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Authors
Littlejohn, R.G.
Kaufman, A.N.
Johnston, G.L.

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Hamiltonian Structure of Particle Motion in an Ideal Helical Wiggler with Guide Field*

Robert G. Littlejohn and Allan N. Kaufman
Lawrence Berkeley Laboratory and Physics Department
University of California, Berkeley, CA 94720

and

George L. Johnston
Plasma Fusion Center, Massachusetts Institute of Technology
Cambridge, MA 02139

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Abstract

The ideal helical wiggler with guide field is shown to possess an integrable Hamiltonian. Explicit generating functions are presented for the canonical transformation to action/angle variables.

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The following is a reduction of the motion of a particle in the magnetic field:

\[ B = B_0 \hat{z} + B_w (x \cos kz + y \sin kz), \]

to integrable form, including a partial reduction to action/angle variables. Essentially the same problem was solved by Roberts and Buchsbaum [1], but without revealing the Hamiltonian structure. The practical use of the calculation presented here is as a basis for a perturbative treatment of more difficult and/or realistic magnetic field models [2].

Choose units so that \( e=m=c=1 \), and the following gauge:

\[ A = B_0 xy + \frac{B_w}{k} (-x \cos kz - y \sin kz). \]

Then the relativistic Hamiltonian (squared) is

\[
H^2(x,p) = (p_x + \frac{B_w}{k} \cos kz)^2 + (p_y - B_0 x + \frac{B_w}{k} \sin kz)^2 + p_z^2 + 1.
\]

Now introduce the following canonical transformation:

\[(x,p) \rightarrow (Q_1,Q_2,Q_3;P_1,P_2,P_3), \]

given by the generating function:

\[
F(x,y,z;P_1,P_2,P_3) = (P_1 + B_0 y)(x - \frac{B_w}{B_0 k} \sin kz) + P_2 (\frac{p_1}{B_0} + y + \frac{B_w}{B_0 k} \cos kz)
\]

\[ + P_3 z - \frac{B_w^2}{4B_0 k^2} \sin 2kz - \frac{B_w^2}{2B_0 k} z. \]

This yields the following transformation equations:
\[ \begin{align*}
p_x &= p_1 + B_0 y, \\
p_y &= p_2 + B_0 x - \frac{B_w}{k} \sin kz, \\
p_z &= p_3 - \frac{B_w}{B_0} (\cos kz) (p_1 + B_0 y) - \frac{B_w}{B_0} (\sin kz)p_2 - \frac{B_w^2}{B_0^2} \cos^2 kz, \\
Q_1 &= p_2/B_0 + x - \frac{B_w}{B_0} \sin kz, \\
Q_2 &= p_1/B_0 + y - \frac{B_w}{B_0} \cos kz, \\
Q_3 &= z.
\end{align*} \]

Untangling this, we obtain
\[ \begin{align*}
x &= Q_1 - p_2/B_0 + \frac{B_w}{B_0} \sin kz, \\
y &= Q_2 - p_1/B_0 - \frac{B_w}{B_0} \cos kz, \\
z &= Q_3, \\
p_x &= B_0 Q_2 - \frac{B_w}{k} \cos kz, \\
p_y &= B_0 Q_1, \\
p_z &= p_3 - \frac{B_w}{B_0} (B_0 Q_2 \cos kz + p_2 \sin kz).
\end{align*} \]

In the new variables, we have
\[ H^2 = p_2^2 + B_0^2 Q_2^2 + [p_3 - \frac{B_w}{B_0} (B_0 Q_2 \cos kz + p_2 \sin kz)]^2 + 1. \]
Note that this $H$ is ignorable in both $Q_1$ and $P_1$, so both of these are invariants. In fact, $Q_1$ and $P_1$ are closely related to a kind of generalized guiding center position, denoted by $X$ and $Y$:

$$X = Q_1, \quad Y = -\frac{P_1}{B_0}$$

The Hamiltonian now has only 2 degrees of freedom. It is still not clear that it is integrable, since it depends on all 4 canonical variables: $Q_2$, $P_2$, $Z$, $P_3$. Notice that it has a harmonic oscillator term in $Q_2$ and $P_2$, with frequency $B_0$. The physical significance of the variables $Q_2$ and $P_2$ is that they are essentially the $x$ and $y$ components of the four-velocity:

$$\dot{x} = B_0 Q_2, \quad \dot{y} = P_2$$

However, the harmonic oscillator term is coupled to the longitudinal motion through the term $[P_3 - ...]^2$.

Introduce action/angle variables for the $Q_2$, $P_2$ variables, i.e. set

$$Q_2 = (2J/B_0)^{\frac{1}{4}} \sin \theta, \quad P_2 = (2J/B_0)^{\frac{1}{4}} \cos \theta,$$

so that the new canonical variables are $(\theta, J; z, P_3)$. The Hamiltonian becomes

$$H^2 = 2B_0 J + [P_3 - B_w (2J/B_0)^{\frac{1}{4}} \sin (\theta + kz)]^2 + 1.$$

Now introduce a new canonical transformation: $(\theta, J; z, P_3) \rightarrow (\phi, J; z, P_3')$, generated by

$$F'(\theta, z; J, P_3') = J (\theta + kz) + z P_3'$$

yielding

$$\phi = \theta + kz, \quad P_3' = P_3 + kJ.$$

Then the Hamiltonian becomes

$$H^2 = 2B_0 J + [P_3' + kJ - B_w (2J/B_0)^{\frac{1}{4}} \sin \phi]^2 + 1.$$
This is now ignorable in \( z \), so that \( P_3 \) is an invariant.

The Hamiltonian now has only one essential degree of freedom, and so is integrable. The integration can be carried out via elliptic functions [3].

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**References**

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