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On velocity-space sensitivity of fast-ion D-alpha spectroscopy


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Abstract
The velocity-space observation regions and sensitivities in fast-ion D$_\alpha$ (FIDA) spectroscopy measurements are often described by so-called weight functions. Here we derive expressions for FIDA weight functions accounting for the Doppler shift, Stark splitting, and the charge-exchange reaction and electron transition probabilities. Our approach yields an efficient way to calculate correctly scaled FIDA weight functions and implies simple analytic expressions for their boundaries that separate the triangular observable regions in (v$_\parallel$, v$_\perp$)-space from the unobservable regions. These boundaries are determined by the Doppler shift and Stark splitting and could until now only be found by numeric simulation.

Keywords: fast-ion D-alpha spectroscopy, FIDA, charge-exchange recombination spectroscopy, fast ions, velocity space

1. Introduction
Fast-ion D$_\alpha$ (FIDA) spectroscopy [1–3] is an application of charge-exchange recombination (CER) spectroscopy [4, 5] based on deuterium [6–11]. Deuterium ions in the plasma are neutralized in charge-exchange reactions with deuterium atoms from a neutral beam injector (NBI). The neutralized deuterium atoms are often in excited states, and hence they can emit D$_\alpha$-photons which are Doppler-shifted due to the motion of the excited atoms. As the excited atoms inherit the velocities of the deuterium ions before the charge-exchange reaction, spectra of Doppler-shifted D$_\alpha$-light are sensitive to the velocity distribution function of deuterium ions in the plasma. The measurement volume is given by the intersection of the NBI path and the line-of-sight of the CER diagnostic. D$_\alpha$-photons due to bulk deuterium ions typically have Doppler shifts of about 1–2 nm whereas D$_\alpha$-photons due to fast deuterium, which is the FIDA light, can have Doppler shifts of several nanometers. This paper deals with FIDA light but as the physics of D$_\alpha$-light due to bulk deuterium ions is the same, our methods also apply to deuterium-based CER spectroscopy. The FIDA or CER-D$_\alpha$ light is sometimes obscured by Doppler shifted D$_\alpha$-light from the NBI, unshifted D$_\alpha$-light from the plasma edge, bremsstrahlung or line radiation from impurities.

FIDA spectra can be related to 2D velocity space by so-called weight functions [2, 3, 12]. Weight functions have been used in four ways: first, they quantify the velocity-space sensitivity of FIDA measurements, and hence they also separate the observable region in velocity space for a particular wavelength range from the unobservable region [2, 3, 13–29]. Second, they reveal how much FIDA light is emitted resolved in velocity space for a given fast-ion velocity distribution function [2, 3, 24–30]. The ions in the regions with the brightest FIDA light are then argued to dominate the measurement. Third, weight functions have been used to calculate FIDA spectra from given fast-ion velocity distribution functions [14, 31–33], eliminating the Monte-Carlo approach of the standard FIDA analysis code FIDASIM [34]. Fourth, recent tomographic inversion algorithms to infer 2D fast-ion velocity distribution...
functions directly from the measurements rely heavily on weight functions \([12, 31–33, 35]\).

Here we present a comprehensive discussion of FIDA weight functions and derive analytic expressions describing them. FIDA weight functions have often been presented in arbitrary units, relative units or without any units \([2, 3, 15–28, 30]\) which is sufficient for their use as indicator of the velocity-space interrogation region or of the velocity-space origin of FIDA light. However, correctly scaled FIDA weight functions, which are necessary to calculate FIDA spectra or tomographic inversions, have only been implemented in the FIDASIM code recently \([13, 14, 29, 31–33]\). Weight functions are traditionally calculated using the FIDASIM code by computing the FIDA light from an ion on a fine grid in 2D velocity space and gyroangle. It is then counted how many photons contribute to a particular wavelength range for a given observation angle and point in velocity space using models for the Doppler shift, Stark splitting, charge-exchange probabilities and electron transition probabilities.

In section 2 we define weight functions and motivate their interpretation in terms of probabilities. Our viewpoint provides insights into functional dependencies between wavelength space and 2D velocity space that are not revealed by the traditional numerical calculation approach using FIDASIM. As a consequence we demonstrate how Doppler shift, Stark splitting, charge-exchange probabilities as well as the electron transition probabilities contribute to the velocity-space sensitivity of FIDA measurements. Section 3 focuses on weight functions implied by the Doppler shift alone as a relatively simple approximation. In section 4 we additionally treat Stark splitting and in section 5 the charge-exchange and the electron transition processes. In section 6 we present full FIDA weight functions accounting for these four effects. In section 7 we deduce exact analytic expressions for the boundaries of FIDA weight functions. We discuss the applicability of our results to CER spectroscopy and other fast-ion diagnostics in section 8 and conclude in section 9.

2. Definitions of weight functions

The velocity-space interrogation or observation regions of FIDA diagnostics are described by weight functions \(w_{\text{vol}}\) which are determined by charge-exchange probabilities, electron transition probabilities, Stark splitting and the Doppler shift. They thereby depend on position space and velocity space. Weight functions are defined to obey \([2, 3, 12]\)

\[
I(\lambda_1, \lambda_2, \phi) = \int_{v_0}^{v} \int_{-\infty}^{\infty} w_{\text{vol}}(\lambda_1, \lambda_2, \phi, v_1, v_2, x) \times f(v_1, v_2, x) \, dv_1 \, dv_2 \, dx.
\]

\(I(\lambda_1, \lambda_2, \phi)\) is the intensity of FIDA light in the wavelength range \(\lambda_1 < \lambda < \lambda_2\) with a viewing angle \(\phi\) between the line-of-sight of the FIDA diagnostic and the magnetic field. \((v_1, v_2)\) denote velocities parallel and perpendicular to the magnetic field, respectively, and \(x\) denotes the spatial coordinates. Here we use \((v_0, v_\perp)\)-coordinates rather than the more widespread \((E, p)\)-coordinates (energy, pitch) since our mathematical expressions are simpler in \((v_0, v_\perp)\)-coordinates. The energy and the pitch are defined as

\[
E = \frac{1}{2} m_0 (v_0^2 + v_\perp^2) \quad (2)
\]

\[
p = -\frac{v_\perp}{v} \quad (3)
\]

where \(m_0\) is the mass of a deuteron and \(v = \sqrt{v_0^2 + v_\perp^2}\) is the velocity magnitude. Note that the pitch is positive for co-current particles as usual. Key expressions are given in \((E, p)\)-coordinates in the appendix. We assume \(w_{\text{vol}}(\lambda_1, \lambda_2, \phi, v_1, v_\perp, x)\) and the fast-ion distribution function \(f(v_1, v_\perp, x)\) to be spatially uniform within the small measurement volume \(V\). This may be violated near the foot of the pedestal where the density gradient length scale could be comparable with the mean free path of the emitters, but it should be fulfilled in the core plasma. With

\[
w(\lambda_1, \lambda_2, \phi, v_1, v_\perp) = V w_{\text{vol}}(\lambda_1, \lambda_2, \phi, v_1, v_\perp, x) \quad (4)
\]

equation (1) becomes

\[
I(\lambda_1, \lambda_2, \phi) = \int_{-\infty}^{\infty} \int_{\lambda_1}^{\lambda_2} w(\lambda_1, \lambda_2, \phi, v_1, v_\perp) f(v_1, v_\perp) \, dv_1 \, dv_\perp.
\]

(5)

Weight functions \(w\) relate the FIDA intensity \(I(\lambda_1, \lambda_2, \phi)\) with units \([N_{ph}/(s \times sr \times m^3)]\) to the 2D fast-ion velocity distribution function with units \([N_i/(m^3 \times (m/s)^2)]\). The units of FIDA weight functions \(w\) are hence \([N_{ph}/(s \times sr \times m^2 \times N_i/m^3)]\) i.e. the FIDA intensity per ion in \(\lambda_1 < \lambda < \lambda_2\) for a viewing angle \(\phi\) as a function of the ion velocity \((v_1, v_\perp)\), i.e. the FIDA intensity per unit ion density in \(\lambda_1 < \lambda < \lambda_2\) for a viewing angle \(\phi\) as function of \((v_1, v_\perp)\). We will split FIDA weight functions \(w\) into a FIDA intensity function \(R(v_1, v_\perp)\) and a probability \(\text{prob}(\lambda_1 < \lambda < \lambda_2 | \phi, v_1, v_\perp)\) according to

\[
w(\lambda_1, \lambda_2, \phi, v_1, v_\perp) = R(v_1, v_\perp) \text{prob}(\lambda_1 < \lambda < \lambda_2 | \phi, v_1, v_\perp).
\]

(6)

\(R(v_1, v_\perp)\) determines the total FIDA intensity for any wavelength of the photons per unit ion density. It depends only on the charge-exchange and electron transition processes, but not on the Doppler shift or Stark splitting that only change the wavelength of the photons. \(\text{prob}(\lambda_1 < \lambda < \lambda_2 | \phi, v_1, v_\perp)\) determines the probability that a randomly selected detected photon has a wavelength in a particular range \(\lambda_1 < \lambda < \lambda_2\) for a given projection angle \(\phi\) and \((v_1, v_\perp)\)-coordinates. The conditioning symbol ‘|’ means ‘given’. The subject of this paper is the derivation of this probability. \(\text{prob}(\lambda_1 < \lambda < \lambda_2 | \phi, v_1, v_\perp)\) depends on the Doppler shift and Stark Splitting as well as on the charge-exchange and electron transition processes which in turn all depend on the gyroangle \(\gamma\) of the ion at the time of the charge-exchange reaction. We treat \(\gamma\) in \([0, 2\pi]\) as a random variable since we do not know the phases of all ions in the plasma, i.e. the initial conditions of any set of equations determining the ion motion are unknown as always in problems with a very large number of degrees of freedom. Since \(\lambda\) is determined by \(\gamma\), it is also treated
as random variable. Probabilities are always dimensionless numbers in the interval [0,1], and hence the FIDA intensity function $R(v_\parallel, v_\perp)$ has the same units as weight functions.

$R(v_\parallel, v_\perp)$ is a common factor of all weight functions for a given $\phi$ at any wavelength. On the contrary, the probability function depends on the wavelength range and the projection angle $\phi$ and hence contains the spectral information. We compute $R(v_\parallel, v_\perp)$ using FIDASIM by modeling the charge-exchange and the electron transition processes. Examples of the FIDA intensity function for NBI Q3 at ASDEX Upgrade, which is used for FIDA measurements, are shown in figure 1(a) in $(v_\parallel, v_\perp)$-coordinates and in figure 1(b) in $(E, p)$-coordinates. The sensitivity of FIDA is low for very large ion energies where few photons are generated per ion. Ions with positive pitch generate more photons per ion than ions with negative pitch for Q3.

Usually one measures spectral or specific intensities $I_\lambda$, i.e. the intensity per wavelength with units [$N_\lambda/(s \times sr \times m^2 \times nm)$]. The intensity and the spectral intensity are related by

$$I_\lambda(\lambda, \phi) = \int_{\lambda_i}^{\lambda_f} I_\lambda(\lambda, \phi) d\lambda.$$ \hspace{1cm} (7)

The spectral intensity $I_\lambda(\lambda, \phi)$ can likewise be related to $f(v_\parallel, v_\perp)$ by a probability density function pdf$(\lambda|\phi, v_\parallel, v_\perp)$ that then leads to a differential weight function $dw$ as

$$I_\lambda(\lambda, \phi) = \int_{\infty}^{0} \int_{-\infty}^{0} dw(\lambda, \phi, v_\parallel, v_\perp) f(v_\parallel, v_\perp) dv_\parallel dv_\perp.$$ \hspace{1cm} (8)

with

$$dw(\lambda, \phi, v_\parallel, v_\perp) = R(v_\parallel, v_\perp) \text{pdf}(\lambda|\phi, v_\parallel, v_\perp).$$ \hspace{1cm} (9)

However, the weight functions we discuss here are related to a wavelength range rather than a particular wavelength since FIDA intensity measurements can only be made for a wavelength range and not for a single wavelength. Mathematically this is reflected in the always finite amplitudes of $w$ whereas $dw$ is singular at its boundary.

3. Doppler shift

An approximate shape of FIDA weight functions can be found by considering only the Doppler shift $\lambda - \lambda_0$ where $\lambda_0 = 656.1 \text{ nm}$ is the wavelength of the unshifted D$_\text{a}$-line and $\lambda$ is the Doppler-shifted wavelength. In this section we derive this approximate shape by neglecting Stark splitting and by assuming that the D$_\text{a}$-photon emission is equally likely for all gyroangles $\gamma$ of the ion at the time of the charge-exchange reaction. The probability density function in $\gamma$ of randomly selected detected D$_\text{a}$ photons is

$$\text{pdf}_{D_a}(\gamma | v_\parallel, v_\perp) = 1/2\pi.$$ \hspace{1cm} (10)

Stark splitting and an arbitrary pdf$_{D_a}$ describing charge-exchange and electron transition probabilities will be introduced into the model in the next two sections. The Doppler shift depends on the projected velocity $v$ of the ion along the line-of-sight according to

$$\lambda - \lambda_0 = u\lambda_0/c$$ \hspace{1cm} (11)

where $c$ is the speed of light. Equation (11) assumes $u \ll c$. Consider a gyrating ion with velocity $(v_\parallel, v_\perp)$ in a magnetic field. The ion is neutralized in a charge-exchange reaction which ultimately leads to emission of a D$_\text{a}$-photon. We define a coordinate system such that for $\gamma = 0$ the velocity vector of the ion is in the plane defined by the unit vector along the line-of-sight $\hat{u}$ and $\hat{B}$ such that $\hat{v} \cdot \hat{u} > 0$. Then the ion velocity is

$$v = v_\parallel \hat{B} + v_\perp \cos \gamma \hat{v}_{\perp,1} - v_\perp \sin \gamma \hat{v}_{\perp,2}$$ \hspace{1cm} (12)

and the unit vector along the line-of-sight is

$$\hat{u} = \cos \phi \hat{B} + \sin \phi \hat{v}_{\perp,1}.$$ \hspace{1cm} (13)

The velocity component $u$ of the ion along the line-of-sight at a projection angle $\phi$ to the magnetic field is then given by [12]

$$u = v \cdot \hat{u} = v_\parallel \cos \phi + v_\perp \sin \phi \cos \gamma.$$ \hspace{1cm} (14)
Stark splitting discussed in section 4.

The projections of the ion velocity $v$ and the unit vector $\hat{v} \times \hat{B}$ (relevant for Stark splitting) onto the line-of-sight in this coordinate system are illustrated in figure 2. Equation (14) shows that $u$ is a random variable which depends on the random variable $\gamma \in [0, 2\pi]$. We now calculate the probability $\text{prob}(u_1 < u < u_2|\phi, v_||, v_\perp)$ that the ion has a projected velocity between $u_1$ and $u_2$ at the time of the charge-exchange reaction and therefore a Doppler-shifted $D_\alpha$ line wavelength between $\lambda_1$ and $\lambda_2$ according to equation (11). For given $(v_||, v_\perp)$ with $v_\perp \neq 0$ and projection angle $\phi \neq 0$, the projected velocity depends on the gyroangle $\gamma$. Conversely, we can calculate the gyroangles that lead to a given projected velocity $u$ by solving equation (14) for $\gamma$:

$$
\gamma = \arccos \frac{u - v_\perp \cos \phi}{v_\perp \sin \phi}.
$$

The arccos function is defined for $0 < \gamma < \pi$, and a second solution in $\pi < \gamma' < 2\pi$ is given by

$$
\gamma' = 2\pi - \gamma.
$$

Using equations (15) and (16) we can calculate gyroangles $\gamma_1$ and $\gamma_2$ and $\gamma'_1$ and $\gamma'_2$ corresponding to the limits $u_1$ and $u_2$ and transform the calculation of the probability to $\gamma'$-space:

$$
\text{prob}(u_1 < u < u_2|\phi, v_||, v_\perp) = \text{prob}(\gamma_2 < \gamma < \gamma_1|v_||, v_\perp) + \text{prob}(\gamma'_2 < \gamma < \gamma'_1|v_||, v_\perp) = \int_{\gamma_2}^{\gamma_1} \text{pdf}_D(\gamma_1|v_||, v_\perp) d\gamma + \int_{\gamma'_2}^{\gamma'_1} \text{pdf}_D(\gamma|v_||, v_\perp) d\gamma.
$$

As we here assume a uniform probability density, we can integrate equation (17) analytically:

$$
\text{prob}(u_1 < u < u_2|\phi, v_||, v_\perp) = \frac{\gamma_1 - \gamma_2}{2\pi} + \frac{\gamma'_2 - \gamma'_1}{2\pi}.
$$

The probability $\text{prob}(u_1 < u < u_2|\phi, v_||, v_\perp)$ is thus the fraction of the gyro-orbit that leads to a projected velocity between $u_1$ and $u_2$. Substitution of $\gamma$ using equation (15) gives

$$
\text{prob}(u_1 < u < u_2|\phi, v_||, v_\perp) = \frac{1}{\pi} \left( \arccos \frac{u - v_\perp \cos \phi}{v_\perp \sin \phi} - \arccos \frac{u_2 - v_\perp \cos \phi}{v_\perp \sin \phi} \right).
$$

Equation (19) is singular for $v_\perp = 0$ or $\phi = 0$. If $\phi = 0$, the projected velocity is just the parallel velocity as equation (14) reduces to $u = v_||$. Then the probability function becomes

$$
\text{prob}(u_1 < u < u_2|\phi = 0, v_||, v_\perp) = \begin{cases} 1 & \text{for } u_1 < v_|| < u_2 \\ 0 & \text{otherwise} \end{cases}
$$

which is identical to equation (19) in the limit $\phi \to 0$. For $v_\perp = 0$, i.e. on the $v_||$-axis corresponding to ions not actually gyrating, equation (14) reduces to $u = v_\perp \cos \phi$, and the probability function becomes

$$
\text{prob}(u_1 < u < u_2|\phi, v_||, v_\perp = 0) = \begin{cases} 1 & \text{for } u_1 / \cos \phi < v_|| < u_2 / \cos \phi \\ 0 & \text{otherwise}. \end{cases}
$$

Lastly, we note that the argument of the arccos function is often outside the range $[-1,1]$. In this case the arccos is complex, and we take the real part to obtain physically meaningful quantities. Equation (19) is a weight function describing just the projection onto the line-of-sight. We have previously derived the corresponding probability density function $\text{pdf}(u|\phi, v_||, v_\perp)$ to describe the velocity-space sensitivity of collective Thomson scattering (CTS) measurements [12]. The pdf can be found from the probability function by letting $u_1, u_2 \to u$:

$$
\text{pdf}(u|\phi, v_||, v_\perp) = \lim_{u_1, u_2 \to u} \frac{\text{prob}(u_1 < u < u_2|\phi, v_||, v_\perp)}{u_2 - u_1} = \frac{1}{\pi v_\perp \sin \phi \sqrt{1 - \left( \frac{u - v_\perp \cos \phi}{v_\perp \sin \phi} \right)^2}}.
$$

Equations (19) to (22) have been used to interpret CTS measurements at TEXTOR [36] and should have great utility.
for CTS measurements at ASDEX Upgrade [37–39], LHD [40,41] or ITER [42–44].

To obtain the probability function in \( \lambda \)-space, we first find the integration limits by substituting \( u \) in equation (15) using equation (11):

\[
\gamma = \arccos \left( \frac{\lambda_1}{\lambda_2} - 1 \right) - v_1 \cos \phi \frac{\pi}{v_1 \sin \phi}. \tag{23}
\]

Hence the probability function in \( \lambda \)-space becomes

\[
\text{prob}(\lambda_1 < \lambda < \lambda_2|\phi, v_\parallel, v_\perp) = \frac{\gamma - \gamma_2}{\pi} \left( \arccos \left( \frac{\lambda_1}{\lambda_2} - 1 \right) - v_\parallel \cos \phi \frac{\pi}{v_\parallel \sin \phi} 
- \arccos \left( \frac{\lambda_2}{\lambda_1} - 1 \right) - v_\parallel \cos \phi \frac{\pi}{v_\parallel \sin \phi} \right). \tag{24}
\]

This is a simple approximation to the probability part of FIDA weight functions neglecting Stark splitting and non-uniformity in \( \text{pdf}_\lambda \) due to charge-exchange and electron transition probabilities.

Figures 3((a)–(d)) show \( \text{prob}(\lambda_1 < \lambda < \lambda_2|\phi, v_\parallel, v_\perp) \) for a narrow wavelength range of 0.1 nm at various Doppler shifts. Figures 3((e)–(h)) show the corresponding probabilities \( \text{prob}(\lambda_1 < \lambda < \lambda_2|\phi, E, p) \). The observable regions or interrogation regions are colored whereas the unobservable regions are white. The viewing angle is \( \phi = 10^\circ \). The wavelength interval width \( \lambda_2 - \lambda_1 = 0.1 \) nm is comparable to the achievable spectral resolution of FIDA measurements at ASDEX Upgrade and is typical for tomographic measurements of 2D fast-ion velocity distribution functions [33]. The shape of the probability functions is triangular and symmetric in \( (v_\parallel, v_\perp) \)-coordinates, but the very tip of the triangle is cut off by the \( v_\parallel \)-axis as we will show more clearly in figure 5. The opening angle of the triangles is \( 2\phi = 20^\circ \) as the two sides have inclination angles of \( \pm \phi \) with respect to the \( v_\perp \)-axis [12]. The location of the interrogation region changes substantially with the magnitude of the Doppler shift. In figures 3((e)–(h)) we show the same probability functions in \((E, p)\)-coordinates since FIDA weight functions are traditionally given in these coordinates. The probability functions have more complicated shapes in \((E, p)\)-coordinates.

In figure 4 we vary the viewing angle \( \phi \). The larger the viewing angle, the larger the opening angle \( 2\phi \) of the triangular regions in \((v_\parallel, v_\perp)\)-space, and the lower the probabilities that a detected photon has a wavelength in the particular wavelength range. These probabilities decrease for increasing projection angle \( \phi \) since the spectrum of projected velocities of the ion and therefore wavelengths of the photons broaden according to equation (11) while the integral over the spectrum is the same.

Figure 5 shows probability functions for broader wavelength ranges up to \( \lambda_2 - \lambda_1 = 1 \) nm typical for the traditional use of weight functions as sensitivity or signal origin indicators. The inclinations of the sides of the triangle are not affected by the larger wavelength range, but a larger tip of the triangle is now cut off by the \( v_\parallel \)-axis as figure 5(d) shows most clearly. The larger the wavelength range, the larger the probabilities become since larger fractions of the ion orbits can produce Doppler shifts within the wavelength limits. In the limit of wavelength ranges covering very large red- and blue-shifts, the probability function becomes unity.

Figures 3–5 show that patterns in the velocity-space sensitivity of FIDA measurements are easier to spot in \((v_\parallel, v_\perp)\)-space where FIDA weight functions always have triangular shapes.

Equations (14), (15) and (16) transform the problem of finding a probability in \( \lambda \)-space into the simpler problem of finding a probability in \( \gamma \)-space. We will use this transformation when we account for Stark splitting and non-uniform charge-exchange and electron transition probabilities in the next two sections.

### 4. Stark splitting

An electron Balmer alpha transition from the \( n = 3 \) to \( n = 2 \) state of a moving D-atom in the magnetic field of a tokamak leads to light emission at 15 distinct wavelengths
probability functions after \((\lambda - \lambda_l)\) equation (24) for various projection angles \(\phi\) and a narrow wavelength range \(\lambda_2 - \lambda_1\) = 659.1 – 659.0 nm = 0.1 nm. The colorbar shows the base ten logarithm of the probability function \(\log_{10}(\text{prob}(\lambda_2 < \lambda < \lambda_1 | \phi, v, v_{\parallel}, v_{\perp}))\).

\[
\lambda_2 - \lambda_1 = \lambda_0 + s_l \tilde{E}
\]

where \(l\) is a number from 1 to 15 corresponding to the 15 lines and the constants \(s_l\) are [45, 46]

\[
s_{l=1,\ldots,15} = (-220.2, -165.2, -137.7, -110.2, -82.64, -55.1, -27.56, 0, 27.57, 55.15, 82.74, 110.3, 138.0, 165.6, 220.9) \times 10^{-19} \text{m}^2 \text{V}^{-1}
\]

Lines 1, 4–6, 10–12 and 15 are so-called \(\pi\)-lines, and lines 2, 3, 7–9, 13 and 14 are so-called \(\sigma\)-lines. Line 8 is the unshifted wavelength with \(s_8 = 0\). The electric field \(\tilde{E}\) in the reference frame of the neutral is

\[
\tilde{E} = \hat{E} + v \times \hat{B}.
\]

where \(\hat{E}\) is the electric field in the lab frame. In components this is

\[
\begin{bmatrix}
\tilde{E}_{\perp 1} \\
\tilde{E}_{\perp 2} \\
\tilde{E}_{\parallel}
\end{bmatrix}
= \begin{bmatrix}
\hat{E}_{\perp 1} \\
\hat{E}_{\perp 2} \\
\hat{E}_{\parallel}
\end{bmatrix} + \begin{bmatrix}
v_{\perp} \cos \gamma \\
v_{\perp} \sin \gamma \\
v_{\parallel}
\end{bmatrix}
\times \begin{bmatrix}
0 \\
0 \\
B
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\hat{E}_{\perp 1} + Bv_{\perp} \sin \gamma \\
\hat{E}_{\perp 2} - Bv_{\perp} \cos \gamma \\
\hat{E}_{\parallel}
\end{bmatrix}.
\]

The magnitude of the electric field in the frame of the neutral is

\[
\tilde{E} = \sqrt{\hat{E}^2 + v_{\perp}^2 B^2 + 2v_{\perp} B(\hat{E}_{\perp 1} \sin \gamma - \hat{E}_{\perp 2} \cos \gamma)}.
\]
Suppose we make a FIDA measurement at a particular wavelength \( \lambda \). The photon could have been emitted from any of the 15 lines with wavelength \( \lambda_l \) that is then Doppler shifted. Each of the 15 lines has a particular Stark wavelength shift corresponding to a particular Doppler shift with projected velocity \( u_l \) to be observable at \( \lambda \). The 15 Doppler shift conditions are
\[
\lambda = \lambda_l \left( 1 + \frac{u_l}{c} \right)
\]
which in combination with equation (25) yields
\[
\lambda = \left( \lambda_0 + s_l \hat{E} \right) \left( 1 + \frac{u_l}{c} \right).
\]
The projected velocity \( u_l \) and the electric field in the frame of the particle \( \hat{E} \) depend on the gyroangle. Substitution of \( \hat{E} \) using equation (29) and of \( u_l \) using equation (14) shows the relation between \( \lambda, \phi, v_\perp, v_\parallel, \gamma_l \) and \( s_l \):
\[
\lambda = \left( \lambda_0 + s_l \sqrt{\hat{E}^2 + v_\perp^2 B^2 + 2v_\perp B (\hat{E}_\perp \sin \gamma_l - \hat{E}_\parallel \cos \gamma_l)} \right) \\
\times \left( 1 + \frac{1}{c} \left( v_\parallel \cos \phi + v_\perp \sin \phi \cos \gamma_l \right) \right).
\]
This relation describes not only the Doppler effect but also Stark splitting, the two effects changing the wavelength of a detectable photon. It can be used to transform integration limits in \( \lambda \) to \( \gamma \)-space where the integration is easier to do. Here we include Stark splitting neglecting any electric field in the laboratory frame of reference. This reveals the most important effects and is often a good approximation in a tokamak as \( |\hat{E}| \ll |v \times B| \), in particular for fast ions with large \( v_\perp \). In FIDASIM simulations this approximation is usually made. If there is no electric field in the laboratory reference frame, the electric field in the reference frame of the particle is
\[
\hat{E} = v_\perp B
\]
and the Stark shift is just proportional to \( v_\perp \):
\[
\lambda_l = \lambda_0 + s_l v_\perp B.
\]
The functional dependence between \( \lambda \) and \( \gamma \) in equation (32) simplifies, and \( \lambda \) becomes a cosine function of \( \gamma \) as in the relation between \( u \) and \( \gamma \) in equation (14). Equation (32) becomes
\[
\lambda = (\lambda_0 + s_l v_\perp B) \left( 1 + \frac{1}{c} \left( v_\parallel \cos \phi + v_\perp \sin \phi \cos \gamma \right) \right).
\]
Equation (35) implies an equation for the exact shape of FIDA weight functions neglecting the electric field in the lab frame but accounting for Stark splitting as we will show in section 7. The inverse function is
\[
\gamma_l = \arccos \left( \frac{\lambda - \lambda_0 - s_l v_\perp B}{v_\perp \cos \phi} \right)
\]
which gives a solution for \( 0 < \gamma < \pi \). A second solution is given by equation (16). These are integration limits in \( \gamma_l \) for each of the 15 lines. The relative intensities \( I_l(\gamma) \) of \( \sigma \)-lines and \( \sigma \)-lines depend on the gyroangle \( \gamma \) and can be written as [14]
\[
\sigma : I_l(\gamma) = C_l(1 + \cos^2(\alpha, \hat{u} \times \hat{B})) = C_l(1 + \sin^2 \phi \sin^2 \gamma)
\]
\[
\pi : I_l(\gamma) = C_l(1 - \cos^2(\alpha, \hat{u} \times \hat{B})) = C_l(1 - \sin^2 \phi \sin^2 \gamma)
\]
where \( \hat{u}, \hat{v} \) and \( \hat{B} \) are unit vectors and the constants \( C_l \) are [14, 45, 47]
\[
C_{l=1,...,15} = \left( 1, 18, 16, 1681, 2304, 729, 1936, 5490, 1936, 729, 2304, 1681, 16, 18, 1 \right).
\]
The expression of the projection of \( \hat{u} \times \hat{B} \) onto the line-of-sight vector \( \hat{u} \) in terms of the gyroangle \( \gamma \) is illustrated in figure 2. The probabilities \( \text{prob}(l | \gamma) \) that a detected photon comes from line \( l \) given the gyroangle \( \gamma \) can be calculated from the relative intensities:
\[
\text{prob}(l | \gamma) = \frac{I_l(\gamma)}{\sum_{l=1}^{15} I_l(\gamma)}.
\]
Since \( \sum_{l=1}^{15} I_l(\gamma) = 18860 \) is a constant independent of \( \gamma \), we can write the probabilities of line \( l \) as
\[
\text{prob}(l | \gamma) = \hat{C}_l(1 \pm \sin^2 \phi \sin^2 \gamma)
\]
where the plus is used for the \( \sigma \)-lines and minus for the \( \pi \)-lines and
\[
\hat{C}_l = \frac{C_l}{\sum_{n=1}^{15} C_l}.
\]
The probability part of full FIDA weight functions accounting for Doppler and Stark effects for arbitrary pdfD\( \alpha \) can now be calculated according to
\[
\text{prob}(\lambda_1 < \lambda < \lambda_2 | \phi, v_\parallel, v_\perp) = \sum_{l=1}^{15} \int_{\gamma_{l_1}}^{\gamma_{l_2}} \text{prob}(l | \gamma) \text{pdfD}_\alpha(\gamma | v_\parallel, v_\perp) \, d\gamma
\]
\[
+ \int_{\gamma_{l_1}}^{\gamma_{l_2}} \text{prob}(l | \gamma) \text{pdfD}_\alpha(\gamma | v_\parallel, v_\perp) \, d\gamma.
\]
We will discuss the nature of the pdfD\( \alpha \) in FIDA measurements in the following section. Here we study basic effects by assuming a uniform pdfD\( \alpha \) = 1/(2\( \pi \)) for which we can solve the integrals in equation (43) analytically:
\[
\text{prob}(\lambda_1 < \lambda < \lambda_2 | \phi, v_\parallel, v_\perp) = \sum_{l=1}^{15} \int_{\gamma_{l_1}}^{\gamma_{l_2}} \hat{C}_l(1 \pm \sin^2 \phi \sin^2 \gamma) \, d\gamma
\]
\[
+ \int_{\gamma_{l_1}}^{\gamma_{l_2}} \hat{C}_l(1 \pm \sin^2 \phi \sin^2 \gamma) \, d\gamma
\]
\[
= \sum_{l=1}^{15} \hat{C}_l \left[ \frac{\gamma_{l_2} - \gamma_{l_1}}{\pi} \pm \frac{\sin^2 \phi}{2} \right.
\]
\[
\left. \times \left( \frac{\gamma_{l_1} - \gamma_{l_2}}{\pi} - \frac{\sin(2\gamma_{l_1}) - \sin(2\gamma_{l_2})}{2\pi} \right) \right].
\]
We leave the probability function in this form as substitution of the gyroangles using equation (36) provides no new insights. The probability function is calculated as a weighted sum over the 15 Stark splitting lines. The first fraction accounts for 15 different probability functions for the uniform distribution where the integration limits change for each Stark splitting line. The second term is a small correction due to the changing relative intensities of the 15 Stark splitting lines over the gyroangle. The corrections due to $\sigma$-lines and $\pi$-lines have different signs and hence partly cancel. For $\phi = 0$ this correction disappears.

Figure 6 demonstrates the effects of Stark splitting for a uniform pdfD, and a magnetic field of 1.74 T. The observation angle is $\phi = 30^\circ$ and the wavelength range is 658.0–658.1 nm in both figures. Stark splitting widens the interrogation region and changes the probabilities. The effect of the 15 Stark splitting lines shows most clearly close to the boundary of the observable region where several local maxima in the probability are formed. Since Stark splitting can be calculated accurately, it actually does not limit the spectral resolution of FIDA measurements as was sometimes asserted \[3, 15, 16, 24, 48, 49\] but rather just changes the velocity-space sensitivities.

5. Charge-exchange reaction and D$_{\alpha}$-emission

The probability density pdfD($\gamma \mid v_{b}, v_{1}$) is in fact not uniform as we assumed until now but is a complicated function depending on the charge-exchange probabilities and the electron transition probabilities and hence ultimately on the particular NBI as well as on the ion and electron temperatures and drift velocities. We hence find pdfD($\gamma \mid v_{b}, v_{1}$) and the FIDA intensity per unit ion density $R(v_{1}, v_{\perp})$ irrespective of the detected wavelength by numeric computation using FIDASIM. Here we discuss the nature of these contributions.

The probability of a charge-exchange reaction between an ion and a neutral depends on their relative velocity as well as on the particular charge-exchange reaction. For an ion with given $(v_{b}, v_{1})$, the probability density of a charge-exchange reaction pdfCX($\gamma \mid v_{1}, v_{\perp}$) therefore depends on the gyroangle $\gamma$. Since FIDA light comes from a fast neutral that has been created from a fast ion in a charge-exchange reaction, FIDA does not sample the gyroangles of the ions uniformly, but favors those gyroangles for which the ion velocity vectors are similar to those of the neutrals. The charge-exchange probability density depends on the distribution of injected neutrals and halo neutrals and therefore on the particular NBI heating geometry.

The gyroangle probability densities that an ion at a particular gyroangle ultimately leads to a detection of a D$_{\alpha}$-photon are further influenced by the electron transition probability densities $pdf_{n\rightarrow m}(\gamma \mid v_{1}, v_{\perp})$ from energy level $m$ to $n$. The $n = 3$ state can be populated and depopulated from any other energy state whereas only the $n = 3 \rightarrow 2$ leads to D$_{\alpha}$-emission. These electron transition probabilities also depend on the velocity due to collisions. The probability density pdf$_{n\rightarrow m}(\gamma \mid v_{b}, v_{\perp})$ is hence found numerically using FIDASIM.

Before we proceed to such a full numeric computation of the relevant charge-exchange reactions and electron transitions, we study essential features using a simplified model. We consider the charge-exchange reaction

\[ D^+ + D(n) \rightarrow D(n = 3) + D^+ \]  \tag{45}

where the donor neutral $D(n)$ is in the $n$th excited state and the product neutral $D(n = 3)$ is in the $n = 3$ state and so can directly emit a D$_{\alpha}$-photon. We emphasize that the $n = 3$ state can also be populated via any electron transition. However, in our simplified model we neglect electron transitions and consider only the direct population of the $n = 3$ state via the charge-exchange reaction. The charge-exchange reaction cross sections $\sigma$ and the reactivities $\sigma v_{\text{rel}}$ strongly depend on the relative velocity $v_{\text{rel}}$ which is usually expressed as the relative energy

\[ E_{\text{rel}} = \frac{1}{2} m v_{\text{rel}}^2. \]  \tag{46}

Figure 7 illustrates the cross sections $\sigma_{m}$ and the reactivities $\sigma_{m} v_{\text{rel}}$ for charge-exchange reactions with a donor neutral in state $m$ directly resulting in an excited $n = 3$ neutral \[14, 50–52\]. In these reactions the donor neutral was in one of the six excited states. The reactivities strongly depend on the relative velocities which in turn depend on the gyroangle.

For simplicity, we treat a single source of injected neutrals neglecting that in reality there are sources at full, half and third injection energy. In the coordinate system from figure 2 the velocity of the beam neutrals is

\[ v_{b} = v_{b,\parallel} \hat{B} + v_{b,\perp 1} \hat{v}_{\perp 1} + v_{b,\perp 2} \hat{v}_{\perp 2} \]  \tag{47}
Here we show rates for reactions with these beam neutrals in the first six excited states directly resulting in an excited neutral in the $n = 3$ state. The density of neutrals $n_{\text{neut},m}$ in the first energy state is by far largest, this charge-exchange reaction often dominates. In figure 7 we show these reaction rates for the six charge-exchange reactions for an energy of $E = 60\text{ keV}$ and pitches of $p = \pm 0.5$. The donor neutral population is here from the full injection energy peak of NBI Q3 while we neglect donor neutrals from half or third injection energies. Here we show rates for reactions with these beam neutrals in the first six excited states directly resulting in an excited neutral in the $n = 3$ state.

The reaction rates per ion are given by
\[ r_m = \sigma_m v_{rel} n_{\text{neut},m}. \] (51)

\[ v_{rel} = \sqrt{(v_{b,\parallel} - v_\parallel)^2 + (v_{b,\perp} - v_\perp \cos \gamma)^2 + (v_{b,\perp} - v_\perp \sin \gamma)^2}. \] (48)

To find extremal values in $v_{rel}$, we set
\[ \frac{dv_{rel}}{d\gamma} = 0 \] (49)

and find that the gyroangle $\gamma$ is then given by
\[ \tan \gamma = \frac{v_{b,\perp}}{v_{b,\parallel}}. \] (50)

If the reactivity $\sigma_m v_{rel}$ were monotonic in the range of interest, the extrema of $\sigma_m v_{rel}$ would correspond to the extrema of $v_{rel}$. However, figure 7 shows that the reactivities in particular of the charge-exchange reactions $1 \rightarrow 3$ and $2 \rightarrow 3$ are not monotonic but have maxima in the energy range of interest. Since the density of neutrals $n_{\text{neut},m=1}$ in the first energy state is by far largest, this charge-exchange reaction often dominates. The reaction rates per ion are given by

and the fast-ion velocity is given by equation (12). The relative velocity is then
\[ v_{rel} = \sqrt{(v_{b,\parallel} - v_\parallel)^2 + (v_{b,\perp} - v_\perp \cos \gamma)^2 + (v_{b,\perp} - v_\perp \sin \gamma)^2}. \] (48)

To find extremal values in $v_{rel}$, we set
\[ \frac{dv_{rel}}{d\gamma} = 0 \] (49)

and find that the gyroangle $\gamma$ is then given by
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\[ r_m = \sigma_m v_{rel} n_{\text{neut},m}. \] (51)

In figure 8 we show these reaction rates for the six charge-exchange reactions for an energy of $E = 60\text{ keV}$ and pitches of $p = \pm 0.5$. The reaction rates strongly depend on the gyroangle and have local maxima and minima. The dashed line shows the minima of the relative velocities given by equation (50) which coincides well with the local minima or maxima in the corresponding reaction rates. An extreme case is illustrated in figure 8(a) where the relative velocity goes to zero for a particular gyroangle. Figure 8(b) illustrates the reaction rates for velocity space coordinates far away from the donor neutral velocities.

Up to now we have not considered electron transition processes. In the following we calculate the full pdf$_{\text{Dne}}$ with FIDASIM where we model the important charge-exchange reactions and electron transitions as well as the beam geometry and energy distribution. Figure 9 shows such numerically calculated pdf$_{\text{Dne}}$ for a few energies and pitches. They often coarsely resemble phase-shifted cosine curves if one disregards local minima and maxima and Monte-Carlo noise. To study the effects of the non-uniform gyroangle distributions by simple models, we assume a model pdf to take the form
\[ \text{pdf}_{\text{Dne}}(\gamma \mid v_1, v_\perp) = 1/2\pi + a \cos(\gamma + \bar{\gamma}) \] (52)

where $a < 1/2\pi$ is an amplitude and $\bar{\gamma}$ is a phase shift. The integrals in equation (17) can be solved assuming pdf$_{\text{Dne}}$ from
FIDASIM. The NBI Q3 has an injection energy of 60 keV and an injection angle of about 120°.

The probability function in this form and do not substitute the gyroangles. The first term in equation (53) also appears for the uniform pdf whereas the second term accounts for cosines shape. It is proportional to the amplitude $a$ and to the cosine of the phase shift $\gamma$. We can also integrate our model pdf accounting for Stark splitting. Equation (43) becomes

$$
\begin{align*}
\text{prob}(\lambda_1 < \lambda < \lambda_2 \mid \phi, v_\parallel, v_\perp) &= \int_{\gamma_1}^{\gamma_2} \left( \frac{1}{2\pi} + a \cos(\gamma + \gamma') \right) d\gamma \\
&\quad + \int_{\gamma_1}^{\gamma_2} \left( \frac{1}{2\pi} + a \cos(\gamma + \gamma') \right) d\gamma' \\
&= \frac{\gamma_1 - \gamma_2}{\pi} + 2a \cos \bar{\gamma} \left( \sin \gamma_1 - \sin \gamma_2 \right).
\end{align*}
$$

(53)

Again we leave the probability function in this form and do not substitute the gyroangles. The first term in equation (53) also appears for the uniform pdf whereas the second term accounts for cosine shape. It is proportional to the amplitude $a$ and to the cosine of the phase shift $\gamma$. We can also integrate our model pdf accounting for Stark splitting. Equation (43) becomes

$$
\begin{align*}
\text{prob}(\lambda_1 < \lambda < \lambda_2 \mid \phi, v_\parallel, v_\perp) &= \sum_{l=1}^{15} \left( \int_{\gamma_{l,1}}^{\gamma_{l,2}} \hat{C}_l \left( 1 \pm \sin^2 \phi \sin^2 \gamma \right) \\
&\quad \times \left( \frac{1}{2\pi} + a \cos(\gamma + \gamma') \right) d\gamma \\
&\quad + \int_{\gamma_{l,1}}^{\gamma_{l,2}} \hat{C}_l \left( 1 \pm \sin^2 \phi \sin^2 \gamma \right) \left( \frac{1}{2\pi} + a \cos(\gamma + \gamma') \right) d\gamma' \right) \\
&= \sum_{l=1}^{15} \hat{C}_l \left( \frac{\gamma_{l,1} - \gamma_{l,2}}{\pi} \pm \frac{\sin^2 \phi}{2} \right) \\
&\quad \times \left( \frac{\gamma_{l,1} - \gamma_{l,2}}{\pi} - \frac{\sin(2\gamma_{l,1}) - \sin(2\gamma_{l,2})}{2\pi} \right)
\end{align*}
$$

Equation (54) contains all terms of equation (44) as well as the term accounting for the cosine shape from equation (53). Additionally, another correction term arises accounting for changing intensities of the Stark splitting lines and varying amplitude due to the cosine function. This term has again different signs for $\sigma$-lines and $\pi$-lines and disappears for $\phi = 0$.

As already mentioned, the phase shift $\gamma$ in equation (52) can be found approximately from geometric considerations. Further, we construct a model for the amplitude so that it increases with energy and decreases with the magnitude of the pitch as motivated by figure 9 where these trends appear:

$$
a = \frac{E}{E_0} (1 - p^2) = \frac{v^2}{v_{\perp,0}^2}.
$$

(55)

This model for the amplitude has $E_0$ as the only free parameter. It has units of energy to non-dimensionalize the energy coordinate. The amplitude $a$ of the cosine function in equation (52) is inversely proportional to $E_0$. For $E_0 = 1$ MeV the amplitudes of the probability density functions roughly correspond to the FIDASIM calculation over the relevant energy range up to 90 keV as we show in figure 10. Figure 11 shows that the typical large-scale cosine-like shape of pdf$_{\phi}$ leads to lopsided probabilities. In this particular case ions close to the right side of the triangular weight functions have higher...
probabilities to emit light in the particular wavelength range than those close to the left side of the triangle. The phase angle $\gamma$ determines how lopsided the probability function becomes. In figure 11(a) we show one of the extreme cases as $\cos(\gamma) = 1$. For $\cos(\gamma) = 0$ the probability function is symmetric and the same as that for the uniform distribution. Figure 11(b) shows the probability function for the model $pdf_\alpha$ given by equation (52) and accounting for Stark splitting. The effect of Stark splitting is similar to that observed for the uniform $pdf_\alpha$. Lastly, we note that any arbitrary $pdf_\alpha$ can be expanded into a Fourier series and then analytical, smooth FIDA weight functions could be constructed from the Fourier components which each can be integrated as in equation (54).

6. Full FIDA weight functions

Substitution of equation (43) into equation (6) gives an analytic expression for full FIDA weight functions accounting for Doppler and Stark effects and allowing for arbitrary $pdf_\alpha$:

$$w(\lambda_1, \lambda_2, \phi, v_\parallel, v_\perp) = R(v_\parallel, v_\perp) \times \sum_{l=1}^{15} \int_{\gamma_{l-1}}^{\gamma_l} \int_{|\parallel|} \int_{|\perp|} \text{prob}(l|\gamma) pdf_{D_\alpha}(\gamma | v_\parallel, v_\perp) d\gamma$$

$$+ \int_{\gamma_{15}}^{\gamma_l} \int_{|\parallel|} \int_{|\perp|} \text{prob}(l|\gamma) pdf_{D_\alpha}(\gamma | v_\parallel, v_\perp) d\gamma.$$  \hspace{1cm} (56)

Equation (56) is general whereas the assumptions of the FIDASIM code are used to calculate $R$ and $pdf_{D_\alpha}$ [14, 34]. In particular, the calculation of the weight functions assumes that the FIDA emission comes from a small volume in configuration space. Practically, $R$ and $pdf_{D_\alpha}$ are calculated from the distribution function $f_{D_\alpha}(\gamma | v_\parallel, v_\perp)$ of the FIDA intensity per unit ion density over $\gamma$ which we calculate numerically using FIDASIM. Then

$$R(v_\parallel, v_\perp) = \int_0^{2\pi} f_{D_\alpha}(\gamma | v_\parallel, v_\perp) d\gamma,$$  \hspace{1cm} (57)

$$pdf_{D_\alpha}(\gamma | v_\parallel, v_\perp) = \frac{f_{D_\alpha}(\gamma | v_\parallel, v_\perp)}{R(v_\parallel, v_\perp)}.$$ \hspace{1cm} (58)

We prefer not to substitute equation (58) into equation (56) to emphasize that $R$ is a factor common to any weight function with any wavelength range. We compare full FIDA weight functions as computed with our formalism with the traditional weight function as computed with FIDASIM in figure 12. The two approaches give the same result within small and controllable discretization errors and Monte Carlo noise from the sampling of the neutral beam particles in FIDASIM below 5%. This shows that our new formalism is consistent with the traditional FIDASIM computation as expected since the physics assumptions are the same. However, our approach provides additional insight into functional dependencies not revealed by the traditional brute-force computation. It also leads to faster computations if weight functions in several wavelength ranges are to be computed since the time-consuming collisional–radiative model only has to be evaluated once to find $R$ and $pdf_{D_\alpha}$, and weight functions.
for any wavelength range can then be computed rapidly using equation (56). Additionally, we compare these full FIDA weight functions based on numerically computed $pdf_{D\alpha}$ using FIDASIM with full weight functions given by the uniform pdf$_{D\alpha}$ and the cosine pdf$_{D\alpha}$ which match the full computation to within 20%.

7. Boundaries of FIDA weight functions

Often it is useful to know the velocity-space interrogation regions of FIDA measurements. Until now these observable regions in velocity space had to be found by numerical simulations with the FIDASIM code. Here we show that these velocity-space interrogation regions are in fact completely determined by a simple analytic expression accounting for the Doppler shift and Stark splitting. The boundaries of FIDA weight functions are found by solving equation (35) for $v_\parallel$ and setting $\cos \gamma = \pm 1$ and $l = 1$ or $l = 15$ which gives the largest possible Doppler shift and Stark splitting wavelength shift, respectively. The boundaries for arbitrary $l$ are

$$v_\parallel = \pm v_\perp \tan \phi + \frac{c}{\cos \phi} \left( \frac{\lambda}{\lambda_0 + s_l v_\perp B} - 1 \right).$$

This is a hyperbolic equation. Nevertheless, for $v_\perp \ll c$ we have $s_l v_\perp B \ll \lambda_0$, and we can expand the right hand side in a Taylor series:

$$v_\parallel \approx \left( \pm \tan \phi - \frac{c}{\cos \phi} \frac{\lambda}{\lambda_0} \frac{s_l B}{\lambda_0} \right) v_\perp + \frac{c}{\cos \phi} \frac{\lambda - \lambda_0}{\lambda_0}.$$

For $v_\perp \ll c$ the FIDA weight functions are thus approximately bounded by straight lines in $(v_\parallel, v_\perp)$-coordinates. The $v_\parallel$-intercept is $\frac{c}{\cos \phi} \frac{\lambda - \lambda_0}{\lambda_0}$ and the slope is given by the term in the bracket. In figure 13 we compare the outer boundary given by equation (60) with the corresponding FIDA weight function in $(E, p)$-coordinates. The outermost boundaries are found for $l = 1$ and $l = 15$. However, since the outermost three lines on each side correspond to Stark lines with tiny intensities (see equation (39)), the effective boundaries of the velocity-space interrogation region could be considered to be defined by $l = 4$ and $l = 12$ as indicated by dashed lines. Stark splitting has always been neglected in previous work where boundaries of weight functions or minimum energies below which the weight function is zero have been discussed [2, 3, 15, 16, 18, 19, 22, 28]. Figure 13 demonstrates that the effect of Stark splitting can be substantial as it decreases the minimum energy below which the weight function is zero by
functions using the available diagnostics [32]. For example, in joint measurements of 2D fast-ion velocity distribution (FILD) [26, 60, 61], fast-ion ASDEX Upgrade is equipped with fast-ion loss detectors FIDA and CTS [12], the fast-ion diagnostics can be combined for the other diagnostics are correctly scaled, as those for and beam emission spectroscopy [59]. If weight functions [2], fast-ion loss detectors [56], neutron spectroscopy [57, 58] to FIDA and have also been given for CTS [12], neutron count rate measurements [2], neutral particle analyzers (NPAs) [66, 67], and $\gamma$-ray spectrometry [68].

10–20 keV depending on whether we define the boundary by $l = 1, 15$ or by $l = 4, 12$. In Figure 13 the thick lines correspond to previous models with no Stark splitting (here $l = 8$). The outermost lines set the interrogation region accounting for Stark splitting ($l = 1, 15$), and dashed lines correspond to the Stark lines $l = 4, 12$.

**8. Discussion**

8.1. Fast-ion studies

ASDEX Upgrade has five FIDA views. Correctly scaled FIDA weight functions, as we present here, allow measurements of 2D fast-ion velocity distribution functions by tomographic inversion [33]. This will allow velocity-space studies of fast-ion distributions which are generated by up to 20 MW of neutral beam injection power and 6 MW of ion cyclotron heating power [53–55]. Moreover, weight functions are not specific to FIDA and have also been given for CTS [12], neutron count rate measurements [2], neutral particle analyzers (NPAs) [2], fast-ion loss detectors [56], neutron spectroscopy [57, 58] and beam emission spectroscopy [59]. If weight functions for the other diagnostics are correctly scaled, as those for FIDA and CTS [12], the fast-ion diagnostics can be combined in joint measurements of 2D fast-ion velocity distribution functions using the available diagnostics [32]. For example, ASDEX Upgrade is equipped with fast-ion loss detectors (FILD) [26, 60, 61], fast-ion $D_\alpha$ (FIDA) [13, 27, 29, 33], CTS [31, 32, 37–39, 62, 63], neutron energy spectrometry [64, 65], NPAs [66, 67], and $\gamma$-ray spectrometry [68].

8.2. CER spectroscopy of the bulk ions

Weight functions describing FIDA diagnostics will also describe $D_\alpha$-based CER spectroscopy of the bulk deuterium ions [6–11] and would then also be applicable to CER spectroscopy based on impurity species [4, 5] if the path of the emitter from the charge-exchange reaction to the photon emission does not curve significantly. Hence we could also show velocity-space interrogation regions of particular wavelength intervals in CER spectroscopy with our approach, estimate where in velocity space most signal comes for a given ion velocity distribution function, calculate spectra, and—perhaps the most interesting application—calculate velocity-space tomographies of bulk-ion velocity distribution functions of the emitting species. A temperature, density and drift parallel to the magnetic field could be found by fitting a 2D Maxwellian to the tomography of the ion distribution functions, and this could provide an alternative to standard methods. This method would be even more interesting if parallel and perpendicular ion temperatures are discrepant as sometimes observed in MAST [69] or JET [70] or if the ions do not have a Maxwellian distribution.

9. Conclusions

The velocity-space sensitivity of FIDA measurements can be described by weight functions. We derive correctly scaled expressions for FIDA weight functions accounting for the Doppler shift, Stark splitting, and the charge-exchange and the electron transition probabilities. Our approach provides insight not revealed by the traditional numerical computation of weight functions implemented in the FIDASIM code. By using simple analytic models we show how these physical effects contribute to the velocity-space sensitivities of FIDA measurements. The Doppler shift determines an approximate shape of the observable region in $(v_\parallel, v_\perp)$-space which is triangular and mirror symmetric. Stark splitting broadens this triangular observable region whereas the charge-exchange and electron transition probabilities do not change the boundaries of FIDA weight functions separating the observable region from the unobservable region in velocity space. Our approach implies exact analytic expressions for these boundaries that take Stark splitting into account and therefore differ by up to 10–20 keV in (energy, pitch)-space from similar expressions in previous work. We show that Stark splitting changes the sensitivity of the measurement, but this does not limit the achievable spectral resolution of FIDA measurements as has sometimes been asserted [3, 15, 16, 24, 48, 49]. Weight functions as we deduce here can be used to rapidly compute synthetic FIDA spectra from a 2D velocity distribution function. This lays the groundwork for the solution of the inverse problem to determine 2D velocity distribution functions from FIDA measurements. Lastly, our methods are immediately applicable to charge-exchange recombination spectroscopy measurements of $D_\alpha$-light from the bulk deuterium population to determine their temperature and drift velocity as well as any anisotropy.

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Appendix

Here we give key expressions in the widespread \((E, p)\) coordinates (Energy, pitch) that are used in the TRANSP code. They can be obtained by substituting 
\[
v_\parallel = -p\sqrt{2E/m} \\
 v_\perp = \sqrt{(1 - p^2)2E/m}
\]
(61)
(62)
into the corresponding expressions in \((v_\parallel, v_\perp)\)-coordinates. Weight functions in \((E, p)\)-coordinates are defined as 
\[
I(\lambda_1, \lambda_2, \phi) = \int \int \int w(\lambda_1, \lambda_2, \phi, E, p) f(E, p) dE dp.
\]
They can be written as 
\[
w(\lambda_1, \lambda_2, \phi, E, p) = R(E, p)\text{prob}\lambda_1 < \lambda < \lambda_2|\phi, E, p).
\]
(63)
The probability function for a uniform gyroangle distribution
\[
\text{prob}\lambda_1 < \lambda < \lambda_2|\phi, E, p) = \frac{1}{\pi}\left(\arccos\frac{\sqrt{2E/m}}{1 - p^2\sin \phi} - 1 + p\cos \phi\right).
\]
and the inverse function is 
\[
\gamma_1 = \arccos\frac{\lambda_0 + s_\lambda B\sqrt{(1 - p^2)2E/m}}{\sqrt{1 - p^2\sin \phi}} + p\cos \phi.
\]
and the probability function becomes 
\[
\text{prob}\lambda_1 < \lambda < \lambda_2|\phi, E, p) = \frac{1}{\pi}\left(\arccos\frac{\sqrt{2E/m}}{1 - p^2\sin \phi} - 1 + p\cos \phi\right) - \arccos\frac{\sqrt{2E/m}}{1 - p^2\sin \phi} + p\cos \phi
\]
(64)
(65)
(66)
(67)
(68)
(69)
For an arbitrary gyroangle distribution and no Stark splitting the probability function becomes 
\[
\text{prob}\lambda_1 < \lambda < \lambda_2|\phi, E, p) = \int_{\gamma_1}^{\gamma_2} pdf_{D_\gamma}(\gamma | E, p) d\gamma + \int_{\gamma_1}^{\gamma_2} pdf_{D_\gamma}(\gamma | E, p) d\gamma.
\]
(70)
The general expression of the probability function for an arbitrary gyroangle distribution and accounting for Stark splitting is 
\[
\text{prob}(\lambda_1 < \lambda < \lambda_2|\phi, E, p) = \sum_{l=1}^{15} \left(\int_{\gamma_1}^{\gamma_2} \text{prob}(l|\gamma)pdf_{D_\gamma}(\gamma | E, p) d\gamma + \int_{\gamma_1}^{\gamma_2} \text{prob}(l|\gamma)pdf_{D_\gamma}(\gamma | E, p) d\gamma\right).
\]
(71)
The general expression of FIDA weight functions is 
\[
w(\lambda_1, \lambda_2, \phi, E, p) = R(E, p) \times \sum_{l=1}^{15} \left(\int_{\gamma_1}^{\gamma_2} \text{prob}(l|\gamma)pdf_{D_\gamma}(\gamma | E, p) d\gamma + \int_{\gamma_1}^{\gamma_2} \text{prob}(l|\gamma)pdf_{D_\gamma}(\gamma | E, p) d\gamma\right)
\]
(72)
and their boundaries are given by 
\[
E \equiv \frac{mc^2(\lambda_0 - \lambda_0)^2}{2(1 - p^2)} \times \left(\lambda_0 \cos \phi \left(\frac{\pm \sec \phi + \frac{\lambda_0 B}{\sqrt{2}E} \sin \phi}{\lambda_0 \cos \phi}\right)\right)^2.
\]
(73)

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