRP),
i.e., in the Brownian model, the efforts of the individual servers can be efficiently combined to act as a single pooled resource or “superserver”. In [21], Harrison and Lopez conjectured that a discrete review scheduling policy (for the original parallel server system), obtained by using the BIGSTEP discretization procedure of Harrison [18], is asymptotically optimal in the heavy traffic limit. They did not prove the conjecture, although, in a slightly earlier work, Harrison [19] did prove asymptotic optimality of a discrete review policy for the case of two servers and two buffers with special distributional assumptions. In [41], it was shown that the CRP condition is equivalent to the server-buffer graph $G$ (consisting of servers and buffers linked by basic activities) being a tree. In that paper Williams proposed a continuous review threshold type control policy for a parallel server system satisfying the CRP condition. Under the CRP condition, Bell and Williams [3, 4] established the asymptotic optimality of this control policy first for the case of two servers and two buffers with an N-type structure for $G$ and later for a general tree structure for $G$. Several other authors have investigated stochastic processing networks that satisfy the CRP condition [21, 41, 3, 4].

Here we explore a parallel server system in which the graph $G$ is more general than a single tree. First we establish that the server-buffer graph $G$ for a parallel server system is a forest of trees, where the number of trees equals the workload dimension, $L$. The proof of this fact takes advantage of a workload formula of Bramson and Williams [8]. The proof of this fact takes advantage of a workload formula of Bramson and Williams [8]. Assuming that the workload dimension $L > 1$, we show that a workload matrix can be chosen to have a block diagonal like structure and to this particular choice of workload matrix corresponds a special control matrix. This choice of a workload matrix is related to a structure of the forest of trees comprising $G$ and it is generally different from the “canonical” choice of workload matrix proposed by Harrison [20]. Next, we consider the graph $H$ that is obtained by adding to the server-buffer graph $G$ the non-basic activities that connect distinct trees in the server-buffer graph. We give conditions on the graph $H$ and the cost function under which a least control process is optimal for the EWF. Under these conditions, we propose a continuous review threshold type control policy which we conjecture is asymptotically optimal in heavy traffic for the original parallel server system. This control policy takes advantage of partial pooling of servers. Finally, to illustrate our theoretical developments we consider a three buffer, three server parallel server system with two-dimensional workload. For this specific example we solve the EWF, we specify a control policy and we prove that this policy is asymptotically optimal in the heavy traffic limit. The proof of asymptotic optimality uses an invariance principle of Kang and Williams [28] and we use the work of Reiman and Williams [35] to simplify the treatment of the situation when both
components of the workload are close to zero. To our knowledge, this is the first instance where asymptotic optimality is proved for a parallel server system with partial pooling in the usual heavy traffic regime.

The paper is organized as follows. In Chapter 2, we describe the model of a parallel server system considered here. In Chapter 3, we introduce a sequence of such systems and define notion of heavy traffic. In Chapter 4, we describe the Brownian Control Problem and the Equivalent Workload Formulation associated with the sequence of parallel server systems. In this chapter we also describe how to reduce the Equivalent Workload Formulation to a simpler control problem called the Reduced Equivalent Workload Formulation (REWF). In Chapter 5, we introduce the server-buffer graph $\mathcal{G}$ and we prove the results relating the structure of $\mathcal{G}$ to the workload dimension. In this chapter we also describe our choices for the workload and control matrices. In Chapter 6, we introduce the extended server-buffer graph $\mathcal{H}$. We give conditions on $\mathcal{H}$, the cost function in the EWF and the control matrix under which the least control process is a solution of the EWF. In Chapter 7, we interpret the solution of the EWF for the original network and we propose a control policy which we conjecture is asymptotically optimal. In Chapter 8, we give a three buffer, three server example of a parallel server system. For this example we carry out the steps from Sections 2-6 and we give a control policy that is a specific instance of the control policy proposed in Chapter 7. In Chapter 8, we also prove that this policy is asymptotically optimal for the original parallel server system. In Appendix A we state large deviation estimates that are used in proving the asymptotic optimality in Chapter 8. In Appendix B we use strong approximations of Csörgő, Horvath and Steinebach [12] to establish estimates for a sequence of $GI/GI/1$ queueing network starting from zero. These are used in Chapter 8. In Appendix C we state results on the existence and uniqueness of least controls proved in [42]. These results are used in Chapter 6.

1.1 Notation and Terminology

The set of non-negative integers is denoted by $\mathbb{N}$ and the value $+\infty$ is denoted by $\infty$. We let $\mathbb{R}_+$ denote $[0, \infty)$. The $m$-dimensional ($m \geq 1$) Euclidean space is denoted by $\mathbb{R}^m$ and the $m$ dimensional positive orthant is denoted by $\mathbb{R}^m_+ = \{ x \in \mathbb{R}^m : x_i \geq 0 \text{ for } i = 1, \ldots, m \}$. Let $|x|$ denote the norm on $\mathbb{R}^m$ given by $|x| = (\sum x_i^2)^{1/2}$. Let $\{e_1, \ldots, e_m\}$ be the standard basis for $\mathbb{R}^m$. A sum over an empty index set is defined to be zero. Vectors in $\mathbb{R}^m$ should be treated as column vectors unless indicated otherwise, inequalities between vectors should be interpreted componentwise, the transpose of a vector $b$ will be denoted by $b'$, the diagonal matrix
with the entries of a vector $b$ on its diagonal will be denoted by $\text{diag}(b)$, and the dot product of two vectors $b$ and $c$ in $\mathbb{R}^m$ will be denoted by $b \cdot c$.

For each positive integer $m$, let $D^m$ be the space of Skorokhod paths in $\mathbb{R}^m$ having time domain $\mathbb{R}_+$, i.e., $D^m$ is the set of all functions $\omega : \mathbb{R}_+ \to \mathbb{R}^m$ that are right continuous on $\mathbb{R}_+$ and have finite left limits on $(0, \infty)$. Let $D^m_+ = \{\omega \in D^m : \omega(0) \geq 0\}$. The member of $D^m$ that stays at the origin in $\mathbb{R}^m$ for all time will be denoted by 0. For $\omega \in D^m$,

$$||\omega||_t = \sup_{s \in [0,t]} |\omega(s)|, \text{ for each } t \geq 0. \quad (1.1)$$

Consider $D^m$ to be endowed with the usual Skorokhod $J_1$-topology. Let $\mathcal{M}^m$ denote the Borel $\sigma$-algebra on $D^m$ associated with the $J_1$-topology. For a non-negative integer $m$, given a probability space $(\Omega, \mathcal{F}, P)$, a $m$-dimensional stochastic process defined on this space is a collection $X = \{X(t) : t \in \mathbb{R}_+\}$ of measurable functions $X(t) : \Omega \to \mathbb{R}^m$ where $\Omega$ has the $\sigma$-algebra $\mathcal{F}$ and $\mathbb{R}^m$ has the Borel $\sigma$-algebra. Such a process $X$ will be said to be non-decreasing, if each of its components is non-decreasing $P$-a.s.. All of the continuous-time stochastic processes in this paper are assumed to have sample paths in $D^m$ for some $m \geq 1$. (We shall frequently use the term process in place of stochastic process.)

Suppose that $\{W^n\}_{n=1}^{\infty}$ is a sequence of processes with sample paths in $D^m$ for some $m \geq 1$. Then we say that $\{W^n\}_{n=1}^{\infty}$ is tight if and only if the probability measures induced by the $W^n$ on $(D^m, \mathcal{M}^m)$ form a tight sequence, i.e., they form a weakly relatively compact sequence in the space of probability measures on $(D^m, \mathcal{M}^m)$. The notation $W^n \Rightarrow W$ as $n \to \infty$, where $W$ is a process with sample paths in $D^m$, will mean that the probability measures induced by the $W^n$ on $(D^m, \mathcal{M}^m)$ converge weakly to the probability measure on $(D^m, \mathcal{M}^m)$ induced by $W$. If, for each $n$, $W^n$ and $W$ are defined on the same probability space, we write $W^n \to W$ uniformly on compact time intervals in probability (u.o.c. in prob.) as $n \to \infty$, if $P(||W^n - W||_t \geq \epsilon) \to 0$ as $n \to \infty$ for each $\epsilon > 0$ and $t \geq 0$. In particular, if $W$ is a continuous deterministic process and $W^n \Rightarrow W$, then $W^n \to W$ u.o.c. in probability. This result is implicitly used several times in the proofs below to combine statements involving convergence in distribution to deterministic processes.

A filtered probability space is a quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\mathcal{F}_t\}$ is a filtration, i.e., a family of sub-$\sigma$-algebras of the $\sigma$-algebra $\mathcal{F}$ indexed by $t \in \mathbb{R}_+$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $0 \leq s \leq t < \infty$. An $m$-dimensional process $X = \{X(t) : t \in \mathbb{R}_+\}$ defined on such a filtered probability space is said to be adapted if for each $t \geq 0$ the function $X(t) : \Omega \to \mathbb{R}^m$ is measurable when $\Omega$ has the $\sigma$-algebra $\mathcal{F}_t$ and $\mathbb{R}^m$. 

has its Borel $\sigma$-algebra. Given a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, a vector $\theta \in \mathbb{R}^m$, an $m \times m$ symmetric, strictly positive definite matrix $\Sigma$, an $\{\mathcal{F}_t\}$-Brownian motion with statistics $(\theta, \Sigma)$ starting at the origin, is an $m$-dimensional process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ such that the following hold under $P$:

(a) $X$ is an $m$-dimensional Brownian motion with continuous sample paths such that $X(0) = 0$ $P$-a.s.,

(b) $\{X_i(t) - \theta_i t, \mathcal{F}_t, t \geq 0\}$ is a martingale for $i = 1, \ldots, m$,

(c) $\{(X_i(t) - \theta_i t)(X_j(t) - \theta_j t) - \Sigma_{ij} t, \mathcal{F}_t, t \geq 0\}$ is a martingale for $i, j = 1, \ldots, m$.

In this definition, the filtration $\{\mathcal{F}_t\}$ may be larger than the one generated by $X$; however for each $t \geq 0$, under $P$, the $\sigma$-algebra $\mathcal{F}_t$ is independent of the increments of $X$ from $t$ onward. The parameter $\theta$ is called the drift of the Brownian motion $X$ and $\Sigma$ is called the covariance matrix of $X$. 
Chapter 2

Parallel Server System

In this chapter we describe our model of a parallel server system. This is the same setup as used by Bell and Williams [4].

2.1 System Structure.

We consider a parallel server system (see Figure 1) consisting of \( I \) infinite capacity buffers (job classes) for holding jobs awaiting service, indexed by \( i = 1, \ldots, I \) and \( K \) non-identical servers working in parallel indexed by \( k = 1, \ldots, K \). Customers arrive to each of the buffers from outside the system. Arrivals to buffer \( i \) are called class \( i \) jobs. Jobs within each buffer are ordered according to their arrival times with the job that arrived the longest time ago being at the head of the line. Each job that enters the system requires a single service before it leaves the system. Service of a given job class \( i \) by a given server \( k \) is called a processing activity. A single server \( k \) may be capable of processing several different job classes. A single job class \( i \) may be capable of being processed by one of several servers. It is assumed that there are \( J \) processing activities, indexed by \( j = 1, \ldots, J \), where \( J \leq I \cdot K \). Each activity \( j \), serves a single buffer \( i(j) \) and is performed by a single server \( k(j) \). Relations between activities and buffers, and activities and servers, are given by two deterministic matrices \( C, A \) where \( C \) is an \( I \times J \) matrix with

\[
C_{ij} = \begin{cases} 
1 & \text{if activity } j \text{ processes class } i, \\
0 & \text{otherwise},
\end{cases}
\quad (2.1)
\]
and $A$ is a $K \times J$ matrix with

$$A_{kj} = \begin{cases} 1 & \text{if server } k \text{ performs activity } j, \\ 0 & \text{otherwise}. \end{cases} \quad (2.2)$$

Each activity $j$ has exactly one class $i(j)$ and one server $k(j)$ associated with it, and so each column of $C$ and each column of $A$ contains the number one exactly once. We assume that each job class is capable of being served by at least one processing activity and each server is capable of performing at least one processing activity, and so each row of $C$ and each row of $A$ contains the number one at least once. Once a job starts being served at a server it remains there until its service is complete, even if its service is suspended for some time. A server may not start on a new job of class $i$ until it has finished any class $i$ job that it is working on or that it has in suspension. When taking a job from a buffer, a server always takes the job at the head of the line. A server may not work unless it has a job to work on. It is assumed that the system is initially empty. For later use, we let $I = \{1, \ldots, I\}$, $J = \{1, \ldots, J\}$ and $K = \{1, \ldots, K\}$. A holding cost for this system will be introduced later in Section 3.3.
2.2 Stochastic Primitives

All random variables and stochastic processes in our parallel server model description are defined on a complete probability space \((\Omega, \mathcal{F}, \mathbf{P})\). The expectation operator under \(\mathbf{P}\) is denoted by \(E\). For each buffer \(i \in \mathcal{I}\), there is a sequence of strictly positive, independent and identically distributed (i.i.d.) random variables \(\{u_i(l), l = 1, 2, \ldots\}\), with mean \(\lambda_i^{-1} \in (0, \infty)\) and squared coefficient of variation \(a_i^2 \in [0, \infty)\). The random variable \(u_i(l)\) represents the interarrival time between the \((l-1)th\) and \(lth\) customer to buffer \(i\); by convention, the “0th arrival” occurs at time zero. Let

\[
\zeta_i(n) = \sum_{l=1}^{n} u_i(l), \quad n = 1, 2, \ldots, \tag{2.3}
\]

and define

\[
A_i(t) = \sup\{n \geq 0 : \zeta_i(n) \leq t\}, \quad \text{for all } t \geq 0. \tag{2.4}
\]

Then \(A_i(t)\) is the number of arrivals to buffer \(i\) that have occurred in \([0, t]\), and \(\lambda_i\) is the long run arrival rate to buffer \(i\). For each activity \(j \in \mathcal{J}\), there is a sequence of strictly positive i.i.d. random variables \(\{v_j(l), l = 1, 2, \ldots\}\) with mean \(\mu_j^{-1} \in (0, \infty)\) and squared coefficient of variation \(b_j^2 \in [0, \infty)\). The random variable \(v_j(l)\) is the amount of service time required by the \(lth\) job processed by activity \(j\), and \(\mu_j\) is the long run rate at which activity \(j\) could process its associated class of job \(i(j)\) if the associated server \(k(j)\) worked continuously and exclusively on this class. For \(j \in \mathcal{J}\), let \(\eta_j(0) = 0\),

\[
\eta_j(n) = \sum_{l=1}^{n} v_j(l), \quad n = 1, 2, \ldots, \tag{2.5}
\]

and

\[
S_j(t) = \sup\{n \geq 0 : \eta_j(n) \leq t\} \quad \text{for all } t \geq 0. \tag{2.6}
\]

Then \(S_j(t)\) is the number of jobs that activity \(j\) could process up to time \(t\) if the server \(k(j)\) worked continuously and exclusively on class \(i(j)\). The interarrival time sequences \(\{u_i(l) : l = 1, 2, \ldots\}, i \in \mathcal{I}\) and service time sequences \(\{v_j(l), l = 1, 2, \ldots\}, j \in \mathcal{J}\) are all assumed to be mutually independent.

2.3 Scheduling Control

The system is controlled by specifying how each server is to allocate its service time to its processing activities. The setup described here is fairly general. It allows for dynamic
sequencing and alternate routing of jobs. For example, if server \( k \) performs more than one activity (i.e., \( A_{kj} \neq 0 \) for more than one \( j \)), then once a service of a job is complete, server \( k \) can make a sequencing decision, i.e., which activity to perform next. If a given job class \( i \) can be processed by more than one activity (i.e., \( C_{ij} \neq 0 \) for more than one \( j \)), then class \( i \) may be serviced by one of a collection of servers and so simple alternate routing capabilities are encompassed here.

Formally, scheduling control is exerted by specifying a \( J \)-dimensional stochastic process, \( T = \{T(t), t \geq 0\} \) where

\[
T(t) = (T_1(t), \ldots, T_J(t))' \quad \text{for} \quad t \geq 0, \quad (2.7)
\]

and \( T_j(t) \) is the cumulative amount of service time devoted to activity \( j \) by server \( k(j) \) in the time interval \([0, t]\). The control process \( T \) must satisfy certain natural constraints that go along with its interpretation (see (8.29)-(2.15)) below. For each \( t \geq 0 \), let

\[
I(t) = 1t - AT(t), \quad (2.8)
\]

where \( 1 \) is the \( K \)-dimensional vector of ones. Then for each \( k \in K \), \( I_k(t) \) is the cumulative amount of time that server \( k \) has been idle up to time \( t \). The cumulative idle-time \( I(\cdot) \) process is continuous and non-decreasing in all components. This implies that \( T \) is Lipschitz continuous with Lipschitz constant equal to one. For each \( j \), \( S_j(T_j(t)) \) is the number of jobs processed by activity \( j \) in the time interval \([0, t]\). For each \( i \in I \), let

\[
Q_i(t) = A_i(t) - \sum_{j=1}^{J} C_{ij} S_j(T_j(t)), \quad (2.9)
\]

which we write in the vector form (with a slight abuse of notation for \( S(T(t)) \)) as

\[
Q(t) = A(t) - CS(T(t)). \quad (2.10)
\]

Then \( Q_i(t) \) is interpreted as the number of class \( i \) jobs that are either in queue or in progress at time \( t \). The following properties are assumed for any scheduling control \( T \) with associated queue-length \( Q \) and idle-time process \( I \). For each \( i \in I, j \in J, k \in K, \)

\[
T_j(t) \in F \quad \text{for each} \quad t \geq 0, \quad (2.11)
\]

\( T_j \) is Lipschitz continuous with a Lipschitz constant of one, \quad (2.12)

\( T_j \) is non-decreasing and \( T_j(0) = 0, \quad (2.13)\)

\( I_k \) is continuous, non-decreasing, and \( I_k(0) = 0, \quad (2.14)\)

\( Q_i(t) \geq 0 \) for all \( t \geq 0. \quad (2.15)\)
For later reference, we collect here the queueing system equations satisfied by $Q$ and $I$:

\begin{align*}
Q(t) &= A(t) - CS(T(t)), \quad t \geq 0, \quad (2.16) \\
I(t) &= 1 - AT(t), \quad t \geq 0, \quad (2.17)
\end{align*}

where $T, Q$ and $I$ satisfy the properties (8.29)-(2.15). In addition to the properties mentioned above one might expect that $T$ should satisfy some additional non-anticipating property. Even though this is a reasonable assumption to make, we have not restricted $T$ a priori in this way. However, the policy that we propose in Chapter 7 is non-anticipating.
Chapter 3

Sequence of Systems, Heavy Traffic and the Cost Function

For the parallel server system described in Chapter 2, the problem of finding a control policy that minimizes a cost associated with holding jobs in the system is notoriously difficult. One possible method for discriminating between policies is to look for policies that outperform others in some asymptotic regime. Here, we use such an approach as proposed by Harrison [20], where the asymptotic regime is the heavy traffic regime.

3.1 Sequence of Systems

Consider a sequence of parallel server systems indexed by \( r \), where \( r \) tends to infinity through a sequence of values in \([1, \infty)\). The \( r^{th} \) system has the same basic structure as described in Chapter 2, except that the arrival and service rates and scheduling control are allowed to vary with \( r \). We denote this dependence on \( r \) by appending a superscript \( r \) to all of the relevant quantities. We assume that the interarrival and service times are given for each \( r \geq 1, i \in I, j \in J \), by

\[
\begin{align*}
\lambda_r \overset{u}{\sim} \hat{u}_i(l), & \quad \overset{v}{\sim} \hat{v}_j(l), & \text{for } l = 1, 2, \ldots,
\end{align*}
\]

(3.1)

where \( \hat{u}_i(l), \hat{v}_j(l) \), are independent of \( r \), with mean 1 and squared coefficient of variation \( \alpha_i^2 \), respectively \( b_j^2 \). The sequences \( \{ \hat{u}_i(l), l = 1, 2, \ldots \} \), \( \{ \hat{v}_j(l), l = 1, 2, \ldots \} \) are mutually independent sequences of i.i.d. random variables. This setup is convenient for allowing the sequence of networks to approach the heavy traffic limit by simply changing arrival and service rates while
keeping the underlying sources of variability \( \bar{u}_i(l), \bar{v}_j(l) \) unaffected. We make the following assumption about the first order parameters for our sequence of systems.

**Assumption 3.1.1.** There are vectors \( \lambda \in \mathbb{R}^I_+, \mu \in \mathbb{R}^J_+ \) such that

(i) \( \lambda_i > 0 \) for all \( i \in \mathcal{I} \), \( \mu_j > 0 \) for all \( j \in \mathcal{J} \),

(ii) \( \lambda^r \to \lambda \) and \( \mu^r \to \mu \), as \( r \to \infty \).

In addition, we make the following exponential moment assumptions to ensure that certain large deviation estimates, which are stated in Appendix A, hold for the renewal processes \( A_i^r, i \in \mathcal{I} \), and \( S_j^r, j \in \mathcal{J} \). These large deviation estimates are used in Chapter 8 to prove the asymptotic optimality of the control policy for a specific example of a parallel server system.

**Assumption 3.1.2.** For \( i \in \mathcal{I}, j \in \mathcal{J} \), for all \( l \geq 1 \), let \( u_i(l) = \frac{1}{\lambda_i} \bar{u}_i(l) \) and \( v_j(l) = \frac{1}{\mu_j} \bar{v}_i(l) \). Assume that there is an open neighborhood \( O \) of \( 0 \in \mathbb{R} \) such that for all \( \ell \in O \),

\[
\Lambda_i^\ell \equiv \log \mathbb{E} \left[ e^{\ell u_i(1)} \right] < \infty \quad \text{for} \quad i \in \mathcal{I},
\]

and

\[
\Lambda_j^\ell \equiv \log \mathbb{E} \left[ e^{\ell v_j(1)} \right] < \infty \quad \text{for} \quad j \in \mathcal{J}.
\]

### 3.2 Heavy Traffic and Fluid Model

In [20], Harrison proposed a notion of heavy traffic for stochastic processing networks with scheduling control. Given the parameters \( \lambda, \mu \) from Assumption 3.1.1, for our sequence of parallel server systems, his notion is the same as Assumption 3.2.1 below. Henceforth, let \( R = C \text{diag}(\mu) \).

**Assumption 3.2.1.** There is a unique optimal solution \((\rho^*, x^*)\) of the linear program:

\[
\begin{align*}
\text{minimize} & \quad \rho \\
\text{subject to} & \quad Rx = \lambda, \quad Ax \leq \rho 1 \quad \text{and} \quad x \geq 0.
\end{align*}
\]

Moreover, that solution is such that \( \rho^* = 1 \) and \( Ax^* = 1 \).

A fluid model solution (with zero initial condition) is a triple of continuous deterministic functions \((\bar{Q}, \bar{T}, \bar{I})\) defined on \([0, \infty)\), where \( \bar{Q} \) takes values in \( \mathbb{R}^I_+ \), \( \bar{T} \) takes values in \( \mathbb{R}^J_+ \), and \( \bar{I} \) takes values in \( \mathbb{R}^K_+ \), such that

\[
\begin{align*}
\bar{Q}(t) &= \lambda t - R\bar{T}(t), \quad t \geq 0, \\
\bar{I}(t) &= 1 t - A\bar{T}(t), \quad t \geq 0,
\end{align*}
\]
and for all $i, j, k$,

- $\bar{T}_j$ is Lipschitz continuous with a Lipschitz constant of one, \hspace{1cm} (3.7)
- $\bar{T}_j$ is non-decreasing, and $\bar{T}_j(0) = 0$, \hspace{1cm} (3.8)
- $\bar{I}_k$ is continuous, non-decreasing, and $\bar{I}_k(0) = 0$, \hspace{1cm} (3.9)
- $\bar{Q}_i(t) \geq 0$ for all $t \geq 0$. \hspace{1cm} (3.10)

A continuous function $\bar{T} : [0, \infty) \to \mathbb{R}_{\geq 0}$ such that (3.5)-(3.10) hold is called a fluid control. The system is said to be balanced under $\bar{T}$ if the associated $\bar{Q}$ is constant in time. In this case, since the system starts empty, that means that $\bar{Q} \equiv 0$. The system is said to incur no idleness under $\bar{T}$, if $\bar{I} \equiv 0$, i.e., $A\bar{T} = 1t$ for all $t \geq 0$.

**Definition 3.2.1.** The fluid model is said to be in heavy traffic if the following two conditions hold:
- i) there is a unique fluid control $\bar{T}^*$ under which the fluid system is balanced, and
- ii) under $\bar{T}^*$, the fluid system incurs no idleness.

In [41], Williams showed that the Assumption 3.2.1 is equivalent to a heavy traffic condition for a fluid model associated with our sequence of parallel server systems. We summarize that result here as the fluid model plays a role in establishing asymptotic optimality of a control policy for our sequence of systems.

**Proposition 3.2.1.** The fluid model is in heavy traffic if and only if Assumption 3.2.1 holds.

We impose the following heavy traffic assumption on our sequence of parallel server systems, henceforth.

**Assumption 3.2.2.** (Heavy Traffic) For the sequence of parallel server systems defined in Section 3.1 satisfying Assumptions 3.1.1 and 3.1.2, assume that Assumption 3.2.1 holds and that there is a vector $\theta \in \mathbb{R}^I$ such that

$$r(\lambda^r - R^r x^*) \to \theta, \quad \text{as} \quad r \to \infty,$$

where $R^r = Cdiag(\mu^r)$ \hspace{1cm} (3.11)

Activities $j$ for which $x^*_j > 0$ in Assumption 3.2.1 are called basic. Activities $j$ for which $x^*_j = 0$ in Assumption 3.2.1 are called non-basic. Let $B$ and $N$ denote the number of basic and non-basic activities, respectively. It is assumed without any loss of generality that the
first B activities are basic and the last N activities are non-basic. We let $B = \{1, \ldots, B\}$ and $\mathcal{N} = \{B + 1, \ldots, J\}$. Let
\[
R = [H, J], \quad A = [B, N],
\] be partitions of $R, A$ according to basic and non-basic activities. Let $K$ be the $(K + N) \times J$ dimensional matrix
\[
K = \begin{pmatrix}
B & N \\
0 & -I
\end{pmatrix},
\] where $-I$ is the negative of the $N \times N$ identity matrix. Later we will implicitly use the following property of $K$.

**Lemma 3.2.1.** Let $K$ be given by (3.13), then $\text{range}(K) = \mathbb{R}^{K+N}$.

**Proof.** It is enough to show that for $l = 1, \ldots, K + N$, $e_l \in \text{range}(K)$. Let $l$ be arbitrary. If $l \leq K$, then since each server serves at least one basic activity, there exists a $j \in B$ such that $l = k(j)$. Then by (2.2), (3.12) and (3.13), $K^j = e_l$. If $K < l \leq K + N$, let $j = l - K + B$. Then $j \in \mathcal{N}$ and by (2.2), (3.12) and (3.13), $K^j_{k(j)} = 1$ and $K^j_1 = -1$. Since $e_{k(j)} \in \text{range}(K)$, it follows that $e_{k(j)} - K^j = e_l \in \text{range}(K)$. This completes the proof.

\[\square\]

### 3.3 Diffusion Scaling and Cost Function

For a fixed $r$, and a scheduling control $T^r$, the associated queue-length and idle-time processes are given by equations (17) and (18) in Chapter 2, where the superscript $r$ is appended to $A, S, Q, I$, and $T$ there. The diffusion scaled queue-length and idle-time processes are defined by
\[
\hat{Q}^r(t) = r^{-1}Q^r(r^2 t), \quad \hat{I}^r(t) = r^{-1}I^r(r^2 t), \quad t \geq 0.
\] For a control $T^r$ and its associated diffusion scaled queue-length process let
\[
\hat{J}^r(T^r) = \mathbb{E} \left( \int_0^\infty e^{-\gamma t} h \cdot \hat{Q}^r(t) dt \right),
\] an expected cumulative discounted holding cost, where $\gamma > 0$ is a fixed constant (discount factor) and $h = (h_1, \ldots, h_I)'$, $h_i > 0$ for all $i \in \mathcal{I}$, is a constant vector of holding cost rates for
the buffers. To write equations for $\hat{Q}_r$, $\hat{I}_r$, it is convenient to consider centered diffusion scaled versions $\hat{A}_r$, $\hat{S}_r$ of the primitive processes $A_r$, $S_r$:

$$\hat{A}_r(t) = r^{-1}(A_r(r^2t) - \lambda r^2t), \quad \hat{S}_r(t) = r^{-1}(S_r(r^2t) - \mu r^2t), \quad t \geq 0, \quad (3.16)$$

and a deviation process $\hat{Y}_r$ that measures normalized deviations of server time allocations from the nominal allocations given by $x^*$:

$$\hat{Y}_r(t) = r^{-1}(x^*r^2t - T_r(r^2t)), \quad t \geq 0. \quad (3.17)$$

Also, we define the fluid scaled allocation process $\bar{T}_r$,

$$\bar{T}_r(t) = r^{-2}T_r(r^2t), \quad t \geq 0. \quad (3.18)$$

Upon substituting the above into the equations for $Q_r$, $I_r$, we obtain for $t \geq 0$:

$$\hat{Q}_r(t) = \hat{X}_r(t) + R_r\hat{Y}_r(t), \quad (3.19)$$

$$\hat{I}_r(t) = A\hat{Y}_r(t), \quad (3.20)$$

$$\hat{X}_r(t) = \hat{A}_r(t) - C\hat{S}_r(\bar{T}_r(t)) + r(\lambda - R_r x^*)t, \quad (3.21)$$

where

$$\hat{I}_k^r$$ is continuous, non-decreasing and $\hat{I}_k^r(0) = 0$, for all $k \in K$, \hspace{1cm} (3.22)

$$\hat{Q}_i^r(t) \geq 0$$ for all $t \geq 0$ and $i \in I$, \hspace{1cm} (3.23)

Combining (3.1) and Assumption 3.1.1 with the mutual independence of the stochastic primitive sequences of i.i.d. random variables $\{\hat{u}_i(l)\}_{l=1}^{\infty}, i \in I, \{\hat{v}_j(l)\}_{l=1}^{\infty}, j \in J$, we may deduce from the functional central limit theorem for renewal processes that

$$(\hat{A}_r, \hat{S}_r) \Rightarrow (\bar{A}, \bar{S}), \quad \text{as} \quad r \to \infty, \quad (3.24)$$

where $\bar{A}$, $\bar{S}$ are independent, $\bar{A}$ is an $\mathbb{I}$-dimensional driftless Brownian motion that starts from the origin and has a diagonal covariance matrix whose $i^{th}$ diagonal entry is $\lambda_i a_i^2$, and $\bar{S}$ is an $\mathbb{J}$-dimensional driftless Brownian motion that starts from the origin and has a diagonal covariance matrix whose $j^{th}$ diagonal entry is $\mu_j b_j^2$.  

Chapter 4

Brownian Control Problem and Equivalent Workload Formulation

In this section, following the method proposed by Harrison et. al [20, 22], we formulate a Brownian Control Problem (BCP) and its Equivalent Workload Formulation (EWF) as formal approximations to the control problem for the sequence of parallel server systems. We also show how the EWF can also sometimes be reduced to an REWF. From hereon, $\bar{T}^*(t) = x^*t$, for all $t \geq 0$.

4.1 Brownian Control Problem (BCP)

Definition 4.1.1. (Admissible control for the BCP) An admissible control for the BCP is a $J$-dimensional, adapted process $\tilde{Y} = \{\tilde{Y}(t), t \geq 0\}$ defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ which supports an $I$-dimensional adapted processes $\tilde{Q}$ and $\tilde{X}$, such that the following three properties hold under $P$:

(i) $\tilde{Q}(t) = \tilde{X}(t) + R\tilde{Y}(t) \in \mathbb{R}^I_+$ for all $t \geq 0$, $P$-a.s.,

(ii) $\tilde{U} \equiv \{K\tilde{Y}(t), t \geq 0\}$ is non-decreasing and $\tilde{U}(0) \geq 0$, $P$-a.s.,

(iii) $\tilde{X}$ is an $I$-dimensional $\{\mathcal{F}_t\}$-Brownian motion starting at the origin, with drift $\theta$ and diagonal covariance matrix $\Sigma$ whose $i^{th}$ diagonal entry is equal to $\lambda_i a_i^2 + \sum_{j=1}^I C_{ij} \mu_j b_j^2 x_j^*$ for $i \in I$.

We call $\tilde{Q}$ the state process, $(\tilde{Q}, \tilde{U})$ the extended state processes and $\tilde{X}$ the Brownian motion, for the Brownian network under the control $\tilde{Y}$.

Remark 4.1.1. Note that the filtered probability space with Brownian motion $\tilde{X}$ upon which $\tilde{Y}$ is defined is part of the specification of $\tilde{Y}$. In other words, this is a weak formulation of the
control problem. The first $K$ components of $\tilde{U}$ are sometimes denoted by $\tilde{I}$ as they correspond to diffusion analogues of idle time.

**Definition 4.1.2.** (Brownian Control Problem-BCP) Determine the optimal value

$$\bar{J}^* = \inf_{\tilde{Y}} \tilde{J}(\tilde{Y}) \quad \text{where} \quad \tilde{J}(\tilde{Y}) \equiv \mathbb{E} \left( \int_0^{\infty} e^{-\gamma t} h \cdot \tilde{Q}(t) dt \right)$$

and the infimum is taken over all admissible controls for the BCP. An admissible control $\tilde{Y}^*$ that achieves the infimum in (4.1) is called an optimal control for the BCP.

The Brownian motion $\tilde{X}$ appearing in the Brownian Control Problem defined above is the formal limit in distribution of $\hat{X}_r$ of (38). The functional central limit theorems for the independent renewal processes $A^r, S^r$ and a time change theorem (together with the assumption that $\bar{T}^r \Rightarrow \bar{T}^*$), are used to derive the covariance matrix for this Brownian motion. The control process $\tilde{Y}$ in the Brownian Control Problem above arises as a formal limit of the deviation processes $\hat{Y}_r$ which measure the deviation of the allocation processes $T^r$ from the nominal fluid levels, i.e., formally $\hat{Y}_r \Rightarrow \tilde{Y}$ as $r \to \infty$.

### 4.2 Equivalent Workload Formulation (EWF)

Harrison and Van Mieghem [22] showed that one can reduce the dimensionality of the Brownian Control Problem to that of an EWF.

**Definition 4.2.1.** (Space of reversible displacements) Let

$$\mathcal{R} = \{ \delta \in \mathbb{R}^I : \delta = Rx \text{ and } Kx = 0, x \in \mathbb{R}^J \}.$$  

(4.2)

One thinks of the vector space $\mathcal{R}$ as follows. Given $\hat{q} \in \mathbb{R}^I_+, \hat{q} > 0$, for any $\delta \in \mathcal{R}$ such that $\hat{q} + \delta > 0$, using allowed controls in the BCP it is possible to instantaneously move the “queue-length” from $\hat{q}$ to $\hat{q} + \delta$ without incurring any idleness nor using any non-basic activities. Since $\mathcal{R}$ is a vector space such changes are reversible. The idea of the Equivalent Workload Formulation is to focus on the non-reversible displacements of the queue-length, i.e., those in $\mathcal{R}^\perp$.

**Definition 4.2.2.** (Workload dimension, workload matrix) Let $L$ be the dimension of $\mathcal{R}^\perp$. Then $L$ is called the workload dimension. Let $M$ be any $L \times I$ dimensional matrix whose rows span $\mathcal{R}^\perp$, then $M$ is called a workload matrix.
The following workload dimension formula was established by Bramson and Williams [8] for Brownian control problems associated with more general stochastic processing networks. That result is applied here for the special case of parallel server systems.

**Proposition 4.2.1.** The workload dimension $L = I + K - B$.

The following result was proved by Harrison and Van Mieghem [22]. It is essential to the reduction of the Brownian Control Problem.

**Proposition 4.2.2.** Let $M$ be an arbitrary workload matrix. Then, there exists an $L \times (K + N)$ matrix $G$ such that

$$MR = GK. \tag{4.3}$$

The choice of $G$ is usually not unique. We refer to $G$ as a control matrix associated with $M$. In the following, we fix a choice for $M$ and an associated $G$. Let $W = M R^I_+$ and define

$$g(w) = \inf\{ h \cdot q : Mq = w, q \in R^I_+, \} , \quad w \in W. \tag{4.4}$$

We consider the following.

**Assumption 4.2.1.** The mapping $g$ given by (4.4) is a well defined, continuous function from $W$ into $R_+$ and the infimum in (4.4) is attained for each $w \in W$. Moreover, there exists a continuous mapping $\phi : W \to R^I_+$ such that for $w \in W$, $\phi(w) \in \{ q \in R^I_+ : Mq = w, h \cdot q = g(w) \}$.

It is known that Assumption 4.2.1 is satisfied if for each $w \in W$ the set $\{ q \in R^I_+ : Mq = w \}$ is compact, see Appendix A.3 in [26]. Later, in Theorem 5.5.1, we show that for our choice of $M$ this condition is satisfied. In fact, in that case $g$ is a linear map.

**Definition 4.2.3.** (Admissible control for the EWF). An admissible control for the EWF is a $(K + N)$-dimensional adapted process $\tilde{U}$ defined on some probability space $(\Lambda, G, \{G_t\}, Q)$, which supports $L$-dimensional adapted processes $\tilde{W}$ and $\tilde{\xi}$, such that the following properties hold under $Q$:

(i) $\tilde{W}(t) = \tilde{\xi}(t) + G\tilde{U}(t) \in W$ for all $t \geq 0$, $Q$-a.s.,

(ii) $\tilde{U}$ is non-decreasing, $\tilde{U}(0) \geq 0$, $Q$-a.s.,

(iii) $\tilde{\xi}$ is an $L$-dimensional $\{G_t\}$-Brownian motion starting at the origin, with drift $M\theta$ and covariance matrix $M\Sigma M'$.

We call $\tilde{W}$ the state process with Brownian motion $\tilde{\xi}$ for the EWF under the control $\tilde{U}$. We let $A$ denote the set of admissible controls for the EWF.
Remark 4.2.1. Since \( \text{range}(K) = \mathbb{R}^{\mathbb{K} + \mathbb{N}} \) (see Lemma 3.2.1), we do not need to add a constraint on the range of \( \tilde{U} \).

Definition 4.2.4. (Equivalent Workload Formulation-EWF) Determine the optimal value
\[
\tilde{J}^* = \inf_{\tilde{U}} \tilde{J}(\tilde{U}) \quad \text{where} \quad \tilde{J}(\tilde{U}) = \mathbb{E} \left( \int_0^{\infty} e^{-\gamma t} g(\tilde{W}(t)) dt \right),
\]
where the infimum is taken over all admissible controls for the EWF. An admissible control that achieves the infimum in (4.5) is called an optimal control for the EWF.

The following proposition shows the equivalence of the BCP to the EWF. For a similar formulation of the BCP and the EWF this equivalence was proved by Harrison and Van Mieghem in [22]. For the formulation used here the equivalence can be proved in the same way as in Harrison and Williams [26].

Theorem 4.2.1. Suppose that Assumption 4.2.1 holds. The optimal value \( \tilde{J}^* \) for the BCP is equal to the optimal value \( \tilde{J}^* \) for the EWF. Furthermore,

(i) if \( \tilde{Y} \) is an admissible control process for the BCP with extended state processes \( (\tilde{Q}, \tilde{U}) \) and Brownian motion \( \tilde{X} \), all defined on some filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}) \), then on this same space, \( \tilde{U} = K\tilde{Y} \) is an admissible control for the EWF with the state process \( \tilde{W} = M\tilde{Q} \) and Brownian motion \( \tilde{\xi} = M\tilde{X} \).

(ii) if \( \tilde{U} \) is an admissible control for the EWF with state process \( \tilde{W} \) and Brownian motion \( \tilde{\xi} \), all defined on some filtered probability space \( (\Lambda, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{Q}) \), then there is a \( \{\mathcal{G}_t\} \)-adapted \( \mathbb{L} \)-dimensional process \( \tilde{Q} \) satisfying \( \mathbb{Q} \)-a.s.,
\[
M\tilde{Q} = \tilde{W} \quad \text{and} \quad \tilde{Q}(t) \in \mathbb{R}^\mathbb{I}_+ \quad \text{for all} \quad t \geq 0,
\]
and an extension \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}) \) of the filtered probability space \( (\Lambda, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{Q}) \) such that on this extended space there is an \( \mathbb{I} \)-dimensional \( \{\mathcal{F}_t\} \)-Brownian motion \( \tilde{X} \) starting at the origin with drift \( \theta \) and covariance matrix \( \Sigma \) that satisfies \( M\tilde{X} = \tilde{\xi} \), and there is an admissible control \( \tilde{Y} \) for the BCP that has extended state processes \( (\tilde{Q}, \tilde{U}) \) and Brownian motion \( \tilde{X} \).

4.3 Reduced EWF (REWF)

In this section we show how the EWF may sometimes be further reduced by reducing the matrix \( G \). Let \( \mathbb{P} = \mathbb{K} + \mathbb{N} \). Given an \( \mathbb{L} \times \mathbb{D} \) matrix \( H \) for \( 1 \leq \mathbb{D} \leq \mathbb{P} \), the cone generated by the matrix \( H \) is defined as follows,
\[
\mathcal{C}(H) = \left\{ v \in \mathbb{R}^\mathbb{P}_+ : v = Hu, u \in \mathbb{R}^\mathbb{D}_+ \right\} = \left\{ \sum_{j=1}^{\mathbb{D}} u_j H_j, u_j \in [0, \infty) \right\} = \mathbb{H} \mathbb{R}^\mathbb{D}_+,
\]
(4.7)
where $H^j$ denotes the $j^{th}$ column of $H$.

**Theorem 4.3.1.** Suppose $\hat{G}$ is an $L \times D$ matrix where $1 \leq D \leq P$ and $C(G) = C(\hat{G})$. Let $\hat{U} \in A$ be arbitrary. Then, there exists a $D$-dimensional adapted process $\hat{U}$ defined on the same filtered probability space as $\hspace{1em}$ such that:

(i) $G\hat{U} = \hat{G}\hat{U}$,  
(ii) $\hat{U}$ is non-decreasing with $\hat{U}(0) \geq 0$ almost surely.

**Proof.** Let $G^j$ be an arbitrary column of $G$. Then, since $C(G) = C(\hat{G})$ there exists $v^j \in \mathbb{R}^P_+$ such that $G^j = \hat{G}v^j$. Let

\[ \hat{U} = v^1\hat{U}_1 + \cdots + v^p\hat{U}_p. \]

(4.8)

It is straightforward to verify the properties (i), (ii) and that $\hat{U}$ is an adapted process. 

**Definition 4.3.1.** (Admissible control for the REWF) For some $1 \leq D \leq P$, let $\hat{G}$ be an $L \times D$ matrix such that $C(\hat{G}) = C(G)$. An admissible control for the REWF associated with $\hat{G}$ is a $D$-dimensional adapted process $\hat{U} = \{\hat{U}(t), t \geq 0\}$ defined on some filtered probability space $(\Lambda, \mathcal{G}, \{\mathcal{G}_t\}, Q)$, which supports $L$-dimensional adapted processes $\hat{W}$ and $\hat{\xi}$, such that the following properties hold under $Q$:

(i) $\hat{W}(t) = \hat{\xi}(t) + \hat{G}\hat{U}(t) \in \mathbb{R}^L_+$ for all $t \geq 0$, $Q$-a.s.,
(ii) $\hat{U}$ is non-decreasing with $\hat{U}(0) \geq 0$, $Q$-a.s.,
(iii) $\hat{\xi}$ is an $L$-dimensional $\{\mathcal{G}_t\}$-Brownian motion starting at the origin, with drift $M\theta$ and covariance matrix $M\Sigma M'$. We let $\hat{A}(\hat{G})$ denote the set of admissible controls for the REWF associated with $\hat{G}$.

**Definition 4.3.2.** (REWF) Suppose that $\hat{G}$ is an $L \times D$ for some $1 \leq D \leq P$ such that $C(G) = C(\hat{G})$. The REWF associated with $\hat{G}$ is to determine the optimal value $\hat{J}^*\hat{G}$, where

\[ \hat{J}^*\hat{G} = \inf_{\hat{U}} \hat{J}^\hat{G}(\hat{U}) \quad \text{where} \quad \hat{J}^\hat{G}(\hat{U}) = \mathbb{E}\left(\int_0^\infty e^{-\gamma t}g(\hat{W}(t))dt\right), \]

(4.9)

where the infimum is taken over all $\hat{U} \in \hat{A}(\hat{G})$. An admissible control $\hat{U}$ that achieves the infimum in (4.9) is called an optimal control for the REWF associated with $\hat{G}$.

Each $L \times D$ matrix $\hat{G}$, with the property that $C(G) = C(\hat{G})$, gives rise to an REWF. The following lemma shows that all REWFs are equivalent to the EWF.

**Theorem 4.3.2.** For some $1 \leq D \leq P$, let $\hat{G}$ be an $L \times D$ matrix such that $C(G) = C(\hat{G})$. Then,

\[ \hat{J}^*\hat{G} = \hat{J}^*, \]

(4.10)
i.e., the optimal value of the REWF associated with \( \hat{G} \) is the same as the optimal value of the EWF associated with \( G \).

**Proof.** By Theorem 4.3.1 for each \( \tilde{U} \in \mathcal{A} \) there exists a \( \hat{U} \in \hat{\mathcal{A}}(\hat{G}) \) defined on the same filtered probability space as \( \tilde{U} \) such that \( \tilde{G} \tilde{U} = \hat{G} \hat{U} \). Then with \( \tilde{\xi} = \hat{\xi} \), \( \tilde{W} = \hat{W} = \tilde{\xi} + G \tilde{U} = \hat{\xi} + \hat{G} \hat{U} = \tilde{\xi} + G \tilde{U} \), and therefore, \( \tilde{J}(\tilde{U}) = \hat{J}(\hat{U}) \). Also, by a similar proof to that for Theorem 4.3.1 (by switching the roles of \( G \) and \( \hat{G} \)), for each \( \hat{U} \in \hat{\mathcal{A}}(\hat{G}) \) there exists a \( \tilde{U} \in \mathcal{A}(G) \) defined on the same probability space as \( \hat{U} \) such that, \( \hat{G} \hat{U} = G \tilde{U} \) and again \( \tilde{J}(\tilde{U}) = \hat{J}(\hat{U}) \). It follows that \( \tilde{J} = \hat{J} \).

\[
4.4 \text{ Canonical Choice of } M \text{ and Dual Program}
\]

Harrison [20] provided an alternative description of \( R^\perp \) and proposed a “cannonical” choice for \( M \) and \( G \). For this we introduce the following.

**Definition 4.4.1.** (Dual Program DP)

\[
\text{maximize } y \cdot \lambda \text{ subject to } y' R \leq z' A, \; z' \cdot 1 \leq 1 \text{ and } z \geq 0. \quad (4.11)
\]

**Proposition 4.4.1.** Let \( \{(y^1, z^1), \ldots, (y^\tilde{L}, z^\tilde{L})\} \) be the set of extremal optimal solutions of the dual program. Let \( \{(y^1, z^1), \ldots, (y^\tilde{L}, z^\tilde{L})\} \) be such that \( \{y^1, \ldots, y^\tilde{L}\} \) is a maximal linearly independent subset of \( \{y^1, \ldots, y^\tilde{L}\} \). Then,

\[
R^\perp = \text{span}\{y^1, \ldots, y^\tilde{L}\}. \quad (4.12)
\]

Proposition 4.4.1 suggests a natural choice for a workload matrix \( M \). In particular, we fix \( \{y^1, \ldots, y^\tilde{L}\} \) and let

\[
M = \begin{pmatrix}
y^1 \\
\vdots \\
y^{\tilde{L}}
\end{pmatrix}, \quad (4.13)
\]

where we abuse notation and we think of \( y^1, \ldots, y^{\tilde{L}} \) as row vectors. Following Harrison [20] we let

\[
G = [\Pi \; \Pi N - MJ], \quad (4.14)
\]

where,

\[
\Pi = \begin{pmatrix}
z^1 \\
\vdots \\
z^{\tilde{L}}
\end{pmatrix}, \quad (4.15)
\]
and vectors $z^1, \ldots, z^L$ are viewed as row vectors that accompany $\{y^1, \ldots, y^L\}$ as in Proposition 4.4.1. Again we abuse notation and we think of $z^1, \ldots, z^L$ as row vectors. Then, the matrix $G$ given by (4.14) satisfies the relation $MR = GK$ in Proposition 4.2.2. Moreover, from the dual program it follows that $G \geq 0$. 
Chapter 5

Workload Dimension and the Server-Buffer Graph $\mathcal{G}$

5.1 Server-Buffer Graph

Definition 5.1.1. (Server-buffer graph $\mathcal{G}$) The graph $\mathcal{G}$ in which servers and buffers form the nodes, and undirected edges between the nodes are given by basic activities, is called the server-buffer graph.

The following theorem was established in two stages. In [21], Harrison and Lopez proved the equivalence of (i)-(iii) below and subsequently Williams [41] observed that (i)-(iii) are equivalent to (iv).

Theorem 5.1.1. The following conditions are equivalent:

(i) the dual program (4.4.1) has a unique solution $(y^*, z^*)$,

(ii) the number of basic activities $B = I + K - 1$,

(iii) all servers communicate via basic activities,

(iv) the graph $\mathcal{G}$ is a tree.

Remark 5.1.1. It follows from Proposition 2 in [21] that for the solution in (i), $y^* > 0$, $z^* > 0$, $y^* H = z^* B$ and $y^* J < z^* N$. 

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A parallel server system that satisfies any of the equivalent conditions (i)-(iv) of Theorem 5.1.1 is said to satisfy the complete resource pooling (CRP) condition. From the results of Bramson and Williams [8] embodied in Proposition 4.2.1, we know that in general $L = I + K - B$, and so Theorem 5.1.1 characterizes the situation when $L = 1$. The following new result generalizes part (iv) of Theorem 5.1.1 to $L > 1$. This result is proved in Section 5.2.

**Theorem 5.1.2.** The workload dimension $L$ is equal to the number of connected components in the server-buffer graph $G$. Indeed, the server-buffer graph $G$ is a forest of $L$ trees.

### 5.2 Enumeration of Servers, Buffers and Activities and Proof of Theorem 5.1.2

To prove Theorem 5.1.2, we introduce the following enumeration scheme. Recall that $I = \{1, \ldots, I\}$, $K = \{1, \ldots, K\}$ and $B = \{1, \ldots, B\}$. Let $M$ be the number of connected components in $G$ denoted by $T_1, \ldots, T_M$. For each $m \in M = \{1, \ldots, M\}$ let $I_m$ be the subset of $I$ that indexes the buffers in $T_m$. Similarly, for each $m \in M$ let $K_m$, respectively $B_m$, be the subset of $K$, respectively $B$, that indexes the servers, respectively the basic activities, in $T_m$. The cardinalities of $I_m$, $K_m$ and $B_m$ are denoted by $I_m$, $K_m$ and $B_m$, respectively. Note that $B_m$ is the number of edges in $T_m$. The set $N = \{B + 1, \ldots, J\}$ of non-basic activities, has cardinality $N$.

For each $m \in M$, let $N^{m,c}$ be the set of non-basic activities that consume material from buffers in $T_m$ and let $N^{m,p}$ be the set of non-basic activities that are processed by servers in $T_m$. For each $m, m' \in M$ let $N^{m,m'}$ be the set of non-basic activities that consume material from buffers in $T_m$ and that are processed by servers in $T_{m'}$. By definition $N^{m}_m$ is a subset of both $N^{m,c}$ and $N^{m,p}$; in fact $N^{m}_m = N^{m,c} \cap N^{m,p}$. Note that $N^{m}_m$ is the set of non-basic activities that consume material from buffers in $T_m$ and that are processed by servers in $T_{m'}$. Let $N^{m,c}$, $N^{m,p}$ and $N^{m,m'}$ be the cardinalities of $N^{m,c}$, $N^{m,p}$, and $N^{m,m'}$, respectively. Then for any $m' \in M$,

$$N^{m,c} = \sum_{m'=1}^{M} N^{m,m'}.$$  

We choose the enumeration of buffers, servers, basic and non-basic activities so that the following properties hold.

**Assumption 5.2.1.**

i) if buffer $i \in I_m$ and buffer $i' \in I_{m'}$ where $m < m'$, then $i < i'$,

ii) if server $k \in K_m$ and server $k' \in K_{m'}$ where $m < m'$, then $k < k'$. 

iii) if $i$ and $i'$ are distinct buffers such that $i < i'$ and if $j$ and $j'$ are basic activities such that $i = i(j)$ and $i' = i(j')$, then $j < j'$.

iv) if $j \in \mathcal{N}_{m,c}$ and $j' \in \mathcal{N}_{m',c}$ where $m < m'$, then $j < j'$.

v) if $j' \in \mathcal{N}_{m}^{m'}$ and $j'' \in \mathcal{N}_{m}^{m''}$ where $m' < m''$, then $j' < j''$.

Assumption 5.2.1 induces the following partitions of $R$ and $A$,

$$R = \begin{pmatrix} H^1 & 0 & \ldots & 0 & J^1 & 0 & \ldots & 0 \\ 0 & H^2 & \ddots & \vdots & 0 & J^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & H^M & 0 & \ldots & 0 & J^M \end{pmatrix}, \quad (5.1)$$

$$A = \begin{pmatrix} B^1 & 0 & \ldots & 0 & N^1_1 & 0 & \ldots & 0 & N^M_1 & 0 & \ldots & 0 \\ 0 & B^2 & \ddots & \vdots & 0 & N^1_2 & \ddots & \vdots & 0 & N^M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & B^M & 0 & \ldots & 0 & N^1_M & 0 & \ldots & 0 & N^M_M \end{pmatrix}, \quad (5.2)$$

where $H^m$ is the $I_m \times B_m$ matrix of rates at which basic activities in $T_m$ consume material from buffers in $T_m$. The $I_m \times N_{m,c}$ matrix $J^m$ is the matrix of average processing rates for non-basic activities in $\mathcal{N}_{m,c}$ that consume material from buffers in $T_m$. The $K_m \times B_m$ matrix $B^m$ has a one in its $j^{th}$ column in the row corresponding to the server that processes the $j^{th}$ basic activity in $T_m$. The matrix $N^m_{m'}$ is a $K_{m'} \times N_{m,c}$ matrix of zeros and ones that signals which servers in $T_{m'}$ process jobs from buffers in $T_m$ using non-basic activities, i.e. it has a one in its $j^{th}$ column for the $j^{th}$ non-basic activity that consumes material from a buffer in $T_m$ and is processed by a server in $T_{m'}$. The row in which the one is positively tells which server performs the processing. For each $m$, the enumeration induces a partition of $J^m$

$$J^m = [J^m_1, \ldots, J^m_M],$$

where $J^m_m$ is the $I_m \times N^{m}_{m'}$ matrix of consumption rates for non-basic activities in $\mathcal{N}_{m}^{m'}$. In the following proof, a vector $x$ depending on the context sometimes denotes a row vector and at other times it denotes a column vector, which is meant will be clear from the context.

(Proof of Theorem 5.1.2) Since each $T_m$ is connected and $I_m + K_m$ is a number of nodes in $T_m$, it follows immediately that

$$B_m \geq I_m + K_m - 1, \quad \text{for} \quad m = 1, \ldots, M. \quad (5.3)$$
Summing over all connected components, we obtain that

\[ B = \sum_{m=1}^{M} B_m \geq \sum_{m=1}^{M} (I_m + K_m - 1) = I + K - M. \]  

(5.4)

Rearranging the terms in the last equation and using Proposition 4.2.1 we obtain that

\[ M \geq I + K - B = L. \]  

(5.5)

Hence the workload dimension \( L \) is less than or equal to the number \( M \) of connected components in \( G \). To establish that \( M = L \), it is enough to show that \( B_m = I_m + K_m - 1 \) for \( 1 \leq m \leq M \).

For an argument by contradiction, suppose that for some index \( m^* \),

\[ B_{m^*} > I_{m^*} + K_{m^*} - 1. \]  

(5.6)

From the heavy traffic Assumption 3.2.1 we have that \( x^* = (x_{m^*}^*, 0_n) \), where \( x_{m^*}^* \) is a positive \( B \)-dimensional vector of nominal rate allocations for basic activities and \( 0_n \) is an \( N \)-dimensional vector of zeros whose entries are nominal allocations for non-basic activities. With a slight abuse of notation, we write

\[ x_{m^*}^* = (x_{1_m}^*, \ldots, x_{M_m}^*), \]  

(5.7)

where \( x_{m_m}^* \) is a \( B_m \)-dimensional vector of allocations for the basic activities in \( B_m, m = 1, \ldots, M \). Similarly, we write

\[ \lambda = (\lambda_1, \ldots, \lambda_M), \]  

(5.8)

where \( \lambda_m \) is an \( I_m \)-dimensional vector of average arrival rates for the buffers in \( I_m \) and

\[ 1_K = (1_1, \ldots, 1_M), \]  

(5.9)

where \( 1_m \) is a \( K_m \)-dimensional vector of ones. From the heavy traffic Assumption 3.2.1 and (5.1)-(5.2),

\[ \lambda_m = H^m x_m^* \quad \text{and} \quad 1_m = B^m x_m^* \quad \text{for} \quad m = 1, \ldots, M, \]  

(5.10)

and in particular \( \lambda_{m^*} = H^{m^*} x_{m^*}^* \) and \( 1_{m^*} = B^{m^*} x_{m^*}^* \). There are two cases to consider.

**Case I:** Suppose that \( I_{m^*} + K_{m^*} < B_{m^*} \).

Let

\[ P^{m^*} = \begin{pmatrix} H^{m^*} \\ B^{m^*} \end{pmatrix}. \]  

(5.11)
Then $P_{m^*}$ is an $(\mathbb{I}_{m^*} + \mathbb{K}_{m^*}) \times \mathbb{B}_{m^*}$ matrix and the null space of $P_{m^*}$ is non-trivial. Thus, there exists a non-zero $\mathbb{B}_{m^*}$-dimensional vector $v$ such that $P_{m^*}v = 0$. Since all of the components of $x_{m^*}$ are strictly greater than zero, there exists a $\delta > 0$ such that all of the components of $x_{m^*} + \delta v$ are strictly greater than zero. Define a $\mathcal{J}$-dimensional vector $\hat{v}$ as follows,

$$
\hat{v} = (0, \ldots, 0_{m^*-1}, v, 0_{m^*+1}, \ldots, 0_{\mathcal{M}}, 0_{\mathcal{N}})',
$$

where $0_m$ is the $\mathbb{B}_m$-dimensional zero vector for $m \neq m^*$ and $0_n$ is the $\mathbb{N}$-dimensional zero vector. Then

$$
\begin{pmatrix} R \\ A \end{pmatrix} \delta \hat{v} = 0.
$$

The first $\mathcal{B}$ components of $x^* + \delta \hat{v}$ are strictly greater than zero, the last $\mathcal{N}$ components are identically zero, and

$$
R(x^* + \delta \hat{v}) = \lambda, \quad A(x^* + \delta \hat{v}) = 1_{\mathcal{K}}.
$$

This violates the uniqueness part of the heavy traffic Assumption 3.2.1.

**Case II:** Suppose that $\mathbb{I}_{m^*} + \mathbb{K}_{m^*} = \mathbb{B}_{m^*}$.

Let $P_{m^*}$ be as in Case I. There are two subcases to consider.

(i) The null space of $P_{m^*}$ is not $\{0\}$. Then we can repeat the argument from Case I.

(ii) The null space of $P_{m^*}$ is $\{0\}$. Then $P_{m^*}$ is invertible. Let $(P_{m^*})^{-1}$ be the inverse of $P_{m^*}$.

Then there exists a $\mathbb{B}_{m^*}$-dimensional vector $u$ such that

$$
P_{m^*}u = \begin{pmatrix} 0 \\ -1_{m^*} \end{pmatrix},
$$

where $0_{m^*}$ is the $\mathbb{I}_{m^*}$-dimensional zero vector and $1_{m^*}$ is the $\mathbb{K}_{m^*}$-dimensional vector of ones. As in Case I, there is a $\delta \in (0, 1)$ such that all of the components of $x_{m^*} + \delta u$ are strictly positive and in the same manner as in (5.12), we can extend $u$ to a $\mathcal{J}$-dimensional vector $\hat{u} > 0$ satisfying

$$
R(x^* + \delta \hat{u}) = \lambda \quad \text{and} \quad A(x^* + \delta \hat{u}) = 1 - \delta 1_{m^*},
$$

where $(1_{m^*})_k = 1$ if $k \in \mathcal{K}_{m^*}$ and $(1_{m^*})_k = 0$ otherwise. This violates the heavy traffic Assumption 3.2.1. Thus, by contradiction we have that,

$$
\mathbb{B}_m = \mathbb{I}_m + \mathbb{K}_m - 1, \text{ for all } m,
$$

(5.17)
Figure 5.1: Server-buffer graph $G$. Trees $T_1, \ldots, T_L$ encircled by dashed lines are enumerated from left to right

as desired. Furthermore, since (5.17) holds, as observed by Williams [41], since the connected graph $T_m$ with $I_m + K_m$ nodes has exactly $I_m + K_m - 1$ edges, it must be a tree, see Theorem 3.1 in [5].

From this point on, the trees in $G$ will be denoted by $T_1, \ldots, T_L$, and we let $L = \{1, \ldots, L\}$.

5.3 Choice of Workload and Control Matrices

In this section we propose a choice of workload matrix $M$ which in general is different from the “canonical” choice proposed by Harrison [20]. We have seen in the previous section that Harrison’s choice for the workload matrix is obtained by finding the extremal optimal solutions of the dual program for the entire network. For our choice, we treat each tree $T_l$, $l \in L$, in isolation as a network with a one-dimensional workload. By collecting workload vectors, one for each tree, we construct a workload matrix for the whole network. In contrast to the situation for Harrison’s choice of workload matrix, the control matrix $G$ that goes with our choice of workload matrix is not necessarily non-negative. In this section, a vector $x$ depending on the context sometimes denotes a row vector and at other times it denotes a column vector, which is meant will be clear from the context.

For $x \in \mathbb{R}_+^J$ let $x = (x_B, x_N)$, where $x_B$ is a $B$-dimensional vector and $x_N$ is an $N$-dimensional vector. Further let $x_B = (x_1, \ldots, x_L)$, where $x_l$ corresponds to basic activities in $T_l$, $l \in L$. For each $l \in L$, we augment $T_l$ to $\tilde{T}_l$ by adding the non-basic activities in $N_l^l$, i.e., the non-basic
activities that consume material from buffers in $T_l$ and that are processed by servers in $T_l$. The first order data for $\tilde{T}_l$ is as follows:

$$R_l = [H^l, J^l], \quad A_l = [B^l, N^l], \quad \lambda_l.$$  \hfill (5.18)

**Lemma 5.3.1.** For $l \in \mathcal{L}$, $\tilde{T}_l$ is in heavy traffic, i.e., $(x^\ast_l, 1)$ is the unique solution of the linear program in Assumption 3.2.1, with $R_l$ in place of $R$, $A_l$ in place of $A$ and $\lambda_l$ in place of $\lambda$.

**Proof.** Suppose that $(x_1^l, \rho^1)$ are such that $R_l x_1^l = \lambda_l, A_l x_1^l \leq \rho^1 \leq 1$ and $x_1^l \neq x^\ast_l$. By the proof of Theorem 5.1.2, in view of (5.1) and (5.2), we can extend $x_1^l$ to a $J$-dimensional vector $\hat{x}^l$ such that $R \hat{x}^l = \lambda, A \hat{x}^l \leq 1$ and $\hat{x}^l \neq x^\ast$. Then, $(\hat{x}^l, 1)$ is another solution of the linear program in Assumption 3.2.1 and this contradicts the uniqueness part of Assumption 3.2.1 for the whole network. \hfill $\square$

In view of Theorem 5.1.2, the isolated “subnetwork” $\tilde{T}_l$ corresponds to a parallel server system with a one-dimensional workload. A workload matrix (vector) for $\tilde{T}_l$ can be obtained, by finding a unique solution (see Theorem 5.1.1) of the following dual program for $\tilde{T}_l$.

**Definition 5.3.1.**

$$\max y^l \cdot \lambda_l \text{ subject to } y^l R^l \leq z^l A^l, z^l \cdot 1 \leq 1 \text{ and } z^l \geq 0.$$  \hfill (5.19)

If $(y^l, z^l)$ is the solution of the dual program for $\tilde{T}_l$, then by $y^l$ is the choice of workload matrix for $\tilde{T}_l$. It follows that $y^l \in \mathcal{R}_l^\perp$ where

$$\mathcal{R}_l = \{ \delta_l \in \mathbb{R}^l : \delta_l = H^l x_l, K^l x_l = 0 \},$$  \hfill (5.20)

is the space of reversible displacements for $\tilde{T}_l$, with

$$K^l = \begin{pmatrix} B^l & N^l \\ 0 & -I^l \end{pmatrix},$$  \hfill (5.21)

where $I^l$ is the $\mathbb{N}^l \times \mathbb{N}^l$ identity matrix. Recall that by Proposition 2 in [21], $y^l > 0$ and $z^l > 0$. Let $\hat{y}^l$ be the $l$-dimensional vector:

$$\hat{y}^l = (0_1, \ldots, 0_{l-1}, y^l, 0_{l+1}, \ldots, 0_n),$$  \hfill (5.22)

which is the augmentation of $y^l$, where $0_l$ is the $l$-dimensional vector of zeros for $l' \neq l$. Let $M$ be the $\mathbb{L} \times \mathbb{I}$ matrix with rows given by $\hat{y}^l, l \in \mathcal{L}$. Then, $M$ has a block diagonal-like structure and the following two lemmas show that $M$ is a valid choice for a workload matrix for the entire network. In particular, the rows of $M$ are linearly independent and they span $\mathcal{R}_l^\perp$. 


Lemma 5.3.2. Given \( y^l \in \mathcal{R}_1^l \) and \( \hat{y}^l \) given by (5.22), we have \( \hat{y}^l \in \mathcal{R}_1^l \). Furthermore the set \( \{ \hat{y}^1, \ldots, \hat{y}^L \} \) is a basis for \( \mathcal{R}_1^L \).

Proof. Let \( w \in \mathcal{R} \) be arbitrary. Then \( w = (w_1, \ldots, w_L) \), where each \( w_l \) is \( l_l \)-dimensional. We would like to show that \( \hat{y}^l \cdot w = 0 \). Since \( w \in \mathcal{R} \) there exists an \( x = (x_1, \ldots, x_l) \in \mathbb{R}^l \) such that \( w = Hx \) and \( Bx = 0 \). By the form of \( H \) and \( B \), this means that \( w_l = H^lx_l \) and \( B^lx_l = 0 \). Therefore \( w_l \in \mathcal{R}_1^l \) and by the assumption on \( y^l \), \( y^l \cdot w_l = 0 \). Hence \( \hat{y}^l \cdot w = 0 \) and since \( w \in \mathcal{R} \) was arbitrary \( \hat{y}^l \in \mathcal{R}_1^l \). The set \( \{ \hat{y}^1, \ldots, \hat{y}^L \} \) is linearly independent by the form (5.22) and \( L \) linearly independent vectors in \( L \) dimensional vector space \( \mathcal{R}_1^L \) constitute a basis for \( \mathcal{R}_1^L \). \qed

It follows from Lemma 5.3.2 that one can use non-zero multiples of \( \{ \hat{y}^1, \ldots, \hat{y}^L \} \) as the rows of a workload matrix. The workload matrix \( M \) has the following form:

\[
M = \begin{pmatrix}
\hat{y}^1 \\
\hat{y}^2 \\
\vdots \\
\hat{y}^{-1} \\
\hat{y}^l
\end{pmatrix} = \begin{pmatrix}
y^1 & 0 & \ldots & \ldots & 0 \\
0 & y^2 & 0 & \ldots & \vdots \\
\vdots & 0 & \ddots & 0 & \vdots \\
\vdots & \ldots & 0 & y^{-1} & 0 \\
0 & \ldots & \ldots & 0 & y^l
\end{pmatrix}.
\]

(5.23)

Note that for \( \hat{W} = M\hat{Q} \), for each \( l \in \mathcal{L} \), \( \hat{W}_l \) is a sum of scaled “queue-lengths” for buffers in \( \mathcal{T}_l \). In this sense, \( \hat{W}_l \) represents the Brownian workload of the tree \( \hat{T}_l \). This is not the case in general for Harrison’s proposal for a choice of workload matrix. The \( l^{th} \) component of workload \( \hat{W}_l \) that goes with Harrison’s “canonical” workload matrix can involve queue-lengths of all the buffers in \( \mathcal{G} \), not only those in a single tree \( \hat{T}_l \).

Proposition 5.3.1. With above choice of \( M \), \( \mathcal{W} = MR_1^L = \mathcal{R}_1^L \).

Proof. For \( l \in \mathcal{L} \), by Proposition 2 in [21], \( y^l > 0 \). The conclusion is immediate by the form of \( M \) in equation (5.23). \qed

For the workload matrix \( M \) described above we find a control matrix \( G \) that will satisfy \( MR = GK \). For this, for each \( l \in \mathcal{L} \), we augment the \( K_l \)-dimensional vector \( z_l \) to a \( K \)-dimensional vector:

\[
z^l = (0_1, \ldots, 0_{l-1}, z^l, 0_{l+1}, \ldots, 0_L),
\]

(5.24)
where $0_l$ is $\mathbb{R}^l$-dimensional zero vector. Let

$$\Pi = \begin{pmatrix} \hat{z}^1 \\ \hat{z}^2 \\ \vdots \\ \hat{z}^{L-1} \\ \hat{z}^L \end{pmatrix} = \begin{pmatrix} z^1 & 0 & \ldots & \ldots & 0 \\ 0 & z^2 & 0 & \ldots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \ldots & \ldots & 0 & z^{L-1} \\ 0 & \ldots & \ldots & 0 & z^L \end{pmatrix} \quad (5.25)$$

**Lemma 5.3.3.** Let $M, \Pi$ be given by (5.23) and (5.25). Define an $L \times \mathbb{P}$ matrix $G$ by

$$G = [\Pi \Pi N - MJ]. \quad (5.26)$$

Then $G$ is a valid choice for a control matrix, i.e., the relation $MR = GK$ is satisfied.

**Proof.** For $l \in L$, since $(y^l, z^l)$ is a unique extremal optimal solution of the dual linear program for the subnetwork $\hat{T}_l$, we have that $y^l H^l = z^l B^l$. By the form of $M$ and $\Pi$ it follows that $\Pi B = MH$ and therefore,

$$GK = [\Pi \Pi N - MJ] \begin{pmatrix} B & N \\ 0 & -I \end{pmatrix} = [\Pi B \Pi N - \Pi N + MJ] \quad (5.27)$$

$$= [MH \; MJ] = M [H \; J] = MR. \quad \square$$

We proceed to describe $G$ more explicitly. First we compute $MJ$:

$$MJ = \begin{pmatrix} y^1 J^1 \\ 0 & y^2 J^2 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & y^L J^L \end{pmatrix} \quad (5.28)$$

$$= \begin{pmatrix} y^1 J^1_1 & \ldots & y^1 J^1_L & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & y^2 J^2_1 & \ldots & y^2 J^2_L & \vdots & \vdots \\ \vdots & \ldots & \vdots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \vdots & \ldots & \vdots & y^L J^L_1 & \ldots & y^L J^L_L \end{pmatrix},$$

where $y^l J^l$ is an $\mathbb{N}^{l,c}$-dimensional vector and $y^l J^l_{m_m}$ is an $\mathbb{N}^{l,m}$-dimensional vector. Then we
compute $\Pi N$:

$$\Pi N = \begin{pmatrix}
z^1 N^1_1 & 0 & \ldots & 0 & \ldots & z^1 N^1_1 & 0 & \ldots & 0 \\
0 & z^2 N^2_1 & \ddots & \ldots & \ldots & 0 & z^2 N^2_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & z^l N^l_1 & \ldots & 0 & \ldots & 0 & z^l N^l_1
\end{pmatrix}, \quad (5.29)$$

where $z^l N^m_l$ is an $N^m_l$-dimensional vector. For the above choices of $M$ and $\Pi$, it is not true in general that $\Pi N - MJ \geq 0$. More specifically, $G$ has the following form:

$$G = [\Upsilon_0, \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_L], \quad (5.30)$$

where

$$\Upsilon_0 = \Pi = \begin{pmatrix}
z^1 & 0 & \ldots & \ldots & 0 \\
0 & z^2 & 0 & \ldots & \vdots \\
\vdots & 0 & \ddots & 0 & \vdots \\
\vdots & \ldots & 0 & z^{l-1} & 0 \\
0 & \ldots & 0 & z^l
\end{pmatrix}, \quad (5.31)$$

and for $l \in \mathcal{L}$,

$$\Upsilon_l = \begin{pmatrix}
z^1 N^1_l & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & z^2 N^2_l & 0 & \ldots & \ldots & \vdots \\
\vdots & 0 & \ddots & 0 & \ddots & 0 & \vdots \\
-y^l J^l_l & -y^l J^l_l & \ldots & z^l N^l_l - y^l J^l_l & -y^l J^l_{l+1} & \ldots & -y^l J^l_l \\
0 & 0 & 0 & 0 & z^{l+1} N^{l+1}_l & 0 & 0 \\
\vdots & \ldots & \ldots & \ldots & 0 & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & z^l N^l_l
\end{pmatrix}. \quad (5.32)$$

For $l, m \in \mathcal{L}$, each column of $N^l_m$ and $J^l_m$ has one positive entry and all other entries are equal to zero. Recall that for each $l \in \mathcal{L}$, $y^l > 0$, $z^l > 0$ and $y^l J^l_l < z^l N^l_l$ (see Remark 5.1.1). It follows that for each $l, m \in \mathcal{L}$, $z^m N^l_m > 0$, $-y^l J^l_m < 0$ and $z^l N^l_l - y^l J^l_l > 0$.

As we will see in Chapter 8, in practice it may be computationally convenient to choose a slightly different workload and control matrix according to the following lemma.
Lemma 5.3.4. For any $c \in \mathbb{R}_{+}^{L}$ such that $c^{l} > 0$ for $l = 1, \ldots, L$, let

$$M^{c} = \begin{pmatrix}
c^{1}y^{1} & 0 & \ldots & \ldots & 0 \\
0 & c^{2}y^{2} & 0 & \ldots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
\vdots & \ldots & 0 & c^{L-1}y^{L-1} & 0 \\
0 & \ldots & \ldots & 0 & c^{L}y^{L}
\end{pmatrix},$$

$$\Pi^{c} = \begin{pmatrix}
c^{1}z^{1} & 0 & \ldots & \ldots & 0 \\
0 & c^{2}z^{2} & 0 & \ldots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
\vdots & \ldots & 0 & c^{L-1}z^{L-1} & 0 \\
0 & \ldots & \ldots & 0 & c^{L}z^{L}
\end{pmatrix},$$

and

$$G^{c} = [\Pi^{c} \Pi^{c}N - M^{c}J].$$

(5.33)

Then, $M^{c}$ and $G^{c}$ are valid choices for a workload and a control matrix.

Proof. Since $c^{l} > 0$ for each $l$, the rows of $M^{c}$ constitute a basis for $\mathbb{R}_{+}^{L}$ by Lemma 5.3.2. By a straightforward computation as in Lemma 5.3.3 we see that $M^{c}R = G^{c}K$. Moreover, the $l$th row of $G^{c}$ is the $l$th row of $G$ multiplied by $c^{l}$.

Note that, if $c$ is the $L$-dimensional vector of ones, then $M^{c} = M$ and $G^{c} = G$.

5.4 Columns of $G$ and Components of the Control $\tilde{U}$

The control process $\tilde{U}$ in the EWF is $\mathcal{P}$-dimensional. The first $K$ components of $\tilde{U}$ are Brownian model analogues of server idle-times. The last $N$ components of $\tilde{U}$ are Brownian model analogues of allocations to non-basic activities. An increase in $\tilde{U}_{j}$ moves the system in the direction $G^{j}$, where $G^{j}$ is the $j$th column of $G$. To better understand the components of $\tilde{U}$ there are three cases to consider.

(i) Increasing $\tilde{U}_{k}$ for $k \in K$, increases the workload $\tilde{W}_{l}$ for $\mathcal{T}_{l}$ when server $k$ belongs to $\mathcal{T}_{l}$ and has no effect on other components of the workload.

(ii) Increasing $\tilde{U}_{j}$ for $j \in N_{l}^{l}$, $l \in \mathcal{L}$, increases the workload $\tilde{W}_{l}$ for $\mathcal{T}_{l}$ and has no effect on other components of the workload.
(iii) Increasing $\tilde{U}_j$ for $j \in \mathcal{N}_m^l$, $m \neq l, m, l \in \mathcal{L}$ corresponds to use of an “external” non-basic activity, and will decrease the workload $\tilde{W}_l$, of $\mathcal{T}_l$ and increase the workload $\tilde{W}_m$ for $\mathcal{T}_m$. This has to do with the fact that $j$ consumes material from a buffer in $\mathcal{T}_l$ and is processed by a server in $\mathcal{T}_m$.

### 5.5 The Cost Function

Recall that $h = (h_1, \ldots, h_L)'$ and by definition, for $w \in \mathbb{R}_+^L$,

$$g(w) = \min \{h \cdot q : Mq = w, q \in \mathbb{R}_+^I \}.$$  \hfill (5.34)

**Theorem 5.5.1.** The cost function function $g$ is linear. In particular, for $w \in \mathbb{R}_+^L$, \hfill

$$g(w) = \kappa \cdot w,$$ \hfill (5.35)

where \hfill

$$\kappa_l = \min_{i \in I_l} \left( \frac{h_i}{y_{li}} \right), \quad l \in \mathcal{L}. \hfill (5.36)$$

Moreover, a continuous selection function $\phi$ associated with $g$ is given by $\phi(w) = q^*(w)$ for $w \in \mathbb{R}_+^L$, where, for each $l \in \mathcal{L}$,

$$q^*_l = \frac{w_l}{y^l_{i^*_l}} \quad \text{and} \quad q^*_i = 0 \quad \text{for} \quad i \in I_l \setminus \{i^*_l\}, \hfill (5.37)$$

and $i^*_l \in I_l$ is such that $\kappa_l = h_{i^*_l}/y^l_{i^*_l}$.

**Proof.** Fix $w \in \mathbb{R}_+^L$. For $q \in \mathbb{R}_+^L$, let $q = (q^1, \ldots, q^L)'$ where $q^l$ is an $\mathbb{R}_{+l}$-dimensional vector. Recall the special block diagonal like structure of the workload matrix $M$. If $Mq = w$, then for each $l \in \{1, \ldots, L\}$,

$$w_l = y^l \cdot q^l, \hfill (5.38)$$

and

$$h \cdot q = \sum_{l=1}^L \sum_{i \in I_l} h_i q^l_i = \sum_{l=1}^L \sum_{i \in I_l} \left( \frac{h_i}{y^l_{i}} \right) y^l_{i} q^l_i \hfill (5.39)$$

$$\geq \sum_{l=1}^L \kappa_l \left( \sum_{i \in I_l} y^l_{i} q^l_i \right) = \sum_{l=1}^L \kappa_l w_l. \hfill (5.40)$$

If $q^*(w)$ is as in (5.37), then $Mq^*(w) = w$ and $h \cdot q^*(w) = \kappa \cdot w$. This completes the proof. \hfill \Box
Remark 5.5.1. If \( M^c \) and \( G^c \) are used in place of \( M \) and \( G \), for each \( l \), \( i^*_l \) can be kept the same, but the values of \( \kappa \) and \( q^* \) need to be adjusted for the fact that \( y^1_l \) is replaced by \( c^l y^1_l \) in (5.36) and (5.38).

5.6 Our Choice of Workload Matrix versus that of Harrison

In this section, for a specific example, we explicitly compute our choices for workload and control matrices, and compare them to Harrison’s canonical choices. As we will see the main difference is that the columns of Harrison’s control matrix cannot be interpreted as in Section 5.4. We consider a parallel server system consisting two buffers, two servers and three activities (see Figure 5.6). First order parameters are as follows:

\[
\lambda_1 = \mu_1 > 0, \quad \lambda_2 = \mu_2 > 0, \quad R = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & \mu_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.\] (5.41)

Assumption 3.2.1 is satisfied with \( x^* = (1, 1, 0) \). Activities 1 and 2 are basic and activity 3 is non-basic. The server-buffer graph \( G \) consists of two trees, \( T_1, T_2 \) and by Theorem 5.1.2 the workload is 2-dimensional. Harrison’s canonical workload matrix is obtained by finding extremal solutions of the dual program:

\[
\text{maximize } y_1 \lambda_1 + y_2 \lambda_2 \text{ subject to: } y_1 \lambda_1 \leq z_1, y_2 \lambda_2 \leq z_2, y_2 \mu_3 \leq z_1, z_1 + z_2 = 1, z_1, z_2 \geq 0.
\]

It is well known that the value of the above dual program equals the value of the primal program in Assumption 3.2.1, which equals one. Using the fact that \( \lambda_1 = \mu_1, \lambda_2 = \mu_2 \), the extremal solutions of this dual program are seen to be \((y^1, z^1)\) and \((y^2, z^2)\) where

\[
y^1 = (1/\lambda_1, 0), \quad z^1 = (1, 0),
\]

\[
y^2 = (\mu_3/(\lambda_1+\lambda_2), 1/(\mu_3+\lambda_2)), \quad z^2 = (\mu_3/(\mu_3+\lambda_2), \lambda_2/(\mu_3+\lambda_2)).
\]

The following is the canonical workload matrix in Harrison’s sense given by equation (4.13):

\[
M = \begin{pmatrix} 1/\lambda_1 & 0 \\ \mu_3/(\lambda_1+\lambda_2) & 1/(\mu_3+\lambda_2) \end{pmatrix},
\] (5.42)

with associated matrices,

\[
\Pi = \begin{pmatrix} 1 \\ \mu_3/(\mu_3+\lambda_2) \end{pmatrix}, \quad N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 \\ \mu_3 \end{pmatrix},
\] (5.43)
so that
\[ G = [\Pi \Pi N - MJ] = \begin{pmatrix}
1 & 0 & 1 \\
\mu_3/(\mu_3 + \lambda_2) & \lambda_2/(\mu_3 + \lambda_2) & 0
\end{pmatrix}. \]
(5.44)

We now proceed to compute our choice for the workload matrix and associated control matrix. Each tree is considered as an isolated parallel server system. We solve systems of inequalities associated with the dual program for each tree:
\[ T_1 : \quad y_1 \lambda_1 = 1, \quad y_1 \mu_1 \leq z_1, \quad z_1 = 1; \]
\[ T_2 : \quad y_2 \lambda_2 = 1, \quad y_2 \mu_2 \leq z_2, \quad z_2 = 1. \]

This yields
\[ \hat{y}^1 = (1/\lambda_1, 0), \quad \hat{z}^1 = (1, 0) \]
\[ \hat{y}^2 = (0, 1/\lambda_2), \quad \hat{z}^2 = (0, 1). \]

Using (5.23), (5.25) and Lemma 5.3.3 we can choose the following matrix \( M \) and associated control matrix \( G \):
\[ M = \begin{pmatrix}
1/\lambda_1 & 0 \\
0 & 1/\lambda_2
\end{pmatrix}, \quad G = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -\mu_3/\lambda_2
\end{pmatrix}. \]
(5.45)

The third column of the control matrix \( G \) has a positive first entry and a negative second entry. As discussed in Section 5.4 this is because activity 3 is non-basic, it is processed in \( T_1 \) and it consumes material from buffer 2 in \( T_2 \). Intuitively, we expect that use of activity 3 will reduce the workload of \( T_2 \) and increase the workload of \( T_1 \). In contrast, Harrison’s \( M \) matrix is not diagonal and the third column of his control matrix has strictly positive components. Thus, the second component of Harrison’s workload does not represent the workload of tree \( T_2 \) (i.e., the workload associated with buffer 2), but it is a sum of scaled workloads for the two trees. Accordingly, use of the non-basic activity 3 will cause an increase in both components of Harrison’s workload.
Figure 5.2: Two trees connected via a single non-basic activity
Chapter 6

Solution of the EWF for a Parallel Server System with Special Graph and Cost Structure

In this section, we prove that under certain conditions on the extended server-buffer graph $H$ defined below, and the cost function $g$, a least control is optimal for a REWF, which enables us to solve the EWF and the BCP. Henceforth, we let $M$ and $G$ denote a choice of $M^c$ and $G^c$ for some $c > 0$ as in Lemma 5.3.4; for convenience we suppress the superscript $c$. When we mention the EWF, we mean the EWF corresponding to these matrices $M$ and $G$.

6.1 Special Graph Structure

Definition 6.1.1. (Extended server-buffer graph with some nonbasic activities) Let $H$ be the graph in which servers and buffers form the nodes and undirected edges between nodes are given by basic activities plus the non-basic activities that connect distinct trees in $G$. The set of non-basic activities connecting distinct trees in $G$ is given by $\bigcup_{l \neq m} N^e_{lm}$ and we denote it by $N^{ext}$, where $ext$ is mnemonic for external. We call the activities in $N^{ext}$ external non-basic activities and we let $N^{ext}$ denote the cardinality of $N^{ext}$. An edge in $H$ is either a basic activity or a non-basic activity in $N^{ext}$ that consumes material from a buffer in some tree $T_l$ and that is processed by a server in some other tree $T_m$, $m \neq l$. Note that, $N^{ext}$ includes all non-basic activities except those that are wholly confined to individual trees $T_l$ for $l \in \mathcal{L}$. We now
consider networks which satisfy the following assumption.

**Assumption 6.1.1.**

(i) For each \( l \in \mathcal{L} \), \( N_l^{\text{tr}} \leq N_l^0 + 1 \).

(ii) \( \mathcal{H} \) is connected.

**Remark 6.1.1.** Part (i) of Assumption 6.1.1 means that for each \( l \in \mathcal{L} \) there is at most one external non-basic activity that is processed by a server in \( \mathcal{T}_l \). Part (i) implies that \( N_{\text{ext}} \leq \mathcal{L} \).

On the other hand, part (ii) implies that \( N_{\text{ext}} \geq \mathcal{L} - 1 \). Together parts (i) and (ii) imply that \( N_{\text{ext}} \) equals \( \mathcal{L} - 1 \) or \( \mathcal{L} \). If \( N_{\text{ext}} = \mathcal{L} - 1 \), the graph \( \mathcal{H} \) is a tree, otherwise \( N = \mathcal{L} \) and \( \mathcal{H} \) has a loop (see [5]), for more structure see the proof of Theorem 6.2.1). If (i) holds, but not (ii), then \( \mathcal{H} \) consists of two or more connected components each of which satisfies parts (i) and (ii) of Assumption 6.1.1 and each component can be treated separately by the methods described below.

For a non-basic activity \( j \in \mathcal{N}_{\text{ext}} \) let \( t^c(j), t^p(j) \) be such that \( j \) consumes material from a buffer in \( \mathcal{T}^c(j) \) and is processed by a server in \( \mathcal{T}^p(j) \). For each \( l \in \mathcal{L} \), let \( a^c(l) \) be the subset of \( \mathcal{N}_{\text{ext}} \) consisting of non-basic activities that consume material from buffers in \( \mathcal{T}_l \), and let \( a^p(l) \) be the subset of \( \mathcal{N}_{\text{ext}} \) consisting of non-basic activities that are processed by servers in \( \mathcal{T}_l \). By part (i) of Assumption 6.1.1, for each \( l \in \mathcal{L} \), the set \( a^p(l) \) consists of at most one element, but it may be empty. If \( N_{\text{ext}} = \mathcal{L} \), then \( a^p(l) \) is not empty for all \( l \in \mathcal{L} \). Otherwise, \( N_{\text{ext}} = \mathcal{L} - 1 \) and there is exactly one index \( l = l^p \) for which \( a^p(l) \) is empty. For notational convenience, without loss of generality, we henceforth adopt the following convention about the enumeration of trees.

**Convention 6.1.1.** If \( N_{\text{ext}} = \mathcal{L} - 1 \), then \( l^p = \mathcal{L} \), i.e., there is no external non-basic activity that is processed in \( \mathcal{T}_L \).

Note that Convention 6.1.1 does not violate our enumeration of servers, buffers, basic and non-basic activities specified in Assumption 5.2.1; it can be thought of as a further refinement of that enumeration. When the set \( a^p(l) \) is not empty, we refer to \( a^p(l) \) as the external non-basic activity processed in \( \mathcal{T}_l \). On the other hand, for a fixed \( l \in \mathcal{L} \), the set \( a^c(l) \) may include several non-basic activities or it may be empty. Also, \( a^c(l) \) may be empty for several \( l \in \mathcal{L} \).

### 6.2 Special Control Matrix \( \tilde{G} \)

We will solve the EWF by solving the REWF associated to a reduced control matrix \( \tilde{G} \) chosen as follows. For this, for each \( l \in \mathcal{L} \), we let \( k^*_l \) be a server that can serve buffer \( i^*_l \)
(defined in Section 5.5) via a basic activity. Also, recall the convention about the enumeration of trees that we adopted in Section 6.1 and how components of $\tilde{U}$ are associated with Brownian analogues of server idle-times and allocations to non-basic activities (see Section 5.4).

**Definition 6.2.1.** Let $\tilde{G}$ be the $L \times L$ matrix defined as follows. There are two cases.

**Case I.** $N_{\text{ext}} = L$.

For $l = 1, \ldots, L$, the $l$th column of $\tilde{G}$, $\tilde{G}^l$, is given by the column of $G$ that corresponds to the component of $\tilde{U}$ associated with the non-basic activity $a^p(l) \in N_{\text{ext}}$, i.e., the external non-basic activity that is processed in $T_l$.

**Case II.** $N_{\text{ext}} = L - 1$.

For $l = 1, \ldots, L - 1$, the $l$th column of $\tilde{G}$, $\tilde{G}^l$, is given by the column of $G$ that corresponds to the component of $\tilde{U}$ associated with the non-basic activity $a^p(l) \in N_{\text{ext}}$, i.e., the non-basic activity that is processed in $T_l$, and the $L$th column of $\tilde{G}$ is given by the column of $G$ that corresponds to the component of $\tilde{U}$ associated with the idle-time of server $k^*_L$.

**Remark 6.2.1.** The matrix $\tilde{G}$ is obtained by deleting some columns of $G$ and reordering the remaining columns. If $N_{\text{ext}} = L$, for each $l$, since $\tilde{G}^l$ corresponds to $a^p(l)$, $\tilde{G}^l$ has a positive $l$th entry, and since $a^p(l)$ consumes material from a buffer in $T_{c(a^p(l))}$, $\tilde{G}^l$ has a negative entry in the position with index $t^{c(a^p(l))}$; all other entries of $\tilde{G}^l$ are zero. If $N_{\text{ext}} = L - 1$, the first $L - 1$ columns of $\tilde{G}$ have the form previously described, while the $L$th column has a positive $L$th entry and all other entries are equal to zero. These observation follow from (5.31)-(5.32) and the fact that $z_L$ is associated with the servers in the tree $\mathcal{T}_L$. Thus,

$$\tilde{G} = \tilde{G}^+ + \tilde{G}^-,$$

where $\tilde{G}^+$ is a diagonal matrix with positive diagonal entries and $\tilde{G}^-$ is a matrix whose non-zero entries are off diagonal and non-positive. Moreover, each row and each column of $\tilde{G}$ and $\tilde{G}^-$ has at most one negative entry.

Henceforth we make the following assumption about the cost function $g$ and the control matrix $\tilde{G}$. Recall the definition of $\kappa$ from Theorem 5.5.1. We note that this condition is the same whether one uses $M^c, G^c$ or the original $M, G$, even though the value of $\kappa$ and $\tilde{G}$ will vary with $c$.

**Assumption 6.2.1.** For each $l \in \mathcal{L}$, $\kappa \cdot \tilde{G}^l > 0$.

Later, in Chapter 8, we give an explicit example where Assumption 6.2.1 is satisfied. The proof of the following theorem is postponed until Section 6.4.
Theorem 6.2.1. Let $\tilde{G}$ be as in Definition 6.2.1. Then $C(G) = C(\tilde{G})$.

With the choice of matrix $\hat{G} = \tilde{G}$ for the REWF, in view of Theorem 4.3.2, we can find an optimal control for the EWF from an optimal control for the REWF by setting components of the control for the EWF that correspond to columns of $G$ that are not columns of $\tilde{G}$ to be identically zero. This is more formally described in the following lemma.

Lemma 6.2.1. Suppose $\hat{U}^*$ is an optimal control for the REWF associated with $\tilde{G}$, i.e., $\hat{J}^* = \hat{J}(\hat{U}^*)$. Then, an optimal control $\tilde{U}^*$ for the EWF is given by setting

$$\tilde{U}^*_j = \hat{U}^*_l$$

if $G^j = \tilde{G}^l$ for some $l$, and $\tilde{U}^*_j = 0$ if $G^j$ is not a column of $\tilde{G}$. (6.2)

Proof. By Theorem 4.3.2, and the optimality of $\hat{U}^*$ for the REWF, $\hat{J}^* = \hat{J}(\hat{U}^*)$. It is easy to check that $\tilde{U}$ is an admissible control for the EWF and $G\tilde{U}^* = \tilde{G}\hat{U}^*$. It follows that, $\tilde{J}(\tilde{U}^*) = \hat{J}(\hat{U}^*) = J^*$.

In view of Theorems 4.3.1, 4.3.2 and Lemma 6.2.1, to simplify notation, henceforth we suppress $\tilde{G}$ in $\hat{A}_G$ and $\hat{J}_{\tilde{G}}$, and we denote them simply by $\hat{A}$ and $\hat{J}$ respectively.

6.3 Solution of the REWF

In this section, we first show that the REWF has a least control. Then, we show that the least control is optimal for the REWF and we give equations characterizing the control. We express a solution of the BCP in terms of the optimal solution of the REWF. Let $\Upsilon$ index the possible filtered probability spaces (with associated Brownian motions) for the REWF.

Definition 6.3.1. For $\upsilon \in \Upsilon$, let $\hat{A}_\upsilon$ denote the set of admissible controls for the REWF associated to $\tilde{G}$ defined on a filtered probability space $(\Omega^\upsilon, \mathcal{F}^\upsilon, \{\mathcal{F}_t^\upsilon\}, P^\upsilon)$ with the same associated $\{\mathcal{F}_t^\upsilon\}$-Brownian motion $\hat{\xi}^\upsilon$ and let

$$\hat{J}^*_\upsilon = \inf_{\hat{U} \in \hat{A}_\upsilon} \hat{J}(\hat{U}).$$

(6.3)

Given $\hat{A}_\upsilon$, a control process $\hat{U}^* \in \hat{A}_\upsilon$ is called a least control process in $\hat{A}_\upsilon$ if for each $l \in \{1, \ldots, L\}$, $\hat{U}^*_l \leq \hat{U}_l$ for all $\hat{U} \in \hat{A}_\upsilon$.

Remark 6.3.1. In view of Definition 6.3.1 we have that $\hat{J}^* = \inf_{\upsilon} \hat{J}^*_\upsilon$ and $\hat{A} = \bigcup_{\upsilon \in \Upsilon} \hat{A}_\upsilon$. 

To begin, we fix a filtered probability space, with an associated Brownian motion \( \hat{\xi}_\upsilon \). We construct a least control process \( \hat{U}_\upsilon^* \) and we show that \( \hat{J}_\upsilon^* = \hat{J}(\hat{U}_\upsilon^*) \). Then, we show that if \( \hat{U}_\upsilon^* \) is a least control on another filtered probability space \( (\Omega^0, \mathcal{F}^0, \{\mathcal{F}^0_t\}, P^0) \), with an associated Brownian motion \( \tilde{\xi}_\upsilon \), then \( \hat{J}_\upsilon^* = \tilde{J}_\upsilon^* \). This implies that \( \hat{J}^* = \tilde{J}^* \).

**Definition 6.3.2.** (S-matrix) An \( m \times m \) matrix \( D \) is an S-matrix if there exists an \( x \in \mathbb{R}^m_+ \) for which \( Dx > 0 \).

**Lemma 6.3.1.** \( \tilde{G} \) is an S-matrix with exactly one positive element in each row.

**Proof.** Since \( G \) has form (86) and II is an S-matrix, \( G \) is an S-matrix. Thus, since \( C(G) = C(\tilde{G}) \), \( \tilde{G} \) is an S-matrix. By construction, diagonal elements of \( \tilde{G} \) are strictly positive and off-diagonal elements of \( \tilde{G} \) are non-positive (see Remark 6.2.1).

By Propositions C.0.5, C.0.6 and C.0.7 of Appendix C, there exists a unique least control process in \( \hat{A}_\upsilon \), in particular it is given by

\[
\hat{U}_\upsilon^* = \Phi(\hat{\xi}_\upsilon),
\]

where the continuous mapping \( \Phi : D^m_+ \rightarrow D^m_+ \) is described in Appendix B.

**Theorem 6.3.1.** Let \( \hat{U}_\upsilon^* \) be the least control in \( \hat{A}_\upsilon \). Then,

\[
\hat{J}(\hat{U}_\upsilon^*) = \hat{J}_\upsilon^*.
\]

**Proof.** Let \( \hat{U} \in \hat{A}_\upsilon \) be arbitrary. Let \( \hat{W} \) be the associated workload:

\[
\hat{W}(t) = \hat{\xi}_\upsilon(t) + \tilde{G}\hat{U}(t) \quad \text{for} \quad t \geq 0.
\]

Let \( \hat{W}_\upsilon^* \) be the workload process associated to the least control \( \hat{U}_\upsilon^* \). By the minimality of \( \hat{U}_\upsilon^* \),

\[
\hat{U}_l(t) \geq \hat{U}_\upsilon^*(t), \quad \text{for all} \quad t \geq 0, \quad \text{and} \quad l \in \mathcal{L}.
\]

Then, for \( t \geq 0 \),

\[
\hat{W}(t) = \hat{\xi}_\upsilon(t) + \tilde{G}\hat{U}_\upsilon^*(t) + \tilde{G}(\hat{U}(t) - \hat{U}_\upsilon^*(t)) = \hat{W}_\upsilon^*(t) + \tilde{G}(\hat{U}(t) - \hat{U}_\upsilon^*(t)),
\]

and by Theorem 5.5.1, for \( t \geq 0 \),

\[
g(\hat{W}(t)) = \kappa \cdot \hat{W}(t)
\]

\[
= \kappa \cdot \left( \hat{W}_\upsilon^*(t) + \tilde{G}(\hat{U}(t) - \hat{U}_\upsilon^*(t)) \right)
\]

\[
\geq \kappa \cdot \hat{W}_\upsilon^*(t)
\]

\[
= g(\hat{W}_\upsilon^*(t)),
\]
where we have used Assumption 6.2.1 and (6.6) for the last inequality. It follows that,

\[
\hat{J}(\hat{U}) = E \left[ \int_0^\infty e^{-\gamma t} g(\hat{W}(t)) dt \right] \\
\geq E \left[ \int_0^\infty e^{-\gamma t} g(\hat{W}(t)) dt \right] = \hat{J}(\hat{U}^*_V).
\]

Since \( \hat{U} \in \hat{A}_V \) was arbitrary, \( \hat{J}(\hat{U}^*_V) = \hat{J}^*_V \). The following theorem shows that \( \hat{U}^*_V \) is in fact optimal for the REWF.

**Theorem 6.3.2.** Let \( v \in \Upsilon \) and \( \hat{U}^*_V \in \hat{A}_V \) be the least control. Then,

\[
\hat{J}_V^* = \hat{J}^*.
\]

**Proof.** Let \( \Psi(x) = x + \hat{G}\Phi(x) \) for \( x \in D^{\alpha}_1 \). The least control process \( \hat{U}^*_V = \Phi(\hat{\xi}^V) \) and the corresponding state process \( \hat{W}^*_V = \Psi(\hat{\xi}^V) \) are uniquely determined by the Brownian motion \( \hat{\xi}^V \). If \( \hat{A}_V \) is a set of admissible controls for some different setup, i.e., filtered probability space \( (\Omega^\Psi, \mathcal{F}^\Psi, \{F_t^\Psi\}, \mathbb{P}^\Psi) \) and Brownian motion \( \hat{\xi}^V \), then \( \hat{U}^*_V = \Phi(\hat{\xi}^V) \) and \( \hat{W}^*_V = \Psi(\hat{\xi}^V) \). The Brownian motions \( \hat{\xi}^V \) and \( \hat{\xi}^V \) are identically distributed. Therefore, \( \Phi(\hat{\xi}^V) \overset{d}{=} \Phi(\hat{\xi}^V) \) and \( \Psi(\hat{\xi}^V) \overset{d}{=} \Psi(\hat{\xi}^V) \). It follows that \( \hat{J}_V^* = \hat{J}^*_V \). Since the setup was arbitrarily chosen and \( \hat{J}^* = \inf_v \hat{J}_V^* \), it follows that \( \hat{J}^* = \hat{J}^*_V \).}

In view of Theorem 6.3.2, we fix a setup \( (v \in \Upsilon) \) and we suppress the subscript \( v \) from \( \hat{U}^*_V \) and \( \hat{W}^*_V \), and denote the optimal control process for the REWF associated with \( \hat{G} \) by \( \hat{U}^* \) and the associated state process by \( \hat{W}^* \). We denote the associated Brownian motion by \( \hat{\xi} \).

We proceed to describe \( \hat{U}^* \) and \( \hat{W}^* \) more explicitly below (in the case when \( N^{ext} = L - 1 \) this can be used to explicitly construct \( \hat{U}^* \)). To accomplish this we introduce some new notation. For \( l \in \mathcal{L} \), let \( \beta_l = \hat{G}^1_l \) and for \( j \in a^c(l) \) let \( \alpha^j_l = \hat{G}^j_{l(j)} \). With the convention that the sum over an empty set is zero we have for each \( l \in \mathcal{L} \) and \( t \geq 0 \),

\[
\hat{W}^*_l(t) = \hat{\xi}^l(t) - \sum_{j \in a^c(l)} \alpha^j_l \hat{U}^*_p(j)(t) + \beta_l \hat{U}^*_l(t),
\]

where by the minimality of \( \hat{U}^* \),

\[
\hat{U}^*_l(t) = \left( -\frac{1}{\beta_l} \inf_{0 \leq s \leq t} \left\{ \hat{\xi}^l(s) - \sum_{j \in a^c(l)} \alpha^j_l \hat{U}^*_p(j)(s) \right\} \right) \lor 0.
\]

Now that we have the solution of the REWF (and hence of the EWF by Lemma 6.2.1), we specify the Brownian queue-length and idle-time processes for the BCP that accompany it. Recall the
definition of \(i_t^l\) for \(l \in \mathcal{L}\) from Section 5.5. There are two cases.

**Case I:** Suppose that \(N^\text{ext} = \mathbb{L} - 1\). Then for \(l \in \mathcal{L}\), \(k \in \mathcal{K}\), and \(t \geq 0\),

\[
\hat{Q}_{i_t^l}^*(t) = \frac{W_{i_t^l}^*(t)}{y_{i_t^l}^*} \quad \text{and} \quad \hat{Q}_{i_t^l}^* = 0 \quad \text{for} \quad i \in \mathcal{I} \setminus \{i_t^l\},
\]

\[
\hat{I}_{k_t^l}^*(t) = \hat{U}_{k_t^l}^*(t), \quad \hat{I}_{k_t^l}^* = 0 \quad \text{for} \quad k \neq k_t^l. \tag{6.13}
\]

**Case II:** Suppose that \(N^\text{ext} = \mathbb{L}\). Then for \(l \in \mathcal{L}\), \(k \in \mathcal{K}\) and \(t \geq 0\),

\[
\hat{Q}_{i_t^l}^*(t) = \frac{W_{i_t^l}^*(t)}{y_{i_t^l}^*}, \quad \hat{Q}_{i_t^l}^* = 0 \quad \text{for} \quad i \in \mathcal{I} \setminus \{i_t^l\}, \quad \hat{I}_{k_t^l}^* = 0 \quad \text{for} \quad k \in \mathcal{K}. \tag{6.14}
\]

We note in particular that in **Case II**, no Brownian server idle-time is accumulated. Also, note that (6.13) and (6.14) are correct if one uses the original \(M\) and \(G\) and if one uses \(M^c\) and \(G^c\), one has to multiply the \(y\)'s in the formulas by the appropriate components of \(c\).

### 6.4 Proof of Theorem 6.2.1

**Definition 6.4.1.** Let \(l \in \mathcal{L}\) be arbitrary and let \(\pi_1 = l\). We inductively define \(\pi_2, \ldots, \pi_f\), where \(f \leq \mathbb{L}\), as follows. If \(N^\text{ext} = \mathbb{L}\) then \(\pi_{j+1} = t^c(a^p(\pi_j))\) for \(j = 1, \ldots, f - 1\) and \(f\) is the first index such that \(t^c(a^p(\pi_f)) = \pi_j\) for some \(j \leq f - 1\). If \(N^\text{ext} = \mathbb{L} - 1\), then assuming that \(\pi_1, \ldots, \pi_j\) have been defined, if \(a^p(\pi_j)\) is defined, let \(\pi_{j+1} = t^c(a^p(\pi_j))\), or if \(a^p(\pi_j)\) is not defined, set \(f = j\) and the induction procedure stops.

**Remark 6.4.1.** If \(N^\text{ext} = \mathbb{L}\), then \(a^p(\pi_j)\) exists for all \(j \leq f\) and since \(\mathcal{H}\) contains a loop there exists a \(j \leq f - 1\) such that \(t^c(a^p(\pi_f)) = \pi_j\). For example, if \(t^c(a^p(\pi_f)) = \pi_1\) and \(f = \mathbb{L}\), then \(\mathcal{H}\) is a “circle of trees.” If \(N^\text{ext} = \mathbb{L} - 1\), then by Convention 6.1.1, \(\pi_f = \mathbb{L}\). The sequence \(\pi_1, \ldots, \pi_f\) consists of distinct entries and depends on the choice of \(l \in \mathcal{L}\) and the original enumeration of trees.

**Proof of Theorem 6.2.1.** We need to show that each column of \(G\) is in \(\mathcal{C}(\hat{G})\). The matrix \(\hat{G}\) includes all of the columns of \(G\) which correspond to non-basic activities in \(N^{\text{ext}}\). Each column of \(G\) that does not correspond to a non-basic activity in \(N^{\text{ext}}\) is a positive multiple of \(e_l\), for some \(l\), where \(e_l\) is the \(l\)th vector in the standard basis for \(R_+^d\). Therefore, we need to show that for each \(l\), \(e_l \in \mathcal{C}(\hat{G})\). It is enough to show that any positive multiple of \(e_l\) is in \(\mathcal{C}(\hat{G})\). There are two cases to consider.

**Case I:** \(N^{\text{ext}} = \mathbb{L} - 1\). Let \(l \in \mathcal{L}\) be arbitrary. Let \(\pi_1, \ldots, \pi_f\) be as in Definition 6.4.1. For each \(j = 1, \ldots, f - 1\), \(\hat{G}^\pi_j\) corresponds to the external non-basic activity \(a^p(\pi_j)\) which is processed...
in $T_{\pi_j}$ and consumes from $T_{\pi_{j+1}}$; accordingly $\tilde{G}_{\pi_j} > 0$, $\tilde{G}_{\pi_{j+1}} < 0$ and all other entries of $\tilde{G}_{\pi_j}$ are zero (see Remark 6.2.1). Furthermore, since $\pi_f = \mathbb{L}$, $\tilde{G}_{\pi_f} > 0$ and all other entries of $\tilde{G}_{\pi_f}$ are zero. It follows by successive cancelation that there are positive constants $d_1, \ldots, d_f$ such that

$$d_1\tilde{G}_{\pi_1} + d_2\tilde{G}_{\pi_2} + \cdots + d_f\tilde{G}_{\pi_f},$$

has a positive $l^{th}$ entry and all other entries equal to zero. 

**Case II:** $N = L$. Let $l \in \mathcal{L}$ be arbitrary and let $\pi_1, \ldots, \pi_f$ be as in Definition 6.4.1. As in Case I, for $j = 1, \ldots, f$, $\tilde{G}_{\pi_j} > 0$, $\tilde{G}_{\pi_{j+1}} < 0$, and all other entries of $\tilde{G}_{\pi_j}$ are zero. Let $1 \leq f' \leq f - 1$ be such that $\pi_{f'} = \pi'_f = \pi_{f'}$. It follows that there are positive constants such that the entry of

$$v = d_{f'}\tilde{G}_{\pi_{f'}} + d_{f'+1}\tilde{G}_{\pi_{f'+1}} + \cdots + d_f\tilde{G}_{\pi_f}$$

with index $\pi_{f'}$ is $d_{f'}\tilde{G}_{\pi_{f'}} + d_f\tilde{G}_{\pi_f}$ and all other entries are zero. Thus,

$$\sum_{j=f'}^f \kappa \cdot \tilde{G}_{\pi_j} d_j = \kappa_{\pi_{f'}} \left( d_{f'}\tilde{G}_{\pi_{f'}} + d_f\tilde{G}_{\pi_f} \right),$$

where the left side is strictly positive by Assumption 6.2.1 and the strict positivity of the $d_j$'s. Since $\kappa_{\pi_{f'}} > 0$, it follows that so too is $d_{f'}\tilde{G}_{\pi_{f'}} + d_f\tilde{G}_{\pi_f}$ and the vector $v$ is a positive multiple of $e_{\pi_{f'}}$. As in Case I it follows by successive cancelation that there are positive constants $c_1, \ldots, c_{f'}$ such that

$$c_1\tilde{G}_{\pi_1} + c_2\tilde{G}_{\pi_2} + \cdots + c_{f'}v,$$

has a positive $l^{th}$ entry and all other entries equal to zero. Since $l$ was arbitrary, this completes the proof. 

\(\square\)
Chapter 7

Proposed Interpretation of Optimal Solution of EWF

In this chapter we describe a proposed interpretation of the solution obtained in the last chapter for the EWF and the BCP. As in Chapter 3, here we consider a sequence of parallel server systems indexed by \( r \in [1, \infty) \), where in particular as \( r \to \infty \) the first order parameters in the \( r^{th} \) network satisfy the heavy traffic Assumption 3.1.1; we also assume that Assumptions 6.1.1 and 6.2.1 are satisfied. Here \( M \) and \( G \) are fixed as in Chapter 6. As shown in Chapter 6, under these assumptions, one can solve the EWF exactly and accordingly the BCP. Nevertheless, interpreting this solution for the original parallel server system is a challenging problem. In Section 7.1 we outline a proposed interpretation and in Section 7.2 we expand on some details of this.

7.1 Overall Description of the Control Policy

We introduce a notion of workload for the \( r^{th} \) parallel server system: \( W^r = MQ^r \). The solution of the EWF and the associated BCP suggest that the bulk of the workload in each tree \( T_l \) should be kept in the cheapest buffer \( i_l^* \). Each tree \( T_l \) can be viewed as a parallel server system with one-dimensional workload. When the workload of each tree in the \( r^{th} \) network is significantly greater than zero, each tree should be controlled using the threshold policy of Williams [41], which achieves a pooling of servers in each tree to minimize the workload in each tree and pushes the bulk of the work in the tree into the cheapest buffer. When the workload for a tree is small there are two cases to consider. For \( N_{ext} = L \), when the workload \( W_l^r \) of \( T_l \) in the \( r^{th} \) network, is close to zero it should be kept non-negative by switching the server \( s_l^* \) that serves
$a^p(l)$ to process that activity. For $\mathbb{N}^{ext} = \mathbb{L} - 1$, for $l = 1, \ldots, \mathbb{L} - 1$, when the workload $W_{r}^l$ of $\mathcal{T}_r$ in the $r^{th}$ network is close to zero, it should be kept non-negative in a similar manner to when $\mathbb{N}^{ext} = \mathbb{L}$, but when the workload $W_r^r$ of $\mathcal{T}_r$ in the $r^{th}$ network is close to zero, it should be kept non-negative by allowing the server $k_r^*$ to idle. We elaborate on this control policy, which is a dynamic threshold type policy, in greater detail below.

### 7.2 Threshold Policy

We first suppose that $\mathbb{N}^{ext} = \mathbb{L}$. For each $l \in \{1, \ldots, \mathbb{L}\}$, let $s_r^l$ be the server in $\mathcal{T}_l$ which performs the non-basic activity $a^p(l)$ and let $b_r^l$ be the buffer in $\mathcal{T}_l$ such that $a^p(l)$ consumes material from $b_r^l$. Let $b^1(a^c(l))$ be the set of buffers in $\mathcal{T}_l$ such that the non-basic activities in $a^c(l)$ consume material from the buffers in $b^1(a^c(l))$. Each buffer $i$ has a threshold. Let $L_i^r \geq 0$ denote the size of the threshold placed on buffer $i$ in the $r^{th}$ network. For each $l$ define,

$$D_l^r = \{q \in \mathbb{R}^i_+ : q_i \leq 2L_i^r \text{ for } i \in b^1(a^c(l)), q_i \leq L_i^r \text{ for } i \in \mathcal{I}_l \setminus b^1(a^c(l))\}. \quad (7.1)$$

The dynamic control policy is described as follows with several cases depending on the position of the queue-length process.

**Case I: $Q^r \in (\cup_{l} D_l^r)^c$.**

On the set $Q^r \in (\cup_{l} D_l^r)^c$, for each $l$, the workload $W_{r}^l \geq \min(y_i^lL_i^r : i \in \mathcal{I}_l)$ and considering $\mathcal{T}_l$ in isolation one uses the control policy there that only involves the use of basic activities. This policy is outlined in Williams [41] and elaborated in [4]. A key to the description of this policy is a hierarchical structure of the server-buffer tree $\mathcal{T}_l$ and an associated protocol for the dynamic allocation of class priorities at each server. This protocol is described in an iterative manner, working from the bottom of the tree up towards the root. It is helpful to consider a server tree $\mathcal{S}_l$ that is obtained by suppressing buffers in $\mathcal{T}_l$. The root of $\mathcal{T}_l$ is the server $k_r^*$ that serves buffer $i_r^*$ via a basic activity. Classes (or buffers) that link one level of servers to those at the next highest level are called transition buffers. Classes that are served by a single server are called terminal classes or terminal buffers. Note that since $\mathcal{T}_l$ is a tree, with the exception of $k_r^*$, there is exactly one transition class immediately above each server. However, unless a given server is at the lowest level, there may be several transition classes immediately below this server. The protocol that we will describe immediately below applies except when buffers in $b^1(a^c(l))$ are being served. These need special treatment since they are capable of being served by servers outside of $\mathcal{T}_l$.

Consider a server at the lowest level. This server gives the lowest priority to the class that is
immediately above it in $T_l$ (there will always be such a class unless the server is the root of the tree). This class is also served by a server in the next level up in $S_l$ and is therefore a transition class. The highest priority is given to terminal classes. The priority ranking for these classes is not so important. For concreteness we rank them in such a way that the lower numbered ones have higher priority over the higher numbered ones, see [4] for details. Now we look at a server at the next level up. This server may serve several transition classes immediately below it and unless it is the root of the tree it also serves a transition class immediately above it. Also, it may serve several terminal classes. The highest priority is given to transition buffers immediately below the server. If there are several such buffers, they are ranked in such a way that the lower numbered ones have higher priority over the higher numbered ones. However, if the number of jobs for a transition buffer associated with such an activity is at or below the threshold for this buffer, service of that activity is suspended. The next priority is given to terminal classes. Again the lower numbered ones receive higher priority over the higher numbered ones. The lowest priority is given to the transition class immediately above the server in $T_l$. This procedure is repeated until the root of the tree is reached. The server at the root behaves as do other servers except that it gives the lowest priority to the cheapest buffer $i^*_l$. If two or more servers simultaneously attempt to serve a particular transition buffer, a tie breaking rule is used to determine which server takes a job first, see [4].

An exception to the above protocol is that when the number of jobs in a buffer $i \in b^l(a^r(l))$ is at or below the level $2L^r_l$, any service of this buffer by any server in $T_l$ is suspended until the queue-length of this buffer is strictly greater than $2L^r_l$.

**Case II:** $Q^r_l \in D^{l,r}$ for some $l$ and $Q^r \notin D^{l',r}$ for all $l' \neq l$.

As soon as $Q^r_l$ hits the outer boundary of $D^{l,r}$, server $s^*_l$ starts performing activity $a^p(l)$ and it continues to do so as long as $Q^r \notin D^{l',r}$ for $l' \neq l$, until $Q^r \notin D^{l,r}$. If the buffer $b^*(l)$ becomes empty due to use of activity $a^p(l)$ before $Q^r \notin D^{l,r}$, server $s^*_l$ idles until the first new arrival to buffer $b^*(l)$. All other servers except server $k^*_l$ operate under the regime described in Case I. Server $k^*_l$ operates according to Case I, unless the number of jobs in the cheapest buffer $i^*_l$ and in terminal buffers for server $k^*_l$ equals zero, in which case server $k^*_l$ starts serving its transition buffers. Every other tree $T_{l'}$ operates according to Case I.

**Case III:** $Q^r \in D^{l,r}$ for more than one $l$.

If $Q^r \notin D^{l',r}$, then each server in $T_{l'}$ operates as in Case I. If $Q^r \in D^{l,r}$, each server in $T_l$ operates according to Case II.

The policy when $N_{ext} = L - 1$ is the same as that described above with one exception.
In particular, Case II is modified as follows. When $Q^r \in D^{I-r}$, server $k^*_1$ idles until $Q^r \in (D^{I-r})^c$.

As in [4], we anticipate that for asymptotic optimality of this control policy, the thresholds $L^r_i$, $i \in I$, should be such that as $r \to \infty$, $\max_{i \in I} L^r_i / r \to 0$ and $\min_{i \in I} L^r_i \to \infty$ and the thresholds should be suitably chosen to be of the order of $\log r$ as $r \to \infty$.

**Remark 7.2.1.** Recall that the non-basic activity $a^p(l)$ consumes material from $b^*_1$. When $Q^r$ reaches the outer boundary of $D^{I-r}$ one has to ensure that with increasingly high probability as $r \to \infty$, $Q^r$ moves back to $(D^{I-r})^c$ before buffer $b^*_1$ becomes empty. This is the reason why the number of jobs in buffer $b^*_1$ is not allowed to dip below the threshold $L^r_{b^*_1}$ unless $a^p(l)$ is being utilized and the threshold $L^r_{b^*_1}$ needs to be large enough for this. The policy for Case III is not so important as the probability that the system is in Case III should tend to zero as $r \to \infty$.

We conjecture that the above policy is asymptotically optimal. As a first step, in the next chapter we prove this for a specific example. We believe this is the first example of a proof of asymptotic optimality for a policy derived from solving the BCP/EWF with non-basic activities when the workload is more than one-dimensional.
Chapter 8

Illustrative Example

8.1 Description and First Order Data

We consider a sequence of parallel server systems indexed by \( r \in [1, \infty) \) where each system has 3 buffers, 3 servers and 5 activities. The first order data for the \( r^{th} \) member of the sequence has

\[
C = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad A = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
\lambda^r = (\lambda_1^r, \lambda_2^r, \lambda_3^r), \quad \mu^r = (\mu_1^r, \mu_2^r, \mu_3^r, \mu_4^r, \mu_5^r),
\]

such that the following holds.

**Assumption 8.1.1.** There are constant vectors \( \lambda = (\lambda_1, \lambda_2, \lambda_3), \mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5), (\theta_1, \theta_2, \theta_3) \) such that as \( r \to \infty, \)

(i) \( \lambda_i^r \to \lambda_i > 0 \) for \( i = 1, 2, 3, \)

(ii) \( \mu_j^r \to \mu_j > 0 \) for \( j = 1, \ldots, 5, \)

where \( \lambda_1 > \mu_1, \lambda_2 < \mu_3, \lambda_3 = \mu_4, \frac{\lambda_1 - \mu_1}{\mu_2} = 1 - \frac{\lambda_2}{\mu_3}, \)

(iii) \( r \mu_2^r \left( \frac{\lambda_1^r - \mu_1^r}{\mu_2^r} - \frac{\lambda_1 - \mu_1}{\mu_2} \right) \to \theta_1, \)
(iv) \( r\mu_3 \left( \frac{\lambda_2}{\mu_3} - \frac{\lambda_2}{\mu_3} \right) \to \theta_2. \)

(v) \( r(\lambda_2 - \mu_4) \to \theta_3. \)

Assumptions 3.2.1, 3.2.2 are satisfied with

\[ x^* = \left( 1, \frac{\lambda_1 - \mu_1}{\mu_2}, \frac{\lambda_2}{\mu_3}, 1, 0 \right). \]  

8.2 Brownian Control Problem

The Brownian control problem associated with the above data is as follows.

**Definition 8.2.1. (Brownian Control Problem).**

\[
\text{minimize} \quad E \left( \int_0^\infty e^{-\gamma t} h \cdot \tilde{Q}(t) dt \right) \tag{8.3}
\]

using a 5-dimensional adapted control process \( \tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4, \tilde{Y}_5) \) defined on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) that supports 3-dimensional adapted processes \( \tilde{Q} \) and \( \tilde{X} \) such
that a.s. for all $t \geq 0$:

\[
\tilde{Q}_1(t) = \tilde{X}_1(t) + \mu_1 \tilde{Y}_1(t) + \mu_2 \tilde{Y}_2(t) \geq 0, \quad (8.4)
\]
\[
\tilde{Q}_2(t) = \tilde{X}_2(t) + \mu_3 \tilde{Y}_3(t) \geq 0, \quad (8.5)
\]
\[
\tilde{Q}_3(t) = \tilde{X}_3(t) + \mu_4 \tilde{Y}_4(t) + \mu_5 \tilde{Y}_5(t) \geq 0, \quad (8.6)
\]
\[
\tilde{I}_1(\cdot) = \tilde{Y}_1(\cdot) + \tilde{Y}_5(\cdot) \text{ is nondecreasing}, \quad \tilde{I}_1(0) \geq 0, \quad (8.7)
\]
\[
\tilde{I}_2(\cdot) = \tilde{Y}_2(\cdot) + \tilde{Y}_3(\cdot) \text{ is nondecreasing}, \quad \tilde{I}_2(0) \geq 0, \quad (8.8)
\]
\[
\tilde{I}_3(\cdot) = \tilde{Y}_4(\cdot) \text{ is nondecreasing}, \quad \tilde{I}_3(0) \geq 0, \quad (8.9)
\]
\[
\tilde{Y}_5(\cdot) \text{ is non-increasing}, \quad \tilde{Y}_5(0) \leq 0, \quad (8.10)
\]

where $\tilde{X}$ is a 3-dimensional $\{F_t\}$-Brownian motion starting at the origin with drift
\[
\theta = (\theta_1, \theta_2, \theta_3)
\]
and with diagonal covariance matrix $\Sigma$ whose $i$th diagonal entry is equal to
\[
\lambda_i a_i^2 + \sum_{j=1}^{5} C_{ij} \mu_j b_i^2 x_j \text{ for } i = 1, 2, 3.
\]

8.3 Equivalent Workload Formulation

According to Theorem 5.1.2, since the server-buffer graph $G$ consists of two trees $T_1$ and $T_2$, the workload is two-dimensional. There are four basic activities and a single non-basic activity. Since $G$ consists of two trees and since there is only one non-basic activity that connects these two trees ($N^{ext} \neq \emptyset$), the extended server-buffer graph $H$ (see Figure 8.1) satisfies Assumption 6.1.1. Following the steps outlined in Sections 4 and 5, we proceed to compute a workload matrix $M$ and control matrix $G$ that goes with it. We solve the dual programs for $T_1$ and $T_2$. The unique solution of the dual program for $T_1$ is given by

\[
\begin{pmatrix}
y_1^1 \\
y_2^1
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{\mu_1 + \mu_2} \\
\frac{\mu_2}{\mu_3 (\mu_1 + \mu_2)}
\end{pmatrix}, \quad \begin{pmatrix}
z_1^1 \\
z_2^1
\end{pmatrix} = 
\begin{pmatrix}
\frac{\mu_1}{\mu_1 + \mu_2} \\
\frac{\mu_2}{\mu_1 + \mu_2}
\end{pmatrix}, \quad (8.11)
\]

and the unique solution of the dual program for $T_2$ is given by

\[
y_1^2 = \frac{1}{\mu_4}, \quad z_1^2 = 1. \quad (8.12)
\]

By Lemma 5.3.4 we can multiply $(y^1, z^1)$ by $\mu_1 + \mu_2$ to obtain a workload matrix

\[
M = 
\begin{pmatrix}
1 & \frac{\mu_2}{\mu_3} & 0 \\
0 & 0 & 1/\mu_4
\end{pmatrix}, \quad (8.13)
\]

and the corresponding control matrix

\[
G = 
\begin{pmatrix}
\mu_1 & \mu_2 & 0 & \mu_1 \\
0 & 0 & 1 & -\mu_5/\mu_4
\end{pmatrix}, \quad (8.14)
\]
and \((\kappa_1, \kappa_2)\) is such that \(\kappa_1 = \min(h_1, h_2\mu_3/\mu_2), \kappa_2 = h_3\mu_4\). By Definition 6.2.1 the control matrix \(\tilde{G}\) that goes with the REWF (see Definitions 4.3.1-4.3.2) is

\[
\tilde{G} = \begin{pmatrix}
\mu_1 & 0 \\
-\mu_5/\mu_4 & 1
\end{pmatrix}.
\]

(8.15)

Given \(\tilde{X}\) as in Definition 8.1, let \(\hat{\xi} = M\tilde{X}\). Then, \(\hat{\xi}\) is a 2-dimensional Brownian motion with drift \(M\theta\) and covariance matrix \(M\Sigma M'\). In terms of \(\hat{\xi}\) the workload \(\hat{W}\) in the REWF is given by

\[
\hat{W}_1(t) = \hat{\xi}_1(t) + \mu_1 \hat{U}_1(t), \quad \hat{W}_2(t) = \hat{\xi}_2(t) + \hat{U}_2(t) - (\mu_5/\mu_4) \hat{U}_1(t).
\]

(8.16)

The correspondence with controls in the BCP is,

\[
\hat{U}_1 = -\hat{Y}_3, \quad \hat{U}_2 = \tilde{I}_3.
\]

Henceforth, we assume that the parameters satisfy the following.

**Assumption 8.3.1.**

(i) \(h_1\mu_2 > h_2\mu_3\),

(ii) \(h_2\mu_1\mu_3 > h_3\mu_2\mu_5\).

The first assumption \((i)\) assumes that buffer 2 is the “cheapest” in the tree \(T_1\). Part \((ii)\) of Assumption 8.3.1 corresponds to Assumption 6.2.1.

The assumptions of Section 6.2 are satisfied and therefore by Section 6.3, the solution of the REWF and BCP are as follows:

\[
\hat{W}_1^*(t) = \hat{\xi}_1(t) + \mu_1 \hat{U}_1^*(t), \quad \hat{U}_1^*(t) = \left(-\inf_{0\leq s\leq t} \{\hat{\xi}_1(s)\}\right) \lor 0,
\]

(8.17)

\[
\hat{W}_2^*(t) = \hat{\xi}_2(t) + \hat{U}_2^*(t) - (\mu_5/\mu_4) \hat{U}_1^*(t),
\]

(8.18)

\[
\hat{U}_2^*(t) = \left(-\inf_{0\leq s\leq t} \{\hat{\xi}_2(s) - (\mu_5/\mu_4) \hat{U}_1^*(s)\}\right) \lor 0,
\]

(8.19)

\[
\hat{Q}_1^*(t) = 0, \quad \hat{Q}_2^*(t) = (\mu_3/\mu_2) \hat{W}_1^*(t), \quad \hat{Q}_3^*(t) = \mu_4 \hat{W}_2^*(t),
\]

(8.20)

\[
\hat{I}_2^*(t) = 0, \quad \hat{I}_1^*(t) = \hat{Y}_1^*(t) + \hat{Y}_5^*(t) = 0, \quad \hat{Y}_5^*(t) = -\hat{Y}_1^*(t) = -\hat{U}_1^*(t),
\]

(8.21)

\[
\hat{Y}_4^*(t) = \hat{I}_3^*(t) = \hat{U}_2^*(t), \quad \hat{Y}_2^*(t) = -\mu_2^{-1}(\hat{X}_1(t) + \mu_1 \hat{Y}_1^*(t)),
\]

(8.22)

\[
\hat{Y}_3^*(t) = \mu_3^{-1}(\hat{Q}_3^*(t) - \hat{X}_2(t)),
\]

(8.23)

and the associated minimum cost is

\[
\hat{J}^* = \mathbb{E}\left(\int_0^\infty e^{-\gamma t} \left((h_2\mu_3/\mu_2)\hat{W}_1^* + h_3\mu_4\hat{W}_2^*\right) dt\right).
\]

(8.24)
In this instance, the optimal control  $\tilde{Y}^*$ for the BCP, is uniquely determined by the optimal control $\hat{U}^*$ for the REWF. In general, this need not be the case (see [22]). When $\hat{W}^*_1$ hits zero,  $\hat{U}^*_1$ increases by a minimal amount in order to keep $\hat{W}^*_1$ non-negative. When $\hat{W}^*_2$ hits zero, $\hat{U}^*_2$ increases by a minimal amount in order to keep $\hat{W}^*_2$ non-negative (see Figure 8.3 for a depiction of the optimal control directions).

### 8.4 Threshold Policy

The policy which follows is a special instance of the policy proposed in Chapter 7. It is described using three regions in the two-dimensional quadrant that are induced by thresholds on buffers 1 and 2.

**Definition 8.4.1.** (Threshold policy) For each $r$, let

$$L_r^1 = \lfloor c \log r \rfloor + 1, \quad L_r^2 = \lfloor (8\mu_3 L_r^1) / \lambda_1 \rfloor + 1 \quad \text{and} \quad L_r^3 = 0$$

where $c$ is chosen such that certain large deviation estimates in Theorems 8.6.1 and 8.6.1 are satisfied. In the $r^{th}$ system, the dynamic threshold policy is as follows:

(i) Server 3 operates whenever possible. In other words server 3 is never idle when there are jobs in buffer 3 or at server 3.

(ii) When the number of class 1 jobs is above the threshold $L_r^1$, server 1 and server 2 both process class 1 jobs. In particular, as soon as the number of class 1 jobs reaches level $L_r^1 + 1$ from below, server 2 suspends any work on class 2 jobs and shifts service to class 1 jobs. As soon as the number of class 1 jobs reaches level $L_r^1$ from above, server 2 suspends any work on class 1 jobs and shifts service to class 2 jobs provided that buffer 2 is non-empty; if there are no class 2 jobs to be served, server 2 continues serving class 1 jobs until the first new arrival of a class 2 job.

(iii) When the number of class 1 jobs is at or below the threshold $L_r^1$ and the number of class 2 jobs is above the threshold $L_r^2$, server 1 works on class 1 jobs and server 2 works on class 2 jobs.

(iv) When the number of class 1 jobs is at or below the threshold $L_r^1$ and the number of class 2 jobs is at or below the threshold $L_r^2$, server 1 works on class 3 jobs provided that buffer 3 is not empty and server 2 works on class 2 jobs provided that buffer 2 is not empty. In this regime, if buffer 3 becomes empty, server 1 idles and, if buffer 2 becomes empty, server 2 works on class 1 jobs.
Figure 8.2: Optimal control: $\hat{U}_1^*$ is the amount of pushing done in the direction $\tilde{G}^1$, and $\hat{U}_2^*$ is the amount of pushing done in the direction $\tilde{G}^2$. 
The policy described above is only one asymptotically optimal policy. It is not necessarily unique, there may be many other policies that are asymptotically optimal.

8.5 Analytic Preliminaries

It is assumed that the $r^{\text{th}}$ network operates under the previously described threshold policy. Let

$$R^r_1(t) = Q^r_1(t) - L^r_1, \quad t \geq 0.$$  \hfill (8.25)

Let $\sigma^r_0 = \inf \{ t \geq 0 : R^r_1(t) \geq 0 \}$. We define recursively for $n \geq 1$, $\sigma^r_{2n-1} = \inf \{ t \geq \sigma^r_{2n-2} : R^r_1(t) \leq -1 \}$ and $\sigma^r_{2n} = \inf \{ t \geq \sigma^r_{2n-1} : R^r_1(t) \geq 0 \}$. We call $[\sigma^r_{2n-1}, \sigma^r_{2n})$ the $n^{\text{th}}$ “down” excursion interval for $R^r_1$.

Let $\mathbb{N}_\infty = \{0, 1, 2, \ldots \} \cup \{\infty\}$. Consider $\mathbb{N}_\infty^5$ to be a partially ordered by componentwise inequality, that is, $p \leq q$ in $\mathbb{N}_\infty^5$ if and only if $p_i \leq q_i$ for $i = 1, \ldots, 5$. In the $i^{\text{th}}$ system, the cumulative interarrival time processes for classes 1 and 2 are denoted by $\zeta^r_1$ and $\zeta^r_2$, and the
The cumulative service time process for activities 1, 2 and 3 are denoted by \( \eta_1, \eta_2, \) and \( \eta_3 \). For each \( p \in \mathbb{N}_\infty^5 \), let
\[
\mathcal{F}_p^\tau = \sigma \{ \zeta_1^r(\cdot \cap (p_1 + 1)), \zeta_2^r(\cdot \cap (p_2 + 1)), \eta_1^r(\cdot \cap (p_3 + 1)) , \eta_2^r(\cdot \cap (p_4 + 1)), \eta_3^r(\cdot \cap (p_5 + 1)) \} \sim,
\]
where \( \sim \) denotes the augmentation by the \( \mathbb{P} \)-null sets in the complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

Then, \( \{ \mathcal{F}_p^\tau : p \in \mathbb{N}_\infty^5 \} \) is a multiparameter filtration. The following definition and the proof of the next two propositions are analogous to those in [3], and so we omit the proofs of the propositions.

**Definition 8.5.1.** A multiparameter stopping time relative to \( \{ \mathcal{F}_p^\tau : p \in \mathbb{N}_\infty^5 \} \) is a random variable \( \tau \) taking values in \( \mathbb{N}_\infty^5 \) such that
\[
\{ \tau = p \} \in \mathcal{F}_p^\tau \quad \text{for all} \quad p \in \mathbb{N}_\infty^5.
\]
The \( \sigma \)-algebra associated with such a stopping time \( \tau \) is
\[
\mathcal{F}_\tau = \{ B \in \mathcal{F} : B \cap \{ \tau = p \} \in \mathcal{F}_p^\tau \quad \text{for all} \quad p \in \mathbb{N}_\infty^5 \}.
\]

**Lemma 8.5.1.** For each \( r \geq 1, \) for \( i = 1, 2 \) and each \( n \geq 1 \),
\[
\tau_n^r = (A_1^r(\sigma_{2n-1}^r), A_2^r(\sigma_{2n-1}^r), S_1^r(T_1^r(\sigma_{2n-1}^r)), S_2^r(T_2^r(\sigma_{2n-1}^r)), S_3^r(T_3^r(\sigma_{2n-1}^r)))
\]
is a (multiparameter) stopping time relative to the filtration \( \{ \mathcal{F}_p^\tau : p \in \mathbb{N}_\infty^5 \} \), where we adopt the convention that each of \( A_i^r(\cdot) \) for \( i = 1, 2 \) and \( S_j^r(T_j^r(\cdot)) \) for \( j = 1, 2, 3 \) takes the value \( \infty \) when evaluated at time \( \infty \).

**Lemma 8.5.2.** Let \( \tau = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \) be a multiparameter stopping time relative to the filtration \( \{ \mathcal{F}_p^\tau : p \in \mathbb{N}_\infty^5 \} \). In the following, for notational convenience, we make the convention that each of \( u_i^r(\cdot) \) for \( i = 1, 2 \) and \( v_j^r(\cdot) \) for \( j = 1, 2, 3 \) takes the value \( \infty \) when its argument takes the value \( \infty \). Then,
\[
(u_1^r(\tau_1 + 1), u_2^r(\tau_2 + 1), v_1^r(\tau_3 + 1), v_2^r(\tau_4 + 1), v_3^r(\tau_5 + 1)) \in \mathcal{F}_\tau^\tau.
\]
and on \( \{ \tau \in \mathbb{N}_\infty^5 \} \), the conditional distribution of
\[
\{(u_1^r(\tau_1 + n), u_2^r(\tau_2 + n), v_1^r(\tau_3 + n), v_2^r(\tau_4 + n), v_3^r(\tau_5 + n)), n = 2, 3, \ldots\}
\]
given \( \mathcal{F}_\tau^\tau \) is the same as the (unconditioned) distribution of the original family of i.i.d. random variables \( \{(u_1^r(n), u_2^r(n), v_1^r(n), v_2^r(n), v_3^r(n)), n = 1, 2, \ldots\} \).
8.6 State Space Collapse

Before we state the main result of this section we will need the diffusion scaled equations for the \( r^{th} \) network. Let \( \tilde{A}^r, \tilde{S}^r, \tilde{Y}^r, \tilde{T}^r \) be given by (3.16), (3.17) and (3.18). From (3.19)-(3.21), for each \( t \geq 0 \), components of the diffusion scaled queue-length and idle-time processes in the \( r^{th} \) network, \( \hat{Q}^r \) and \( \hat{I}^r \), are as follows:

\[
\begin{align*}
\hat{Q}_1^r(t) &= \hat{X}_1(t) + \mu_1 \hat{Y}_1(t) + \mu_2 \hat{Y}_2(t), \\
\hat{Q}_2^r(t) &= \hat{X}_2(t) + \mu_3 \hat{Y}_3(t), \\
\hat{Q}_3^r(t) &= \hat{X}_3(t) + \mu_4 \hat{Y}_4(t) + \mu_5 \hat{Y}_5(t), \\
\hat{I}_1^r(t) &= \hat{Y}_1(t) + \hat{Y}_3(t), \\
\hat{I}_2^r(t) &= \hat{Y}_2(t) + \hat{Y}_5(t), \\
\hat{I}_3^r(t) &= \hat{Y}_4(t),
\end{align*}
\]

where

\[
\begin{align*}
\hat{X}_1^r(t) &= \tilde{A}_1^r(t) - \tilde{S}_1^r(T_1^r(t)) - \tilde{S}_2^r(T_2^r(t)) + r \mu_2 \left( \frac{\lambda_1 - \mu_1}{\mu_2} - \frac{\lambda_1 - \mu_1}{\mu_2} \right) t, \\
\hat{X}_2^r(t) &= \tilde{A}_2^r(t) - \tilde{S}_3^r(T_3^r(t)) + r \mu_3 \left( \frac{\lambda_2}{\mu_3} - \frac{\lambda_2}{\mu_3} \right) t, \\
\hat{X}_3^r(t) &= \tilde{A}_3^r(t) - \tilde{S}_4^r(T_4^r(t)) - \tilde{S}_5^r(T_5^r(t)) + r (\lambda_5 - \mu_4) t.
\end{align*}
\]

Let

\[
M^r = \begin{pmatrix} 1 & \mu_2^r/\mu_3^r & 0 \\ 0 & 0 & 1/\mu_4^r \end{pmatrix}, \quad W^r = M^r Q^r,
\]

and

\[
\hat{W}^r = M^r \hat{Q}^r,
\]

the diffusion scaled nominal workload for the \( r^{th} \) network. For \( \hat{X}^r = M^r \hat{X}^r \), we have for each \( t \geq 0 \),

\[
\begin{align*}
\hat{W}_1^r(t) &= \hat{X}_1^r(t) + \mu_2 \hat{I}_2^r(t) + \mu_4 \hat{Y}_4(t), \\
\hat{W}_2^r(t) &= \hat{X}_2^r(t) + \hat{I}_3^r(t) + (\mu_2^r/\mu_3^r) \hat{Y}_5^r(t).
\end{align*}
\]

Note that \( W^r \) as defined above is slightly different from the process in Chapter 7. As we shall see, the difference between the diffusion scaled versions of these processes vanishes, since \( \hat{W}^r \) of (8.41) converges in distribution and by Assumption 8.1.1, \( M^r \to M \) as \( r \to \infty \). The version of \( W^r \) used here simplifies the proofs. The following is the main result of this section.
**Theorem 8.6.1.** Consider the sequence of parallel server systems indexed by $r$, where the $r$th system operates under the threshold policy described above. Then

$$(\hat{Q}_r^1, \hat{T}_r^1, \hat{I}_r^1) \Rightarrow (0, 0, 0) \text{ as } r \to \infty.$$ (8.44)

To prove Theorem 8.6.1 we will need the next lemma.

**Lemma 8.6.1.** Let $\sigma_0^r = \inf\{t \geq 0 : R_r^1(t) \geq 0\} = \inf\{t \geq 0 : Q_r^1(t) \geq L_r^1\}$. Then, for any $\epsilon > 0$ and $t > 0$,

(i) $P(I_r^2(\sigma_0^r) \geq \epsilon) \rightarrow 0 \text{ as } r \to \infty,$ (8.45)

(ii) $P(\sup_{\sigma_0^r \leq s \leq r^2t} |R_r^1(s)| \geq L_r^1 - 2) \rightarrow 0 \text{ as } r \to \infty.$ (8.46)

**Proof.** (i) According to the threshold policy, for $t \leq \sigma_0^r$, $T_r^2(t) \leq \int_0^t 1_{\{Q_r^1(s) = 0\}} ds \leq t.$ (8.47)

It suffices to prove (8.45) for all $\epsilon$ sufficiently small. Fix $0 < \epsilon < \min(\mu_2/2, \mu_3/2, \lambda_1/2, \lambda_2/2, \mu_3 - \lambda_2, 1), \quad \nu_r = 1 - (\lambda_2 - \epsilon)/\mu_3 > \nu/2 > 0,$ (8.50)

where $\nu = 1 - (\lambda_2 - \epsilon)/(\mu_3 + \epsilon).$ Let $t^r = \max(2/\epsilon, 8L_1^r/\mu_1)$,

$s^r = t^r \nu^r + \delta L_1^r(1 - \nu^r)/(\mu_3 + \epsilon)$,

where $\delta L_1^r = \delta L_1^r$, and for all $r \geq r_e$,

$$0 < \delta < \min \left( \frac{\mu_3 + \epsilon}{\mu_2 + \epsilon} \frac{8(\mu_3^r - \epsilon)}{\lambda_1^r + \epsilon} \frac{t^r}{2L_1^r} (1 - \nu^r)(\mu_3^r + \epsilon) \right).$$ (8.51)
Note that by the choice of $\delta$, $t^r > s^r$ and $L_2^{r,\dagger} < L_1^r$ for $r \geq r_e$. Also, by the choice of $\delta$, $t^r - s^r \to \infty$ as $r \to \infty$. In addition to (8.49)-(8.51), $r_e$ is chosen such that for $r \geq r_e$, $s^r > 2/\epsilon$, (A.4) and (A.5) are satisfied with $\mu_j, \mu_j^†, j = 1, 2, 3, 4, 5$ and $\lambda_i, \lambda_i^†, i = 1, 2, 3$ in place of $\nu, \nu^r$. Let

$$A_i^r = \{ A_i^r(t^r) \geq (\lambda_i^r - \epsilon)t^r, S_i^r(s^r) \leq (\mu_i^r + \epsilon)s^r, T_i^r(t^r) \leq s^r, \max_{0 \leq s \leq t^r} Q_i^r(s) < L_2^{r,\dagger}\}.$$  

Then, for $r \geq r_e$, on $A_i^r$, for $0 \leq t \leq \sigma_0^r \land t^r$, since $Q_i^r(t) < L_2^r, T_i^r(t) = 0$ and therefore

$$Q_i^r(t) = A_i^r(t) - S_i^r(T_i^r(t)).$$

Thus, for $r \geq r_e$,

$$\mathbf{P}(I_2^r(\sigma_0^r) > t^r) \leq \mathbf{P}(\sigma_0^r > t^r) \leq \mathbf{P}(A_i^r(t^r) - S_i^r(T_i^r(t^r)) < L_1^r, A_i^r)$$  

$$+ \mathbf{P}((A_i^r)^c, \sigma_0^r > t^r) \leq \mathbf{P}((\lambda_i^r - \epsilon)t^r - (\mu_i^2 + \epsilon)s^r < L_1^r, A_i^r)$$  

$$+ \mathbf{P}(A_i^r(t^r) < (\lambda_i^r - \epsilon)t^r) + \mathbf{P}(S_i^r(s^r) > (\mu_i^2 + \epsilon)s^r)$$  

$$+ \mathbf{P}(T_i^r(t^r) > s^r, \sigma_0^r > t^r) + \mathbf{P}(\max_{0 \leq s \leq t^r} Q_i^r(s) \geq L_2^{r,\dagger}, \sigma_0^r > t^r).$$

For $r \geq r_e$,

$$\frac{|\lambda_2^r - \lambda_2^r - \epsilon|}{\mu_3^2 + \epsilon} \leq \frac{8(\lambda_2 + \mu_3)}{\mu_3^2},$$

and combining this with (8.49) and the definitions of $s^r, L_2^{r,\dagger}$, we have

$$(\lambda_i^r - \epsilon)t^r - (\mu_i^2 + \epsilon)s^r$$  

$$= t^r((\lambda_i^r - \epsilon) - (\mu_i^2 + \epsilon)\nu^r) - (\mu_i^2 + \epsilon)\frac{L_2^{r,\dagger}}{\mu_3^2 + \epsilon}$$  

$$\geq t^r((\lambda_i^r - 2\epsilon) - (\mu_i^2 + 2\epsilon)\bigg((1 - \frac{\lambda_2}{\mu_3}) + \epsilon\bigg(1 + \frac{8(\lambda_2 + \mu_3)}{\mu_3^2}\bigg)\bigg)) - L_1^r$$

$$= t^r((\lambda_i^r - 2\epsilon) - (\mu_i^2 + 2\epsilon)\bigg((\frac{\lambda_1 - \mu_1}{\mu_2}) + \epsilon\bigg(1 + \frac{8(\lambda_2 + \mu_3)}{\mu_3^2}\bigg)\bigg)) - L_1^r$$

$$= t^r(\mu_1 - 2\epsilon - \mu_2\frac{8(\lambda_2 + \mu_3)}{\mu_3^2})$$  

$$- 2\epsilon\bigg(\frac{(\lambda_1 - \mu_1)}{\mu_2} + \epsilon\bigg(1 + \frac{8(\lambda_2 + \mu_3)}{\mu_3^2}\bigg)\bigg) - L_1^r$$

$$= t^r(\mu_1 - p_1(\epsilon)) - \frac{8L_1^r}{\mu_1} \geq \frac{3L_1^r}{2} > L_1^r.$$
where we have used the choice of \( \delta \) in the third line, the fact that \( (\lambda_1 - \mu_1)/\mu_2 = 1 - (\lambda_2/\mu_3) \) (heavy traffic) in the fourth line and the choice of \( \epsilon \) in the last line. Therefore the first probability in the right member of the last inequality in (8.52) is zero. By Proposition A.0.3 in the Appendix, for all \( r \geq r_e 
olimits \):

\[
P(A^r_1(t^r) < (\lambda_1^r - \epsilon)t^r) \leq \exp\left(- (\lambda_1 - 2\epsilon)t^r \Lambda_1^{a,s}\left(\frac{1}{\lambda_1}\left(1 + \frac{\epsilon}{2\lambda_1}\right)\right)\right) + \exp\left(- \frac{\ell_0 t^r}{2\lambda_1} + \Lambda_1^r(\ell_0)\right),
\]

for any \( 0 < \ell_0 \in \mathcal{O} \), and

\[
P(S^{r}_{2}(s^r) > (\mu_2^r + \epsilon)s^r) \leq \exp\left(- (\mu_2 s^r - 1)\Lambda_2^{s,s}\left(\frac{1}{\mu_2}\left(1 + \frac{\epsilon}{3\mu_2}\right)\right)\right).
\]

Now,

\[
P(T^r_2(t^r) > s^r, \sigma_0^r > t^r) \leq P(T^r_2(t^r) > s^r, \max_{0 \leq s \leq t^r} Q^r_2(s) < L_2^{r,t}, \sigma_0^r > t^r) + P(\max_{0 \leq s \leq t^r} Q^r_2(s) \geq L_2^{r,t}, \sigma_0^r > t^r),
\]

where

\[
P(\max_{0 \leq s \leq t^r} Q^r_2(s) \geq L_2^{r,t}, \sigma_0^r > t^r) \leq \frac{C}{r^3}, \tag{8.58}
\]

for all \( r \) sufficiently large, by Lemma B.0.3 in Appendix B, where \( C \) is a positive constant independent of \( r \). The probability in (8.58) is also the last probability in (8.52). On the other hand, for

\[
A^r_2 = \{ T^r_2(t^r) > s^r, \max_{0 \leq s \leq t^r} Q^r_2(s) < L_2^{r,t},
A^r_2(t^r) \geq (\lambda_2^r - \epsilon)t^r, S^r_3(t^r - s^r) \leq (t^r - s^r)(\mu_3^r + \epsilon)\},
\]

\[
P(T^r_2(t^r) > s^r, \max_{0 \leq s \leq t^r} Q^r_2(s) < L_2^{r,t}, \sigma_0^r > t^r)
\leq P(A^r_2, \sigma_0^r > t^r) + P(A^r_2(t^r) < (\lambda_2^r - \epsilon)t^r)
+ P(S^r_3(t^r - s^r) > (t^r - s^r)(\mu_3^r + \epsilon)).
\]
Then, on $\mathcal{A}_2^r$, since $t^r \geq T_2^r(t^r) + T_3^r(t^r)$, $T_3^r(t^r) \leq t^r - s^r$. Also, on $\mathcal{A}_2^r \cap \{\sigma_0^r > t^r\}$,

$$
\begin{align*}
L_2^{r,1} &> Q_2^r(t^r) = A_2^r(t^r) - S_3^r(T_3^r(t^r)) \\
&\geq (\lambda_2^r - \epsilon)t^r - S_3^r(t^r - s^r) \\
&\geq (\lambda_2^r - \epsilon)t^r - (t^r - s^r)(\mu_3^r + \epsilon) \\
&= (\mu_3^r + \epsilon)t^r \left( \frac{\lambda_2^r - \epsilon}{\mu_3^r + \epsilon} - 1 + \frac{s^r}{t^r} \right) \\
&= (\mu_3^r + \epsilon)t^r \left( -\nu^r + \nu^r + \frac{L_2^{r,1}}{(\mu_3^r + \epsilon)t^r} \right) = L_2^{r,1}.
\end{align*}
$$

(8.61)

It follows that $\mathcal{A}_2^r \cap \{\sigma_0^r > t^r\}$ is empty and so the first probability after the inequality in (8.60) is zero. Again, by Proposition A.0.3 in the Appendix, for all $r \geq r_\epsilon$,

$$
P(A_2^r(t^r)) < (\lambda_2^r - \epsilon)t^r \leq \exp \left( - (\lambda_2 - 2\epsilon)t^r \Lambda_2^{a,s} \left( \frac{1}{\lambda_2} \left( 1 + \frac{\epsilon}{2\lambda_2} \right) \right) \right)
$$

(8.62)

$$
+ \exp \left( - \frac{\ell_0 c t^r}{2\lambda_2} + A_2^r(\ell_0) \right),
$$

(8.63)

for any $0 < \ell_0 \in \mathcal{O}$, and

$$
P(S_3^r(t^r - s^r) > (\mu_3^r + \epsilon)(t^r - s^r))
\leq \exp \left( - (\mu_3^r(t^r - s^r) - 1)\Lambda_3^{a,s} \left( \frac{1}{\mu_3} \left( \frac{1}{1 + \epsilon/3\mu_3} \right) \right) \right).
$$

(8.64)

By Appendix A,

$$
\Lambda_i^{a,s} \left( \frac{1}{\lambda_i} \left( 1 + \frac{\epsilon}{2\lambda_i} \right) \right) > 0, \text{ for } i = 1, 2 \text{ and } \Lambda_j^{a,s} \left( \frac{1}{\mu_j} \left( \frac{1}{1 + \epsilon/3\mu_j} \right) \right) > 0, \text{ for } j = 2, 3.
$$

Since, as $r \to \infty$, $t^r \to \infty$, $s^r \to \infty$, and $(t^r - s^r) \to \infty$, it follows that the probabilities in (8.55),(8.56),(8.63) and (8.64) all tend to zero as $r \to \infty$. Since $t^r < r\epsilon$ for all $r$ sufficiently large, the result in (8.45) follows.

(ii) Let $\epsilon > 0$ be such that

$$
0 < \epsilon < \min(\mu_2/4, \mu_3/4, \lambda_1/4, \lambda_2/4, (\lambda_1 - \mu_1)/8, \mu_3 - \lambda_2, 1),
$$

(8.65)

and for $p_1(\epsilon)$ given by (8.48), $p_1(\epsilon) < \mu_1/4$. It suffices to prove (8.46) for $t > 2/\epsilon$. Fix such a $t$.

Let $r_\epsilon$ be such that (8.49)-(8.50) hold, and (A.4)-(A.5) hold with $\mu_j, \mu_j^r, j = 1, 2, 3, \lambda_i, \lambda_i^r, i = 1, 2$ in place of $\nu, \nu^r$ for all $r \geq r_\epsilon$. For (8.46) we will show that

$$
P \left( \inf_{0 \leq s \leq r^2t} R_1^r(s) \leq -(L_1^r - 2) \right) \to 0 \text{ as } r \to \infty.
$$

(8.66)
The proof that
\[ P \left( \sup_{0 \leq s \leq r^2 t} R_1^r(s) \geq L_1^r - 2 \right) \to 0, \quad \text{as } r \to \infty, \]
is similar and we omit it (see [3] for details). For any \( n \geq 1 \), consider the \( n^{th} \) down excursion \([\sigma_{2n-1}^r, \sigma_{2n}^r]\) of buffer 1. During this time interval, server 1 works on class 1 jobs unless buffer 1 is empty or buffer 2 is at or below its threshold. If buffer 2 is at or below its threshold, server 1 serves class 3 jobs. Server 2 works on class 2 jobs unless there are none available to serve, in which case server 2 serves class 1 jobs. Shifted processes are defined as follows. For each \( n \geq 1 \), on \( \{\sigma_{2n-1}^r < \infty\} \), define for each \( s \geq 0 \),

\[
\begin{align*}
A_i^{r,n}(s) &= A_i^r(\sigma_{2n-1}^r + s) - A_i^r(\sigma_{2n-1}^r), \quad i = 1, 2, \\
S_j^{r,n}(s) &= S_j^r(T_j^r(\sigma_{2n-1}^r) + s) - S_j^r(T_j^r(\sigma_{2n-1}^r)), \quad j = 1, 2, 3, \\
T_j^{r,n}(s) &= T_j^r(\sigma_{2n-1}^r + s) - T_j^r(\sigma_{2n-1}^r), \quad j = 1, 2, 3, \\
I_k^{r,n}(s) &= I_k(\sigma_{2n}^r + s) - I_k(\sigma_{2n}^r), \quad k = 1, 2, \\
\bar{A}_i^{r,n}(s) &= \sup \{ m \geq 0 : \zeta_i^r(A_i^r(\sigma_{2n-1}^r) + m) - \zeta_i^r(A_i^r(\sigma_{2n-1}^r)) \leq s \}, \quad i = 1, 2, \\
\bar{S}_j^{r,n}(s) &= \sup \{ m \geq 0 : \zeta_j^r(S_j^r(T_j^r(\sigma_{2n-1}^r)) + m + 1) \leq s \}, \quad j = 1, 2, 3.
\end{align*}
\]

For concreteness, on \( \{\sigma_{2n-1}^r = \infty\} \), define all of the above processes to be identically zero. Then on \( \{\sigma_{2n-1}^r < \infty\} \), for \( 0 \leq s \leq \sigma_{2n}^r - \sigma_{2n-1}^r \)
\[ R_1^r(\sigma_{2n-1}^r + s) = -1 + A_1^{r,n}(s) - S_1^{r,n}(T_1^{r,n}(s)) - S_2^{r,n}(T_2^{r,n}(s)), \quad \text{(8.68)} \]
which takes account of the fact that \( R_1^r(\sigma_{2n-1}^r) = -1 \) since this level is reached either from \( R_1^r = 1 \) due to simultaneous completions of service by servers 1 and 2, or from \( R_1^r = 0 \) due to a single completion of service by server 1, or due to a single completion of service by server 2 while buffer 2 is empty. Taking account of the fact that one or two departures, which may be due to one or both of activities \( j \in \{1, 2\}, \) occur at \( \sigma_{2n-1}^r < \infty \) and a job may have partially arrived to both buffers, we have that

\[ \bar{A}_i^{r,n}(s) \leq A_i^{r,n}(s), \quad i = 1, 2 \quad \text{and} \quad \bar{S}_j^{r,n}(s) + 1 \geq S_j^{r,n}(s), \quad j = 1, 2, 3. \quad \text{(8.69)} \]

Every excursion of buffer one below its threshold requires at least one arrival to buffer one before it is completed. The number of arrivals to buffer one during an interval of length \( r^2 t \) is bounded
above by \( n'_1 = \lfloor (\lambda'_1 + \epsilon)r^2 t \rfloor + 2 \) with high probability, as \( r \to \infty \). In particular using Proposition A.0.3 of Appendix A,

\[
P(\sigma^2_{2n_1 - 1} \leq r^2 t) \leq P(\Lambda'_1(r^2 t) > (\lambda'_1 + \epsilon)r^2 t) \\
\leq \exp \left( - (\lambda_1 r^2 t - 1)A^{a,s}_{n_1} \left( \frac{1}{\lambda_1} \left( \frac{1}{1 + \epsilon/3\lambda_1} \right) \right) \right),
\]

for all \( r \geq r_s \). Then,

\[
P(R^r_1(s) \leq -(L^r_1 - 2) \text{ some } s \in [0, r^2 t]) \\
\leq P(\sigma^2_{2n_1 - 1} \leq r^2 t) \\
\leq P(\sigma^2_{2n_1 - 1} > r^2 t, R^r_1(s) \leq -(L^r_1 - 2) \text{ some } s \in [0, r^2 t])
\leq P(\sigma^2_{2n_1 - 1} \leq r^2 t)
\leq \sum_{n=1}^{n'_1} P(R^r_1(s) \leq -(L^r_1 - 2) \text{ some } s \in (\sigma^r_{2n-1}, \sigma^r_{2n}), \sigma^r_{2n-1} \leq r^2 t).
\]

Let \( t'_1 = c_1 L^r_1 \) where

\[
c_1 = \min \left( \frac{2}{\lambda_1}, \frac{1}{8(\mu_1 + \mu_2)} \right),
\]

and let \( s^r_1 = \nu^r t'_1 + L^r_2/(\mu_3 + \epsilon) \), where \( L^r_2 = \delta L^r_1 \) and

\[
0 < \delta < \min \left( \frac{c_1 \mu_1 \mu_3}{8\mu_2}, \frac{c_1 \mu_3 \lambda_2 - 2\epsilon}{2\mu_3 + 2\epsilon} \right).
\]

Note that \( \delta \) in (8.73) is different from \( \delta \) in the part (i) of the proof. By the choice of \( \delta \), \( t'_1 - s^r_1 > \frac{\lambda_2 - 2\epsilon}{2(\mu_3 + 2\epsilon)}t'_1 > 0 \) and \( t'_1 - s^r_1 \to \infty \), as \( r \to \infty \). For each positive integer \( n \), let

\[
A^{r,n}_1 = \{ A^{r,n}_1(t'_1) \geq (\lambda'_1 - \epsilon)t'_1 \text{ for } i = 1, 2, A^{r,n}_2(t'_1) \leq (\lambda'_2 + \epsilon)t'_1, S^{r,n}_1(t'_1) \leq (\mu'_2 + \epsilon)s^r_1, S^{r,n}_2(s^r_1) \leq (\mu'_2 + \epsilon)s^r_1, \sigma^r_{2n-1} \leq r^2 t \},
\]

and

\[
R^{r,n} = \{ R^r_1(s) \leq -(L^r_1 - 2) \text{ some } s \in (\sigma^r_{2n-1}, \sigma^r_{2n}) \}.
\]

For each \( n \geq 1 \), on \( \{ \sigma^r_{2n-1} < \infty \} \), let \( \Delta^r_n = \sigma^r_{2n} - \sigma^r_{2n-1} \). Then,

\[
P(R^{r,n}, \sigma^r_{2n-1} \leq r^2 t) \\
\leq P(R^{r,n}, (A^{r,n})^c, \sigma^r_{2n-1} \leq r^2 t) \\
+ P(R^{r,n}, A^{r,n}) \\
= P(R^{r,n}, (A^{r,n})^c, \sigma^r_{2n-1} \leq r^2 t) \\
+ P(R^{r,n}, A^{r,n}, \Delta^r_n \leq t'_1) \\
+ P(R^{r,n}, A^{r,n}, \Delta^r_n > t'_1).
\]
By making $r_\epsilon$ larger if necessary, we can ensure that for $r \geq r_\epsilon$,
\[
\min \left( (\lambda_1 - \mu_1 - 4\epsilon)c_1, \frac{\mu_1}{2c_1}, \frac{\mu_3}{\lambda_1}, \frac{c_1\mu_3}{2}, \frac{1}{4}, c_1\epsilon \right) L_1^r > 3, \text{ and } L_2^r > 8. \tag{8.76}
\]

Henceforth we only consider $r \geq r_\epsilon$. On $\mathcal{A}_3^{r,n} \cap \{ \Delta_n^r \leq t_1^r \}$, for any $s \in [\sigma_{2n-1}^r, \sigma_{2n}^r]$,
\[
Q_1^r(s) \geq L_1^r - 1 - (\mu_1^r + \epsilon)t_1^r - (\mu_2^r + \epsilon)t_1^r
\]
\[
\geq L_1^r - 1 - (\mu_1 + \mu_2 + 4\epsilon)c_1 L_1^r
\]
\[
\geq L_1^r/2 - \frac{2(\mu_1 + \mu_2)}{8(\mu_1 + \mu_2)}L_1^r \tag{8.77}
\]
\[
\geq L_1^r/2 - L_1^r/4 = L_1^r/4 > 2,
\]
where we have used (8.65), (8.72) and (8.76), since (8.77) implies that $R_1^r(s) > -(L_1^r - 2)$. It follows that
\[
P(\mathcal{R}^{r,n}, \mathcal{A}_3^{r,n}, \Delta_n^r \leq t_1^r) = 0. \tag{8.78}
\]

We have that
\[
P(\mathcal{R}^{r,n}, \mathcal{A}_3^{r,n}, \Delta_n^r > t_1^r) = P(\mathcal{R}^{r,n}, \mathcal{A}_3^{r,n}, \Delta_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) > L_2^r/2)
\]
\[
+ P(\mathcal{R}^{r,n}, \mathcal{A}_3^{r,n}, \Delta_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) \leq L_2^r/2). \tag{8.79}
\]

On $\mathcal{A}_3^{r,n} \cap \{ \Delta_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) > L_2^r/2 \}$, for any $s \in [\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_1^r]$,
\[
Q_2^r(s) \geq L_2^r/2 - (\mu_3^r + \epsilon)t_2^r
\]
\[
\geq 4\mu_3 L_1^r/\lambda_1 - 2((\mu_3 + 2\epsilon)/\lambda_1))L_1^r \tag{8.80}
\]
\[
\geq (4\mu_3/\lambda_1 - 3\mu_3/\lambda_1)L_1^r
\]
\[
= \mu_3 L_1^r/\lambda_1 > 3,
\]
where we have used (8.72) in the second, (8.65) in the third and (8.76) in the last inequality. Therefore, on $\mathcal{A}_3^{r,n} \cap \{ \Delta_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) > L_2^r/2 \}$, for any $s \in [\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_1^r], T_2^{r,n}(s) = 0, S_2^{r,n}(T_2^{r,n}(s)) = 0$, and so by (8.68),
\[
R_1^r(\sigma_{2n-1}^r + t_1^r) = -1 + A_1^{r,n}(t_1^r) - S_1^{r,n}(T_1^{r,n}(t_1^r))
\]
\[
\geq -1 + (\lambda_1^r - \epsilon)t_1^r - (\mu_1^r + \epsilon)t_1^r \tag{8.81}
\]
\[
\geq -1 + (\lambda_1 - \mu_1 - 4\epsilon)t_1^r > 2,
\]
where we have used (8.76) to obtain the last inequality. It follows that $\triangle_n^r \leq t_1^r$ and so

$$P(R_{3n}^r, A_3^{rn}, \triangle_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) > L_2^r/2) = 0.$$ (8.82)

On the other hand, on $A_3^{rn} \cap \{\triangle_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) \leq L_2^r/2\}$ and for $s \in [\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_1^r]$, we have

$$Q_2^r(s) \leq L_2^r/2 + (\lambda_2^r + \epsilon)t_1^r$$

$$\leq 4\mu_3 L_1^r/\lambda_1 + 3\lambda_2^r L_1^r + 1$$

$$\leq 7\mu_3 L_1^r/\lambda_1 + 1$$

$$= \frac{7L_2^r}{8} + 1 < L_2^r,$$

where we have used the fact that $\lambda_2/\mu_3 < 1$, (8.65), (8.72) and (8.76). Therefore, on $A_3^{rn} \cap \{\triangle_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) \leq L_2^r/2\}$, for $p_2(\epsilon) = p_1(\epsilon) - 2\epsilon$,

$$R_1^r(\sigma_{2n-1}^r + t_1^r) = -1 + A_1^{rn}(t_1^r) - S_2^{rn}(T_2^{rn}(t_1^r))$$

$$\geq -1 + (\lambda_1 - 2\epsilon)t_1^r - (\mu_2^r + \epsilon)t_1^r - (\lambda_1 - \mu_1 + p_2(\epsilon))t_1^r - \frac{(\mu_2 + 2\epsilon)\nu}{c_1\mu_3}t_1^r$$

$$\geq -1 + (\lambda_1 - \mu_1(\epsilon) - \frac{(\mu_2 + 2\epsilon)c_1\mu_3}{8c_1\mu_3\mu_2})t_1^r$$

$$\geq -1 + (\mu_1/2)t_1^r$$

$$\geq 2,$$

where we have used (8.48), (8.49), (8.53), (8.73), (8.76) and the fact that $p_1(\epsilon) < \mu_1/4$. Therefore, on $A_3^{rn} \cap \{\triangle_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) \leq L_2^r/2\}$, $\triangle_n^r \leq t_1^r$, a contradiction. It follows that

$$P(R_{3n}^r, A_3^{rn}, \triangle_n^r > t_1^r, Q_2^r(\sigma_{2n-1}^r) \leq L_2^r/2) = 0.$$ (8.85)

Now,

$$P(R_{3n}^r, (A_3^{rn})^c, \sigma_{2n-1}^r \leq r^2t)$$

$$\leq \sum_{i=1}^2 P(A_i^{rn}(t_1^r) < (\lambda_i^r - \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t)$$

$$+ P(A_2^{rn}(t_1^r) > (\lambda_2^r + \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t)$$

$$+ \sum_{j=1}^3 P(S_j^{rn}(t_1^r) > (\mu_j^r + \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t)$$

$$+ P(S_2^{rn}(s_j^r) > (\mu_2^r + \epsilon)s_j^r, \sigma_{2n-1}^r \leq r^2t)$$

$$+ P(R_{3n}^r, T_2^{rn}(t_1^r) > s_j^r, S_j^{rn}(t_1^r) \leq (\mu_j^r + \epsilon)t_1^r, j = 1, 2, 3, A_2^{rn}(t_1^r) \leq (\lambda_2^r + \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t).$$ (8.86)
Recall the definition of \( \tau_n^r \) from (8.29). The set \( \{ \sigma_{2n-1}^r < \infty \} \) is contained in \( \{ \tau_n^r \in \mathbb{N}^5 \} \) by the convention that the arrival and service renewal processes are finite everywhere on \( \Omega \), and so for \( i = 1, 2, \)

\[
P(\bar{A}_i^{r,n}(t_1^r) < (\lambda_i^r - \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t) \leq \mathbb{E}(1_{\{\tau_n^r \in \mathbb{N}^5\}}P(\bar{A}_i^{r,n}(t_1^r) < (\lambda_i^r - \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t|\mathcal{F}_{\tau_n^r})) \quad (8.87)
\]

where on \( \{ \tau_n^r \in \mathbb{N}^5 \} \), \( \bar{A}_i^{r,n} \) is the counting process defined from the sequence of interarrival time random variables \( \{u_i^r(A_i^r(\sigma_{2n-1}^r) + l), l = 1, 2, \ldots \} \), where by Lemmas 8.5.1 and 8.5.2, the conditional distribution of this sequence from \( l = 2 \) onwards given \( \mathcal{F}_{\tau_n^r} \) is the same as the distribution of \( \{u_i^r(l) : l = 2, 3, \ldots \} \). Hence for \( i = 1, 2, \) \( \delta_i^r = \epsilon/2\lambda_i^r \) we have a.s. on \( \{ \tau_n^r \in \mathbb{N}^5 \} \), by a similar argument to that for Proposition A.0.3 (modified to include the event \( \sigma_{2n-1}^r \leq r^2t \)),

\[
P(\bar{A}_i^{r,n}(t_1^r) < (\lambda_i^r - \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t)^{r^2t|\mathcal{F}_{\tau_n^r}} \leq \exp \left( - (\lambda_i - 2\epsilon)t_1^r \Lambda_i^{a,s} \left( \frac{1}{\lambda_i^r} \left( 1 + \frac{\epsilon}{2\lambda_i^r} \right) \right) \right) + P(\bar{u}_i(A_i^r(\sigma_{2n-1}^r) + 1) > \delta_i^r t_1^r, \sigma_{2n-1}^r \leq r^2t|\mathcal{F}_{\tau_n^r}) \quad (8.88)
\]

For any \( \ell_0 \in \mathcal{O} \) such that \( \ell_0 > 0 \) we have, for \( i = 1, 2, \) by Proposition A.0.3 in Appendix A,

\[
P(\bar{u}_i^r(A_i^r(\sigma_{2n-1}^r) + 1) > \delta_i^r t_1^r, \sigma_{2n-1}^r \leq r^2t) \leq P\left(\max_{l=1}^{[\lambda_i^r r^2t]} u_i^r(l) > \delta_i^r t_1^r\right) \leq P(A_i^r(r^2t) > (\lambda_i^r + \epsilon)r^2t) + P\left(\max_{l=1}^{[\lambda_i^r r^2t]} u_i^r(l) > \delta_i^r t_1^r\right) \quad (8.89)
\]

Also, by Proposition A.0.3 in Appendix A,

\[
P(A_2^{r,n}(t_1^r) > (\lambda_2^r + \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t) = \mathbb{E}\left(1_{\{\tau_n^r \in \mathbb{N}^5\}}P\left(A_2^{r,n}(t_1^r) > (\lambda_2^r + \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t|\mathcal{F}_{\tau_n^r}\right)\right) \leq \exp \left( - (\lambda_2t_1^r - 1) \Lambda_2^{a,s} \left( \frac{1}{\lambda_2^r} \left( 1 + \frac{\epsilon}{3\lambda_i^r} \right) \right) \right). \quad (8.90)
\]

For \( j = 1, 2, 3, \) we have

\[
P\left(S_j^{r,n}(t_1^r) > (\mu_j^r + \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t\right) \leq \mathbb{E}\left(1_{\{\tau_n^r \in \mathbb{N}^5\}}P\left(S_j^{r,n}(t_1^r) + 1 > (\mu_j^r + \epsilon)t_1^r, \sigma_{2n-1}^r \leq r^2t|\mathcal{F}_{\tau_n^r}\right)\right), \quad (8.91)
\]
where on \( \{ \tau^r_n \in \mathbb{N} \} \), \( S^r_n \) is the counting process defined by a sequence of service time random variables \( \{ v^r_j(S^r_j(T^r_j(\sigma^r_{2n-1})) + l) : l = 2,3,\ldots \} \) where the conditional distribution of the sequence on \( \{ \tau^r_n \in \mathbb{N} \} \) given \( F^r_{2n} \) is that of the original sequence of service time random variables \( \{ v^r_j(l) : l = 2,3,\ldots \} \). By similar reasoning to that for Proposition A.0.3 of the Appendix, for \( j = 1,2,3 \),

\[
\Pr \left( S^r_n \left( t^r_1 \right) > (\mu^r_j + \epsilon)t^r_1, \sigma^r_{2n-1} \leq r^2 t \middle| F^r_{2n} \right) \\
\leq \exp \left( - (\mu^r_j t^r_1 - 1) \Lambda^{r,s} \left( \frac{1}{\mu_j} \left( \frac{1}{1 + \epsilon/3\mu_j} \right) \right) \right). \tag{8.92}
\]

Similarly,

\[
\Pr \left( S^r_n \left( t^r_1 \right) > (\mu^r_2 + \epsilon)s^r_1, \sigma^r_{2n-1} \leq r^2 t \middle| F^r_{2n} \right) \leq \exp \left( - (\mu^r_2 s^r_1 - 1) \Lambda^{r,s} \left( \frac{1}{\mu_2} \left( \frac{1}{1 + \epsilon/3\mu_2} \right) \right) \right). \tag{8.93}
\]

We proceed to estimate the last probability in (8.86). For each \( n \geq 1 \) let

\[ D^r_n = \{ T^r_n(t^r_1) > s^r_1, S^r_n(t^r_1) \leq (\mu^r_j + \epsilon)t^r_1, j = 1,2,3, A^r_n(t^r_1) \leq (\Lambda^r_j + \epsilon)t^r_1 \}. \tag{8.94} \]

We have that,

\[
\Pr (R^r_n, D^r_n, \sigma^r_{2n-1} \leq r^2 t) = \Pr (R^r_n, D^r_n, \Delta^r_n \leq t^r_1, \sigma^r_{2n-1} \leq r^2 t) + \Pr (R^r_n, D^r_n, \Delta^r_n > t^r_1, \sigma^r_{2n-1} \leq r^2 t). \tag{8.95} 
\]

On \( D^r_n \cap \{ \Delta^r_n \leq t^r_1 \} \), for every \( s \in [\sigma^r_{2n-1}, \sigma^r_{2n}] \),

\[
Q^r_1(s) \geq (L^r_1 - 1) - (\mu^r_1 + \mu^r_2 + 2\epsilon)t^r_1 \\
\geq L^r_1/2 - \mu_1 + \mu_2 + 4\epsilon \overline{L}^r_1 \\
\geq L^r_1/2 - \frac{2(\mu_1 + \mu_2)}{8(\mu_1 + \mu_2)}L^r_1 \\
> L^r_1/4 > 3.
\]

where we have used (8.72) and (8.76). It follows that \( D^r_n \cap \{ \Delta^r_n \leq t^r_1 \} \cap R^r_n = \emptyset \), and therefore

\[
\Pr (R^r_n, D^r_n, \Delta^r_n \leq t^r_1, \sigma^r_{2n-1} \leq r^2 t) = 0. \tag{8.97}
\]
On \( \{ \sigma^r_{2n-1} < \infty \} \), let \( \hat{\sigma}^r_{2n-1} = \inf \{ s \geq \sigma^r_{2n-1} : Q^r_2(s) = 0 \} \) and \( \bar{\Delta}^r_n = \hat{\sigma}^r_{2n-1} - \sigma^r_{2n-1} \). Now,

\[
\mathbf{P}(R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \sigma^r_{2n-1} \leq r^2t) \\
= \mathbf{P}(R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \bar{\Delta}^r_n \leq t'_1 - s^r_1, \sigma^r_{2n-1} \leq r^2t) \\
\quad + \mathbf{P}(R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \bar{\Delta}^r_n > t'_1 - s^r_1, \sigma^r_{2n-1} \leq r^2t). \tag{8.98}
\]

Now, on \( D^{r,n} \cap \{ \Delta^r_n > t'_1, \bar{\Delta}^r_n > t'_1 - s^r_1 \} \), since \( T^{r,n}_2(\hat{\sigma}^r_{2n-1}) = 0 \) and \( \sigma^r_{2n-1} + t'_1 - \hat{\sigma}^r_{2n-1} \leq t'_1 - \bar{\Delta}^r_n < s^r_1 \), then \( T^{r,n}_2(t'_1) \leq s^r_1 \) and it follows that,

\[
\mathbf{P}(R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \bar{\Delta}^r_n > t'_1 - s^r_1, \sigma^r_{2n-1} \leq r^2t) = 0. \tag{8.99}
\]

Then,

\[
\mathbf{P}(R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \bar{\Delta}^r_n \leq t'_1 - s^r_1, \sigma^r_{2n-1} \leq r^2t) \\
\leq \mathbf{P}(R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \bar{\Delta}^r_n \leq t'_1 - s^r_1, \sigma^r_{2n-1} \leq r^2t, Q^r_2(\sigma^r_{2n-1}) \leq L^r_0/2) \\
\quad + \mathbf{P}(R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \bar{\Delta}^r_n \leq t'_1 - s^r_1, \sigma^r_{2n-1} \leq r^2t, Q^r_2(\sigma^r_{2n-1}) > L^r_2/2). \tag{8.100}
\]

By a similar computation to that in (8.80), the last probability above is zero. Let

\[
R^{r,n}_1 = \{ R^{r,n}, D^{r,n}, \Delta^r_n > t'_1, \sigma^r_{2n-1} \leq r^2t, Q^r_2(\sigma^r_{2n-1}) \leq L^r_2/2 \}, \tag{8.101}
\]

and let

\[
0 < \tilde{\gamma} < \min \left( \frac{\delta}{2c_1 \lambda_2}, 1 \right). \tag{8.102}
\]

Now,

\[
\mathbf{P}(R^{r,n}_1, \bar{\Delta}^r_n \leq t'_1 - s^r_1) \leq \mathbf{P}(R^{r,n}_1, \bar{\Delta}^r_n \leq \tilde{\gamma}(t'_1 - s^r_1)) \\
\quad + \mathbf{P}(R^{r,n}_1, \bar{\Delta}^r_n > \tilde{\gamma}(t'_1 - s^r_1)). \tag{8.103}
\]

On \( R^{r,n}_1 \), by a similar argument to that for (8.83), for any \( s \in [\sigma^r_{2n-1}, \hat{\sigma}^r_{2n-1}] \), \( T^{r,n}_2(s) = T^{r,n}_2(s) = 0 \) and so \( Q^r_2(s) \geq L^r_1 - 1 \) for all \( s \in [\sigma^r_{2n-1}, \hat{\sigma}^r_{2n-1}] \), and then the \( n \)th down excursion would end if there were an arrival to buffer 1 during this interval, and since the \( n \)th down excursion is at least \( t'_1 \) in length, it follows that there are no arrivals to buffer 1 during the time interval \( [\sigma^r_{2n-1}, \hat{\sigma}^r_{2n-1}] \cap [\sigma^r_{2n-1}, \sigma^r_{2n-1} + t'_1] \). Hence, for \( r \) sufficiently large,

\[
\mathbf{P}(R^{r,n}_1, \bar{\Delta}^r_n > \tilde{\gamma}(t'_1 - s^r_1)) \\
\quad \leq \mathbf{P}(A^r_1(\tilde{\gamma}(t'_1 - s^r_1)) \leq \lambda_1 \sigma^r_{2n-1} \leq r^2t) \tag{8.104}
\]

\[
\quad \leq \mathbf{P}(A^r_1(\tilde{\gamma}(t'_1 - s^r_1)) \leq (\lambda_1 - \tilde{\gamma})(t'_1 - s^r_1), \sigma^r_{2n-1} \leq r^2t),
\]
where

\[
P(A_1^{r,n}(\bar{\gamma}(t^*_1 - s^*_1)) \leq (\lambda_1^*_r - \epsilon)(t^*_1 - s^*_1), \sigma^2_{2n-1} < r^2 t) \\
\leq \exp \left( - (\lambda_1 - 2\epsilon)\bar{\gamma}(t^*_1 - s^*_1)\Lambda_{A_1}^{a^*} \left( \frac{1}{\lambda_1^*} \left( 1 + \frac{\epsilon}{2\lambda_1} \right) \right) \right) \\
+ \exp \left( - (\lambda_1 r^2 t - 1)\Lambda_{A_1}^{a^*} \left( \frac{1}{\lambda_1^*} \left( 1 + \frac{\epsilon}{3\lambda_1} \right) \right) \right) \\
+ \exp \left( \log((\lambda_1 + \epsilon)r^2 t + 1) - \frac{\ell_0\bar{\gamma}(t^*_1 - s^*_1)}{2\lambda_1^*} + \Lambda_{A_1}^{a^*}(\ell_0) \right),
\]

by a similar argument as in (8.87)-(8.89). Let

\[
\mathcal{R}_{2}^{r,n} = \mathcal{R}_{1}^{r,n} \cap \{ \Delta_{n}^{r} \leq \bar{\gamma}(t^*_1 - s^*_1) \}.
\]

On \( \{ \sigma^2_{2n-1} < \infty \} \), for each \( r, n \) let

\[
\begin{align*}
\tilde{A}_{i}^{r,n}(s) &= A_{i}^{r}(\sigma^2_{2n-1} + s) - A_{i}^{r}(\sigma^2_{2n-1}), \text{ for } i = 1, 2, \\
\tilde{S}_{j}^{r,n}(s) &= S_{j}^{r}(T_{j}(\sigma^2_{2n-1} + s)) - S_{j}^{r}(T_{j}(\sigma^2_{2n-1})), \text{ for } j = 1, 2, 3, \\
\tilde{T}_{j}^{r,n}(s) &= T_{j}^{r}(\sigma^2_{2n-1} + s) - T_{j}^{r}(\sigma^2_{2n-1}), \text{ for } j = 1, 2, 3,
\end{align*}
\]

and let \( \bar{\tau}^*_n \equiv (A_1^r(\sigma^2_{2n-1}), A_2^r(\sigma^2_{2n-1}), S_1^r(T_1^r(\sigma^2_{2n-1})), S_2^r(T_2^r(\sigma^2_{2n-1})), S_3^r(T_3^r(\sigma^2_{2n-1}))) \). Then, similar to Lemma 8.5.1 \( \bar{\tau}^*_n \) is a stopping time relative to a filtration \( \{ \mathcal{F}_p : p \in \mathbb{N}_{\infty}^5 \} \). Also, let

\[
\begin{align*}
A_{I}^{r,n} &= \{ \tilde{A}_{2}^{r,n}(t^*_1 - \bar{\gamma}(t^*_1 - s^*_1)) < (\lambda^*_r - \epsilon)(t^*_1 - \bar{\gamma}(t^*_1 - s^*_1)) \}, \\
\tilde{S}_{3}^{r,n}(t^*_1 - s^*_1) &\leq (t^*_1 - s^*_1)(\mu^*_r + \epsilon).
\end{align*}
\]

We have that,

\[
P(\mathcal{R}_{2}^{r,n}) = P(\mathcal{R}_{2}^{r,n}, Q_2^r(s) \leq L^r_{2,n} / 2 \text{ for all } s \in (\bar{\sigma}_{2n-1}, \sigma^2_{2n-1} + t^*_1)) \leq \frac{C}{r^m},
\]

as \( r \to \infty \), by Lemma B.0.3 of Appendix B, where \( C > 0 \) is independent of \( r \) (this argument is similar to that used for (8.58)). On \( \mathcal{R}_{2}^{r,n}, T_{2}^{r,n}(t^*_1) > s^*_1 \) and since \( t^*_1 \geq T_{2}^{r,n}(t^*_1) + T_{3}^{r,n}(t^*_1) \), then
$T_{3}^{r,n}(t_{r}^*) \leq t_{r}^* - s_{r}^*$ and in particular $\hat{T}_{3}^{r,n}(t_{r}^* - \hat{\Delta}_{n}^r) \leq t_{r}^* - s_{r}^*$. We have,

$$
P(\mathcal{R}_{2}^{r,n}, Q_{2}^{r}(s) \leq L_{2}^{r,\frac{1}{2}} \text{ for all } s \in (\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_{1}^*))$$

$$= E \left( 1_{(\tau_{n}^r \in \mathbb{N})} P \left( \mathcal{R}_{2}^{r,n}, Q_{2}^{r}(s) \leq L_{2}^{r,\frac{1}{2}} \text{ for all } s \in (\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_{1}^* | F_{1}^{r,n} \right) \right).$$

Now,

$$P \left( \mathcal{R}_{2}^{r,n}, Q_{2}^{r}(s) \leq L_{2}^{r,\frac{1}{2}} \text{ for all } s \in (\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_{1}^* | F_{1}^{r,n} \right)$$

$$\leq P(\mathcal{R}_{2}^{r,n}, Q_{2}^{r}(s) \leq L_{2}^{r,\frac{1}{2}} \text{ for all } s \in (\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_{1}^*), \mathcal{A}_{4}^{r,n} | F_{1}^{r,n} \right) \quad (8.111)$$

$$+ P(\hat{A}_{2}^{r,n}(t_{1}^* - \hat{\gamma}(t_{1}^* - s_{1}^*)) \leq (\lambda_{2}^r - \epsilon)(t_{1}^* - \hat{\gamma}(t_{1}^* - s_{1}^*)) | F_{1}^{r,n}$$

$$+ P(\hat{S}_{3}^{r,n}(t_{1}^* - s_{1}^*) > (t_{1}^* - s_{1}^*)(\mu_{3}^r + \epsilon) | F_{1}^{r,n}).$$

Then, on $\mathcal{A}_{4}^{r,n} \cap \mathcal{R}_{2}^{r,n} \cap \{ Q_{2}^{r}(s) \leq L_{2}^{r} \text{ for all } s \in (\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_{1}^*) \},$

$$L_{2}^{r,\frac{1}{2}} \geq Q_{2}^{r}(\sigma_{2n-1}^r + t_{1}^*)$$

$$\begin{align*}
&= \hat{A}_{2}^{r,n}(t_{1}^* - \hat{\Delta}_{n}^r) - \hat{S}_{3}^{r,n}(\hat{T}_{3}^{r,n}(t_{1}^* - \hat{\Delta}_{n}^r)) \\
&\geq (\lambda_{2}^r - \epsilon)(t_{1}^* - \hat{\gamma}(t_{1}^* - s_{1}^*)) - \hat{S}_{3}^{r,n}(t_{1}^* - s_{1}^*) \\
&\geq (\lambda_{2}^r - \epsilon)t_{1}^* - (\lambda_{2}^r - \epsilon)\hat{\gamma}(t_{1}^* - s_{1}^*) - (\mu_{3}^r + \epsilon) \left( 1 - \nu^{r} \right) t_{1}^* - \frac{L_{2}^{r,\frac{1}{2}}}{\mu_{3}^r + \epsilon} \\
&= (\lambda_{2}^r - \epsilon)t_{1}^* - (\lambda_{2}^r - \epsilon)\hat{\gamma}(t_{1}^* - s_{1}^*) - (\lambda_{2}^r - \epsilon)t_{1}^* + L_{2}^{r,\frac{1}{2}} \\
&= - (\lambda_{2}^r - \epsilon)\delta t_{1}^* + L_{2}^{r,\frac{1}{2}} \\
&\geq - (\lambda_{2}^r - \epsilon)\delta t_{1}^* - L_{2}^{r,\frac{1}{2}} \\
&= - \frac{\delta}{2c_{1}\lambda_{2}} c_{1}L_{1}^{r} - L_{2}^{r,\frac{1}{2}} \\
&= - L_{2}^{r,\frac{1}{2}} + L_{2}^{r,\frac{1}{2}} = L_{2}^{r,\frac{1}{2}}/2,
\end{align*}$$

where we have used (8.50), (8.65), the definitions of $t_{1}^*, s_{1}^*, L_{2}^{r,\frac{1}{2}}$ and (8.102). This is a contradiction; it follows that

$$P(\mathcal{R}_{2}^{r,n}, Q_{2}^{r}(s) \leq L_{2}^{r,\frac{1}{2}} \text{ for all } s \in (\sigma_{2n-1}^r, \sigma_{2n-1}^r + t_{1}^*), \mathcal{A}_{4}^{r,n} | F_{1}^{r,n}) = 0. \quad (8.113)$$
We further have that,

\[
P(\tilde{A}_2^n(t_1^r - \tilde{\gamma}(t_2^r - s_1^r)) \leq (\lambda_2 - \epsilon)(t_1^r - \tilde{\gamma}(t_1^r - s_1^r)) | \mathcal{F}_{\tilde{\tau}^n})
\]

\[= \mathbb{E}(1_{(\tilde{\tau}^n_1 \in \mathbb{N}^n)} P(\tilde{A}_2^n(t_1^r - \tilde{\gamma}(t_1^r - s_1^r)) \leq (\lambda_2 - \epsilon)(t_1^r - \tilde{\gamma}(t_1^r - s_1^r)), \sigma_{2n-1}^r \leq r^2 t | \mathcal{F}_{\tilde{\tau}^n}) )
\]

\[\leq \exp\left(- (\lambda_2 - 2\epsilon)(t_1^r - \tilde{\gamma}(t_1^r - s_1^r)) \Lambda_2^a \left( \frac{1}{\lambda_2} \left( 1 + \frac{\epsilon}{2\lambda_2} \right) \right) \right)
\]

\[+ \exp\left(- (\lambda_2 r^2 t - 1) \Lambda_2^a \left( \frac{1}{\lambda_2} \left( 1 + \frac{\epsilon}{3\lambda_2} \right) \right) \right)
\]

\[+ \exp\left( \log((\lambda_2 + \epsilon)r^2 t + 1) - \frac{\ell_0 \epsilon(t_1^r - \tilde{\gamma}(t_1^r - s_1^r))}{2\lambda_2} + \Lambda_2^a(\ell_0) \right),
\]

by a similar argument as in (8.87)-(8.90) and (8.91)-(8.93).
Combining everything from (8.70) onwards we have for all $r$ sufficiently large,

$$
P(R_i^r(s) \leq -(L_i^r - 2) \text{ some } s \in [0, r^2t])
\leq \exp \left( -(\lambda_1 r^2t - 1) \Lambda_1^{a,s} \left( \frac{1}{\Lambda_1} \left( \frac{1}{1 + \epsilon/3\lambda_1} \right) \right) \right)
+ n_1^r \left\{ \sum_{i=1}^2 \left( \exp \left( -(\lambda_1 - 2\epsilon)t_i^r \Lambda_1^{a,s} \left( \frac{1}{\Lambda_1} \left( \frac{1}{1 + \epsilon/2\lambda_1} \right) \right) \right) + \exp \left( \log((\lambda_1 + \epsilon)r^2t + 1) - \frac{\ell_0\epsilon t_i^r}{2\lambda_1} + \Lambda_i^a(\ell_0) \right) \right) + \exp \left( -(\lambda_2 t_i^r - 1) \Lambda_2^{a,s} \left( \frac{1}{\Lambda_2} \left( \frac{1}{1 + \epsilon/2\lambda_2} \right) \right) \right) + \exp \left( \log((\lambda_1 + \epsilon)r^2t + 1) - \frac{\ell_0\epsilon s_i^r}{2\lambda_1} + \Lambda_i^a(\ell_0) \right) 
\right)
+ \exp \left( -(\lambda_2 r^2t - 1) \Lambda_2^{a,s} \left( \frac{1}{\Lambda_2} \left( \frac{1}{1 + \epsilon/3\lambda_2} \right) \right) \right)
+ \exp \left( \log((\lambda_2 + \epsilon)r^2t + 1) - \frac{\ell_0\epsilon\gamma(t_i^r - s_i^r)}{2\lambda_2} + \Lambda_i^a(\ell_0) \right) + \exp \left( -(\mu_2 s_i^r - 1) \Lambda_2^{a,s} \left( \frac{1}{\mu_2} \left( \frac{1}{1 + \epsilon/3\mu_2} \right) \right) \right) + \exp \left( -(\mu_3(t_i^r - s_i^r) - 1) \Lambda_3^{a,s} \left( \frac{1}{\mu_3} \left( \frac{1}{1 + \epsilon/3\mu_3} \right) \right) \right) + \frac{C'}{r^3} \right\}. \tag{8.116}
$$

Since $n_i^r = O(r^2)$, provided that $c$ is large enough, in particular

$$c > \max \left( \frac{20(\mu_3 + 2\epsilon)\lambda_3}{\ell_0\epsilon \gamma (\lambda_2 - 2\epsilon)c_1}, \frac{10\lambda_i}{\ell_0\epsilon c_1}, i = 1, 2; \frac{10\lambda_2}{\ell_0\epsilon(1 - \gamma)c_1} \right),$$

and in addition $c$ satisfies (B.3) and (B.11) with $c_1$ in place of $c$ (recall that $L_i^r = [c \log r] + 1$), and that $r$ is large enough, (8.116) tends to 0 as $r \to \infty$. \qed
For the proof of Theorem 8.6.1, we will need the following result, which is proved in the next section. In the \( r^{th} \) network, let \( \bar{T}_{r}^{*} \) be the fluid scaled allocation process under the threshold policy.

**Lemma 8.6.2.** For the fluid scaled allocation processes, \( \bar{T}_{j}^{r*}, \) for \( j = 1, \ldots, 5, \) we have,

\[
\bar{T}_{j}^{r*} \xrightarrow{\text{ as } r \to \infty} \bar{T}^{*},
\]

(8.117)

where

\[
\bar{T}^{*}(t) = \left( t, \frac{\lambda_{1} - \mu_{1}}{\mu_{2}}, \frac{\lambda_{2}}{\mu_{3}}, t, 0 \right), \quad t \geq 0.
\]

(8.118)

**Proof of Theorem 8.6.1.** Fix \( t \geq 0 \) and let \( \epsilon > 0 \) be given. According to Lemma 8.6.1 and properties of \( L_{r}^{1}, \) there exists an \( r_{\epsilon} \geq 1 \) such that for all \( r \geq r_{\epsilon}, \) \( 2L_{r}^{1}/r < \epsilon \) and

\[
P(I_{r}^{2}(\sigma^{r}_{0}) \geq r\epsilon) < \epsilon,
\]

(8.119)

\[
P\left( \sup_{\sigma^{r}_{0} \leq s \leq r^{2}t} |R_{1}^{r}(s)| \geq L_{1}^{r} - 2 \right) < \epsilon.
\]

(8.120)

Note that under the threshold policy, server 2 is idle only if both buffer 1 and buffer 2 are empty and so \( I_{r}^{2} \) can increase only if \( Q_{1}^{r} \leq 1. \) In particular, if \( I_{r}^{2} \) increases in \( (\sigma^{r}_{0}, r^{2}t], \) that is \( I_{r}^{2}(r^{2}t) - I_{r}^{2}(\sigma^{r}_{0}) > 0 \) then \( R_{1}^{r}(s) = Q_{1}^{r}(s) - L_{1}^{r} \leq -(L_{1}^{r} - 2) \) for some \( s \in [\sigma^{r}_{0}, r^{2}t]. \) We have for all \( r \geq r_{\epsilon}, \)

\[
P \left( ||\hat{Q}_{1}^{r}||_{t} \geq \epsilon \text{ or } ||\hat{I}_{2}^{r}||_{t} \geq \epsilon \right)
\]

\[
= P \left( ||\hat{Q}_{1}^{r}||_{r^{2}t} \geq r\epsilon \text{ or } ||\hat{I}_{2}^{r}||_{r^{2}t} \geq r\epsilon \right)
\]

\[
\leq P \left( \sup_{\sigma^{r}_{0} \leq s \leq r^{2}t} Q_{1}^{r}(s) \geq 2L_{1}^{r} \text{ or } \inf_{\sigma^{r}_{0} \leq s \leq r^{2}t} Q_{1}^{r}(s) \leq 1 \text{ or } I_{r}^{2}(\sigma^{r}_{0}) \geq r\epsilon \right)
\]

\[
\leq P \left( \sup_{\sigma^{r}_{0} \leq s \leq r^{2}t} |R_{1}^{r}(s)| \geq L_{1}^{r} - 2 \right) + P \left( I_{r}^{2}(\sigma^{r}_{0}) \geq r\epsilon \right) < 2\epsilon.
\]

(8.121)

It follows from this that \( (\hat{Q}_{1}^{r}, \hat{I}_{2}^{r}) \Rightarrow 0 \) as \( r \to \infty. \) Next, we proceed to show that \( \hat{I}_{1}^{r} \Rightarrow 0 \) as \( r \to \infty. \) We have from (8.42),(8.43) and (8.34),

\[
\hat{W}_{1}^{r}(t) = \tilde{\xi}_{1}^{r}(t) + \mu_{r}^{\beta} \hat{I}_{2}^{r}(t) + \mu_{r}^{\beta} (\hat{I}_{1}^{r}(t) - \hat{Y}_{0}^{r}(t)),
\]

\[
\hat{W}_{2}^{r}(t) = \tilde{\xi}_{2}^{r}(t) + \hat{I}_{2}^{r}(t) + (\mu_{r}^{\beta}/\mu_{r}^{\beta}) \hat{Y}_{0}^{r}(t).
\]

(8.122)

By Lemma 8.6.2, the heavy traffic conditions in Assumption 8.1.1, the functional central limit theorem, assumption (3.24), \( (\tilde{\xi}_{1}^{r}, \tilde{\xi}_{2}^{r}) \Rightarrow (\tilde{\xi}_{1}, \tilde{\xi}_{2}) \) as \( r \to \infty \) where \( \tilde{\xi} \) has the properties described
in Definition 4.2.3 (iii). Also, as above, \((\hat{Q}_1^r, \hat{I}_2^r) \implies (0, 0)\) as \(r \to \infty\). Under the threshold policy there are two ways that \(I_1^r\) can increase. Either class 1 and class 2 levels are at or below their thresholds and class 3 is nearly empty (at or below level one), or class 1 is nearly empty (at or below level one) and class 2 is above its threshold. This leads to a description of \(\hat{I}^r\) such that

\[
\hat{I}_1^r(t) = \hat{I}_1^{r,1}(t) + \hat{I}_1^{r,\dagger}(t),
\]

where

\[
\hat{I}_1^{r,1}(u) \leq \int_0^u 1_{\{Q_1^r(s) \leq L_1^r/r, \hat{Q}_2^r(s) \leq L_2^r/r, \hat{Q}_3^r(s) \leq 1/r\}} \, ds,
\]

and

\[
\hat{I}_1^{r,\dagger}(u) \leq \int_0^u 1_{\{Q_1^r(s) \leq 1/r, \hat{Q}_2^r(s) > L_2^r/r\}} \, ds.
\]

It follows from the proof of Lemma 8.6.1 (in particular by (8.78),(8.82),(8.85),(8.86)-(8.111)), that \(\hat{I}_1^{r,\dagger} \Rightarrow 0\) as \(r \to \infty\). Then,

\[
\hat{W}^r(t) = \hat{\xi}^r(t) + \begin{pmatrix} 1 & 0 \end{pmatrix} \left( \mu_2^r \hat{P}_{2}^r(t) + \mu_1^r \hat{I}_1^{r,\dagger}(t) \right)
\]

\[
+ \begin{pmatrix} \mu_1^r & -\mu_2^r/\mu_4^r \\ -\mu_2^r/\mu_4^r & \mu_4^r \end{pmatrix} \hat{V}_{1}^r(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{V}_{2}^r(t),
\]

where \(\hat{V}_{1}^r(t) = \hat{I}_1^{r,\dagger}(t) - \hat{\gamma}_{1}^r(t), \hat{V}_{2}^r(t) = \hat{I}_2^r(t) + (\mu_2^r/\mu_4^r) \hat{I}_1^{r,\dagger}(t)\). Under the threshold policy \(\hat{V}_{1}^r\) can increase only if \(\hat{W}_{1}^r \leq (L_1^r + (\mu_2^r/\mu_4^r)L_2^r)/r\) and \(\hat{V}_{2}^r\) can increase only if \(\hat{W}_{2}^r \leq 1/\mu_4^r r\) (We have used the fact that \(-\hat{\gamma}_{1}^r\) can increase only when \(Q_1^r \leq L_1^r/r\) and \(Q_2^r \leq L_2^r/r\)). Therefore, we have that

\[
\hat{V}_{1}^r(t) = \int_0^t 1_{\{W_1^r(s) \leq (L_1^r + (\mu_2^r/\mu_4^r)L_2^r)/r\}} \, d\hat{V}_{1}^r(s),
\]

\[
\hat{V}_{2}^r(t) = \int_0^t 1_{\{W_2^r(s) \leq 1/\mu_4^r r\}} \, d\hat{V}_{2}^r(s).
\]

Since \(\mu_2^r \hat{P}_{2}^r + \mu_1^r \hat{I}_1^{r,\dagger} \Rightarrow 0\) as \(r \to \infty\),

\[
\hat{G}^r = \begin{pmatrix} \mu_1^r & 0 \\ -\mu_2^r/\mu_4^r & \mu_4^r \end{pmatrix} \to \hat{G},
\]

as \(r \to \infty\) where \(\hat{G}\) is a non-singular \(M\)-matrix, i.e., of Harrison-Reiman type [23], and \(L_1^r/r, L_2^r/r \to 0\) as \(r \to \infty\), by Theorems 4.2 and 4.3 in [28], \(\{(\hat{W}^r, \hat{\xi}^r, \hat{V}^r)\}\) is \(C\)-tight and any
weak limit point of this sequence, \((\tilde{W}, \tilde{\xi}, \tilde{V})\), defines an SRBM with data \((\mathbf{R}_+^2, M\theta, M\Sigma M', \tilde{G})\), starting at the origin. In particular, \(\tilde{W}\) is an \(\{\mathcal{F}_t\}\)-adapted, 2-dimensional process defined on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})\) such that \(\mathbf{P}\)-a.s.: 

(i) \(\tilde{W}\) has continuous paths, \(\tilde{W}(t) = \bar{\xi}(t) + GV(t) \in \mathbf{R}_+^2\), for all \(t \geq 0\),

(ii) \(\tilde{\xi}\) is a 2-dimensional, \(\{\mathcal{F}_t\}\)-Brownian motion with drift \(M\theta\) and covariance \(M\Sigma M'\), starting at the origin,

(iii) \(\tilde{V}\) is an \(\{\mathcal{F}_t\}\)-adapted, 2-dimensional, continuous and non-decreasing process with \(\tilde{V}_j(0) = 0\) for \(j = 1, 2\). In addition, \(\tilde{V}_j\) can increase only if \(\tilde{W}_j = 0\), i.e.,

\[
\tilde{V}_j(t) = \int_0^t 1_{\{\tilde{W}_j(s) = 0\}} d\tilde{V}_j(s), \quad t \geq 0.
\]

Since \(\tilde{G}\), given by (8.15) is a matrix of Harrison-Reiman type [23], the law of \((\tilde{W}, \tilde{\xi}, \tilde{V})\) is unique and so in fact \((\tilde{W}^r, \tilde{\xi}^r, \tilde{V}^r) \Rightarrow (\hat{W}^*, \hat{\xi}, \hat{V}^*)\) satisfying (8.17)-(8.19). By Theorem 1 in [35], we have that \(\mathbf{P}\)-a.s.,

\[
\int_0^\infty 1_{\{\tilde{W}^*_1(s) = 0, \tilde{W}^*_2(s) = 0\}} d\tilde{U}^*_j(s) = 0, \quad \text{for } j = 1, 2. \tag{8.130}
\]

For each \(\epsilon > 0\) there exists a continuous function \(f^r : \mathbf{R}_+^2 \mapsto \mathbf{R}_+\), such that for \(z \in \mathbf{R}_+^2\), \(f^r(z) = 1\) if \(|z| \leq \epsilon\), \(f^r(z) = 0\) if \(|z| \geq 2\epsilon\) and \(0 < f^r(z) < 1\) if \(\epsilon < |z| < 2\epsilon\). Let \(r_\epsilon \geq 1\) be such that for \(r \geq r_\epsilon\), \((L_1^r + (\mu_2^r/\mu_3^r)L_2^r + 1/\mu_4^r)/r < \epsilon/2\). Then, for each \(t \geq 0\), for \(r \geq r_\epsilon\),

\[
0 \leq \tilde{f}^{-1}_1(t) \leq \int_0^t 1_{\{\tilde{W}^*_1(s) \leq (L_1^r + (\mu_2^r/\mu_3^r)L_2^r)/r, \tilde{W}^*_2(s) \leq 1/(\mu_4^r)\}} d\tilde{U}^*_1(s) \\
\leq \int_0^t 1_{\{\tilde{W}^*_1(s) \leq (L_1^r + (\mu_2^r/\mu_3^r)L_2^r)/r, \tilde{W}^*_2(s) \leq 1/(\mu_4^r)\}} d\tilde{V}^*_1(s) \\
\leq \int_0^t f^r(\tilde{W}^r(s)) d\tilde{V}^*_1(s),
\]

where we have used (8.124), (8.128), (8.40) and the choice of \(r_\epsilon\). Since \(\tilde{V}^*_1\) is non-decreasing and \((\tilde{W}^r, \tilde{V}^*_1) \Rightarrow (\hat{W}^*, \hat{V}^*_1)\) as \(r \to \infty\), by appealing to the Skorokhod representation theorem and the real analysis Lemma A.4 in [28], it follows that as \(r \to \infty\),

\[
\int_0^t f^r(\tilde{W}^r(s)) d\tilde{V}^*_1(s) \Rightarrow \int_0^t f^r(\hat{W}^*(s)) d\hat{V}^*_1(s). \tag{8.131}
\]

As \(\epsilon \to 0\),

\[
\int_0^t f^r(\tilde{W}^r(s)) d\tilde{U}^*_1(s) \overset{a.s.}{\Rightarrow} \int_0^t 1_{\{\tilde{W}^*_1(s)=0, \tilde{W}^*_2(s)=0\}} d\tilde{U}^*_1(s) = 0. \tag{8.132}
\]

For any \(\eta > 0\), there exists \(\epsilon = \epsilon(\eta)\) such that

\[
\mathbf{P}\left(\int_0^t f^r(\hat{W}^*(s)) d\hat{U}^*_1(s) > \eta\right) < \eta. \tag{8.133}
\]
For $r$ sufficiently large and all, but perhaps countably many $\eta$,
\[
\mathbb{P}\left(\int_0^t f^*(\hat{W}^r(s))d\hat{V}_1^r(s) > 2\eta\right) \leq \mathbb{P}\left(\int_0^t f^*(\hat{W}^*(s))d\hat{U}_1^*(s) > \eta\right) + \eta < 2\eta, \tag{8.134}
\]
and so
\[
\mathbb{P}\left(\hat{I}_1^{r\uparrow}(t) \leq 2\eta\right) \geq \mathbb{P}\left(\int_0^t f^*(\hat{W}^r(s))d\hat{V}_1^r(s) \leq 2\eta\right) \geq 1 - 2\eta. \tag{8.135}
\]
Since $\eta > 0$ was arbitrary (except possibly for a countable set) and $\hat{I}_1^{r\uparrow}$ is non-decreasing, it follows that $\hat{I}_1^{r\uparrow}$ converges to the zero process in probability as $r \to \infty$ and therefore, $\hat{I}_1^r \Rightarrow 0$. We conclude that as $r \to \infty$,
\[
\hat{Y}_s^r \Rightarrow -\hat{U}_1^*, \hat{I}_3^r \Rightarrow \hat{U}_2^*.
\]

The following corollary follows from the proofs of Lemmas 8.6.1 and 8.6.2 and Theorem 8.6.1.

**Corollary 8.6.1.** Consider the sequence of parallel server systems indexed by $r$, where the $r$th system operates according to the threshold policy described in Section 8.4. Then,
\[
(Q^r, \hat{W}^r, \hat{Y}^r, \hat{I}^r) \Rightarrow (\hat{Q}^*, \hat{W}^*, \hat{Y}^*, \hat{I}^*), \tag{8.136}
\]
where the members on the right are given by (8.17)-(8.23).

### 8.7 Weak Convergence under the Threshold Policy

In this section we prove Lemma 8.6.2. The proof below is very similar to the proof of Lemma 8.1 in [3].

**Proof of Lemma 8.6.2.** For all $t \geq 0$, let
\[
\begin{align*}
\hat{I}_1^r(t) &\equiv r^{-1}\hat{I}_1^r(t) = t - \hat{T}_1^r(t) - \hat{T}_3^r(t), \\
\hat{I}_2^r(t) &\equiv r^{-1}\hat{I}_2^r(t) = t - \hat{T}_2^r(t) - \hat{T}_3^r(t), \\
\hat{I}_3^r(t) &\equiv r^{-1}\hat{I}_3^r(t) = t - \hat{T}_4^r(t), \\
\hat{Q}_1^r(t) &\equiv r^{-1}\hat{Q}_1^r(t) - r^{-1}\hat{S}_1^r(T_1^r(t)) - r^{-1}\hat{S}_2^r(T_2^r(t)) + \lambda_1^r t - \mu_1^r \hat{T}_1^r(t) - \mu_2^r \hat{T}_2^r(t), \\
\hat{Q}_2^r(t) &\equiv r^{-1}\hat{Q}_2^r(t) - r^{-1}\hat{S}_3^r(T_3^r(t)) + \lambda_2^r t - \mu_2^r \hat{T}_3^r(t), \\
\hat{Q}_3^r(t) &\equiv r^{-1}\hat{Q}_3^r(t) - r^{-1}\hat{S}_4^r(T_4^r(t)) + r^{-1}\hat{S}_5^r(T_5^r(t)) + \lambda_3^r t - \mu_3^r \hat{T}_4^r(t) - \mu_5^r \hat{T}_5^r(t).
\end{align*}
\]
By the functional central limit theorem,
\[ r^{-1} \hat{A}_r^*(\cdot) \rightarrow 0 \quad \text{for} \quad i = 1, 2, 3 \quad \text{as} \quad r \rightarrow \infty, \quad (8.138) \]
\[ r^{-1} \hat{S}_r^*(\cdot) \rightarrow 0 \quad \text{for} \quad j = 1, \ldots, 5 \quad \text{as} \quad r \rightarrow \infty. \quad (8.139) \]

Since \( \tilde{T}_j^{*,r}(t) \leq t \) for each \( t \geq 0 \) and all \( j \),
\[ \sup_{0 \leq s \leq t} |\tilde{S}_j^*(\tilde{T}_j^{*,r}(s))| \leq \sup_{0 \leq s \leq t} |\tilde{S}_j^*(s)|, \quad \text{for all} \quad t \geq 0 \quad \text{and all} \quad j. \quad (8.140) \]

From this it follows that
\[ r^{-1} \tilde{S}_j^*(\tilde{T}_j^{*,r}(\cdot)) \rightarrow 0, \quad \text{for} \quad j = 1, \ldots, 5. \quad (8.141) \]

Recall that \( \hat{X}_1^r, \hat{X}_2^r, \hat{X}_3^r \) are defined by (8.37)-(8.39). Then,
\[ (\hat{X}_1^r, \hat{X}_2^r, \hat{X}_3^r) = \left( r^{-1} \hat{X}_1^r(\cdot), r^{-1} \hat{X}_2^r(\cdot), r^{-1} \hat{X}_3^r(\cdot) \right) \rightarrow (0, 0, 0), \quad (8.142) \]
as \( r \rightarrow \infty \), and therefore
\[ \left( r^{-1} \hat{\xi}_1^*(\cdot), r^{-1} \hat{\xi}_2^*(\cdot) \right) \rightarrow (0, 0), \quad \text{as} \quad r \rightarrow \infty. \quad (8.143) \]

The fact that \( \tilde{T}_j^{*,r} \) are Lipschitz continuous with Lipshitz constant less than or equal to one, implies that \( \tilde{T}_j^{*,r} = (\tilde{T}_1^{*,r}, \tilde{T}_2^{*,r}, \tilde{T}_3^{*,r}, \tilde{T}_4^{*,r}, \tilde{T}_5^{*,r}) \) is tight. By Lemma 8.6.1, \( (\hat{Q}_1^r, \hat{I}_2^r) \Rightarrow (0, 0) \) and therefore, \( (\hat{Q}_1^r, \hat{I}_2^r) \Rightarrow (0, 0) \). Combining the last two statements with Assumption 8.1.1 and letting \( r \) go to infinity along a subsequence indexed by \( r' \) we obtain a.s., for all \( t \geq 0, \)
\[ \tilde{I}_1^r(t) = t - \tilde{T}_1^r(t) - \tilde{T}_5^r(t), \quad (8.144) \]
\[ \tilde{I}_2^r(t) = t - \tilde{T}_2^r(t) - \tilde{T}_3^r(t) = 0, \quad (8.145) \]
\[ \tilde{I}_3^r(t) = t - \tilde{T}_4^r(t), \quad (8.146) \]
\[ \hat{Q}_1^r(t) = \lambda_1 t - \mu_1 \tilde{T}_1^r(t) - \mu_2 \tilde{T}_2^r(t) = 0, \quad (8.147) \]
\[ \hat{Q}_2^r(t) = \lambda_2 t - \mu_3 \tilde{T}_3^r(t), \quad (8.148) \]
\[ \hat{Q}_3^r(t) = \lambda_3 t - \mu_4 \tilde{T}_4^r(t) - \mu_5 \tilde{T}_5^r(t), \quad (8.149) \]

where \( \tilde{T} = (\tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4, \tilde{T}_5) \) is a weak limit of \( \tilde{T}_j^{*,r} \) along \( r' \), \( \hat{Q} = (\hat{Q}_1^r, \hat{Q}_2^r, \hat{Q}_3^r) \) is a weak limit of \( \hat{Q}_r^r \) along \( r' \) and \( \tilde{I} = (\tilde{I}_1^r, \tilde{I}_2^r, \tilde{I}_3^r) \) is a weak limit of \( \tilde{I}_r^r \) along \( r' \). Let \( \tilde{W}^r = M \hat{Q} \). In the limit we have that \( \tilde{W} = M \hat{Q} \) satisfies
\[ \tilde{W}_1^r(t) = \hat{Q}_1^r(t) + (\mu_2/\mu_3) \hat{Q}_2^r(t) \]
\[ = \lambda_1 t - \mu_1 \tilde{T}_1^r(t) - \mu_2 \tilde{T}_2^r(t) + \frac{\mu_2 \lambda_2}{\mu_3} t \]
\[ = \lambda_1 t - \mu_1 \tilde{T}_1^r(t) + \frac{\mu_2 \lambda_2}{\mu_3} t - \mu_2 t + \mu_1 \tilde{I}_1^r(t) + \mu_1 \tilde{T}_5^r(t) + \mu_2 \tilde{I}_2^r(t) \]
\[ = \mu_1 \tilde{I}_1^r(t) + \mu_1 \tilde{T}_5^r(t), \]
where we have used Assumption 8.1.1 and the fact that $\bar{I}_2 \equiv 0$. Now, $\bar{W}_1 \geq 0$ since $\bar{W}_1^r \geq 0$ for all $r$ and this property is preserved in the limit. On the other hand, since $\mu^*_1 \bar{I}_1 + \mu^*_1 \bar{T}_5$ is non-negative, continuous, non-decreasing with continuous, non-decreasing with $I$ that $\bar{W}_1 \equiv 0$ and $\bar{W}_1 \equiv 0$, we have used Assumption 8.1.1 and the fact that for all $r$, $\bar{W}_1 \equiv 0$ and the property is preserved in the limit. On the other hand, since $\bar{W}_1 \equiv 0$ and this property is preserved in the limit. Also, since $\bar{I}_1 \equiv 0$ and $\bar{T}_5 \equiv 0$, it follows that $\bar{I}_1 \equiv 0$ and $\bar{T}_5 \equiv 0$. Hence, for each $t \geq 0$, $\bar{T}_5(t) = \bar{T}_5^*(t) = 0$. Substituting 0 for $\bar{T}_5(t)$ and $\bar{I}_1(t)$ in (8.144) yields

$$0 = t - \bar{T}_1(t), \quad t \geq 0,$$

(8.150)

from which it follows that $\bar{T}_1(t) = \bar{T}_1^*(t) = 0$ for all $t \geq 0$. This combined with (8.147) gives

$$\bar{T}_2(t) = (\lambda_1 - \mu_1)t/\mu_2 = \bar{T}_2^*(t),$$

for all $t \geq 0$. Substituting $(\lambda_1 - \mu_1)t/\mu_2$ for $\bar{T}_2(t)$ in equation (8.145) and using Assumption 8.1.1 yields that $\bar{T}_3(t) = (\lambda_2/\mu_3)t = \bar{T}_3^*(t)$ for all $t \geq 0$. Now, for each $t \geq 0$,

$$\bar{W}_2(t) = (1/\mu_4)\bar{Q}_3(t) = \bar{I}_3(t),$$

(8.151)

where we have used the fact that $\lambda_3 = \mu_4$ and the fact that $\bar{T}_5 \equiv 0$. Again, $\bar{W}_2 \geq 0$ since $\bar{W}_2^r \geq 0$ for each $r$ and this property is preserved in the limit. Also, since $\bar{I}_3$ is non-negative, continuous, non-decreasing with $\bar{I}_3(0) = 0$ and $\bar{I}_3^*$ can increase only if $\bar{W}_2^r \leq 1/r$ it follows that $\bar{I}_3$ is non-negative, continuous, non-decreasing with $\bar{I}_3(0) = 0$ and that $\bar{I}_3$ can increase only if $\bar{W}_2 = 0$. Again, since $(\bar{W}_2, \bar{I}_3)$ is the solution of the Skorokhod problem with zero driving process, we conclude that,

$$0 \equiv \bar{W}_2 = \bar{I}_3,$$

(8.152)

This combined with (8.146) yields that $\bar{T}_4(t) = t = \bar{T}_4^*(t)$ for all $t \geq 0$. It follows that $\bar{T} = \bar{T}^*$, and since $\bar{T}$ was an arbitrary weak limit point of $\bar{T}^r$ we conclude that $\bar{T}^r \Rightarrow \bar{T}^*$.

The proof of the following theorem is very similar to the proof of Theorem 5.2 in [3].

**Theorem 8.7.1.** Consider the sequence of parallel server systems indexed by $r$, where the $r^{th}$ system operates according to the threshold policy described in Section 8.4. Then the associated normalized queue-length and idle-time processes satisfy

$$(\bar{Q}_1^r, \bar{Q}_2^r, \bar{Q}_3^r, \bar{I}_1^r, \bar{I}_2^r, \bar{I}_3^r) \Rightarrow (0, \bar{Q}_2^r, \bar{Q}_3^r, 0, 0, \bar{I}_3^r), \quad as \quad r \to \infty,$$

(8.153)
where \( \tilde{Q}_2^*, \tilde{Q}_3^* \) are as in (8.20).

Proof. The result is already established for \( \hat{Q}_1^r, \hat{I}_1^r, \hat{I}_2^r, \hat{I}_3^r \) in Theorem 8.6.1 and Corollary 8.6.1. By the Corollary 8.6.1, as \( r \to \infty \),

\[
\hat{Q}_1^r + (\mu_2^r/\mu_3^r)\hat{Q}_2^r = \hat{W}_1^* \Rightarrow \tilde{W}_1^*
\]  

(8.154)

and therefore,

\[
\hat{Q}_2^r \Rightarrow \hat{Q}_2^* = (\mu_3^r/\mu_2^r)\hat{W}_1^* \text{, as } r \to \infty,
\]  

(8.155)

where we have used (8.20) and Assumption 8.1.1. Also, by Corollary 8.6.1, as \( r \to \infty \),

\[
(1/\mu_4^r)\hat{Q}_3^r = \hat{W}_2^* \Rightarrow \tilde{W}_2^*
\]  

(8.156)

and therefore,

\[
\hat{Q}_3^r \Rightarrow \hat{Q}_3^* = \mu_4\hat{W}_2^* \text{, as } r \to \infty,
\]  

(8.157)

where we have used (8.20) and Assumption 8.1.1. This completes the proof. \( \square \)

8.8 Asymptotic Optimality

The purpose of this section is to establish the asymptotic optimality of the threshold policy described in Section 8.4. In this section, \( T = \{T^r\} \) will be any sequence of scheduling control policies (one for each member of the sequence of parallel server systems). The associated queue-length and idle-time processes will be denoted by \( Q^r, I^r \) and the fluid and diffusion scaled versions of these processes will be denoted by \( \bar{Q}^r, \bar{I}^r \) and \( \hat{Q}^r, \hat{I}^r \), respectively. We also let

\[
\hat{J}(T) = \liminf_{r \to \infty} \hat{J}^r(T^r).
\]

(8.158)

where \( \hat{J}^r(T^r) \) is defined by (3.15). When the sequence of threshold policies \( \{T^{r,*}\} \) is used, we append a superscript * to the queue-length, idle-time processes etc., e.g., \( Q^{r,*}, I^{r,*} \), etc..

Definition 8.8.1. (\( C \)-tightness) In the following, a sequence of processes with paths in \( D^m \) for some \( m \geq 1 \) is called \( C \)-tight if it is tight in \( D^m \) and any weak limit point of the sequence (obtained as a weak limit along a subsequence) has continuous paths almost surely.

The following two propositions are very similar to Lemmas 9.2 and 9.3 in [3]. We prove them here for the sake of completeness. For \( t \geq 0 \), let

\[
\bar{A}^r(t) = \frac{1}{r^2} A^r(r^2 t), \bar{S}^r(t) = \frac{1}{r^2} A^r(r^2 t).
\]

(8.159)
Proposition 8.8.1. Let \( \{T^r\} \) be any sequence of scheduling control policies (one for each member of the parallel server systems). Then
\[
\left\{ \left( \bar{Q}^r, \bar{A}^r, \bar{S}^r, \bar{T}^r, \bar{I}^r \right) \right\}
\] (8.160)
is \( C \)-tight.

Proof. From (8.138) and (8.139) it follows that,
\[
(\bar{A}^r(\cdot), \bar{S}^r(\cdot)) \Rightarrow (\lambda(\cdot), \mu(\cdot)) \quad \text{as} \quad r \to \infty,
\] (8.161)
where \( \lambda(t) = \lambda t \) and \( \mu(t) = \mu t \) for all \( t \geq 0 \). Since they represent cumulative allocations of time, each of the five components of \( T^r \) is uniformly Lipschitz continuous with a Lipschitz constant less than or equal to 1 and this property is preserved by the fluid scaled processes \( \bar{T}^r \). It follows immediately from this and (8.161) that \( \left\{ \left( \bar{A}^r, \bar{S}^r, \bar{T}^r \right) \right\} \) is \( C \)-tight (cf. Theorem 15.5 in [6]). From equations (3.5) and (3.6) for queue-length and idle-time we have for all \( t \geq 0 \),
\[
\begin{align*}
\bar{Q}^r_1(t) &= \bar{A}^r_1(t) - \bar{S}^r_1(\bar{T}^r_1(t)) - \bar{S}^r_2(\bar{T}^r_2(t)), \\
\bar{Q}^r_2(t) &= \bar{A}^r_2(t) - \bar{S}^r_3(\bar{T}^r_3(t)), \\
\bar{Q}^r_3(t) &= \bar{A}^r_3(t) - \bar{S}^r_3(\bar{T}^r_3(t)) - \bar{S}^r_5(\bar{T}^r_5(t)), \\
\bar{I}^r_1(t) &= t - \bar{T}^r_1(t) - \bar{T}^r_5(t), \\
\bar{I}^r_2(t) &= t - \bar{T}^r_2(t) - \bar{T}^r_3(t), \\
\bar{I}^r_3(t) &= t - \bar{T}^r_4(t).
\end{align*}
\] (8.162–8.167)
Combining these with the \( C \)-tightness established above and a random time change theorem (cf. [6], page 145) yields the desired result. \( \square \)

Proposition 8.8.2. Let \( T = \{T^r\} \) be a sequence of scheduling control policies such that \( J(T) < \infty \). Consider a subsequence \( \{T^{r'}\} \) of \( \{T^r\} \) along which the \( \liminf \) in the definition of \( J(T) \) is achieved; that is
\[
\lim_{r' \to \infty} \hat{J}^{r'}(T^{r'}) = J(T).
\] (8.168)

Then,
\[
(\bar{Q}^{r'}(\cdot), \bar{A}^{r'}(\cdot), \bar{S}^{r'}(\cdot), \bar{T}^{r'}(\cdot), \bar{I}^{r'}(\cdot)) \Rightarrow (0, \lambda(\cdot), \mu(\cdot), \bar{T}^*(\cdot), 0) \quad \text{as} \quad r' \to \infty,
\] (8.169)
where \( T^* \) is defined in (8.118), \( 0 \) denotes the constant process that stays at the origin in \( \mathbb{R}^4_+ \), and \( \lambda(t) = \lambda t, \mu(t) = \mu t \) for all \( t \geq 0 \).
Proof. From Proposition 8.8.1 it follows that
\[
\{(\bar{Q}^r, \bar{A}^r, \bar{S}^r, \bar{T}^r, \bar{I}^r)\}
\] (8.170)
is C-tight. Thus, it suffices to show that all weak limit points of this sequence are given by the right member of (8.169). For this, suppose that
\[
(\bar{Q}, \bar{A}, \bar{S}, \bar{T}, \bar{I})
\] (8.171)
is obtained as a weak limit of (8.170) along a subsequence indexed by \( r'' \). Without loss of generality, by appealing to the Skorokhod representation theorem (cf. [13], Theorem 3.1.8), we may choose an equivalent distributional representation (for which we use the same symbols) such that all of the random processes in (8.170) indexed by \( r \), as well as the limit (8.171), are defined on the same probability space and the convergence in distribution is replaced by almost sure convergence on compact time intervals, so that a.s.
\[
(\bar{Q}^{r''}, \bar{A}^{r''}, \bar{S}^{r''}, \bar{T}^{r''}, \bar{I}^{r''}) \to (\bar{Q}, \bar{A}, \bar{S}, \bar{T}, \bar{I}),
\] (8.172)
u.o.c. as \( r'' \to \infty \). From (8.161) we have that a.s., \( \bar{A}(\cdot) = \lambda(\cdot) \) and \( \bar{S}(\cdot) = \mu(\cdot) \). Next, we show that a.s., \( \bar{Q} \equiv 0 \). Combining the fact that \( \lim_{r'' \to \infty} \hat{J}^{r''}(T^{r''}) = J(T) < \infty \), with (8.172) and Fatou’s lemma, we have
\[
0 = \inf_{r'' \to \infty} \frac{1}{r''} \hat{J}^{r''}(T^{r''}) \geq \mathbb{E} \left( \int_0^\infty e^{-\gamma t} \liminf_{r'' \to \infty} \left( h \cdot \bar{Q}^{r''}(t) \right) \right) = \mathbb{E} \left( \int_0^\infty e^{-\gamma t} h \cdot \bar{Q}(t) dt \right).
\] (8.173)
Since \( h_i > 0 \) for \( i = 1, 2, 3 \), and a.s., \( \bar{Q} \) has continuous path in \( \mathbb{R}_+^3 \), it follows from the above that a.s., \( \bar{Q} \equiv 0 \). By letting \( r'' \to \infty \) in (8.162)–(8.167) and using (8.161), (8.172), (8.173), we have a.s., for each \( t \geq 0 \)
\[
0 = \lambda_1 t - \mu_1 \bar{T}_1(t) - \mu_2 \bar{T}_2(t),
\] (8.174)
\[
0 = \lambda_2 t - \mu_3 \bar{T}_3(t),
\] (8.175)
\[
0 = \lambda_3 t - \mu_4 \bar{T}_4(t) - \mu_5 \bar{T}_5(t),
\] (8.176)
\[
\bar{I}_1(t) = t - \bar{T}_1(t) - \bar{T}_5(t),
\] (8.177)
\[
\bar{I}_2(t) = t - \bar{T}_2(t) - \bar{T}_3(t),
\] (8.178)
\[
\bar{I}_3(t) = t - \bar{T}_4(t).
\] (8.179)
Multiplying (8.173) by 1, and adding to \( \mu_2/\mu_3 \) times (8.174), using Assumption 8.1.1 we obtain a.s.,

\[
\mu_1 \bar{I}_1 + \mu_2 \bar{T}_5 + \mu_2 \bar{I}_2 \equiv 0. \tag{8.180}
\]

Since a.s., \( \bar{I}_1, \bar{I}_2, \bar{T}_5 \) inherit the property that they are non-negative at all times for \( \bar{T}_1'', \bar{T}_2'', \bar{T}_5'' \), it follows that a.s.,

\[
\bar{I}_1 = \bar{I}_2 = \bar{T}_3 \equiv 0, \tag{8.181}
\]

and so by (8.174), (8.176), (8.177), a.s. for all \( t \geq 0, \bar{T}_1(t) = t = \bar{T}_1^*(t), \bar{T}_2(t) = (\lambda_2/\mu_3)t = \bar{T}_2^*(t) \) and

\[
\bar{T}_3(t) = \left( 1 - \frac{\lambda_2}{\mu_3} \right) t = \frac{\lambda_1 - \mu_1}{\mu_2} t = \bar{T}_3^*(t).
\tag{8.182}
\]

Then from (8.175), a.s. for all \( t \geq 0, \)

\[
\bar{T}_4(t) = \frac{\lambda_3}{\mu_4} t = \bar{T}_4^*(t),
\tag{8.183}
\]

and then by (8.178), \( \bar{I}_3 \equiv 0 \text{ a.s.} \)

\[\square\]

**Theorem 8.8.1.** Let \( \{T^r\} \) be any sequence of scheduling control policies (one for each member of the sequence of parallel server systems). Then

\[
\liminf_{r \to \infty} \hat{J}^r(T^r) \geq J^* = \lim_{r \to \infty} \hat{J}^r(T^r,*)
\tag{8.184}
\]

and \( J^* < \infty \).

**Proof.** Let \( T = \{T^r\} \) be any sequence of scheduling control policies. To prove the inequality in (8.184) without loss of generality we assume that \( J(T) < \infty \). Recall that

\[
\hat{W}^r = M^r \hat{Q}^r \quad \text{where} \quad M^r = \begin{pmatrix}
1 & \mu_2^r/\mu_3^r & 0 \\
0 & 0 & 1/\mu_4^r
\end{pmatrix}.
\]

Then by (8.42), (8.43) and (8.34),

\[
\hat{W}^r = \hat{\xi}^r + \bar{G}^r \hat{U}^r,
\tag{8.185}
\]

where

\[
\bar{G}^r = \begin{pmatrix}
\mu_1^r & 0 \\
-\mu_5^r/\mu_4^r & 1
\end{pmatrix}.
\]
and where
\[
\begin{align*}
\hat{U}_1 &= -\hat{Y}_1 + \hat{I}_1 + (\mu_2 / \mu_4) \hat{I}_2, \\
\hat{U}_2 &= \hat{I}_3 + (\mu_5 / \mu_4) \hat{I}_1 + (\mu_5 \mu_2 / \mu_3 \mu_4) \hat{I}_2.
\end{align*}
\]

For all \( r \) sufficiently large, \( \hat{G}^r \) is an \( S \)-matrix with exactly one positive element in each row and therefore by Appendix C, there exists a least control process, \( \Phi(\hat{\xi}^r) = (\hat{U}_1^r, \hat{U}_2^r) \) that goes with \( \hat{\xi}^r \) and \( \hat{G}^r \). Let \( \hat{W}^r, \hat{\xi}^r = \hat{G}^r \Phi(\hat{\xi}^r) \). Now, we have that a.s., \( \hat{U}_1^r \leq \hat{U}_1^r \) and \( \hat{U}_2^r \leq \hat{U}_2^r \). Also, for each \( t \geq 0 \),
\[
\begin{align*}
\hat{U}_1^r(t) &= -\left( \inf_{0 \leq s \leq t} \{ \hat{\xi}_1'(s) \} \right) \lor 0 \quad (8.186) \\
\hat{U}_2^r(t) &= -\left( \inf_{0 \leq s \leq t} \{ \hat{\xi}_2'(s) - (\mu_5 / \mu_4) \hat{U}_1^r(s) \} \right) \lor 0. \quad (8.187)
\end{align*}
\]

Recall the definition of \( \kappa \) in Section 8.3 and let
\[
\kappa_1^r = \min \left( h_1, h_2 \mu_3 / \mu_2 \right), \quad \kappa_5^r = h_3 \mu_4. \quad (8.188)
\]

Let \( r_1 \) be such that for all \( r \geq r_1 \), Assumption 8.3.1 holds with \( \mu_j^r \), in place of \( \mu_j \) for \( j = 1, \ldots, 5 \).

Then for \( r \geq r_1 \), by a similar argument as in the proof of Theorem 6.3.1, \( \kappa^r \cdot \hat{W}^r \geq \kappa^r \cdot \hat{W}^r, \hat{\xi}^r \).

Then, for each \( t \geq 0 \), for \( r \geq r_1 \),
\[
\begin{align*}
\hat{h} \cdot \hat{Q}^r(t) &= \left( h_1 \hat{Q}_1^r(t) + h_2 \frac{\mu_5^r \mu_2^r}{\mu_2^r \mu_3^r} \hat{Q}_2^r(t) \right) \lor \hat{h}_3 \hat{Q}_3^r(t) \\
&\geq \kappa_1^r \hat{W}_1^r(t) + \kappa_5^r \hat{W}_2^r(t) \quad (8.189) \\
&= \kappa^r \cdot \hat{W}^r \geq \kappa^r \cdot \hat{W}^r, \hat{\xi}^r.
\end{align*}
\]

Now, let \( \{r' \} \) be a subsequence of \( \{r \} \) such that \( \lim_{r' \to \infty} \hat{J}^r(T^{r'}) = J(T) \). Then, by Proposition 8.8.2 and (3.24), as \( r' \to \infty \)
\[
(\hat{A}^{r'}(\cdot), \hat{S}^{r'}(\cdot), T^{r'}(\cdot)) \implies (\hat{A}(\cdot), \hat{S}(\cdot), T^*(\cdot)). \quad (8.190)
\]

Using this, Assumption 8.1.1 and the continuous mapping theorem, it follows that for \( \hat{\xi}^r \) given by (8.37)-(8.39), \( \hat{\xi}^r = M^r \hat{\xi}^r, \hat{U}^r, \hat{\xi}^r \) given by (8.186)-(8.187), and \( \hat{W}^r, \hat{\xi}^r \) given by (8.185) with \( \hat{U} = \hat{U}^r, \hat{\xi}^r \), we have together with (8.190) that as \( r' \to \infty \),
\[
(\hat{X}^{r'}, \hat{\xi}^{r'}, \hat{W}^{r'}, \hat{U}^{r'}, \hat{\xi}^{r'}) \Rightarrow (\hat{X}, \hat{\xi}, \hat{W}^*, \hat{U}^*), \quad (8.191)
\]

where for \( t \geq 0 \),
\[
\begin{align*}
\hat{X}_1(t) &= \hat{A}_1(t) - \hat{S}_1(T_1^r(t)) - \hat{S}_2(T_2^r(t)) + \theta_1 t, \quad (8.192) \\
\hat{X}_2(t) &= \hat{A}_2(t) - \hat{S}_3(T_3^r(t)) + \theta_2 t, \quad (8.193) \\
\hat{X}_3(t) &= \hat{A}_3(t) - \hat{S}_4(T_4^r(t)) - \hat{S}_5(T_5^r(t)) + \theta_3 t. \quad (8.194)
\end{align*}
\]
\( \hat{\xi} = M \hat{X} \) and \((\hat{W}^*, \hat{U}^*)\) are given by (8.17)-(8.19). By invoking the Skorokhod representation theorem, we may assume that there is joint convergence, a.s. uniform on compact time intervals:

\[
(\hat{A}^{r'}(\cdot), \hat{S}^{r'}(\cdot), \hat{T}^{r'}(\cdot), \hat{X}^{r}(\cdot), \hat{\xi}^{r'}(\cdot), \hat{W}^{r} \cdot \hat{\xi}^{r'}(\cdot), \hat{U}^{r} \cdot \hat{\xi}^{r'}(\cdot)) \rightarrow (\hat{A}(\cdot), \hat{S}(\cdot), \hat{T}^{*}(\cdot), \hat{X}(\cdot), \hat{\xi}(\cdot), \hat{W}^{*}(\cdot), \hat{U}^{*}(\cdot)), \quad \text{as } r \rightarrow \infty. \quad (8.195)
\]

Then, by Fatou’s lemma

\[
\mathcal{J}(T) = \lim_{r' \to \infty} \hat{\mathcal{J}}^{r'}(T^{r'}) \geq \mathbb{E} \left( \int_0^\infty e^{-\gamma t} \liminf_{r' \to \infty} \left( h \cdot \hat{Q}^{r'}(t) \right) dt \right). \quad (8.196)
\]

Now we claim that a.s., for \( t \geq 0 \),

\[
\liminf_{r' \to \infty} \left( h \cdot \hat{Q}^{r'}(t) \right) \geq h \cdot \hat{Q}^{*}(t), \quad (8.197)
\]

where \( \hat{Q}^{*} \) is given by (8.20). To show this, we fix \( \omega \in \Omega \) and we assume that \( \omega \) is in the set of probability one where the convergence in (8.195) holds a.s. u.o.c., and we fix \( t > 0 \). Without loss of generality we may assume that the left member of the inequality above is finite and there exists a further subsequence \( r'' \) (possibly depending on \( \omega \) and \( t \)) such that

\[
\lim_{r'' \to \infty} \left( h \cdot \hat{Q}^{r''}(t, \omega) \right) = \liminf_{r' \to \infty} \left( h \cdot \hat{Q}^{r'}(t, \omega) \right) < \infty. \quad (8.198)
\]

Now, since \( h_i > 0 \) for \( i = 1, 2, 3 \) and \( \hat{Q}^{r''}_i(t, \omega) \geq 0 \) for \( i = 1, 2, 3 \), it follows that \( \hat{Q}^{r''}_i(t, \omega) \) is bounded as \( r'' \to \infty \) for \( i = 1, 2, 3 \). Then,

\[
\lim_{r'' \to \infty} \left( h \cdot \hat{Q}^{r''}(t, \omega) \right) \geq \lim_{r'' \to \infty} \kappa \cdot \hat{W}^{r''} \cdot \hat{\xi}^{r''}(t, \omega) = \kappa \cdot \hat{W}^{*}(t, \omega) = h \cdot \hat{Q}^{*}(t, \omega),
\]

where we have used (8.189). We conclude that

\[
\mathcal{J}(T) \geq \mathbb{E} \left( \int_0^\infty e^{-\gamma t} h \cdot \hat{Q}^{*}(t) dt \right) = J^{*}. \quad (8.199)
\]

This proves the inequality in (8.184) with this definition of \( J^{*} \). It remains to prove the equality on the right hand side. For this, we assume that the threshold policy \( \{T^{r''}\} \) is used in the \( r^{th} \) parallel server system. By Corollary 8.6.1 and again by appealing to the Skorokhod representation theorem, we may assume that

\[
\hat{Q}^{r''} \rightarrow \hat{Q}^{*} \ \text{u.o.c as } \ r \rightarrow \infty. \quad (8.200)
\]
Again, following the notation in [3], we let
\[ \hat{H}^{r,*} = h \cdot \hat{Q}^{r,*} \quad \text{and} \quad \hat{H}^* = h \cdot \hat{Q}^*. \] (8.201)

Then,
\[ \hat{H}^{r,*} \to \hat{H}^* \quad (m \times P) - \text{a.e.}, \] (8.202)

where \( dm = \gamma e^{-\gamma t} dt \). Since \((\mathbb{R}^+ \times \Omega, \mathcal{B}_+ \times \mathcal{F}, m \times P)\) is a probability space, to establish that
\[ \hat{J}^{r,t} = \mathbb{E} \left( \int_0^\infty e^{-\gamma t} \hat{H}^{r,*}(t)^2 dt \right) \to J^* < \infty \quad \text{as} \quad r \to \infty, \] (8.203)

it suffices to show that
\[ \limsup_{r \to \infty} \mathbb{E} \left( \int_0^\infty e^{-\gamma t} \hat{H}^{r,*}(t)^2 dt \right) < \infty, \] (8.204)

which implies the required uniform integrability. We have that
\[ \hat{H}^{r,*} = h \cdot \hat{Q}^{r,*} \leq \left( h_1 + \frac{h_2 \mu_5}{\mu_5^2} \right) \hat{W}^{r,*} + \mu_5^2 h_3 \hat{W}^{r,*}, \] (8.205)

where \( \hat{W}^{r,*} \) satisfies (8.42), (8.43). Under the threshold policy \( \hat{Y}^{r,*}_1 \) is non-decreasing and can increase only if \( \hat{Q}^{r,*}_1 \leq L_2^*/r \) and \( \hat{Q}^{r,*}_2 \leq L_1^*/r \). Also, under the threshold policy, \( \hat{I}^{r,*}_3 \) can increase only if \( \hat{Q}^{r,*}_3 \leq 1/r \). It follows from the oscillation inequality for perturbed Skorokhod problems [39], (8.34) and the fact that \( \hat{Y}^{r}_5 \) is decreasing, that for each \( t \geq 0 \),

\[ \mu_1 \hat{Y}^{r,*}_1(t) \leq \sup_{0 \leq s \leq t} \left| \hat{X}^{r,*}_1(s) + \left( \frac{\mu_2}{\mu_3^2} \right) \hat{X}^{r,*}_2(s) \right| + \mu_3^2 \hat{I}^{r,*}_2(t) + (L_1^* + L_2^*)/r, \]

\[ \hat{I}^{r,*}_3(t) \leq \sup_{0 \leq s \leq t} \left| \left( 1/\mu_4^2 \right) \hat{X}^{r,*}_3(s) \right| + \mu_4^2 \left( \hat{Y}^{r,*}_1(t) - \hat{I}^{r,*}_3(t) \right) + 1/r. \] (8.206)

Hence to establish the desired uniform integrability it suffices to show that as functions of \( t \) the following are all in a bounded subset of \( L^1(m) \equiv L^1(\mathbb{R}_+, \mathcal{B}_+, m) \) for all \( r \) sufficiently large:

\[ \mathbb{E} \left( \sup_{0 \leq s \leq t} (\hat{A}^i_r(s))^2 \right), \quad \mathbb{E} \left( \sup_{0 \leq s \leq t} (\hat{S}^j_r(\hat{I}^{r,*}_j(s)))^2 \right), \quad \mathbb{E} \left( (\hat{I}^{r,*}_k(t))^2 \right), \] (8.207)

for \( i = 1, 2, 3, j = 1, 2, 3, 4 \) and \( k = 2 \). We establish estimates to show this for the very last expectation in (8.207). The estimates for the first two expectations are nearly identical to
estimates which have been established in [3] and for that reason are not included. We have that,

\[
E((\hat{I}_2^r(t))^2) = \int_0^\infty P((\hat{I}_2^r(t))^2 > s) ds = \int_0^{r^2t} P(I_2^r(r^2t) > r\sqrt{s}) ds
\]

\[
\leq \int_0^{r^2t} \left\{ P(I_2^r(\sigma^*_0) > r\sqrt{s}) + P\left( \sup_{\sigma^*_0 \leq u \leq r^2t} |R^r(u)| \geq L^*_1 - 2 \right) \right\} ds
\]

\[
\leq \left( \frac{t^r}{r} \right)^2 + \int_{(r/r^2)^2}^{r^2t} P(I_2^r(\sigma^*_0) > t^r) ds
\]

\[
+ r^2t^2 \left( \sup_{\sigma^*_0 \leq u \leq r^2t} |R^r(u)| \geq L^*_1 - 2 \right),
\]

where \( t^r = O(\log r) \) and we have used the fact that \( I_2^r(\sigma^*_0) = 0 \) if \( \sup_{\sigma^*_0 \leq u \leq r^2t} |R^r(u)| \leq L^*_1 - 2 \). By the proof of the Lemma 8.6.1, there are constants \( C_1, C_2 \) such that for all \( r \) sufficiently large,

\[
P(I_2^r(\sigma^*_0) > t^r) \leq C_1 r^{-3},
\]

and

\[
P\left( \sup_{\sigma^*_0 \leq u \leq r^2t} |R^r(u)| \geq L^*_1 - 2 \right) \leq C_2 r^{-3}.
\]

Substituting these estimates into (8.208) we see that for \( r \) large enough, the above is a dominated by a sequence of functions in \( L^1(m) \). \( \square \)
Appendix A

Large Deviation Bounds for Renewal Processes

What follows is a summary of Appendix A in [3] and Section 6.6 in [4]. Let \( \{\rho(i)\}_{i=1}^{\infty} \) be a sequence of strictly positive, independent random variables such that \( \{\rho(i)\}_{i=2}^{\infty} \) are identically distributed, with finite mean \( 1/\nu \), for some \( \nu \in (0, \infty) \), and where \( \rho(1) \) may have a different distribution from \( \rho(i) \) for \( i \geq 2 \). It is assumed that there is a nonempty open neighborhood \( \mathcal{O} \) of the origin such that for \( i = 2, 3, \ldots \),

\[
\Lambda(\ell) \equiv \log \mathbb{E}[e^{\ell \rho(1)}] < \infty \quad \text{for all} \quad \ell \in \mathcal{O}.
\]

(A.1)

Let the values of \( r \geq 1 \) range throughout a sequence that increases to infinity. For each \( r \), let \( \nu_r \equiv \frac{\nu}{r} \) and suppose that \( \lim_{r \to \infty} \nu_r = \nu \). For each \( r \) and \( i = 1, 2, 3, \ldots \), let

\[
\rho^r(i) = \frac{\nu_r}{\nu} \rho(i).
\]

(A.2)

It follows that \( \rho^r(i) \) for \( i \geq 2 \) has finite mean \( 1/\nu_r = E[\rho^r(2)] \). Given \( 0 < \epsilon < \nu/2 \), let \( r_\epsilon \geq 1 \) be such that for \( r \geq r_\epsilon \),

\[
\frac{\nu^r}{\nu} \left(1 + \frac{\epsilon}{\nu} + \frac{\nu}{2} \right) \leq \frac{1}{\nu} \left(1 + \frac{\epsilon}{\nu} + \frac{\nu}{2} \right) < \frac{1}{\nu},
\]

(A.4)

\[
\frac{1}{\nu} \left(1 + \frac{\epsilon}{2(\nu^r - \nu)}\right) \geq \frac{1}{\nu} \left(1 + \frac{\epsilon}{2\nu}\right) > \frac{1}{\nu}.
\]

(A.5)

For \( n = 0, 1, 2, \ldots \), define

\[
Z^r(n) = \sum_{i=1}^{n} \rho^r(i) \quad \text{and} \quad \tilde{Z}^r(n) = \sum_{i=2}^{n} \rho^r(i).
\]

(A.6)
For each \( r \geq 1, t \geq 0 \), let
\[
N^r(t) = \sup\{n \geq 0 : Z^r(n) \leq t\}, \tag{A.7}
\]
the renewal process associated with \( Z^r \).

**Proposition A.0.3.** For a fixed \( \epsilon \) let \( r_\epsilon \) be such that (A.3),(A.4) and (A.5) are satisfied for all \( r \geq r_\epsilon \). Then

i) for \( r \geq r_\epsilon \) and for \( t > 2/\epsilon \)
\[
P(Z^r(t) > (\nu^r + \epsilon)t) \leq \exp\left(-((\nu^r + \epsilon)t - 1)\Lambda^*\left(\frac{1}{\nu}\left(\frac{1}{1 + \epsilon/3\nu}\right)\right)\right)
\leq \exp\left(-(\nu t - 1)\Lambda^*\left(\frac{1}{\nu}\left(\frac{1}{1 + \epsilon/3\nu}\right)\right)\right), \tag{A.8}
\]

ii) for \( r \geq r_\epsilon \) and \( t \geq 0 \)
\[
P(Z^r(t) < (\nu^r - \epsilon)t) \leq \exp\left(-(\nu^r - \epsilon)t\Lambda^*\left(\frac{1}{\nu}\left(1 + \frac{\epsilon}{2\nu}\right)\right)\right)
+ P\left(\rho^r(1) > \frac{\epsilon}{2\nu^r}t\right)
\leq \exp\left(-(\nu - 2\epsilon)t\Lambda^*\left(\frac{1}{\nu}\left(1 + \frac{\epsilon}{2\nu}\right)\right)\right)
+ P\left(\rho^r(1) > \frac{\epsilon}{2\nu^r}t\right), \tag{A.9}
\]

where
\[
\Lambda^*(x) \equiv \sup_{\ell \in \mathbb{R}}(\ell x - \Lambda(\ell)), \quad \text{for} \quad x \geq 0, \tag{A.10}
\]
and where the values of the quantities involving \( \Lambda^* \) above are strictly positive. (The function \( \Lambda^* \) is called the Legendre-Fenchel transform of \( \Lambda \).) Furthermore, if \( \rho(1) \) has the same distribution as \( \{\rho(i)\}_{i=2}^{\infty} \), then for each \( r \geq 1, t \geq 0 \) and \( 0 < \ell_0 \in \mathcal{O} \), for any \( n \geq 1 \),
\[
P\left(\max_{i=1}^{n} \rho^r(i) > \frac{\epsilon}{2\nu^r}t\right) \leq n \exp\left(-\frac{l_0\ell t}{2\nu} + \Lambda(\ell_0)\right). \tag{A.11}
\]
Appendix B

Estimates for a GI/GI/1 queue

In this section we give a justification for the bounds on probabilities in (8.58) and (8.109) in the proof of Lemma 8.6.1. We believe that this is a well known result, but we give a proof for completeness. In this section we will reuse some notation in a different way from that in the main text. In the first part of this section, we use the same setup as in Appendix A except that \( \{\rho(i)\}_{i=1}^{\infty} \) are i.i.d. and we let \( \text{Var}(\rho(i)) = \sigma^2 \) for \( \sigma \in (0, \infty) \). Since we have imposed an exponential moment assumption on \( \{\rho(i)\}_{i=1}^{\infty} \), the following result is due to Csörgö, Horvath and Steinebach (see Corollary 4.2 in [12]).

**Lemma B.0.1.** The renewal process, \( \{N(t), t \geq 0\} \), associated to \( \{\rho(i)\}_{i=1}^{\infty} \), can be realized on a probability space, which supports a standard one-dimensional Brownian motion \( \{B(s), s \geq 0\} \), such that

\[
P \left( \sup_{0 \leq s \leq t} \left| N(s) - sv - \sigma \nu^{3/2} B(s) \right| > \tilde{C}_1 \log t + x \right) \leq \tilde{C}_2 \exp\{-\tilde{C}_3 x\}, \tag{B.1}
\]

for all \( x > 0 \) and \( t \geq 4 \), where \( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \) are strictly positive.

Let \( \kappa^r = (\nu^r)^{3/2} \sigma^r \), where \( \sigma^r \geq 0 \) such that \( (\sigma^r)^2 = \text{Var}(\rho^r(i)) \), and let \( \kappa = \nu^{3/2} \sigma \). Let

\[
L^r = c \log r, \quad t^r = \tilde{c} L^r, \quad x^r = \delta L^r, \tag{B.2}
\]

where \( c > 0, \tilde{c} > 0, \delta > 0 \) and

\[
c > \max \left( \frac{8}{\delta \tilde{C}_3}, \frac{108 \tilde{c} \kappa^2}{\delta^2} \right). \tag{B.3}
\]

In our applications \( \delta \) will depend on \( \tilde{c} \). Let \( r_1 \geq 1 \) be such that for all \( r \geq r_1 \),

\[
t^r > 4, \quad \log r > 1, \quad \nu^r / \nu < 2, \quad |\kappa^r - \kappa| < \min(1, \kappa). \tag{B.4}
\]

For each \( t \geq 0 \), let \( N^r(t) \) be defined as in (A.7). Note that, \( N^r(t) = N(\frac{\nu^r}{\nu} t) \).
Lemma B.0.2. Let \( t^r \) and \( x^r \) be as in \((B.2)\). Let
\[
C_1 = 2\tilde{C}_1, \quad C_2 = 2\max \left( \frac{24\sqrt{2}\kappa}{\sqrt{\pi c_0}}, \tilde{C}_2 \exp(\tilde{C}_1 \log 2) \right).
\]

Then, there exists a standard one-dimensional Brownian motion \( \{B(s), s \geq 0\} \) such that for all \( r \geq r_1 \),
\[
P \left( \sup_{0 \leq r \leq t} |N^r(s) - \nu^r s - \sigma^r (\nu^r)^{3/2} B(s)| > C_1 \log t^r + x^r \right) \leq \frac{C_2}{r^\delta},
\]
where \( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \) are as in Lemma B.0.1.

For the proof of Lemma B.0.2 we will need the following fact about standard one-dimensional Brownian motion (see [27], page 96, Problem 8.2).

Proposition B.0.4. For a standard one-dimensional Brownian motion \( \{B(s), s \geq 0\} \),
\[
P \left[ \max_{0 \leq s \leq t} |B(s)| > z \right] \leq \sqrt{\frac{4}{2\pi z}} e^{-z^2/2t},
\]
for all \( z > 0 \) and \( t > 0 \).

Proof of Lemma B.0.2. Let \( B \) be a standard one-dimensional Brownian motion as in Lemma B.0.1. Then, for \( r \geq r_1 \), and all \( t > 0 \),
\[
\sup_{0 \leq s \leq t} |N^r(s) - \nu^r s - \kappa^r B(s)| \leq \sup_{0 \leq s \leq t} |N(\nu^r s/\nu) - \nu^r s - \kappa^r B(s) - \kappa B(\nu^r s/\nu) + \kappa B(\nu^r s/\nu)|
\]
\[
\leq \sup_{0 \leq s \leq t} |N(\nu^r s/\nu) - \nu^r s - \kappa B(\nu^r s/\nu)| + \sup_{0 \leq s \leq t} |\kappa B(\nu^r s/\nu) - \kappa B(s)|
\]
\[
\leq \sup_{0 \leq s \leq 2t} |N(s) - \nu s - \kappa B(s)| + \sup_{0 \leq s \leq t} |3\kappa B(s)|,
\]
where we used a substitution \( u = (\nu^r/\nu)s \) and the fact that \( (\nu^r/\nu) < 2 \). It follows that
\[
P \left( \sup_{0 \leq s \leq t} |N^r(s) - \nu^r s - \kappa^r B(s)| > C_1 \log t^r + x^r \right)
\]
\[
\leq P \left( \sup_{0 \leq s \leq 2t} |N(s) - \nu s - \kappa B(s)| > \tilde{C}_1 \log t^r + x^r/2 \right)
\]
\[
+ P \left( \sup_{0 \leq s \leq 2t} |3\kappa B(s)| > \tilde{C}_1 \log t^r + x^r/2 \right).
\]
Now, by Lemma B.0.1,
\[
P \left( \sup_{0 \leq s \leq 2t} |N(s) - \nu s - \kappa B(s)| > \tilde{C}_1 \log t^r + x^r/2 \right) \leq \tilde{C}_2 e^{-\tilde{C}_3 t^r}, \quad \text{(B.5)}
\]
where \( y^r = -\tilde{C}_1 \log 2 + x^r / 2 \). Then,

\[
\tilde{C}_2 \exp(-\tilde{C}_3 y^r) \leq \tilde{C}_2 \exp(\tilde{C}_1 \log 2) \exp(-\tilde{C}_3 x^r / 2) \leq \tilde{C}_2 \exp(\tilde{C}_1 \log 2) r^{-(c_6 \delta^2 / (36 \tilde{c}_6 \kappa^2))} \leq \frac{C_2}{2r^3},
\]

where we have used (B.3) and the choice of \( C_2 \). We have that,

\[
\begin{align*}
\text{P} \left( \sup_{0 \leq s \leq 2t^r} |B(s)| > \frac{\tilde{C}_1 \log t^r + x^r / 2}{3\kappa} \right) \\
\leq \text{P} \left( \sup_{0 \leq s \leq 2t^r} |B(s)| > \frac{x^r}{6\kappa} \right) \\
= \text{P} \left( \sup_{0 \leq s \leq 2t^r} |B(s)| > \frac{\delta t^r}{6\tilde{c}_6 \kappa} \right) \\
= \text{P} \left( \sup_{0 \leq s \leq 2t^r} |B(t^r s)| > \frac{\delta}{6\tilde{c}_6 \kappa} \sqrt{t^r} \right) \\
\leq \frac{24\tilde{c}_6 \kappa}{\sqrt{\pi \delta}} \sqrt{cc \log r} r^{-(c_6 \delta^2 / (36 \tilde{c}_6 \kappa^2))} \\
\leq \frac{C_2}{2r^3},
\end{align*}
\]

where we have used (B.2), (B.3), (B.4), Brownian scaling, choice of \( C_2 \) and Proposition B.0.4.

This completes the proof. \( \Box \)

Now, we consider a sequence of single buffer, single server (GI/GI/1) queueing systems indexed by \( r \in [1, \infty) \). The \( r^{th} \) network has the basic structure described in Chapter 2. In particular, the inter-arrival times in the \( r^{th} \) network are given by a sequence of i.i.d. random variables, \( \{u^r(i)\}_{i=1}^\infty \) with mean \( 1/\lambda^r \) and coefficient of variation \( a \). The service times in the \( r^{th} \) network are given by a sequence of i.i.d. random variables, \( \{v^r(i)\}_{i=1}^\infty \) with mean \( 1/\mu^r \) and coefficient of variation \( b \). We assume these are defined from a common sequence \( \{\tilde{u}(i)\}_{i=1}^\infty, \{\tilde{v}(i)\}_{i=1}^\infty \) not depending on \( r \), satisfying finite exponential moment conditions as in (3.1) and Assumption 3.1.2. The server processes jobs in FIFO order and does not idle unless the buffer is empty. We assume that, as \( r \to \infty \),

\[
\mu^r \to \mu > 0, \quad \lambda^r \to \lambda > 0, \quad \text{where} \quad \lambda < \mu.
\]

(B.7)

When the system starts empty, the queue-length in the \( r^{th} \) system at time \( t \geq 0 \) is given by,

\[
Q^r(t) = A^r(t) - S^r(T^r(t)),
\]

(B.8)
where $A^r(t)$ is the number of arrivals up to time $t$ and $S^r(T^r(t))$ is the number of jobs served up to time $t$. For $t \geq 0$, let, $X^r(t) = A^r(t) - S^r(t)$ and $U^r(t) = S^r(t) - S^r(T^r(t))$. Then, for $t \geq 0$,

$$Q^r(t) = X^r(t) + U^r(t). \tag{B.9}$$

Now, since $U^r$ is non-negative, non-decreasing, and $U^r$ can increase only if $Q^r$ is zero, we have that for $t \geq 0$,

$$U^r(t) = -\left(\inf_{0 \leq s \leq t} \{X^r(s)\}\right) \vee 0. \tag{B.10}$$

Let $\tilde{C}^a_1, \tilde{C}^a_2, \tilde{C}^a_3$ (resp. $\tilde{C}^{a\,\prime}_1, \tilde{C}^{a\,\prime}_2, \tilde{C}^{a\,\prime}_3$) be the constants such that Lemma B.0.1 holds for $\{\tilde{u}(i)/\lambda\}_{i=1}^\infty$ (resp. $\{\tilde{v}(i)/\mu\}_{i=1}^\infty$) in place of $\{\rho(i)\}_{i=1}^\infty$. Let $\kappa_\alpha = \lambda^{1/2} / a, \kappa_\ast = \mu^{1/2} / b$. Let

$$C_1 = 2 \max(\tilde{C}^a_1, \tilde{C}^{a\,\prime}_1), \quad C_2 = 2 \max\left(\frac{24 \sqrt{\beta}(\kappa_\alpha \vee \kappa_\ast)}{\sqrt{\pi C \delta}}, (\tilde{C}^a_2 \vee \tilde{C}^{a\,\prime}_2) \exp\left((\tilde{C}^a_1 \vee \tilde{C}^{a\,\prime}_1) \log 2\right)\right).$$

Let

$$\beta^r = \left(a^2 \lambda^r + b^2 \mu^r\right)^{1/2} \quad \text{and} \quad \beta = \left(a^2 \lambda + b^2 \mu\right)^{1/2}.$$

Let $L^r = c \log r$, where $c$ is such that

$$c > \max\left(\frac{12 \beta^2 \tilde{c}}{\delta^2}, \frac{8}{\delta \tilde{C}^a_3}, \frac{8}{\delta \tilde{C}^{a\,\prime}_3}, \frac{108 \kappa_\alpha^2}{\delta^2}, \frac{108 \kappa_\ast^2}{\delta^2}\right). \tag{B.11}$$

Let $t^r, x^r$ be given by (B.2). By Lemmas B.0.1 and B.0.2, the independent renewal processes $A^r$ and $S^r$ can be realized on a common probability space which supports standard independent one-dimensional Brownian motions, $\{B_1(s), s \geq 0\}, \{B_2(s), s \geq 0\}$ such that for all $r$ sufficiently large,

$$P\left(\sup_{0 \leq s \leq t^r} |A^r(s) - \lambda^r s - a^r(\lambda^r)^{1/2} B_1(s)| > C_1 \log t^r + x^r\right) \leq C_2 r^{-3}, \tag{B.12}$$

and

$$P\left(\sup_{0 \leq s \leq t^r} |S^r(s) - \mu^r s - b^r(\mu^r)^{1/2} B_2(s)| > C_1 \log t^r + x^r\right) \leq C_2 r^{-3}. \tag{B.13}$$

Let $r_2 \geq r_1$ be large enough that (B.12)-(B.13) hold for all $r \geq r_2$ as well as

$$\beta^r \leq 2 \beta, \quad \mu^r - \lambda^r > (\mu - \lambda)/2, \quad 2C_1 \log t^r < x^r. \tag{B.14}$$
Lemma B.0.3. Then, for \( r \geq r_2 \),
\[
\mathbb{P} \left( \sup_{0 \leq s \leq t^r} Q^r(s) \geq 7x^r \right) \leq \frac{C}{r^3},
\]
where \( C = 3 \max \left( C_2, \frac{4c\beta}{\sqrt{2\pi\delta}} \right) \).

Proof. Let
\[
A^a_r = \{ \sup_{0 \leq s \leq t^r} |A^r(s) - \lambda^r s - a^r(\lambda^r)^{1/2}B_1(s)| \leq C_1 \log t^r + x^r \},
\]
and
\[
A^s_r = \{ \sup_{0 \leq s \leq t^r} |S^r(s) - \mu^r s - b^r(\mu^r)^{1/2}B_2(s)| \leq C_1 \log t^r + x^r \},
\]
and let \( \mathcal{A}_r = A^a_r \cap A^s_r \). Then,
\[
\mathbb{P} \left( \sup_{0 \leq s \leq t^r} Q^r(s) \geq 7x^r \right) = \mathbb{P} \left( \sup_{0 \leq s \leq t^r} Q^r(s) \geq 7x^r, \mathcal{A}_r \right) + \mathbb{P} \left( \sup_{0 \leq s \leq t^r} Q^r(s) \geq 7x^r, \mathcal{A}_r^c \right) \leq \mathbb{P} \left( \sup_{0 \leq s \leq t^r} Q^r(s) \geq 7x^r, \mathcal{A}_r \right) + \mathbb{P}((A^a_r)^c) + \mathbb{P}((A^s_r)^c),
\]
where by (B.12), (B.13), and the choice of \( C \),
\[
\mathbb{P}((A^a_r)^c) + \mathbb{P}((A^s_r)^c) \leq \frac{2C}{3r^3}.
\]
For \( s \geq 0 \), let
\[
\chi^r(s) = (\lambda^r - \mu^r)s + a^r(\lambda^r)^{1/2}B_1(s) - b^r(\mu^r)^{1/2}B_2(s).
\]
Then, for \( s \geq 0 \),
\[
|X^r(s) - \chi^r(s)| \leq |A^r(s) - \chi^r(s)| + |S^r(s) - \mu^r s - b^r(\mu^r)^{1/2}B_2(s)|.
\]
It follows that, on \( \mathcal{A}_r \),
\[
\sup_{0 \leq s \leq t^r} |X^r(s) - \chi^r(s)| \leq 2C_1 \log t^r + 2x^r,
\]
and therefore,
\[
- \inf_{0 \leq s \leq t^r} \{ X^r(s) \} \leq - \inf_{0 \leq s \leq t^r} \{ \chi^r(s) \} + 2C_1 \log t^r + 2x^r.
\]
Moreover, on \( \mathcal{A}_r \), for \( 0 \leq t \leq t^r \),
\[
Q^r(t) \leq \chi^r(t) - \inf_{0 \leq s \leq t} \{ \chi^r(s) \} + 4C_1 \log t^r + 4x^r.
\]

For each \( s \geq 0 \), let \( \eta^r(s) = a^r(\lambda^r)^{1/2}B_1(s) - b^r(\mu^r)^{1/2}B_2(s) \). Then, \( \eta^r \) is a one-dimensional Brownian motion. In particular, \( \eta^r = \beta^r B_3 \), where \( \{ B_3(s), s \geq 0 \} \) is a standard one-dimensional Brownian motion. Now, by (B.14), for all \( r \geq r_2 \) and \( s \geq 0 \), \( \eta^r(s) - \chi^r(s) = (\mu^r - \lambda^r)s \) is increasing. It follows by Lemma B.0.4 below and since \( \chi^r, \eta^r \) start from the origin that for each \( t \geq 0 \),
\[
\chi^r(t) - \inf_{0 \leq s \leq t} \{ \chi^r(s) \} \leq \eta^r(t) - \inf_{0 \leq s \leq t} \{ \eta^r(s) \} \leq 2 \sup_{0 \leq s \leq t} |\eta^r(s)|.
\]

Then, on \( \mathcal{A}_r \), for \( r \geq r_2 \),
\[
\sup_{0 \leq s \leq t^r} Q^r(s) \leq 2 \sup_{0 \leq s \leq t^r} |\eta^r(s)| + 5x^r,
\]
where we have used (B.14). Then,
\[
P \left( \sup_{0 \leq s \leq t^r} Q^r(s) \geq 7x^r, \mathcal{A}_r \right) \leq P \left( \sup_{0 \leq s \leq t^r} |\eta^r(s)| \geq x^r \right)
\]
\[
= P \left( \sup_{0 \leq s \leq t^r} |B_3(s)| \geq \frac{\delta t^r}{\sqrt{c^r \beta^r}} \right)
\]
\[
\leq P \left( \sup_{0 \leq s \leq t^r} \frac{|B_3(t^r s)|}{\sqrt{t^r}} \geq \frac{\delta}{2c^r \beta^r} \sqrt{t^r} \right)
\]
\[
= P \left( \sup_{0 \leq s \leq 1} |B_3(s)| \geq \frac{\delta}{2c^r \beta^r} \sqrt{t^r} \right)
\]
\[
\leq \frac{4c^r \beta^r}{\sqrt{2\pi \delta}} \exp \left( -\frac{\delta^2}{4c^r \beta^r} \log r \right)
\]
\[
\leq C \frac{1}{3r^3},
\]
where we have used (B.11), (B.14), Brownian scaling, Proposition B.0.4 and the definition of \( C \).

This completes the proof. \( \square \)

For \( z \in \mathbb{D}_+ \) and \( t \geq 0 \), let \( \varphi(z)(t) = z(t) + \left( -\inf_{0 \leq s \leq t} \{ z(s) \} \right) \lor 0 = \sup_{0 \leq s \leq t} (z(t) - z(s)) \lor z(t) \).

**Lemma B.0.4.** Let \( u, v \in \mathbb{D}_+ \) be such that \( w = u - v \geq 0 \) is non-decreasing. Then, for each \( t \geq 0 \),
\[
\varphi(u)(t) \geq \varphi(v)(t).
\]
Proof. For $t \geq 0$, we have

$$
\varphi(u)(t) = \left( \sup_{0 \leq s \leq t} (u(t) - u(s)) \right) \vee u(t) \\
= \sup_{0 \leq s \leq t} (v(t) - v(s) + w(t) - w(s)) \vee (v(t) + w(t)) \\
\geq \sup_{0 \leq s \leq t} (v(t) - v(s)) \vee v(t) \\
= \varphi(v)(t).
$$

Since $t \geq 0$ was arbitrary, the proof is complete. \qed
Appendix C

Existence of a Least Control Process

In this section of the appendix we summarize the main results in [42] without proving them. These are used to show the existence of a least control process for the matrix $\tilde{G}$ and a driving Brownian motion in Chapter 6. Given an $m \times m$ matrix $D$, for each $x \in \mathbb{R}^m_+$ let

$$ U(x) = \{ u \in \mathbb{R}^m_+ : x + Du \geq 0, u(0) = 0, u(\cdot) \text{ is non-decreasing} \}, $$

be the functional polyhedral set in $\mathbb{R}^m_+$. The set $U(x)$ is said to have a least element if it is nonempty and if there exists an element $u^0 \in U(x)$ such that $u^0 \leq u$ for every $u \in U(x)$. If a least element exists, then it is clearly unique. If $U(x)$ has a least element, we let $\Phi(x)$ denote the least element of $U(x)$. When defined, for each $x \in \mathbb{R}^m_+$, the mapping $\Phi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is called a least element mapping associated with the functional polyhedral sets $\{U(x) : x \in \mathbb{R}^m_+\}$.

**Definition C.0.2.** (S-matrix) An $m \times m$ matrix $D$ is an S-matrix if there exists an $x \in \mathbb{R}^m_+$ for which $Dx > 0$.

**Proposition C.0.5.** The set $U(x)$ has a least element for each $x \in \mathbb{R}^m_+$, if and only if $D$ is an S-matrix with exactly one positive element in each row.

**Proposition C.0.6.** The least element mapping $\Phi$ is continuous in the Skorokhod topology. In particular, if $x^k \Rightarrow x$, then $\Phi(x^k) \Rightarrow \Phi(x)$. Moreover, if $x \in \mathbb{R}^m_+$ is continuous, $\Phi(x)(\cdot)$ is continuous.

We now use the above results to show the existence of least control processes for our application. For this we fix a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $X = \{X(t), t \geq 0\}$ on this space. An $m$-dimensional control process $U = \{U(t) : t \geq 0\}$ is admissible for $X$ and $D$ if the following are satisfied:
(i) for each $t \geq 0$, $U(t)$ is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t = \sigma\{X(s) : 0 \leq s \leq t\}$, and $\mathbb{P}$-a.s.,

(ii) $U(\cdot)$ is nondecreasing and right continuous with $U(0) \geq 0$,

(iii) for each $t \geq 0$, $X(t) + DU(t) \geq 0$.

Assuming $D$ satisfies the hypothesis of Proposition C.0.5, let $\mathcal{U}(X)$ denote the set of admissible controls for $X$ and $D$. For each $\omega \in \Omega$, let

$$\Phi(X)(\cdot, \omega) = \Phi(X(\cdot, \omega)).$$

(C.2)

This is well defined since $X(\cdot, \omega) \in D^m_+$ for all $\omega \in \Omega$.

**Proposition C.0.7.** The control process $\Phi(X)$ given by (C.2) satisfies the following properties:

(i) for each $t \geq 0$, $\Phi(X)(t)$ is $\mathcal{F}_t$-adapted,

(ii) $\Phi(X)(\cdot) \in D^m_+$, $\Phi(X)(\cdot)$ is non-decreasing and $\Phi(X)(0) = 0$, $\mathbb{P}$-a.s.,

(iii) $X(t) + D\Phi(X)(t) \geq 0$ for all $t \geq 0$, $\mathbb{P}$-a.s.,

(iv) $\mathbb{P}(\omega : \Phi(X)(t, \omega) \leq U(t, \omega) \text{ for all } t \geq 0) = 1$, for any $U \in \mathcal{U}(X)$.

We call $\Phi(X)$ the least control process associated to $X$ and $D$ and for $t \geq 0$, we let $\Psi(X)(t) = X(t) + D\Phi(X)(t)$. 

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