Title
Solutions to Monthly Problems 11456 and 11457

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Solutions to Monthly Problems 11456
and 11457

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The Monthly problem #11456 [1] asks to evaluate

\[ \alpha = \lim_{n \to \infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right). \]

Numerical computations, using, say, \( n = 10^9 \), yields the numerical value 3.6898333.... Using this value as input to the Inverse Symbolic Calculator 2.0 tool (available at http://glooscap.ca.dal.ca:8087, one of the output results is the tantalizingly simple expression

\[ \alpha = \frac{e^\pi + e^{-\pi}}{2\pi}. \]

Indeed, this result can be established directly by typing the Maple command

\[ n \ast \text{product}(1 - 1/m + 5/(4*m^2), m = 1..n); \]

which yields the expression

\[ \frac{n \Gamma (n + 1/2 - i) \Gamma (n + 1/2 + i)}{\Gamma^2(n + 1) \Gamma(1/2 - i) \Gamma(1/2 + i)}. \]

After typing \texttt{limit(%,n=infinity)}; this reduces to

\[ \frac{1}{\Gamma(1/2 - i) \Gamma(1/2 + i)} \]

which, after \texttt{simplify(%)}, yields the final result:

\[ \frac{\cosh \pi}{\pi}. \]

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The Monthly problem #11457 [2] asks to evaluate

\[ F(a, b) = \int_a^b \arccos \left( \frac{x}{\sqrt{(a + b)x - ab}} \right) \, dx \]

Here again, computer experimentation (using either Mathematica or Maple) yields a number of specific results:

\[
\begin{align*}
F[0, b] & = \frac{b \pi}{4} \quad \text{for } b \geq 0, \\
F[1, b] & = \frac{(b - 1)^2 \pi}{4(b + 1)} \quad \text{for } b \geq 1, \\
F[2, b] & = \frac{(b - 2)^2 \pi}{4(b + 2)} \quad \text{for } b \geq 2, \\
F[3, b] & = \frac{(b - 3)^2 \pi}{4(b + 3)} \quad \text{for } b \geq 3,
\end{align*}
\]

which quickly suggest the “obvious” answer:

\[ F[a, b] \equiv \frac{(a - b)^2 \pi}{4(a + b)} \]

This result can be established directly by the Maple command

`factor(int(arccos(x/sqrt((a+b)*x - a*b)), x=a..b)) assuming a>0, a<b;`

References
