The Intervertebral Joint as a Stiffness Matrix: Theory, Practice, and Application

by

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A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Engineering – Mechanical Engineering in the Graduate Division of the University of California, Berkeley

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Abstract

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The relationship between the motion of a pair of vertebrae and the ensuing loads exerted by the intervertebral joint have been studied extensively. In the realm of small and quasi-static motion, this relationship is approximately linear and can be modeled using a $6 \times 6$ stiffness matrix. Efforts to determine the elements of these stiffness matrices as well as to apply them in models of the joint have been underway since this joint model was first proposed in the early 1970s. Meanwhile, a considerable amount of theoretical work has been conducted in identifying linear mappings that relate increments in force and moment components acting on a system of rigid bodies to the infinitesimal motions that produced them. Most prominent among these is the Cartesian stiffness matrix parameterization. Alongside these developments, advancements in musculoskeletal software modeling has led to the developments of various software platforms capable of performing the underlying computations necessary to study musculoskeletal structures. The motion of articulating bodies in this case is defined using joints. In the case of the spine, bushing elements – similar to the rubber bushings employed in vehicular dynamics – are a popular alternative. Unfortunately, numerous difficulties exist in adapting these bushing elements to mimic the stiffness matrix model of the intervertebral joint.

In this dissertation, we connect the inter-related subjects above to pave the way for a more comprehensive model of the intervertebral joint. To do this, we derive the Cartesian stiffness matrix associated with the joints of the spine using arguments based on energetics consideration and the Euler angle parameterization of rotations. We next show how this Cartesian stiffness matrix can be related to elements of the experimentally measured stiffness matrix of the intervertebral joint. Finally, we apply the resulting stiffness matrices in a model of the lumbar spine developed in OpenSim using a stiffness matrix plugin we created specific to the intervertebral joint.
## Contents

### Acknowledgments

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>The Cartesian Stiffness Matrix</td>
<td>3</td>
</tr>
<tr>
<td>1.2</td>
<td>Bushing Forces</td>
<td>4</td>
</tr>
<tr>
<td>1.3</td>
<td>Dissertation Outline</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>Relative Motions of Rigid Bodies</td>
<td>7</td>
</tr>
<tr>
<td>2.1</td>
<td>Background and Notation</td>
<td>7</td>
</tr>
<tr>
<td>2.2</td>
<td>A Single Rigid Body</td>
<td>8</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Euler’s Representation</td>
<td>10</td>
</tr>
<tr>
<td>2.2.2</td>
<td>The Euler-Rodrigues Symmetric Parameter Representation</td>
<td>11</td>
</tr>
<tr>
<td>2.2.3</td>
<td>The Euler Angle Representation</td>
<td>12</td>
</tr>
<tr>
<td>2.3</td>
<td>Multibody Systems</td>
<td>16</td>
</tr>
<tr>
<td>2.4</td>
<td>Closing Remarks</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>Conservative Forces and Moments from the Potential Function: Application to a Single Rigid Body</td>
<td>20</td>
</tr>
<tr>
<td>3.1</td>
<td>Relevant Background</td>
<td>20</td>
</tr>
<tr>
<td>3.2</td>
<td>Conservative Forces and Moments</td>
<td>21</td>
</tr>
<tr>
<td>3.3</td>
<td>The Cartesian Stiffness Matrix</td>
<td>23</td>
</tr>
<tr>
<td>3.3.1</td>
<td>The Stiffness Matrices $\mathbf{m}_1 \mathbf{K}$, $\mathbf{m}_1 \mathbf{\hat{K}}$, $\mathbf{c}_1 \mathbf{K}$ and $\mathbf{c}_1 \mathbf{\hat{K}}$</td>
<td>25</td>
</tr>
<tr>
<td>3.3.2</td>
<td>The Stiffness Matrix $\mathbf{\check{K}}$</td>
<td>29</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Remarks</td>
<td>29</td>
</tr>
<tr>
<td>3.4</td>
<td>The Asymmetric Parts of the Stiffness Matrices</td>
<td>30</td>
</tr>
<tr>
<td>3.5</td>
<td>Examples</td>
<td>33</td>
</tr>
<tr>
<td>3.5.1</td>
<td>The Planar Case</td>
<td>34</td>
</tr>
<tr>
<td>3.5.2</td>
<td>The Stewart-Gough Platform</td>
<td>35</td>
</tr>
<tr>
<td>3.6</td>
<td>Closing Remarks</td>
<td>41</td>
</tr>
</tbody>
</table>
4 Conservative Forces and Moments from a Potential Function: Application to a System of Rigid Bodies

4.1 Potential Energy Functions

4.1.1 The Case \( U = \hat{U}_2 \)

4.1.2 The Cases \( U = \hat{U}_1 \) and \( U = \hat{U}_4 \)

4.1.3 Bodies Connected in Series

4.1.4 Incorporating Constraints

4.2 The Cartesian Stiffness Matrices for Multibody Systems

4.2.1 The Stiffness Matrices \( m^1K, m^1\hat{K}, \hat{\hat{\hat{K}}} \) and \( \hat{\hat{\hat{\hat{K}}} \) for Multibody Systems

4.2.2 The Stiffness Matrices \( m^2K \) and \( m^2\hat{K} \) for Multibody Systems

4.2.3 The Stiffness Matrix \( m^5K \) for Multibody Systems

4.3 Simplifications to \( m^5\hat{K} \)

4.3.1 A Pair of Rigid Bodies

4.3.2 The Case \( U = \hat{U}_5^# = \hat{U}_5^1(\mathbf{R}_1, \mathbf{y}_1^1) + \ldots + \hat{U}_5^N(\mathbf{R}_N, \mathbf{y}_N^1) \)

4.3.3 The Stiffness Matrix \( m^5\hat{K} \) for the Two-Body Constrained System

4.4 Relating the Asymmetry in the Stiffness Matrices to the Conservative Forces and Moments

4.4.1 The Asymmetric Parts of the Cartesian Stiffness Matrices \( m^1K, m^1\hat{K} \) and \( m^2\hat{K} \)

4.4.2 Skew-Symmetry Associated With \( m^5\hat{K} \)

4.5 Closing Remarks

5 The Cartesian Stiffness Matrix for an Intervertebral Joint in the Lumbar Spine

5.1 The Cartesian Stiffness Matrix of the Intervertebral Joint \( c^K_J \)

5.2 Experimental Stiffness Matrix Measurements

5.3 Relating the Experimental Stiffness Matrix to the Cartesian Stiffness Matrix

5.4 Examples

5.4.1 Results and Discussion

5.5 Closing Remarks

6 Applying the Cartesian Stiffness Matrix to a Musculoskeletal Model of the Lumbar Spine

6.1 Introduction

6.2 The Bushing Element in Musculoskeletal Software Platforms

6.2.1 The Six Frames

6.2.2 The Bushing Forces and Moments

6.2.3 Comments on Experimental Data

6.3 A SpineBushing Element

6.3.1 The Stiffness Matrix of the SpineBushing Element

6.3.2 The SpineBushing Forces and Moments

6.4 Application
6.5 Results ..................................................... 98
6.6 Conclusion ................................................. 100

7 Closing Remarks ............................................. 104

References ......................................................... 107
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Chapter 1

Introduction

The ability to accommodate multi-dimensional coupled motion of the spine in a non-linear manner makes the intervertebral joint one of the most complex joints in the body (Fig. 1.1). Consequently, accurate models of the joint tend to be computationally expensive while simple, numerically non-intensive models fail to capture its true behavior. These modeling issues hinder attempts at better understanding various existing spinal disorders and designing the most appropriate treatments for those afflicted.

From an engineering perspective, it is convenient to think of the spine as a vertical column consisting of rigid bodies connected by joints. The joints both permit and constrain the relative motion between the bodies by exerting conservative forces and moments on the two bodies it connects (Fig. 1.2). In the early 1970s, Panjabi posited that these conservative forces and moments could be captured using a 6 dimensional stiffness matrix [1]. A few years later, Panjabi and two of his colleagues published a seminal paper proposing a stiffness matrix $K_F$ for the intervertebral joints of the thoracic spine [2]. In this and later works, the number of unique components in the stiffness matrix were reduced from 36 to 12 using symmetry arguments and restricting attention to infinitesimal rotations. Measurements of these 12 elements for the lumbar spinal column have been reported in Stokes et al. [3–6]. A related stiffness matrix for the lumbar spine was later proposed by McGill and Norman [7], and in subsequent works, the potential energy of the muscle and external forces were incorporated into this matrix (see [8–10] and references cited therein).

Unfortunately, the stiffness matrices proposed by Panjabi et al. [2] and McGill and Norman [7] have several restrictions which limit their utility. The most problematic is the inability to accommodate finite rotations and the non-linear force-displacement
Fig. 1.1: Images of the human spine. Shown on the top left are the sacrum, lumbar, and thoracic components of the human spine, together with a number of spinal muscle fibers. The motion of each vertebra relative to its neighbors is specified by the joints situated in between the two bodies (top right). These joints permit not just uniaxial motion (lower figure), but are also responsible for the complex coupled motion evidenced in the spine.
response of the joint. Motivated by the work of O’Reilly et al. \cite{11}, we elaborate upon a Cartesian stiffness matrix that addresses these deficiencies.

![Image: Lumbar spine approximated as a system of rigid bodies connected by joints with stiffnesses along six degrees of freedom.](image)

**Fig. 1.2:** The lumbar spine (left) can be approximated as a system of rigid bodies connected by joints with stiffnesses along all six degrees of freedom. The conservative forces and moments exerted by the joints can be equated with a six degree-of-freedom stiffness matrix (right).

1.1 The Cartesian Stiffness Matrix

Effort to relate the increments in force and moment components acting on a system of rigid bodies to the infinitesimal motions that produced them have been numerous and ongoing. In the early 1990s, Duffy, Griffis and Pigoski \cite{12,13} produced a number of papers discussing examples of a linear mapping of the increments to the conservative force and moment components acting on a rigid body with the infinitesimal displacements and rotations which produced them. The linear mapping was a stiffness matrix $K_O$ which had the unusual feature of being asymmetric. To distinguish this matrix from the Hessian $H$ of a potential energy, $K_O$ is known as the “Cartesian stiffness matrix.” Griffis and Duffy’s examples in \cite{12} featured rigid bodies tethered to a fixed surface using linear springs (cf. Fig. 1.3). In the event that the springs were unstretched in the state of the body of interest, then the asymmetry of $K_O$ was seen to vanish. Subsequent works by Ciblak and Lipkins \cite{14} clarified aspects of the asymmetry. Howard et al. \cite{15} and Žefran and Kumar \cite{16}, using a Lie group approach, considered the more general case of a rigid body in a potential field, and established several representations for the Cartesian stiffness matrix. Other
researchers, such as [17–20], have extended the formulation of the Cartesian stiffness matrices to a range of mechanical systems.

Fig. 1.3: Schematic of the planar rigid body example first introduced in Griffis and Duffy [12] to illustrate and verify the computation of the Cartesian stiffness matrix. The body is attached to the ground by springs and the changes in the forces and moments related to the infinitesimal translation and rotations using the Cartesian stiffness matrix $K_O$.

Here we follow [21] and use an argument based on energetic considerations to establish the resultant conservative force and moment acting on a single rigid body, as well as a system composed of multiple rigid bodies. With the help of a Taylor series expansion, multiple examples of the resulting Cartesian stiffness matrices are then established and discussed. The potential biomechanical applications of this matrix [2, 4, 11, 22] are exemplified by using the Cartesian stiffness matrix to model the intervertebral joints of the lumbar spine. We supplement the stiffness matrix model of the spinal joint first proposed in 1979 by Panjabi et al. [2] so as to accommodate finite rotations. Further, we show how to relate the Cartesian stiffness matrix presented here to stiffness matrices measured experimentally and shed light on its use in musculoskeletal models of the spine.

1.2 Bushing Forces

Encouragingly, recent musculoskeletal software models of the spine built using, for example, LifeMOD Biomechanics Modeler (MSC Software, Santa Ana, CA), visualNastran 4D (MSC Software, Santa Ana, CA), and APOLLO [23] have started to employ spinal joints with stiffness and damping elements. This is done using a bushing element capable of exerting forces and moments in the same direction as the displacement between the bushing frames.\(^1\) These bushing elements can be likened to diagonal stiffness and damping matrices connected in series. Unfortunately, the diagonal nature of these bushing elements fail to replicate the coupled motion that is a

\(^1\)The term bushing is an artifact of its more common use in vehicle dynamics (see, e.g., [24, 25]).
defining feature of the intervertebral joint. Further complicating matters is the lack of agreement between how the elements are measured experimentally and how they are applied in musculoskeletal models of the spine. Consequently, erroneous kinematics and reaction forces may result. We address this problem by constructing a SpineB-

![Fig. 1.4: Schematic of a bushing element. By definition, the forces and moments exerted on the vertebral bodies due to the bushings are applied at the bushing frames (shown in blue). From the two figures above, it is apparent that the point of application of the bushing forces and moments will significantly alter the ensuing kinematics. To ensure that the computed force and moments (red arrows) agree with their experimental counterparts, it is important to position these frames correctly.](image)

ushing element specific to the intervertebral joints of the spine using OpenSim, an open source musculoskeletal software platform [26]. In contrast to existing bushing elements, the SpineBushing function permits coupling between the six degrees-of-freedom of the spine by the incorporation of non-zero off-diagonal stiffness matrix components. In addition, a number of small, yet highly significant differences with regards to the point of application of the bushing forces and moments are made to ensure compatibility between the computed SpineBushing forces and those obtained experimentally (Fig. 1.4). In Chapter 6, we show how this small change significantly alters the moments exerted on the vertebrae, and the resulting kinematics. Note that in designing the SpineBushing, we focused only on small, quasi-static motion of the individual vertebra. Consequently, no damping terms were incorporated, but can be easily included if necessary. Keeping the motions infinitesimal also allowed us to use functions that were much less numerically expensive, making the overall computational cost comparable to joints with existing bushing elements.

1.3 Dissertation Outline

This dissertation is organized as follows: in the next chapter, we present a brief introduction to the relative motion of rigid bodies and introduce some important
notation. In Chapter 3, four distinct representations for the potential energy function of a single rigid body are detailed (Section 3.2) and the corresponding conservative force and moment expressions associated with each of these potential energies derived. Expressions for the resulting Cartesian stiffness matrices relating incremental changes in the motion to changes in the forces and moments are given in Section 3.3 and some interesting features of the skew-symmetric components of the Cartesian stiffness matrix further discussed in Section 3.4. Sections 3.5.1 and 3.5.2 then illustrates the application of the Cartesian stiffness matrix by looking at the example of a planar body and a six degree-of-freedom Stewart-Gough platform.

In Chapter 4, we extrapolate upon the results of Chapter 3 to multibody systems. In particular, Section 4.1 details six possible expressions for the potential energy function that will be used to determine the conservative forces and moments acting on the system. Taylor series expansions of these forces and moments are then performed in Section 4.2 to determine the distinct Cartesian stiffness matrices associated with each potential energy function. The relationship between the skew-symmetric components of the ensuing Cartesian stiffness matrices and the conservative forces and moments are elaborated upon in Section 4.4.

The developments presented in Chapters 2, 3, and 4 provide the necessary background material for the establishment of the Cartesian stiffness matrix associated with the intervertebral joints of the lumbar spine. These will be described in Chapter 5. An example of a quadratic potential energy function suited to the lumbar spine joints is detailed in 5.4 and the Cartesian stiffness matrix for this potential is computed and discussed. Section 5.4.1 further illuminates the contributions of each of the different components of the Cartesian stiffness matrix to the resulting change in conservative forces and moments.

In Chapter 6, we first explain the use of bushing elements in musculoskeletal models of the spine. We then introduce a SpineBushing element - constructed in the OpenSim software platform [26] - that is specific to the intervertebral joint. This function features a stiffness matrix with off-diagonal elements, thus permitting more accurate replication of spinal kinematics. In addition, the joint forces and moments are exerted on the bodies themselves instead of at the joint. With the relevant groundwork in place (cf. Sections 6.2 and 6.3), the computed SpineBushing forces and moments are compared to those due to a typical bushing element in Section 6.5 with the help of three different models of the lumbar spine introduced in Section 6.4.

Finally, Chapter 7 briefly summarizes the results of this dissertation as well as avenues for further study.
Chapter 2

Relative Motions of Rigid Bodies

In this chapter, we will discuss several methods which are used to parameterize the relative motion of rigid bodies. In order to do this, we first present an overview of vectors and tensors, as well as elaborate upon our notation. With these basics in place, we then discuss the quantification of rigid body motion involving a single rigid body in Section 2.2. Here, we elaborate upon several different representations of rotations with particular emphasis placed on the Euler angle representation in Section 2.2.3. Section 2.3 details the generalization of the results of Section 2.2 to the parameterization of the relative motions between multiple rigid bodies. The definitions presented here will be integral to the developments presented in Chapters 3 and 4.

2.1 Background and Notation

We base much of our discussion on rotations on the authoritative review by Shuster [27] and follow the lead of J. W. Gibbs and others by writing our results using tensors. This gives us the freedom of expressing the results in either the reference or rotated frame and allows for a much smoother exposition of the theoretical developments.

However, in order to compute things numerically, it is necessary to use matrices to represent tensors. To minimize confusion, arrays and matrices of real numbers will be denoted by san-serif roman letters, such as $a, A$, etc, vectors and tensors will be denoted by bold-faced roman letters, e.g., $\mathbf{a}$ and $\mathbf{A}$, the indices $i, j, k, l, n, m, r, p, q$, and $s$ range from 1 to 3 and capitalized indices $I, J, K, L$ range from 1 to $N$. In the interest of brevity, we will utilize the Einstein summation convention.

To elaborate, if $\mathbf{A}$ is an arbitrary tensor, $\mathbf{a}$ is an arbitrary vector, and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$
and \{e_1, e_2, e_3\} are the set of fixed and moving right-handed orthonormal basis vectors respectively, then

\[
A = A_{ik} E_i \otimes E_k = \tilde{A}_{ik} e_i \otimes e_k = \tilde{A}_{ik} E_i \otimes e_k, \quad \mathbf{a} = A_i E_i = a_k e_k. \quad (2.1)
\]

The column vector expression for the vector \(\mathbf{a}\) in the fixed and moving frames are

\[
\begin{bmatrix}
\mathbf{a} \cdot \mathbf{E}_1 \\
\mathbf{a} \cdot \mathbf{E}_2 \\
\mathbf{a} \cdot \mathbf{E}_3
\end{bmatrix} =
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix}, \quad
\begin{bmatrix}
\mathbf{a} \cdot \mathbf{e}_1 \\
\mathbf{a} \cdot \mathbf{e}_2 \\
\mathbf{a} \cdot \mathbf{e}_3
\end{bmatrix} =
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}, \quad (2.2)
\]

and the matrix expression of the tensor \(A\) when projected to the fixed frame is

\[
A_{EE} =
\begin{bmatrix}
A_{11} = (AE_1) \cdot E_1 \\
A_{12} = (AE_2) \cdot E_1 \\
A_{13} = (AE_3) \cdot E_1 \\
A_{21} = (AE_1) \cdot E_2 \\
A_{22} = (AE_2) \cdot E_2 \\
A_{23} = (AE_3) \cdot E_2 \\
A_{31} = (AE_1) \cdot E_3 \\
A_{32} = (AE_2) \cdot E_3 \\
A_{33} = (AE_3) \cdot E_3
\end{bmatrix}. \quad (2.3)
\]

We also take this opportunity to note the following tensorial identities:

\[
(a \otimes b)c = a(b \cdot c), \quad c(a \otimes b) = b(a \cdot c), \quad (2.4)
\]

for \(a, b\) and \(c \in \mathbb{E}^3\), where \(\mathbb{E}^3\) is Euclidean three-space. In addition, we introduce the skew-symmetric tensor \(-(\varepsilon a)\) with axial vector \(a\), such that

\[
-(\varepsilon a)c = a \times c, \quad ax(\varepsilon a) = a \quad (2.5)
\]

These identities will be used extensively throughout this paper.

### 2.2 A Single Rigid Body

A rigid body \(\mathcal{B}\) consists of a collection of material points \(X\) where the distance between any of these points remains constant. As shown in Figure 2.1, it is convenient to define a fixed reference configuration \(\kappa_0\) of this body. This configuration occupies a fixed region of Euclidean three-space \(\mathbb{E}^3\). The position vector, relative to a fixed origin \(O\), of a material point \(X\) in this configuration is defined by the position vector \(X\). In a similar manner, the present (or current) configuration \(\kappa_t\) of \(\mathcal{B}\) can be defined and the position vector of a material point \(X\) in this configuration is denoted by \(x\).

The position vector of the center of mass and the material point \(A\) of \(\mathcal{B}\) in \(\kappa_t\) will thus have the representations

\[
\begin{align*}
\bar{x} &= X_1 E_1 + X_2 E_2 + X_3 E_3 = x_1 e_1 + x_2 e_2 + x_3 e_3, \\
x_A &= X_{A_1} E_1 + X_{A_2} E_2 + X_{A_3} E_3 = x_{A_1} e_1 + x_{A_2} e_2 + x_{A_3} e_3. \quad (2.6)
\end{align*}
\]
The motion of the rigid body can be characterized by a rotation $Q$ of the body about its center of mass, followed by a displacement $d$. That is, if $x$ is the position vector of any point on the body, then

$$x = Q(t)X + d(t), \quad \bar{x} = Q(t)\bar{X} + d(t).$$

(2.7)

It is useful to consider the rotation tensor $Q$ as the proper-orthogonal tensor that transforms vectors from the fixed frame into vectors in the corotating frame. Thus, we use it to define the corotational basis in terms of the fixed basis:

$$e_i = Q E_i = Q_{ki} E_k.$$

(2.8)

From (2.8), we can write $Q$ as

$$Q = Q_{ik} E_i \otimes E_k = Q_{ik} e_i \otimes e_k = e_i \otimes E_i.$$

(2.9)

It is apparent that, $Q$ has the same components in the bases $\{E_i \otimes E_k\}$ and $\{e_i \otimes e_k\}$, where $e_i$ is given by (2.8).

---

1 A number of other representations for rigid body motion are commonly used in the literature. For instance, using Chasles’ theorem, a rigid body motion can be defined as a translation along a line, followed by a rotation about a line [28]. This representation is commonly termed the helical axis parameterization of rigid body motion and is used in, for example, [29]. However, as was recently shown in [30], existing algorithms currently employed in the biomechanics community to determine the associated helical axis parameters of interest are highly inaccurate and may lead to erroneous results, even in the case of simple planar motion of bodies. For this reason, we find it convenient to use the parameterization given by equation (2.7).
A comprehensive exposition on the different parameterizations of $Q$ are discussed in Shuster [27] and the interested reader is directed to that reference for a more in-depth discussion. Here, we elaborate briefly on the following three parameterizations commonly used in rigid body dynamics:

1. Euler’s representation.

2. The Euler-Rodrigues symmetric parameter representation.

3. The Euler angle representation.

Each of these representations have their advantages and disadvantages; the best choice of parameterization is often dependent on the problem of interest, analysis employed, and the amount of information available. Regardless of the representation, we have the following identities for any rotation tensor $Q$:

$$\det (Q) = 1, \quad Q^T Q = Q Q^T = I. \quad (2.10)$$

2.2.1 Euler’s Representation

Euler’s representation for the rotation tensor that produces a counterclockwise rotation $\alpha$ about an axis of rotation $p$ is

$$Q = L(\alpha, p) = \cos(\alpha)I + (1 - \cos(\alpha))(p \otimes p) - \sin(\alpha)(\varepsilon p)$$

$$= \cos(\alpha)(I - p \otimes p) - \sin(\alpha)(\varepsilon p) + p \otimes p. \quad (2.11)$$

where $(\varepsilon p)$ is the skew-symmetric tensor defined by equation (2.5) and $p \cdot p = 1$.

To derive this well known representation [27,31], we consider a vector $v_O$ rotated about the axis of rotation $p$ to a new vector $v_f$. The component of $v_O$ that is parallel to the $p$ remains unaltered while the component of $v_O$ that is perpendicular to $p$ is rotated by amount $\alpha$ to its new value. Thus, denoting $v_{O,\parallel}$ and $v_{O,\perp}$ as the parallel and perpendicular decompositions of $v_O$, we have the identities

$$v_{O,\parallel} = v_{f,\parallel}, \quad v_{f,\perp} = \cos(\alpha)v_{O,\perp} + \sin(\alpha)(p \times v_{O,\perp}). \quad (2.12)$$

Using the identities (2.4) and (2.5) we can write $v_f$ as,

$$v_f = v_{f,\parallel} + v_{f,\perp} = (v_{O,\parallel}) + (\cos(\alpha)v_{O,\perp} + \sin(\alpha)(p \times v_{O,\perp}))$$

$$= (v_O \cdot p)p + \cos(\alpha)(v_O - (v_O \cdot p)p) + \sin(\alpha)(p \times (v_O - (v_O \cdot p)p))$$

$$= (\cos(\alpha)I + (1 - \cos(\alpha))(p \otimes p) - \sin(\alpha)(\varepsilon p))v_O. \quad (2.13)$$

Expression (2.11) for the Euler representation for the rotation tensor follows immediately from the underbraced term.
Further, if $Q_{ik}$ are the components of the tensor $Q = L(\alpha, p)$ expressed in matrix form, then we also have the expressions

$$
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
= \cos(\alpha) \begin{bmatrix}
1-p_1^2 & -p_1p_2 & -p_1p_3 \\
-p_1p_2 & 1-p_2^2 & -p_2p_3 \\
-p_1p_3 & -p_2p_3 & 1-p_3^2
\end{bmatrix}
+ \sin(\alpha) \begin{bmatrix}
0 & -p_3 & p_2 \\
p_3 & 0 & -p_1 \\
-p_2 & p_1 & 0
\end{bmatrix},
$$

(2.14)

where

$$p = p_i E_i = p_i e_i.$$  

(2.15)

To see (2.15), note that $Qp = p$. Furthermore, as $Q = Q_{ik} E_i \otimes E_k = Q_{ik} e_i \otimes e_k$, the components of $Q$ are similar regardless of whether we choose to express $Q$ in terms of the fixed, or corotational basis. We also mention that the computation of the inverse of $Q$ is trivial due to the identities

$$L^T(\alpha, p) = L(-\alpha, p) = L(\alpha, -p).$$  

(2.16)

2.2.2 The Euler-Rodrigues Symmetric Parameter Representation

The Euler-Rodrigues symmetric parameter representation is very closely related to the Euler representation, as will be shown below. Briefly, given an axis and angle of rotation $(p, \alpha)$, the four Euler-Rodrigues symmetric parameters, $q_0$ and $\bm{q}$ can be defined as,

$$q_0 = \cos\left(\frac{\alpha}{2}\right), \quad \bm{q} = \sin\left(\frac{\alpha}{2}\right) \, p.$$  

(2.17)

As a consequence of their definition, the parameters are subject to what is known as the Euler parameter constraint [32–34]:

$$q_0^2 + \bm{q} \cdot \bm{q} = 1.$$  

(2.18)

Expressing $p$ and $\alpha$ in terms of the parameters $q_0$ and $\bm{q}$ in (2.14), we obtain the

---

2The four parameters, $q_0$ and the three components of $\bm{q}$, are often considered to be the four components of a quaternion $\bm{q} = q_0 + q_1 i + q_2 j + q_3 k$, where $q_i$ are the components of $\bm{q}$ relative to a right-handed orthonormal basis, and $i, j,$ and $k$ are bases vectors for the quaternion. Consequently, Euler-Rodrigues symmetric parameters are sometimes referred to as (unit) quaternions.
Euler-Rodrigues symmetric parameter representation for a rotation tensor:

\[
Q = \bar{Q} (q_0, q) = Q_{ik} \mathbf{E}_i \otimes \mathbf{E}_k \\
= (q_0^2 - q \cdot q) \mathbf{I} + 2q \otimes q - 2q_0 (\varepsilon q).
\]  

(2.19)

With the help of (2.17), we find that the components of the matrix \( Q = \bar{Q}(q_0, q) \) are

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
= (q_0^2 - q_1^2 - q_2^2 - q_3^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
+ \begin{bmatrix} 2q_1^2 & 2q_1q_2 & 2q_1q_3 \\ 2q_1q_2 & 2q_2^2 & 2q_2q_3 \\ 2q_1q_3 & 2q_2q_3 & 2q_3^2 \end{bmatrix}
+ \begin{bmatrix} 0 & -2q_0q_3 & 2q_0q_2 \\ 2q_0q_3 & 0 & -2q_0q_1 \\ -2q_0q_2 & 2q_0q_1 & 0 \end{bmatrix}.
\]  

(2.20)

Note that as in (2.14), the matrix representation for \( Q = \bar{Q}(q_0, q) \) given by (2.20) can be decomposed into the sum of two symmetric matrices and a skew-symmetric matrix. We also note that

\[
\bar{Q}(q_0, q) = \bar{Q}(-q_0, -q),
\]  

(2.21)

and

\[
\bar{Q}^T(q_0, q) = \bar{Q}(-q_0, q) = \bar{Q}(q_0, -q).
\]  

(2.22)

2.2.3 The Euler Angle Representation

The Euler angle representation of the rotation tensor dates back to works by Euler first presented in 1751 [35, 36]. They provide a means of representing the orientation of the moving frame relative to the fixed frame using a composition of three rotations about three distinct axes:

\[
Q = \tilde{Q}(\nu^1, \nu^2, \nu^3) = \mathbf{L}(\nu^3, \mathbf{g}_3)\mathbf{L}(\nu^2, \mathbf{g}_2)\mathbf{L}(\nu^1, \mathbf{g}_1),
\]  

(2.23)

where \( \{\nu^i\} \) are the Euler angles, \( \{\mathbf{g}_i\} \) are the set of unit vectors known as the Euler basis, and \( \mathbf{L}(\nu^i, \mathbf{g}_i) \) is defined by (2.11).

As may be seen from Figure 2.2, the manner in which the Euler angles parametrize the rotation is easily visualized by imagining two intermediate bases \( \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} \), \( \{\mathbf{e}''_1, \mathbf{e}''_2, \mathbf{e}''_3\} \). For the 3-2-1 Euler angle sequence, the first angle \( \nu^1 \) represents the rotation of \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) about \( \mathbf{E}_3 = \mathbf{g}_1 \) to their respective transformed values \( \mathbf{e}'_1 \) and \( \mathbf{e}'_2 \). Similarly, the second rotation through the angle \( \nu^2 \) about the vector \( \mathbf{e}_2 = \mathbf{g}_2 \) transforms \( \mathbf{e}_3 \) and \( \mathbf{e}'_2 \) into \( \mathbf{e}''_3 \) and \( \mathbf{e}''_2 \), respectively while the third rotation is through the angle \( \nu^3 \) about the vector \( \mathbf{e}_1 = \mathbf{g}_3 \) and transforms \( \mathbf{e}'_3 \) and \( \mathbf{e}''_3 \) into \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \).
respectively. Thus, for any rigid body motion, the final orientation of the body is found by first rotating it by an amount $\nu^1$ about the (fixed) axis $g_1$, followed by a rotation about the (intermediate) axis $g_2$ by $\nu^2$ and finally, by $\nu^3$ about the (body=fixed) axis $g_3$. Notice that $g_2$ is a function of only $\nu^1$ while $g_3$ is a function of both $\nu^1$ and $\nu^2$ but not $\nu^3$.

![Diagram](image.png)

**Fig. 2.2:** The 3-2-1 Euler angle rotation sequence for a rectangular rigid body. Here, $e'_i$ and $e''_i$ are the intermediate basis vectors after the rigid body has been rotated by $\nu^1$ and $\nu^2$ about $E_3$ and $e'_i$ respectively. The vectors $g_1 = E_3$, $g_2 = e'_2 = \cos(\nu^1)E_2 - \sin(\nu^1)E_1$, and $g_3 = e_1 = \cos(\nu^2)e'_1 + \sin(\nu^2)E_3$ form the Euler basis.

Following $[21, 36, 37]$, the dual Euler basis is defined as the set $\{g^1, g^2, g^3\}$ such that

$$g^i \cdot g_k = \delta^i_k,$$  \hspace{1cm} (2.24)

where $\delta^i_k$ is the Kronecker delta: $\delta^i_k = 1$ when $i = k$ and is otherwise 0. We can think of the Euler basis $\{g_1, g_2, g_3\}$ and the dual Euler basis $\{g^1, g^2, g^3\}$ as basis vectors in the tangent and cotangent spaces respectively of the manifold $SO(3)$. The connection coefficients associated with the Euler angles are defined as

$$\gamma^i_{jk} = \frac{\partial g^i_j}{\partial \nu^k} \cdot g^j.$$  \hspace{1cm} (2.25)

In contrast to the contravariant basis vectors $a_i = \frac{\partial r}{\partial q^i}$ for a curvilinear coordinate system $\{q^1, q^2, q^3\} \in \mathbb{R}^3$, the following result

$$\frac{\partial g_j}{\partial \nu^k} \cdot g^i = \frac{\partial g_i}{\partial \nu^k} \cdot g^j,$$  \hspace{1cm} (2.26)

is not always true. Using the identity (2.24), we can find an alternative representation for the connection coefficients:

$$\gamma^i_{jk} = -\frac{\partial g^i_j}{\partial \nu^k} \cdot g_j.$$  \hspace{1cm} (2.27)
Further details on the role played by connection coefficients and their relationship to Christoffel symbols can be found in [38–40].

The 3-2-1 Set of Euler Angles

The 3-2-1 set of Euler angles is commonly used in biomechanics and vehicle dynamics, and is discussed in numerous textbooks. Here, we recall some results for this set from [36, 37] to aid in the developments presented in the remainder of this thesis.

Fig. 2.3: Schematic of the 3-2-1 set of Euler angles and the individual rotations these angles represent. The rotation tensor \( Q \) that they parameterize transforms \( E_i \) to \( e_i \).

Recalling that \( e_k = Q_{ik} E_i \) and \( Q_{ik} = e_k \cdot E_i \), one can compute the components of the matrix \( Q = Q(\nu^1, \nu^2, \nu^3) \) for the 3-2-1 Euler angle sequence:

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
= \begin{bmatrix}
c(\nu^1) & -s(\nu^1) & 0 \\
s(\nu^1) & c(\nu^1) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c(\nu^2) & 0 & -s(\nu^2) \\
0 & 1 & 0 \\
-s(\nu^2) & 0 & c(\nu^2)
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & c(\nu^3) & -s(\nu^3) \\
0 & s(\nu^3) & c(\nu^3)
\end{bmatrix},
\]

(2.28)

where \( c(x) \equiv \cos(x) \) and \( s(x) \equiv \sin(x) \).
The Euler basis vectors \( \{ \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \} \) have the representations

\[
\begin{bmatrix}
\mathbf{g}_1 \\
\mathbf{g}_2 \\
\mathbf{g}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 \\
- \sin(\nu^1) & \cos(\nu^1) & 0 \\
\cos(\nu^1) \cos(\nu^2) & \sin(\nu^1) \cos(\nu^2) & - \sin(\nu^2)
\end{bmatrix}
\begin{bmatrix}
\mathbf{E}_1 \\
\mathbf{E}_2 \\
\mathbf{E}_3
\end{bmatrix}
\] (2.29)

The components \( G^i_k = \mathbf{g}^i \cdot \mathbf{E}_k \) of the matrix \( \mathbf{G} \) can be inferred from the first of the two representations below for the dual Euler basis vectors:

\[
\begin{bmatrix}
\mathbf{g}_1 \\
\mathbf{g}_2 \\
\mathbf{g}_3
\end{bmatrix} =
\begin{bmatrix}
\cos(\nu^1) \tan(\nu^2) & \sin(\nu^1) \tan(\nu^2) & 1 \\
- \sin(\nu^1) & \cos(\nu^1) & 0 \\
\cos(\nu^1) \sec(\nu^2) & \sin(\nu^1) \sec(\nu^2) & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{E}_1 \\
\mathbf{E}_2 \\
\mathbf{E}_3
\end{bmatrix}
\] (2.31)

\[
\begin{bmatrix}
\mathbf{g}_1 \\
\mathbf{g}_2 \\
\mathbf{g}_3
\end{bmatrix} =
\begin{bmatrix}
0 & \sin(\nu^3) \sec(\nu^2) & \cos(\nu^3) \sec(\nu^2) \\
0 & \cos(\nu^3) & - \sin(\nu^3) \\
1 & \sin(\nu^3) \tan(\nu^2) & \cos(\nu^3) \tan(\nu^2)
\end{bmatrix}
\begin{bmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{bmatrix}
\] . (2.32)

The matrix featuring on the right-hand side of (2.29) is \( \mathbf{G}^{-T} \). As expected, \( \mathbf{g}^i \cdot \mathbf{g}_k = \delta^i_k \), where \( \delta^i_k \) is the Kronecker delta. Similar expressions for the components \( \hat{G}^i_k = \mathbf{g}^i \cdot \mathbf{e}_k \) of the matrix \( \hat{\mathbf{G}} \), and its associated inverse, \( \hat{\mathbf{G}}^{-T} \) can be deduced from equations (2.30) and (2.32).

The angular velocity vector of the rigid body \( \mathbf{\omega} \) is defined as the axial vector of the skew-symmetric tensor \( \hat{\mathbf{Q}} \hat{\mathbf{Q}}^T \). This vector has the representations

\[
\mathbf{\omega} = \mathbf{a} \times (\hat{\mathbf{Q}} \hat{\mathbf{Q}}^T) = \dot{\nu}^k \mathbf{a} \times \left( \frac{\partial \hat{\mathbf{Q}}}{\partial \nu^k} \hat{\mathbf{Q}}^T \right) = \dot{\nu}^k \mathbf{g}_k.
\] (2.33)

It is apparent from representation (2.33), that \( \mathbf{g}_k \) is the axial vector of the skew-symmetric tensor \( \mathbf{\Omega}_k = \left( \frac{\partial \hat{\mathbf{Q}}}{\partial \nu^k} \hat{\mathbf{Q}}^T \right) \). Hence, the derivative of \( \mathbf{g}_k \) can also be written as,

\[
\frac{\partial \mathbf{g}_k}{\partial \nu^j} = \mathbf{a} \times \left( \frac{\partial^2 \hat{\mathbf{Q}}}{\partial \nu^j \partial \nu^k} \hat{\mathbf{Q}}^T \right) + \mathbf{a} \times \left( \frac{\partial \hat{\mathbf{Q}}}{\partial \nu^k} \frac{\partial \hat{\mathbf{Q}}^T}{\partial \nu^j} \right)
\]

\[
= \mathbf{a} \times \left( \frac{\partial^2 \hat{\mathbf{Q}}}{\partial \nu^j \partial \nu^k} \hat{\mathbf{Q}}^T \right) + \mathbf{a} \times \left( \frac{\partial \hat{\mathbf{Q}}}{\partial \nu^k} \hat{\mathbf{Q}}^T \frac{\partial \hat{\mathbf{Q}}^T}{\partial \nu^j} \right)
\]

\[
= \mathbf{a} \times \left( \frac{\partial^2 \hat{\mathbf{Q}}}{\partial \nu^j \partial \nu^k} \hat{\mathbf{Q}}^T \right) + \mathbf{a} \times (\mathbf{\Omega}_k \mathbf{\Omega}_j^T).
\] (2.34)
Invoking the identities \( \frac{\partial^2 Q}{\partial \nu \partial \nu} = \frac{\partial^2 Q}{\partial \nu' \partial \nu'} \) and \( \Omega_j^T = -\Omega_j \), it follows that

\[
\begin{align*}
\frac{\partial g_k}{\partial \nu} - \frac{\partial g_j}{\partial \nu} &= \text{ax} (\Omega_j \Omega_k - \Omega_k \Omega_j). \tag{2.35}
\end{align*}
\]

Noting that the axial vectors of \( \Omega_j \) and \( \Omega_k \) are, respectively, \( g_j \) and \( g_k \), and with the help of a well known identity\(^3\) for the axial vector of a product of the form \( \Omega_j \Omega_k - \Omega_k \Omega_j \), we conclude that

\[
\frac{\partial g_k}{\partial \nu} - \frac{\partial g_j}{\partial \nu} = g_j \times g_k. \tag{2.36}
\]

This identity will be used to establish (3.55), (3.57) in Section 3.4, and (4.83) and (4.85) in Section 4.4.

### 2.3 Multibody Systems

Suppose now that our system is composed of \( N \) rigid bodies with positions of the center of mass and rotation tensor associated with the present configuration of the \( K^{th} \) body denoted by \( \bar{x}^K \) and \( Q^K \) respectively. That is, the corotational basis vectors fixed to body \( K \) is given by \( e^K_i = Q^K E_i \) (cf. Figure 2.4). Echoing the development in Section 2.2, we have the representations,

\[
\begin{align*}
\bar{x}^K &= X^K_1 E_1 + X^K_2 E_2 + X^K_3 E_3 = x^K_1 e^K_1 + x^K_2 e^K_2 + x^K_3 e^K_3, \\
x^K_A &= X^K_A_1 E_1 + X^K_A_2 E_2 + X^K_A_3 E_3 = x^K_{A_1} e^K_{A_1} + x^K_{A_2} e^K_{A_2} + x^K_{A_3} e^K_{A_3}, \quad (K = 1, \ldots, N), \tag{2.37}
\end{align*}
\]

for the center of mass, and the material point \( A \) in the present configuration of the \( K^{th} \) rigid body. We also define the relative positions between the center of mass of two adjacent bodies as

\[
\bar{y}^K = \bar{x}^K - \bar{x}^{K-1} = Y^K_i E_i, \quad K = 1, 2, \ldots, N, \tag{2.38}
\]

where \( \bar{x}^0 = 0 \).

To aid in the developments presented in this section, we will denote the Euler angles used to parameterize the rotation tensor \( Q_K \) of the \( K^{th} \) rigid body by \( (\nu^1_K, \nu^2_K, \nu^3_K) \). Likewise, \( (\beta^1_K, \beta^2_K, \beta^3_K) \) will be utilized to represent the three Euler angles used to characterize the relative rotation between the two bodies:

\[
Q_K = Q_K(\nu^1_K, \nu^2_K, \nu^3_K), \quad R_K = R(\beta^1_K, \beta^2_K, \beta^3_K) = Q_K(Q_{K-1})^T, \quad K = 1, 2, \ldots, N. \tag{2.39}
\]

In the interest of conciseness, for the first rigid body, we assume that \( Y^1 = x^1, \ R_1 = Q_1, Q_0 = I, \ \bar{x}^0 = 0, \ \text{and} \ \nu_1^1 = \beta_1^1 \), where necessary.

---

\(^3\) See Example A.7 in [41] or Equation (A.3) in [42].
Fig. 2.4: An example of a rigid body system composed of two bodies. The functional spinal unit shown consists of the sacrum \( S \), the fifth lumbar vertebra \( L_5 \) and the intervertebral disc \( I \). The basis vectors \( \{ e_1^1, e_3^1, e_3^3 \} \) and \( \{ e_1^2, e_3^2, e_3^3 \} \) are attached to the body \( S \) and \( L_5 \) respectively.

It follows from (2.39) that the angular velocity vectors of the \( K^{th} \) rigid body has the representations

\[
\omega_K = \nu_K^1 g_K^1 + \nu_K^2 g_K^2 + \nu_K^3 g_K^3, \\
\omega_K = \omega_{K-1} + \omega_K \\
= (\nu_{K-1}^1 g_{K-1}^1 + \nu_{K-1}^2 g_{K-1}^2 + \nu_{K-1}^3 g_{K-1}^3) + \left( \beta_K^1 g_K^1 + \beta_K^2 g_K^2 + \beta_K^3 g_K^3 \right). 
\]

(2.40)

Here, \( \{ g_1^K, g_2^K, g_3^K \} \) is the Euler basis (with dual basis \( \{ g_1^K, g_2^K, g_3^K \} \)) of the \( K^{th} \) rigid body, and \( \{ g_1^K, g_2^K, g_3^K \} \) is the Euler basis of the relative rotation \( R_K \), between the \( K^{th} \) and \( (K-1)^{st} \) body (with dual basis \( \{ g_1^K, g_2^K, g_3^K \} \)).

Using distinct sets of the 3-2-1 Euler angles to parameterize each \( Q_K \) and \( R_K \),

\(^4\) It is important to note that the Euler angles are not additive: \( \nu_K^i \neq \nu_{K-1}^i + \beta_K^i \) (\( i = 1, 2, 3 \)).
equations (2.32) and (2.30) of Section 2.2.3 then take the form,

\[
\begin{bmatrix}
    g_1^K \\
g_2^K \\
g_3^K
\end{bmatrix} =
\begin{bmatrix}
    0 & 0 & 1 \\
    -\sin(\nu_1^K) & \cos(\nu_1^K) & 0 \\
    \cos(\nu_1^K) \cos(\nu_2^K) & \sin(\nu_1^K) \cos(\nu_2^K) & -\sin(\nu_2^K)
\end{bmatrix}
\begin{bmatrix}
    E_1 \\
    E_2 \\
    E_3
\end{bmatrix}
\] (2.41)

and

\[
\begin{bmatrix}
    g_{K,1} \\
g_{K,2} \\
g_{K,3}
\end{bmatrix} =
\begin{bmatrix}
    \cos(\nu_1^K) \tan(\nu_2^K) & \sin(\nu_1^K) \tan(\nu_2^K) & 1 \\
    -\sin(\nu_1^K) & \cos(\nu_1^K) & 0 \\
    \cos(\nu_1^K) \sec(\nu_2^K) & \sin(\nu_1^K) \sec(\nu_2^K) & 0
\end{bmatrix}
\begin{bmatrix}
    E_1 \\
    E_2 \\
    E_3
\end{bmatrix}
\] (2.43)

\[
\begin{bmatrix}
    g_{K,1} \\
g_{K,2} \\
g_{K,3}
\end{bmatrix} =
\begin{bmatrix}
    0 & \sin(\nu_1^K) \sec(\nu_2^K) & \cos(\nu_1^K) \sec(\nu_2^K) \\
    0 & \cos(\nu_1^K) & -\sin(\nu_2^K) \\
    1 & \sin(\nu_1^K) \tan(\nu_2^K) & \cos(\nu_1^K) \tan(\nu_2^K)
\end{bmatrix}
\begin{bmatrix}
    E_1 \\
    E_2 \\
    E_3
\end{bmatrix}
\] (2.44)

That is,

\[
G_{K,j}^i = g_{K,i}^j \cdot E_j, \quad \tilde{G}_{K,j}^i = g_{K,i}^j \cdot e_j^K,
\]

\[(G^-T)_{K,j}^i = g_{K,i}^j \cdot E_j, \quad (\tilde{G}^-T)_{K,j}^i = g_{K,i}^j \cdot e_j^K.
\] (2.45)

We also take this opportunity to introduce the matrices

\[
\tilde{G}_{K,j}^i = \tilde{g}_{K,i}^j \cdot E_j, \quad \tilde{G}_{K,j}^i = \tilde{g}_{K,i}^j \cdot e_j^K,
\]

\[(\tilde{G}^-T)_{K,j}^i = \tilde{g}_{K,i}^j \cdot E_j, \quad (\tilde{G}^-T)_{K,j}^i = \tilde{g}_{K,i}^j \cdot e_j^K.
\] (2.46)

which relate the Euler and dual Euler basis of the relative rotation between bodies \(K\) and \(K'\) to the inertial and body-fixed basis vectors respectively. As

\[
\tilde{G}_K = \tilde{G}_K R_{K-1} R_{K-2} \ldots R_2 R_1 = \tilde{G}_K (\beta_1^1, \beta_1^2, \ldots, \beta_K^2, \beta_K^3),
\] (2.47)

the associated expressions for \(\tilde{G}_K\) and \((\tilde{G}_K)^-T\) can be determined in a straightforward manner from those of \(G_K\) and \((G_K)^-T\).

### 2.4 Closing Remarks

In this chapter, we introduced a number of important notations and concepts. We first showed how to parameterize the motion of a system composed of a single rigid body in Section 2.2. Focusing on the Euler angle parameterization of rotation, we
introduced the Euler basis $\{g_1, g_2, g_3\}$ and its associated dual $\{g_1, g_2, g_3\}$ (cf. Section 2.2.3). These concepts were then extended to the case where the system was composed of multiple rigid bodies in Section 2.3. Various different matrix representations for the components of the Euler basis and its dual were also presented in the same section. These developments will be used extensively in the following chapters.
Recall that a force $\mathbf{F}$ and moment $\mathbf{M}$ acting on a body are *conservative* if they can both be written as the gradient of a potential energy function. In this chapter, we explore different representations of this potential function for a single rigid body. Four distinct representations for the potential energy function of a single rigid body are detailed in Section 3.2 and the corresponding conservative force and moment expressions associated with each of the associated potential energies are derived. Expressions for the resulting Cartesian stiffness matrices relating incremental changes in the displacements and rotations to small changes in the forces and moments are further detailed in Sections 3.3.1 and 3.3.2. In Section 3.4, we show how the skew-symmetric part of the Cartesian stiffness matrix are related to the conservative forces and moments. Following these developments, two examples, the first featuring a planar rigid body and the second, the 3-dimensional Stewart Gough platform, are described in Sections 3.5 and 3.5.2.

### 3.1 Relevant Background

As mentioned in Section 2.2 that the motion of a single rigid body can be characterized by the rotation $\mathbf{Q}$ of the body and the position vector $\mathbf{x}$ of a material point
on the body:
\[ x = Q(t)X + d(t), \]
where \( Q(t) \) and \( d(t) \) are functions of time. Here, we use Euler angles to parameterize \( Q \). Thus, the angular velocity of the rigid body is given by
\[ \omega = \dot{\nu}^1 g_1 + \dot{\nu}^2 g_2 + \dot{\nu}^3 g_3, \] (3.1)
where \( \{g_1, g_2, g_3\} \) is the Euler basis and is dual to the dual Euler basis vectors \( \{\bar{g}_1, \bar{g}_2, \bar{g}_3\} \).

Our work in this is aided considerably by the fact that we are only considering a single rigid body.

### 3.2 Conservative Forces and Moments

Motivated by the developments in [21], we assume that the potential energy function \( U \) of a rigid body can be expressed as a function of the position vector of the center of mass \( \bar{x} \) and rotation tensor \( Q \). Among others, such a function encompasses the situation where the conservative field is supplied by springs tethering the body to a fixed surface, and a Newtonian gravitational force field attracting the rigid body to a fixed body. Alternatively, we can also express \( U \) as a function of the Cartesian coordinates of a point on the rigid body and the Euler angles.\(^1\) In fact, we can readily establish several distinct representations for \( U \):

\[
U = U(Q, \bar{x}) = U_1(\nu^1, \nu^2, \nu^3, x_1, x_2, x_3) = U_2(\nu^1, \nu^2, \nu^3, X_1, X_2, X_3) = U_3(\nu^1, \nu^2, \nu^3, x_{A_1}, x_{A_2}, x_{A_3}) = U_4(\nu^1, \nu^2, \nu^3, X_{A_1}, X_{A_2}, X_{A_3}).
\] (3.2)

We obtain \( U_1 \) from \( U \) by expressing the components of \( Q \) in terms of the Euler angles and the vector \( \bar{x} \) in terms of its components \( x_k \) in the bases vectors \( e_k \): \( \bar{x} = x_k e_k \). The vector \( \bar{x} \) can also be expressed in terms of the fixed basis \( \{E_1, E_2, E_3\} \) and this leads to the representation \( U_2 \). Related comments apply for the two potential functions \( U_3, U_4 \).

To prescribe the conservative force \( \mathbf{F} \) and moment (relative to the center of mass) \( \mathbf{M} \) acting on the body, we identify the mechanical power of these quantities with the negative of the time rate of change of \( U \):
\[ -\dot{U} = \mathbf{F} \cdot \dot{\bar{x}} + \mathbf{M} \cdot \dot{\omega}. \] (3.3)

\(^1\)Related developments utilizing the quaternion parameterization of rigid body rotation have been elaborated upon in [43]. In this dissertation, we will focus on the Euler angle parameterization of rigid body rotation as this more commonly used in the biomechanics community.
Following [21], and with the help of the dual Euler basis, we find the following representations for $F$ and $M$:

$$F = -\frac{\partial U_2}{\partial X_k} E_k, \quad M = -\frac{\partial U_2}{\partial \nu} g^k. \quad (3.4)$$

We emphasize that the force in this expression is assumed to act at the center of mass and the moment $M$ is taken relative to the center of mass (cf. Figure 3.1).

Now suppose we wish to consider moments relative to other points. There are two cases of primary interest: a point $A$ on the body and a fixed point $O$. With the help of the well-known identities for the resultant moments relative to $A$ and $O$,

$$M_A = M - (x_A - \bar{x}) \times F, \quad M_O = M + \bar{x} \times F, \quad (3.5)$$

and using the fact that $A$ is a point on the body,

$$\dot{x}_A = \dot{x} + \omega \times (x_A - \bar{x}), \quad (3.6)$$

we find that

$$F \cdot \dot{x} + M \cdot \omega = F \cdot \dot{x}_A + M_A \cdot \omega = F \cdot (\dot{x} - \omega \times \bar{x}) + M_O \cdot \omega. \quad (3.7)$$

Invoking (3.3) and noting that

$$\dot{x} - \omega \times \bar{x} = \dot{x}_k e_k, \quad (3.8)$$
we conclude that
\[ F = -\frac{\partial U_4}{\partial X_{A_k}} E_k, \quad M_A = -\frac{\partial U_4}{\partial \nu^k} g^k, \]  
\( \text{(3.9)} \)

and
\[ F = -\frac{\partial U_1}{\partial x_k} e_k, \quad M_O = -\frac{\partial U_1}{\partial \nu^k} g^k. \]  
\( \text{(3.10)} \)

The contrasts between (3.4), (3.9), and (3.10) are illuminating. The representation (3.10) indicates that when moments about a fixed point \( O \) are considered, the natural representation for the conservative force is with respect to the corotational basis. This is in surprising contrast to the case where the moments are taken relative to a material point on the body (cf. (3.9)).

**Fig. 3.2:** Two configurations \( \kappa_t \) and \( \kappa_{t'} \) of a rigid body \( B \). The kinematic quantities associated with the configuration \( \kappa_{t'} \) are distinguished by a superscript \( ' \) from those associated with the configuration \( \kappa_t \): e.g., \( \bar{x}' = \bar{x}(t') \).

### 3.3 The Cartesian Stiffness Matrix

For any of the representations of the conservative forces and moments, a stiffness matrix can be defined. This matrix relates the changes in the components of a pair of forces and moments in two configurations of the rigid body to the components of the infinitesimal displacement of a point on the rigid body and the infinitesimal rotation between the configurations. Early works distinguishing the Hessian of the potential
energy function from the stiffness matrix were first presented in the 1990s by Duffy, Griffis and Pigoski [12, 13], and were soon followed by a slew of other works relating the infinitesimal generalized displacements of a conservative system to the ensuing changes in the conservative forces and moments acting on the system [14–16, 18]. In contrast to these authors who employ screw theory to parameterize the rigid body motion of the system however, here, we derive the ensuing stiffness matrices when the rigid body motion is given by expression (2.7), with the rotation tensor \( \mathbf{Q} \) given as a function of the Euler angles.

The resulting stiffness matrix can be represented in terms of basis vectors fixed to the moving body or, alternatively, with respect to the Cartesian coordinates in the inertial frame. As the latter is a function of the former, we first derive the expression for the non-Cartesian stiffness matrix \( m \mathbf{K} \) in terms of the basis vectors \( \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3 \} \) (or, \( \{ \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3 \} \)). The Cartesian stiffness matrix \( c \mathbf{K} \) can then be determined from \( m \mathbf{K} \) in a straightforward manner. Note however, that the Cartesian stiffness matrix is often easier to determine experimentally although the stiffness matrix with components expressed in the corotational basis is often more intrinsically related to the stiffness properties of the rigid body of interest. In particular, the Hessian component of the stiffness matrix can be used to obtain useful information with regards to the stability of the system.

To elaborate, consider two configurations of a rigid body \( \kappa_t \) and \( \kappa_t' \). We distinguish quantities associated with \( \kappa_t' \) with a superscript \( ' \). The motion between these configurations can be defined with the help of (2.7):

\[
\mathbf{x}' = \mathbf{x} (t') = \mathbf{Q} (t') \mathbf{Q}^T (t) \mathbf{x} (t) + \mathbf{z}, \quad \mathbf{z} = \mathbf{d} (t') - \mathbf{Q}^T (t) \mathbf{d} (t). \tag{3.11}
\]

We shall assume that the two configurations differ by an infinitesimal rigid body motion. Thus,

\[
\Delta \mathbf{x} = \mathbf{x}' - \mathbf{x} = O (\epsilon), \quad \mathbf{I} + \Delta \mathbf{Q} = \mathbf{Q} (t') \mathbf{Q}^T (t), \quad \Delta \mathbf{Q} = O (\epsilon), \tag{3.12}
\]

where \( \epsilon \) is a small number and \( \mathbf{I} \) is the identity tensor. As the rotation \( \Delta \mathbf{Q} \) is infinitesimal, \( \Delta \mathbf{Q} \) is skew-symmetric [27]. If \( \nu^k \) denote the values of the Euler angles associated with \( \mathbf{Q} (t') \), then a lengthy, but straightforward calculation, shows that the axial vector \( \Delta \nu \) of \( \Delta \mathbf{Q} \) has the representation (cf. (2.33))

\[
\Delta \nu = (\nu'^k - \nu^k) \mathbf{g}_k + O (\epsilon^2), \tag{3.13}
\]

where \( \mathbf{g}_k \) are the Euler basis vectors associated with \( \mathbf{Q} (t) \). It follows that

\[
\nu'^k - \nu^k = \Delta \nu \cdot \mathbf{g}_k^* = G_i^k \Delta \nu \cdot \mathbf{E}_i, \tag{3.14}
\]

where we used (2.31) to express the dual Euler basis vectors in terms of the Cartesian basis vectors.
To first order in \( \epsilon \), the displacement vector \( \Delta \mathbf{x} \) has the representations

\[
\Delta \mathbf{x} = (X'_k - X_k) \mathbf{E}_k = (x'_r - x_r) \mathbf{e}_r + \Delta \mathbf{\nu} \times \mathbf{x},
\]

where \( x'_k = \mathbf{x}' \cdot \mathbf{e}'_k = \mathbf{x}' \cdot \mathbf{e}_k + \mathbf{x}' \cdot (\Delta \mathbf{Qe}_k) \). Consequently,

\[
X'_k - X_k = \Delta \mathbf{x} \cdot \mathbf{E}_k, \\
x'_r - x_r = \Delta \mathbf{x} \cdot (Q_{rk} \mathbf{E}_k) - (\Delta \mathbf{\nu} \times \mathbf{x}) \cdot \mathbf{e}_r.
\]

The presence of the term \( - (\Delta \mathbf{\nu} \times \mathbf{x}) \cdot \mathbf{e}_r \) in (3.16) reflects the difference in the vectors \( \mathbf{e}'_r \) and \( \mathbf{e}_r \):

\[
e'_r = \Delta \mathbf{Qe}_r = \Delta \mathbf{\nu} \times \mathbf{e}_r.
\]

We are now in a position to define the stiffness matrix relating the infinitesimal changes in forces and moments to the infinitesimal displacements and rotations. Based on the four functions discussed earlier, there are four possible matrices and we distinguish them by a left subscript. All of the stiffness matrices are obtained by performing a Taylor series expansion of the expressions for the appropriate conservative forces and moments. In addition, the developments for the stiffness matrices associated with the potential energies \( U'_3 \) and \( U'_4 \) are similar to those presented for \( U'_1 \) and \( U'_2 \), respectively. In the interests of brevity, they are omitted.

### 3.3.1 The Stiffness Matrices \( ^m \mathbf{K}, ^c \mathbf{K} \) and \( {\mathbf{K}} \) and \( \hat{\mathbf{K}} \)

The first stiffness matrix, which we denote by \( ^m \mathbf{K} \) in the moving frame and \( {\mathbf{K}} \) in the (fixed) Cartesian frame, relates the differences in the components of the force \( \mathbf{F} \) and moment \( \mathbf{M}_O \) in the configurations \( \mathbf{\kappa}' \) and \( \mathbf{\kappa} \) to the components of the infinitesimal displacement vectors \( \Delta \mathbf{x} \) and \( \Delta \mathbf{\nu} \). The matrices \( ^m \mathbf{K} \) and \( {\mathbf{K}} \) are defined by the identities

\[
^m \Delta \mathbf{F} = -^m \mathbf{K} \Delta \mathbf{x} + O(\epsilon^2), \\
{\mathbf{K}} \Delta \mathbf{F} = -{\mathbf{K}} \Delta \mathbf{x} + O(\epsilon^2), \tag{3.18}
\]

where

\[
^m \Delta \mathbf{F} = \begin{bmatrix}
(F' - F) \cdot \mathbf{e}_1 \\
(F' - F) \cdot \mathbf{e}_2 \\
(F' - F) \cdot \mathbf{e}_3 \\
(M'_O - M_O) \cdot \mathbf{g}_1 \\
(M'_O - M_O) \cdot \mathbf{g}_2 \\
(M'_O - M_O) \cdot \mathbf{g}_3
\end{bmatrix}, \\
^m \Delta \mathbf{x} = \begin{bmatrix}
\Delta \mathbf{x} \cdot \mathbf{e}_1 \\
\Delta \mathbf{x} \cdot \mathbf{e}_2 \\
\Delta \mathbf{x} \cdot \mathbf{e}_3 \\
\Delta \mathbf{\nu} \cdot \mathbf{g}_1 \\
\Delta \mathbf{\nu} \cdot \mathbf{g}_2 \\
\Delta \mathbf{\nu} \cdot \mathbf{g}_3
\end{bmatrix}, \\
^m \Delta \mathbf{s} = \begin{bmatrix}
(\Delta \mathbf{x} - \Delta \mathbf{\nu} \times \mathbf{x}) \cdot \mathbf{e}_1 \\
(\Delta \mathbf{x} - \Delta \mathbf{\nu} \times \mathbf{x}) \cdot \mathbf{e}_2 \\
(\Delta \mathbf{x} - \Delta \mathbf{\nu} \times \mathbf{x}) \cdot \mathbf{e}_3 \\
\Delta \mathbf{\nu} \cdot \mathbf{g}_1 \\
\Delta \mathbf{\nu} \cdot \mathbf{g}_2 \\
\Delta \mathbf{\nu} \cdot \mathbf{g}_3
\end{bmatrix}.
\]

\[
(3.19)
\]
and

\[
\begin{bmatrix}
(F' - F) \cdot E_1 \\
(F' - F) \cdot E_2 \\
(F' - F) \cdot E_3 \\
(M'_{O} - M_O) \cdot E_1 \\
(M'_{O} - M_O) \cdot E_2 \\
(M'_{O} - M_O) \cdot E_3
\end{bmatrix},
\begin{bmatrix}
\Delta \chi \cdot E_1 \\
\Delta \chi \cdot E_2 \\
\Delta \chi \cdot E_3 \\
\Delta \nu \cdot E_1 \\
\Delta \nu \cdot E_2 \\
\Delta \nu \cdot E_3
\end{bmatrix},
\begin{bmatrix}
(\Delta \chi - \Delta \nu \times \Delta \chi) \cdot E_1 \\
(\Delta \chi - \Delta \nu \times \Delta \chi) \cdot E_2 \\
(\Delta \chi - \Delta \nu \times \Delta \chi) \cdot E_3
\end{bmatrix}.
\]

(Note that terms with the left superscript \(m\) are defined with respect to the basis \(\{e_1, e_2, e_3, g_1, g_2, g_3\}\) while terms with the left superscript \(c\) are written in terms of the bases \(\{E_1, E_2, E_3, E_1, E_2, E_3\}\). Should we instead seek to express the generalized force as a function of \(\Delta s\), then equation (3.18) assumes the form

\[
\begin{align*}
m \Delta F &= -m \hat{K}(m \Delta s) + O(\epsilon^2), \\
c \Delta F &= -c \hat{K}(c \Delta s) + O(\epsilon^2).
\end{align*}
\]

The expressions (3.21) and (3.21) permits a simpler expression for the stiffness matrices and will allow us to compare our work with those of Ciblak and Lipkin [14] and others who use screw theory as opposed to (2.7).

To obtain a representation for \(m \hat{K}\), we perform Taylor series expansions of the expressions for \(F\) and \(M_O\) about the configuration \(\kappa_t\) (cf. (3.10)). After ignoring terms of order \(\epsilon^2\), we find that

\[
\begin{align*}
F' - F &= -\frac{\partial}{\partial x_k} \left( \frac{\partial U_1}{\partial x_i} e_i \right) (x'_k - x_k) - \frac{\partial}{\partial v^k} \left( \frac{\partial U_1}{\partial x_i} e_i \right) (v'^k - v^k), \\
M'_O - M_O &= -\frac{\partial}{\partial x_k} \left( \frac{\partial U_1}{\partial v^i} g^i \right) (x'_k - x_k) - \frac{\partial}{\partial v^k} \left( \frac{\partial U_1}{\partial v^i} g^i \right) (v'^k - v^k).
\end{align*}
\]

(3.22)

Employing the notation (2.26) introduced in Chapter 2, we write

\[
\begin{align*}
\hat{\gamma}^{k}_{ij} &= -\frac{\partial e_k}{\partial v^j} \cdot e_i = \frac{\partial e_i}{\partial v^j} \cdot e_k, \\
\hat{\gamma}^{k}_{ij} &= -\frac{\partial g^k}{\partial v^j} \cdot g_i = \frac{\partial g_i}{\partial v^j} \cdot g^k, \quad i, j, k = 1, 2, 3.
\end{align*}
\]

(3.23)
equation (3.22) can be written as

\[
F' - F = - \left( \frac{\partial^2 U_1}{\partial x_k \partial x_i} \right) e_i (x'_k - x_k) - \left( \frac{\partial^2 U_1}{\partial x_i \partial x_k} - \frac{\partial U_1}{\partial x_i} \partial e_i \partial x_k \right) \left( \nu^{k'} - \nu^k \right)
\]

\[
= - \left( \frac{\partial^2 U_1}{\partial x_k \partial x_i} \right) (x'_k - x_k) - \left( \frac{\partial^2 U_1}{\partial x_i \partial x_k} - \frac{\partial U_1}{\partial x_i} \xi_{ik}^{j} \right) \left( \nu^{k'} - \nu^k \right) e_i,
\]

\[
M'_O - M_O = - \left( \frac{\partial^2 U_1}{\partial x_k \partial \nu^i} \right) g^i (x'_k - x_k) - \frac{\partial}{\partial \nu^i} \left( \frac{\partial U_1}{\partial \nu^i} g^i + \frac{\partial U_1}{\partial \nu^i} \partial e_i \partial \nu^k \right) \left( \nu^{k'} - \nu^k \right)
\]

\[
= - \left( \frac{\partial^2 U_1}{\partial x_k \partial \nu^i} \right) (x'_k - x_k) - \left( \frac{\partial^2 U_1}{\partial \nu^i \partial x_k} - \frac{\partial U_1}{\partial \nu^i} \xi_{ik}^{j} \right) \left( \nu^{k'} - \nu^k \right) g^i.
\]

Taking the \(e_i\) and \(g^i\) components of the force and moment vectors respectively, we have the following representation for the stiffness matrix:

\[
\begin{bmatrix}
\hat{m} K_{ij} & \hat{m} C_{ij} & \hat{m} D_{ij}
\end{bmatrix}
\]

(3.25)

Here, \(\hat{m} H\) is the Hessian of the potential energy function \(U_1\):

\[
\hat{m} H = \begin{bmatrix}
\hat{m} H_1 & \hat{m} H_2 \\
\hat{m} H_2^T & \hat{m} H_3
\end{bmatrix},
\]

(3.26)

with

\[
\begin{align*}
\hat{m} H_1_{ij} &= \frac{\partial^2 U_1}{\partial x_i \partial x_j}, \\
\hat{m} H_2_{ij} &= \frac{\partial^2 U_1}{\partial x_i \partial \nu^j}, \\
\hat{m} H_3_{ij} &= \frac{\partial^2 U_1}{\partial \nu^i \partial x_j}.
\end{align*}
\]

(3.27)

The components of the \(3 \times 3\) matrices \(\hat{m} C\) and \(\hat{m} D\) are, respectively,

\[
\begin{align*}
\hat{m} C_{jk} &= -\xi_{jk}^{i} \frac{\partial U_1}{\partial x_i} \partial e_i \partial x_k, \\
\hat{m} D_{jk} &= -\xi_{jk}^{i} \frac{\partial U_1}{\partial \nu^i} \partial g^i \partial \nu_k \cdot g_j.
\end{align*}
\]

(3.28)

We emphasize that the partial derivatives and vectors in the expressions for the components of \(\hat{m} K\) are all evaluated using the values \(\bar{x}\) and \(\nu^k\) associated with the configuration \(\kappa_t\).

Notice that, if we seek to use expression (3.18) to define the stiffness matrix, then the expression for the stiffness matrix assumes the form

\[
\begin{bmatrix}
\hat{m} K_{ij} & \hat{m} Y_{ij} & \hat{m} Z_{ij}
\end{bmatrix}
\]

(3.29)
with
\[ m^1 Y = 1^1 H^1 S, \quad m^1 Z = (1^1 H^3)^T S, \] (3.30)

where the skew-symmetric matrix \( S \) with components
\[ S_{mn} = \bar{x} \cdot (e_n \times e_m), \] (3.31)

accounts for the difference between the unit vectors \( e'_r \) and \( e_r \) (cf. equation (3.17)).

To determine the Cartesian stiffness matrices \( \hat{\kappa} K \) and \( \kappa K \), we use the identities
\[ e_k = Q_{ki} E_i, \quad g^k = G^k_i E_i, \] (3.32)
to obtain
\[ \hat{\kappa} K = \left[ \begin{array}{cc} Q^T & 0 \\ 0 & G^T \end{array} \right] \left\{ \left[ \begin{array}{cc} 1^1 H^1 & 1^1 H^3 \\ (1^1 H^3)^T & 1^1 H^2 \end{array} \right] + \left[ \begin{array}{cc} 0 & m^1 C \\ 0 & m^1 D \end{array} \right] \right\} \left[ \begin{array}{cc} Q & 0 \\ 0 & G \end{array} \right] \] (3.33)
\[ \kappa K = \left[ \begin{array}{cc} Q^T & 0 \\ 0 & G^T \end{array} \right] \left\{ \left[ \begin{array}{cc} 1^1 H^1 & 1^1 H^3 \\ (1^1 H^3)^T & 1^1 H^2 \end{array} \right] + \left[ \begin{array}{cc} 0 & m^1 C \\ 0 & m^1 D \end{array} \right] + \left[ \begin{array}{cc} 0 & m^1 Y \\ 0 & m^1 Z \end{array} \right] \right\} \left[ \begin{array}{cc} Q & 0 \\ 0 & G \end{array} \right] \] (3.34)

where the matrices \( m^1 C, m^1 D, m^1 Y, \) and \( m^1 Z \) are given by (3.28) and (3.30), and
\[ \kappa C_{pq} = \frac{\partial Q_{ip}}{\partial \nu^k} \frac{\partial U^1}{\partial \nu^k} G^k_q = Q_{ipq} (m^1 C_{ik}) G^k_q, \quad \kappa C = Q^T (m^1 C) G, \]
\[ \kappa D_{pq} = \frac{\partial G^i_p}{\partial \nu^k} \frac{\partial U^1}{\partial \nu^k} G^k_q = G^i_p (m^1 D_{ik}) G^k_q, \quad \kappa D = G^T (m^1 D) G. \] (3.35)

The matrix \( \hat{\kappa} K \) agrees with the definition of the stiffness matrix in the work of Ciblak and Lipkin [14] and others where screw theory was used. Further, as mentioned in [14], at equilibrium, the matrix \( \hat{\kappa} K \) can be asymmetric even when \( \kappa K \) is symmetric, due to the vanishing of the gradient of the potential energy function.
3.3.2 The Stiffness Matrix \( \frac{\partial^2}{\partial \Delta x^2} \)K

A second Cartesian stiffness matrix can be defined relating the components of \( F \cdot E_k \) and \( M \cdot E_k \) to the vector \( \Delta x \):

\[
\Delta \bar{F} = -\frac{\partial}{\partial \Delta x} \frac{\partial^2}{\partial \Delta x^2} \Delta x + O \left( \epsilon^2 \right),
\]

where

\[
\Delta \bar{F} = \begin{bmatrix}
(F' - F) \cdot E_1 \\
(F' - F) \cdot E_2 \\
(F' - F) \cdot E_3 \\
(M' - M) \cdot E_1 \\
(M' - M) \cdot E_2 \\
(M' - M) \cdot E_3
\end{bmatrix}.
\]

The derivation of \( \frac{\partial^2}{\partial \Delta x^2} \)K closely follows the developments in the previous subsection and is omitted in the interests of brevity. In summary, we find that

\[
\frac{\partial^2}{\partial \Delta x^2} \Delta x = \begin{bmatrix}
0 & 0 & 0 \\
0 & G^T & 0 \\
0 & 0 & G
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Here, \( \frac{\partial^2}{\partial \Delta x^2} \Delta x \) is the Hessian of the potential energy function \( U_2 \):

\[
\frac{\partial^2}{\partial \Delta x^2} \Delta x = \begin{bmatrix}
\frac{\partial^2}{\partial X_i \partial X_j} U_2 \\
\frac{\partial^2}{\partial \nu^i \partial \nu^j} U_2 \\
\frac{\partial^2}{\partial \nu^i \partial X_j} U_2
\end{bmatrix}.
\]

With

\[
\frac{\partial^2}{\partial X_i \partial X_j} U_2 = \frac{\partial^2}{\partial \nu^i \partial \nu^j} U_2 = \frac{\partial^2}{\partial \nu^i \partial X_j} U_2,
\]

and

\[
\frac{\partial}{\partial \nu^j} \frac{\partial}{\partial \nu^i} U_2 = -\frac{\partial}{\partial \nu^j} \frac{\partial}{\partial \nu^i} U_2 = \frac{\partial}{\partial \nu^j} \frac{\partial}{\partial X_i} U_2.
\]

In the previous expression, \( \frac{\partial}{\partial \Delta x} \frac{\partial^2}{\partial \Delta x^2} \Delta x \) are the components of the \( 3 \times 3 \) matrix \( \frac{\partial^2}{\partial \Delta x^2} \Delta x \). It will be shown in Section 3.4 that the skew-symmetric part of \( \frac{\partial^2}{\partial \Delta x^2} \Delta x \) has an axial vector \( \frac{1}{2} \tilde{M} \cdot E_i \).

3.3.3 Remarks

As noted by several authors (e.g., [18]), it is important to distinguish the Cartesian stiffness matrix \( K \) from the stiffness matrix or Hessian \( h \) in analytical dynamics.
Indeed, it is transparent from (3.33) and (3.38), how $\xi K$ is a function of, and distinct from, $H$.

The matrices $\xi K$ and $\xi K$ both provide expressions for $F' - F$ in terms of $\Delta \vec{x}$ and $\Delta \nu$. Consequently, 18 of the 36 components of $\xi K$ and $\xi K$ are identical:

$$Q^T H^1 Q = 2H^1, \quad Q^T (H^3 + \frac{m}{1} C + \frac{m}{1} Y) G = 2H^3 G.$$  \hspace{1cm} (3.42)

This result will be used in Section 3.5.2 to validate our numerical computations of $\xi K$ and $\xi K$ for a specific mechanism.

Now suppose we were to consider a point $P$ which is not a material point of the body. In this case, $v_t \neq \dot{x} + \omega \times (x_p - \dot{x})$. Unless $P$ is a fixed point, it is not possible to establish identities of the form (3.7) featuring $F$ and the resultant moment relative to $P$, $M_P$. As a result, if we wish to establish an expression for a stiffness matrix relative to a point $P$ on the helical axis of motion, we would need to consider a fixed point $O$ which instantaneously coincides with the point $P$ of interest. The stiffness matrix would then be $\xi K$. If $P$ were to move, then we would need to relocate $O$, recompute $\vec{x} \cdot e_i$, and reevaluate $\xi K$.

The stiffness matrices in (3.33) and (3.38) relate the $E_i$ components of the infinitesimal displacements to the increments in the $E_i$ components of the conservative forces and moments. It is possible to define yet another Cartesian stiffness matrix where the $E_i$ components are replaced by the components with respect to $e_i(t)$. As an example,

$$\begin{pmatrix}
(F' - F) \cdot e_1(t) \\
(F' - F) \cdot e_2(t) \\
(F' - F) \cdot e_3(t) \\
(M' - M) \cdot e_1(t) \\
(M' - M) \cdot e_2(t) \\
(M' - M) \cdot e_3(t)
\end{pmatrix} = - \begin{pmatrix}
Q & 0 \\
0 & Q
\end{pmatrix} \xi K \begin{pmatrix}
Q^T & 0 \\
0 & Q^T
\end{pmatrix} \begin{pmatrix}
\Delta \vec{x} \cdot e_1(t) \\
\Delta \vec{x} \cdot e_2(t) \\
\Delta \vec{x} \cdot e_3(t) \\
\Delta \nu \cdot e_1(t) \\
\Delta \nu \cdot e_2(t) \\
\Delta \nu \cdot e_3(t)
\end{pmatrix}.$$  \hspace{1cm} (3.43)

From this equation, it is easy to infer that the Cartesian stiffness matrix in this case is a transformation of the stiffness matrix when the fixed basis is used. Similar expressions in terms of $\xi K$, $\xi K$, and $\xi K$ can be obtained in a straightforward manner.

### 3.4 The Asymmetric Parts of the Stiffness Matrices

As is the case with the situations discussed in [12, 14–16], the asymmetry of $\xi K$ and $\xi K$ arises because of the presence of a non-zero gradient of $U$ for the configuration $k_i$. A similar situation arises for $\xi K$ and $\xi K$. For $\xi K$ and $\xi K$, the non-zero gradient of $U_1$ then combines with the dependency of the basis vectors $e_k$ and $g^k$ on the Euler angles to yield asymmetric contributions to the $\xi K$, $\xi K$, and the Cartesian stiffness matrix. On the other hand, for $\xi K$ only the moment $M$ contributes to the asymmetry.
of this matrix. If the body is in equilibrium under the sole action of conservative forces and moments in the configuration $\kappa_t$, then the gradient of $U$ will be zero. In this case, the stiffness matrices $m^nK$, $m^iK$, and $m^zK$ will be symmetric, however the matrices $m^nK$ and $m^iK$ may still be asymmetric (due to the presence of non-zero $m^iY$ and $m^iZ$).

As previously alluded to, the asymmetry of $m^iK$, $m^iK$ and part of the asymmetry of $m^nK$ and $m^iK$ is due to the presence of the matrices $m^nC$, $m^nD$, $m^iC$, and $m^iD$. These matrices have several unusual features. In particular $m^iC$ is skew-symmetric, and the skew-symmetric parts of $m^iC$ and $m^iD$ can be related to the force $F$ and moment $M_O$, respectively. In a similar manner, for the stiffness matrix $m^iK$, the skew-symmetric part of $m^iD$ is related to the moment $M$. Our results for the matrix $m^iK$ and $m^iK$ in this respect are the analogues of Theorem 1 of Ciblak and Lipkin [14], Corollary 1 in Howard et al. [15], and Proposition 4.2 in Žefran and Kumar [16].

We start with the matrix $m^iC$. This matrix has a strong dependency on the change in the corotational basis vectors with respect to the Euler angles: $\frac{\partial e_i}{\partial \nu_k}$. To prove the skew-symmetry of $m^iC$, we first observe that the components of $m^iC$ can be used to form a tensor $\overset{1}{C}$:

$$\overset{1}{C} = \frac{\partial U_1}{\partial x_i} \frac{\partial e_i}{\partial \nu^k} \otimes g^k = \overset{i}{C}_{pq} E_p \otimes E_q.$$  (3.44)

We now invoke the pair of identities

$$\frac{\partial e_i}{\partial \nu^k} = g_k \times e_i, \quad \frac{\partial U_1}{\partial x_i} = -F \cdot e_i,$$  (3.45)

to obtain

$$\overset{1}{C} = \left( g_k \times \left( \frac{\partial U_1}{\partial x_i} e_i \right) \right) \otimes g^k = -\left( g_k \times F \right) \otimes g^k.$$  (3.46)

A direct calculation shows that, for any vector $a$,

$$-\left( (g_k \times F) \otimes g^k \right) a = F \times a.$$  (3.47)

Thus, we conclude that the matrix $m^iC$ is skew-symmetric and that

$$\overset{i}{C}_{32} = -\overset{i}{C}_{23} = F \cdot E_1, \quad \overset{i}{C}_{13} = -\overset{i}{C}_{31} = F \cdot E_2, \quad \overset{i}{C}_{21} = -\overset{i}{C}_{12} = F \cdot E_3.$$  (3.48)

This result can be considered as a generalization of Theorem 1 of Ciblak and Lipkin [14] to systems where the elastic element can also supply pure moments.

It is tempting to conclude that $m^iD$ will also be skew-symmetric, but this is not the case. The skew-symmetric part of this matrix is in direct correspondence with the components $M_O \cdot E_k$. To arrive at this result, we note that the components of $m^iD$ can
be used to form a tensor $iD$:

$$iD = \frac{\partial U_1}{\partial \nu^i} \frac{\partial g_i}{\partial \nu^k} \otimes g^k = iD_{pq} E_p \otimes E_q.$$  \hspace{1cm} (3.49)

However, as (cf. (2.26) and (3.10))

$$\tilde{\gamma}^i_{jk} g_i = \frac{\partial g_i}{\partial \nu^k}, \quad M_O \cdot g_i = -\frac{\partial U_1}{\partial \nu^i},$$  \hspace{1cm} (3.50)

we find that the expression for $iD$ simplifies to

$$iD = \left( M_O \cdot \frac{\partial g_i}{\partial \nu^k} \right) g^j \otimes g^k.$$  \hspace{1cm} (3.51)

Thus,

$$iD - iD^T = \left( M_O \cdot \left( \frac{\partial g_i}{\partial \nu^k} - \frac{\partial g_k}{\partial \nu^j} \right) \right) g^j \otimes g^k.$$  \hspace{1cm} (3.52)

We next appeal to the identity (2.36) and conclude that

$$iD - iD^T = \left( M_O \cdot (g_k \times g_j) \right) g^j \otimes g^k.$$  \hspace{1cm} (3.53)

To compute the axial vector of this tensor, we note that, for any vector $a = a^r g_r$,

$$\left( \left( M_O \cdot (g_k \times g_j) \right) g^j \otimes g^k \right) a = \left( M_O \cdot (a^k g_k \times g_j) \right) g^j = \left( M_O \times a \right) g^j = M_O \times a.$$  \hspace{1cm} (3.54)

We conclude that $M_O$ is the axial vector of $iD - iD^T$. Hence, from

$$iD_{32} - iD_{23} = M_O \cdot E_1, \quad iD_{13} - iD_{31} = M_O \cdot E_2, \quad iD_{21} - iD_{12} = M_O \cdot E_3,$$  \hspace{1cm} (3.55)

and (3.33), the skew-symmetric part of $i\dot{K}$ can be shown to have the representation

$$\frac{1}{2} (i\dot{K} - i\dot{K}^T) = \frac{1}{2} \begin{bmatrix} 0 & iC \\ -iC & (\dot{i}D - (iD)\dot{i}) \\ \end{bmatrix}.$$  \hspace{1cm} (3.56)

Thus, we surmise that the force $F$ and moment $M_O$ contribute equally to the skew-symmetric components of $i\dot{K}$.

For the stiffness matrix $\frac{1}{2}K$, the only matrix which contributes to its skew-symmetric
part is \( \frac{c}{2}D \) (cf. (3.38) and (3.41)). Paralleling the development of (3.55), we find that
\[
\frac{c}{2}D_{32} - \frac{c}{2}D_{23} = M \cdot E_1, \quad \frac{c}{2}D_{13} - \frac{c}{2}D_{31} = M \cdot E_2, \quad \frac{c}{2}D_{21} - \frac{c}{2}D_{12} = M \cdot E_3. \tag{3.57}
\]
We emphasize that in contrast to \( \frac{c}{2}K \) and \( K_O \), \( F \) does not contribute to the skew-symmetric part of \( \frac{c}{2}K \). In Section 3.5.2, examples of the identities (3.48), (3.55), and (3.57) will be shown.

Note that, as the components of \( \frac{m}{i}C \) and \( \frac{m}{i}D \) are given by
\[
\frac{m}{i}C_{pq} = e_p \left( \frac{\partial U_1}{\partial x_i} \frac{\partial e_i}{\partial \nu^k} \otimes g^k \right) g_q \neq - \frac{\partial U_1}{\partial x_i} \epsilon_{ipq},
\]
\[
\frac{m}{i}D_{pq} = g_p \left( \frac{\partial U_1}{\partial \nu^i} \frac{\partial g^i}{\partial \nu^k} \otimes g^k \right) g_q \neq - \frac{\partial U_1}{\partial \nu^i} \epsilon_{ipq},
\]
where \( \epsilon_{ipq} \) is the Levi-Civita symbol. It is thus erroneous to write
\[
\frac{m}{i}C_{32} = - \frac{m}{i}C_{23} = F \cdot e_1, \quad \frac{m}{i}C_{13} = - \frac{m}{i}C_{31} = F \cdot e_2, \quad \frac{m}{i}C_{21} = - \frac{m}{i}C_{12} = F \cdot e_3, \tag{3.58}
\]
or
\[
\frac{m}{i}D_{32} - \frac{m}{i}D_{23} = M_O \cdot g_1, \quad \frac{m}{i}D_{13} - \frac{m}{i}D_{31} = M_O \cdot g_2, \quad \frac{m}{i}D_{21} - \frac{m}{i}D_{12} = M_O \cdot g_3. \tag{3.59}
\]
That is,
\[
\frac{m}{i}C \neq \begin{bmatrix} 0 & -F \cdot e_3 & F \cdot e_2 \\ F \cdot e_3 & 0 & -F \cdot e_1 \\ -F \cdot e_2 & F \cdot e_1 & 0 \end{bmatrix}, \tag{3.60}
\]
and
\[
\frac{1}{2} \left( \frac{m}{i}D - \left( \frac{m}{i}D \right)^T \right) \neq \begin{bmatrix} 0 & -M_O \cdot g_3 & M_O \cdot g_2 \\ M_O \cdot g_3 & 0 & -M_O \cdot g_1 \\ -M_O \cdot g_2 & M_O \cdot g_1 & 0 \end{bmatrix}. \tag{3.61}
\]

3.5 Examples

In this section, we will present two examples highlighting the utility and application of the Cartesian stiffness matrix to a single rigid body. In Section 3.5.1, an example of a rigid body undergoing planar motion is illustrated. Following this, a more complicated example featuring the Stewart-Gough platform is featured in 3.5.2. The latter example features prominently in the developments of the Cartesian stiffness matrix given in [12, 14, 16].

33
3.5.1 The Planar Case

It is of interest to restrict attention to rigid bodies undergoing planar motions in the $E_1 - E_2$ plane. An example of such a system is shown in Figure 3.3. For the planar case, the sole angle of rotation is $\nu^1$. Additionally, the dual Euler basis is not needed and the axis of rotation is simply $E_3$. Further,

$$U = U_1 (\nu^1, x_1, x_2) = U_2 (\nu^1, X_1, X_2) = U_3 (\nu^1, x_{A_1}, x_{A_2}) = U_4 (\nu^1, X_{A_1}, X_{A_2}) .$$

(3.62)

As shall shortly become apparent that the Cartesian stiffness matrix $\underline{\underline{K}}$ will be symmetric, while the matrix $\underline{\underline{K}}$ can still retain an asymmetric component provided that the gradient of $U_1$ doesn’t vanish.

The expression for the stiffness matrix simplifies dramatically in the planar case. First, the stiffness matrix is now defined by the relations

$$\begin{bmatrix}
(F' - F) \cdot E_1 \\
(F' - F) \cdot E_2 \\
(M' - M_0) \cdot E_3
\end{bmatrix} = -\underline{\underline{K}} \begin{bmatrix}
(x' - \bar{x}) \cdot E_1 \\
(x' - \bar{x}) \cdot E_2 \\
\nu' - \nu^1
\end{bmatrix} .$$

(3.63)

Paralleling the developments in Section 3.3.1, we find that the Cartesian stiffness
matrix has the representation

$$\bar{\bar{K}} = Q^T \begin{bmatrix} \frac{\partial^2 U_1}{\partial x_1 \partial x_1} & \frac{\partial^2 U_1}{\partial x_1 \partial x_2} & \frac{\partial^2 U_1}{\partial x_1 \partial \nu^i} \\ \frac{\partial^2 U_1}{\partial x_2 \partial x_1} & \frac{\partial^2 U_1}{\partial x_2 \partial x_2} & \frac{\partial^2 U_1}{\partial x_2 \partial \nu^i} \\ \frac{\partial^2 U_1}{\partial \nu^i \partial x_1} & \frac{\partial^2 U_1}{\partial \nu^i \partial x_2} & \frac{\partial^2 U_1}{\partial \nu^i \partial \nu^i} \end{bmatrix} Q + \begin{bmatrix} 0 & 0 & F \cdot E_2 \\ 0 & 0 & -F \cdot E_3 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, the rotation matrix $Q$ is

$$Q = \begin{bmatrix} \cos(\nu^1) & \sin(\nu^1) & 0 \\ -\sin(\nu^1) & \cos(\nu^1) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

In writing (3.64), we choose to express the skew-symmetric components of $\bar{\bar{K}}$ in terms of the force components.

If the force $F$ and moment $M$ are used, then we need to repeat the calculation with the function $U_2$. In this case, we simply find a symmetric Cartesian stiffness matrix:

$$\begin{bmatrix} (F' - F) \cdot E_1 \\ (F' - F) \cdot E_2 \\ (M' - M) \cdot E_3 \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2 U_2}{\partial x_1 \partial x_1} & \frac{\partial^2 U_2}{\partial x_1 \partial x_2} & \frac{\partial^2 U_2}{\partial x_1 \partial \nu^i} \\ \frac{\partial^2 U_2}{\partial x_2 \partial x_1} & \frac{\partial^2 U_2}{\partial x_2 \partial x_2} & \frac{\partial^2 U_2}{\partial x_2 \partial \nu^i} \\ \frac{\partial^2 U_2}{\partial \nu^i \partial x_1} & \frac{\partial^2 U_2}{\partial \nu^i \partial x_2} & \frac{\partial^2 U_2}{\partial \nu^i \partial \nu^i} \end{bmatrix} \begin{bmatrix} (x' - x) \cdot E_1 \\ (x' - x) \cdot E_2 \\ \nu^i - \nu^i \end{bmatrix}.$$

The $3 \times 3$ matrix in this equation is a symmetric Cartesian stiffness matrix $\bar{\bar{K}}$. The symmetry of this matrix is independent of the value of the gradient of $U_2$.

### 3.5.2 The Stewart-Gough Platform

To illustrate the previous developments, we turn to the example of the Stewart-Gough platform. As shown in Figure 3.4, the realization of this system for the purposes of this paper is that of a rigid platform in the shape of an equilateral triangle which is attached by six springs to a rigid base. The springs define a conservative force field for the platform, and in the sequel we compute its potential energy and Cartesian stiffness matrices. This platform is a featured example in several other works on the Cartesian stiffness matrix [12,14,16]. For the purpose of comparison, we consider the same parameter values as these works.
Fig. 3.4: Schematic of a realization of a six degree-of-freedom system known as a Stewart-Gough platform. Here, a rigid body which undergoes planar motions is attached to three fixed points $O, P, Q$ by linear springs. This example is identical to one considered by Griffis and Duffy [12] and the related publications [14, 16].
Preliminary Kinematic Considerations

The springs have stiffnesses of $k_1, \ldots, k_6$ and unstretched lengths of $l_{01}, \ldots, l_{06}$, respectively. We follow [12, 14, 16] and specify the parameter values

$$
l_{01} = 11, \ l_{02} = 12, \ l_{03} = 13, \ l_{04} = 14, \ l_{05} = 15, \ l_{06} = 16,
$$

$$
k_1 = 10, \ k_2 = 20, \ k_3 = 30, \ k_4 = 40, \ k_5 = 50, \ k_6 = 60.
$$

The lengths are prescribed in centimeters and the stiffnesses are prescribed in N/cm. The position vectors of the two points $P$ and $Q$ are

$$
r_P = 7\mathbf{E}_1, \quad r_Q = 3.5\mathbf{E}_1 + 3.5\sqrt{3}\mathbf{E}_2.
$$

The configuration $\kappa_t$ of the platform is defined by the position vector $\bar{x}$ of the center of mass and the set of 3-2-1 Euler angles values:

$$
\bar{x} = 12.8457\mathbf{E}_1 + 4.3709\mathbf{E}_2 + 14.8457\mathbf{E}_3,
$$

$$
\nu^1 = -33.0826^\circ, \quad \nu^2 = -39.9638^\circ, \quad \nu^3 = 202.701^\circ.
$$

For the configuration $\kappa_t$ of interest, the position vectors of the points $R$, $S$ and $T$ are

$$
x_R = \bar{x} - 3.5\mathbf{e}_1 + \frac{3.5}{\sqrt{3}}\mathbf{e}_2, \quad x_S = \bar{x} - \frac{7}{\sqrt{3}}\mathbf{e}_2, \quad x_T = \bar{x} + 3.5\mathbf{e}_1 + \frac{3.5}{\sqrt{3}}\mathbf{e}_2.
$$

The reader is referred to Figure 3.4 for an illustration of some of these vectors. Representations for the corotational basis vectors in the configuration $\kappa_t$ are obtained using (2.28):

$$
e_1 = 0.642198\mathbf{E}_1 - 0.41832\mathbf{E}_2 + 0.642332\mathbf{E}_3,
$$

$$
e_2 = -0.29577\mathbf{E}_1 - 0.9083\mathbf{E}_2 - 0.295824\mathbf{E}_3,
$$

$$
e_3 = 0.707179\mathbf{E}_1 - 0.707035\mathbf{E}_3.
$$

With the help of (2.31), representations for the dual Euler basis vectors can be found:

$$
g^1 = -0.702167\mathbf{E}_1 + 0.457433\mathbf{E}_2 + \mathbf{E}_3,
$$

$$
g^2 = 0.545847\mathbf{E}_1 + 0.837885\mathbf{E}_2,
$$

$$
g^3 = 1.0932\mathbf{E}_1 - 0.712176\mathbf{E}_2.
$$

The potential energy function for the platform can be obtained by adding the
potential energies of each of the springs:

\[ V = \sum_{j=1}^{6} \frac{k_J}{2} (l_J - l_{0J})^2, \quad (3.73) \]

where \( l_1, \ldots, l_6 \) are the stretched lengths of the springs. The configuration \( \kappa \) is held in equilibrium by a force \( \mathbf{F} \) acting at the center of mass and a moment \( \mathbf{M} \) relative to the center of mass. These quantities are obtained using the potential energy function \( V \) and the representations (3.9):

\[ \mathbf{F} = -\sum_{k=1}^{3} \frac{\partial V_2}{\partial x_k} \mathbf{e}_k = -304.649 \mathbf{E}_1 - 59.3016 \mathbf{E}_2 - 505.968 \mathbf{E}_3, \]

\[ \mathbf{M} = -\sum_{k=1}^{3} \frac{\partial V_2}{\partial \nu_k} \mathbf{g}^k = -200.324 \mathbf{g}^1 + 545.558 \mathbf{g}^2 - 47.945 \mathbf{g}^3 \]
\[ = 386.039 \mathbf{E}_1 + 399.625 \mathbf{E}_2 - 200.324 \mathbf{E}_3. \quad (3.74) \]

The system (3.74) is equipollent to a force \( \mathbf{F} \) and a moment \( \mathbf{M}_O \) where

\[ \mathbf{F} = -\sum_{k=1}^{3} \frac{\partial V_1}{\partial x_k} \mathbf{e}_k = -495.82 \mathbf{e}_1 + 293.658 \mathbf{e}_2 + 142.333 \mathbf{e}_3 \]
\[ = -304.649 \mathbf{E}_1 - 59.3016 \mathbf{E}_2 - 505.968 \mathbf{E}_3, \]

\[ \mathbf{M}_O = -\sum_{k=1}^{3} \frac{\partial V_1}{\partial \nu_k} \mathbf{g}^k = 369.493 \mathbf{g}^1 + 1475.27 \mathbf{g}^2 - 1363.84 \mathbf{g}^3 \]
\[ = -945.122 \mathbf{E}_1 + 2376.42 \mathbf{E}_2 + 369.493 \mathbf{E}_3. \quad (3.75) \]
**Stiffness Matrices**

It is straightforward to compute the stiffness matrices associated with the potential energy $V$ for the configuration $\kappa_t$. With the help of (3.33) and (3.34), we find that

$$\hat{\kappa} = \begin{bmatrix}
80.0014 & 5.20958 & 75.5599 & 206.947 & -202.424 & -180.249 \\
5.20958 & 39.3191 & 5.20994 & -75.5072 & 5.269 & 212.017 \\
75.5599 & 5.20994 & 150.613 & 407.245 & -532.151 & -212.216 \\
206.947 & -75.5072 & 407.245 & 4779.91 & -1693.32 & -3895.24 \\
-180.249 & 212.017 & -212.216 & -3895.24 & -336.504 & 3264.02
\end{bmatrix},$$

$$+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 505.968 & -59.3016 \\
0 & 0 & 0 & -505.968 & 0 & 304.649 \\
0 & 0 & 0 & 59.3016 & -304.649 & 0 \\
0 & 0 & 0 & -2638.32 & -401.422 & 2376.42 \\
0 & 0 & 0 & -31.9282 & 1402.01 & 945.122 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (3.76)$$

$$\hat{\kappa} = \hat{\kappa} + \begin{bmatrix}
0 & 0 & 0 & -252.925 & -217.057 & 282.757 \\
0 & 0 & 0 & 560.947 & -10.4145 & -482.311 \\
0 & 0 & 0 & -580.968 & 812.987 & 263.339 \\
0 & 0 & 0 & -2900.98 & 2159.07 & 1874.49 \\
0 & 0 & 0 & 2404.2 & -3830.72 & -952.461 \\
0 & 0 & 0 & 4075.12 & -50.1405 & -3511.36
\end{bmatrix}. \quad (3.77)$$

It is interesting to note that the Hessian of $V_1$ has the following value:

$$\begin{bmatrix}
-340.763 & -76.4696 & 22.5743 & 3264.02 & -2408.16 & -264.238 \\
-164.594 & 116.32 & 118.039 & -2408.16 & 1774.58 & -1344.02 \\
253.051 & -210.66 & -173.972 & -264.238 & -1344.02 & 1668.71
\end{bmatrix}. \quad (3.78)$$

This Hessian has a spectrum

$$[\lambda_1, \ldots, \lambda_6] = [-711.826, 20.9161, 49.4755, 117.917, 2343.64, 5157.12]. \quad (3.79)$$
Due to the non-zero values of $\mathbf{F}$ and $\mathbf{M}_O$, the matrix $\hat{K}$ is asymmetric. We also observe that the components of the skew-symmetric matrix $\hat{C}$ and the matrix $\hat{D}$ which feature in (3.76) satisfy the identities (3.48) and (3.55). The value for $\hat{K}$ in (3.76) is identical to the expression for the stiffness matrix $\hat{K}_O$ recorded in equation (50) of Ciblak and Lipkin [14] although their methods are different to ours.

The stiffness matrix $\hat{K}$ is distinct from $\hat{K}$. With the help of (3.38), it is straightforward to compute that

$$\hat{K} = \begin{bmatrix}
80.0014 & 5.20958 & 75.5599 & -45.9777 & 86.4865 & 43.2066 \\
5.20958 & 39.3191 & 5.20994 & -20.5279 & -5.14554 & 34.355 \\
75.5599 & 5.20994 & 150.613 & -114.422 & -23.812 & 51.1232 \\
86.4865 & -5.14554 & -23.812 & -293.389 & -1004.59 & 57.4772 \\
43.2066 & 34.355 & 51.1232 & 242.607 & 57.4772 & -499.799
\end{bmatrix}$$

$$+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -148.617 & 385.404 & 399.625 & 0 \\
0 & 0 & 185.08 & -308.574 & -386.039 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  \tag{3.80}

Clearly, this matrix is asymmetric. We also observe that the components of the skew-symmetric part of the matrix $\hat{D}$ which feature in (3.80) satisfy the identities (3.57). In addition, the values of the matrices $\hat{K}$ and $\hat{K}$ in (3.76) and (3.80) satisfy the identities (3.42).

The Hessian of $V_2$ has the value

$$\hat{H} = \begin{bmatrix}
80.0014 & 5.20958 & 75.5599 & 43.2066 & 47.3689 & -37.9579 \\
75.5599 & 5.20994 & 150.613 & 51.1232 & -82.4086 & -30.6826 \\
43.2066 & 34.355 & 51.1232 & -499.799 & 180.586 & -189.268 \\
47.3689 & -15.5164 & -82.4086 & 180.586 & -1097.41 & 231.658 \\
\end{bmatrix}.$$  \tag{3.81}

with corresponding spectrum

$$[\mu_1, \ldots, \mu_6] = [-1231.46, -492.751, -128.017, 38.0753, 41.9597, 218.474].$$  \tag{3.82}

As the spectra of the Hessians of $V_1$ and $V_2$ are distinct (cf. (3.79) and (3.82)) they
cannot be related by a similarity transformation.

3.6 Closing Remarks

In this chapter, it was shown how various representations for Cartesian stiffness matrices $^cK$ would be obtained for a wide range of pairs of resultant forces and moments. The selection of the pair of forces and moments is not arbitrary: rather it is related by a work argument to the functional representation of the potential energy function (see (3.3) and (3.7)):

\[
(F, M) \leftrightarrow U_1 (\nu_1, \nu_2, \nu_3, x_1, x_2, x_3), \\
(F, M) \leftrightarrow U_2 (\nu_1, \nu_2, \nu_3, X_1, X_2, X_3), \\
(F_A, M_O) \leftrightarrow U_3 (\nu_1, \nu_2, \nu_3, x_{A1}, x_{A2}, x_{A3}), \\
(F_A, M_A) \leftrightarrow U_4 (\nu_1, \nu_2, \nu_3, X_{A1}, X_{A2}, X_{A3}). \tag{3.83}
\]

In Sections 3.3.1 and 3.3.2, we showed how to derive the Cartesian stiffness matrices

\[
\hat{m}_1^K = H \begin{bmatrix} 0 & m_1^C \\ 0 & m_1^D \end{bmatrix}, \quad m_1^K = \hat{m}_1^K \begin{bmatrix} 0 & m_1^Y \\ 0 & m_1^Z \end{bmatrix},
\]

\[
\hat{c}_1^K = Q^T \begin{bmatrix} 0 & 0 & G^T \end{bmatrix} H \begin{bmatrix} Q & 0 \\ 0 & 0 & G \end{bmatrix} + \begin{bmatrix} 0 & \hat{c}_1^C \\ 0 & \hat{c}_1^D \end{bmatrix}, \quad c_1^K = \hat{c}_1^K \begin{bmatrix} 0 & c_1^Y \\ 0 & c_1^Z \end{bmatrix},
\]

and

\[
\hat{c}_2^K = \begin{bmatrix} 1 & 0 \\ 0 & G^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \hat{c}_2^D \end{bmatrix},
\]

by performing Taylor series expansions of $U_1$ and $U_2$, and taking the dot product of the resulting tensor with the appropriate basis vectors. The relationship between the skew-symmetric components of the Cartesian stiffness matrices $\hat{c}_1^K, \hat{c}_1K$ and $\hat{c}_2K$ and the conservative forces and moments were elaborated upon in Section 3.4. The direct correspondence between the two is the reason it is often advantageous to employ the Cartesian representation when the forces and moments are known. Further, since the Euler basis vectors are difficult to measure physically, it may not be feasible to derive any physical meaning from the components $m_1^K_{ij}$ and $m_1^K_{ij}$ for $i = 1, \ldots, 6$, $j = 4, 5, 6$ featured in Section 3.3.1.

Finally, it is important to note that the resulting Cartesian stiffness matrix is a function of the parameterization used to quantify the motion. In this dissertation, the Euler angle representation was employed and, thus, the use of the dual Euler basis to calculate $^cK$ was an essential component of the formulation. Should a quaternion
or Euler-Rodrigues symmetric parameter representation of the rotation be used, then it is possible to extend the formulation presented in this chapter to that case. The formulation would use representations for the conservative moments that can be found in [43] and would result in different expressions for the components of the Cartesian stiffness matrix. However, the numerical value of the matrix for a given configuration should be independent of the representation. For instance, our results on the value of $\mathcal{C}_1\mathbf{K}$ given by (3.76) are identical to the value for the stiffness matrix in Griffis and Duffy [12], even though our representations are different.
As alluded to previously, the spinal column can be approximated as a system of rigid bodies, each connected by joints with stiffness and damping properties (Fig. 4.1). In this dissertation, we propose modeling the stiffness component of the joints using Cartesian stiffness matrices. To accomplish this, the developments detailed in Chapter 3 will need to be extended to multibody systems. First, six possible potential energy expressions for a multibody system are identified in Section 4.1. The advantages of using the different representations depending on the type of multibody system being analyzed are discussed and the Cartesian stiffness matrix associated with some of these potential energy functions are subsequently derived in Section 4.2.

4.1 Potential Energy Functions

Depending on the system of interest, several different representations of the potential energy function for a system of multiple rigid bodies is possible. In general, the potential energy function depends on position vectors of points on each body and the rotation tensors of each body. Thus, we need to consider several distinct
Fig. 4.1: The lumbar spine (left) can be approximated as a system of rigid bodies connected by joints with stiffnesses along all six degrees of freedom. The conservative forces and moments exerted by the joints can be equated with a 6 degree-of-freedom stiffness matrix (right).

representations for $U$:

$$
U = \hat{U}_1 \left( \mathbf{Q}_1, \ldots, \mathbf{Q}_N, \mathbf{x}_1^1, \ldots, \mathbf{x}_N^N \right) \\
= \hat{U}_2 \left( \mathbf{Q}_1, \ldots, \mathbf{Q}_N, \mathbf{x}_1^1, \ldots, \mathbf{x}_N^N \right) \\
= \hat{U}_3 \left( \mathbf{Q}_1, \ldots, \mathbf{Q}_N, \mathbf{x}_A^1, \ldots, \mathbf{x}_A^N \right) \\
= \hat{U}_4 \left( \mathbf{Q}_1, \ldots, \mathbf{Q}_N, \mathbf{x}_A^1, \ldots, \mathbf{x}_A^N \right) \\
= \hat{U}_5 \left( \mathbf{R}_1 (= \mathbf{Q}_1), \ldots, \mathbf{R}_N, \mathbf{y}_1^1 (= \mathbf{x}_1^{1}), \ldots, \mathbf{y}_N^N \right) \\
= \hat{U}_6 \left( \mathbf{R}_1 (= \mathbf{Q}_1), \ldots, \mathbf{R}_N, \mathbf{x}_A^1, \ldots, \mathbf{x}_A^N \right). \tag{4.1}
$$

These representations have the respective component forms,

$$
U = U_1 \left( \nu_1^1, \nu_1^2, \ldots, \nu_N^2, \nu_N^3, x_1^1, x_2^1, \ldots, x_N^N, x_3^N \right) \\
= U_2 \left( \nu_1^1, \nu_1^2, \ldots, \nu_N^2, \nu_N^3, x_1^1, x_2^1, \ldots, x_N^N, x_3^N \right) \\
= U_3 \left( \nu_1^1, \nu_1^2, \ldots, \nu_N^2, \nu_N^3, x_{A_1}^1, x_{A_2}^1, \ldots, x_{A_2}^N, x_{A_3}^N \right) \\
= U_4 \left( \nu_1^1, \nu_1^2, \ldots, \nu_N^2, \nu_N^3, x_{A_1}^1, x_{A_2}^1, \ldots, x_{A_2}^N, x_{A_3}^N \right) \\
= U_5 \left( \beta_1^1, \beta_1^2, \ldots, \beta_N^2, \beta_N^3, y_1^1, y_2^1, \ldots, y_N^N, y_3^N \right) \\
= U_6 \left( \beta_1^1, \beta_1^2, \ldots, \beta_N^2, \beta_N^3, x_{A_1}^1, x_{A_2}^1, \ldots, x_{A_2}^N, x_{A_3}^N \right). \tag{4.2}
$$
Fig. 4.2: The general case of two rigid bodies connected to each other and the ground by springs is shown in (a). The potential energy function can then be written as $U = \hat{U}_1$, $U = \hat{U}_2$, $U = \hat{U}_3$, $U = \hat{U}_4$ or $U = \hat{U}_6$ (cf. equation (4.1)). If the bodies are connected at point $A$ by a joint (b), then, choosing the point $A$ on each body to coincide with the joint allows us to write the potential energy as a simplified form of $U = \hat{U}_6$ (cf. Section 4.1.4). The simplest representation of the potential energy function is given by $U = \hat{U}_5$ which arises when the pair of rigid bodies are isolated from the surroundings (c).
In addition to the four distinct representations given by (3.2), we have introduced two more possible representations \( \hat{U}_5 \) and \( \hat{U}_6 \) unique to multibody systems (cf. Fig. 4.2 for examples of this). We will soon show how, in certain instances, either of these representations will permit certain simplifications with respect to deriving the conservative forces and moments. We have also made use of the identities \( R_i = Q_1 \), \( \nu_i^1 = \beta_i \) and \( y^1 = \bar{x}^1 \) in (4.1) in (4.2). Notationally, this will allow for a more tractable determination of the resulting conservative forces and moments.

4.1.1 The Case \( U = \hat{U}_2 \)

The expression \( U = \hat{U}_2 (\nu_1^1, \nu_2^1, \ldots, \nu_N^1, X_1^1, X_2^1, \ldots, X_N^N) \) for the potential function is a direct extension of the potential \( U_2 \) presented for the single rigid body case. To determine the conservative forces \( F_K \) and moments (taken relative to the center of mass of the individual bodies) \( M_K \) we once again follow [21] and write

\[
-\dot{U} = \sum_{K=1}^{N} F_K \cdot \dot{x}_K^K + M_K \cdot \omega_K. \tag{4.3}
\]

Expressing \( \dot{U} \) as,

\[
\dot{U} = \sum_{K=1}^{N} \left( \frac{\partial U_2}{\partial X_i^K} \dot{X}_i^K + \frac{\partial U_2}{\partial \nu_i^K} \dot{\nu}_i^K \right), \tag{4.4}
\]

it can be concluded that

\[
F_K = -\frac{\partial U_2}{\partial X_i^K} E_i, \quad \text{and} \quad M_K = -\frac{\partial U_2}{\partial \nu_i^K} g^{K,i}. \tag{4.5}
\]

We emphasize that the force \( F_K \) in this expression is assumed to act at the center of mass \( X^K \) and the moment \( M_K \) is taken relative to the center of mass of the \( K^{th} \) rigid body.

4.1.2 The Cases \( U = \hat{U}_1 \) and \( U = \hat{U}_4 \)

If we wish to consider moments relative to the points \( x_A^K \), or the origin \( O \), equations (3.5) and (3.6) of Section 3.2 can be used to show that

\[
\sum_{K=1}^{N} \left( F_K \cdot \dot{x}_K^K + M_K \cdot \omega_K \right) = \sum_{K=1}^{N} \left( F_K \cdot \dot{x}_A^K + M_{K,A} \cdot \omega_K \right) = \sum_{K=1}^{N} \left( F_K \cdot \left( \dot{x}_K^K - \omega_K \times \bar{x}_K^K \right) + M_{K,O} \cdot \omega_K \right). \tag{4.6}
\]
Notice that the corotational derivative of $\bar{x}^K$ in (4.6) is given in terms of basis vectors fixed to the respective bodies:

$$\dot{\bar{x}}^K - \omega_K \times \bar{x}^K = \dot{x}_i^K e_i^K.$$  

(4.7)

Thus, invoking (4.3), we can establish the identities

$$F_K = -\frac{\partial U_4}{\partial X^K_{\alpha_i}} e_i^K, \quad M_{K,A} = -\frac{\partial U_4}{\partial \nu^K_{i}} \bar{g}^{K,i},$$  

(4.8)

and

$$F_K = -\frac{\partial U_1}{\partial x^K_{i}} e_i^K, \quad M_{K,O} = -\frac{\partial U_1}{\partial \nu^K_{i}} \bar{g}^{K,i}.$$  

(4.9)

In contrast to (4.5), equation (4.9) illustrates, once again, how the natural representation for the conservative force associated with the rigid body of interest is with respect to the corotational basis of that body when moments about a fixed point $O$ are being considered. This is a direct consequence of equation (4.7).

### 4.1.3 Bodies Connected in Series

In certain instances, the rigid bodies in our multibody system of interest may be connected in series. An example of this is the vertebral column of the human spine (cf. Figure 4.3). For systems of this type, it is often advantageous to write the potential energy function as a function of the relative position and rotations of the $K^{th}$ rigid body with respect to the $(K-1)^{th}$ body. I.e., $U = \hat{U}_5 (R_1, \ldots, R_N, y^1, \ldots, y^N)$. Taking the time derivative of the potential and paralleling the developments in Section 4.1.2, we have,

$$F_K = -\left( \frac{\partial \hat{U}_5}{\partial Y^K_{i}} - \frac{\partial \hat{U}_5}{\partial Y^{K+1}_{i}} \right) E_i, \quad M_K = -\left( \frac{\partial \hat{U}_5}{\partial \beta^K_{i}} \bar{g}^{K,i} - \frac{\partial \hat{U}_5}{\partial \beta^{K+1}_{i}} \bar{g}^{K+1,i} \right).$$  

(4.10)

If there are only two rigid bodies in the system, and the potential energy function is independent of the surroundings (cf. Figure 4.2c), the potential energy simplifies dramatically to $U = U_5 (R, y)$. In this instance, (4.10) results in the expressions

$$F_2 = -F_1 = -\frac{\partial U_5}{\partial Y^1_{i}} E_i, \quad M_2 = -M_1 = -\frac{\partial U_5}{\partial \beta^1_{i}} \bar{g}^{2,i},$$  

(4.11)

for the conservative forces and moments acting on each of the rigid bodies respectively. These expressions are consistent with Newton’s third law.

In Section 4.3.1, we will show that the Cartesian stiffness matrix for the case $U = U_5$ can be expressed as a $6 \times 6$ matrix which relates the components of $F_2$ and $M_2$ to the increments in $\Delta \beta$ and $y \cdot E_i$, where $\Delta \beta$ is the axial vector of $\Delta R$. In addition,
we show in Section 4.3.2, and elaborate further upon in Chapter 5, the significant simplifications to the Cartesian stiffness matrix that ensue when the potential energy function \( U_5 \) can be written as a linear sum of the potential energies associated with each individual body (cf. equation (4.71)).

\subsection{4.1.4 Incorporating Constraints}

In certain situations, some of the rigid bodies may be connected by joints (e.g. as in Figure 4.2b). This situation can be accommodated by appropriately selecting the point \( A \) on each body to coincide with the joint:

\[ x_A = x_A^1 = x_A^2. \quad (4.12) \]

and, if needed, for instance in the case of a pin joint, constraining the angles \( \beta_1^1 \) and \( \beta_2^2. \)

To establish the stiffness matrix for this case, we consider \( U = \dot{U}_6 (R_1 = Q_1, R_2, x_A). \)

Simplifying (4.3) with the help of (4.6), we seek solutions of

\[ -\dot{U}_6 = F_1 \cdot \dot{x}_A^1 + F_2 \cdot \dot{x}_A^2 + M_{1,A} \cdot \omega_1 + M_{2,A} \cdot \omega_2, \quad (4.13) \]

\footnote{In the interest of brevity, we focus here on the two-body system. Similar results for the constrained multibody system can be inferred from the derivations presented in this section.}
subject to the constraints
\[
\dot{x}_A^1 = \dot{x}_A^2. \tag{4.14}
\]

Using a standard procedure, the solution is
\[
\begin{align*}
F_1 + F_2 &= -\frac{\partial U_6}{\partial x_{A,k}} E_k, \\
M_{1,A} &= -\frac{\partial U_6}{\partial \beta_1^k} \tilde{g}_1^{1,k} + \frac{\partial U_6}{\partial \beta_2^k} \tilde{g}_2^{2,k}, \\
M_{2,A} &= -\frac{\partial U_6}{\partial \beta_2^k} \tilde{g}_2^{2,k}. \tag{4.15}
\end{align*}
\]

The Cartesian stiffness matrix in this case will be a 9 x 9 matrix which will relate the Cartesian components \((F_1 + F_2) \cdot E_k, M_{1,A} \cdot E_k, \) and \(M_{2,A} \cdot E_k\) to the Cartesian components of \(x_A\), and the axial vectors of \(\Delta R_1 = \Delta Q_1\) and \(\Delta R_2\).

**Remarks**

It is important to note that the forces and moments derived from the functions \(U_1, U_2, \ldots, U_6\) are not always equal to the total force and moment acting on the individual rigid bodies in the system. For example, in the exposition presented in Section 4.1.4 for the constrained two-body case, the bodies connected by joints will experience reaction forces \(N_1\) and \(N_2\), in addition to the conservative forces \(F_1\) and \(F_2\). Depending on the type of joint employed, the bodies may experience reaction (or constraint) moments \(T_1\) and \(T_2\). Hence, we should remain cognizant of the fact that the net forces and moments acting on any of the rigid bodies in the system are prescribed by
\[
\begin{align*}
F_{tot,K} &= F_K + N_K, \\
M_{tot,K} &= M_K + T_K. \tag{4.16}
\end{align*}
\]

Needless to say, if the bodies are subject only to conservative forces, then \(N_K\) and \(T_K\) are zero.

**4.2 The Cartesian Stiffness Matrices for Multibody Systems**

As detailed in Section 3.3, a stiffness matrix can be defined that relates the infinitesimal changes in the components of a pair of forces and moments to the infinitesimal displacement and rotations undergone by a rigid body going from one configuration to the next. This idea can be extended to the case of a multibody system, with the resulting stiffness matrix connecting small changes in the (conservative) forces and moments acting on each body in the system to the incremental configurational changes.
To develop this further, consider again two configurations of a multibody system \( \kappa_t \) and \( \kappa_t' \). We distinguish quantities associated with \( \kappa_t' \) with a superscript '. The motion between these configurations can be defined with the help of (2.7):

\[
x^{K'} = x^K(t) = Q_K(t') Q_K^T(t) x^K(t) + z^K, \quad z^K = d^K(t') - Q_K^T(t) d^K(t).
\]  

(4.17)

We shall assume that the two configurations differ by an infinitesimal rigid body motion. Thus,

\[
\Delta \bar{x}^{K} = \bar{x}^{K'} - \bar{x}^{K} = O(\epsilon), \quad \mathbf{I} + \Delta Q_{K} = Q_{K}(t') Q_{K}^{T}(t), \quad \Delta Q_{K} = O(\epsilon),
\]

(4.18)

where \( \epsilon \) is a small number and \( \mathbf{I} \) is the identity tensor.

Paralleling the developments of Section 3.3, \( \Delta Q_{K} \) is infinitesimal and skew-symmetric, and, if \( \nu_{i}^{K} \) denotes the values of the Euler angles associated with \( Q_{K}(t') \), then

\[
\Delta \theta_{K} = (\nu_{i}^{K'} - \nu_{i}^{K}) \mathbf{g}_{i}^{K} + O(\epsilon^2)
\]

(4.19)

is the axial vector of \( \Delta Q_{K} \). Using (2.45) to express the dual Euler basis vectors in terms of the Cartesian basis vectors, we find,

\[
\nu_{i}^{K'} - \nu_{i}^{K} = \Delta \theta \cdot \mathbf{g}_{K,i}^{i} = G_{K,i}^{ij} \Delta \theta \cdot \mathbf{E}_{j}.
\]

(4.20)

To first-order in \( \epsilon \), the displacement vector \( \Delta \bar{x}^{K} \) has the representations

\[
\Delta \bar{x}^{K} = (X_{i}^{K'} - X_{i}^{K}) \mathbf{E}_{i} = (x_{r}^{K'} - x_{r}^{K}) \mathbf{e}_{r}^{K} + \Delta \theta_{K} \times \bar{x}^{K},
\]

(4.21)

where

\[
x_{r}^{K'} = \bar{x}^{K'} \cdot \mathbf{e}_{r}^{K'} = \bar{x}^{K'} \cdot \mathbf{e}_{r}^{K} + \bar{x}^{K'} \cdot (\Delta Q_{K} \mathbf{e}_{r}^{K}).
\]

(4.22)

Consequently,

\[
X_{r}^{K'} - X_{r}^{K} = \Delta \bar{x}^{K} \cdot \mathbf{E}_{r},
\]

\[
x_{r}^{K'} - x_{r}^{K} = \Delta \bar{x}^{K} \cdot (\mathbf{Q}_{r} \mathbf{E}_{i}) - (\Delta \theta_{K} \times \bar{x}^{K}) \cdot \mathbf{e}_{r}^{K},
\]

(4.23)

where, as before, the presence of the term \(- (\Delta \theta_{K} \times \bar{x}^{K}) \cdot \mathbf{e}_{r}^{K}\) in (4.23)\(_2\) reflects the difference in the vectors \( \mathbf{e}_{r}^{K'} \) and \( \mathbf{e}_{r}^{K} \):

\[
\mathbf{e}_{r}^{K'} = \Delta Q_{K} \mathbf{e}_{r}^{K} = \Delta \theta_{K} \times \mathbf{e}_{r}^{K}.
\]

(4.24)

We are now in a position to define the stiffness matrix relating the infinitesimal changes in forces and moments to the infinitesimal displacements and rotations. Based on the six functions discussed earlier, there are a number of possible matrices - distinguished by a left subscript - that are obtained by performing a Taylor series expansion of the expressions for the appropriate conservative forces and moments.
4.2.1 The Stiffness Matrices $^mK$, $^iK$, $^i\hat{K}$ and $^i\hat{\hat{K}}$ for Multibody Systems

The first stiffness matrix, which we denote by $^iK$ in the moving frame and $^i\hat{K}$ in the (fixed) Cartesian frame, relates the differences in the forces $F_1, \ldots, F_N$ and moments $M_{1,0}, \ldots, M_{N,O}$ in the configurations $\kappa_t$ and $\kappa_t$ to the infinitesimal displacement vectors $\Delta \hat{x}^1, \ldots, \Delta \hat{x}^N$ and $\Delta \theta^K, \ldots, \Delta \theta^N$. The matrices $^iK$, $^i\hat{K}, ^i\hat{\hat{K}}$ and $^i\hat{\hat{\hat{K}}}$ are defined by the identities

\[
^m \Delta F = -^iK^m \Delta x + O(\epsilon^2)
\]
\[
= -^i\hat{K}^m \Delta s + O(\epsilon^2),
\]
\[
c\Delta F = -^cK^c \Delta x + O(\epsilon^2)
\]
\[
= -^c\hat{K}^c \Delta s + O(\epsilon^2),
\]

where

\[
^m \Delta F = \begin{bmatrix}
^m \Delta F_1 \\
^m \Delta F_2 \\
\vdots \\
^m \Delta F_{N-1} \\
^m \Delta F_N
\end{bmatrix},
\]
\[
^m \Delta x = \begin{bmatrix}
^m \Delta x^1 \\
^m \Delta x^2 \\
\vdots \\
^m \Delta x^{N-1} \\
^m \Delta x^N
\end{bmatrix},
\]
\[
^m \Delta s = \begin{bmatrix}
^m \Delta s^1 \\
^m \Delta s^2 \\
\vdots \\
^m \Delta s^{N-1} \\
^m \Delta s^N
\end{bmatrix}.
\]

Here, each of the $^m \Delta F_K, ^m \Delta x^K$, and $^m \Delta s^K$ are defined by equation (3.19) but with the respective variables and basis vectors replaced with their related counterparts associated with the $K^{th}$ rigid body. A similar definition holds for $^c \Delta F, ^c \Delta x$, and $^c \Delta s$, with the $K^{th}$ components of $^c \Delta F_K, ^c \Delta x^K$, and $^c \Delta s^K$ given by (3.20) instead.

A representation for $^iK$ can be obtained using a Taylor series expansions of the expressions for $F_K$ and $M_{K,O}$ about the configuration $\kappa_t$ (cf. (4.9)). After ignoring terms of order $\epsilon^2$, we find that

\[
F_K' - F_K = - \sum_{L=1}^N \left( \frac{\partial}{\partial x_p^L} \left( \frac{\partial U_1}{\partial x_q^K} e^K_q \right) \delta x_p^L + \frac{\partial}{\partial \nu_p^L} \left( \frac{\partial U_1}{\partial x_q^K} e^K_q \right) \delta \nu_p^L \right),
\]
\[
M_{K,O}' - M_{K,O} = - \sum_{L=1}^N \left( \frac{\partial}{\partial x_p^L} \left( \frac{\partial U_1}{\partial \nu^K_q} \gamma^K_{q} \right) \delta x_p^K + \frac{\partial}{\partial \nu_p^L} \left( \frac{\partial U_1}{\partial \nu^K_q} \gamma^K_{q} \right) \delta \nu_p^K \right),
\]

where the abbreviations,

\[
\delta x_p^L = x_p^L' - x_p^L, \quad \delta \nu_p^L = \nu_p^L' - \nu_p^L,
\]

\[\text{(4.28)}\]
have been utilized in the interest of conciseness. Setting
\[ \tilde{\gamma}_{ij}^{K,n} = -\frac{\partial e_{n}^{K}}{\partial v_{j}^{K}} \cdot e_{i}^{K}, \]
\[ \tilde{\gamma}_{ij}^{K,n} - \frac{\partial g_{n}^{K}}{\partial v_{j}^{K}} \cdot g_{i}^{K}, \quad i, j, n = 1, 2, 3, \quad K = 1, \ldots, N, \quad (4.30) \]
and, noting that
\[ \frac{\partial e_{n}^{K}}{\partial v_{L}^{K}} = 0, \quad \frac{\partial g_{n}^{K}}{\partial v_{L}^{K}} = 0, \quad \text{for} \quad K \neq L, \quad (4.31) \]
\((4.28)\) can be written as
\[
F'_{K} - F_{K} = -\sum_{L=1}^{N} \left( \left( \frac{\partial^{2} U_{1}}{\partial x_{p}^{L} \partial x_{q}^{K}} \right) \delta x_{p}^{K} e_{q}^{K} + \left( \frac{\partial^{2} U_{1}}{\partial v_{p}^{L} \partial x_{q}^{K}} e_{q}^{K} + \frac{\partial U_{1}}{\partial x_{q}^{K}} \frac{\partial e_{q}^{K}}{\partial v_{L}^{K}} \right) \delta v_{L}^{p} \right)
\]
\[ = -\sum_{L=1}^{N} \left( \left( \frac{\partial^{2} U_{1}}{\partial x_{q}^{L} \partial x_{q}^{K}} \right) \delta x_{p}^{L} g_{q}^{K} + \left( \frac{\partial^{2} U_{1}}{\partial v_{p}^{L} \partial v_{q}^{K}} g_{q}^{K} + \frac{\partial U_{1}}{\partial v_{q}^{K}} \frac{\partial g_{q}^{K}}{\partial v_{L}^{p}} \right) \delta v_{L}^{p} \right) e_{q}^{K}, \]
\[ M'_{K,O} - M_{K,O} = \sum_{L=1}^{N} \left( \left( \frac{\partial^{2} U_{1}}{\partial x_{p}^{L} \partial v_{q}^{K}} \right) \delta x_{p}^{L} g_{q}^{K} + \left( \frac{\partial^{2} U_{1}}{\partial v_{p}^{L} \partial v_{q}^{K}} g_{q}^{K} + \frac{\partial U_{1}}{\partial v_{q}^{K}} \frac{\partial g_{q}^{K}}{\partial v_{L}^{p}} \right) \delta v_{L}^{p} \right) g_{q}^{K}. \quad (4.32) \]
Taking the \(e_{i}^{K}\) and \(g_{i}^{K}\) components of the force and moment vectors in \((4.32)\) respectively, we have the following representation for the stiffness matrix:
\[ m^{K} = \mathbb{I} + m CD. \quad (4.33) \]
Here, \(\mathbb{I}\) is the Hessian of the potential energy function \(U_{1}\):
\[
\mathbb{I} = \begin{bmatrix}
\mathbb{I}_{1,1} & \cdots & \mathbb{I}_{1,N} \\
\vdots & \ddots & \vdots \\
\mathbb{I}_{N,1} & \cdots & \mathbb{I}_{N,N}
\end{bmatrix}, \quad \mathbb{I}_{I,J} = \begin{bmatrix}
\mathbb{I}_{I,J} & \mathbb{I}_{I,J} \\
\mathbb{I}_{I,J} & \mathbb{I}_{I,J}\n\end{bmatrix}, \quad (I, J = 1, \ldots, N), \quad (4.34)
with components,

$$1H_{1,J}^1 = \begin{bmatrix} \frac{\partial^2 U_i}{\partial x_1^i \partial x_1^j} & \ldots & \frac{\partial^2 U_i}{\partial x_1^i \partial x_N^j} \\ \frac{\partial^2 U_i}{\partial x_2^i \partial x_1^j} & \ldots & \frac{\partial^2 U_i}{\partial x_2^i \partial x_N^j} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U_i}{\partial x_N^i \partial x_1^j} & \ldots & \frac{\partial^2 U_i}{\partial x_N^i \partial x_N^j} \end{bmatrix}, \quad 1H_{1,J}^2 = \begin{bmatrix} \frac{\partial^2 U_i}{\partial \nu_1^i \partial \nu_1^j} & \ldots & \frac{\partial^2 U_i}{\partial \nu_1^i \partial \nu_N^j} \\ \frac{\partial^2 U_i}{\partial \nu_2^i \partial \nu_1^j} & \ldots & \frac{\partial^2 U_i}{\partial \nu_2^i \partial \nu_N^j} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U_i}{\partial \nu_N^i \partial \nu_1^j} & \ldots & \frac{\partial^2 U_i}{\partial \nu_N^i \partial \nu_N^j} \end{bmatrix}, \quad (4.35)$$

and the matrix $^{m}_1CD$ is defined as

$$^{m}_1CD = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial^2 U_i}{\partial x_1^i \partial x_1^j} & \ldots & \frac{\partial^2 U_i}{\partial x_1^i \partial x_N^j} \\ \frac{\partial^2 U_i}{\partial x_2^i \partial \nu_1^j} & \frac{\partial^2 U_i}{\partial x_2^i \partial \nu_2^j} & \ldots & \frac{\partial^2 U_i}{\partial x_2^i \partial \nu_N^j} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial^2 U_i}{\partial x_N^i \partial \nu_1^j} & \frac{\partial^2 U_i}{\partial x_N^i \partial \nu_2^j} & \ldots & \frac{\partial^2 U_i}{\partial x_N^i \partial \nu_N^j} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\partial^2 U_i}{\partial \nu_1^i \partial \nu_1^j} & \ldots & \frac{\partial^2 U_i}{\partial \nu_1^i \partial \nu_N^j} \\ \frac{\partial^2 U_i}{\partial \nu_2^i \partial \nu_1^j} & \frac{\partial^2 U_i}{\partial \nu_2^i \partial \nu_2^j} & \ldots & \frac{\partial^2 U_i}{\partial \nu_2^i \partial \nu_N^j} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial^2 U_i}{\partial \nu_N^i \partial \nu_1^j} & \frac{\partial^2 U_i}{\partial \nu_N^i \partial \nu_2^j} & \ldots & \frac{\partial^2 U_i}{\partial \nu_N^i \partial \nu_N^j} \end{bmatrix}, \quad (4.36)$$

with elements of the $3 \times 3$ matrices $^{m}_1C_K$ and $^{m}_1D_K$ given by

$$^{m}_1C_{K,ij} = -\gamma_{ij}^{K,n} \frac{\partial U_i}{\partial x_n^K} = \frac{\partial \mathbf{e}_n^K}{\partial \nu_i^K} \cdot \mathbf{e}_i^K,$$

$$^{m}_1D_{K,ij} = -\gamma_{ij}^{K,n} \frac{\partial U_i}{\partial \nu_n^K} = \frac{\partial \mathbf{g}_n^K}{\partial \nu_i^K} \cdot \mathbf{g}_i^K. \quad (4.37)$$

An alternative stiffness matrix $^{m}_1K$ can be defined from expression (4.25). In this case, the stiffness matrix assumes the form

$$^{m}_1K = ^m\bar{K} + ^mYZ, \quad (4.38)$$

where,

$$^{m}_1YZ = \begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} ^mY_{1,1} \\ ^mZ_{1,1} \\ \vdots \\ ^mY_{N,1} \\ ^mZ_{N,1} \end{bmatrix}, \quad (4.39)$$
Here,
\[ mY_{J,K} = H_{J,K}^1 S_K, \quad mZ_{J,K} = H_{J,K}^2 S_K. \] (4.40)

The skew-symmetric matrices \( S_K \) with components
\[ S_{K, mn} = x^K \cdot (e^K_n \times e^K_m) \] (4.41)
takes into account the difference between the unit vectors \( e^K_r \) and \( e^K_r' \) (cf. (4.24)).

To determine the Cartesian stiffness matrices \( \hat{c}_K \) and \( \check{c}_K \), we use the pair of identities
\[ e^K_p = Q_{K, pq} E_q, \quad g^K_{p q} = G^K_{p q} E_q, \] (4.42)
to obtain
\[ \hat{c}_K = (\hat{Q} G)^T \hat{m} \hat{K} (\hat{Q} G), \]
\[ \check{c}_K = (\hat{Q} G)^T \check{m} \hat{K} (\hat{Q} G), \]
\[ \hat{c}_C_K = Q_{K, (1)^m} C_K G_K, \]
\[ \check{c}_C_K,pq = \frac{\partial Q_{K, 1 p}}{\partial \nu_K^j} \frac{\partial U_1}{\partial x_K^i} G_{K, q}^j = Q_{K, 1 p} (\hat{m} C_{K, ij}) G_{K, q}^j, \]
\[ \hat{c}_D_K = G_{K, (1)^m} D_K G_K, \]
\[ \check{c}_D_K,pq = \frac{\partial G_{K, 1 p}}{\partial \nu_K^j} \frac{\partial U_1}{\partial \nu_K^i} G_{K, q}^j = G_{K, 1 p} (\hat{m} D_{K, ij}) G_{K, q}^j. \]

The matrices \( \hat{m} \) \( C_K \) and \( \check{m} \) \( D_K \), are defined by,

and the matrix \( \hat{Q} G \) is
\[ \hat{Q} G = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_N \\ 0 & 0 & G_N \end{bmatrix}. \] (4.45)

Note that we have made use of the identity (4.31) in writing (4.36), (4.37), and (4.45).
4.2.2 The Stiffness Matrices $\mathbf{\bar{K}}$ and $\mathbf{\hat{K}}$ for Multibody Systems

A second Cartesian stiffness matrix can be defined relating the components of $\mathbf{F}_K \cdot \mathbf{E}_i$ and $\mathbf{M} \cdot \mathbf{E}_i$ to the vector $c \Delta \mathbf{x}$:

$$
c \Delta \mathbf{F} = -2K \ c \Delta \mathbf{x} + O \ (c^2), \quad (4.46)
$$

where

$$
c \Delta \mathbf{F} = \begin{bmatrix}
c \Delta \mathbf{F}_1 \\
c \Delta \mathbf{F}_2 \\
\vdots \\
c \Delta \mathbf{F}_{N-1} \\
c \Delta \mathbf{F}_N
\end{bmatrix}, \quad c \Delta \mathbf{F}_K = \begin{bmatrix}
(F'_K - F_K) \cdot \mathbf{E}_1 \\
(F'_K - F_K) \cdot \mathbf{E}_2 \\
(F'_K - F_K) \cdot \mathbf{E}_3 \\
(M'_K - M_K) \cdot \mathbf{E}_1 \\
(M'_K - M_K) \cdot \mathbf{E}_2 \\
(M'_K - M_K) \cdot \mathbf{E}_3
\end{bmatrix},
$$

and each of the $m \Delta \mathbf{x}^K$ are defined by equation (3.20) but with the respective variables replaced by their related counterparts associated with the $K$th rigid body.

The derivation of $\mathbf{\bar{K}}$ closely parallels the derivation in the previous subsection. In the interest of brevity, we merely summarize the results:

$$
\mathbf{\bar{K}} = (\hat{\mathbf{G}})^T_2 \mathbf{H} (\hat{\mathbf{G}}) + \mathbf{\hat{D}}, \quad (4.48)
$$

where $\mathbf{H}$ is the Hessian of $U_2$:

$$
\mathbf{H} = \begin{bmatrix}
2H_{1,1} & \ldots & 2H_{1,N} \\
\vdots & \ddots & \vdots \\
2H_{N,1} & \ldots & 2H_{N,N}
\end{bmatrix}, \quad 2H_{I,J} = \begin{bmatrix}
2H_{I,J}^1 & 2H_{I,J}^2 \\
2H_{I,J}^1 & 2H_{I,J}^2
\end{bmatrix}, \quad (I, J = 1, \ldots, N),
$$

with components

$$
2H_{I,J}^1 = \begin{bmatrix}
\frac{\partial^2 U_2}{\partial X_1^I \partial X_1^J} & \frac{\partial^2 U_2}{\partial X_1^I \partial X_2^J} \\
\frac{\partial^2 U_2}{\partial X_2^I \partial X_1^J} & \frac{\partial^2 U_2}{\partial X_2^I \partial X_2^J}
\end{bmatrix}, \quad 2H_{I,J}^2 = \begin{bmatrix}
\frac{\partial^2 U_2}{\partial \nu_1^I \partial \nu_1^J} & \frac{\partial^2 U_2}{\partial \nu_1^I \partial \nu_2^J} \\
\frac{\partial^2 U_2}{\partial \nu_2^I \partial \nu_1^J} & \frac{\partial^2 U_2}{\partial \nu_2^I \partial \nu_2^J}
\end{bmatrix}.
$$

As the potential is written in terms of the vectors $(X^1, \ldots, X^N)$, we do not have to take into account the change in the basis vectors (cf. (4.24)). Hence, it is unnecessary to define the second displacement $c \Delta \mathbf{s}$.
\[ 2H^3_{I,J} = \begin{bmatrix} \frac{\partial^2 U_2}{\partial X_1 \partial \nu_j} & \cdots & \frac{\partial^2 U_2}{\partial X_1 \partial \nu_I} \\ \frac{\partial^2 U_2}{\partial X_2 \partial \nu_j} & & \frac{\partial^2 U_2}{\partial X_2 \partial \nu_I} \\ \vdots & & \vdots \\ \frac{\partial^2 U_2}{\partial X_N \partial \nu_j} & \cdots & \frac{\partial^2 U_2}{\partial X_N \partial \nu_I} \end{bmatrix}, \quad 2H^4_{I,J} = \begin{bmatrix} \frac{\partial^2 U_2}{\partial X_1^2} & \cdots & \frac{\partial^2 U_2}{\partial X_N^2} \\ \frac{\partial^2 U_2}{\partial X_1 \partial \nu_1} & & \frac{\partial^2 U_2}{\partial X_1 \partial \nu_N} \\ \vdots & & \vdots \\ \frac{\partial^2 U_2}{\partial X_N \partial \nu_1} & \cdots & \frac{\partial^2 U_2}{\partial X_N \partial \nu_N} \end{bmatrix}. \]

The matrix \(\xi 0D\) is defined as
\[
\xi 0D = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \xi 2D_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},
\tag{4.50}
\]
with elements
\[ \xi D_{K,pq} = \frac{\partial G_K^i}{\partial v_K^j} \frac{\partial U_2}{\partial v_K^i} G_K^j_{K,q}, \]
and the matrix \(\hat{G}\) is
\[
\hat{G} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & G_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & G_N \end{bmatrix},
\tag{4.51}
\]

The block diagonal nature of the matrix \(\xi 0D\) is a direct result of the identities (4.31).

4.2.3 The Stiffness Matrix \(\xi K\) for Multibody Systems

The derivation of the stiffness matrix \(\xi K\) is significantly more involved than that of either \(\eta K\) or \(\zeta K\). To proceed, recall from (4.10) that the conservative forces and moments associated with the \(K^{th}\) body are prescribed by
\[
F_K = -\left( \frac{\partial U_5}{\partial Y_i^K} - \frac{\partial U_5}{\partial Y_i^{K+1}} \right) E_i, \quad M_K = -\left( \frac{\partial U_5}{\partial \beta_i^K} g_i^{K,i} - \frac{\partial U_5}{\partial \beta_i^{K+1}} g_i^{K+1,i} \right).
\]
Performing a Taylor series expansions of the expressions for the $F_K$ and $M_K$ about the configuration $\kappa_t$ (cf. (4.10)), and ignoring terms of order $\epsilon^2$, we find that

$$F'_K - F_K = - \sum_{L=1}^{N} \left( \frac{\partial}{\partial Y_p^L} \left( \frac{\partial U_5}{\partial Y^K_q} - \frac{\partial U_5}{\partial Y^K_{q+1}} \right) \delta Y^K_p + \frac{\partial}{\partial \beta^p_L} \left( \frac{\partial U_5}{\partial Y^K_q} - \frac{\partial U_5}{\partial Y^K_{q+1}} \right) \delta \beta^p_L \right) E_q,$$

(4.52)

and

$$M'_K - M_K = - \sum_{L=1}^{N} \frac{\partial}{\partial Y_p^L} \left( \frac{\partial U_5}{\partial \beta^p_L} \tilde{g}^{K,q} - \frac{\partial U_5}{\partial \beta^p_{K+1}} \tilde{g}^{K+1,q} \right) \delta Y^K_p$$

$$- \sum_{L=1}^{N} \frac{\partial}{\partial \beta^p_L} \left( \frac{\partial U_5}{\partial \beta^p_L} \tilde{g}^{K,q} - \frac{\partial U_5}{\partial \beta^p_{K+1}} \tilde{g}^{K+1,q} \right) \delta \beta^p_L,$$

(4.53)

where,

$$\delta Y^K_p = Y^K_p' - Y^K_p, \quad \delta \beta^p_L = \beta^p_L' - \beta^p_L.$$  

(4.54)

Rearranging, using $\tilde{G}^{i}_{K,j} = \tilde{g}^{K,i} \cdot E_j$ (cf. (2.46)), and noting that,

$$\delta \beta^p_L = \beta^p_L' - \beta^p_L = \Delta \beta_L \cdot \tilde{g}^{L,p} = \tilde{G}^{p}_{L,q} \Delta \beta_L \cdot E_q,$$

(4.55)

where $\Delta \beta_L$ is the axial vector of $\Delta R_L$, we can write (4.52) as

$$F'_K - F_K = - \sum_{L=1}^{N} \left( \frac{\partial^2 U_5}{\partial Y_p^L \partial Y^K_q} - \frac{\partial^2 U_5}{\partial Y_p^L \partial Y^K_{q+1}} \right) \Delta Y^K \cdot E_q$$

$$- \sum_{L=1}^{N} \left( \frac{\partial^2 U_5}{\partial \beta^p_L \partial Y^K_q} - \frac{\partial^2 U_5}{\partial \beta^p_L \partial Y^K_{q+1}} \right) \tilde{G}^{p}_{L,s} \Delta \beta_L \cdot E_s,$$

(4.56)

and (4.53) as

$$M'_K - M_K = - \sum_{L=1}^{N} \left( \frac{\partial^2 U_5}{\partial Y_p^L \partial \beta^q_K} \tilde{g}^{K,q} - \frac{\partial^2 U_5}{\partial Y_p^L \partial \beta^q_{K+1}} \tilde{g}^{K+1,q} \right) \Delta Y^K \cdot E_p$$

$$- \sum_{L=1}^{N} \left( \frac{\partial^2 U_5}{\partial \beta^p_L \partial \beta^q_K} \tilde{g}^{K,q} - \frac{\partial^2 U_5}{\partial \beta^p_L \partial \beta^q_{K+1}} \tilde{g}^{K+1,q} \right) \tilde{G}^{p}_{L,s} \Delta \beta_L \cdot E_s,$$

(4.57)
For this specific potential energy representation, we specify \( \hat{\xi} K \) by the identity
\[
\hat{\xi} \Delta \bar{F} = -\hat{\xi} K \hat{\xi} \Delta \bar{y} + O(\epsilon^2),
\]
where \( \hat{\xi} \Delta \bar{F} \) is defined by equation (4.47) in Section 4.2.2, and
\[
\hat{\xi} \Delta \bar{y} = \begin{bmatrix}
\hat{\xi} \Delta y^1 \\
\hat{\xi} \Delta y^2 \\
\vdots \\
\hat{\xi} \Delta y^{N-1} \\
\hat{\xi} \Delta y^N
\end{bmatrix}, \quad \hat{\xi} \Delta \bar{y}^K = \begin{bmatrix}
\Delta \bar{y}^K \cdot \mathbf{E}_1 \\
\Delta \bar{y}^K \cdot \mathbf{E}_2 \\
\Delta \bar{y}^K \cdot \mathbf{E}_3 \\
\Delta \beta_K \cdot \mathbf{E}_1 \\
\Delta \beta_K \cdot \mathbf{E}_2 \\
\Delta \beta_K \cdot \mathbf{E}_3
\end{bmatrix}.
\] (4.58)

The resulting representation for the stiffness matrix is then obtained by taking the \( E_n \) components of the force and moment vectors in (4.56) and (4.57),
\[
\hat{\xi} \hat{\xi} K = \left( (I\hat{G})^T \hat{\xi} H - (I\hat{G}^T)^T \hat{\xi} H^* \right) (I\hat{G}) + \hat{\xi} \xi D.
\] (4.59)

Here, \( \xi H \) is the Hessian of the potential energy function \( U_5 \):
\[
\xi H = \begin{bmatrix}
5H_{1,1} & \cdots & 5H_{1,N} \\
\vdots & \ddots & \vdots \\
5H_{N,1} & \cdots & 5H_{N,N}
\end{bmatrix}, \quad 5H_{I,J} = \begin{bmatrix}
5H_{I,J}^1 \\
\vdots \\
5H_{I,J}^N
\end{bmatrix}, \quad (I, J = 1, \ldots, N),
\] (4.60)

with components
\[
5H_{I,J}^1 = \begin{bmatrix}
\frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_N^J} \\
\frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_N^J} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U_5}{\partial \beta_N^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_N^I \partial \beta_N^J}
\end{bmatrix}, \quad 5H_{I,J}^2 = \begin{bmatrix}
\frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_N^J} \\
\frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_2^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_N^J} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U_5}{\partial \beta_N^I \partial \beta_N^J}
\end{bmatrix},
\] (4.61)

\[
5H_{I,J}^3 = \begin{bmatrix}
\frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_N^J} \\
\frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_N^J} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U_5}{\partial \beta_N^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_N^I \partial \beta_N^J}
\end{bmatrix}, \quad 5H_{I,J}^4 = \begin{bmatrix}
\frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_1^I \partial \beta_N^J} \\
\frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_2^I \partial \beta_N^J} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U_5}{\partial \beta_N^I \partial \beta_1^J} & \cdots & \frac{\partial^2 U_5}{\partial \beta_N^I \partial \beta_N^J}
\end{bmatrix}.
\]
while $5H^*$ is the reduced Hessian of the potential energy function $U^5$:

$$
5H^* = \begin{bmatrix}
5H_{2,1} & \cdots & 5H_{2,N} \\
\vdots & \ddots & \vdots \\
5H_{N,1} & \cdots & 5H_{N,N} \\
0 & \cdots & 0
\end{bmatrix}. 
$$

(4.62)

Unlike the sparse expressions for the matrices $CD$ and $0D$ associated with $\xi K$ and $\xi^2 K$ respectively, the matrix $50D$ of $\xi K$ is determined by the much denser expression:

$$
\xi 0D = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}, 
$$

(4.63)

with

$$
\xi D_{KL,pq} = \left( \frac{\partial \tilde{G}_{K,p}^i}{\partial \beta_L^i} \frac{\partial U^5}{\partial \beta_K^i} - \frac{\partial \tilde{G}_{K+1,p}^i}{\partial \beta_L^i} \frac{\partial U^5}{\partial \beta_K^{i+1}} \right) \tilde{G}_{L,q}^i. 
$$

(4.64)

Expression (4.64) is a consequence of (2.47). Finally, the matrices $\tilde{G}$ and $\tilde{G}^*$ are given by

$$
\tilde{G} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \tilde{G}_1 & \cdots & 0 \\
0 & \cdots & 0 \\
0 & 0 & \cdots & \tilde{G}_N
\end{bmatrix}, 
\tilde{G}^* = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \tilde{G}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \tilde{G}_N
\end{bmatrix}. 
$$

(4.65)
4.3 Simplifications to $\tilde{\mathbf{K}}$

The main disadvantage to writing the potential energy as a function of the relative displacements and rotations of the rigid bodies is the more complicated expression for the Cartesian stiffness matrix. In contrast to (4.31), the relative Euler basis $\{\tilde{g}_1^i, \tilde{g}_2^i, \tilde{g}_3^i\}$ and its associated dual, $\{\tilde{g}^{K,1}_i, \tilde{g}^{K,2}_i, \tilde{g}^{K,3}_i\}$ are now functions of all of the first $3K$ relative Euler angles $(\beta_{1}^{1}, \beta_{1}^{2}, \ldots, \beta_{K}^{1}, \beta_{1}^{2}, \ldots, \beta_{K}^{3})$:

$$\frac{\partial \mathcal{G}_{K,p}^{i}}{\partial \beta_{j}^{l}} \neq 0 \quad K \geq L. \quad (4.66)$$

In addition, the dependence of the force $F_K$ and moment $M_K$ on the partial derivative of the potential energy with respect to $\beta_{K+1}$ necessitates the introduction of the matrix $\mathbf{H}^*$. However, in certain instances writing the potential energy function as

$$U_5(\beta_{1}^{1}, \beta_{1}^{2}, \ldots, \beta_{N}^{1}, \beta_{2}^{1}, \ldots, \beta_{N}^{3}, Y_{1}^{1}, Y_{2}^{1}, \ldots, Y_{N}^{2}, Y_{3}^{N}) \quad (4.67)$$

can be particularly advantageous. We describe two of these here.

4.3.1 A Pair of Rigid Bodies

When the multibody system is composed of a pair of rigid bodies and the potential energy function of the system is independent of the surroundings (cf. Fig. 4.4),

$$U = \check{U}_5(\mathbf{R}_2, \mathbf{y}_2), \quad (4.68)$$

the conservative forces and moments, given by (4.10), reduce to (4.11):

$$\mathbf{F}_2 = -\mathbf{F}_1 = -\frac{\partial U_5}{\partial Y_i} \mathbf{E}_i,$$

$$\mathbf{M}_2 = -\mathbf{M}_1 = -\frac{\partial U_5}{\partial \beta_i} \mathbf{g}_i.$$ 

In the equation above, the variables $\beta_i$ and $Y_i$ in the expression above parameterize the relative rotation between the 2 bodies, and $\mathbf{g}_i$ is the dual Euler basis associated with this relative rotation.\(^3\)

Using $K = L = 2$, in (4.56) and (4.57), we see that the load-displacement relationship of the system can be parameterized by a single $6 \times 6$ Cartesian stiffness matrix,

$$^c \mathbf{\Delta} \bar{\mathbf{F}}_2 = -^c \mathbf{\Delta} \bar{\mathbf{F}}_1 = -^c \kappa K ^c \mathbf{\Delta} \mathbf{y} + O(\epsilon^2),$$

\(^3\) Henceforth, we will suppress the super/subscripts 2 in writing the associated relative position and rotation parameters unless it is not obvious.
Fig. 4.4: An example of a rigid body system composed of two rigid bodies with potential energy independent of its surroundings. The functional spinal unit shown consists of the sacrum $S$, the fifth lumbar vertebra $L_5$ and the intervertebral disc $I$. The basis vectors $\{e_1^1, e_1^2, e_1^3\}$ and $\{e_2^1, e_2^2, e_2^3\}$ are attached to the body $S$ and $L_5$ respectively. The conservative forces ($F_1$ and $F_2$) and moments ($M_1$ and $M_2$) exerted by the intervertebral joint on the two vertebral units is given by equation (4.11).

where $\xi \mathbf{K}$ takes the form

$$
\xi \mathbf{K} = (\tilde{\mathbf{G}})^T \begin{bmatrix}
\xi \mathbf{H}^1 & \xi \mathbf{H}^3 \\
(\xi \mathbf{H}^3)^T & \xi \mathbf{H}^2
\end{bmatrix} (\tilde{\mathbf{G}}) + 
\begin{bmatrix}
0 & 0 \\
0 & \xi \mathbf{D}
\end{bmatrix}.
$$

(4.69)

Here, the underbraced term is the Hessian of the potential energy function $U_5$ with

$$
\xi \mathbf{H}^1_{ij} = \frac{\partial^2 U_5}{\partial Y_i \partial Y_j}, 
\xi \mathbf{H}^2_{ij} = \frac{\partial^2 U_5}{\partial \beta^i \partial \beta^j}, 
\xi \mathbf{H}^3_{ij} = \frac{\partial^2 U_5}{\partial \beta^i \partial Y_j},
$$

(4.70)

and the components of $\xi \mathbf{D}$ are given by

$$
\xi \mathbf{D}_{pq} = \frac{\partial \tilde{G}_p^q}{\partial \beta^i} \frac{\partial U_5}{\partial \beta^j} \tilde{G}_q^i.
$$

Note that when one of the bodies is fixed, the system simplifies even further and any of the other potential energy representations may also be suitable. The developments featured in Chapter 3 can be used as well with minimal modification.
4.3.2 The Case $U = \hat{U}_5^\# = \hat{U}_5^1(R_1, y^1) + \ldots + \hat{U}_5^N(R_N, y^N)$

The representation of the potential energy function $U_5$ is particularly advantageous if $U_5$ can be written as a linear sum of the potential energies associated with each individual body:

$$\hat{U}_5(R_1, \ldots, R_N, y^1, \ldots, y^N) = \hat{U}_5^1(R_1, y^1) + \ldots + \hat{U}_5^N(R_N, y^N).$$

(4.71)

Then, the conservative forces and moments acting on the $K^{th}$ rigid body are

$$\mathbf{F}_K = -\left(\frac{\partial U_5^\#}{\partial Y_k^i} - \frac{\partial U_5^\#}{\partial Y_{K+1}^i}\right) \mathbf{E}_i,$$

$$\mathbf{M}_K = -\left(\frac{\partial U_5^\#}{\partial \beta_k} - \frac{\partial U_5^\#}{\partial \beta_{K+1}}\right) \mathbf{g} - \left(\frac{\partial U_5^\#}{\partial \beta_k} - \frac{\partial U_5^\#}{\partial \beta_{K+1}}\right) \mathbf{g}^T.$$

(4.72)

Paralleling the developments used to derive $\hat{H}$ in Section 4.2.3, it is straightforward to show that

$$\hat{\epsilon}^K = \left((\hat{\mathbf{g}})T\hat{\epsilon}^K - (\hat{\mathbf{g}}^*)^T(\hat{\epsilon}^K)^*\right) (\hat{\mathbf{g}}) + \hat{\epsilon}^D.$$

(4.73)

Here, $\hat{\epsilon}^K$ is the Hessian of the potential energy function $U_5^\#:

$$\hat{\epsilon}^K = \begin{bmatrix}
\hat{\epsilon}_{1,1}^K & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \hat{\epsilon}_{N,N}^K
\end{bmatrix},$$

(4.74)

with components:

$$\left(\hat{\epsilon}^K\right)_{K,K}^1 = \begin{bmatrix}
\frac{\partial^2 U_5^\#}{\partial Y_1^i \partial Y_1^j} & \ldots & \frac{\partial^2 U_5^\#}{\partial Y_1^i \partial Y_N^j} \\
\frac{\partial^2 U_5^\#}{\partial Y_2^i \partial Y_1^j} & \ldots & \frac{\partial^2 U_5^\#}{\partial Y_2^i \partial Y_N^j} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U_5^\#}{\partial Y_N^i \partial Y_1^j} & \ldots & \frac{\partial^2 U_5^\#}{\partial Y_N^i \partial Y_N^j}
\end{bmatrix},$$

$$\left(\hat{\epsilon}^K\right)_{K,K}^2 = \begin{bmatrix}
\frac{\partial^2 U_5^\#}{\partial \beta_1 \partial \beta_1} & \ldots & \frac{\partial^2 U_5^\#}{\partial \beta_1 \partial \beta_N} \\
\frac{\partial^2 U_5^\#}{\partial \beta_2 \partial \beta_1} & \ldots & \frac{\partial^2 U_5^\#}{\partial \beta_2 \partial \beta_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U_5^\#}{\partial \beta_N \partial \beta_1} & \ldots & \frac{\partial^2 U_5^\#}{\partial \beta_N \partial \beta_N}
\end{bmatrix}.$$

(4.75)

Here, $\hat{\mathbf{g}}$ and $\hat{\mathbf{g}}^*$ are defined in (4.65) while $(\hat{\epsilon}^K)^* \hat{\epsilon}^D$, and their respective constituents can be inferred from equations (4.62), (4.63), and (4.64) of Section 4.2.3.
Note that, while the Hessian term given by (4.74) simplifies into a bidiagonal block diagonal form, the matrix $\xi_0D$ does not (cf. equation 4.63).

In Chapter 5, we present an example that utilizes the Cartesian stiffness matrix derived here to model the intervertebral joint. There, we also go into further detail with regards to how it is related to the stiffness matrices measured experimentally, as well as present an example featuring a potential energy function that’s a linear sum of quadratic functions of the relative orientations.

### 4.3.3 The Stiffness Matrix $[^6cK]$ for the Two-Body Constrained System

The $9 \times 9$ Cartesian stiffness matrix $[^6cK]$ relating the components of $F_1 + F_2$ and $M_{1,A}$ and $M_{2,A}$ to the vector $[^cA] \Delta x_A$ is defined by the identity

$$[^cA] \Delta \tilde{F}M = -[^6cK] [^cA] \Delta x_A + O(\epsilon^2),$$

where

$$[^cA] \Delta \tilde{F}M = \begin{bmatrix}
(F_1 + F_2)' - (F_1 + F_2) \cdot E_1 \\
(F_1 + F_2)' - (F_1 + F_2) \cdot E_2 \\
(F_1 + F_2)' - (F_1 + F_2) \cdot E_3 \\
(M_{1,A} - M_{1,A}) \cdot E_1 \\
(M_{1,A} - M_{1,A}) \cdot E_2 \\
(M_{1,A} - M_{1,A}) \cdot E_3 \\
(M_{2,A} - M_{2,A}) \cdot E_1 \\
(M_{2,A} - M_{2,A}) \cdot E_2 \\
(M_{2,A} - M_{2,A}) \cdot E_3 
\end{bmatrix},
[^cA] \Delta x_A = \begin{bmatrix}
\Delta x_A \cdot E_1 \\
\Delta x_A \cdot E_2 \\
\Delta x_A \cdot E_3 \\
\Delta \beta_1 \cdot E_1 \\
\Delta \beta_1 \cdot E_2 \\
\Delta \beta_1 \cdot E_3 \\
\Delta \beta_2 \cdot E_1 \\
\Delta \beta_2 \cdot E_2 \\
\Delta \beta_2 \cdot E_3 
\end{bmatrix}. \tag{4.76}$$

A first-order Taylor expansion for the conservative forces and moments of the two-body system of Section 4.1.4 (cf. equation (4.15)) leads to

$$(F_1 + F_2)' - (F_1 + F_2) = -\left(\frac{\partial^2 U_6}{\partial X_{A,k} \partial X_{A,i}}\right) E_i (\delta X_{A,k}) - \left(\frac{\partial^2 U_6}{\partial \beta^k L \partial X_{A,i}}\right) E_i (\delta \beta^k L),$$

$$M'_{A,1} - M_{A,1} = -\left(\frac{\partial^2 U_6}{\partial X_{A,k} \beta^k_1}\right) \tilde{g}^i_1(\delta X_{A,k}) + \left(\frac{\partial^2 U_6}{\partial \beta^k L \partial \beta^k_1}\tilde{g}^i_1 + \frac{\partial U_6}{\partial \beta^k_1} \frac{\partial \tilde{g}^i_1}{\partial \beta^k L}\right) (\delta \beta^k L),$$

$$+ \left(\frac{\partial^2 U_6}{\partial X_{A,k} \beta^k_2}\right) \tilde{g}^i_2(\delta X_{A,k}) + \left(\frac{\partial^2 U_6}{\partial \beta^k L \partial \beta^k_2}\tilde{g}^i_2 + \frac{\partial U_6}{\partial \beta^k_2} \frac{\partial \tilde{g}^i_2}{\partial \beta^k L}\right) (\delta \beta^k L),$$

$$M'_{A,2} - M_{A,2} = -\left(\frac{\partial^2 U_6}{\partial X_{A,k} \beta^k_2}\right) \tilde{g}^i_2(\delta X_{A,k}) + \left(\frac{\partial^2 U_6}{\partial \beta^k L \partial \beta^k_2}\tilde{g}^i_2 + \frac{\partial U_6}{\partial \beta^k_2} \frac{\partial \tilde{g}^i_2}{\partial \beta^k L}\right) (\delta \beta^k L). \tag{4.77}$$
As \( \frac{\partial \tilde{g}_i}{\partial y^2} = 0 \), the associated Cartesian stiffness matrix can be written as
\[
\epsilon K = \left( (\tilde{I}G)^T \epsilon H - (\tilde{I}G^*)^T \epsilon H^* \right) (\tilde{I}G) + \epsilon \tilde{0} D,
\]
where \( \epsilon \tilde{0} D \), \( \tilde{I}G \), and \( \tilde{I}G^* \) can be inferred from (4.63) and (4.65) respectively. Likewise, a relatively straightforward modification of (4.60), (4.61), and (4.62) can be used to obtain the expressions for \( \epsilon H \) and \( \epsilon H^* \).

4.4 Relating the Asymmetry in the Stiffness Matrices to the Conservative Forces and Moments

4.4.1 The Asymmetric Parts of the Cartesian Stiffness Matrices \( \epsilon K \), \( \epsilon \hat{K} \) and \( \epsilon \bar{K} \)

Paralleling the developments of Section 3.4, the matrices \( \epsilon \tilde{CD} \), \( \epsilon \tilde{0} D \), and \( \epsilon \tilde{0} D \) can be related to the force and moment vectors acting on the rigid body. We begin with the matrix \( \epsilon \tilde{CD} \). Using the components of \( \epsilon \tilde{C}_K \) to form the tensor \( \epsilon \tilde{C}_K \), we have
\[
\epsilon \tilde{C}_K = \frac{\partial U_1}{\partial x^K} \frac{\partial e^K_q}{\partial v^L_q} \otimes g^{L,p} = \epsilon C_{ik} E_i \otimes E_k.
\]
Extending the identities given in (3.45) to multibody systems,
\[
\frac{\partial e^K_p}{\partial v^L_q} = g^L_q \times e^K_p \delta^K_L, \quad \frac{\partial U_1}{\partial x^K} = -F_K \cdot e^K_p,
\]
we can write
\[
\epsilon \tilde{C} = \left( g^L_p \times \left( \frac{\partial U_1}{\partial x^K} e^K_q \right) \right) \otimes g^{L,p} = - \left( g^L_p \times F_K \right) \otimes g^{L,p}.
\]
Echoing the derivation in Section 3.4, it is straightforward to show that \( F_K \) is the axial vector of the skew-symmetric tensor \( \epsilon \tilde{C}_K \), and that
\[
\epsilon \tilde{C}_{K,pq} = -\epsilon_{pqr} F_K \cdot E_r.
\]
Further, it can be shown that $M_{K,O}$ is the axial vector of the skew-symmetric tensor $D_K - D^T_K$. Hence, the skew-symmetric part of $c_{1CD}$ can be written as

$$(c_{1CD} - (c_{1CD})^T) = \left(\hat{c}_K - \hat{c}_K^T\right)$$

$$= \begin{bmatrix}
0 & F_1^x & 0 & 0 \\
-F_1^x & M_{1,O}^x & 0 & 0 \\
0 & \cdots & 0 & 0 \\
0 & 0 & 0 & F_N^x \\
-F_N^x & M_{N,O}^x & \cdots & 0
\end{bmatrix}, \quad (4.83)$$

where, for any vector $A = A_1E_1 + A_2E_2 + A_3E_3$,

$$A^* = \begin{bmatrix}
0 & -A_3 & A_2 \\
A_3 & 0 & -A_1 \\
-A^2 & A_1 & 0
\end{bmatrix}. \quad (4.84)$$

Similarly, the skew-symmetric part of $\hat{c}_K$ has the representation

$$(\hat{c}_{0D} - (\hat{c}_{0D})^T) = \left(\hat{c}_K - \hat{c}_K^T\right) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & M_1^x & 0 & 0 \\
0 & \cdots & 0 & 0 \\
0 & 0 & 0 & M_N^x
\end{bmatrix}. \quad (4.85)$$

### 4.4.2 Skew-Symmetry Associated With $\hat{c}_K$

The relationship between the moments $M_K$ and the skew-symmetric components of the stiffness matrix $\hat{c}_K$ is not apparent. Essentially, as $g_{i,i+1}^K = \delta_i^j$ if and only if $K = L$,

$$M_K \cdot \hat{g}_i^K = -\left(\frac{\partial U_5}{\partial \beta_K^i} \hat{g}_i^K - \frac{\partial U_5}{\partial \beta^i_{K+1}} \hat{g}_i^{K+1}\right) \cdot \hat{g}_i^K \neq -\frac{\partial U_5}{\partial \beta_K^i}$$

$$M_K \cdot \hat{g}_i^{K+1} = -\left(\frac{\partial U_5}{\partial \beta_K^i} \hat{g}_i^K - \frac{\partial U_5}{\partial \beta^i_{K+1}} \hat{g}_i^{K+1}\right) \cdot \hat{g}_i^{K+1} \neq -\frac{\partial U_5}{\partial \beta_K^i}$$

$$M_K \cdot \hat{g}_i^L = -\left(\frac{\partial U_5}{\partial \beta_K^i} \hat{g}_i^K - \frac{\partial U_5}{\partial \beta^i_{K+1}} \hat{g}_i^{K+1}\right) \cdot \hat{g}_i^L \neq 0 \quad K \neq L,$$
Further, denoting the tensor associated with the \( KL \) components of the matrix \( \xi D \) by (cf. (4.64)),

\[
5D_{KL} = \left( \frac{\partial U_5}{\partial \beta^i_K} \frac{\partial g^{K_i}}{\partial \beta^j_L} - \frac{\partial U_5}{\partial \beta^i_{K+1}} \frac{\partial g^{K+1,i}}{\partial \beta^j_L} \right) \otimes \tilde{g}^{L,j},
\]

the skew-symmetric component of \( 5D_{KL} \) can be written as

\[
5D_{KL} - 5D^T_{KL} = \left( \frac{\partial U_5}{\partial \beta^i_K} \frac{\partial g^{K_i}}{\partial \beta^j_L} - \frac{\partial U_5}{\partial \beta^i_{K+1}} \frac{\partial g^{K+1,i}}{\partial \beta^j_L} \right) \otimes \tilde{g}^{L,j}
- \tilde{g}^{L,j} \otimes \left( \frac{\partial U_5}{\partial \beta^i_K} \frac{\partial g^{K_i}}{\partial \beta^j_L} - \frac{\partial U_5}{\partial \beta^i_{K+1}} \frac{\partial g^{K+1,i}}{\partial \beta^j_L} \right)
= M_K \cdot \left( \frac{\partial g^{K_i}}{\partial \beta^j_L} \otimes g^{L,j} - \frac{\partial g^{K+1,i}}{\partial \beta^j_L} \otimes g^{K,j} \right)
- \frac{\partial U_5}{\partial \beta^i_K} \frac{\partial g^{K+1,i}}{\partial \beta^j_L} \otimes \tilde{g}^{L,j} + \frac{\partial U_5}{\partial \beta^i_{K+1}} \frac{\partial g^{K,i}}{\partial \beta^j_L} \otimes \tilde{g}^{K+1,j},
\]

and, thus, we cannot conclude that \( M_K \) is the axial vector of \( 5D_{KL} - 5D^T_{KL} \) as we were able to surmise for \( jD \) and \( 2D \). This is true even if we decide to split the components of \( 5D \) into two parts: one related to the partial derivative of the potential with respect to \( \beta^i_K \), and the other containing terms due to \( \frac{\partial \beta^i_K}{\partial \beta^j_{K+1}} \). However, if the potential energy of the system can be (and is) expressed as a linear sum of the potential energies associated with each of the joints connecting the bodies of the system (cf. equation (4.71) of Section 4.3.2), then some of the ensuing skew-symmetric parts of the components of the matrix \( 5D \) can be related to the components of the moments in the fixed frame. This is discussed at length in Chapter 5.

An additional interesting scenario is when the system is composed of only two rigid bodies and the potential can be written as \( U = U_5(R, y) \) (cf. Section 4.3.1). Then, the tensor \( 5D \) associated with the matrix \( 5D \) can be written as

\[
5D = \frac{\partial g^{j}}{\partial \beta^j} \frac{\partial g^{i}}{\partial \beta^i} \otimes \tilde{g}^{i} = D_{pq} E_p \otimes E_q.
\]

As we have only a single set of Euler basis vectors (and associated dual), the expression
for $5D$ simplifies to

$$5D = \left( M_2 \cdot \frac{\partial \tilde{g}_i}{\partial \beta^j} \right) \tilde{g}^i \otimes \tilde{g}^j, \quad (4.87)$$

with the skew-symmetric part of $5D$ given by

$$5D - 5D^T = \left( M_2 \cdot \left( \frac{\partial \tilde{g}_i}{\partial \beta^j} - \frac{\partial \tilde{g}_j}{\partial \beta^i} \right) \right) \tilde{g}^i \otimes \tilde{g}^j. \quad (4.88)$$

Using the same argument used to establish (2.36), we can also show that

$$\frac{\partial \tilde{g}_i}{\partial \beta^j} - \frac{\partial \tilde{g}_j}{\partial \beta^i} = \tilde{g}_j \times \tilde{g}_i. \quad (4.89)$$

Hence

$$5D - 5D^T = (M_2 \cdot (\tilde{g}_j \times \tilde{g}_i)) \tilde{g}^i \otimes \tilde{g}^j, \quad (4.90)$$

and we can conclude that, for the two-body system with potential $U_5$ that is a function only of the relative position and relative rotation between the two bodies, $M_2$ is the axial vector of $5D - 5D^T$:

$$5D_{32} - 5D_{23} = M_2 \cdot E_1, \quad 5D_{13} - 5D_{31} = M_2 \cdot E_2, \quad 5D_{21} - 5D_{12} = M_2 \cdot E_3. \quad (4.91)$$

### 4.5 Closing Remarks

In this chapter, it was first shown how different functional representations of the potential energy function $U$ could be used to derive expressions for the conservative forces and moments acting on the multibody system (see (4.3), (4.6), and (4.13)):

$$(F_1, M_{1,O}, \ldots, F_N, M_{N,O}) \leftrightarrow U_1 (\nu_1^2, \ldots, \nu_N^2, \nu_N^3, x_1^2, x_2^2, \ldots, x_2^N, x_3^N),$$

$$(F_1, M_1, \ldots, F_N, M_N) \leftrightarrow U_2 (\nu_1^2, \ldots, \nu_N^2, \nu_N^3, X_1^1, X_2^1, \ldots, X_2^N, X_3^N)$$

$$(F_1, M_{1,A}, \ldots, F_N, M_{N,A}) \leftrightarrow U_4 (\nu_1^2, \ldots, \nu_N^2, \nu_N^3, X_{A_1}^1, X_{A_2}^1, \ldots, X_A^N, X_A^N),$$

$$(F_1, M_1, \ldots, F_N, M_N) \leftrightarrow U_5 (\beta_1^2, \ldots, \beta_N^2, \beta_N^3, Y_1^1, Y_2^1, \ldots, Y_2^N, Y_3^N)$$

$$(F_1, M_{1,A}, \ldots, F_N, M_{N,A}) \leftrightarrow U_6 (\beta_1^2, \ldots, \beta_N^2, \beta_N^3, X_{A_1}^1, X_{A_2}^1, \ldots, X_A^N, X_A^N). \quad (4.92)$$

Taylor series expansions of these forces and moments were then used to derive the Cartesian stiffness matrices $\frac{m}{m}{K_1}$, $\frac{m}{m}{K_1}$, $\frac{q}{q}{K_1}$, and $\frac{q}{q}{K_1}$ associated with $U_1$, $\frac{q}{q}{K_1}$, and $\frac{q}{q}{K_1}$ related to $U_2$, and $\frac{q}{q}{K}$ and $\frac{q}{q}{K}$ from $U_5$ and $U_6$ respectively. Expressions for these stiffness matrices are given in Sections 4.2.1, 4.2.2, 4.2.3, and 4.3.3, and details on the correspondence between the skew-symmetric components of the stiffness matrices to the conservative forces and moments acting on the system elucidated upon in Section
We also showed how the expressions for the Cartesian stiffness matrix $\mathbf{K}$ simplified dramatically for the systems described in Sections 4.3.1 and 4.3.2. In the following chapter, we show how the latter simplification, with potential function,

$$\hat{U}_g \left( \mathbf{R}_1, \ldots, \mathbf{R}_N, \mathbf{y}^1, \ldots, \mathbf{y}^N \right) = \hat{U}_g^1 \left( \mathbf{R}_1, \mathbf{y}^1 \right) + \ldots + \hat{U}_g^N \left( \mathbf{R}_N, \mathbf{y}^N \right).$$

can be used to model the intervertebral joints of the lumbar spine. We also describe briefly how the elements of the experimental stiffness matrix are measured, and how one would use those values to study finite motions of the spine.
The intervertebral joints of the spine are unique due to their ability to control motion about all six degrees of freedom, as well as their load-, time-, and orientation-dependent viscoelastic properties. Efforts to quantify changes in the intervertebral joint elements – consisting of the intervertebral disc, facets, and surrounding ligaments – in healthy and degenerate lumbar spines have involved various metrics. Rousseau et al. [44,45] looked at changes in the position and orientation of the instantaneous axis of rotation, and, more recently, Wachowski [46] studied the migration of the helical axis of motion in an attempt to understand the underlying motion patterns in lumbar spine kinematics.

A complementary effort involves quantifying intervertebral joint kinematics using a stiffness matrix. This is done in Section 5.1 where we introduce the Cartesian stiffness matrix associated with the intervertebral joint $cK^J$. In Section 5.2, we explain how the elements of experimental stiffness matrices are measured followed by Section 5.3 detailing the relationship between the experimental and the Cartesian stiffness matrix. In this section we also posit a potential energy that is a quadratic function of the relative positions and orientations between two adjacent vertebrae. To illustrate the theoretical developments of the previous three sections, two examples: flexion-extension and lateral bending of the lumbar spine are studied in Section 5.4 with the resulting Cartesian stiffness matrices of the joints given in Section 5.4.1. The contribution of the components of $cK^J$ to the computed forces and moments are also
elaborated upon in further detail. Finally, we conclude with some comments and possible extensions to the potential energy function used in Section 5.4.

5.1 The Cartesian Stiffness Matrix of the Intervertebral Joint $^cK^J$

In Section 4.3.2, we posited that the potential energy function associated with the spine could be written as a linear sum of the potential energies associated with each joint (cf. (4.71)):

$$\hat{U}_5(R_1, \ldots, R_N, y^1, \ldots, y^N) = \hat{U}_5^1(R_1, y^1) + \ldots + \hat{U}_5^N(R_N, y^N).$$

We then derived expressions for the conservative forces and moments acting on the $K$th rigid body (equation (4.72)) as well as the resulting Cartesian stiffness matrix (equation 4.73)). As our interest in $K$ lies in its potential biomechanical application, it is appropriate to separate the force and moment exerted on the $K$th vertebral body into the portions due to the joint above and below it as depicted in Fig. 5.1,

$$F_K = F^J_K - F^J_{K+1} = -\left(\frac{\partial U^K}{\partial Y^K_i} E_i\right) - \left(-\frac{\partial U^{K+1}}{\partial Y^{K+1}_i} E_i\right), \quad (5.1)$$
$$M_K = M^J_K - M^J_{K+1} = -\left(\frac{\partial U^K}{\partial \beta^K} \bar{g}^{K,i}\right) - \left(-\frac{\partial U^{K+1}}{\partial \beta^{K+1}} \bar{g}^{K+1,i}\right), \quad (5.2)$$

where, in the interest of brevity, we have suppressed the subscript 5, superscript #, and hats over $\hat{K}$. Denoting the $6 \times (6N)$ Cartesian stiffness matrix due to the $K$th
joint by $c_{KJ}^J$, we can write

$$
\Delta E_K^J = -c_{KJ}^J \Delta y.
$$

(5.3)

with $\Delta y$ defined by expression (4.58). The components of $c_{KJ}^J$ can be inferred from the matrix expressions for $c_{KJ}^J$, $c_{KJ}^J$, and $c_{KJ}^J$ in a system composed of $N = 3$ bodies,

$$
c_{KJ}^J_1 = \begin{bmatrix}
(\tilde{I_G}_1)^T H_1 (\tilde{I_G}_1)
+ c_{0D_{11}}^J
0
0
\end{bmatrix},
$$

$$
c_{KJ}^J_2 = \begin{bmatrix}
0D_{21}^J
(\tilde{I_G}_2)^T H_2 (\tilde{I_G}_2)
+ c_{0D_{22}}^J
0
\end{bmatrix},
$$

and

$$
c_{KJ}^J_3 = \begin{bmatrix}
0D_{31}^J
0D_{32}^J
(\tilde{I_G}_3)^T H_3 (\tilde{I_G}_3)
+ c_{0D_{33}}^J
\end{bmatrix},
$$

(5.4)

respectively. In the expressions above, $\tilde{I_G}_K$ and $c_{0D_{KL}}^J$ are given by

$$
\tilde{I_G}_K = \begin{bmatrix}
1
0
0
\tilde{G}_K
\end{bmatrix},
$$

$$
c_{D_{KL}}^J = \left( \frac{\partial \tilde{G}_K^i}{\partial \beta_L^j} \frac{\partial U_K}{\partial \beta_L^j} \tilde{G}_{L,q}^j \right),
$$

(5.5)

while

$$
H_K = \begin{bmatrix}
H_{K}^1 & H_{K}^3
(H_{K}^1)^T & H_{K}^2
\end{bmatrix},
$$

(5.6)
and corresponds to the Hessian of the potential energy function,

\[
\begin{align*}
H^1_K &= \begin{bmatrix}
\frac{\partial^2 U^K}{\partial Y_1^K \partial Y_1^K} & \cdots & \frac{\partial^2 U^K}{\partial Y_1^K \partial Y_3^K} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U^K}{\partial Y_3^K \partial Y_1^K} & \cdots & \frac{\partial^2 U^K}{\partial Y_3^K \partial Y_3^K}
\end{bmatrix}, \\
H^2_K &= \begin{bmatrix}
\frac{\partial^2 U^K}{\partial \beta_1^K \partial \beta_1^K} & \cdots & \frac{\partial^2 U^K}{\partial \beta_1^K \partial \beta_3^K} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U^K}{\partial \beta_3^K \partial \beta_1^K} & \cdots & \frac{\partial^2 U^K}{\partial \beta_3^K \partial \beta_3^K}
\end{bmatrix}, \\
H^3_K &= \begin{bmatrix}
\frac{\partial^2 U^K}{\partial Y_1^K \partial \beta_1^K} & \cdots & \frac{\partial^2 U^K}{\partial Y_1^K \partial \beta_3^K} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U^K}{\partial Y_3^K \partial \beta_1^K} & \cdots & \frac{\partial^2 U^K}{\partial Y_3^K \partial \beta_3^K}
\end{bmatrix}, \\
H^4_K &= \begin{bmatrix}
\frac{\partial^2 U^K}{\partial \beta_1^K \partial Y_1^K} & \cdots & \frac{\partial^2 U^K}{\partial \beta_1^K \partial Y_3^K} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 U^K}{\partial \beta_3^K \partial Y_1^K} & \cdots & \frac{\partial^2 U^K}{\partial \beta_3^K \partial Y_3^K}
\end{bmatrix}.
\end{align*}
\]

(5.7)

For completeness, we remark that for the lumbar column consisting of the five vertebrae L1 through L5, the 30 × 30 Cartesian stiffness matrix specified by equation (4.73) is very simply given by

\[
\begin{align*}
\xi K^# &\equiv K = \begin{bmatrix}
c K_1^J \\
c K_2^J \\
c K_3^J \\
c K_4^J \\
c K_5^J \\
0
\end{bmatrix} - \begin{bmatrix}
c K_2^J \\
c K_3^J \\
c K_4^J \\
c K_5^J \\
0
\end{bmatrix}.
\end{align*}
\]

(5.8)

5.2 Experimental Stiffness Matrix Measurements

Typical experimental determinations of the stiffness matrix elements associated with the intervertebral joint have involved pairs of vertebral bodies. Most commonly, the lower vertebra is fixed, the upper vertebra subjected to infinitesimal motion, and the forces and moments due to the ensuing deformation measured [3–6]. Alternatively, a force or moment is applied and the resulting deformations measured [2, 47] (Fig. 5.2). Variations of these protocols have also been tested using larger deformations [11]. The elements of the experimental stiffness matrix \(K^E\) can then be determined by comparing the changes in the forces and moments to the relative motion between the upper and lower vertebra. That is,

\[
\Delta F = -K^E \Delta y,
\]

(5.9)

where \(\Delta F\) and \(\Delta y\) correspond to the generalized force acting on the upper body and the relative displacement vectors respectively. The components of \(\Delta F\) and \(\Delta y\) can be easily inferred from expressions (4.47) and (4.58) if they are expressed in the Cartesian frame. As shown in Chapter 4, various other expressions exist, and we refer the reader there for further detail.
5.3 Relating the Experimental Stiffness Matrix to the Cartesian Stiffness Matrix

One of the simplest potential energy function for the intervertebral joint is the quadratic function

\[ U_5 = \sum_{K=1}^{5} U_5^K, \quad U_5^K = (u^K - u^K_0)^T H_K (u^K - u^K_0), \]

(5.10)

with \( H_K \) set to a constant matrix, \( u^K_0 \) corresponding to the configuration of \( u^K \) in the neutral posture:

\[ u^K = \begin{bmatrix} \bar{y}^K \cdot E_1 \\ \bar{y}^K \cdot E_2 \\ \bar{y}^K \cdot E_3 \\ \beta_K \cdot \bar{g}^{K,1} \\ \beta_K \cdot \bar{g}^{K,2} \\ \beta_K \cdot \bar{g}^{K,3} \end{bmatrix}, \quad \Delta u^K = \begin{bmatrix} \Delta y^K \cdot E_1 \\ \Delta y^K \cdot E_2 \\ \Delta y^K \cdot E_3 \\ \Delta \beta_K \cdot \bar{g}^{K,1} \\ \Delta \beta_K \cdot \bar{g}^{K,2} \\ \Delta \beta_K \cdot \bar{g}^{K,3} \end{bmatrix}. \]

(5.11)
Equation (5.11) also defines the incremental change in the configuration $\Delta u^K$, used to compute the forces and moments exerted due to infinitesimal relative motion between the vertebrae,

$$
\Delta y^K \cdot E_i = (\bar{y}^{K'} - \bar{y}^{K}) \cdot E_i, \quad \Delta \beta_K \cdot \tilde{g}^{K,i} = \left(\beta^{K'} - \beta^K\right) \cdot \tilde{g}^{K,i}, \quad i = 1, 2, 3. \quad (5.12)
$$

Notice that in contrast to equation (4.58), the orientation components of $u^K$ and $\Delta u^K$ are parameterized using the relative Euler angles $\beta_K$. This parameterization guarantees that the resulting moments due to the potential energy function are conservative.

In the realm of infinitesimal rotations employed in experiments, the relative Euler basis vectors, the body fixed basis vectors, and the Cartesian basis vectors converge and the need to keep track of the Euler angles in computing the experimental stiffness matrix elements disappears. Further, imposing the symmetry restrictions used in, e.g., [2, 4], the experimental stiffness matrix $K^E$ reduces to the Hessian of the Cartesian stiffness matrix detailed in Section 4.3.1 featuring a pair of rigid bodies with potential function independent of its surroundings (Fig. 5.3).

![Figure 5.3: A pair of vertebral bodies, $V_1$ and $V_2$, connected by an intervertebral disc, $I$. The disc, facets ($F_1, F_2$), and surrounding tissues and ligaments comprise the intervertebral joint. As the vertebra move from their neutral position, the joint exerts equal and opposite conservative forces and moments on the vertebra ($F_K, M_K$). These force-moment pairs can be related to the relative motion between the vertebra using a Cartesian stiffness matrix, derived from a Taylor series expansion of $U = \hat{U}_5(R, y)$.](image)

To study the entire lumbar column of the spine, we assume that the potential energy of the system is given by the sum of the potential energies associated with each vertebral pair as shown in equation (4.71) of Section 4.3.2, Chapter 4. We can then use the joint Cartesian stiffness matrices $\hat{K}_J^e$ detailed in Section 5.1 to relate changes in the motion to the ensuing forces and moments.
5.4 Examples

Fig. 5.4: The Cartesian basis vectors \( \{E_1, E_2, E_3\} \). In the neutral position, the basis vectors affixed to the \( K \)th vertebral body, \( V_K \), point forward, up, and to the right respectively.

To continue, we first define the basis vectors associated with the \( K \)th vertebra \( \{e_1^K, e_2^K, e_3^K\} \) such that, in the neutral posture, \( e_1^K \) points forward, \( e_2^K \) points up, and \( e_3^K \) points to the right (Fig. 5.4). In addition, we refer to Table 1 of [4] to estimate the components of the Hessian matrix as

\[
H_K = \begin{bmatrix}
500,000 & 0 & 0 & 11,000 & 0 & 0 \\
0 & 2,500,000 & 0 & -5,000 & 0 & 0 \\
0 & 0 & 500,000 & 0 & 13,500 & -12,000 \\
11,000 & -5,000 & 0 & 600 & 0 & 0 \\
0 & 0 & 13,500 & 0 & 850 & -300 \\
0 & 0 & -12,000 & 0 & -300 & 400
\end{bmatrix}, \quad K = 1, \ldots, 5,
\]

where \( H_{1,K} \), \( H_{2,K} \), and \( H_{3,K} \) are expressed in N/m, N/rad and Nm/rad respectively. Following [48], the motion of each of the lumbar vertebrae is prescribed as a linear function of the coordinate of interest. That is

\[
\Delta u_{K,i} = \alpha_{K,ij} \Delta u_{tot,j}, \quad i, j = 1, \ldots, 6, \quad K = 1, \ldots, 5.
\]

In (5.13), \( \Delta u_{tot,j} \) is the total change in configuration in the \( j \)th degree of freedom, \( \Delta u_{K,i} \) is the change in the \( i \)th generalized coordinate of the \( K \)th vertebral body, and \( \alpha_{K,ij} \) is the slope of the linear function relating the two. For example, flexion-extension of the lumbar column \( (j = 4) \) has been shown to be accompanied by displacements in the anterior-posterior and axial directions \( (i = 1 \text{ and } 2 \text{ respectively}) \) [49]. Hence, \( \alpha_{K,14} \) and \( \alpha_{K,24} \) are non-zero in addition to \( \alpha_{K,44} \). This is illustrated in Fig. 5.5.

For illustrative purposes, we will consider two different motions. The first consists of planar motion in the sagittal plane, while the second - lateral bending of the lumbar
Fig. 5.5: Schematic showing extension of the lumbar spine and the accompanying motion of each vertebra $V_K$ about its instantaneous axis of rotation $IAR_K$. In the equilibrium position shown on the left, $\Delta u_{K,i}$ and $\Delta u_{\text{tot},4}$ both equal zero. As the spine extends however, $\Delta u_{\text{tot},4}$ increases and each of the lumbar vertebral bodies $V_K$ rotates and displaces anteriorly and axially. The motion is specified by the linear function (5.13) with the values for $\alpha_{K,14}$, $\alpha_{K,24}$ and $\alpha_{K,44}$ given in Table 5.1.

Planar Motion: Flexion and Extension of the Lumbar Spine

Fig. 5.6: Flexion and extension of the lumbar spine. The red lines depict some of the muscles used to move the spine in the sagittal plane.

Coupling between the different coordinates of the intervertebral joint is evident even for planar motion.\(^1\) For example, flexion and extension of the lumbar spine

---

\(^1\)An in-depth discussion of this can be found in the excellent text on the subject by Professor Nikolai Bogduk, one of the leading researchers on the subject [49].
is often accompanied by coupled axial and and anterior-posterior translation (Figs. 5.5 and 5.6) [49]. To compute the components of $\kappa^J_k$ for this motion, we used data from [50] for the slopes $\alpha_{k,44}$. The values of $\alpha_{k,14}$ and $\alpha_{k,24}$ were estimated as 0.005 due to the lack of conclusive data on the subject while the remaining slopes, $\alpha_{k,34}$, $\alpha_{k,54}$, and $\alpha_{k,64}$, were set to zero under the assumption that the motion was symmetric in the coronal and axial plane (cf. Table 5.1).

<table>
<thead>
<tr>
<th>Joint $(K)$</th>
<th>$\alpha_{k,44}$</th>
<th>$\alpha_{k,14}$</th>
<th>$\alpha_{k,24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_5/S_1$ $(K = 1)$</td>
<td>0.125</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>$L_4/L_5$ $(K = 2)$</td>
<td>0.185</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>$L_3/L_4$ $(K = 3)$</td>
<td>0.204</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>$L_2/L_3$ $(K = 4)$</td>
<td>0.231</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>$L_1/L_2$ $(K = 5)$</td>
<td>0.255</td>
<td>0.005</td>
<td>0.005</td>
</tr>
</tbody>
</table>

**Table 5.1:** Slope $\alpha_{k,44}$ defining the relationship between the total lumbar flexion-extension and the ensuing vertebral motion.

Non-Planar Motion: Lateral Bending of the Lumbar Spine

![Fig. 5.7: Lateral bending of the lumbar spine. The red lines depict some of the muscles used to move the spine in the coronal plane.](image)

Lateral bending of the lumbar column has been shown to be accompanied by rotations about the other degrees of freedom [49,51–53]. Table 5.2 lists the amount of coupling evidenced at each lumbar level as specified in Table 1 of [53].

In the interest of brevity, the remainder of our analysis will be confined to the lower three levels of the lumbar spine, namely the $L_5/S_1$, $L_4/L_5$ and $L_3/L_4$ joints.
Joint \((K)\) & \(\alpha_{K,66}\) & \(\alpha_{K,46}\) & \(\alpha_{K,56}\) \\
\hline
\(L_5/S_1\) \((K = 1)\) & 0.1355 & 0.1215 & 0.1093 \\
\(L_4/L_5\) \((K = 2)\) & 0.1811 & 0.0824 & 0.0882 \\
\(L_3/L_4\) \((K = 3)\) & 0.2453 & 0.0865 & 0.0857 \\
\(L_2/L_3\) \((K = 4)\) & 0.2501 & 0.0929 & 0.1115 \\
\(L_1/L_2\) \((K = 5)\) & 0.1879 & 0.1090 & 0.0990 \\

Table 5.2: Slope \(\alpha_{K,66}\) defining the relationship between the lateral bending motion of the lumbar column and the motion of each vertebra.

5.4.1 Results and Discussion

The forces and moments exerted by the intervertebral joints can be easily computed using equation (5.3). Table 5.3 lists the moments exerted on the center of mass of the \(K^{th}\) vertebra by the intervertebral joint directly below it. Equal but opposite moments are exerted on the center of mass of the \((K-1)^{th}\) vertebra due to the relative motion between the two bodies. One can then relate the incremental changes in the Cartesian components of the forces and moments exerted by the joint to the motions about this configuration using equations (5.3), (5.4), and (5.5).

<table>
<thead>
<tr>
<th>Joint (Body (K))</th>
<th>Motion</th>
<th>(M_K \cdot E_1) (Nm)</th>
<th>(M_K \cdot E_2) (Nm)</th>
<th>(M_K \cdot E_3) (Nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_5/S_1) ((K = 1))</td>
<td>5° Flexion</td>
<td>0.00</td>
<td>0.00</td>
<td>13.53</td>
</tr>
<tr>
<td></td>
<td>15° Flexion</td>
<td>0.00</td>
<td>0.00</td>
<td>40.58</td>
</tr>
<tr>
<td></td>
<td>5° Bending</td>
<td>-7.53</td>
<td>-11.74</td>
<td>-6.36</td>
</tr>
<tr>
<td>(L_4/L_5) ((K = 2))</td>
<td>5° Flexion</td>
<td>0.00</td>
<td>0.00</td>
<td>16.67</td>
</tr>
<tr>
<td></td>
<td>15° Flexion</td>
<td>0.00</td>
<td>0.00</td>
<td>50.00</td>
</tr>
<tr>
<td></td>
<td>5° Bending</td>
<td>-8.50</td>
<td>-11.38</td>
<td>-4.37</td>
</tr>
<tr>
<td>(L_3/L_4) ((K = 3))</td>
<td>5° Flexion</td>
<td>0.00</td>
<td>0.00</td>
<td>17.66</td>
</tr>
<tr>
<td></td>
<td>15° Flexion</td>
<td>0.00</td>
<td>0.00</td>
<td>52.99</td>
</tr>
<tr>
<td></td>
<td>5° Bending</td>
<td>-10.60</td>
<td>-12.92</td>
<td>-4.70</td>
</tr>
</tbody>
</table>

Table 5.3: Conservative moments exerted by the \(K\) joints, as determined from equation (5.10), under 5° and 15° flexion as well as 5° lateral bending to the right.

For example, for 5° of flexion, the components of \(\alpha K^J\) are
\[
\begin{bmatrix}
500000 & 0 & 0 & 0 & 0 & 11000 \\
0 & 2500000 & 0 & 0 & 0 & 5000 \\
0 & 0 & 500000 & 12146.55 & 13368.30 & 0 \\
0 & 0 & 147.26 & 406.60 & 304.84 & 0 \\
0 & 0 & 13499.20 & 304.84 & 843.40 & 0 \\
11000 & 50000 & 0 & 0 & 0 & 600
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
500000 & 0 & 0 & 0 & 0 & 11000 \\
0 & 2500000 & 0 & 0 & 0 & 5000 \\
0 & 0 & 500000 & 12360.77 & 13170.47 & 0 \\
0 & 0 & 365.17 & 416.55 & 311.73 & 0 \\
0 & 0 & 13495.06 & 311.73 & 833.45 & 0 \\
11000 & 50000 & 0 & 0 & 0 & 600
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
500000 & 0 & 0 & 0 & 0 & 11000 \\
0 & 2500000 & 0 & 0 & 0 & 5000 \\
0 & 0 & 500000 & 12593.27 & 12948.34 & 0 \\
0 & 0 & 605.34 & 427.78 & 318.95 & 0 \\
0 & 0 & 13486.42 & 318.95 & 822.22 & 0 \\
11000 & 50000 & 0 & 0 & 0 & 600
\end{bmatrix}
\]

79
while, for 15° flexion,

\[
\begin{bmatrix}
500000 & 0 & 0 & 0 & 0 & 11000 \\
0 & 2500000 & 0 & 0 & 0 & 5000 \\
0 & 0 & 500000 & 12435.28 & 13100.14 & 0 \\
0 & 0 & 441.71 & 420.10 & 314.07 & 0 \\
0 & 0 & 13492.77 & 314.07 & 829.90 & 0 \\
11000 & 50000 & 0 & 0 & 0 & 600 \\
\end{bmatrix}
\]

(5.14)
\[ {^cH_3} = \begin{bmatrix} 500000 & 0 & 0 & 0 & 0 & 11000 \\ 0 & 2500000 & 0 & 0 & 0 & 5000 \\ 0 & 0 & 500000 & 13702.67 & 11768.05 & 0 \\ 0 & 0 & 1811.15 & 487.87 & 349.03 & 0 \\ 0 & 0 & 13377.96 & 349.03 & 762.13 & 0 \\ 11000 & 50000 & 0 & 0 & 0 & 600 \end{bmatrix}, \]

\[ {^cD_{33}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7.04 & -52.03 & 0 & 0 \\ 0 & 0 & 0.95 & 7.04 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ \text{(5.15)} \]

In the above expressions, we used \( {^cH_K} \) to denote \( \tilde{\text{IG}} \left( _5H_K \right) \tilde{\text{IG}}_K \).

The \( {^cD_{KK}} \) and \( {^cH_K} \) matrices become much denser when the motion is not confined to a plane. This is evident from the computed elements of \( {^cK^T_K} \) for lateral bending to the right by 5°,

\[ {^cH_1} = \begin{bmatrix} 500000 & 0 & 0 & 104.92 & 1.11 & 11000 \\ 0 & 2500000 & 0 & 47.69 & 0.51 & 5000 \\ 0 & 0 & 500000 & 11856.74 & 13626.48 & 0 \\ 104.92 & 47.69 & -143.14 & 393.78 & 295.18 & 5.72 \\ 1.11 & 0.51 & 13499.24 & 295.18 & 856.31 & 0.06 \\ 11000 & 50000 & 0 & 5.72 & 0.06 & 600 \end{bmatrix}, \]

\[ {^cD_{11}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.18 & 6.43 & -11.74 & 0 \\ 0 & 0 & 0.07 & 0.07 & 7.53 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[
\begin{pmatrix}
500000 & 0 & 0 & 190.94 & -127.44 & 10997.93 \\
0 & 2500000 & 0 & 86.79 & -57.93 & 4999.06 \\
0 & 0 & 500000 & 11759.26 & 13710.44 & 47.11 \\
190.94 & 86.79 & -238.66 & 389.70 & 291.76 & 10.07 \\
-127.44 & -57.93 & 13496.94 & 291.76 & 860.49 & 0.32 \\
10997.93 & 4999.06 & 160.54 & 10.07 & 0.32 & 599.86
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.28 & 4.51 & -11.38 & \\
0 & 0 & 0.15 & -0.02 & 8.46 & \\
0 & 0 & 0 & -0.04 & -0.21 & 
\end{pmatrix},
\]

\[
\begin{pmatrix}
500000 & 0 & 0 & 276.21 & -299.13 & 10992.77 \\
0 & 2500000 & 0 & 125.55 & -135.97 & 4996.71 \\
0 & 0 & 500000 & 11657.94 & 13795.68 & 172.28 \\
276.21 & 125.55 & -336.80 & 385.59 & 288.03 & 16.16 \\
-299.13 & -135.97 & 13490.57 & 288.03 & 864.76 & 2.29 \\
10992.77 & 4996.71 & 375.56 & 16.16 & 2.29 & 599.70
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.44 & 4.96 & -12.91 & \\
0 & 0 & 0.26 & -0.17 & 10.50 & \\
0 & 0 & 0.01 & -0.09 & 0.52 & 
\end{pmatrix}.
\]

Overall, the contribution of \(c^{D_J}\) to incremental changes in the forces and moments depends on both the type and magnitude of motion. When the motion is planar, computing the elements of \(c^{0D_J}\) is unnecessary as \(c^{0D_J} \Delta y = 0\). The matrices \(c^{0D_K}\) become relevant when the motion is non-planar as is evident from the lateral bending example. Referring to the matrices featured in (5.16), the \(c^{0D_{KK,46}}\) and \(c^{H_{K,46}}\) terms counteract each other resulting in negligible moment about \(E_1\) for rotations about \(E_3\). On the other hand, the \(c^{0D_{KK,56}}\) term creates a moment about the \(E_2\) axis of magnitude 10Nm/rad for \(\Delta \Phi_K \cdot E_3 \neq 0\).

Of particular interest in this dissertation is the ability of the forces and moments computed using (5.3) to approximate the actual force and moment increments given
by

$$c\Delta F_{ex}^K = \left[ \left( F_J^T - F_J^T K \right) \cdot E_i \right] = \left[ -\left( \frac{\partial U_K^a}{\partial y_i} - \frac{\partial U_K^a}{\partial y_i} \right) \cdot E_i \right] = \left[ \begin{array}{c} -\left( \frac{\partial U_K^a}{\partial y_i} - \frac{\partial U_K^a}{\partial y_i} \right) \cdot E_i \end{array} \right] \cdot E_i \right] \tag{5.17}$$

to the forces and moments determined from (5.3). To do this, we look at lateral bending of the lumbar from 5° to 5.5°. To better appreciate the contribution of each element of the $cK^{J}$ matrix, we also define the variables

$$c\Delta F_{H}^K = -cH_K \Delta y^K, \quad c\Delta F_{HD}^K = -\left( cH_K + cD_{KK} \right) \Delta y^K, \quad c\Delta F_{HDD}^K = -\left( cH_K + cD_{KL} \Delta y^K \right), \quad (K \neq L). \tag{5.18}$$

The force components all agree with each other - a consequence of writing the potential energy as a function of the Cartesian components of the relative displacements. However, the values for the resulting moment components of $c\Delta F_{ex}^K$, $c\Delta F_{H}^K$, $c\Delta F_{HD}^K$, and $c\Delta F_{HDD}^K$ are shown in Table 5.4.²

<table>
<thead>
<tr>
<th>Joint $(K)$</th>
<th>Components</th>
<th>$c\Delta F_{ex}^K$</th>
<th>$c\Delta F_{HDD}^K$</th>
<th>$c\Delta F_{HD}^K$</th>
<th>$c\Delta F_{H}^K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_5/S_1$ $(K = 1)$</td>
<td>$(M_J^T - M_J^T) \cdot E_1$</td>
<td>-0.7459</td>
<td>-0.7465</td>
<td>-0.7465</td>
<td>-0.7528</td>
</tr>
<tr>
<td></td>
<td>$(M_J^T - M_J^T) \cdot E_2$</td>
<td>-1.1824</td>
<td>-1.1816</td>
<td>-1.1816</td>
<td>-1.1735</td>
</tr>
<tr>
<td></td>
<td>$(M_J^T - M_J^T) \cdot E_3$</td>
<td>-0.6362</td>
<td>-0.6362</td>
<td>-0.6362</td>
<td>-0.6362</td>
</tr>
<tr>
<td>$L_4/L_5$ $(K = 2)$</td>
<td>$(M_J^T - M_J^T) \cdot E_1$</td>
<td>-0.8366</td>
<td>-0.8379</td>
<td>-0.8456</td>
<td>-0.8505</td>
</tr>
<tr>
<td></td>
<td>$(M_J^T - M_J^T) \cdot E_2$</td>
<td>-1.1494</td>
<td>-1.1484</td>
<td>-1.1446</td>
<td>-1.1385</td>
</tr>
<tr>
<td></td>
<td>$(M_J^T - M_J^T) \cdot E_3$</td>
<td>-0.4424</td>
<td>-0.4418</td>
<td>-0.4367</td>
<td>-0.4366</td>
</tr>
<tr>
<td>$L_3/L_4$ $(K = 3)$</td>
<td>$(M_J^T - M_J^T) \cdot E_1$</td>
<td>-1.0368</td>
<td>-1.0390</td>
<td>-1.0532</td>
<td>-1.0596</td>
</tr>
<tr>
<td></td>
<td>$(M_J^T - M_J^T) \cdot E_2$</td>
<td>-1.3071</td>
<td>-1.3057</td>
<td>-1.3002</td>
<td>-1.2922</td>
</tr>
<tr>
<td></td>
<td>$(M_J^T - M_J^T) \cdot E_3$</td>
<td>-0.4888</td>
<td>-0.4869</td>
<td>-0.4702</td>
<td>-0.4699</td>
</tr>
</tbody>
</table>

Table 5.4: Comparison of the changes in moments as determined from equation (5.17) to the quantities $c\Delta F_{ex}^K$, $c\Delta F_{HDD}^K$, and $c\Delta F_{HD}^K$ defined by equations (5.18) and (5.19) respectively for lumbar bending from 5° to 5.5°.

As we include first, the term involving the diagonal components of $c0D$, and then, the off-diagonal components, the computed values for the changes in the forces and

²The force components are not shown as they are all similar.
moments begins to better approximate their true value. This effect becomes increasingly pronounced as we move upwards along the spine (cf. $\Delta M_K$ for $K = 2, 3$ in Table 5.4).

5.5 Closing Remarks

In this chapter, we first modified the Cartesian stiffness matrix developments featured in Section 4.3.2 of Chapter 4 to obtain the Cartesian stiffness matrices $^cK^J_K$ associated with each of the 5 lumbar joints. We then explained how the elements of stiffness matrices are measured experimentally in Section 5.2 and showed how, if the experimental motions were small and uniaxial, $K^E$ presented a good approximation of the Hessian $H_K$ of the potential energy function,

$$
\hat{U}_5 (R_1, \ldots, R_N, y^1, \ldots, y^N) = \hat{U}_5^1 (R_1, y^1) + \ldots + \hat{U}_5^N (R_N, y^N).
$$

For the purposes of illustration, we looked at two examples: planar and non-planar motion. Some interesting features with regards to the components of the computed $^cK^J_K$ matrix were identified. In particular, we showed how the $^c0D_{KL}$ matrix could contribute to more accurate estimates of the moments exerted by the joints.

While the quadratic form of the potential energy function used to derive the conservative forces and moments is highly tractable and has been alluded to in [11], the constant nature of the Hessian matrix renders it unable to account for the dependence of the intervertebral joint on the amount of applied preload [3, 54–56], vertebral level [4, 57], degeneration grade [58, 59], load history [60], loading frequency [47, 59–62] and loading pattern [61, 63, 64]. To overcome this, one could elect to write the matrices $H_K$ in $U^K_5 = (u^K - u^K_O)^T H_K (u^K - u^K_O)$ as functions of the specified parameters of interest. Inspiration for this can be found in, for example, the potential energy functions mentioned in [65] and [8]. Paralleling their developments, one could posit a potential energy function with Hessian

$$
H_{K,ij} = \frac{\partial^2 U^K_5}{\partial u^K_i \partial u^K_j} = \left( a_{K,ij}^{b_{K,j}} \left( e^{b_{K,i}} u^K_i \right) + \kappa_{K,ij} \right) + \left( f_{K,ij} \left( 1 - e^{g_{K,i}} F_a \right) \right) \delta_{ij}, \tag{5.20}
$$

where the coefficients $a, b, f,$ and $g$ are functions of age, degeneration grade, vertebral level, and loading frequency, and $F_a$ is the amount of applied preload. The form of the underbraced term reflects the theory that, as the amount of preload is increased beyond a certain value, the diagonal terms in the stiffness matrix begin to approach a constant [56, 66]. Equation (5.20) highlights the difficulties present in applying theory to practice. Determining all the coefficients required to form a comprehensive potential energy function is challenging; the large number of experiments required to do so may compromise the integrity of the specimens being tested.

Other functions relating pure rotational motion to the applied forces and moments have also been posited in [8, 65, 67, 68]. Unfortunately, none of the experiments used
to determine these functions were conducted with an applied preload which has been shown to affect the stiffness of the joint significantly. In some instances, the functions were computed by fitting best-fit curves to experimental data. Consequently, the conservative nature of the joint forces and moments is not guaranteed and one is unable to derive a Cartesian stiffness matrix using a Taylor series expansion of the forces and moments. Thus, while the constant nature of the Hessian matrices $H_K$ featured in the potential energy function does not capture many of the elements that influence the stiffness of the intervertebral joint, it presents a feasible foundation upon which to build more complex models.

In the next chapter, we will apply the stiffness matrix elaborated upon here in a simple example of the fourth lumbar vertebra moving relative to the fifth – built in OpenSim [26] – albeit with some necessary and justifiable simplifications.
Chapter 6

Applying the Cartesian Stiffness Matrix to a Musculoskeletal Model of the Lumbar Spine

6.1 Introduction

Recent medical and technological improvements have led to an increased demand for user-friendly, computationally inexpensive numerical models of the human body free of the ethical and logistical constraints imposed by experimental analyses. The existence of these models is anticipated to help pave the way for easy, cost effective, and comprehensive studies on the efficacies of differing physical therapy options and the evaluation of prosthetic devices.

Unfortunately, the complexities surrounding the joint anatomy and function have presented obstacles in accurately modeling it in musculoskeletal models of the spine. Following the lead set by Panjabi and White in [48] some researchers have assumed that the motion of each spinal vertebra can be written as a function of the net motion of the lumbar spine [69–72]. The suitability of this approach to studying true spinal motion however is questionable as there is a large amount of wide inter-subject variability [73]. The numerical cost of constraint functions is also an inhibiting factor especially for larger models. The Cartesian stiffness matrix parameterization of the intervertebral joint presents a viable alternative.

Interestingly, Panjabi was also one of the first champions of employing a 6x6 stiffness matrix to model the intervertebral joints of the spine. Extrapolating upon theory initially developed as part of his thesis [74], he presented an elegant elucidation of its
application to the joints of the spine in [1]. Three years later, Panjabi, White, and a third colleague, Brand, published a paper proposing a stiffness matrix $K_P$ to model the joint [2]. Using symmetry arguments and restricting attention to infinitesimal rotations, the number of stiffnesses in the matrix $K_P$ were reduced from 36 to 12 and experiments conducted to determine these elements (cf. Fig. 6.1).

![Fig. 6.1: Schematic of the stiffness matrix testing apparatus used in Panjabi et al.’s seminal paper [2]. The lower vertebra, $V_1$ was fixed by bolting it to a polyester cast. Load vector components, one at a time, were then applied to the upper vertebra $V_2$ via a threaded pin and the displacements measured using three displacement gauges. The elements of the experimental stiffness matrix $K_P$ used to model the intervertebral joint $J$ were computed from the resulting load-displacement curves.](image)

Subsequent work, primarily by Gardner-Morse, Stokes and colleagues have measured these 12 parameters [4–6, 56]. However, over twenty years elapsed since Panjabi’s initial work before detailed spinal models featuring a complete stiffness matrix were introduced. The first of these, presented in [75], used stiffness matrices obtained in [76]. Further refinements on this model have focused on improving muscle architecture and optimization routines for predicting muscle activation patterns [56, 77–79]. Little mention has been made with regards to using newer data that capture the effects of preload and various experimental loading schemes on the ensuing stiffness matrix [3–5, 11, 54, 55, 66].

Concurrent with these developments, advances in computational modeling have led to the development of software capable of creating and analyzing musculoskeletal models of the human body. Of these, software platforms such as the LifeMOD Biomechanics Modeler (MSC Software, Santa Ana, CA), visualNastran 4D (MSC Software, Santa Ana, CA), APOLLO [23], SIMM (www.musculographics.com), and
OpenSim [26], have adapted the bushing elements used in vehicle dynamics to model the properties of the intervertebral joint. These elements generate forces and moments between a pair of vertebrae equivalent to those that would be exerted by diagonal stiffness and damping matrices. Indeed, the approximate similarity between the bushing element and the intervertebral joint has led to its increased use in musculoskeletal models of the spine [80–86].

This chapter serves three aims. The first is to provide a concise introduction to the bushing force function available in some musculoskeletal software platforms. These details - based on the bushing force function of the open-source software OpenSim - are given in Section 6.2. The second aim is driven by the need to ensure that the parameters used as input for the bushing force function agree with their experimental counterparts. Section 6.2.3 discusses this, and the specific modifications necessary to use a bushing element to model the intervertebral joint. The third motivating factor is the creation of a bushing element capable of producing load-motion coupling along all six degrees of freedom of the joint. This was done by enhancing the function to allow for the incorporation of a $6 \times 6$ stiffness matrix and is elaborated upon in Section 6.3. To reduce the amount of computation required, we utilize a simplified version of the Cartesian stiffness matrix $\mathbf{cKJ}$ introduced in Section 5.1. The element, which we refer to as a SpineBushing was created in OpenSim [26] as it is an open-source software platform with a model of the lumbar spine readily available from https://simtk.org/home/lumbarspine/ [72]. (The SpineBushing plugin itself can be obtained from https://simtk.org/home/spinebushing/.) The changes in the computed forces and moments are presented in Section 6.5 with the help of examples mentioned in Section 6.4.

We emphasize that the SpineBushing element proposed here is not meant to be a comprehensive model of the intervertebral joint. Rather our intention is to provide a solid and well-documented basis for future, more detailed models. Particularly useful additions include the incorporation of a damping element akin to the stiffness matrix element detailed here as well as the addition of contact forces upon articulation of the vertebral facet faces.

Comments on Notation

Four different stiffness matrices associated with the intervertebral joint are examined. The first is the Cartesian stiffness matrix associated with the intervertebral joint $\mathbf{cKJ}$ while the second is the stiffness matrix determined experimentally $\mathbf{KE}$. A third stiffness matrix, $\mathbf{KB}$ is introduced to specify the stiffness matrix of the bushing element detailed in Section 6.2. Finally, we use $\mathbf{KS}$ to denote the stiffness matrix used in the SpineBushing element we developed specific to the intervertebral joint.

When discussing a system composed of $N$ vertebrae, we attach a capitalized subscript $K$ to the stiffness matrix associated with the $K^{th}$ vertebra, e.g., $\mathbf{cKJ}$ and $\mathbf{KE}$. Where appropriate however, we confine our analysis to a system composed of 2 vertebrae connected by an intervertebral joint. To further simplify our notation, we also
assume that the experimental stiffness matrices are computed using the forces and motions exerted at the geometric centers of the adjacent vertebrae and that these points coincide with the vertebral centers of mass.

There is a considerable diversity in the choices of coordinate axes and Euler angles used in spine mechanics. In this chapter, we use a 1-2-3 set of Euler angles to parameterize the relative rotation tensor $R$ and let the basis vectors $\{e^K_1, e^K_2, e^K_3\}$ point forward, up, and to the right, respectively (cf. Fig. 2.4). As a result, we have the following correspondences:

$$
\beta^K_1 \equiv \text{lateral bending},
\beta^K_2 \equiv \text{axial rotation},
\beta^K_3 \equiv \text{flexion-extension with } \beta_3 > 0 \text{ corresponding to extension}.
$$

We emphasize that these angles of rotation pertain to the rotation of the upper vertebra $V_K$ relative to the lower one $V_{K-1}$.

6.2 The Bushing Element in Musculoskeletal Software Platforms

The bushing element commonly used in musculoskeletal software platforms computes the forces and moments proportional to the relative motions of two frames. This element is identical in function to applying a diagonal stiffness and damping matrix between the frames. Here, we elaborate on the algorithm behind the bushing element, and show how it can be used to accommodate a stiffness matrix. We base our analysis on the bushing force function available in OpenSim as it is an open-source software platform with readily available documentation. In the interest of brevity, our discussion will be limited to the stiffness portion of the bushing element: the corresponding developments for damping are easily inferred.

6.2.1 The Six Frames

To elaborate, recall that a frame is defined by the position vector of the frame and the set of basis vectors associated with these points. To discuss the bushing function, it is necessary to introduce three sets of frames. The first of these sets are the two body frames $\mathcal{F}_1$ and $\mathcal{F}_2$ situated at the origins of the vertebral bodies:

$$
\mathcal{F}_1 = \{ \bar{x}^1, \{e^1_1, e^1_2, e^1_3\} \} , \quad \mathcal{F}_2 = \{ \bar{x}^2, \{e^2_1, e^2_2, e^2_3\} \} .
$$

For convenience, we have assumed that the center of mass is coincident with the frame origins. The second set of frames correspond to the joint frames:

$$
\mathcal{J}_1 = \{ x^j, \{j^1_1, j^1_2, j^1_3\} \} , \quad \mathcal{J}_2 = \{ x^j, \{j^2_1, j^2_2, j^2_3\} \} .
$$
Fig. 6.2: The frames associated with the bushing element. (a) the frames in the neutral position, and (b) the frames in a displaced position. Once set, the orientation between the frames \( \{ F_K, J_K, B_K \} \) remain fixed.

As the joint frames \( J_1 \) and \( J_2 \) are used to characterize the motion of \( V_2 \) relative to \( V_1 \), they are coincident in the neutral position:

\[
x_j^1 = x_j^2, \quad j_i^1 = j_i^2, \quad (k = 1, 2, 3).
\]

but deviate from each other as the bodies move. The third set of frames are known as the bushing frames and are associated with the lower and upper vertebra \( V_1 \) and \( V_2 \), respectively:

\[
\mathbb{B}_1 = \{ x_{iB}^1, \{ b_{1i}^1, b_{2i}^1, b_{3i}^1 \} \}, \quad \mathbb{B}_2 = \{ x_{iB}^2, \{ b_{1i}^2, b_{2i}^2, b_{3i}^2 \} \}.
\]

These six frames are depicted in Fig. 6.2. It is important to note that \( j_i^1 \) and \( b_i^1 \) corotate with \( V_1 \) and \( j_i^2 \) and \( b_i^2 \) corotate with \( V_2 \).

In addition to the rotation tensors \( Q_1 \) and \( Q_2 \) associated with the respective frames \( F_1 \) and \( F_2 \), several additional rotation tensors must be defined. Here, we limit our discussion to the rotation tensor \( Q_1^B \) associated with the frame \( \mathbb{B}_1 \) as the remaining tensors can be easily inferred. This tensor has the representation

\[
Q_1^B = b_{1i}^1 \otimes E_i + b_{2i}^1 \otimes E_2 + b_{3i}^1 \otimes E_3.
\]

Because the basis vectors \( \{ b_{1i}^1, b_{2i}^1, b_{3i}^1 \} \) corotate with \( \mathbb{B}_1 \), it can be shown that \( Q_1 \) and
\( Q_1^B \) are related:

\[
Q_1 = T_1^B Q_1^B
\]  \( \quad (6.6) \)

where \( T_1^B \) is a rotation tensor: \( e_1^i = T_1^B b_1^i \). Hence,

\[
R = Q_2 Q_1^T = T_2^B Q_2^B (T_1^B Q_1^B)^T, \quad R^B = Q_2^B (Q_1^B)^T.
\]  \( \quad (6.7) \)

When the three frames \( F_K, J_K \) and \( B_K \) are aligned, \( Q_K^B = Q_K \) and \( R = R^B \). In addition, it is crucial to note that the configuration between the three frames does not change once it has been set.

Following the notation introduced in Chapter 2, we define the relative position between the bushing frames using the relative position vector \( y_B = x_2^B - x_1^B \),

\[
y_B = \begin{bmatrix}
y_B \cdot b_1^1 \\
y_B \cdot b_1^2 \\
y_B \cdot b_1^3 \\
\beta^B \cdot \bar{g}^{B,1} \\
\beta^B \cdot \bar{g}^{B,2} \\
\beta^B \cdot \bar{g}^{B,3}
\end{bmatrix}
\]

and use the set of Euler angles \( \{ \beta_1^B, \beta_2^B, \beta_3^B \} \) to parameterize \( R^B \):

\[
R^B = R^B \left( \beta_1^B, \beta_2^B, \beta_3^B, \bar{g}_1^B, \bar{g}_2^B, \bar{g}_3^B \right).
\]  \( \quad (6.9) \)

Unlike the joint frames \( J_1 \) and \( J_2 \), the bushing frames \( B_1 \) and \( B_2 \) need not be coincident in the neutral position:

\[
y_B \neq 0, \quad R^B \neq I.
\]  \( \quad (6.10) \)

In the next section, we show how this may result in non-zero bushing forces even when the vertebras are in their equilibrium posture.

**6.2.2 The Bushing Forces and Moments**

It is convenient to define the following variables:

\[
y_B = \begin{bmatrix}
y_B \cdot b_1^1 \\
y_B \cdot b_1^2 \\
y_B \cdot b_1^3 \\
\beta^B \cdot \bar{g}^{B,1} \\
\beta^B \cdot \bar{g}^{B,2} \\
\beta^B \cdot \bar{g}^{B,3}
\end{bmatrix}, \quad F_K^B = \begin{bmatrix}
F_K^B \cdot b_1^1 \\
F_K^B \cdot b_1^2 \\
F_K^B \cdot b_1^3 \\
M_K^B \cdot \bar{g}_1^B \\
M_K^B \cdot \bar{g}_2^B \\
M_K^B \cdot \bar{g}_3^B
\end{bmatrix}
\]  \( \quad (6.11) \)

where \( F_K^B \) and \( M_K^B \) indicate the force and moment vectors exerted by the bushing element on the respective bushing frames \( B_K \). The generalized displacement vector \( y_B \) is used in combination with the stiffness matrix \( K^B \) to determine the bushing forces.
Fig. 6.3: Schematic of the bushing element computations for the $\mathcal{L}_4$-$\mathcal{L}_5$ motion segment. The forces and moments due to the bushing element - exerted at the frames $\mathbb{B}_1$ and $\mathbb{B}_2$ shown on the right - are related to the relative motion $(y_B, \beta^B)$ between the two bushing frames using the bushing stiffness matrix $K^B$ by equations (6.12) and (6.14).

and moments acting on the two bushing frames:

$$ F^B_2 = -K^B y_B. $$  \hspace{1cm} (6.12)

and,$^1$

$$ F^B_1 = -F^B_2. $$  \hspace{1cm} (6.14)

This is illustrated in Fig. 6.3. The generalized force vectors $F^B_1$ and $F^B_2$ in (6.12) have force components expressed in the frame $\{b^1_1, b^1_2, b^1_3\}$ fixed to $\mathbb{B}_1$ and moments expressed in the dual Euler basis associated with the relative rotation tensor $R^B$. This can be inferred from expressions (6.11) and (6.12). Notice that the relative displacement and orientation between the bushing frames is used as opposed to the change in relative position and orientation. Conceptually, this is similar to a spring with a zero resting length. Consequently, $F^B_2$ and $F^B_1$ are non-zero in the neutral position if the bushing frames are not initially coincident as shown in Fig. ??.

\footnote{In OpenSim, the generalized force due to the bushing element $F^B = -F^B_B$ is actually exerted on the bushing frame $\mathbb{B}_2$ and an additional transformation applied to account for the shift from $\mathbb{B}_2$ to $\mathbb{B}_1$. Consequently, the generalized forces acting on the frame $\mathbb{B}_1$ is not equal and opposite to that exerted on the frame $\mathbb{B}_2$ by the bushing element. Rather, the force and moment vectors exerted on $\mathbb{B}_1$ by the bushing element are given by

$$ F^B_1 = -F^B_2, \quad M^B_1 = -M^B_2 + (x^2_B - x^1_B) \times F^B_1. $$  \hspace{1cm} (6.13)

We will refrain from using this convention.}

92
6.2.3 Comments on Experimental Data

A number of notable differences exist between the forces and moments exerted by the bushing element and the experimental stiffness matrix associated with the intervertebral joint. If a set of stiffness data for $K_E$, such as those found in [2, 4], are being used to prescribe the components of $K_B$, then the relative position vectors $\bar{x}_1 - x_1^B$ and $\bar{x}_2 - x_2^B$, and the relationships between $R$ and $R^B$ must first be established. This is to ensure that the increments to the pairs $(F_1, M_1)$ and $(F_2, M_2)$ determined using $K_E$ are equipollent to the increments to the pairs $(F_1^B, M_1^B)$ and $(F_2^B, M_2^B)$ computed using $K_B$. Explicitly, the force-moment pairs should satisfy

\begin{align}
F_1 &= F_1^B, \\
F_2 &= F_2^B,
\end{align}

(6.15)

and

\begin{align}
M_1 &= M_1^B + (\bar{x}_1^B - x_1^B) \times F_1^B, \\
M_2 &= M_2^B + (\bar{x}_2^B - x_2^B) \times F_2^B.
\end{align}

(6.16)

to ensure compatibility between the experimental stiffness matrices $K_B$ and $K_E$.

The simple planar example shown in Fig. 6.4 exemplifies this point further. Here, the bodies are connected by a spring whose line of action passes through the geometric centers. However, when this spring is implemented using a bushing element, the forces are exerted on the bushing frames $B_1$ and $B_2$ resulting in a spurious moment on the geometric centers of the respective bodies. In addition, as the relative position and not the change in relative position between the bushing frames is used, the forces in the neutral posture can be non-zero unless the frames are initially coincident.

6.3 A SpineBushing Element

The simplest method of ensuring that (6.15) and (6.16) are satisfied is to place the bushing frames at the landmark points $x_1^A$ and $x_2^A$ used to determine the elements of $K_E$, and to employ the relative displacement and rotation vectors (from the neutral posture), to compute the ensuing forces and moments. We do precisely this using a SpineBushing function we recently developed in OpenSim. In addition, the SpineBush-ushing element permits the analysis of coupling between the motion and loads along all six degrees of freedom of the spine by the incorporation of non-zero off-diagonal stiffness matrix components.

6.3.1 The Stiffness Matrix of the SpineBushing Element

To increase the tractability of the SpineBushing element, a few simplifications were made to the Cartesian stiffness matrix described in Chapter 5. Often times, \footnote{Recall that in both these studies, the motion and loads at the vertebral centers of geometry ($\approx$ vertebral center of mass) were used to determine the elements of $K_E$.}
Fig. 6.4: A simple two-dimensional example illustrating the importance of using the correct parameters in computing the bushing force. In (a), a spring is used to connect the two blocks $B_1$ and $B_2$ and the forces exerted at the centers measured and used to determine the spring constant $k^E \approx k$. $k^E$ is then (incorrectly) used in a model consisting of the two bodies connected by a bushing element (b). The resulting forces and moments – applied at the bushing frames – result in spurious forces and moments about the geometric center.

one does not have a potential energy function $U$ for the joint. For this reason, the SpineBushing uses the constant stiffness matrix $K^E$, determined experimentally in place of the Hessian $H$ of a potential energy function. Next, the 0D terms were removed. These simplifications reduced the stiffness matrix of the SpineBushing element $K^S_K$, to the $6 \times 6$ matrix:

$$K^S_K = (\mathbf{I}G)^T_K K^E_K (\mathbf{I}G)_K. \quad (6.17)$$

In contrast to the Hessian portion of the Cartesian stiffness matrix $K^J_K$,

$$K^J_K = H^T_K (\mathbf{I}G)_K, \quad (6.18)$$

The motivation for this was as follows: to compute the elements of $^cD^J$, one needs to know the conservative forces and moments exerted by the potential energy function in a given posture. This is not commonly reported. In addition, the existence of the 0D$_{KL}$ matrices necessitates a $6 \times (6N)$ Cartesian stiffness matrix to model the joint. For large systems, such as the human spine, we found this to be computationally prohibitive. Furthermore, the additional accuracy provided by the inclusion of the 0D$_{KL}$ terms were dwarfed by the standard deviations of the elements of $K$ [4].
Fig. 6.5: Schematic of the SpineBushing element computations for the $L_1$-$L_5$ motion segment for the case where the SpineBushing frames $S_1$ and $S_2$ are situated at the frame of origins of the vertebral bodies $F_1$ and $F_2$. The forces and moments due to the element are exerted at the origins of the $S_1$ and $S_2$ frames and are related to the increments in relative motion between the two frames $(\Delta y, \Delta \beta)_S$ using the SpineBushing stiffness matrix $K^S$ (cf. (6.21) and (6.22)).

The matrix $(I\ddot{\mathbf{g}})_K$ defined by

$$(I\ddot{\mathbf{g}})_K = \begin{bmatrix} 1 & 0 \\ 0 & \dddot{\mathbf{g}}_K \\ \end{bmatrix}, \quad \dddot{\mathbf{g}}_{K,j} = \dddot{\mathbf{g}}_K^i \cdot e_j^K,$$

is used to relate $K^E_K$ to $K^S_K$ in (6.17).

6.3.2 The SpineBushing Forces and Moments

We now define the following variables:

$$
\Delta y^S = \begin{bmatrix}
\Delta y^S \cdot e_1^{K-1} \\
\Delta y^S \cdot e_2^{K-1} \\
\Delta y^S \cdot e_3^{K-1} \\
\Delta \beta^S \cdot \dddot{\mathbf{g}}_K^{K,1} \\
\Delta \beta^S \cdot \dddot{\mathbf{g}}_K^{K,2} \\
\Delta \beta^S \cdot \dddot{\mathbf{g}}_K^{K,3}
\end{bmatrix}, \quad F^K_S = \begin{bmatrix}
F^K_S \cdot e_1^{K-1} \\
F^K_S \cdot e_2^{K-1} \\
F^K_S \cdot e_3^{K-1} \\
M^K_S \cdot \dddot{\mathbf{g}}_K^1 \\
M^K_S \cdot \dddot{\mathbf{g}}_K^2 \\
M^K_S \cdot \dddot{\mathbf{g}}_K^3
\end{bmatrix}.
$$

The superscripts $S$ in (6.20) stress that the forces and moments are exerted by the SpineBushing. The generalized force exerted at the SpineBushing frame $S_2$ on the
upper vertebra is then
\[ F^S_K = -K^S \Delta y_S. \] (6.21)

With the help of Newton’s third law, it is easy to see that an equal and opposite force and moment is exerted at the SpineBushing frame \( S_1 \) of the lower vertebra:
\[ F^S_{K-1} = -F^S_K. \] (6.22)

Implicit in this is the assumption that the elements of \( K^E_K \) are determined using the Cartesian components of the forces and displacements, the relative rotations and increments in moments measured in the Euler and dual Euler basis respectively, and the SpineBushing frames \( S_1 \) and \( S_2 \) are situated at the landmark points used to determine the elements of \( K^E_K \). If a different set of basis vectors are used, then appropriate modifications need to be made to the expressions for \( (I\hat{G})_K, \Delta F^S_K, \) and \( \Delta y^S \) to ensure compatibility with \( K^E_K \).

We remark that employing \( \Delta y^S \) as opposed to \( y^S \) in computing the SpineBushing forces and moments is a key feature of the SpineBushing element as this ensures that no forces and moments are exerted at equilibrium.

In the interest of notational simplicity, and without any loss of generality, we will henceforth assume that these landmark points coincide with the geometric centers of the adjacent vertebrae. We also position the frames of origin of the vertebrae \( F_K \) at the vertebral centers of geometry (cf. Fig. 6.5).

6.4 Application

To illustrate the issues raised in the preceding discussion, three simple models of the lumbar spine were built in OpenSim. All three models consisted of the vertebrae \( L_1 \) through \( L_5 \), with adjacent bodies connected using bushing elements and vertebral centers of mass situated at the vertebral geometric centers (Fig. 6.6).

The first, Model 1, features a bushing element with a diagonal stiffness matrix and bushing frames situated coincident with the joint frames. Model 2 utilizes SpineBushing elements with a diagonal stiffness matrix connecting the centers of mass of the upper and lower vertebra while Model 3 is similar to Model 2, but features stiffness matrices with non-zero off-diagonal elements. To clarify the distinctions between the bushing elements used in the three models, no actuators were included to model the lumbar muscles.

Similar to Chapter 5, we estimated the elements of the stiffness matrix with off-diagonal elements \( K_{OD} \) based on the experimentally computed stiffness matrix values
Fig. 6.6: Schematic of the three models used to illustrate the application of the SpineBushing element. Model 1 and Model 2 both feature diagonal stiffness matrices but differ in the point of application of the forces and moments (red and blue points). Conversely, the SpineBushing elements in Model 2 and Model 3 exert their forces and moments at the centers of mass of the adjacent bodies, but differ due to the stiffness matrix used (circular and square points). Also shown is the set of body fixed basis vectors attached to the L5 vertebra.

reported in [4]. That is,

\[
K_{OD} = \begin{bmatrix}
500000 & 0 & 0 & 0 & 0 & 11000 \\
0 & 2500000 & 0 & 0 & 0 & -5000 \\
0 & 0 & 500000 & -12000 & 13500 & 0 \\
0 & 0 & -12000 & 400 & -300 & 0 \\
0 & 0 & 13500 & -300 & 850 & 0 \\
11000 & -5000 & 0 & 0 & 0 & 600
\end{bmatrix}
\]  \tag{6.23}

with units given in N, m and rad. For ease of comparison, the non-zero components

\footnote{The coefficients associated with their reported stiffness matrices were determined by performing highly controlled motion in one direction, measuring the ensuing forces and moments and then using a least-squares fit to the experimental data by the method specified in [5].}
of $K_D$ were set similar to $K_{OD}$’s diagonal elements,

$$K_D = \begin{bmatrix} 500000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2500000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 500000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 400 & 0 & 0 \\ 0 & 0 & 0 & 0 & 850 & 0 \\ 0 & 0 & 0 & 0 & 0 & 600 \end{bmatrix}, \quad (6.24)$$

and the same stiffness matrix used for each pair of vertebra. These are detailed in Table 6.1.

<table>
<thead>
<tr>
<th>Model</th>
<th>Connection to Bushing Matrix</th>
<th>Equation</th>
<th>Applied At</th>
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<tbody>
<tr>
<td>1</td>
<td>$K_{R}^S = K_D$</td>
<td>-</td>
<td>Joint Frames</td>
</tr>
<tr>
<td>2</td>
<td>$K_{K}^S = (I\tilde{G})<em>{K}^{T}K</em>{K}^E(I\tilde{G})<em>{K}$, $K</em>{K}^E = K_D$</td>
<td>6.17</td>
<td>Centers of Mass</td>
</tr>
<tr>
<td>3</td>
<td>$K_{K}^S = (I\tilde{G})<em>{K}^{T}K</em>{K}^E(I\tilde{G})<em>{K}$, $K</em>{K}^E = K_{OD}$</td>
<td>6.17</td>
<td>Centers of Mass</td>
</tr>
</tbody>
</table>

**Table 6.1:** Summary of the three models used to illustrate the differences caused by (1) applying the stiffness matrix at the joint connecting the bodies instead of at the vertebral centers of mass, and (2), the incorporation of off-diagonal elements. In all cases, the forces and moments exerted on the centers of mass, $\bar{X}_{K}$ of the upper vertebral body were reported.

A number of different motions were examined. Pure sinusoidal translations in the anterior-posterior ($E_1$) and axial ($E_2$) directions were applied to Models 1 and 2 to reveal the importance of applying the bushing forces at the correct point. We also looked at two separate flexion-extension motions: one involving pure flexion-extension and one coupled with axial and anterior-posterior translation (cf. Fig. 6.7). In both cases, the magnitude of the rotation of each lumbar segment was prescribed as a linear function of the total rotation [48] which, in turn, was limited to $10^\circ$. The coefficients of this linear function were determined from Wong et al.’s videofluoroscopic imaging study (cf. Figs. 10 and 11 of [50]). Table 6.2 summarizes the motions studied.

### 6.5 Results

The additional moment experienced by the vertebrae due to applying the stiffness matrix at the joint was examined by subjecting Models 1 and 2 to pure sinusoidal translations in the anterior-posterior and axial directions. As expected, both models experienced similar forces in the axial and anterior-posterior directions but an additional moment was exerted on the vertebral bodies of Model 1. This was a conse-
Table 6.2: The four motions used to illustrate the differences caused by (1) applying the stiffness matrix at the joint connecting the bodies instead of at the vertebral centers of mass, and (2), the incorporation of off-diagonal elements. Only small amplitude motions were tested to agree with the values used by [61] and [4] in their experimental determination of the stiffness matrices elements.

<table>
<thead>
<tr>
<th>Joint</th>
<th>$L_5/S_1$</th>
<th>$L_4/L_5$</th>
<th>$L_3/L_4$</th>
<th>$L_2/L_3$</th>
<th>$L_1/L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure Translation along $E_1$</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Amplitude (mm)</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Pure Translation along $E_2$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>Amplitude (mm)</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>Pure Flexion-Extension</td>
<td>1.25</td>
<td>1.85</td>
<td>2.04</td>
<td>2.31</td>
<td>2.55</td>
</tr>
<tr>
<td>Coupled Flexion-Extension</td>
<td>1.25</td>
<td>1.85</td>
<td>2.04</td>
<td>2.31</td>
<td>2.55</td>
</tr>
<tr>
<td>Flexion (°)</td>
<td>1.25</td>
<td>1.85</td>
<td>2.04</td>
<td>2.31</td>
<td>2.55</td>
</tr>
<tr>
<td>$e_1^k$ Translation (mm)</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>$e_2^k$ Translation (mm)</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The consequence of applying the forces at the joint frames rather than the centers of mass (Fig. 6.8). We stress that this moment is not due to the coupling between the different degrees of freedom evidenced in the intervertebral joint, but rather a consequence of an erroneous application of the bushing element.

To reveal the contribution of the coupling terms in the SpineBushing stiffness matrices, we compared Models 2 and 3. Under pure lumbar flexion-extension rotation, the off-diagonal terms in Model 3’s SpineBushing resulted in non-zero forces in the axial and anterior-posterior directions not present in Model 2. This is evident in Fig 6.9. Coupling the flexion-extension motion to the other degrees of freedom (cf. Table 6.2) also resulted in differing forces and moments. More specifically, larger flexion-extension moments and anterior-posterior shear forces but smaller axial forces were exerted at the centers of mass of the vertebral bodies in Model 3 compared to Model 2 (Fig. 6.10). In both figures, only the joint loads exerted by the joint on the upper vertebra are shown as equal and opposite loads are applied to the lower bodies (equations (6.21) and (6.22)). In addition, these forces and moments are expressed in the frame of reference affixed to the individual vertebra.
Fig. 6.7: Schematic showing flexion-extension of the lumbar spine and the accompanying motion of each vertebra about its joint $IVJ_K$. The lumbar column is initially in the neutral position shown in (a). It is then subjected to two different sinusoidal flexion-extension motions. The first involves pure flexion-extension (b) while the second flexion-extension motion is accompanied by axial and anterior-posterior displacements (c). The amplitudes of the motion are given in Table 6.2.

6.6 Conclusion

In this chapter, several concepts novel to this dissertation were discussed. We described the application and load computations of bushing elements sometimes used to model the intervertebral joint in Section 6.2. We also noted how it was important to ensure that the bushing frame locations and bushing stiffness matrices were chosen such that the resulting forces and moments agreed with those that would be measured experimentally to ensure compatibility between the stiffness matrices $K^B$ and $K$ (cf. equations (6.15) and (6.16) in Section 6.2.3). To remedy some of the difficulties present in using typical bushing elements to accurately model the intervertebral joint, we introduced and described a SpineBushing element in Section 6.3. This SpineBushing element featured a stiffness matrix

$$K^S_K = (I\mathbf{G}_K)^T J^E_K (I\mathbf{G}_K),$$

that was a simplified version of the Cartesian stiffness matrix $^5\mathbf{K}^J$ elaborated upon in Chapter 5. The SpineBushing element also presents two significant improvements over existing bushing functions. First, we ensured that the bushing forces and moments as well as the relative motion were measured using the same points on the vertebral bodies as those used to compute the experimental stiffness matrix. This eliminated any artificial moments due to the point of application of the forces as shown in Fig. 6.4. For brevity, we assumed that these points were coincident with the vertebral
Fig. 6.8: *Spurious flexion-extension moment at the center of mass of the upper vertebra in Model 1 caused by the pure translations in the anterior-posterior and axial directions. The additional moment is due to the cross product term specified in equation (6.16).*

centers of mass. Second, we enhanced the conventional bushing element by allowing for the incorporation of stiffness matrices with off-diagonal components. This was a straightforward yet highly valuable addition as it permits the signature load-motion coupling between the various degrees of freedom of the intervertebral joint.

To examine the changes generated by these two improvements, we looked at three different models and four different motion sequences in Section 6.4. The results, visible in Figs. 6.8, 6.9, and 6.10 of Section 6.5 exemplify the necessity of an element such as the SpineBushing for accurately modeling and analyzing spinal kinematics and dynamics. Finally, we note that as our intention was merely to introduce the SpineBushing element to the biomedical community, the analysis was purposely kept elementary to stress that the differences in the bushing forces and moments were due solely to the two previously discussed enhancements.
Fig. 6.9: Forces and moments exerted by SpineBushing elements with diagonal ($K_D$) and dense ($K_{OD}$) stiffness matrices due to the pure flexion-extension motion specified in Table 6.2. The forces shown are exerted on the centers of mass of the upper vertebra, and computed using equation (6.21). In both cases, Model 3 experienced additional anterior-posterior and axial forces due to the off-diagonal terms not present in Model 2’s stiffness matrix.
Fig. 6.10: Forces and moments exerted by SpineBushing elements with diagonal ($K_D$) and dense ($K_{OD}$) stiffness matrices due to the coupled flexion-extension motion specified in Table 6.2. The forces and moments shown are exerted on the centers of mass of the upper vertebra, and computed using equation (6.21).
Chapter 7

Closing Remarks

In this thesis, it was shown how various representations for Cartesian stiffness matrices $\mathbf{K}$ can be obtained for a wide range of pairs of resultant forces and moments. The selection of the pair of forces and moments is not arbitrary; rather it is related by a work argument to the functional representation of the potential energy function. For a single rigid body, (see (3.3) and (3.7)):

\[(\mathbf{F}, \mathbf{M}_O) \leftrightarrow U_1(\nu^1, \nu^2, \nu^3, x_1, x_2, x_3),\]
\[(\mathbf{F}, \mathbf{M}) \leftrightarrow U_2(\nu^1, \nu^2, \nu^3, X_1, X_2, X_3),\]
\[(\mathbf{F}, \mathbf{M}_O) \leftrightarrow U_3(\nu^1, \nu^2, \nu^3, x_{A1}, x_{A2}, x_{A3}),\]
\[(\mathbf{F}, \mathbf{M}_A) \leftrightarrow U_4(\nu^1, \nu^2, \nu^3, X_{A1}, X_{A2}, X_{A3}),\]

while, for a system of rigid bodies,

\[(\mathbf{F}_1, \mathbf{M}_{1,O}, \ldots, \mathbf{F}_N, \mathbf{M}_{N,O}) \leftrightarrow U_1(\nu^1_1, \nu^2_1, \ldots, \nu^2_N, \nu^3_1, x_{11}, x_{12}, \ldots, x_{21}, x_{31}),\]
\[(\mathbf{F}_1, \mathbf{M}_1, \ldots, \mathbf{F}_N, \mathbf{M}_N) \leftrightarrow U_2(\nu^1_1, \nu^2_1, \ldots, \nu^2_N, \nu^3_1, X_{11}, X_{12}, \ldots, X_{21}, X_{31}),\]
\[(\mathbf{F}_1, \mathbf{M}_{1,O}, \ldots, \mathbf{F}_N, \mathbf{M}_{N,O}) \leftrightarrow U_3(\nu^1_1, \nu^2_1, \ldots, \nu^2_N, \nu^3_1, x_{A1}, x_{A2}, \ldots, x_{A2}, x_{A3}),\]
\[(\mathbf{F}_1, \mathbf{M}_{1,A}, \ldots, \mathbf{F}_N, \mathbf{M}_{N,A}) \leftrightarrow U_4(\nu^1_1, \nu^2_1, \ldots, \nu^2_N, \nu^3_1, X_{A1}, X_{A2}, \ldots, X_{A2}, X_{A3}).\]

For the case of rigid bodies connected in series, as in the case of the lumbar spine, it is advantageous to write the potential energy function in terms of the relative
displacements and angles of rotation,

\[ (F_1, M_1, \ldots, F_N, M_N) \leftrightarrow U_5(\beta_1^1, \beta_1^2, \ldots, \beta_N^1, \beta_N^2, Y_1^1, Y_1^2, \ldots, Y_2^N, Y_3^N). \]

Constrained systems, whereby the bodies are connected at a coincident point can also be addressed by a judicious selection of the point \( x_A \) on the bodies such that the resulting system can be written as a function of a fewer variables. Taylor series expansions of any of these conservative forces and moments can then be used to derive stiffness matrices relating changes in the displacement and rotations to increments in the forces and moments.

In experimental situations, it is frequently easier to determine \( F \cdot E_i \) and \( M \cdot E_i \) using a load cell. Thus, the Cartesian stiffness matrices (e.g., \( ^cK \) and \( ^\hat{c}K \)) for both the single and multibody rigid body system may be easier to estimate than the stiffness matrices \( ^mK \) and \( ^m\hat{K} \). We also remark that the use of the dual Euler basis to calculate \( ^cK \) was an essential component of the formulation. Should a quaternion or Euler-Rodrigues symmetric parameter representation of the rotation be used, then it is possible to extend the formulation presented in this paper to that case. The formulation would use representations for the conservative moments that can be found in [43].

The application of the stiffness matrix in quantifying the dynamics of the motion are wide and diverse, ranging from parallel linked manipulators to the intervertebral joints connecting the vertebrae of the spinal column. In the latter, it is particularly useful if the potential energy function \( U = U_5 \) can be written as the sum of the potential energy functions associated with each joint:

\[ U_5(\beta_1^1, \beta_1^2, \ldots, Y_2^N, Y_3^N) = U_5(\beta_1^1, \beta_1^2, \ldots, Y_2^1, Y_3^1) + \ldots + U_5(\beta_5^1, \beta_5^2, \ldots, Y_2^5, Y_3^5). \]

The Cartesian stiffness matrix associated with each joint \( ^cK_J \), can then be determined using a Taylor series expansion of the conservative forces and moments such that

\[ \Delta F_J^y = -^cK_J^y \Delta y. \]

In the absence of a suitable potential energy function \( U \), it is possible to estimate the Hessian term \( H \) of \( U \) by a stiffness matrix determined experimentally using infinitesimal uniaxial motion \( K^E \). Likewise, the \( D \) matrix can be approximated by performing a Taylor series expansion of the conservative forces and moments exerted by the joint in the neutral posture.

Often times however, only the Hessian portion is reported. Consequently, one is unable to compute \( D \) and the Cartesian stiffness matrix has to be approximated as

\[ ^cK_J \approx (\overline{IG})_K (K^E_J) (\overline{IG})_K. \]

In musculoskeletal software platforms, the stiffness matrix is frequently assumed to
be synonymous with a *bushing* element. However, the subtle differences between the two yield significantly different forces and moments unless careful attention is made to ensure its proper application. Unfortunately, this is not always the case.

To address the aforementioned issues with bushing elements, we introduced a SpineBushing element that uses a simplified version of the Cartesian stiffness matrix associated with the intervertebral joint. This element was constructed in the open-source software platform OpenSim and is downloadable from

http://simtk.org/home/spinebushing.

The SpineBushing features two modifications to OpenSim’s existing bushing force function. Specifically, the loads and increments in motion were measured and applied at the same points used to measure the components of the experimental stiffness matrix. This eliminated any extraneous moments not due to the joint action itself. The ability to study coupling between the degrees of freedom was also added by the incorporation of stiffness matrices with off-diagonal components. These two improvements were forces and moments that better reflect those exerted by the joint, even for small motions. Consequently, we anticipate that its use in spinal models with greater complexity will lead to physiologically more accurate results.
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110


