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Jacobians and branch points of real analytic open maps

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Summary. The main result is that the Jacobian determinant of an analytic open map \( f : \mathbb{R}^n \to \mathbb{R}^n \) does not change sign. A corollary of the proof is that the set of branch points of \( f \) has dimension \( \leq n - 2 \).

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Introduction

The main object of this paper is to prove the following result:

**Theorem 1.** The Jacobian of a real analytic open map \( f : \mathbb{R}^n \to \mathbb{R}^n \) does not change sign.

One of the referees kindly pointed out that the special case of polynomial maps was proved by Gamboa and Ronga [3]:

**Theorem 2** (Gamboa and Ronga). A polynomial map in \( \mathbb{R}^n \) is open if and only if point inverses are finite and the Jacobian does not change sign.

The proof of Theorem 1 is very similar to methods in [3], which are easily adapted to analytic maps; but as Theorem 1 does not seem to be known, a direct proof may be useful.

\( f : \mathbb{R}^n \to \mathbb{R}^n \) denotes a (real) analytic map in Euclidean \( n \)-space. We always assume \( f \) is open, that is, \( f \) maps open sets onto open sets. Denote the Jacobian matrix of \( f \) at \( p \in \mathbb{R}^n \) by \( df_p = \left[ \frac{\partial f_i}{\partial x_j} (p) \right] \). The rank of \( df_p \) is called the rank of \( f \) at \( p \), denoted by \( \text{rk}_p f \); the determinant of \( df_p \) is the Jacobian of \( f \) at \( p \), denoted by \( Jf(p) \). When the analytic function \( Jf : \mathbb{R}^n \to \mathbb{R} \) is everywhere non-negative or everywhere non-positive (in a set \( X \)), we say \( Jf \) does not change sign (in \( X \)).

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The following sets are defined for any $C^1$ map $g : M \to N$ between $n$-manifolds (without boundary):

- the set $R_k = \{ p \in M : \operatorname{rk}_p g \leq k \}$
- the critical set, $C = R_{n-1}$
- the branch set, $B = \{ p \in U : g$ is not a local homeomorphism at $p \}$

Note that $B \subset C$ by the inverse function theorem. When $g$ is analytic, we also define:

- the critical analytic hypersurface $H \subset C$, comprising those points having a neighborhood in $C$ that is an analytic submanifold of dimension $n - 1$
- the constant rank analytic hypersurface $V \subset H$, at which $g|_H$ has locally constant rank

The following results are byproducts of the proof of Theorem 1:

**Theorem 3.**

(i) The restricted map $f|V$ has rank $n - 1$,

(ii) $f$ is a local homeomorphism at every point of $V$,

(iii) $B \subset R_{n-2}$,

(iv) $\dim R_{n-2} \leq n - 2$.

When $n = 2$, conclusions (iii) and (iv) imply $B$ is a closed discrete set; thus in this case $f$ is light, i.e., point inverses are 0-dimensional. From Stoilow [4], which topologically characterizes germs of light open surface maps, we obtain:

**Corollary 4.** When $n = 2$, the germ of $f$ at any point is topologically equivalent to the germ at 0 of the complex function $z^d$ for some integer $d \neq 0$.

A key role in our proofs is played by the following result, Theorem 1.4 of Church [2]:

**Theorem 5** (Church). If $g : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ and open with rank $\geq n - 1$ at every point, then $g$ is a local homeomorphism.

Our results are close to some of those obtained by Church for $C^n$ maps. It is interesting to compare Theorem 3 (iii) and Corollary 4 to the following results from paragraphs 1.5 to 1.8 of his paper [2]:

**Theorem 6** (Church). Let $g : M \to N$ be a $C^n$ map between $n$-manifolds.

(i) If $M = N = \mathbb{R}^n$ and $g$ is light, the following conditions are equivalent:

(a) $g$ is open,

(b) $Jg$ does not change sign,

(c) $B \subset R_{n-2}$.

(ii) If $M$ is compact and $g$ is open, then $g$ is light.
Proofs

Lemma 7. Assume the critical set of \( f \) is \( C = Jf^{-1}(0) = \mathbb{R}^{n-1} \times \{0\} \), and \( f|C \) has constant rank \( k \), \( 0 \leq k \leq n-1 \). Then \( f \) is a local homeomorphism, \( k = n-1 \), and \( Jf \) does not change sign in \( \mathbb{R}^n \).

Proof. It suffices to prove that the conclusion holds in some neighborhood of each point, which we may take to be the origin.

It is convenient to denote points of \( \mathbb{R}^n \) as \((y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}\).

By the rank theorem we assume that in some open cubical neighborhood \( N \) of the origin,
\[
f_i(y, 0) \equiv 0, \quad i = k + 1, \ldots, n. \tag{1}
\]
Identifying \( N \) with \( \mathbb{R}^n \) by an analytic diffeomorphism, we assume this holds for all \( y \in \mathbb{R}^n \).

Because \( f \) is analytic and open, there is a dense open set \( \Lambda \subset \mathbb{R}^{n-1} \) such that for every \( y \in \Lambda \), the map \( t \mapsto f_n(y, t) \) is not constant on any interval. For each \( y \in \Lambda \) there exists a maximal integer \( \mu(y) \geq 0 \) such that
\[
0 < j < \mu(y) \implies \left( \frac{\partial}{\partial t} \right)^j f_n(y, 0) = 0,
\]
Fix \( y_* \in \Lambda \) such that the function \( \mu: \Lambda \to \mathbb{N} \) takes its minimum value \( m \) at \( y_* \). Then \( \mu = m \) in a precompact open neighborhood \( W \subset \Lambda \) of \( y_* \).

By Taylor’s theorem there exists \( \epsilon > 0 \) such that for \((y, t) \) in the open set
\[
N = W \times ] - \epsilon, \epsilon [ \subset \mathbb{R}^{n-1} \times \mathbb{R}
\]
we have
\[
f_n(y, t) = t^m H(y, t), \quad H(y, t) \neq 0. \tag{2}
\]
Claim. If \( k \leq n - 2 \) and \((y_0, t_0) \in N \) is such that \( f_n(y_0, t_0) = 0 \), then \( f_{n-1}(y_0, t_0) = 0 \). For \( t_0 = 0 \) by (2), and \( k \leq n - 2 \) implies \( f_{n-1}(y_0, 0) = 0 \) by (1).

Now we assume \( k \leq n - 2 \) and reach a contradiction. Since \( f(N) \) is open and contains
\[
f(y_*, 0) = (a_1, \ldots, a_{n-2}, 0, 0),
\]
f\( (N) \) also contains points \((a_1, \ldots, a_{n-2}, \delta, 0) \) with \( \delta > 0 \). But this contradicts the claim.

As \( f \) has rank \( n - 1 \) at every point of the critical set, \( f \) must be a local homeomorphism by Theorem 5. Therefore for every \( p \), the induced homomorphism of homology groups
\[
\mathbb{Z} = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(p)\}) = \mathbb{Z}
\]
is an isomorphism, hence is multiplication by a number \( \delta(p) \in \{+1, -1\} \).
Homology theory implies that each of the two level sets of $\delta: \mathbb{R}^n \to \{+1, -1\}$ is open. As $\mathbb{R}^n$ is connected, $\delta(p)$ is constant. As $\delta(p)$ is the sign of $Jf(p)$ if $Jf(p) \neq 0$, we have proved $Jf$ does not change sign.

**Proof of Theorem 1.** For any set $Y \subset \mathbb{R}^n$, we say the local theorem holds in $Y$ if every point of $Y$ has a neighborhood in $Y$ in which $Jf$ does not change sign.

**Lemma 8.** If the local theorem holds in a connected set $Y$, then $Jf$ does not change sign in the closure $\overline{Y}$.

**Proof.** It suffices to prove $Jf$ does not change sign in $Y$, because $Jf$ is continuous. Define $Y_+, Y_-$ to be the subsets of $Y$ where $Jf$ is respectively $\geq 0$ and $\leq 0$. These sets are closed in $Y$ by continuity of $Jf$, and open in $Y$ by hypothesis. As $Y$ is connected, either $Y = Y_+$ or $Y = Y_-$. The local theorem obviously holds in the set $\mathbb{R}^n \setminus C$ of noncritical points. By Lemma 7, it also holds in the relatively open analytic hypersurface $V \subset C$ defined in the introduction. It remains to prove that every point of $C \setminus V$ has a neighborhood in which $Jf$ does not change sign.

**Lemma 9.** Every point $p \in C \setminus V$ has a neighborhood $X_p \subset C \setminus V$ that is an analytic variety of dimension $\leq n - 2$.

**Proof.** Write $C \setminus V = (C \setminus H) \cup (H \setminus V)$

Suppose $p \in C \setminus H$. In this case we take $X_p$ to be the union of the variety $C_{\text{sing}}$ of singular points of $C$ and those connected components of $C \setminus C_{\text{sing}}$ having dimension $\leq n - 2$.

Suppose $p \in H \setminus V$, or equivalently: $p \in H$ and some minor determinant of $df$ vanishes at $p$ but not identically in any neighborhood of $p$ in $H$. We take $X_p$ to be the intersection of $C$ with the union of the zero sets of such minors.

Now consider any point $p \in C \setminus V$. By Lemma 9, $p$ has a neighborhood $X_p \subset C \setminus V$ that is the union of finitely many smooth submanifolds of $\mathbb{R}^n$ having dimensions $\leq n - 2$.

Choose a connected open neighborhood $N_p \subset \mathbb{R}^n$ of $p$ such that $N_p \cap (C \setminus V) = N_p \cap X_p$. Then $N_p \setminus X_p \subset (\mathbb{R}^n \setminus C) \cup V$. Therefore the local theorem holds in $N_p \setminus X_p$.

Now $N_p \setminus X_p$ is connected, by a standard general position argument. Therefore from Lemma 8, with $Y = N_p \setminus X_p$, we infer that $Jf$ does not change sign in $N_p \setminus X_p$, which equals $X_p$ because $X_p$ is nowhere dense. This completes the proof of Theorem 1.
Proof of Theorem 3. Parts (i) and (ii) of Theorem 2 are proved by applying Lemma 7 locally. Lemma 9 implies (iii), because $B \subset C \setminus V$ by (ii). For (iv), suppose $\dim R_{n-2} = n - 1$. Then the variety $R_{n-2}$ contains an analytic hypersurface, which must meet $V$. As $R_{n-2} \subset C$, this implies $R_{n-2} \cap V \neq \emptyset$, contradicting (i).

References


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