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RIGOROUS FORMULAE FOR THE STATISTICAL ERRORS ON
THE ZEROS AND THE PARTIAL WAVES IN TWO-BODY REACTIONS

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ABSTRACT
The probability densities of the zeros and of the
phase shifts, in spin 0 spin 0 and spin 1/2 spin 0
reactions, are derived rigorously.

INTRODUCTION

In physics we deal very often with polynomial approximations
of experimental data. For instance this is the case when we perform
a classical phase shift analysis of a scattering experiment. While
we are not usually interested in the roots of these polynomials, it
has recently become clear that they are really useful to look at.
Our aim is to find the rigorous probability densities functions of
these roots. The reasons for a priori lengthy calculations are the
following:

--Instead of making the usual linear approximation of the
propagation of errors, we will show that it can be both not too
difficult and very instructive to know the actual probability
densities of the estimated parameters (zeros, partial wave or whatever),
in order, for instance, to define when the linear approximation is
totally wrong.

--These apparently academic calculations give important
physical results. Especially in spin 1/2 spin 0 reactions where we
will see that the linear approximation breaks down for cases of
great theoretical and practical importance.

In the first section the probability densities of the zeros
and of the partial waves in spin 0-spin 0 reactions are derived.
The second section is devoted to the extension of these results to
spin 1/2 spin 0 reactions. The third section illustrates by computer
simulation, the most striking features of our rigorous results.
Practical formulae are given in Section IV.
I. Spin 0-Spin 0 REACTIONS

1. Generalities and Notations

The spin 0-spin 0 case is not academic, in the sense that reactions like $\pi^- p \to \pi^- p$, $K^- K^0 \to K^- K^0$, $\pi^- n \to \pi^- n$, ... can be considered as examples of spin 0 spin 0. Furthermore the calculations are easier than for spin $\frac{1}{2}$ spin 0 and will prepare us for this later case.

Let's call:

$$ z = \cos \theta_{\text{CM}} \quad \theta_{\text{CM}} \text{ is the center of mass scattering angle} $$

$$ \sigma(z) \quad \text{the differential cross section} $$

$$ f(z) \quad \text{the scattering amplitude} $$

When we want to deal with analytical continuations we write:

$$ \tilde{\sigma}(z) = f(z) \overline{f(\tilde{z})} \quad \tilde{z} \text{ means complex conjugate of } z. $$

If $\sigma(z)$ is approximated by a polynomial we write:

$$ \sigma(z) = a_0 + a_1 z + \ldots + a_{2n} z^{2n} = a_{2n} \prod_{i=1}^{n} (z - z_i)(\overline{z} - z_i) $$

The coefficients $a_i$ are real and $a_{2n} > 0$ because $\sigma(\cos \theta) > 0$.

The fact that we choose an even number of roots for $\sigma(z)$ is dictated by the requirement that we do not want to introduce unnecessary singularities in the scattering amplitude. This would be the case when keeping an odd number of roots for $\sigma(z)$. For instance in the case of one root:

$$ \sigma(z) = a_1 (z - z_1) $$

$\bar{z}_1$ is necessarily real and:

$$ f(z) = \sqrt{a_1} e^{i\phi(z)} \sqrt{\overline{z} - \overline{z}_1}. $$

This introduces a branch point on the real axis. On the contrary when the number of roots is even, we can build $f(z)$ as a polynomial by picking $n$ roots among the $2n$ mirror pairs of $\sigma(z)$.

Remark

We can, of course, expand $\sigma(z)$ on another set of polynomials (Legendre polynomials as it is often done), but our results would not change fundamentally since the coefficients in both expansions are linearly related.

2. Some Useful Mathematics

Suppose we know the probability density of the coefficients of our polynomial approximation (multinormal for instance). Since the roots of this polynomial are functions of these coefficients, our problem is to find a procedure to convert one set of random variables $(a_i)$ to another set of random variables $(z_j)$ given a functional relation between the two.$^4$

We will first consider the case of one random variable (r.v.) function of another r.v. Then we give the generalization of $n$ r.v. function of $n$ r.v. From now on, to avoid possible confusion, the absolute value of a real number $x$ will be noted $|x|$, and the modulus of a complex number $z$, will be noted $||z||$.

Let's assume that the r.v. $y$ is related to the r.v. $x$ by the function $g$. We call $p_x(x)$ the probability density function (p.d.f.) of the r.v. $x$, and $p_y(y)$ for $y$. 

If \( y = g(x) \) has a countable number of roots: \( x_1, x_2, \ldots \) we have:

\[
p_y(y) = \frac{p_x(x_1)}{|g'(x_1)|} + \frac{p_x(x_2)}{|g'(x_2)|} + \ldots \quad \text{with} \quad g'(x) = \frac{dg}{dx}.
\]

For example consider the Figure 1, where we have two roots for the particular \( y \) value chosen. The total probability contained in \( (dx_1) \) and \( (dx_2) \) is:

\[
p_x(x_1)|dx_1| + p_x(x_2)|dx_2|
\]

this is the total probability contained in \( dy \) thus:

\[
p_y(y)|dy| = p_x(x_1)|dx_1| + p_x(x_2)|dx_2|
\]

but:

\[
dx_1 = \frac{dy}{g'(x_1)} \quad dx_2 = \frac{dy}{g'(x_2)}
\]

which gives the expected formula.

Generalization

When we have \( n \) r.v. \( X = (x_1, \ldots, x_n) \) and \( n \) r.v. \( Y = (y_1, \ldots, y_n) \) functionally dependent, we are interested in the transformation of the hypervolume \( dx_1 \cdots dx_n \) to \( Y \) space. We use the jacobian to perform such a transformation.

\[
\frac{\partial (y_1, \ldots, y_n)}{\partial (x_1, \ldots, x_n)} \equiv J(x_1, \ldots, x_n) \equiv \det \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}
\]

As in the case of one dimension we get

\[
P_Y(y_1, \ldots, y_n) = \frac{P_x(x_1)}{|J(x_1)|} + \frac{P_x(x_2)}{|J(x_2)|} + \ldots
\]

where \( x_1, x_2, \ldots \) are the countable roots of \( Y = G(x) \).

Two useful mathematical theorems

1. Let \( a_0 + a_1 z + \ldots + a_n z^n \) be a complex polynomial and \((z_1, \ldots, z_n)\) its roots. Then:

\[
det\left(\frac{\partial a_i}{\partial z_j}\right) = (-)^n a_{n} \prod_{1 \leq i < j \leq n} (z_i - z_j)
\]

This is shown in Appendix 1.

2. Let \((\phi_1, \ldots, \phi_n)\) be \( n \) complex functions of \((z_1, \ldots, z_n) \in \mathbb{C}^n\), such that

\[
\frac{\partial \phi_i}{\partial z_j} = 0, \quad z_j \text{ means complex conjugate of } z_j.
\]

then

\[
\det\left(\frac{\partial \phi_i^R}{\partial z_j^R}\frac{\partial \phi_i^I}{\partial z_j^I}\right) \equiv \det\left(\frac{\partial \phi_i^R}{\partial z_j} + \frac{\partial \phi_i^I}{\partial z_j}\right)^2
\]

\[
\phi_i^R \text{ means Real part of } \phi_i,
\]

\[
\phi_i^I \text{ means Imaginary part of } \phi_i.
\]

This is very useful for two reasons: instead of handling a \( 2n \times 2n \) determinant we have only all \( n \times n \) to look at, and the
partial derivatives \( \frac{\partial \lambda}{\partial z_j} \) are always easier to compute than the.

partial derivatives \( \frac{\partial \lambda}{\partial z_j} \).

3. Density Probability of the Zeros

Among the \( 2n \) roots of \( \sigma(z) \), only \( n \) are independent. Call these roots \( (z_1, \ldots, z_n) \). We have \( 2n+1 \) real random variables (the polynomial coefficients) and are trying to determine only \( 2n \) real rv (the \( z_i \)). We use a general trick which consists of adding another rv to the set \( (z_i) \). We choose \( a_{2n} \), since it does not depend on the roots.

Let's call \( p_a(a_0, \ldots, a_{2n}) \) the pdf for the coefficients and \( p_z(z_1, z_1, \ldots, z_n, z_n, a_{2n}) \) the pdf we wish to find.

From Section I.2. we know that:

\[
p_z = \sum \frac{p_a}{\frac{\partial \sigma}{\partial a}} \quad \text{the sign \( \Sigma \) stands for the sum over all the roots of \( z = z(a) \).}
\]

In fact we do it the other way around

\[
p_a = \sum \frac{p_z}{\frac{\partial \sigma}{\partial z}} \quad \text{We do know the expression for the coefficients as a function of the roots.}
\]

and also it is easy to see that there are exactly \( n! \) possibilities giving the same set \( (a_i) \). This result is obtained by permutation of the roots among themselves (the coefficients are symmetric functions of the roots). So:

\[
p_a = n! \frac{p_z}{\frac{\partial \sigma}{\partial z}}
\]

or

\[
p_z = \frac{1}{n!} \frac{\partial (\sigma)}{\partial z} \quad p_a
\]

Let's work out the jacobian

\[
\frac{\partial (a_0, \ldots, a_{2n})}{\partial (z_1, z_1, \ldots, z_n, z_n, a_{2n})} = \frac{\partial (a_0, \ldots, a_{2n-1})}{\partial (z_1, z_1, \ldots, z_n, z_n)}
\]

Now:

\[
\frac{\partial a_j}{\partial z_k} = \frac{\partial a_j}{\partial z_k} + \frac{\partial a_j}{\partial z_k} \quad \text{and} \quad \frac{\partial a_j}{\partial z_k} = i \left[ \frac{\partial a_j}{\partial z_k} - \frac{\partial a_j}{\partial z_k} \right]
\]

Dividing by \( i \) and adding the corresponding column we get:

\[
J = i \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots \\
2 \frac{\partial a_j}{\partial z_k} & \frac{\partial a_j}{\partial z_k} & \frac{\partial a_j}{\partial z_k} & \ldots \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix} = 2i \begin{bmatrix}
\frac{\partial a_j}{\partial z_k} & -\frac{\partial a_j}{\partial z_k} & \frac{\partial a_j}{\partial z_k} & \ldots \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

Doing such a manipulation for the whole determinant we find:
By the theorem (1) of 1.2 we thus have:

\[ J = \prod_{j=1}^{2n} \prod_{j<k}^{2n} (z_j - z_k) \]

\( z_j \) means that now we take the set \((z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)\). The last product can be expressed as:

\[ \prod_{j<k}^{2n} (z_j - z_k) = \prod_{j<k}^{n(n-1)/2} \left| |z_j - z_k| |z_j - \bar{z}_k| \right|^2 \prod_{i=1}^{2n} (z_i - \bar{z}_i) \]

Taking the absolute value we get for \( p_z \):

\[ p_z(z_1, a_{2n}) = \frac{2n}{n!} \left| \prod_{j=1}^{n} \text{Im} z_j \right| \prod_{j<k}^{2n} \left| |z_j - z_k| |z_j - \bar{z}_k| \right|^2 p_a(a_1, a_{2n}) \]

We can rewrite this expression in another form using

\[ \frac{2\sigma(z)}{\sigma_1} = -a_{2n}(z - z_1)(z - \bar{z}_1) \cdots (z - z_i)(z - \bar{z}_i) \cdots (z - \bar{z}_n) \]

\[ \left( \frac{1}{z - z_1} \frac{\partial}{\partial z_1} \right)_{z=z_1} = -a_{2n}(z_1 - z_1) \cdots \left( \frac{z - \bar{z}_i}{z - \bar{z}_i} \right)(z_1 - \bar{z}_1) \cdots (z_1 - \bar{z}_n) \]

and

\[ \frac{3\sigma}{\sigma_1} = \sum_{j=1}^{2n} \frac{\sigma_z}{z_j} \]

so

\[ p_z(z_1, a_{2n}) = \frac{a_{2n}}{n!} 2n \sqrt{\frac{2n}{\sigma_1}} \prod_{i=1}^{2n} \left| \frac{\sigma}{\sigma_1} \right| p_a(a_1, a_{2n}) \]

or since

\[ \frac{3\sigma}{\sigma_1} = \frac{3\sigma}{\sigma_{\bar{z}_1}} \]

\[ p_z(z_1, a_{2n}) = \frac{a_{2n}}{n!} 2n \sqrt{\frac{2n}{\sigma_1}} \prod_{i=1}^{2n} \left| \frac{\sigma}{\sigma_{\bar{z}_1}} \right| p_a(a_1, a_{2n}) \] (1.2)

The final density probability for the zeros is obtained by integrating over \( a_{2n} \)

\[ p_z(z_1) = \int_0^{\infty} p_z(z_1, a_{2n}) \, da_{2n} \]

These formulae are discussed in Section III.
4. Density Probability for the Partial Waves

In building the amplitude we have to choose our set 
\( z_1, \ldots, z_n \) from the set \( \{z_i\} \) and therefore, lead to the 
discrete Barrelet's ambiguity.\(^2\)

The second well-known ambiguity is the global phase. We will 
assume a phase \( \phi(z) \) chosen once for all. This phase is not affected 
by statistical errors since it is not determined by the coefficients 
\( a_i \) (except at the point \( z = 1 \) by using the optical theorem).
Furthermore, at the zeros this phase is absolutely arbitrary since 
there \( f(z_i) = 0 \). We will write \( f(z) \) as:

\[
f(z) = \sqrt{2n} \ e^{i\phi(z)} \prod_{1 \leq j \leq n} (z - z_j)^{1/2n}.
\]

\[
f(z) = b_0 + b_1 z + \ldots + b_n z^n \quad ||b_n|| = \sqrt{2n}
\]

and

\[
f(z) = \sum_{k=0}^{n} (2z + 1) T_k p_k(z) \quad T_k = \frac{n! e^{2iz_k} - 1}{2ik}
\]

where \( k \) is the center of mass momentum. Occasionally we will 
use \( k \) as an indice and we hope there will be no possible confu-
sion. Let's call \( p_b(b_0, b_1, \ldots, ||b_n||) \) the pdf for the \( b \) 
coefficients.

\[
p_b = n! \frac{p(z_1, ||b_n||)}{p(z_1, ||b_n||)}.
\]

Using theorem 2 of 7.1, because the \( b_i \) do not depend on \( z_i \) we get:

\[
p_b(b_0, b_1, \ldots, ||b_n||) = \frac{n! p_z(z_1, ||b_n||)}{||b_n||^{2n+1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2}.
\]

The \( T_k \) are linearly related to the \( b_i \) by:

\[
T_k = \frac{1}{2} \int_{-1}^{+1} [b_0 + \ldots + b_n z^n] p_k(z) dz
\]

this introduces a multiplicative constant:

\[
\left| \det \left( \frac{\partial T_k}{\partial b_i} \right) \right|
\]

in the pdf, so we won't bother about it. In the Appendix we give 
this constant when \( \phi(z) = C = \) constant.

\[
p_b(b_0, b_1, \ldots, ||b_n||) = \frac{2^{2n+1} ||b_n||^{2n+1} \prod_{j=1}^{n} \text{Im} z_j}{\prod_{j<k} |z_j - z_k|^2 p_a(a_k)}.
\]

It is possible to express \( p_b \) in another interesting way.
In Appendix 2 we show another way of getting this expression without using \( p_z \) as an intermediary step.

II. Spin 0-Spin 1/2 REACTIONS

1. Density Probability of the Barrelet's Zeros

For spin 0 spin 1/2 reactions there are two amplitudes to deal with, and we must make a choice of a representation in which to express these amplitudes and their zeros. We will consider Barrelet's representation since it provides a close resemblance to the spin 0-spin 0 case, and also because it deals with the zeros of the eigenvalues of the transition matrix. In Barrelet's representation, \( \lambda \) the amplitude is written \( A(w) \) and is the analogue of \( f(z) \). He defines a transverse section \( I(w) \) which corresponds to \( o(z) \). For a given energy we have:

\[
I(w) = A(w) \bar{\lambda}(\bar{w}^{-1}) = a \sum_{i=1}^{n} \left( w - w_i \right) \left( w_i^{-1} - \bar{w}_i \right)
\]

\( a \) is equal to \( \frac{\Sigma(\theta = 0)}{\sum_{i=1}^{n} ||1 - w_i||^2} \). \( a \) is positive because \( \Sigma(\theta) \) is positive.

The remarks of Section I.1 apply here too.

We can rewrite \( \Sigma(w) \) as

\[
\Sigma(w) = \frac{a(-1)^n}{\omega^n} \left( \bar{w}_1 \right) \left( w - w_i \right) \left( w_i^{-1} - \bar{w}_i \right)
\]

\[
= \frac{1}{\omega^n} \left[ a_0 + a_1 \omega + \cdots + a_n \omega^n \right]
\]

Because \( \Sigma(w) = \bar{\Sigma}(\bar{w}^{-1}) \) we have:

\[
a_0 = \bar{a}_2n, \cdots, a_k = \bar{a}_{2n-k}, \cdots, a_n = \bar{a}_n
\]

Let's call \( \omega_i^M = 1/\omega_i \). \( M \) means mirror (with respect to the unit circle).
we will take as independent variables:

\((a_0, \ldots, a_n) \text{ and } (\omega_1, \ldots, \omega_n, k)\).

As in Section I, we are interested in the Jacobian:

\[
J = \frac{\partial \left( a_0, a_0, \ldots, a_n \right)}{\partial \left( \omega_1, \omega_1, \ldots, \omega_n, \omega_n, a \right)}
\]

\[
\frac{3a^R}{\partial \omega} = \frac{1}{2} \left( \frac{3a + 3a}{\partial \omega} \right)^R;
\]

\[
\frac{3a^R}{\partial \omega} = -\left( \frac{3a}{\partial \omega} \frac{3a}{\partial \omega} \right)^I
\]

\[
\frac{3a^I}{\partial \omega} = \left( \frac{3a}{\partial \omega} \frac{3a}{\partial \omega} \right)^I, \quad \frac{3a^I}{\partial \omega} = \left( \frac{3a}{\partial \omega} \frac{3a}{\partial \omega} \right)^R
\]

After some manipulations we get for the determinant:

\[
J = \begin{vmatrix}
\frac{3a_k}{\partial \omega_j} & \frac{3a_k}{\partial \omega_j} & \frac{3a_k}{\partial \omega_j} \\
\frac{3a_k}{\partial \omega_j} & \frac{3a_k}{\partial \omega_j} & \frac{3a_k}{\partial \omega_j} \\
\frac{3a_k}{\partial \omega_j} & \frac{3a_k}{\partial \omega_j} & \frac{3a_k}{\partial \omega_j}
\end{vmatrix}
\]
with rearrangement of rows and columns and without looking at the sign since we have to take the absolute value, we end up with:

\[
\begin{vmatrix}
3a_0 & \cdots & 3a_0 & \cdots & 3a_0 & 3a_0 \\
3a_0 & \cdots & 3a_0 & \cdots & 3a_0 & 3a_0 \\
\vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
3a_n & \cdots & 3a_n & \cdots & 3a_n & 3a_n \\
\vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
3a_0 & \cdots & 3a_0 & \cdots & 3a_0 & 3a_0 \\
\end{vmatrix}
\]

J = \left[ \prod_{i=1}^{n} (w_i M) \right] \det \left[ \begin{array}{cccc}
3a_0 & \cdots & 3a_0 & 3a_0 \\
3a_0 & \cdots & 3a_0 & 3a_0 \\
\vdots & \cdots & \vdots & \vdots \\
3a_n & \cdots & 3a_n & 3a_n \\
\vdots & \cdots & \vdots & \vdots \\
3a_0 & \cdots & 3a_0 & 3a_0 \\
\end{array} \right]

\left[ \prod_{i=1}^{n} (w_i M) \right] \det \left[ \begin{array}{cccc}
3a_0 & \cdots & 3a_0 & 3a_0 \\
3a_0 & \cdots & 3a_0 & 3a_0 \\
\vdots & \cdots & \vdots & \vdots \\
3a_n & \cdots & 3a_n & 3a_n \\
\vdots & \cdots & \vdots & \vdots \\
3a_0 & \cdots & 3a_0 & 3a_0 \\
\end{array} \right]

\begin{align*}
\frac{\partial^2 n}{\partial w_1} &= 0 \quad \text{and} \quad \frac{\partial^2 n}{\partial w_2} = 0; \quad \text{furthermore:} \quad \frac{\partial^2 n}{\partial a} = \frac{(-)^n}{n!} \prod_{i=1}^{n} w_i^n.
\end{align*}

Again we make use of theorem (1) of 1.2.

\[
J = \left[ \prod_{j=1}^{n} (v_j M) \right] \det \left[ \begin{array}{cccc}
3a_1 & \cdots & 3a_1 \\
3a_1 & \cdots & 3a_1 \\
\vdots & \cdots & \vdots \\
3a_n & \cdots & 3a_n \\
\vdots & \cdots & \vdots \\
3a_1 & \cdots & 3a_1 \\
\end{array} \right]
\left[ \prod_{j=1}^{n} (v_j M) \right] \det \left[ \begin{array}{cccc}
3a_1 & \cdots & 3a_1 \\
3a_1 & \cdots & 3a_1 \\
\vdots & \cdots & \vdots \\
3a_n & \cdots & 3a_n \\
\vdots & \cdots & \vdots \\
3a_1 & \cdots & 3a_1 \\
\end{array} \right]
\]

where ñ stands for the set \( \{v_1, \ldots, v_n, v_1^M, \ldots, v_n^M \} \)

and the final formula is

\[
p_w(\omega_1, \omega_2, \ldots, \omega_n, \omega_1, \ldots, \omega_n, \alpha) = \frac{1}{n!} \prod_{i<j}^n |\omega_i - \omega_j|^2 |1 - \omega_i^2||^2 \int_{1}^{n} 1 - |\omega_i|^2|^{2n} \nu_a .
\]

(II.1)

In Section III we will discuss this formula. In a slightly different form we can rewrite this pdf as a function of \( x(\omega) \).

\[
p_w(\omega, \alpha) = \frac{\alpha}{1!} \prod_{i=1}^{n} (w_i M) \prod_{i=1}^{n} \left| \frac{d^2 x(\omega)}{d\omega} \right|_{\omega=\omega_i} \mu_a
\]

but

\[
a_{2n} = \frac{(-)^n}{n!} \frac{\alpha}{1!} \prod_{i=1}^{n} (w_i M) = \left| \prod_{i=1}^{n} \left| w_i M \right| \right|^0 = \left| a_{0} \right|^n .
\]

Thus

\[
p_w(\omega, \alpha) = \frac{\alpha}{1!} \prod_{i=1}^{n} (w_i M) \prod_{i=1}^{n} \left| \frac{d^2 x(\omega)}{d\omega} \right|_{\omega=\omega_i} \mu_a
\]

(II.2)

Again we get \( p_w(\omega) \) by an integration over \( \alpha \)

\[
p_w(\omega_1, \ldots, \omega_n) = \int_{0}^{w} p_w(\omega_1, \ldots, \omega_n, \alpha) \mu_a .
\]
2. Density Probability of the Partial Waves in
Spin \( \frac{1}{2} \) Spin 0 Reactions

Choosing \( n \) zeros among the \( 2n \) given by our approximation of \( Z(w) \), leads to the same \( 2^n \) discrete ambiguities as in spin 0-spin 0. Also we have to make a choice of a global phase \( \phi \). A third ambiguity appears because it's possible to multiply the amplitude by any power of \( w \), since:

\[
    w^n (w^{-1})^n = 1.
\]

The main effect of this ambiguity is to bring about high partial waves. The amplitude \( A(w) \) can be written as:

\[
    A(w) = w^{-\frac{1}{2}} \sqrt{a} e^{i\phi} \prod_{i=1}^{n} (w - w_i^*)
\]

or

\[
    A(w) = w^{-\frac{1}{2}}(b_0 + b_1 w + \ldots + b_n w^n)
\]

\[
    A(w) = \sum_{J_k, \ell} T_{J_k, \ell} R_{J_k}(w)
\]

\[
    T_{J_k} = \frac{n_{J_k} e^{2\pi iJ_k - 1}}{2\pi k}
\]

\( k \) is in this formula the center of mass momentum. The pseudopolynomials \( R_{J_k}(w) \) have been introduced in Ref. 2 and will not be described here.

As for spin 0 spin 0, the pdf of the \( T_{J_k} \) is related to the pdf of the \( b_i \) by a multiplicative constant since the relationship between \( T_{J_k} \) and \( b_i \) is linear. This is detailed in Appendix 2.

Because the calculus goes along very similar lines as in Section I, we will omit it and state the result.

\[
    p_b(b_0^R, b_0^I, \ldots, b_{n-1}^R, b_{n-1}^I; \sqrt{a}) = \frac{n!}{(\sqrt{a})^{2n}} \prod_{i<j} ||w_i^* - w_j||^2
\]

so that

\[
    p_b(b, \sqrt{a}) = 2(\sqrt{a})^{2n} \prod_{i<j} ||1 - w_i^* w_j||^2 \prod_{i} ||1 - |w_i||^2 ||^2
\]

As in the spin 0 spin 0 case we can rewrite this equation in terms of the amplitude \( A(w) \).

\[
    p_b = 2||b||^{n+1} \prod_{i=1}^{n} \left| w_i^* \right|^{n+1} \left| A(w_i^*) \right|^2 \prod_{i=1}^{n-1} \left| w_i \right|^2
\]
\[ S(\text{statistic}) = a_2^n \cdot p_a. \]

\( D \) is independent of the pdf of the coefficients \( a_n \) and depends only on the roots of the physical reaction under study, thus the term dynamic. \( S \) is dependent on the statistics of the experiment and is very close to a multinomial variable. A maximum in \( S \) does not correspond to a maximum in \( p = SD z \) because the term \( D \) leads to a bias which tends to push the roots away from each other.

Consider, as a matter of illustration, two roots \( z_1, z_2 \).
Suppose \( z_1 \) fixed and vary \( z_2 \). Figure 2 shows qualitatively the phenomenon of the bias. One striking consequence of this phenomenon is that a double zero in the amplitude will appear as two distinct zeros. There is a splitting of double roots.

The phenomenon of the bias and the splitting of double roots are shown in Figures 3b and Figure 3c. This is a computer simulation of 100 experiments of 40,000 events each. The physical region in \( \cos \theta \) is split into 25 bins. The underlying amplitude for these simulated experiments is made of two zeros represented by open circles. The dots are the zeros estimated from each set of simulated data.

2. The formulae for the partial waves (I.3), (I.4), and (II.3), (II.4), differ in two aspects from those for the zeros. First no integration is involved, but this is not surprising since when we consider the zeros we are left with one parameter less than the approximations for the data. In that sense the formula are simpler for the partial waves. Second the pdf for the zeros can be expressed as a function of the cross section and the pdf of the partial waves can be expressed in terms of the amplitude. This indicates again that

the fundamental difference between the two sets of parameters is that the zeros are directly related to the cross section (the data points), while the partial waves are directly related to the amplitude.

3. When we consider the energy dependence of the zeros already extracted from the data, it seems unlikely that any two initially distinct zeros could merge into a double root. However, the data indicate that some zeros' trajectories do cross the physical region. We know that the zeros come in mirror pairs (with respect to the real axis for spin 0, spin 0, or the unit circle for spin \( \frac{1}{2} \) spin 0) in the cross section, so that when one zero touches the physical region it coalesces with its mirror image leading to a double root. These zeros on the physical region are of theoretical and practical importance. Theoretically an amplitude with a zero crossing the physical region and an amplitude with a zero bouncing back are very different. For instance, they give opposite signs for the parity of resonances. Practically these crossing points lead to an asymmetry parameter of one (in spin \( \frac{1}{2} \) spin 0), allowing an absolute calibration for a polarized target.
IV. APPLICATIONS

Instead of considering various cases for $p_{a}(a)$ like a multinormal distribution, we think it is more fruitful to derive practical formulae of the confidence domains and of the biases for the zeros.

1. Spin 0 Spin 0

When going from the experimental data to the estimated roots we follow the path below.

- **Data:** $\sigma_{i}$ -- $\sigma(z)$ is mathematically approximated by a polynomial that we call also $o(z) = a_{0} + \ldots + a_{2n}z^{2n}$
- **Linear estimators of $a_{j}$** -- the coefficients $a_{j}$ are estimated from the experimental values $\sigma_{j}$ by the method of moments. This is linear and no bias is introduced. The estimated polynomial is then: $\hat{o}(z) = \sum_{j=1}^{2n} \hat{a}_{j} z^{j}$.
- **Nonlinear estimators of $z_{k}$** -- The estimators $\hat{z}$ of the roots of $\hat{o}(z)$ are taken as the roots of $\hat{o}(z)$. This is nonlinear so it is biased.

Approximation of $\text{cov}(z_{1}^{i}, z_{1}^{j})$

\[
\sigma(z) = a_{0} + a_{1}z + \ldots + a_{2n}z^{2n} = \sum_{j=1}^{2n} \hat{a}_{j} z^{j}
\]

with $\hat{a} = (a_{0}, \ldots, a_{2n})$ and

\[
\hat{p}^{T}(z) = \begin{pmatrix}
1 \\
z \\
\vdots \\
z^{2n}
\end{pmatrix}
\]

we saw in 1.3 that

\[
\frac{\partial \sigma}{\partial z_{1}}|_{z=z_{1}} = -\frac{\partial \sigma}{\partial z}|_{z=z_{1}}
\]

Let's define $\sigma_{1}^{i} = \frac{\partial \sigma}{\partial z}|_{z=z_{1}}$ then $\hat{\delta}z_{1} = \frac{\sigma_{1}^{1}}{\sigma_{1}^{i}}$

But

\[
\delta\sigma(z = z_{1}) = \delta\hat{p}^{T}(z_{1})
\]

\[
\text{cov}(z_{1}^{i}, z_{1}^{j}) = \langle \hat{\delta}z_{1}, \hat{\delta}z_{1} \rangle = \begin{pmatrix}
\frac{\delta\hat{a}_{1}^{T}(z_{1})}{\sigma_{1}^{i}} & \frac{\delta\hat{a}_{2}^{T}(z_{1})}{\sigma_{1}^{i}} \\
\frac{\delta\hat{a}_{1}^{T}(z_{1})}{\sigma_{1}^{i}} & \sigma_{1}^{i}
\end{pmatrix}
\]

but, $\delta\hat{a}_{1}^{T}(z_{1}), \delta\hat{a}_{2}^{T}(z_{1})$ can be written:

\[
(1, z_{1}, \ldots, z_{1}^{2n}) \begin{pmatrix}
\delta a_{0} \\
\delta a_{1} \\
\vdots \\
\delta a_{2n}
\end{pmatrix} = (1, z_{1}, \ldots, z_{1}^{2n}) \begin{pmatrix}
\delta a_{0} \\
\delta a_{1} \\
\vdots \\
\delta a_{2n}
\end{pmatrix}
\]
so the formulae are very simple since we do not need a matrix inversion.

If we want to look at covariance between real parts or imaginary parts we have:

\[
\text{cov}(z_i, z_j) = \frac{P_k(z_i)}{\sigma_i} \text{cov}(a_k, a_j) \frac{P_k^T(z_i)}{\sigma_j}, \quad \sigma_i' = a_{2n} \sum_{k=1, k \neq i}^{2n} (z_i - \bar{z}_k)
\]

\[
\text{cov}(z_i^R, z_j^R) = \left( \frac{P(z_i)}{\sigma_i} \right)^R \text{cov}(a, a) \left( \frac{P(z_j)}{\sigma_j} \right)^R
\]

which gave us such a simple formula for the covariances.

Approximation of the biases

To avoid too many indexes we will search bias(z_i) this being good for any roots. To the first order \( z_1 - \langle z_1 \rangle = \frac{\partial^2 \bar{z}_1}{\partial z_i^2} (z \rightarrow z_i) \) and \( \langle z_1 - \langle z_1 \rangle \rangle = 0. \) Now in order to find the bias we have to develop \( z_1 - \langle z_1 \rangle \) or \( \frac{\delta z_1}{\sigma_1} \) up to the second order.

We know that:

\[
\sum_{i=1}^{n} \frac{\partial \bar{z}_1}{\partial \bar{z}_1} \delta z_1 = \frac{\partial \bar{z}_1}{\partial \bar{z}_j} \delta z_1
\]

is null except when \( i = 1 \) and \( j \neq 1, \) or \( j = 1 \) and \( i \neq 1, \) so that:

\[
\sum_{i, j}^{2n} \frac{\partial \bar{z}_1}{\partial \bar{z}_i} \delta z_i \delta z_j = \frac{\partial \bar{z}_1}{\partial \bar{z}_j} \delta z_1 \delta z_j
\]

so

\[
\delta \bar{z}(z - z_1) = \frac{\delta z_1}{\sigma_1} \left[ 1 - \sum_{j=2}^{2n} \frac{\delta z_j}{z_1 - \bar{z}_j} \right]
\]

and for any root \( z_j, \)
bias(z_j) = \left< \frac{\partial \sigma_i}{\partial a_j} \right> = - \sum_{j=1}^{2n} \frac{\text{cov}(z_j, z_i)}{z_i - \bar{z}_i} \tag{IV.2}

where again \( \bar{z}_i \in \{z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n \} \). This formula can be interpreted as follows.

The root \( z_1 \), for instance, has a bias which depends on all the other roots, but the farther the root the less the influence and the less these roots are correlated the less is their influence on the bias. As an illustration, let's take the example of only one zero (Figure 4)

\[
\text{bias}(z_1) = - \frac{\text{cov}(z_1, \bar{z}_1)}{z_1 - \bar{z}_1} \quad z_1 = x_1 + iy_1
\]

\[
\text{cov}(z_1, \bar{z}_1) = \langle \|z_1\|^2 \rangle - \langle |z_1| \rangle^2
\]

is a positive real number.

This formula can be applied to the root \( z_1 \). As an illustration, let's take the example of only one zero (Figure 4)

\[
\text{bias}(z_1) = - \frac{\text{cov}(z_1, \bar{z}_1)}{z_1 - \bar{z}_1} \quad z_1 = x_1 + iy_1
\]

The root \( z_1 \) has a bias which depends on all the other roots, but the farther the root the less the influence and the less these roots are correlated the less is their influence on the bias. As an illustration, let's take the example of only one zero (Figure 4)

\[
\text{bias}(z_1) = - \frac{\text{cov}(z_1, \bar{z}_1)}{z_1 - \bar{z}_1} \quad z_1 = x_1 + iy_1
\]

The root \( z_1 \) has a bias which depends on all the other roots, but the farther the root the less the influence and the less these roots are correlated the less is their influence on the bias. As an illustration, let's take the example of only one zero (Figure 4)

\[
\text{bias}(z_1) = - \frac{\text{cov}(z_1, \bar{z}_1)}{z_1 - \bar{z}_1} \quad z_1 = x_1 + iy_1
\]

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\[
\text{bias}(z_1) = - \frac{\text{cov}(z_1, \bar{z}_1)}{z_1 - \bar{z}_1} \quad z_1 = x_1 + iy_1
\]
\[ \Sigma(w) = a \prod_{j=1}^{n} (w - w_j)(w^{-1} - \bar{w}_j) = a^T \Phi(w) \]

\[ \Phi(w) = (w^{-n}, \ldots, 1, \ldots, w^n) \]

we find a very similar formula:

\[ \text{cov}(w_1, w_2) = \frac{P_I(w_1)}{E_I(w_1)} \cdot \text{cov}(a_j, a_j) \cdot \frac{P_T(w_2)}{E_I(w_2)} \]  \hspace{1cm} (IV.3)

with

\[ \Sigma'(w_1) = a(w_1^{-1} - \bar{w}_1) \prod_{j=1}^{n} (w_1 - w_j)(w_1^{-1} - \bar{w}_j) \]

in spin 0-spin 0 we had:

\[ \sigma'(z) = \sigma'(\bar{z}) \]

now

\[ \Sigma(w) = A(w) A^M(w) \Rightarrow d\Sigma(w) = d\Sigma^M(w) \]

\[ \Sigma'(w)dw = \Sigma'(w)dw^M \]

but

\[ \delta w^M = -\frac{1}{w^2} \delta w \Rightarrow \Sigma'(w) = -\langle w^2 \rangle^2 \Sigma'(w) \]

Now for the covariances we have:

\[ \langle \delta w_1 \delta w_2 \rangle = \frac{\langle \delta w_1^M \delta w_2^M \rangle}{\langle w_1^M \rangle^2 \langle w_2^M \rangle^2} \]

**Biases**

\[ \delta \Sigma = -a \prod_{j=1}^{n} (w - w_j) dw_1 + \sum_{j=1}^{n} a \prod_{j=1}^{n} (w - w_j)(w - \bar{w}_j) \delta w_1 \delta w_j \]

\[ \text{bias}(w_1) = -\sum_{i=2}^{n} \frac{\text{cov}(w_1, w_i)}{w_1 - w_i} - \sum_{i=1}^{n} \frac{\text{cov}(w_1, \bar{w}_i)}{\bar{w}_1 - \bar{w}_i} \]

We can rewrite this expression in another way since:

\[ \text{cov}(w_1, \bar{w}_1) = -\frac{\text{cov}(w_1, w_i)}{w_1^i + 2} \]

Thus for any root \( w_j \):

\[ \text{bias}(w_j) = -\sum_{i=1}^{n} \frac{\text{cov}(w_j, w_i)}{w_j - w_i} - \sum_{i=1}^{n} \frac{w_i \text{cov}(w_j, \bar{w}_i)}{\bar{w}_j - \bar{w}_i} \]  \hspace{1cm} (IV.4)

This formula differs from spin 0 spin 0 because the distances are deformed in the \( w \) plane.

Let's see what happens for only one zero. \( w_1 = re^{i\phi} \)

\[ \text{bias}(w_1) = -\frac{\text{cov}(w_1, \bar{w}_1)}{w_1^{-1} - \bar{w}_1} \]
bias(w₁) = \left< \left| \omega_{w₁} \right|^{2} \right> e^{14 \rho - \frac{1}{\rho}}

when \( \rho > 1 \) we have a bias which is along Ow₁ and tends to push w₁ away from the unit circle. (Figure 6.)

Approximate confidence domains

As in spin 0 spin 0, we can draw the ellipses containing 70% CL. Another set of variables, more appropriate to the w plane, is

\( w = re^{i\theta} \Rightarrow \frac{dw}{w} = \frac{dr}{r} + i d\theta \)

so that

\[ \sigma_r^2 = 2 \text{ cov}(r, r) = \text{cov}(w, \bar{w}) + \text{Re}(e^{-2i\theta} \text{cov}(w, w)) \]

\[ \sigma_l^2 = 2 \text{ cov}(l, l) = \text{cov}(w, \bar{w}) - \text{Re}(e^{-2i\theta} \text{cov}(w, w)) \]

\[ dt = rd\theta \text{ we can verify easily that} \]

\[ \sigma_1/r = \frac{1}{r^2} \sigma_r \quad \text{and} \quad \sigma_1/l = \frac{1}{l^2} \sigma_l \]

3. Approximation of the Bias When Two Zeros Cross Each Other

We saw in Fig. 3c the splitting of a double zero. We will now show how to estimate this splitting for both spin 0 spin 0 and spin \( \frac{1}{2} \) spin 0 scattering. Let's call \( z₁ \) the position of the double root. We have for the statistical term S (Section III.2) a confidence domain approximated by an ellipse (Fig. 5). Let's take as coordinate axis the axis of this ellipse, and let's call r,\( \theta \) the polar coordinates. The pdf for the zeros is approximated by:

\[ p_z = \left| z_1 - z_2 \right|^2 \]

\[ p_e = \left[ \frac{r^2 \left( \cos^2 \theta + \sin^2 \theta \right)}{\sigma_1^2 + \sigma_2^2} \right] \]

\[ \frac{dp_e}{dr} = \frac{2}{r} \left[ \cos \theta \sin \theta + \sin \theta \cos \theta \right] \]

Thus we have maximums for

\[ \theta = 0, \pi \quad \text{and} \quad r = \sqrt{2} \sigma_1 \]

\[ \theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad r = \frac{\sqrt{2}}{r} \sigma_2 \]

The maximum bias is equal to roughly one standard deviation and a half. This indicates a practical way of detecting ambiguous points. As soon as the confidence domains, defined as in Fig. 5, of two zeros, are in contact, we must admit the possibility of a double root.

4. How Good Are Our Approximate Formulae?

In order to test the validity of our approximate formulae we performed a computer simulation of a hundred scattering experiments with a given polynomial amplitude. For instance, in Fig. 3a the zeros of the underlying amplitude are \((-0.5 + 2i)\) and \((0.5 + 2i)\). They are represented by open circles. For each simulated scattering
experiment we generated 40,000 events and considered 25 bins equally spaced in \( \cos^0_{\text{CM}} \). The dots are the estimated zeros from each experiment. We drew the rectangles in which our elliptic confidence domains are inscribed. These rectangles should give approximately an 80% confidence level. This is true in Fig. 3a. Figure 3b differs from Fig. 3a in that we choose the zeros of the underlying amplitude closer to each other, namely: \((-0.2 + 2i)\) and \((0.2 + 2i)\). We begin to see, as expected, a bias and the failure of our approximate confidence domains (see III.1). Clearly if we had generated 160,000 events for each simulated experiment, the clouds of estimated zeros would have shrunk by a factor of two, giving us, again, a clear separation and a good behavior of our approximate formulae. It is clear that the notion of two zeros coalescing depends on our information or our statistics.

We tested our approximations for real data: \( \pi^+ p \) elastic at \( p_{\text{lab}} = 1439 \) MeV/c. The clouds of points in Fig. 7 correspond to the following simulation: In each error bar of the data points, we vary the value of the cross sections according to a gaussian distribution whose standard deviation is given by this error bar. The elliptic confidence domains are drawn and we verify that they are in good agreement with the simulation.

CONCLUSION

One of the aims of this article was to show the usefulness of deriving the rigorous probability density functions, instead of doing the usual linear error propagation. Indeed, for zeros and partial waves in spin 0 spin 0 and spin \( \frac{1}{2} \) spin 0 reactions we found that very strong biases and total breakdown of linear formulae are expected for two cases of great theoretical and practical importance: first, when two zeros cross each other, and second, when a zero crosses the physical region. As of now, in \( \pi N \) reactions the first case seems very unlikely\(^2\). The second case is of real importance, as we saw in Section III.3. The existence of strong biases implies that one must consider the energy dependence of a zero near its supposed crossing point in order to determine whether the trajectory actually crosses the physical region.

For all other configurations our computer simulation, both with real and purely artificial data, shows that the practical approximations, given in Section IV, are sufficient. The biases in these usual cases are almost negligible and can be reliably estimated.

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APPENDIX 1

Given a complex polynomial \( a_0 + a_1z + \cdots + a_nz^n = a_n \prod_{j=1}^{n} (z - z_j) \) we want to calculate:

\[
\det \left( \frac{\partial a_i}{\partial z_j} \right)
\]

The coefficients \( a_i \) are symmetrical functions of the roots \( z_j \).

\[
a_0 = (-)^n a_n z_1 \cdots z_n , \ldots , a_{n-1} = -a_n (z_1 + \cdots + z_n) .
\]

Let us call

\[
S_i^n = \frac{a_i}{a_n} (-)^{n-1} .
\]

This for \((z_1 \cdots z_n)\). Now we define symmetrical functions

\[
S_i^{n-1}(z_2 , \cdots , z_n)
\]

in the same way. We have

\[
S_p^n(z_1 , \cdots , z_n) = z_1 S_p^{n-1}(z_2 , \cdots , z_n) + S_p^{n-1}(z_2 , \cdots , z_n)
\]

As

\[
\frac{\partial a_i}{\partial z_j} = a_n \cdot \begin{vmatrix}
\frac{\partial S_0^n}{\partial z_0} & \cdots & \frac{\partial S_0^n}{\partial z_n} \\
\frac{\partial S_1^n}{\partial z_0} & \cdots & \frac{\partial S_1^n}{\partial z_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial S_{n-1}^n}{\partial z_0} & \cdots & \frac{\partial S_{n-1}^n}{\partial z_n}
\end{vmatrix}
\]

\[
\frac{\partial a_i}{\partial z_j} = a_n \cdot \begin{vmatrix}
\frac{n(n+1)}{2} & \cdots & \frac{n(n+1)}{2} \\
\frac{(-1)^2}{2} & \cdots & \frac{(-1)^2}{2} \\
\vdots & \ddots & \vdots \\
\frac{n(n+1)}{2} & \cdots & \frac{n(n+1)}{2}
\end{vmatrix}
\]

\[
\det \frac{\partial a_i}{\partial z_j} = a_n^n \cdot \begin{vmatrix}
\frac{n(n+1)}{2} & \cdots & \frac{n(n+1)}{2} \\
\frac{(-1)^2}{2} & \cdots & \frac{(-1)^2}{2} \\
\vdots & \ddots & \vdots \\
\frac{n(n+1)}{2} & \cdots & \frac{n(n+1)}{2}
\end{vmatrix}
\]

but

\[
\frac{\partial S_k^n}{\partial z_1} = \frac{\partial S_k^{n-1}(z_2 , \cdots , z_n)}{\partial z_1} .
\]

It's easy to show that for \( n = 2, D_z = z_2 - z_1 \). We will proceed by recurrence.

Suppose thus that:

\[
D_0(z_1 , \cdots , z_n) = D_{n-1}(z_2 , \cdots , z_n) \prod_{i=2}^{n}(z_i - z_1)
\]

\[
D_{n+1}(z_1 , \cdots , z_n) = S_{n+1}^n(z_2 , \cdots , z_n)
\]

Subtracting the last column to the others we get

\[
D_{n+1} = \begin{vmatrix}
S_0^n(z_1) - S_0^n(z_{n+1}) & \cdots & S_0^n(z_{n+1}) \\
\vdots & \ddots & \vdots \\
S_{n-1}^n(z_1) - S_{n-1}^n(z_{n+1}) & \cdots & S_{n-1}^n(z_{n+1})
\end{vmatrix}
\]

\[
S_{n+1}^n(z_1) \] means that we take the set \((z_1 , \cdots , z_{n+1})\) etc. The first column gives:

\[
S_k^n(z_1) - S_k^n(z_{n+1}) = z_{n+1} S_k^{n-1}(z_2 , \cdots , z_n) - z_1 S_k^{n-1}(z_2 , \cdots , z_n)
\]

we have thus \( z_{n+1} - z_1 \) in common factor and we are left with \( S_k^{n-1} \).

All the columns, but the last one, give the \( D_{n-1} \) we were looking for.

Hence we have that:
APPENDIX 2

Spin 0-0

As we said in 1.4 the pdf for the partial waves is proportional to the pdf of the b's.

\[ T_k = \frac{1}{2} \int_{-1}^{+1} (b_0 + \cdots + b_n z^n) P_k(z) dz \]

where \( P_k(z) \) are the Legendre polynomials.

We will use the bracket notation \( \langle b | g \rangle = \int_{-1}^{+1} h g dz \) and recall \( \langle z^k | P_k \rangle = 0 \) if \( k < k \).

Now we will calculate the multiplicative constant when the global phase \( \phi(z) \) is taken as a constant.

\[ 2T_k = b_k \langle z^k | P_k \rangle + \cdots + b_{n-1} \langle z^{n-1} | P_1 \rangle + \langle b_n | \langle z^n | P_0 \rangle \]

we look for:

\[
\frac{\det\left( \frac{3T_k}{\bar{b}_j} R, I, \ldots, R, I, b_0, \ldots, b_{n-1}, \bar{b}_n \right)}{\det\left( \frac{3T_k}{\bar{b}_j} R, I, \ldots, R, I, b_0, \ldots, b_{n-1}, \bar{b}_n \right)} = \left| \det\left( \frac{3T_k}{\bar{b}_j} R, I, \ldots, R, I, b_0, \ldots, b_{n-1}, \bar{b}_n \right) \right|^2 \quad \text{(because} \; T_k \text{ does not depend on} \; \bar{b}_j \text{)}
\]

\[
= 2^{-2n} \sum_{i=0}^{n-2} \left| \langle z_i | P_i \rangle \right|^2
\]
When we consider n zeros we have for the maximum total angular momentum

\[ J_{\text{MAX}} = \frac{1}{2} + E\left(\frac{n}{2}\right) \]

E(x) means the integer part of x.

\[ \epsilon_{\text{MAX}} = (-)^n \]

\( \epsilon \) is not the parity, \( \epsilon \) is the naturality.

\[ L = J - \frac{\epsilon}{2} \]

\[ T_{J\epsilon} = \frac{1}{2(2J + 1)} \int_{\gamma} A(w) \tilde{R}_{J\epsilon}(w) 2|w - \frac{1}{w}||d||w|| \]

\( \int_{\gamma} \) is the integral along the unit circle, and we will introduce also a bracket notation for the scalar product. This should not be mixed with the similar notation of spin 0-spin 0.

In order to have the lowest partial waves we must choose

\[ q = L = E\left(\frac{n + 1}{2}\right) \]

\[ A(w) = w^{-L}\sqrt{2\pi} e^{i\phi(w)} \sum_{n=0}^{n} (w - w_1) = w^{-L}(b_0 + \cdots + b_n w^n) \]

Again the multiplicative constant will be calculated when \( \phi(w) = c \).

Here there is a difference with spin 0-spin 0. We have to consider separately odd and even number of zeros.

n even

\[ L = \frac{n}{2} \]

\[ P_T(T_1) = \frac{2(2L + 2)}{4(2p + 2)} \frac{P_b(b_1)}{4(2p + 2)^2} \]

n odd

\[ L = \frac{n + 1}{2} \]

\[ P_T(T_4) = \frac{(4L)^2 |d^p 1|}{\langle R_{p+b} |w^-L \rangle^2} \]

Another way to find \( p_b(b_1, |b_n|) \) in spin 0-spin 0 reactions.
In the theory of equations this kind of determinant is known as a Sylvester's determinant. This determinant is equal to the resultant of \( f(z) \) and \( \tilde{f}(\tilde{z}) \).

\[
\text{Res}(\tilde{f}(\tilde{z}), f(z)) = b_n f(\tilde{z}_1) \cdots f(\tilde{z}_n).
\]

But:

\[
\prod_{i=1}^{n} f(\tilde{z}_i) = b_n \prod_{i,j} (\tilde{z}_i - z_j)
\]

so we end up with

\[
p_b = p_a 2^{n+1} \cdot \prod_{i} f(\tilde{z}_i)
\]

because \( \tilde{z}_j - z_j = i \) and \( p_b(b_1, ||b_n||) = p_b(b_1, a_{2n}) 2 ||b_n|| \)

which is the same equation as in I.4.
FOOTNOTES AND REFERENCES

* This work was supported by the Energy Research and Development Administration.


4. To our knowledge this general problem is not often studied in most textbooks. In this respect the reference (5) gives a lot of simple examples for the case of two random variables function of two random variables.
7. We use the covariances of complex quantities. This is not a problem and usually they are defined in the following manner:
\[ \text{cov}(z_1, z_2) = (z_1 - \langle z_1 \rangle)(\bar{z}_2 - \langle z_2 \rangle) = \langle z_1 \bar{z}_2 \rangle - \langle z_1 \rangle \langle z_2 \rangle \]
This is so because \( \text{cov}(z, z) = \langle |z|^2 \rangle - ||z||^2 \) is real.
But we choose another convention. That is
\[ \text{cov}(z_1, z_2) = \langle z_1 z_2 \rangle - \langle z_1 \rangle \langle z_2 \rangle. \]
8. We have \( \text{bias}(z_1) = \text{bias}(z_1) \) as we should.
9. When \( \phi \) is not a constant we have:
\[ 2T_\lambda = \sum_{n=0}^{\infty} \langle \xi(z) \rangle \sum_{n=0}^{\infty} \langle p(z) \rangle + \sum_{n=0}^{\infty} \langle \xi(z) \rangle \sum_{n=0}^{\infty} \langle p(z) \rangle \]
with \( b_j e^{i\phi(z)} = b_j \).
FIGURE CAPTIONS

Fig. 1. Example of two random variables $x$ and $y$ linked by a functional dependence. $y = g(x)$ has two roots, $x_1, x_2$ in this case.

Fig. 2. Illustration of the origin of the biases. The upper part shows the pdf of the two factors $S$ and $D$ (see Section III.1). $S$ is centered and maximum for $z = \langle z \rangle$, the product $SD$ is shown in the lower part and displays a bias.

Fig. 3. (a) Computer simulation of 100 experiments whose scattering amplitude has two zeros at $-0.5 + 2i$ and $0.5 + 2i$, represented by open circles. On this case we check the validity of our approximate formulae for the confidence domains (rectangles). See Section IV.4. (b) Same as for 3a, except that the roots are $-0.2 + 2i$ and $0.2 + 2i$. We begin getting into troubles with our approximate confidence domains (rectangles). No longer 80% of the dots (the estimated zeros) stands inside and the bias is clearly visible by the depopulation of the region between the actual roots. (c) Same as for (a) and (b) but we have a double root at $2i$. There is, clearly, a big hole where the double root is located (open circle).

Fig. 4. In the complex cost plane, the actual zero $z_1$ and the estimated zero. The bias tends to push apart $z_1$ from $\bar{z}_1$.

Fig. 5. Approximate confidence domains for the zero $z_1$.

$$ o_1^2 = 2 \text{ cov}(x_1, x_1), \quad o_2^2 = 2 \text{ cov}(y_1, y_1), $$

$$ r = 2 \text{ cov}(x_1, y_1)/o_1o_2. $$

Inside such an ellipse and with our definitions of $o_1, o_2$ we expect a 70% confidence level.

Fig. 6. Bias of a single root in the $w$ plane. Again the bias tends to push $w_1$ apart from its mirror image $w_1^M = 1/\bar{w}_1$.

Fig. 7. The approximations of the confidence domains match very well the simulation of the experimental data of $\pi^+ p$ elastic scattering at $p_{\text{lab}} = 1439$ MeV/c. As it is predicted by the rigorous calculus, when the zeros are close to the physical region, we cannot trust our approximations any more.
Fig. 1

\[ y = g(x) \]

\[ y + \delta y \]

\[ x_1, x_1 + \delta x_1, x_2, x_2 + \delta x_2 \]
Figure 5 illustrates a bivariate distribution with an ellipse representing the dispersion of data points. The ellipse is centered around the mean values $\langle x_1 \rangle$ and $\langle y_1 \rangle$. The standard deviations along the $x_1$ and $y_1$ axes are denoted as $\sigma_1$ and $\sigma_2$, respectively. The sensitivity coefficients $\eta_1 = \frac{\sigma_1}{\sqrt{1 - \rho^2}}$ and $\eta_2 = \frac{\sigma_2}{\sqrt{1 - \rho^2}}$ are also shown, where $\rho$ represents the correlation coefficient between the variables.
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