Title
Asymptotic phase, shadowing and reaction-diffusion systems

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0 Introduction

It is with pleasure and gratitude that we honor Professor Larry Markus for his contributions to mathematics. In 1956 he published *Asymptotically autonomous differential systems* [6] in a series called *Contributions to the Study of Nonlinear Oscillations*—a subject which today would be called “Dynamical Systems”. The present article is a direct descendant of Markus’ influential paper, through Conway, Hoff and Smoller [2].

Consider a smooth flow \( \{\Phi_t\} \) having an attracting limit cycle \( \gamma \). It is well known that if \( \gamma \) is a hyperbolic attractor— all Floquet exponents having negative real parts— then every trajectory \( \Phi_t x \) attracted to \( \gamma \) is asymptotic with the trajectory of a unique point of \( \gamma \). If \( \gamma \) is parameterized by the interval \([0, 2\pi]\) then \( y \) can be interpreted as an angle, called the *asymptotic phase* of \( x \).

I abstract this notion as follows. Consider a trajectory \( \Phi_t x \) attracted to some positively invariant set \( A \). If \( y \in A \) is such that \( \lim_{t \to \infty} ||\Phi_t x - \Phi_t y|| = 0 \) then I call \( y \) an *asymptotic phase* for \( x \). (For clarity I use \( ||a - b|| \) to denote the distance between points \( a, b \) in any metric space.) Notice that uniqueness of \( y \) is not required here.

If \( A \) is not negatively invariant it may happen that \( x \) does not have an asymptotic phase in \( A \), but that \( \Phi_s x \) does, for some \( s > 0 \). In this case I say \( x \) has an *eventual asymptotic phase* in \( A \).
It is frequently incorrectly assumed that every orbit approaching an attractor has an eventual asymptotic phase in the attractor. A common situation is that of a cascade of two systems, that is, a system of the form:

\[
\begin{align*}
\frac{dx}{dt} &= F(x, y) \\
\frac{dy}{dt} &= G(y).
\end{align*}
\]

If \((x(t), y(t))\) is a particular solution such that \(y(t) \to c\), it is often asserted without justification that \(x(t)\) is asymptotic to a solution of \(dz/dt = F(z, c)\). A simple counter-example in the plane is:

\[
\begin{align*}
\frac{dx}{dt} &= xy \\
\frac{dy}{dt} &= -y^3.
\end{align*}
\]

The goal of this paper is to find conditions ensuring existence of an eventual asymptotic phase.

1 Main Results

Let \(a = a(q)\) denote a nonnegative real-valued function of a variable \(q\) whose domain is understood to be a terminal segment of either the positive reals or the positive integers. Define

\[
\mathcal{R}a = \mathcal{R}_{q \to \infty} a(q) = \limsup_{q \to \infty} a(q)^{1/q}.
\]

Then

\[
\mathcal{R}(a + b) = \max(\mathcal{R}a, \mathcal{R}b),
\]

and for any constant \(\kappa > 0\):

\[
\mathcal{R}(\kappa a) = \mathcal{R}(a).
\]

Let \(F = \{F_t\}_{t \geq 0}\) be a flow (more precisely, a partial semiflow) on a metric space \(X\) (usually a Banach space). For clarity the distance between points \(x, y \in X\) is denoted \(||x - y||\). In applications \(X\) is usually a subset of a Banach space. I shall always assume the maps \(F_t\) have the following local Lipschitz property: For any \(t_0 \geq 0, x_0 \in X\) there exist \(L \geq 0\) and neighborhoods \(N \in \mathbb{R}_+\) of \(t_0\) and \(U \in X\) of \(x_0\) such that

\[
||F_{t_0}x - F_{t_0}y|| \leq L||x - y||
\]

for all \(t \in N, y \in U\).

Denote by \(A \subset X\) a closed subspace having the following properties:
(a) $A$ is positively invariant under $F$.

(b) $A$ has the structure of a Riemannian manifold without boundary homeomorphically embedded in $X$. The norm of a tangent vector $Y$ in the Riemannian metric is denoted by $||Y||$.

(c) There is a smooth $(C^1)$ tangent vector field $G$ on $A$ whose flow $\Phi = \{\Phi_t\}_{t \in \mathbb{R}}$ coincides with $F_t|A$ for $t \geq 0$.

Let $K \subset A$ denote a nonempty compact set positively invariant under $F$, so that $K$ is also $\Phi$-invariant. The *inset* of $K$ under $F$ is the set

$$In(K) = In(K, F) = \{x \in X : \lim_{t \to \infty} \text{dist}(F_t x, K) = 0\}.$$ 

For $x \in In(K)$ define the *rate of approach to $K$ of $x$ to $K$ under $F$* to be the number

$$\mathcal{P}(x, K, F) = R_{t \to \infty} \text{dist}(F_t x, K).$$

Evidently $0 \leq \mathcal{P}(x, K, F) \leq 1$. If $\mathcal{P}(x, K, F) < 1$ I say $x$ is *exponentially attracted to $K$*

Fix a Riemannian metric on $A$. The closed ball in $A$ with radius $\rho \geq 0$ centered at $x \in A$ is denoted by $B(\rho, x)$.

For a diffeomorphism $h$ between open subsets of $A$, the *expansion constant of $h$ at $x \in A$* is the positive number

$$EC(h, x) = ||T_x h^{-1}||^{-1} = \min_{||Y|| = 1} ||T_x h(Y)||.$$ 

Here $Y$ denotes tangent vectors to $A$ at $x$, and $||T_x h||$ denotes the operator norm of the differential of $h$ at $x$ (defined by the Riemannian metric). Thus $EC(h, x) \geq \mu$ iff $||T_x z|| \geq \mu ||z||$ for all $z \in T_x$.

Now for any compact subset $K \subset A$ define

$$EC(h, K) = \min_{x \in K} EC(h, x)$$

If $EC(h, K) > \nu > 0$ then it is not hard to see that there exists $\rho_* > 0$ such that if $x \in K$ and $0 < \rho \leq \rho_*$ then

$$hB(\rho, x) \supset B(\mu \rho, h(x));$$

see Hirsch and Pugh [4].
The expansion rate of $\Phi$ at $K$ is the nonnegative number

$$\mathcal{E}(\Phi, K) = \sup_{t > 0} EC(\Phi_t, K)^{\frac{1}{t}}.$$ 

Since $[T_x \Phi_t]^{-1} = T_{\Phi_t(x)} \Phi_{-t}$, we have

$$\mathcal{E}(\Phi, K) = \sup_{t > 0} \min_{x \in K} \left| T_{\Phi_t(x)} \Phi_{-t} \right|^{-\frac{1}{t}}.$$ 

The expansion rate is is the largest $\mu > 0$ having the following property: If $0 < \nu < \mu$ then there exist $s > 0, \rho_* > 0$ such that

$$\Phi_s B(\rho, x) \supset B(\nu^s \rho, \Phi_s x)$$

provided $x \in K$ and $0 < \rho \leq \rho_*$. 

The expansion rate depends on the dynamics and the Riemannian metric. In some cases it is possible to estimate it from a formula for the vector field, from the dynamics of its flow, or from estimates using other metrics. Here are several such estimates.

(i) Assume that $A = \mathbb{R}^n$ with the standard inner product $\langle \cdot, \cdot \rangle$, and denote $T_x \Phi_t$ by $D\Phi_t(x)$. The variational equation along orbits of the reversed time flow $\Phi_{-t}$, generated by the vector field $-G$ on $A$, gives the following matrix differential equation:

$$\frac{d}{dt} D\Phi_{-t}(x) = -DG(\Phi_{-t}x)D\Phi_{-t}(x)$$

Therefore for every nonzero vector $Y \in \mathbb{R}^n$ and every $t \geq 0, y \in K$ we have, setting $y = \Phi_t x \in K$:

$$\frac{d}{dt} ||D\Phi_{-t}(y)Y|| = ||D\Phi_{-t}(y)Y||^{-1}(-DG(\Phi_{-t}y)D\Phi_{-t}(y)Y, D\Phi_{-t}(y)Y)$$

The inner product on the right hand side is bounded above by $-\beta ||D\Phi_{-t}(y)Y||^2$ where $\beta = \beta(G, K)$ denotes the minimum over $x \in K$ and unit vectors $\xi \in \mathbb{R}^n$ of $\langle DG(x)\xi, \xi \rangle$. Equivalently, $\beta$ equals the smallest eigenvalue of the symmetric matrix $\frac{1}{2}[DG(x)+DG(x)^T]$ where $T$ denotes the transpose of a matrix. Therefore

$$\frac{d}{dt} ||D\Phi_{-t}(x)|| \leq \beta ||D\Phi_{-t}(x)||,$$

whence

$$||D\Phi_{-t}(x)|| \leq e^{-t\beta}.$$ 

This proves $EC(\Phi_t, x) \geq e^{t\beta}$ for all $t \geq 0, x \in K$. We get the convenient estimate:

$$\mathcal{E}(\Phi, K) \geq e^{\beta(G,K)}.$$ (1)
(ii) Another estimate is obtained by noticing that

$$|\beta| \leq M = M(G, K) = \max_{x \in K} ||DG(x)||$$

(using the Schwarz inequality) so that $\beta \geq -M$. This yields the estimate:

$$\mathcal{E}(\Phi, K) \geq e^{-M(G,K)}.$$  \hspace{1cm} (2)

which will be used in Section 2.

(iii) A different estimate can be obtained in case all forward and backward trajectories in $K$ are attracted to hyperbolic periodic orbit (possibly stationary). Suppose that the real parts of the Floquet exponents of these periodic orbits are all $\geq \gamma \in \mathbb{R}$. Then it can be proved that:

$$\mathcal{E}(\Phi, K) \geq e^\gamma$$  \hspace{1cm} (3)

Suppose for example that the flow in $A$ is the gradient flow of a function $g : A \to \mathbb{R}$ having a finite set of critical points, and $K$ is a compact attractor containing all the critical points. Then $\gamma$ is the minimum of the eigenvalues of the Hessian of $g$ at critical points in $K$.

(iv) More generally, it can be shown that if $L \subset K$ is a compact set containing all alpha and omega limit points in $K$, then $\mathcal{E}(\Phi, K) = \mathcal{E}(\Phi, L)$. The reason is that any semi-trajectory in $K$ spends all but a finite amount of time in any given neighborhood of $L$.

(v) If $K$ is a smooth submanifold and the flow in $K$ is isometric for some Riemannian metric, then $\mathcal{E}(\Psi, K) = 1$. This is the case, for example, when $K$ is a periodic orbit; when $K$ is a smooth submanifold consisting of stationary points; or when the $K$ is an $n$-dimensional torus and the flow is translation by a one parameter subgroup.

(vi) It seems reasonable to conjecture that if $\Psi$ is generated by a vector field $H$ on $A$ of the form $H(x) = c(x)G(x)$ where $c$ is a positive function on $A$, then $\mathcal{E}(\Phi, K) = \mathcal{E}(\Psi, K)$.

It would be very useful to know that $\mathcal{E}(\Phi, K)$ is preserved, or at least well controlled, by a smooth or continuous reparameterization of the trajectories, or by a topological conjugacy between flows. A key test case is a $C^2$ flow on a 2-torus without periodic orbits: Is the expansion rate equal to 1?
Clearly $\mathcal{E}(\Phi, K) \geq R_{t \to \infty} EC(\Phi_t, K)$. The latter number is easier to estimate and in some ways is more natural. For example it is easy to prove that it is independent of the Riemannian metric on $A$.

The main result says roughly that $x$ is exponentially attracted to $K$ at rate $\lambda$, while the expansion rate at $K$ of the flow in $A$ is $\mu > \lambda$, then $x$ is eventually asymptotic at rate $\lambda$ to a unique trajectory in $A$:

**Theorem 1.1** Let $\mathcal{E}(\Phi, K) = \mu$. Suppose $x \in In(K)$ approaches $K$ at rate

$$\mathcal{P}(x, K, F) = \lambda < \min(1, \mu).$$

Then:

(a) There exists $r, y \in A$ such that

$$R_{t \to \infty} ||\Phi_{t+r}x - \Phi_{t}y|| = \lambda.$$

(b) Let $y$ be as in (a). Suppose $l > 0, z \in A$ are such that

$$R_{t \to \infty} ||\Phi_{t+l}x - \Phi_{t}z|| \leq \lambda.$$

Then $z$ and $y$ are on the same orbit of $\Phi$.

This is proved in Section 3 below. The same argument yields the analogous result for mappings.

The proof of the following corollary is left to the reader:

**Corollary 1.2** If $\mathcal{P}(x, K, F) = \lambda < \min(1, \mathcal{E}(\Phi, K))$, then $x$ has an eventual asymptotic phase $y \in K$. If $\mathcal{E}(\Phi, K) \geq 1$ then the $\Phi$-trajectory of such a $y$ is unique.

As a simple example illustrating Theorem 1.1, consider a smooth flow in some manifold $A$ having an invariant $n$-torus $K = T^n = (R/2\pi Z)^n$ in which the flow is quasiperiodic, the generating vector field $G$ in $T^n$ being covered by a constant vector field in $R^n$. It is clear that $\mathcal{E}(\Phi, T^n) = 1$, using the Riemannian metric covered by the Euclidean metric on $R^n$. Therefore by Theorem 1.1, any orbit attracted to $T^n$ at a rate of approach less that 1 has an asymptotic phase in $T^n$. It is not hard to show that the same conclusion holds if the flow in $T^n$ is generated by $gG$ where $g$ is any smooth real-valued function on $T^n$. The proof is based on the fact that orbits of the lifted flow in $R^n$ stay in parallel lines.

**Remark 1.3** Suppose $K$ is a normally hyperbolic submanifold, or a hyperbolic subset, for the flow in $A$ (see [3, 4, 5, 7]). Then any point $x \in A$ attracted to $K$ belongs to the strong stable manifold of some $y \in K$. Therefore $x$ is exponentially asymptotic with $y$. 
Remark 1.4 The main results apply equally to discrete-time systems, i.e. to a
mapping $f$ from an open subset $X_0 \subset X$ to $X$. Everything makes sense if $t$ is
restricted to the natural numbers, $F_t$ is the $t$'th iterate of $f$, and $\Phi$ is replaced by the
iterates of the map $h = f|A \cap X_0$, assumed to be a diffeomorphism from $A_0 = A \cap X_0$
on to a neighborhood of $K$ in $A$. In fact the main part of the proof of the main
theorem in Section 3 consists of a proof of the discrete-time case; this is applied to
the mapping $f = F_s$ for suitable $s > 0$.

2 Reaction Diffusion Systems

Theorem 1.1 is applied to reaction diffusion systems of the following kind. Let $\bar{\Omega} \subset R^n$ be a smooth (i.e. $C^1$) compact submanifold with interior $\Omega$. We look for a
continuous function $u(x,t), x \in \bar{\Omega}, t \geq 0$ with values in $R^n$ satisfying for $t > 0$

$$\frac{\partial u}{\partial t} = B\Delta u + \sum_{j=1}^{m} C_j(x,u) \frac{\partial u}{\partial x_j} + f(u), \quad (4)$$

$$\frac{\partial u}{\partial \nu} = 0. \quad (5)$$

Here $\Delta$ is the Laplacean in the spatial variable $x \in \bar{\Omega}$, operating on each component
$u_j$ of $u$; $B$ is a positive definite $n \times n$ matrix; each $n \times n$ matrix-valued function $C_j$ is
continuous in $(x,u)$; $f$ is a smooth vector field on $R^n$; $\nu$ is the inward pointing unit
vector field normal to the boundary of $\Omega$.

It is known that solutions to this system form a solution semiflow $S = \{S_t\}_{t \geq 0}$
in the Sobolev space $H^1(\bar{\Omega}, R^n)$: The solution taking initial values $u(x,0) = v(x)$ is
$u(x,t) = (S_t v)(x)$.

Let $A \subset H^1(\bar{\Omega}, R^n)$ denote the linear subspace of constant maps $\bar{\Omega} \to R^n$, and
identify $A$ with $R^n$ in the natural way. The form of Equation (4) shows $A$ is positively
invariant under $S$.

A trajectory of $S$ in $A$ defines a spatially homogeneous solution to Equations
(4), (5). Such a solution has the form $u(x,t) = y(t)$ where $y$ is a solution to the
autonomous system $dy/dt = f(y)$.

The restriction to $A$ of the solution flow $S$ of (4), (5) coincides for $t \geq 0$ with the
flow $\Phi$ obtained by integrating the vector field $f$.

Suppose from now on that $\Gamma \subset R^n$ is a compact invariant rectangle\footnote{More generally, $\Gamma$ can be an invariant region as defined in Conway, Hoff and Smoller [2].} (the product
of $n$ nondegenerate compact intervals.) We identify $\Gamma$ with a compact subset of $A$,
namely the constant functions with values in $\Gamma$. Invariance means that if the initial
map \( v : \overline{\Omega} \to \mathbb{R}^n \) takes values in \( \Gamma \) then the same holds for every map \( S_t v \). When \( B \) is a diagonal matrix, invariance holds provided that for every \( y \) on the boundary of \( \Gamma \), the vector \( f(y) \) does not point out of \( \Gamma \).

In [2] a condition is given ensuring that \( \Gamma \) attracts every initial \( v \in H^1(\overline{\Omega}, \mathbb{R}^n) \) taking values in \( \Gamma \), or in other words, that the set \( X = H^1(\overline{\Omega}, \Gamma) \) lies in the inset of \( \Gamma \). This condition is given in terms of the real parameter

\[
\sigma = b\lambda - M - c\sqrt{m\lambda}
\]

(6)
defined in terms of the following constants: The positive number \( b \) is the smallest eigenvalue of the positive definite matrix \( B \); \( \lambda \) (also positive) is the smallest eigenvalue of \(-\Delta \) on \( \Omega \) with homogeneous Neumann boundary conditions (5); \( c \) is the maximum matrix operator norm \( ||C_j(x, y)|| \), \( 1 \leq j \leq m, x \in \overline{\Omega}, y \in \Gamma \); and as before, \( M = \max_{y \in \Gamma} ||Df(y)|| \).

It will also be convenient to consider the slightly different parameter:

\[
\sigma_2 = \sigma - M = b\lambda - 2M - c\sqrt{m\lambda}
\]

(7)

For each \( v \in X \) set \( v_t = S_t v \), and denote by \( \overline{v}_t \in \mathbb{R} \) the average of \( v_t \) over \( \Omega \). Notice that \( \overline{v}_t \) is a curve in \( X \), but it need not be a trajectory of the flow \( S \), that is, \( \overline{v}_t(x) \) need not be a solution to Equations (4, 5).

Let \( ||\cdot||_\infty \) denote the \( L_\infty(\overline{\Omega}, \mathbb{R}^n) \) norm.

The following result is a corollary of Theorem 3.1 of [2].

**Theorem 2.1 (Conway, Hoff, Smoller [2])**

Assume \( \sigma > 0 \) and let \( v \in X = H^1(\overline{\Omega}, \Gamma) \). Then:

(a) There is a constant \( c_1 > 0 \) such that \( ||v_t - \overline{v}_t||_1 \leq c_1 e^{-\sigma t} \) for all \( t \geq 0 \).

(b) If the matrices \( C_1, \ldots, C_n \) are zero, or if \( C_1, \ldots, C_n \) and \( B \) are diagonal, then there is a constant \( c_2 > 0 \) such that \( ||v_t - \overline{v}_t||_\infty \leq c_2 e^{-\frac{2\sigma t}{m}} \) for all \( t \geq 0 \).

This says that when \( \sigma \) is positive, in the appropriate norm trajectories of the reaction-diffusion system approach spatially homogeneous functions. In fact in [2] it is proved that the spatial averages \( \overline{v}_t \) satisfy a nonautonomous system \( d\overline{v}_t/dt = f(\overline{v}_t) + g(t) \) with \( ||g(t)||_1 \leq c_3 e^{-\sigma t} \) for some constant \( c_3 \geq 0 \). Conway, Hoff and Smoller say that "because of a result of Markus [6] it follows that the asymptotic behavior of \( \overline{v}_t \) is determined only by \( f \)."

In the terminology of Section 1 we have:

\[ -\frac{\sigma}{m} \]

The statements of Theorem 2.1 are proved but not stated in this form. The exponent in (b) is given as \(-\frac{\sigma}{m} t\), I think incorrectly.
Corollary 2.2 Under the same hypothesis as Theorem 2.1:

(a) \( R_{t \to \infty} (||v_t - \overline{v}_t||_1) \leq e^{-\sigma} \).

(b) If the matrices \( C_1, \ldots, C_n \) are zero, or if \( C_1, \ldots, C_n \) and \( B \) are diagonal, then also \( R_{t \to \infty} (||v_t - \overline{v}_t||_\infty) \leq e^{-\frac{2\sigma}{m}} \).

While the Conway-Hoff-Smoller theorem provides much information about such systems, it leaves open the question of whether trajectories have an asymptotic phase in \( A \). The following result gives a sufficient condition for this.

Let \( \mu = \mathcal{E}(\Phi, \Gamma) \), the expansion rate in \( \Gamma \) of the flow in \( A = \mathbb{R}^n \) defined by \( \frac{dy}{dt} = f(y) \).

Theorem 2.3 Assume \( \sigma > 0 \) and \( e^{-\sigma} < \mu \). Let \( v \in H^1(\Omega, \mathbb{R}^n) \) take values in the invariant rectangle \( \Gamma \subset \mathbb{R}^n \). Then the trajectory \( S_t v \) in \( \text{In}(\Gamma) \) of the solution flow in \( H^1(\Omega, \mathbb{R}^n) \) of the reaction-diffusion system (4), (5) has an eventual asymptotic phase in the space \( A \) of constant maps. More precisely, if \( S_t v(x) = u(x, t) \) then for every sufficiently large \( s \geq 0 \) there is a unique solution to \( \frac{dy}{dt} = f(y) \) such that:

(a) \( R_{t \to \infty} (||u(\cdot, t+s) - y(t)||_1) \leq e^{-\sigma} \).

Moreover, if the matrices \( C_1, \ldots, C_n \) are zero, or if \( C_1, \ldots, C_n \) and \( B \) are diagonal, then:

(b) \( R_{t \to \infty} (||u(\cdot, t) - y(t)||_\infty) \leq e^{-\frac{2\sigma}{m}} \).

Corollary 2.4 If \( \sigma_2 > 0 \) then the conclusions of Theorem 2.3 hold.

Proof Corollary 2.2(a) implies \( v \) has rate of approach \( \leq e^{-\sigma} \) to \( \Gamma \). Therefore Theorem 2.3 follows from Theorem 1.1 (with \( K = \Gamma \)) and the assumption \( e^{-\sigma} < \mu \).

To prove Corollary 2.4, assume \( \sigma_2 > 0 \). Then \( \sigma > 0 \) and \( e^{-\sigma} < e^{-M} \) (see (7)). Since estimate (2) therefore implies \( e^{-M} \leq \mathcal{E}(\Phi, \Gamma) \), the corollary is a consequence of Theorem 2.3.

QED

3 Shadowing

The main theorem will be derived from the results of this section. The same notations and assumptions as in Section 1 are in force, although at first the setting is quite general.

Let \( X_0 \subset X \) be any subset and let \( g : X_0 \to X \) be a map (\( g = \) some \( F_t \) in the application). Let \( 0 \leq \lambda < 1 \). I call a sequence \( \{y_k\} \) in \( K \) a \( \lambda \)-pseudoorbit for \( g \) if

\( R_{k \to \infty} ||g(y_{k-1}) - y_k|| \leq \lambda. \)
Lemma 3.1 Suppose $g$ is $\alpha$-Hölder, $0 < \alpha \leq 1$. Let $\{y_k\}$ be a sequence in $X$ which is $\lambda$-shadowed by a point $u \in X_0$. Then $\{y_k\}$ is a $\lambda^\alpha$-pseudoorbit for $h$. In particular if $g$ is Lipschitz then $\{y_k\}$ is a $\lambda$-pseudoorbit.

Proof Fix $C > 0$ such that $||g(a) - g(b)|| \leq C||a - b||^\alpha$. Observe that

$$||g(y_{m+k-1}) - y_{m+k}|| \leq ||g(y_{m+k-1}) - g^k u|| + ||g^k u - y_{m+k}||$$

$$\leq C||y_{m+k-1} - g^{k-1} u||^\alpha + ||g^k u - y_{m+k}||.$$ 

Therefore (see Section 1)

$$R_{k \to \infty} C||g(y_{k-1}) - y_k|| \leq \max(R_{k \to \infty}||y_{m+k-1} - g^{k-1} u||^\alpha, R_{k \to \infty}||g^k u - y_{m+k}||)$$

$$\leq \max(\lambda^\alpha, \lambda) = \lambda^\alpha.$$ 

QED

Now set $A_0 = A \cap X_0$, assume $g(A_0) \subset A$ and $g(K) \subset K$. Set $g|A_0 = h$ and assume from now on that $h$ is a $C^1$ diffeomorphism of $A_0$ onto some neighborhood of $K$ in $A$.

A point $u \in A_0$ (or its orbit) is said to $\lambda$-shadow the sequence $\{y_k\}$ in case $h^k(u)$ is defined for all $k \in \mathbb{N}$, and:

$$R_{k \to \infty}||h^k(u) - y_{k+m}|| \leq \lambda$$

for some $m \geq 0$.

Theorem 3.2 Assume the expansion rate of $h$ in $K$ is $EC(h, K) = \mu > 0$. Let $\{y_k\}$ be a $\lambda$-pseudoorbit in $K$ such that

$$0 < \lambda < \min(1, \mu).$$

Then:

(a) There exists $z \in A_0$ which $\lambda$-shadows $\{y_k\}$.

(b) If $z, w \in A_0$ both $\lambda$-shadow $\{y_k\}$ then there exist natural numbers $l, r$ such that $h^l z = h^r w$.

Remark 3.3 The proof shows that $z$ in the theorem can be chosen in $K$ if $K$ is a smooth compact submanifold without boundary, or if $K$ is an attractor for $h$, or if the pseudoorbit $\{y_k\}$ is eventually bounded away from the boundary of $K$ in $A$. In any case the forward orbit of $z$ is attracted to $K$ and its omega limit set is in $K$.  

Remark 3.4 The theorem is valid under the more general hypothesis where $\mu$ denotes $\sup_{k \geq 0} EC(h^k, K)^{1/k}$.

Proof Fix $\rho_* > 0$ so small that if $0 \leq \rho \leq \rho_*$ then

$$hB(\rho, x) \supset B(\mu \rho, h(x))$$

(8)

for all $x \in K$, where $B$ refers to closed balls in $A$. Then this also holds for all $x$ in some neighborhood $N \subset A_0$ of $K$, since $K$ is compact.

Choose $\nu$ such that

$$0 < \lambda < \nu < \min(1, \mu).$$

Pick $\delta$ such that

$$\nu < \delta < \min(1, \mu).$$

I claim that for all sufficiently large positive integers $k$ we have:

$$hB(\delta^{k-1}, y_{k-1}) \supset B(\delta^k, y_k).$$

(9)

To see this observe that $\delta^j < \rho$ and $B(\delta^{k-1}, y_{k-1}) \subset N$ for large $j$. Therefore by (8) it suffices to prove for sufficiently large $k$ that

$$B(\mu \delta^{k-1}, h(y_{k-1})) \supset B(\delta^k, y_k).$$

(10)

And this last will hold by the triangle inequality provided we show

$$\mu \delta^{k-1} \geq \delta^k + ||h(y_{k-1}) - y_k||.$$  

(11)

Because $\{y_k\}$ is a $\lambda$-pseudoorbit, for large $k$ we have

$$||h(y_{k-1}) - y_k|| < \nu^k.$$  

(12)

Therefore it suffices to show

$$\mu \delta^{k-1} \geq \delta^k + \nu^k$$

(13)

or equivalently

$$\mu \geq \delta + \left(\frac{\nu}{\delta}\right)^{k-1} \nu$$

(14)

for sufficiently large $k$. This is true, say for $k \geq m$, because $\mu > \delta > \nu$.

Therefore estimate (9) holds for $k \geq m$. This implies that for $n \geq m$ the set

$$Q_n = \cap_{i \geq 0} (hB(\delta^n, y_n))^{-i} B(\delta^{i+n}, y_{i+n})$$

is not empty, and the orbit of any point in $Q_n$ $\lambda$-shadows $\{y_k\}$. This proves statement (a) of the theorem.
From the assumption \( EC(h, K) > \lambda \) it follows easily that \( Q_n \) is a singleton for every \( n \geq m \). This implies (b). \( \text{QED} \)

**Proof of Theorem 1.1** With the notation and assumptions of Theorem 1.1, fix \( r > 0 \) so that
\[
EC(\Phi_r, K) = \mu_0 > \lambda.
\]
Set \( h = \Phi_r : A_0 \rightarrow A \) where \( A_0 \) denotes the domain of \( \Phi_r \) — a neighborhood of \( K \) in \( A \). For \( k \in \mathbb{N} \) let \( y_k \in K \) be a point nearest to \( h^k(x) \). It then follows from Lemma 3.1(a) with \( u = x \) and \( g = F_r \), and the standing assumption that each \( F_i \) is Lipschitz, that \( \{ y_k \} \) is a \( \lambda \)-pseudoorbit for \( h \). By Theorem 3.2 \( \{ y_k \} \) is \( \lambda \)-shadowed by the orbit of some \( z \in A_0 \). It follows that for some \( m \geq 0 \) we have:
\[
\mathcal{R}_{k \rightarrow \infty} \| \Phi_{k+m}x - \Phi_k z \| = \lambda \ (k \in \mathbb{N}).
\]
Continuity of the flow now implies:
\[
\mathcal{R}_{t \rightarrow \infty} \| \Phi_{t+m}x - \Phi_t z \| = \lambda \ (t \in \mathbb{R}).
\]
This proves part (a) of Theorem 1.1.

Part (b) follows similarly from part (b) of Theorem 3.2. \( \text{QED} \)

**Remark 3.5** The connection between asymptotic phase and shadowing is more extensive. For simplicity consider a diffeomorphism \( h \). Suppose the orbit of some point \( x \) is attracted to a compact invariant set \( K \), not necessarily at an exponential rate. By choosing \( y_k \in K \) to be a point nearest to \( h^k(x) \) we obtain a sequence \( \{ y_k \} \) in \( K \) with the property that \( ||h(y_{k-1}) - y_k|| \rightarrow 0 \). If \( h|K \) has the property of unique shadowing, described below, then it is easy to see that \( \{ y_k \} \) is asymptotic to the orbit of a unique point \( z \in K \). Such a \( z \) would therefore be an asymptotic phase for \( x \).

To say the map \( h|K \) has unique shadowing means the following. For \( \delta > 0 \), \( \{ y_k \} \) is an \( \delta \)-pseudoorbit in case \( ||h(y_{k-1}) - y_k|| < \delta \). “Unique shadowing” means that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( \delta \)-pseudoorbit \( \{ y_k \} \) there is a unique \( z \in K \) such that \( ||y_k - h^k(z)|| < \varepsilon \), or in other words \( \{ y_k \} \) is \( \varepsilon \)-shadowed by \( z \).

R. Bowen [1] showed that if \( K \) is a hyperbolic invariant set, then \( h|K \) has unique shadowing. Suppose for example that \( V \) is a compact smooth invariant submanifold of \( A \) and that \( h|V \) is an Axiom A diffeomorphism in the sense of Smale [7]. If \( x \in A \) is attracted to \( V \) then it is easy to see that in fact \( x \) is attracted to what Smale calls a basic set \( K \) for \( h|V \), which is by definition a hyperbolic invariant set. Therefore Bowen’s theorem implies that \( x \) has an asymptotic phase in \( K \), hence also in \( V \).
References


