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ON THE FIBRE HOMOTOPY TYPE OF NORMAL BUNDLES

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1. INTRODUCTION

It was proved by Atiyah [1] that the fibre homotopy type of the stable normal sphere bundle of a manifold \( M \) is an invariant of the homotopy type of \( M \). Theorem A below (discovered before I learned of Atiyah's proof) gives an elementary proof of this fact, and also applies to nonstable cases. (See [5], for example. One purpose of the present work is to supply item 6 in the bibliography of [5].) The situation considered is a homotopy-commutative diagram

\[
\begin{array}{c}
M_0 \\
\downarrow s_0
\end{array}
\xrightarrow{f} 
\begin{array}{c}
M_1 \\
\downarrow s_1
\end{array}
\quad \quad \quad \quad U_1
\]

with \( f \) a homotopy equivalence, \( g_i : M_i \rightarrow V \) embeddings \((i = 0, 1)\), and \( U_1 \) a closed tubular neighborhood of \( g_1(M_1) \). Theorem A implies that the normal sphere bundles of \( g_0 \) and \( g_1 \) are fibre-homotopically equivalent. Theorem B applies Theorem A to the problem of choosing \( g_0 \) (given \( f \) and \( g \)) so that it will have as many independent normal vector fields as \( g_1 \).

The proof of Theorem A in the case \( \dim V \geq \dim M + 3 \) depends on Lemma 2, due to Milnor, which states that if \( U_0 \) is a closed tubular neighborhood of \( g_0(M_0) \) inside \( \text{int} U_1 \), then \( U_1 - \text{int} U_0 \) is an h-cobordism between the boundaries \( bU_1 \) and \( bU_0 \). This Lemma is no longer universally true if \( \dim V = \dim M + 2 \); Theorem C (which is independent from Theorems A and B) exhibits a special case where it is true. An immediate corollary is that if \( M_0 \times \mathbb{R}^k \) is diffeomorphic to \( M_1 \times \mathbb{R}^k \), then \( M_0 \times S^{k-1} \) is h-cobordant to \( M_1 \times S^{k-1} \). (The interesting case is \( k = 2 \).)

All manifolds, immersions, and embeddings are smooth.

Throughout the paper, \( M_0 \) and \( M_1 \) are compact unbounded manifolds of dimension \( m \), and \( V \) is a Riemannian manifold of dimension \( v \).

2. FIBRE HOMOTOPY TYPE

If \( \alpha \) and \( \beta \) are bundles, then \( \alpha \sim \beta \) indicates that \( \alpha \) and \( \beta \) are isomorphic, while \( \alpha \simeq \beta \) means that \( \alpha \) and \( \beta \) have the same fibre homotopy type. For this concept, the reader is referred to Dold [3].

Let \( f : M \rightarrow V \) be an immersion. If \( \nu \) is the normal vector space bundle of \( f \), then \( \nu \) will denote the normal sphere bundle of \( f \), and conversely.

**THEOREM A.** Let \( g_i : M_i \rightarrow V \) be embeddings \((i = 0, 1)\). Let \( U_1 \subset V \) be a closed tubular neighborhood of \( g_1(M_1) \) such that \( g_0(M_0) \subset U_1 \). Let \( f : M_0 \rightarrow M_1 \) be a homotopy equivalence making a homotopy-commutative diagram.

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Let \( \hat{\nu}_i \) be the normal sphere bundle of \( g_i \). Then

(a) \( f^* \hat{\nu}_1 \sim \hat{\nu}_0 \);

(b) \( f^* \nu_1 \sim \nu_0 \) if \( v - m \leq 2 \).

Remark. If \( v \geq \frac{3}{2}(m + 1) \), and \( f, g_1 \) are given, then \( g_0 \) always exists, by Haefliger [4].

The proof of Theorem A occupies the rest of this section.

In Lemma 1, all the spaces involved are to be CW-complexes.

**LEMMA 1.** Let \( p_i : E_i \to B_i \) be fibre spaces (\( i = 0, 1 \)). A map \( \phi : E_0 \to E_1 \) is a fibre-homotopy equivalence provided

(a) \( \phi \) is a homotopy equivalence,

(b) \( \phi \) covers a homotopy equivalence \( B_0 \to B_1 \).

**Proof.** See Dold [3].

Now let \( U_0 \subset \text{int} \ U_1 \) be a closed tubular neighborhood of \( g_0(M_0) \). (We may assume \( g_0(M_0) \subset \text{int} \ U_1 \).) Put \( N_i = bU_i \).

**LEMMA 2 (Milnor).** If \( v \geq m + 3 \), then \( N_0 \) and \( N_1 \) are deformation retracts of \( U_1 - \text{int} \ U_0 \).

**Proof.** See Lemma 2 of [6].

**LEMMA 3.** If \( N_0 \) and \( N_1 \) are deformation retracts of \( U_1 - \text{int} \ U_0 \), then

\( f^* \hat{\nu}_1 \sim \hat{\nu}_0 \).

**Proof.** Consider the diagram

\[
\begin{align*}
N_0 & \xrightarrow{j} U_1 - \text{int} U_0 & \xrightarrow{r} N_1 \\
\downarrow p_0 & \xrightarrow{g_0} U_1 & \xrightarrow{q_1} \downarrow p_1 \\
M_0 & \xrightarrow{f} M_1
\end{align*}
\]

The maps are defined as follows:

(i) Unlabelled maps are inclusions, and so is \( j \).

(ii) Identify \( U_1 \) with the normal disk bundle of \( g_1 \). Then \( q_1 : U_1 \to M_1 \) is the bundle projection, and \( p_1 = q_1 | N_1 \) is the bundle projection of the normal sphere bundle \( \hat{\nu}_1 \).

(iii) \( r \) is a homotopy inverse to the inclusion \( N_1 \to U_1 - \text{int} U_0 \), which exists by hypothesis.

Each triangle is homotopy-commutative.

The covering homotopy property of the fibre space \( p_1 : N_1 \to M_1 \) implies that \( rj : N_0 \to N_1 \) is homotopic to a map covering \( f : M_0 \to M_1 \). By hypothesis \( rj \) is a homotopy equivalence; consequently Lemma 1 implies Lemma 2.
Part (a) of Theorem A is proved by the lemmas. To prove part (b), first observe that
\[(iv) \quad f^\# w_1(\nu_1) = w_1(\nu_0)\;.
\]
$f^\#$ is the induced homomorphism of cohomology, and $w_1$ denotes the first Stiefel-Whitney class. This follows from the homotopy commutativity of the diagram in Theorem A, the Whitney sum theorem, and the fact that $f^\# w_1(M_1) = w_1(M_0)$. If $v = m + 1$, then (iv) suffices to prove that $f^* \nu_1 \sim \nu_0$.

There remains the case $v = m + 2$.

Let $\Gamma$ be a local system of groups on a space $Y$. For each map $f: X \to Y$ there is an induced local system $f^{-1}\Gamma$ on $X$. Let
\[
\tag{11}
f^\#: H_i(X; f^{-1}\Gamma) \to H_i(Y; \Gamma) \quad \text{and} \quad f^\#: H^i(Y; \Gamma) \to H^i(X; f^{-1}\Gamma)
\]
be the induced homomorphisms. If $f$ is the inclusion of the open subset $X$ into $Y$, and $\mathcal{K}$ denotes homology based on infinite chains, there exists an induced homomorphism $f^1: \mathcal{K}_i(Y; \Gamma) \to \mathcal{K}_i(X; f^{-1}\Gamma)$.

Let $F_i$ be the local system on $M_i$ determined by orientations of the fibres of $\nu_i$. As was shown earlier, $f^\# w_1(\nu_i) = w_1(\nu_0)$. Equivalently, $f^{-1}\Gamma_1 = \Gamma_0$. It is known that $f^* \nu_1 \sim \nu_0$ provided that $f^\# X_i = X_0$, where $X_i \in H^2(M_i; F_i)$ is the Euler class of $\nu_i$. We proceed to prove the last equality.

Let $V_i = \text{int } U_i$. Since $f_i: M_i \to V_i$ is a homotopy equivalence, there exist unique local systems $\tilde{F}_i$ on $V_i$ such that $f_i^{-1}\tilde{F}_i = F_i$. Let $G_i$ be the local system corresponding to orientation of the tangent planes of $M_i$, and let $\tilde{G}_i$ be the local system on $V_i$ such that $\tilde{f}_i^{-1}\tilde{G}_i = G_i$. Since $g_0 \simeq f g_1$ in $V_1$, it follows that if $\phi: U_0 \to U_1$ is the inclusion, then $\phi^{-1}G_1 = G_0$.

The local system of orientations of $V_i$ is easily seen to be $\tilde{F}_i \otimes \tilde{G}_i$. Therefore there is an isomorphism of Poincaré duality $D_i: \mathcal{K}(V_i; \tilde{G}_i) \to H^2(V_i; \tilde{F}_i)$; see [2, p. 4].

It is known that the following diagram commutes:
\[
\begin{array}{ccccccc}
H_m(M_1; G_1) & \xrightarrow{g_1^\#} & \mathcal{K}_m(V_1; \tilde{G}_1) & \xrightarrow{D_1} & H^2(V_1; \tilde{F}_1) & \xrightarrow{g_1^\#} & H^2(M_1; F_1) \\
\uparrow f^\# & & \uparrow \phi^1 & & \uparrow \phi^\# & & \uparrow f^\# \\
H_m(M_0; G_0) & \xrightarrow{g_0^\#} & \mathcal{K}_m(V_0; \tilde{G}_0) & \xrightarrow{D_0} & H^2(V_0; \tilde{F}_0) & \xrightarrow{g_0^\#} & H^2(M_0; F_0)
\end{array}
\]

Let $m_i \in H_m(M_i; G_i)$ be generators such that $f_\#(m_0) = m_1$. A theorem of Thom [7] states that $g_0^\# D_1 g_1^\#(m_0) = X_i$. The commutativity of the diagram shows that $f^\# X_i = X_0$; it follows that $f^* \nu_1 = \nu_0$. Theorem A is proved.

3. NORMAL FRAME FIELDS

Let $g: M \to V$ be an immersion of a manifold in $V$. We say $g$ is $q$-framable if $g$ admits $q$ linearly independent normal vector fields.
THEOREM B. Let a positive integer \( q \) satisfy the condition \( 2(v - q) \geq 3m + 1 \). Let \( f : M_0 \to M_1 \) be a homotopy equivalence and \( g_1 : M_1 \to V \) an immersion or embedding that is q-framable.

(a) If \( g_1 \) is an embedding and \( U \) a tubular neighborhood of \( g_1(M_1) \), there exists a q-framable embedding \( g_0 : M_0 \to U \) that is homotopic to \( g_1 f \) in \( U \).

(b) If \( g_1 \) is an immersion, there exists an immersion \( g_0 : M_0 \to V \) that is homotopic to \( g_1 f \).

(c) In both (a) and (b), \( f^* \tilde{\nu}_1 \simeq \tilde{\nu}_0 \), where \( \tilde{\nu}_1 \) is the normal sphere bundle of \( g_1 \).

Proof. (a) To say that the embedding \( g_1 : M_1 \to V \) is q-framable means that the normal \((v - m)\)-plane bundle \( \nu_1 \) is a Whitney sum \( \nu_1 = \mu_1 \oplus \varepsilon^{q-1} \), where \( \varepsilon^{q-1} \) is the trivial \((q - 1)\)-plane bundle, and the \((v - m - q + 1)\)-plane bundle \( \mu_1 \) has a non-zero section. Factor \( g_1 \) thus:

\[
\begin{align*}
h_1 & : M_1 \to E\mu_1 \subset E\nu_1 \to U, \\
h_0 & : M_0 \to E\mu_1 \subset E\nu_1 \to U,
\end{align*}
\]

where \( E \) indicates total space, \( h_1 \) is the zero cross-section, and \( e \) is a diffeomorphism. The dimension of \( E\mu_1 \) is \( v - q + 1 \). By hypothesis, \( \dim E\mu_1 \geq 3(m + 1)/2 \). Therefore the embedding theory of Haefliger [4] is applicable and says that because the map \( h_1 : M_0 \to E\mu_1 \) is a homotopy equivalence, it is homotopic to an embedding \( h_0 : M_0 \to E\mu_1 \). Let \( \tilde{h}_0 \) be the normal sphere bundle of \( h_0 \), and \( \tilde{\mu}_1 \) that of \( h_1 \). By Theorem A, \( f^* \tilde{\mu}_1 \simeq \tilde{\mu}_0 \). Since \( \tilde{\mu}_1 \) has a section, so has \( \tilde{\mu}_0 \). Clearly the normal \((v - m)\)-plane bundle \( \nu_0 \) of the composite embedding \( g_0 \), defined to be

\[
\begin{align*}
h_0 & : M_0 \to E\mu_1 \subset E\nu_1 \to U,
\end{align*}
\]

has the form \( \alpha \oplus \varepsilon^1 \oplus \varepsilon^{q-1} \). Therefore \( g_0 \) is q-framable, and this proves (a).

Part (b) follows from (a), since an immersion \( g_1 : M_1 \to V \) can be factored:

\[
\begin{align*}
h_1 & : M_1 \to E\nu_1 \to V, \\
h_0 & : M_0 \to E\nu_1 \to V,
\end{align*}
\]

where \( \nu_1 \) is the normal \((v - m)\)-plane bundle of \( g_1 \), \( h_1 \) is an embedding, and \( e \) is an immersion.

Part (c) is a consequence of Theorem A.

4. HOMEOMORPHISMS OF BUNDLES

Let \( E_i \) be the total space of a smooth orthogonal \( k \)-plane bundle over \( M_i \) \((i = 0, 1)\). Let \( B_i \subset E_i \) be the corresponding unit ball bundle. For each \( t > 0 \) put \( t B_i = \{ tx \mid tx \in E_i, \ x \in B_i \} \); let \( N_i = b^2 B_i \) and \( t N_i = b(t B_i) \).

THEOREM C. Let \( \psi : E_0 \to E_1 \) be a diffeomorphism such that \( \psi B_0 \subset \operatorname{int} B_1 \).

Then \( B_1 - \operatorname{int} hB_0 \) is an h-cobordism between \( \psi N_0 \) and \( N_1 \).

Remark. This theorem actually has little to do with manifolds. It can be reformulated so as to apply to bundles over paracompact spaces, with essentially the same proof.

Proof. By compactness of \( M_i \), we can choose \( r, s > 0 \) such that

\[
B_1 \subset \operatorname{int} \psi(rB_0) \quad \text{and} \quad \psi(rB_0) \subset \operatorname{int} sB_1.
\]
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(i) \( B_1 - \text{int} \, \psi B_0 \) is diffeomorphic to \( s B_1 - \text{int} \, \psi (r B_0) \).

To prove (i), choose \( t < 1 \) so that \( \psi B_0 \subset \text{int} \, t B_1 \). Let \( \phi : E_1 \rightarrow E_1 \) be a diffeomorphism taking \( B_1 \) onto \( s B_1 \), leaving \( t B_1 \) fixed. Then choose \( u > r \) so that \( \psi (u B_0) \subset \text{int} \, s B_1 \), and let \( \theta : E_0 \rightarrow E_0 \) be a diffeomorphism taking \( B_0 \) onto \( r B_0 \), leaving \( E_0 - \text{int} \, u B_0 \) fixed. Then \( \psi \theta \psi^{-1} \phi : E_1 \rightarrow E_1 \) is a diffeomorphism taking \( B_1 - \text{int} \, \psi B_0 \) onto \( s B_1 - \text{int} \, \psi (r B_0) \).

(ii) \( \psi N_0 \) is a deformation retract of \( B_1 - \text{int} \, \psi B_0 \).

This follows easily from the facts that \( \psi N_0 \) is a deformation retract of \( \psi r B_0 - \text{int} \, \psi B_0 \), and \( B_1 \) is a retract of \( s B_1 \). Similarly,

(iii) \( s N_1 \) is a deformation retract of \( s B_1 - \text{int} \, \psi (r B_0) \).

From (i) and (iii) it follows that \( N_1 \) is a deformation retract of \( B_1 - \text{int} \, \psi B_0 \). This together with (ii) proves Theorem C.

REFERENCES


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