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ARBITRAGE q: AN EQUILIBRIUM THEORY OF INVESTMENT

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Key words: q theory, Modigliani-Miller theorem, arbitrage.

Abstract

This paper presents an arbitrage theory of investment. It shows that under the assumptions of the Modigliani-Miller theorem, the financial value of the firm equals the value of the firm's capital. This means average (Tobin's) q always equals one, and marginal q equals average q even if the firm's plant technology is strictly concave and there is a cost to adjusting capital.

The data are not kind to average or marginal q models of investment. Previous work focused on technologies that drove a wedge between marginal and average q. The theory presented in this paper shows that technologies should not drive a wedge between marginal and average q. Theory and empirical evidence indicate the source of the failure is not in the specification of technologies. This paper suggests that factors that break the Modigliani-Miller theorem, such as agency costs, are probably important in explaining the empirical relationship between financial and real values.

JEL Classification: 023, 520
Introduction

A "q" theory of investment relates investment in physical capital to the cost (market price) of the capital and the financial valuation of the firm's earnings stream. This paper presents an arbitrage q theory of investment. I use an arbitrage argument to show that given the assumptions of the Modigliani-Miller theorem, the financial value of the firm equals the market value of its capital stock. That is, no arbitrage profit opportunities implies that average (Tobin's) q equals one, and (contrary to previous work) that marginal q always equals average q.

The innovation in this paper is a focus on market, or general, equilibrium conditions rather than technologies that constrain firm behavior. Since the data are not kind to the restrictions imposed by either average or marginal q the major contribution of this paper is to suggest a potentially more fruitful direction to search for the source of the empirical failures in aggregate investment equations.

Tobin and Brainard (1968), and Tobin (1969), introduced a q theory of investment as a long run equilibrium condition linking the financial and real sectors in their macroeconomic models. They defined q as the ratio of the financial valuation of the firm to the replacement cost of its physical assets. Tobin and Brainard hypothesize that two markets value the firm. The asset market provides an
evaluation of the present value of the firm's earning potential—the equity plus debt value of the firm—while the commodity market gives the cost of the firm—the replacement value of the firm's physical capital. The Tobin/Brainard q theory of investment posits that a firm invests if q exceeds one. The flow of investment drives the valuation in the two markets together to eliminate the short-run disequilibrium.

A number of economists sought a more concrete microfoundation for q theory. In contrast to the market (general) equilibrium approach of Tobin and Brainard, traditional macroeconomic investment demand equations (and all the macro sectorial demand equations) are partial equilibrium structural demand equations derived from an explicit model of optimizing individual behavior. Jorgenson (1963, 1965) derived the popular neoclassical investment formulation using a value maximizing model of firm behavior. Even Tobin and Brainard (1977 p243), elaborating on their theory, question its microfoundation, stating "Clearly it is the q-ratio on the margin that matters for investment. The crucial value for marginal q is 1, but this is consistent with average values of q quite different from 1."

Abel (1980), Hayashi (1982), and Yoshikawa (1980) added a cost to adjusting capital to the neoclassical model of the firm and interpreted the modified neoclassical model as the microfoundation for q. Adjustment costs are a mathematically
tractable proxy for the myriad of real world complications associated with capital transactions and installation.\(^1\)

Adjustment costs make the optimal capital accumulation path from an initial condition to a steady state equilibrium endogenous. Endogenizing the adjustment path eliminates the arbitrary and inelegant distinction between long and short run behavior. The firm invests until the present value of the payoff stream from the marginal unit of capital (increment to the financial value of the firm) equals the market price of a unit of capital. Abel and Hayashi, quite naturally, labeled the theory marginal \(q\).

Marginal \(q\) type models, derived from a clearly specified model of firm behavior, now seem widely accepted as the correct approach to the theory of investment. However, despite sophisticated representations of firm technology, and advances in stochastic modelling, the firm-based \(q\) models of investment still fail to explain the data. Abel and Blanchard (1986) in an excellent recent empirical study conclude, "that the data are not sympathetic to the basic restrictions imposed by the \(q\) theory."

This paper takes a different approach to \(q\) theory focusing on the elimination of arbitrage profit opportunities, rather than firm behavior. The argument is similar to Modigliani-
Miller. Modigliani and Miller (1958) showed that if a financial restructuring of the firm changes its market value, then an opportunity exists for arbitrage profits. I show that if marginal $q$ does not equal average $q$ a restructuring of production leads to an increase in the market value of the firm, so an arbitrage profit opportunity exists (Modigliani-Miller applied to production plans). This no arbitrage profit condition also implies an indeterminacy (irrelevance) proposition for the distribution of capital among plants.

The arbitrage $q$ theory imposes the restriction that the value of a firm's physical capital equals the financial value of the firm (debt plus equity), so capital and financial value follow the same stochastic process. The data are not sympathetic to the restriction imposed by arbitrage $q$ theory. But the theory provides a new direction to search for the source of the empirical failures. The partial equilibrium approach naturally led to a search for technologies that caused marginal $q$ to deviate from average $q$. The arbitrage theory shows that a deviation based on technology violates the no arbitrage profit condition. Since marginal $q$ models that allow for arbitrage profits in terms of production restructuring do not perform significantly

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2 The theory does not imply that net investment is zero. In a stochastic equilibrium error realizations trigger the execution of the firm's conditional investment plans and determine market valuation of the earning stream given the realization.
better than average q models that impose the no profit from restructuring restriction, it seems likely that some of the basic assumptions of the Modigliani-Miller theorem are also violated. And the violation of these assumptions is important in explaining the relationship between financial and real values.

The Modigliani-Miller theorem rests on two stringent assumptions. It requires perfect competitive markets, i.e., no distortions due to transactions costs, tax wedges, or sales constraints. And, it requires that agents possess full information about the activities of the firm, i.e., they have rational expectations about the firm's earnings and that information is symmetrically distributed. The investment demand equations derived from optimal firm behavior almost satisfy these conditions. These models assume a competitive environment (and they implicitly assume the markets are complete) where agents have full information. Most of these models are deterministic, so asymmetric information and, in fact, risk, are not considered. Lucas and Prescott (1971), Abel (1985), and Abel and Blanchard use stochastic models where agents have homogeneous rational expectations, so risk, but not asymmetric information, affects the firm's decisions. Many of the firm-based models consider the effect of taxes on the firm's investment decisions, but they ignore any tax wedge effects on financial asset valuation. It seems likely that the full range of real world problems that make
debt-equity ratios relevant (asymmetric information and monitoring costs—anything under the general rubric of agency costs—tax wedges, or transactions costs and sales constraints) also affect the valuation of physical capital with a vengeance.

The paper is organized as follows. Section 1 sets up the households' and firms' objective functions in a simple general equilibrium model. It reviews the Modigliani-Miller theorem, and shows the equilibrium valuation of equities and bonds. Section 2 gives the investment results. It presents the arbitrage $q$ theory, the indeterminacy proposition, and marginal $q$. Section 2 extends (1) the derivation of marginal $q$, and (2) Hayashi's proposition that marginal $q$ equals average $q$ with a constant returns to scale technology, to stochastic models. Section 3 contains the conclusions. The appendix shows that replacing the capital adjustment cost technology with the "time to build" specification introduced by Kydland and Prescott (1982) does not change the results.
Section 1: A Simple General Equilibrium Model

This Section presents the households' and firms' decision problems in a simple Arrow-Debreu complete contingent claim markets economy. The complete contingent claim markets set-up makes stronger assumptions than are necessary to establish the results in the paper. But it is a familiar framework that makes the evaluation of risky assets easy. And partial equilibrium models of the firm that assume the manager's objective is to maximize the market value of the firm are valid only if markets are essentially complete. The last portion of this section shows sufficient conditions to transform the model into a "more realistic" rational expectations economy with a sequence of spot markets.

Assume there is a single commodity that can be consumed or added to the capital stock. Firms produce the commodity, sell claims against current and future production, and buy claims on current and future labor services. Households purchase claims for the commodity and sell claims on their labor services. The claims are contingent on the realization of an exogenous stochastic shock \( s_t \in S_t \), the so-called "state of nature" which buffets the economy. Let \( s(t) = (s_1, s_2, \ldots, s_t) \) denote a particular sequence of shocks. The

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vector \( s(t) \) is an element of the set of vectors of potential sequences, \( s(t) \in S(t) \).

A contingent claim is a financial asset that obligates the seller of the claim to deliver one unit of the commodity or service to the buyer in period \( t \) if the sequence of shocks \( s(t) \) is realized—otherwise the claim has no value. A contingent claim is an option. Let, \( p(s(t)) \) denote the price of a consumption claim conditional on the sequence \( s(t) \), and let \( c(s(t)) \) denote the number of these claims held by households.\(^2\) Let, \( w(s(t)) \) denote the price of a labor claim, and let \( z(s(t)) \) denote the number of labor claims held by firms. All contingent claim trading takes place in period one; delivery of real goods and services takes place in the period specified in the contract.

Households

Households own all the resources in this economy, but decision making is decentralized. The firm owners delegate production planning to firm managers.

\(^2\) \( c(s(t)) \) is the number of units of consumption that must be delivered if the sequence \( s(t) \) occurs. An individual household can borrow by selling consumption claims, but the household sector holds a long position in consumption contracts.
The objective of the \( i^{th} \) household is to maximize the expected value of,

\[
\sum_{t=1}^{\infty} \sum_{s(t) \in S(t)} \pi(s(t))U(c(s(t)), 1-z(s(t)))
\]

a strictly concave, time additive utility function. \( D \) denotes one plus the household rate of time preference, and \( 1-z(s(t)) \) leisure. \( \pi(s(t)) \) denotes the probability that the particular sequence \( s(t) \) occurs.\(^4\) The budget constraint limits the value of household purchases,

\[
\sum_{t=1}^{\infty} \sum_{s(t)} \ p(s(t))c(s(t)) = \sum_{j} p(s(1)) \sum \ m(j)MV(j, s(1)) + \sum_{j} w(s(t))z(s(t)),
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\( p(s(t)) \) to the value of its non-human wealth, \( \Sigma m(j)MV(j, s(1)) \), plus the revenue from the sale of labor services claims (human wealth). \( m(j) \) denotes the fraction of firm \( j \) owned by the household, and \( MV(j) \) is the market value of the \( j^{th} \) firm.

Given the vector of prices, the household buys consumption claims until the present value of the marginal utility of consumption in each state weighted by the probability that the claim gets exercised,

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to the value of its non-human wealth, $\Sigma m(j)MV(j, s(1))$, plus the revenue from the sale of labor services claims (human wealth). $m(j)$ denotes the fraction of firm $j$ owned by the household, and $MV(j)$ is the market value of the $j^{th}$ firm.

Given the vector of prices, the household buys consumption claims until the present value of the marginal utility of consumption in each state weighted by the probability that the claim gets exercised,

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is proportional to the claim price. The constant of proportionality is a Lagrangian multiplier which I normalize at one. The household sells claims on labor services until,

\[ 1.4 \pi(s(t))D^{-1}U_{t-1}(s(t)) = w(s(t)). \]

At a maximum the first-order conditions 1.3 and 1.4 hold for all potential sequences \( s(t) \)ES(\( t \)).

The left-hand side of equation 1.3 is the current shadow price of a unit of future consumption in a particular state and date, i.e., the increase in expected utility from an additional unit of future consumption. The shadow prices contain a discount for time preference (the amount paid today for delivery in the future, \( D^{-1} \)) and for uncertainty (the probability that the sequence \( s(t) \) will occur \( \pi(s(t)) \). If these events fail to occur the contingent claim has no value). In an Arrow-Debreu complete markets equilibrium the contingent claim prices equal the agents' shadow prices.

Firms

Define,

\[ 1.6 d(j,s(t)) \equiv c(j,s(t)) - (w(s(t))/p(s(t)))z(j,s(t)) \]

as net real earnings from the \( j^{th} \) firm's production in state \( s(t) \). The firm sells claims on current and future production and it buys claims for current and future labor inputs. The owners of the firm own a share of the time stream of the firm's net real earnings. The owners instruct the firm's
manager to maximize the market value of the firm which maximizes their wealth,

\[ 1.7 \; MV(j,s(1)) = \max_{s} \sum_{t=1}^{\infty} p(s(t))d(j,s(t)) / p(s(1)). \]

In a complete Arrow-Debreu claims market economy the firm realizes the market value of the entire stream of earnings in the period one.

A concave production technology constrains the firm's feasible contingent claim sales. Output in \( t \) depends on the (predetermined) beginning-of-period capital stock, \( k(j,s(t-1)) \), current investment, \( I(j,s(t)) = k(j,s(t)) - k(j,s(t-1)) \), the labor input, \( z(j,s(t)) \), and the realization of the state of nature, \( s_\epsilon \). Output can be used to cover claim obligations, or added to capital,

\[ 1.8 \; c(j,s(t)) = F(j,k(j,s(t-1)),I(j,s(t)),z(j,s(t)),s_\epsilon) - I(j,(s(t)) \]

If there is a cost to adjusting capital, \( F_i |_{i>0} < 0 \), new investment uses up part of current output. The manager's job is to select a production plan that maximizes the market value of the earnings stream.

The Arrow-Debreu set-up makes the role of the firm manager extremely clear, and not terribly important. The manager is the owners' agent, but the manager only makes technical decisions. He neither bears nor evaluates risk.
Equilibrium

As is well-known, if a competitive equilibrium exists in this economy \( \leq \), it is Pareto-optimal. Exogenous random shocks create uncertainty about future outcomes, but the complete set of contingent claim markets allows agents to make consistent plans that result in optimal risk sharing and give a Pareto-efficient allocation of resources when executed.

Firm Ownership: Debt and Equity

Ownership of the firm is the right to a share of the firm's stream of net earnings.\(^6\) I divide owners' claims into two types of financial assets; equity and debt. Debtholders get first claim on the net earnings. For simplicity, assume they get a fixed payment \( b^* \) each period.\(^7\) Equity holders get the remainder, say \( d^*(s(t+\tau)) \equiv d(s(t+\tau)) - b^* \), in dividends. Equities and debt are financial assets that promise to deliver a stream of future payoffs. The value of the equity or debt is the present market value of the stream of payoffs. Any two assets that deliver the same payoff stream (have the same probability distribution for the

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5 The existence of an equilibrium requires some technical conditions. I assume an equilibrium exists.
6 This includes selling the capital and liquidating the firm.
7 This definition makes debt a risk free asset. With more notation we could make debt risky, e.g., \( b(s(t))=b^* \) if earnings are greater than \( b^* \), or \( b(s(t)) \) equals \( d(s(t)) \) plus the residual value of the firm if earnings are less than \( b^* \). The demonstration of Modigliani-Miller and consumption-asset pricing evaluation equations does not require a risk free asset.
payoff stream) are perfect substitutes, and in equilibrium arbitrage forces them to have the same price. Therefore, in a complete contingent claim markets economy one can easily infer the value of the firm's equity or debt from a portfolio of consumption claims that delivers the same payoff stream.

A portfolio of claims that guarantees delivery of $b^*$ units of consumption each future period, whatever state of nature occurs, requires $b^*$ consumption claims for each potential state of nature $s(t+\tau) \in S(t+\tau)$ and for each time period, $\tau=1,2,\ldots$. The market value of the portfolio of contingent claims that guarantees the payoff stream $b^*$ in perpetuity,

$$1.10 \ b^* \sum_{\tau=1}^{\infty} \sum_{s(t+\tau)} p(s(t+\tau))/p(s(t)) = B(s(t)),$$

is simply the sum of the values of the individual consumption claims that make up the portfolio. An agent would have to give up $B(s(t))$ units of consumption in state $s(t)$ (or $B$ consumption contingent claims for state $s(t)$) to purchase the portfolio. Since the debtholders' claim on the firm's earnings delivers the identical payoff stream $b^*, b^*_{t-1}, \ldots$, in equilibrium arbitrage forces equality between the value of the firm's debt in any state $s(t)$ and the value of the equivalent portfolio of consumption contingent claims. ($B$)

8 \( p(s(t))/\sum_{s(t+\tau)} p(s(t+\tau)) \) $-1$ is the $\tau$ period risk free interest rate.
The same arbitrage argument applies to the valuation of equity holders' claims on the firm. Equity holders own a claim to the state-dependent residual net earnings stream, \( d^*(s(t+1)), d^*s(t+2), \ldots \). A portfolio of contingent claims that delivers the same state-dependent stream of consumption,

\[
1.11 \ V(s(t)) = \sum_{\tau=1}^{\infty} \sum_{s(t+\tau)} \left( \frac{p(s(t+\tau))}{p(s(t))} \right) d^*(s(t+\tau)),
\]

costs \( V(s(t)) \) units of consumption in state \( s(t) \). Therefore in equilibrium the equity value of the firm is \( V(s(t)) \).

Modigliani-Miller

The famous Modigliani-Miller theorem states that with perfect competitive capital markets if agents have full information about the payoff stream, the firm's financing decision is irrelevant, i.e., the debt/equity ratio is of no consequence. Modigliani and Miller's original proof was one of the first applications of an arbitrage pricing argument in finance, and the Modigliani-Miller theorem remains one of the pillars of modern finance theory. They proved the theorem by showing that if a financial restructuring changed the market value of the firm an arbitrage profit opportunity existed. Hirschleifer (1966) and Lucas (1984) used the Arrow-Debreu complete contingent claim markets set-up to present a more general, but very simple and elegant,
demonstration of the Modigliani-Miller theorem. The following is based on Lucas (1984).

The market value of the j-th firm, at any time \( t \), is the present value of the stream of earnings the firm delivers to its owners. In equilibrium, the value of this stream,

\[
1.15 \quad MV(j, s(t)) = \max \left\{ \sum_\tau \left( \sum_{s(t+\tau)} \frac{p(s(t+\tau))}{p(s(t))} d(j, s(t+\tau)) \right) | s(t+\tau) \right\}
\]

equals the value of an equivalent portfolio of contingent claims that delivers the same payoff stream \( d(s(t)) \), \( d(s(t+1)) \), \ldots. Decomposing the contingent claim portfolio into two portfolios (a financial restructuring) cannot change the value of their sum. This is the Modigliani-Miller theorem.

In terms of the example above, suppose bondholders get \( b^* \) each period and stockholders get the remainder. Then,

\[
1.16 \quad V(j, s(t)) + B(j, s(t)) + d(j, s(t)) = MV(j, s(t))
\]

\[
= \max \left\{ \sum_\tau \left( \sum_{s(t+\tau)} \frac{p(s(t+\tau))}{p(s(t))} \right) | \left\{ d(j, s(t+\tau)) - b^* \right\} + b^* \right\}
\]

the equity value, \( V(j, s(t)) \), plus the debt, \( B(j, s(t)) \), plus current earnings \( d(s(t)) \) \( \langle 9 \rangle \) equals the market value of the firm. The Modigliani-Miller theorem says that \( V \) and \( B \) are not uniquely determined.

\( \langle 9 \rangle \) I followed the real world convention that stocks are valued ex dividend, and I also valued bonds ex coupon; this means current earnings must be added to equity and debt to get the market value.
\[ V(j,s(t)) \equiv MV(j,s(t)) - \{B(j,s(t)) + d(j,s(t))\}, \]

or the market value of the firm is independent of the debt to equity ratio. Furthermore, notice that the household is indifferent as to whether it receives its share of the value of the firm in the form of stock or bond certificates (see the budget constraint 1.3). So the debt/equity ratio affects neither households' nor firms' real allocation decisions.

The Arrow-Debreu complete markets representation makes the Modigliani-Miller demonstration almost trivial. However, the central concept that any two assets that offer the same perceived probability distribution of payoffs are perfect substitutes, is as fundamental as the law of one price.

Relaxing the Complete Contingent Claim Markets Assumption:

The Consumption-Asset Pricing Model

This paper values equity and debt in terms of an observable (in a complete contingent claim markets economy) portfolios of contingent claims with the same payoff streams. The deservedly celebrated consumption-capital-asset pricing model makes the "more realistic" assumption that agents trade commodities, services, and financial assets in a sequence of spot markets. It assumes that agents rationally expect the asset's payoff stream (ie, they know the probability distribution of the asset's payoffs). Agents value assets in terms of the shadow value of the payoff stream.
Suppose there are \( J \) shares of stock outstanding (where \( J \) is a large number) in the \( j \)th firm, and the price of a share of stock in state \( s(t) \) is \( V(j, s(t))/J \). \( V(j, s(t))/J \) is the spot market price of a share of stock in terms of the numeraire consumption good. Buying a share of stock entitles the household to \( 1/J \) of the firm's dividend stream, or

\[
d^*(j, s(t)) \times J \text{ units of consumption in state } s(t+\tau) \in S(t+\tau), \tau = 1, 2, \ldots \text{.}
\]

The shadow value of a payoff in a state \( s(t+\tau) \) measured in units of current consumption equals,

\[
\langle D^{-\tau} \pi(s(t+\tau)) U_e(s(t+\tau))/U_e(s(t)) \rangle d^*(j, s(t+\tau))/J,
\]

the shadow price of consumption for state \( s(t+\tau) \) times the payoff divided by the current shadow price (marginal utility of current consumption). So the shadow value of the entire portfolio of payoffs,

\[
\sum_{t=1}^{\infty} \sum_{s(t+\tau)} \langle D^{-\tau} \pi(s(t+\tau)) U_e(s(t+\tau))/U_e(s(t)) \rangle d^*(j, s(t+\tau))/J
\]

must equal the spot market price of a share of stock in equilibrium. Multiplying by the current marginal utility of consumption and rewriting the infinite sum in recursive form,

\[
1.14' U_e(s(t)) V(j, s(t))/J
\]

\[
= \sum_{t=1}^{\infty} D^{-\tau} \pi(s(t+\tau)) U_e(s(t+\tau)) d^*(j, s(t+\tau))/J
\]

\[
= D^{-s} E_e[U_e(s(t+1)) \{ V(j, s(t+1)) + d^*(j, s(t+1)) \}]/J,
\]
gives the popular consumption-asset pricing equation for equities. The left-hand side shows the reduction in current household utility from purchasing an additional share of stock; the right-hand side gives the present value of the increase in expected utility from owning an additional share.\textsuperscript{10} Agents can rearrange their consumption pattern and share risk by trading financial assets in spot markets.

Ross (1987, 1976, 1978) shows that if a unique shadow price vector with all positive elements exists, then spot markets are complete (they span the space) and the unique shadow price vector equals the Arrow-Debreu contingent claim price vector.\textsuperscript{11} If sufficient linearly independent spot markets exist agents can trade securities and commodities so that their relative shadow prices (loosely speaking their marginal rates of substitution) are equal in all states and no mutually beneficial gains from trade remain unexploited.

If markets are incomplete, and nonunique all-positive shadow price vectors exist, then no arbitrage profit opportunities exist so the Modigliani-Miller theorem still holds, see Ross (1978). Agents trade assets until the their shadow value of asset payoff stream equals the market price, but they put different weights (shadow prices) on payoffs in

\textsuperscript{10} A similar formula holds for bonds where b* replaces d*.
\textsuperscript{11} In the consumption-asset pricing model the nonsatiation condition insures an all positive shadow price vector. Lucas' (1978) renowned derivation of the consumption-asset pricing model assumes a representative agent, so the shadow prices are unique and support the Arrow-Debreu equilibrium.
different states. If more markets existed they would trade to equalize their shadow prices in each state. With incomplete markets, the owners of the firm cannot agree on a production plan that changes the probability distribution of the payoff stream. The owners have different shadow prices for the same state of nature, so they value the payoffs differently and there are not enough markets (contracts) for individuals to equalize their shadow prices. With incomplete markets a conflict exists between the owners, so a simple instruction to the firm manager to maximize the market value of the firm does not maximize the individual owner's shadow value of wealth or welfare. Ginrols (1984) contains an excellent discussion of the production problem with incomplete markets and the constrained Pareto-optimal solution.

Partial equilibrium models of the firm in the investment literature assume the firm manager's objective is to maximize the market value of the firm. This objective function implicitly restricts the analysis to complete markets (either Arrow-Debreu contingent claims, or rich spot markets that span the space), or assumes that shareholder unanimity is unimportant in production planning. In the next section on production planning, I continue to use the Arrow-Debreu contingent claim price notation, but a rational expectations complete spot markets economy would give the same results.
Section 2: Investment

This section presents the arbitrage q theory, marginal q, and the indeterminacy results. It shows that under the assumptions of the Modigliani-Miller theorem the financial value of a firm equals the value of its capital. I first present the simplest form of the hypothesis where buying capital is buying a firm. This is a straight application of Modigliani-Miller. Then I review traditional marginal q theory where a cost to adjusting capital prevents entry and decreasing returns to scale drives a wedge between marginal and average q. I show that marginal q less than average q is inconsistent with the no arbitrage profit opportunities condition (Modigliani-Miller applied to production restructuring).

A q model of investment relates investment to the financial value of the firm and the market value of its physical capital. Tobin's q is frequently interpreted as a relative price in the firm's investment demand equation. Hayashi p218 states, "Once q is known ... the firm can decide the optimal rate of investment." And a number of papers use q as regressor in aggregate investment demand equations, eg Summers (1981), von Furstenberg (1977), Cicollo (1975). Arbitrage q, in contrast, is an equilibrium condition equating the value of two assets that promise the same stream (probability distribution) of payoffs. The opportunity for arbitrage profit drives the relationship.
Arbitrage q: An Equilibrium Model of Investment

Equity holders in the firm own a residual claim (after satisfying debt obligations) on the firm's physical assets, \( k(s(t)) \). The owners can exercise their claim by receiving the stream of payouts the firm generates, or they can sell the physical asset. So in equilibrium the present value of the payout stream must equal the value of the asset.

Proposition 1: Under the assumption of the Modigliani-Miller theorem, if purchasing capital is equivalent to purchasing a firm (free entry), then in a competitive equilibrium the debt plus equity value of a firm equals the value of its capital (physical assets), i.e.:

\[
2.1.1 \ V(s(t)) + B(s(t)) = k(s(t)).
\]

Proof:

The proof is obvious. Given the capital stock, \( k(s(t)) \), the firm generates a self-financing stream of payoffs \( d(s(t+\tau)) \).

The market value of this stream equals the value of an equity that promises the same (probability distribution) of payoffs,

\[
2.1.2 \ V(s(t)) = \sum_{\tau=1}^{\infty} \sum_{s(t+\tau)} p(s(t+\tau)/p(s(t))) d(s(t+\tau).
\]

1 Hall (1977) conjectured that Tobin's q should always equal one in equilibrium. The arbitrage condition is a precise statement of this conjecture. Arbitrage forces equality between the value of the optimal end of period capital stock and the equity value of the firm.
If the equity value of the payoff stream exceeds the value of capital, an arbitrager could borrow to buy the capital, $BA(s(t)) \equiv k(s(t))$, and sell the residual stream of payouts, $\omega$

$$2.1.3 \quad VA(s(t)) = \sum \sum_{\tau=1}^\omega p(s(t+\tau)) \{d(s(t+\tau) - ba(s(t+\tau))/p(s(t)) \}

where,

$$BA(s(t)) = \sum \sum_{\tau=1}^\omega p(s(t+\tau)) ba(s(t+\tau))/p(s(t)) = k(s(t))$$

for an arbitrage profit of $VA$. Conversely, if the value of capital exceeds the financial value of the payoff stream an arbitrage profit opportunity also exists. An arbitrager could sell capital short, borrow to buy out an existing firm's equity and debt holders, deliver the capital and pay off the debt, and retain the residual profit. Q.E.D.

Proposition 1 sits on the unreasonably stringent assumption that purchasing capital is equivalent to buying a firm, and that agents trade financial and physical capital in organized markets with no (or insignificant) transactions costs. Few would argue that this describes actual trades.

Partial equilibrium models of investment approximate the many real world problems in trading capital and setting up a firm with the mathematically tractable assumption that there is a cost to adjusting capital. Even if capital trades in a competitive market (as assumed), so agents can buy or sell physical capital, the internal cost of adjusting capital implies that buying capital is not equivalent to buying a
firm. An agent can purchase capital at the market price, but he must pay an adjustment cost to set up the firm. So capital adjustment costs may provide an effective barrier to entry allowing firms to earn rent. Capital adjustment costs also drive a wedge between the market price of investment and the firm's shadow price, so the firm's investment decision depends on the shadow price. Capital adjustment costs seem like a good proxy for the complications associated with capital transactions.

A Partial Equilibrium Model of Investment Demand

Most aggregate investment models focus on the firm's optimal decision rule. The advantage to this approach is that the derived decision rule is a structural demand equation. Many of these models (especially the marginal q models) emphasize a technology with an internal cost to adjusting the capital stock.

The manager's objective is to choose a production plan that maximizes the market value of the firm. The maximization methods used to derive the firm's investment demand decision implicitly constrain the manager's decision to the choice of factor inputs. This follows the time honored Walrasian tradition of assuming an exogenously fixed number of plants (firms); but, as shown in Proposition 2, it may also violate the another time honored tradition that no arbitrage profit opportunities exist in equilibrium.
The first-order conditions partial out the contribution of the marginal factor inputs. The first-order condition for labor is,

$$2.2.1 \quad F_z(s(t)) = \frac{w(s(t))}{p(s(t))},$$

that the marginal product of labor equal the real wage. And the first-order condition for capital is,

$$2.2.2 \quad (1 - F_z(s(t))) =$$

$$\sum \left[ \left\{ \frac{p(s(t+1))}{p(s(t))} \right\} \{F_k(s(t)) + (1 - F_z(s(t+1)))\} \right]_{s(t+1)}$$

Equation 2.2.2 states that at a maximum the cost of acquiring an additional unit of capital—the price plus the internal marginal cost of adjusting the capital stock \((F_z(s)|_{s(t)}>0<0)\) equals the present value of the marginal product of capital plus the asset value of an additional unit of capital. The first product on the right-hand side gives the present value of the payoff in increased production from an additional unit of capital; the marginal physical product of capital in each state of nature weighted by the present value of an additional unit of sales in that state of nature. The second term gives the present sales value of an additional unit of capital. \(\sum p(s(t+1))/p(s(t))\) is the reciprocal of one plus the return on a risk free bond, so the second term is the present value of selling a unit of capital including the marginal internal cost of changing capital.

Marginal q
Abel (1980), and Hayashi derived marginal $q$ by applying the maximum principle in a deterministic model to solve for the firm's optimal investment decision. Integrating equation 2.2.2 forward in time (recursively substituting for
\[ \frac{p(s(t+\tau))/p(s(t))}{1-F(s(t+\tau),\tau)} \text{ gives the stochastic version of marginal } q, \]

2.2.3 \[ (1-F(s(t+\tau),\tau)) = \sum_{\tau=1}^{\infty} \left( \frac{p(s(t+\tau))/p(s(t))}{F(s(t+\tau-1))} \right) s(t+\tau) \]

The right-hand side of 2.2.3 gives the present value of the stream of earnings from an additional unit of capital relative to the market price of a unit of capital. Abel and Hayashi labelled the right-hand side of 2.2.3, marginal $q$, 3

2.2.4 \[ \sum_{\tau=1}^{\infty} \left( \frac{p(s(t+\tau))/p(s(t))}{F(s(t+\tau-1))} \right) \equiv mq(s(t)). \]

Solving equation 2.2.4 gives the optimal investment decision,

2.2.5 \[ I(s(t)) = F(s(t),q^{-1}(mq(s(t))-1)). \]

In an Arrow-Debreu economy all the claim prices are observable, so the manager's decision is only a technical decision. In a complete markets rational expectations economy the manager weights future payoffs (the marginal product of capital) with the unique shadow price vector, so

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2 Appendix 2 shows that dynamic programming gives the "neoclassical" form of the investment decision rule.
3 Equation 2.2.4 is analogous to equation 5b, p252, in Abel and Blanchard.
the manager still only makes technical decisions. If markets are incomplete the manager, or decision maker, plays a much more important role since the owners do not agree on investment plans, eg, see Grinols.

Marginal $q$ and Average $q$

Marginal $q$ offers a compelling and intuitive economic story. It equates the cost (market value) of the last (marginal) unit of capital with the marginal increase in the market value of the firm due to the investment as Tobin and Brainard suggested. In fact, all neoclassical economics and Pareto efficiency require equating costs and benefits at the margin, not on average. Equities and debt, on the other hand, promise a proportional claim on the firm's net earnings. If there are diminishing returns the marginal product of capital is less than the net average product of capital, but a share of stock (roughly) pays the net average product of capital. So average $q$ exceeds one since equity valuation capitalizes the rent stream.

2.3 Marginal $q$ equals Arbitrage $q$

This portion extends proposition 1 to an environment with a cost to adjusting capital. It shows that in a competitive

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4 When markets are complete the calculation of the shadow price vector is conceptually straightforward. For empirical work, however, the correct shadow price calculations are quite complicated, eg, see Abel and Blanchard especially footnote 2.

5 Measured average $q$ has been less than one for most of the post-WWII period which makes one wonder if rents are the key to explaining the data.
equilibrium the financial value of the firm equals the value
of the capital stock. There are two cases. (1) Constant
returns to scale. In this case the initial conditions
determine the plant size. (2) Decreasing returns to scale.
In this case the plant size is indeterminate.

Case 1: Constant Returns to Scale
Hayashi showed that average $q$ equals marginal $q$ in a
deterministic model with a linearly homogeneous production
technology. The following extends Hayashi’s Proposition 1 to
the stochastic model.

If there are constant returns to scale, then marginal $q$
equals average $q$. The initial conditions determine firm
(plant) size.

Intuitively, Hayashi’s proposition follows from the constant
returns to scale property that if factors receive their
physical product in compensation it exhausts output, so no
rents accrue to the firm owners.

If there are constant returns to scale, then, $\delta$

\[ 2.3.1 \mathcal{F}(k(s(t-1), \Omega I(s(t)), \Omega z(s(t)), \Omega z_s), \]

\[ \delta \text{ Notice this specification scales the stochastic shocks so}
\text{that an increase in the scale of the inputs increases the}
magnitude of the shocks by the same amount, ie}
\[ \mathcal{F}(k, I, z, s) = \mathcal{F}(k, \alpha I, \alpha z) s. \text{ In any other specification the}
\text{riskiness of the payoffs is scale dependent. For example, an}
\text{additive shock, } \mathcal{F}(\ldots)+s \text{ means that the variance of output is}
\text{independent of the scale of operation.} \]
\[ = \theta F(k(s(t-1), I(s(t)), z(s(t)), s_t), \text{and,} \]
\[ = \theta (F_k k(s(t-1)) + F_I I(s(t)) + F_z z(s(t))). \]

Using the constant returns to scale properties net real earnings in any state \( s(t+1) \) are,

\[ 2.3.2 \ d(s(t+1)) = (F_k k(s(t)) + F_I I(s(t+1)) - I(s(t+1)), \]
since the wage bill equals labor's contribution to output,

\[ [F_k - w(s(t+1))/p(s(t+1))]z(s(t+1)) = 0. \]

Substituting the investment identity, \( I(s(t+1)) = k(s(t+1) - k(s(t)) \) and rearranging 2.3.2 gives,

\[ 2.3.3 \ F_k k(s(t)) + (1 - F_I I(s(t+1)) \]
\[ = [d(s(t+1)) + (1 - F_I I(s(t+1)))k(s(t+1))]/k(s(t)). \]

The term on the left-hand side of 2.3.3 is the term in braces on the right-hand side of the firm's first order condition for capital accumulation, equation 2.2.2. So eliminating that term in the firm's first order condition and multiplying by \( k(s(t)) \) gives,

\[ 2.3.4 \ (1-F_I I(s(t+1)))k(s(t)) = \sum_{s(t+1)} \frac{[d(s(t+1))/p(s(t))]d(s(t+1))}{s(t+1)} \]
\[ + \sum_{s(t+1)} \frac{[d(s(t+1))/p(s(t))(1-F_I I(s(t+1)))k(s(t+1))]}{s(t+1)}. \]

Solving the difference equation 2.3.4 forward in time (recursively eliminating

\[ (p(s(t+\tau))/p(s(t))) (1-F_I) k(s(t+\tau)), \tau=1,2,\ldots \]
gives,

\[ 2.3.5 \ (1-F_I I(s(t+\tau)))k(s(t)) \]
\[ = \sum_{\tau=1} \sum_{s(t+1)} \frac{[d(s(t+1))/p(s(t))]d(s(t+1))}{s(t+1)}. \]

The right-hand of 2.3.5 is the present value of the net earnings stream. Evaluated at the maximum it equals the
financial value of the firm, \( V(s(t)) + B(s(t)) \). So dividing by \( k(s(t)) \) and rearranging gives,

\[
F_x(s(t)) = 1 - aq(s(t)), \quad \text{or} \quad I(s(t)) = F_x(s(t))^{-1} \frac{1 - aq(s(t))}{k(s(t))},
\]

where \( aq(s(t)) \equiv \frac{V(s(t)) + B(s(t))}{k(s(t))} \),

the average q (aq) version of marginal q.

The optimal end-of-period capital stock, \( k(s(t)) \), produces a self-financing stream (probability distribution) of net earnings whose capitalized value equals an equity (or debt) that offers the same payout stream. Evaluated at the optimal capital stock marginal and average q equal one, and the payoffs to the marginal unit of capital equal the net payoffs to the average unit of capital.

The initial distribution of capital \( k(j, 0) \) determines firm sizes. The model begins with an initial allocation of capital to firms which bears no adjustment cost. The right-hand side of 2.3.5 gives the value of the initial capital.

In subsequent periods, capital adjustment costs \( \{F_x(j, s(t)) \} \) use up part of gross output and reduce net earnings. The forward looking capital valuation equation, however, includes future adjustment costs along any optimal path. The adjustment cost from current investment was anticipated and reflected in the valuation of previous capital. The adjustment cost prevents an outsider from entering the industry. The market cost of the capital plus the adjustment
cost, \( F_{ik}(s(t)) \), to set up the firm exceed the present value of the self-financing stream of future earnings, \( V(s(t)) + B(s(t)) \), the capital produces. The sizes and distribution of firms are determined by the initial distribution of capital, ie, by historical accident.

Remark:
This version of average \( q \) incorporates the partial equilibrium notion that the financial markets' valuation of the firm's earnings stream is exogenous. It implies agents (investors in financial assets) have full information about the activities of the firm and evaluate the firm's earning stream at the maximum. The firm manager uses the asset market's valuation as a signal and invests until average \( q \) equals one. The manager bears no risk, makes only technical decisions, and can rely on observables as signals. This representation of a firm manager's decision role in a rational expectations spot market economy is completely analogous to the role of a firm manager in the Arrow-Debreu formulation.

Case 2: Diminishing Returns to Scale
If there are diminishing returns to scale the payoff to an additional unit of capital at any plant is less than the net payoff to the average unit.
Proposition 2: Under the assumptions of the Modigliani-Miller theorem the financial value of the firm equals the value of its capital in a competitive equilibrium with no arbitrage profit opportunities.

Remark: This is an application of the Modigliani-Miller theorem on financial restructuring to production restructuring, and it gives the standard zero profit condition for a competitive equilibrium.

Proof:
If there are diminishing returns to scale, then
2.3.7 \[ F(k(s(t-1), I(s(t)), z(s(t)), s')) \]
\[ > F_w k(s(t-1)) + F_I I(s(t)) + F_z z(s(t)). \]
And the average payoff to the owners, say,
2.3.8 \[ d(s(t)) / (V(s(t-1)) + B(s(t-1))) \]
\[ \equiv [1 - I(s(t)) - w(s(t)) / p(s(t)) z(s(t))] / (V(s(t-1)) + B(s(t-1))) \]
exceeds the marginal payoff, \( < 7 > \)
2.3.9 \[ \delta d(s(t)) = F_w k(s(t-1)) + (F_I - 1) I(s(t)). \]
The financial value of the firm includes the present value of rents accruing to the owners.

A financial restructuring will not change the market value of the firm, but a production restructuring will increase the value of the firm so a profit opportunity exists.

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7 The marginal product of labor equals the real wage so,
\[ [F_w - w(s(t)) / p(s(t))] z(s(t)) = 0. \]
Suppose the firm organizes production in many small plants, instead of one large plant, so that,

\[ 2.3.10 f(i, k(i, s(t-1), I(i, s(t))), z(i, s(t)), s_t) = f_u k(i, s(t-1)) + f_I I(i, s(t)) + f_z z(i, s(t)), \]

then, net earnings from production in plant 1 are,

\[ 2.3.11 d(i, s(t)) = f_u k(i, s(t-1)) + (f_I - I) I(i, s(t)). \]

Now, given the same initial capital stock, and using the same quantity of factor inputs along a production path the firm has a higher market value, since

\[ 2.3.11 d'(s(t+\tau)) \equiv \sum_i d(i, s(t+\tau)) = d(s(t+\tau)). \]

Production restructuring increases the market value of the firm and it allows an arbitrager to enter the industry using the small plant technology eliminating excess profit opportunities.

In a competitive equilibrium, the present value of the payoff stream from any plant equals the value of the capital in that plant. The financial value of the firm equals the value of its capital. If there are diminishing returns to scale, the plant sizes approach zero and the number of plants is indeterminate. Q.E.D.

Remark 1: This is the intuitive story for capital allocation. Capital gets allocated across industries, firms
and plants to equalize its risk adjusted return in any activity.

Remark 2: Diminishing returns to scale often has the sensible interpretation that the specification omits a factor (e.g., management) since we don't usually observe production organized in many very small plants (8). Including the omitted factor gives a constant returns to scale when all the factors vary proportionally. If any of the factors are fixed (predetermined) the firm has increasing (short-run) marginal costs. When making a capital decision it seems unreasonable that firm managers—or owners—consider the number of plants a fixed factor.

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8 We do observe firms with more than one plant, however.
Conclusion

A q theory of investment relates investment to the financial value of the firm and the market value of its physical assets. Investment demand equations are crucial links between the financial and real sectors in any macroeconomic model. The persistent empirical failures in aggregate investment demand equations remain a puzzle and a nagging thorn in the side of applied economists. The best work in this area explores technological constraints, capital adjustment costs and decreasing returns to scale, that drive a wedge between average and marginal q. The theory in this paper shows that under the assumptions of the models—the assumptions of the Modigliani-Miller theorem coupled with the complete markets assumption implicit in value maximizing models of the firm—that technology should not drive a wedge between average and marginal q. The major contribution of this paper is to point out a new and potentially more fruitful directions to look for the source of the empirical failures.

There is a long list of potential candidates. Aggregation problems and bad data plague any empirical study. These problems are real, but probably intractable, and micro data-based q models of investment do not work particularly well.

This paper suggests that the theory of the firm forming the microfoundation for the investment demand equations is
inadequate. The Modigliani-Miller assumptions make financing
decisions irrelevant. The complete markets assumption,
imPLICIT in the objective function of maximizing the market
value of the firm, reduces the manager's role to
technician's job.

Much of the work in modern finance theory looks at
conditions that break the Modigliani-Miller theorem.
Asymmetric information and monitoring costs (agency costs)
make debt-equity ratios and management relevant. The only
work on investment (that I'm aware of) that considers these
factors are recent papers by Bernanke and Gertler (1986) who
include bankruptcy costs and inside information, and Chrinko
(1987) who assigns a signalling function to debt.

Incomplete markets and tax wedges also give management a
significant decision making role. This is a much more
difficult area since the solution depends on the
distribution of ownership and the incidence of taxes. Like
aggregation, these problems may be intractable.
References


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Appendix 1: Time to Build Technology

Kydland and Prescott introduced a "time to build" technology in their aggregated general equilibrium model. They wanted to build an aggregated real business cycle model that replicated the time series properties of observed aggregate real variables. Their model assumes the Modigliani-Miller theorem holds, otherwise the financial structure has real effects omitted from their specification. The time to build technology is an attractive alternative to capital adjustment costs. Either makes the dynamics richer.

Time to build models consider one aspect of the traditional thorny aggregation problem that plagues all aggregate studies. Time to build models replace the capital adjustment cost proxy with a fixed installation lag (time to build). New capital does not add to productive capacity until it gets installed. As a consequence the present value of a new unit of capital is less than an installed unit since its payoff stream starts later. This means capital is not homogeneous and aggregation is important.

Consider a newly purchased unit of capital, say \( I(s(t)) \), that produces a stream of net payoffs starting \( L \) periods in the future, say \( d(I(s(t)), s(t+\tau)), \tau=L, L+1, \ldots \). So installation takes \( L \) periods. \( p(s(t))I(s(t)) \) is the price of the investment in consumption claims,
\[ A1.1 \quad P(s(t))I(s(t)) = \sum_{\tau=L}^{\infty} \sum_{s(t+\tau)} P(s(t+\tau)d(I(s(t)),s(t+\tau)) = P(s(t))V(I(s(t)),L) \]

and \( P(s(t))V(I(s(t)),L) \) denotes the present financial value of the payoff stream. \( \langle 1 \rangle \)

Installed capital (investment at least \( L \) periods ago), say \( k(s(t-L)) \), produces a payoff stream starting next period, say \( d(s(t+\tau)) \), \( \tau=1,2,... \). So the value of installed capital,

\[ A1.2 \quad P(s(t))k(s(t-L)) = \sum_{\tau=1}^{\infty} \sum_{s(t+\tau)} P(s(t+\tau)d(s(t+\tau)) = P(s(t))V(k(s(t-L))) \]

exceeds the value of new capital since its payoff stream starts sooner. This says the plans plus materials and labor costs to construct a house sell for less than a house. The service flows from the two purchases start in different periods, so they are different commodities.

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1 If installation uses any resources, say building a house versus letting trees in an orchard grow, then the present value of the installation cost also must be deducted from the value of the payoff stream.
The owners of the firm have a claim on the net earnings from all the firm's capital--installed and in the pipeline. So the financial value of the firm,

\[ V(s(t)) + B(s(t)) = V(k(s(t-L)) + \sum_{i=1}^{L-1} V(I(s(t-(L-i))) \]

equals the net present value of all the payoffs which equals the market value of their vintage weighted capital stock.

The financial value of the firm equals the value of the firm's physical assets and no arbitrage profit opportunities exist. However, capital is not a homogeneous commodity so aggregation is important. The market value of an existing plant exceeds the market value of a plant under construction. Time to build models do not affect any of the results in this paper as long as the aggregation is done properly. In this model an analytically clean way to handle the aggregation is to introduce another sector, the construction sector. The construction sector produces plants, and sells ready to operate plants to the consumption goods industry. For empirical work temporal aggregation is one of many aggregation headaches that create well-known, but not easily solved, problems.
Appendix 2: The Neoclassical Formulation

In the early 60's Jorgenson (1963,1965) popularize the neoclassical model of investment. Lucas (1967), Gould (1968) and others added capital adjustment costs to Jorgenson's neoclassical formulation and derived the optimal investment decision rule.

Dynamic programming gives the "neoclassical" form of the investment rule in this model. Suppose we write the firm's objective function in a recursive dynamic programming form as,

\[ A2.1 \quad MV(s(t)) = \max \{ d(s(t)) + \sum \left( p(s(t+1)/p(s(t)) \right) MV(s(t+1)) \} \]

The dynamic programming "value function" is a recursive relationship that says the maximum market value of the firm in state \( s(t) \) equals the maximum of the current dividend plus the present value of the maximum of the market value of the firm next period. The factor inputs, \( k(s(t)), z(s(t)) \), are the current decision variables. The partial derivative of A2.1 with respect to capital is the first-order condition 2.2.1. And, equation 2.2.1 is a recursive form that can be solved for \( k(s(t)) \) given the future decisions.\(^1\) Using the maximum principle, or solving the first-order condition

\(^1\) The first-order condition for capital collapses to the traditional neoclassical rule that the marginal product of capital equals the rental rate when there is no uncertainty or adjustment costs. For example if there is no uncertainty, then \( p_t / p_{t+1} \) equals one plus the risk free return, and if there is no cost to adjusting capital then, \( F_t = 0 \), so equation 2.2.1 becomes \( p_t / p_{t+1} = 1 + F_{t+1} \).
forward in time, gives marginal q. The dynamic programming first-order condition may actually present fewer estimation problems than the marginal q form Abel and Blanchard used.
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