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ABSTRACT

We present a theoretical formalism which allows the generation of a large class of exact Vlasov-Maxwell equilibria with sheared magnetic fields. All quantities are assumed to vary only in one spatial direction, x, and the magnetic field has components only in the y and z directions. The Vlasov equations are solved by making the distribution functions depend only on constants of the motion. The Maxwell equations are then reduced to finding the motion of a pseudo-particle in a two dimensional potential. Three examples corresponding to sheet-like, sheath-like, and wave-like equilibria are presented.

II. GENERAL FORMALISM

Since the electric field is taken to be zero, we require exact charge neutrality:

\[ N_i(x) = N_e(x) \]  \hspace{1cm} (1)

The magnetic field can be derived from a vector potential, \( \mathbf{A} \), and

\[ B_y = \frac{d A_y}{dx} \quad B_z = -\frac{d A_z}{dx} \]  \hspace{1cm} (2)

The Maxwell's equations for the magnetic field become

\[ \frac{d^2 A_y}{dx^2} = \frac{4 \pi J_y}{c} \]  \hspace{1cm} (3)

is on the order of an ion gyroradius. In this paper we present a theoretical formalism which allows us to generate a large class of exact Vlasov-Maxwell equilibria with sheared magnetic fields.

For simplicity we consider a situation in which all quantities vary only in the x direction, and the magnetic field has components \( B_y \) and \( B_z \) in the y and z directions. The equilibrium is characterized by a zero electric field. To find a self-consistent equilibrium, we must solve the coupled Vlasov-Maxwell equations.

The Vlasov equations are easily satisfied by making the distribution functions depend only on constants of the motion. Maxwell's equations are then a coupled set of nonlinear integro-differential equations. We will find a large, but not complete, class of solutions to these equations.
\[ \frac{d^2 A_s}{dx^2} = -\frac{4\beta_s}{c} \tag{4} \]

where \( J(x) \) is the current density.

The constants of the motion for particles of species \( s \) (\( s = i \) or \( e \)), are the Hamiltonian

\[ H_s = \frac{m_s}{2} \left( v_x^2 + v_y^2 + v_z^2 \right), \tag{5} \]

and the \( y \) and \( z \) components of momentum,

\[ P_{ys} = m_s v_y + \frac{q_s A_y}{c}, \tag{6} \]

\[ P_{zs} = m_s v_z + \frac{q_s A_z}{c}, \tag{7} \]

where \( m_s, q_s \) are the mass and charge of particles of species \( s \).

In order to satisfy the Vlasov equation for species \( s \), the distribution function must be a function of the constants of the motion. We assume it is of the form

\[ f_s = e^{-\beta_s H} g_s(P_{ys}, P_{zs}) \tag{8} \]

where \( \beta_s \) are constants and \( g_s \) are functions to be determined. This form for \( f_s \) is arbitrary but is motivated by physical reasonableness and by the considerable mathematical simplicity which follows from the chosen dependence on \( H_s \).

The number densities of ions and electrons are easily seen to be given by

\[ N_i(x) = \frac{1}{m_i^2} \sqrt{\frac{2\pi}{m_i^2 B_i}} \int e^{-\frac{\beta_i}{2m_i} \left( \frac{P_y}{c} \right)^2 + \left( \frac{P_z}{c} \right)^2} \times g_i(P_y, P_z) dP_y dP_z. \tag{9} \]

The current density can be written

\[ J_{ys} = \frac{q_i}{m_i} \sqrt{\frac{2\pi}{m_i^2 B_i}} \int \left( \frac{P_y - q_i A_y}{c} \right) e^{-\frac{\beta_i}{2m_i} \left( \frac{P_y}{c} \right)^2 + \left( \frac{P_z}{c} \right)^2} \times g_i(P_y, P_z) dP_y dP_z, \tag{10} \]

\[ J_{zs} = \frac{q_i}{m_i} \sqrt{\frac{2\pi}{m_i^2 B_i}} \int \left( \frac{P_z - q_i A_z}{c} \right) e^{-\frac{\beta_i}{2m_i} \left( \frac{P_y}{c} \right)^2 + \left( \frac{P_z}{c} \right)^2} \times g_i(P_y, P_z) dP_y dP_z. \tag{11} \]

Let us observe that

\[ J_{ys} = \frac{c}{\beta_i} \frac{\partial N_i}{\partial y}, \tag{12} \]

\[ J_{zs} = \frac{c}{\beta_i} \frac{\partial N_i}{\partial z}. \tag{12} \]

Let us now assume that \( N_i \) and \( N_e \) are equal not only as functions of \( x \) but also as functions of \( A_y \) and \( A_z \); i.e.,

\[ N_i(A_y, A_z) = N_e(A_y, A_z). \tag{13} \]
This is a restrictive assumption but despite it we are able to find many equilibria. The total current can now be written as

\[ J_y = c \left( \frac{1}{B_1} + \frac{1}{B_e} \right) \frac{3N_1}{3A_y} \]
\[ J_z = c \left( \frac{1}{B_1} + \frac{1}{B_e} \right) \frac{3N_1}{3A_z} \]  \hspace{1cm} (14)

Note, by the way, that the ratio of ion current to electron current is

\[ \frac{|J_i|}{|J_e|} = \frac{B_e}{B_1} \]  \hspace{1cm} (15)

Since \( B_e \) is the inverse temperature, Equation (15) is what we would expect to be the case.

If we define

\[ U(A_y, A_z) = 4\pi \left( \frac{1}{B_1} + \frac{1}{B_e} \right) N_1 \] \hspace{1cm} (16)

then Equations (3) and (4) become

\[ \frac{d^2 A_y}{dx^2} = -\frac{3U}{3A_y} \] \hspace{1cm} \[ \frac{d^2 A_z}{dx^2} = -\frac{3U}{3A_z} \]  \hspace{1cm} (17)

These are just the Hamiltonian equations for a pseudo-particle with coordinates \((A_y, A_z)\) moving in the potential \(U(A_y, A_z)\). Equations (17) can be derived from the Hamiltonian

\[ H_A = \frac{P_y^2 + P_z^2}{2} + U(A_y, A_z) \] \hspace{1cm} (18)

Since

\[ B_z = \frac{dA_y}{dx} = \frac{3H_A}{3P_y} = P_y \] \hspace{1cm} \[ -B_y = \frac{dA_z}{dx} = \frac{3H_A}{3P_z} = P_z \]  \hspace{1cm} (19)

we note that the constancy of \( H_A \) in \( x \) is just the equation of total pressure balance.

We have reduced the equations for the fields to a two-dimensional potential problem. Typically, however, instead of knowing the distribution functions from which we can derive the fields we have some idea of what the fields are and want to find the distribution functions. Thus, usually we know the fields and can, by solving Equations (2), find the trajectory of the pseudo-particle in the \((A_y, A_z)\) plane. We then want to find a potential, \(U(A_y, A_z)\), which will produce this trajectory, a problem which, in many cases, is easy to solve qualitatively. Given the potential, \(U(A_y, A_z)\), we must then find the distribution functions. Using Equations (9), (13), and (16), we find that the distribution functions satisfy

\[ \frac{1}{m_e} \sqrt{\frac{2\pi}{m_e \beta_s}} \int \frac{\beta_s}{e} \left[ \left( \frac{q_s P_y}{c} \right)^2 + \left( \frac{q_s P_z}{c} \right)^2 \right] g_s(P_y, P_z) W y dP_y = \frac{\beta_s \beta_1 U(A_y, A_z)}{4\pi (B_e - B_1)} \] \hspace{1cm} (20)

Equations (20) are integral equations for \( g_s \); the distribution functions are then given by Equation (8).

Once the trajectory of the pseudo-particle, i.e., the fields, is known, the potential, \(U(A_y, A_z)\), can be changed, without changing the magnetic fields, literally on any set which does not intersect
the trajectory. However, the distribution functions, given by Equation (20), depend on $U(A_y, A_z)$ for all $(A_y, A_z)$ and thus, there are arbitrarily many distribution functions which produce a given set of fields. This arbitrariness in the potential can be used to produce a variety of features, such as asymmetric momentum distributions, in the distribution functions.

Note also that an overall constant can be added to the potential without changing the fields. The freedom to add this constant must sometimes be used to insure that the distribution functions, which are solutions of Equation (20), are everywhere non-negative. For convenience, the potential can be translated arbitrarily in the $(A_y, A_z)$ plane without changing the fields.

Motivated by the much wider class of situations to which it might be applied, we have attempted to extend this formalism to cylindrical geometry, but have found that a straightforward extension isn't possible.

III. EXAMPLES

In this section we give three examples, each of which illustrates a different way of solving Equation (20).

(a) Unsheared Sheath

Consider a situation in which the magnetic field is unidirectional; we can take $A_z = 0$. Equation (20) then becomes

$$\frac{2\pi}{m_s^2 g_s} \int e^{2\pi i \left( \frac{p_y}{q_y} \right)} g_s(p_y) dp_y = \frac{\beta_s \beta_1}{4\pi (\beta_e + \beta_1)} U(A_y),$$

where we have assumed

$$g_s = g_s(p_y),$$

and

$$U = U(A_y).$$

Let us now assume that

$$U(A_y) = D e^{-\gamma A_y^2}$$

where $D$ and $\gamma$ are constants, so that the potential now resembles a "hill". We can easily choose the velocity (i.e., magnetic field) at $-\infty$ to be such that the pseudo-particle just manages to roll to the top of the hill; i.e., we choose

$$B_z(-\infty) = \frac{dA_y(-\infty)}{dx} = \sqrt{2D} = B_0.$$ 

Thus, the magnetic field and, from the constancy in $x$ of $H_A$ (see Eq. (18)), particle density are as shown in Figure 1; this is a sheath.

To find the distribution function we must solve Equation (21) with $U(A_y)$ given by Equation (24). We Fourier transform Equation (21) and, denoting transformed functions by a tilde, obtain

$$\frac{2\pi}{m_s^2 g_s} \sqrt{\frac{2\pi}{B_s}} \tilde{g}_s(w) e^{-2\pi i \frac{2\beta_s}{B_s} w} = \frac{\beta_s \beta_1 D}{(\beta_e + \beta_1) 8\pi} e^{-\frac{w^2}{4\gamma}}.$$ 

Solving for $\tilde{g}_s$, we find

$$\tilde{g}_s(w) = \frac{8\beta_1 D m_s^2}{16\pi^3 (\beta_e + \beta_1)} \sqrt{\frac{\beta_s}{2\gamma m_s}} e^{-\frac{w^2}{4\gamma} 2\beta_s}.$$
In order to be able to invert the Fourier transform we must require

\[ \gamma < \frac{n \pi}{2 \pi} \left( \frac{\beta_e \beta_i}{2\pi \varepsilon s} \right) \]  

(28)

This simply says that if the sheath is too narrow then charge neutrality cannot be maintained. If we let

\[ \delta_s = \frac{I}{I_s} - \frac{m_e}{p_s} \]  

(29)

then the distribution functions are given by

\[ f_e(H, P_s) = \frac{m_e 2 \pi x_0}{4\pi^2 \sqrt{-4\pi \delta_s - \beta_s H^2}} \]  

(30)

where \( N_0 \) is the density at \( x = \infty \).

(b) Sheared Sheet

Let us assume that

\[ U(A_y, A_z) = D e^{\gamma A_y A_z} \]  

(31)

where \( D \) and \( \gamma \) are constants. Then Equation (17) becomes

\[ \frac{d^2 A_y}{dx^2} = -D e^{\gamma A_y A_z} \]  

(32)

\[ \frac{d^2 A_z}{dx^2} = -D e^{\gamma A_y A_z} \]  

(33)

We observe that

\[ \frac{d^2 A_y}{dx^2} = \frac{d^2 A_z}{dx^2} \]  

(34)

Equation (34) can be immediately integrated twice to give

\[ A_y = A_z + E_1 x + E_2 \]  

(35)

where \( E_1 \) and \( E_2 \) are constants. Combining Equations (33) and (35) gives

\[ \frac{d^2 A_y}{dx^2} = -D' e^{2 \gamma A_z + \gamma E_1 x} \]  

(36)

where

\[ D' = D e^{\gamma E_2} \]  

(37)

If we define

\[ G = 2 \gamma A_z + \gamma E_1 x \]  

(38)

then Equation (36) becomes

\[ \frac{d^2 G}{dx^2} = -2 \gamma D' e^{G} \]  

(39)

Multiplying by \( \frac{dG}{dx} \) and integrating gives

\[ \left( \frac{dG}{dx} \right)^2 = -4 \gamma D' e^G + E_3 \]  

(40)

where \( E_3 \) is a constant. Equation (40) can be easily integrated to give
Using Equation (38) we find

$$A_z(x) = -\frac{1}{\gamma} \ln \left( \cosh \frac{\sqrt{E_3^2}}{2} \right) + \frac{1}{2\gamma} \ln \left( \frac{-E_3}{4\gamma D^2} \right) - \frac{E_3 x}{2} .$$

Combining Equations (39) and (42), we get

$$A_y(x) = -\frac{1}{\gamma} \ln \left( \cosh \frac{\sqrt{E_3^2}}{2} \right) + \frac{E_3 x}{2} + E_2 + \frac{1}{2\gamma} \ln \left( \frac{-E_3}{4\gamma D^2} \right) .$$

If we require

$$B_y(\pm) = 0 \quad (44)$$

$$B_y(\pm) = B_0$$

then, dropping additive constants, we find

$$A_z(x) = -\frac{1}{\gamma} \ln \left( \cosh \frac{\gamma B_0 x}{2} \right) - \frac{B_0 x}{2} \quad (45)$$

$$A_y(x) = -\frac{1}{\gamma} \ln \left( \cosh \frac{\gamma B_0 x}{2} \right) + \frac{B_0 x}{2} \quad (46)$$

The trajectory of the pseudo-particle is shown in Fig. 2. We can now find the magnetic fields from Equation (2);

$$B_z = \frac{B_0}{2} - \frac{B_0}{2 \tanh \frac{\gamma B_0 x}{2}} \quad (47)$$

$$B_y = \frac{B_0}{2} + \frac{B_0}{2 \tanh \frac{\gamma B_0 x}{2}} .$$

In order to find the distribution functions, we must solve Equation (20) with the potential given by Equation (31). Although we cannot use Fourier transforms in this case, the solution is easily seen to be, by inspection,

$$g_0(p_y, p_z) = \frac{\beta e^2 m_0}{4\pi (\beta_0 + \beta_1)} \frac{m_0 \beta_0}{m_1} \frac{\gamma^2 e^2 m_0}{\beta^3} \frac{\gamma e(p_0 + p_z)}{Q} . (49)$$

The distribution function is now given by Equation (8). The number density and magnetic fields for this equilibrium are shown in Figure 3. This is a plasma sheet in a sheared magnetic field.

(c) Wave-like Solution

In our previous examples the pseudo-particle's trajectory went to infinity. If the potential, \( U(A_y, A_z) \), increases as \( A_y, A_z \) go to infinity then the pseudo-particle will be confined and periodic motion can result. Thus, let us assume that the potential is

$$U(A_y, A_z) = D_1 + \frac{D_2}{2} (A_y^2 + A_z^2) \quad (50)$$

Equation (17) is then

$$\frac{d^2 A_y}{dx^2} = -D_2 A_y \quad (51)$$

$$\frac{d^2 A_z}{dx^2} = -D_2 A_z . \quad (52)$$
The solutions of Equations (52) and (53) are clearly

\[ A_y = j_0 \sin(\sqrt{2}x + \delta_1) \]

(53)

\[ A_z = j_1 \sin(\sqrt{2}x + \delta_2) \]

(54)

where \( \delta_0, \delta_1, \delta_2 \) are real constants. The magnetic fields are

\[ B_z = \sqrt{2} A_j \cos(\sqrt{2}x + \delta_1) \]

(55)

\[ B_y = -\sqrt{2} A_j \cos(\sqrt{2}x + \delta_2) \]

(56)

We have found a stationary wave solution. By transforming to a moving frame of reference, so that the magnetic field becomes both an electric and a magnetic field, we produce a travelling electromagnetic wave that is an exact solution of the Vlasov-Maxwell equations. (Note that by choosing a potential, \( U(A_y, A_z) \), which depends on higher powers of \( A_y \) and \( A_z \) we could produce waves with nonsinusoidal shapes.)

We can solve Equation (20) for the distribution function by inspection, but we choose instead to illustrate another technique.

Note that

\[ -\frac{\beta_e}{2m_e} \left( \frac{q_x A_y}{c} \right)^n = \sum_m \frac{\beta_e}{2m_e} H_m \left( \sqrt{2\beta_s \frac{A_y}{c}} \right) \frac{1}{m!} \left( \frac{\beta_s A_y}{2m_e} \right)^m \]

(57)

where \( H_n \) is the \( n \)th hermite polynomial. Using expansion (57) in both variables in Equation (20), we find

\[ \frac{\beta_e}{2m_e} \left( \frac{q_x A_y}{c} \right)^n \sum_m \frac{\beta_e}{2m_e} \left( \sqrt{2\beta_s \frac{A_y}{c}} \right) \frac{1}{m!} \left( \frac{\beta_s A_y}{2m_e} \right)^m \sum_{m,n} \frac{1}{m!n!} \left( \frac{\beta_s}{2m_e} \right)^{m+n} \]

(58)

Let us also expand \( g_s \) as

\[ g_s(P_y, P_z) = \sum_{k,l} C(s) H_k \left( \sqrt{2\beta_s \frac{P_y}{m_e}} \right) H_l \left( \sqrt{2\beta_s \frac{P_z}{m_e}} \right) \]

(59)

Using Equation (59) in Equation (58) we find

\[ U(A_y, A_z) = \frac{4\pi^2 \beta_e}{\beta_e \beta_i m_2} \sqrt{2\pi} \sum_{m,n} \left( \frac{2\beta_s}{m_2} \right)^{m+n} \]

(60)

This equation determines \( C(s) \) in terms of the \( (m,n) \) coefficient of the Taylor series for \( U(A_y, A_z) \). Equation (60) is particularly useful when, as is the case in Equation (50), \( U(A_y, A_z) \) is a polynomial. Thus Equations (50) and (60) yield:
Inserting the expressions for the hermite polynomials, we find

\[ g_s(P_y, P_z) = \frac{\beta_s \beta_1 m_s^2 \sqrt{m_s} \sqrt{m_s}}{2\pi (e_1 + \beta_s) \beta_s q_s^2} \sqrt{D_1 + \frac{m_e c^2}{\beta_s q_s^2} D_2 (P_y^2 + P_z^2 - 1)} . \]  

Requiring \( g_s \geq 0 \) implies

\[ D_1 \geq D_2 \max_s \left( \frac{m_e c^2}{\beta_s q_s^2} \right) . \]  

Equation (64) simply says that there must be enough particles present to produce the required currents.

The hermite polynomial expansion can be used to solve Equation (20) whenever the potential can be expanded in a convergent power series; in fact, our second example could have been solved in this fashion.

We could easily construct other examples of Vlasov-Maxwell equilibria. Because of the intuitive nature of two dimensional potential problems, choosing a potential, \( U(A_y, A_z) \), that will produce the desired magnetic fields is generally easy, even though simple analytic solutions of Equations (3) and (4) do not, in general, exist. The solution of Equation (20) for the distribution functions is more difficult, but, if the potential, \( U(A_y, A_z) \), can be chosen to be a real analytic function the hermite polynomial expansion method can be used to find the distribution functions.
NOTES AND REFERENCES

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FIGURE CAPTIONS

Figure 1: Ratio of magnetic field to maximum magnetic field, $B/B_0$, and ratio of particle density to maximum particle density, $N/N_0$, as a function of $x$ for the unsheared sheath of section III(a). We have taken $\gamma = .25$ in Equation (24), with $A_y(0) = -2$.

Figure 2: Trajectory of the pseudoparticle with coordinates $(A_y, A_z)$ as given by Equations (45) and (46). The components of velocity of the pseudoparticle are related to the magnetic field by Equation (2).

Figure 3: Ratio of $B_z$ to maximum $B_z$, $B_z/B_0$, ratio of $B_y$ to maximum $B_y$, $B_y/B_0$, and ratio of particle density to maximum particle density, $N/N_0$, as a function of $x$ for the equilibrium given by Equations (47), (48), and (49). In Equations (47) and (48), we have taken $\gamma B_0 = 2$. As can be seen, this corresponds to a sheared sheet.
Fig. 1

\[ \frac{B}{B_0} \]

\[ \frac{N}{N_0} \]
Fig. 2
Fig. 3
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